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Abstract: The Fibonacci differential equation is defined with analogy from the Fibonacci difference equation. The linear second order differential equation is solved under suitable initial conditions. The solutions constitute spirals in polar coordinates. The properties of the spirals with respect to the Fibonacci numbers and the differences between the new spirals and classical spirals are discussed.

1 Introduction

The Fibonacci sequence is one of the oldest and most popular sequences in applied mathematics. Apart from the beauty of the mathematical properties of this sequence, it has been applied extensively to understand some natural phenomena also [1]. One of the important properties of the sequence is that, the ratio of the $n+1$'th term to the n'th term approaches the so-called golden ratio [2]. Usually, a rectangle with its side ratio being golden ratio is taken as a reference, a quarter circular arc is drawn using the inscribed square, then another quarter circular arc with a radius which is obtained by dividing the initial radius by the golden ratio is drawn. The process is repeated and a spiral is formed with the original rectangle being the envelope of it. See the web sites [3, 4] in the references for applications of this process to storms, seashells, galaxies, flower heads, human body, aesthetics and historical architectural designs.

In this work, the Fibonacci difference equation is taken as a reference point to generate the Fibonacci difference derivative equation. The difference derivative equation leads to the associated differential equation by analogy which is named in this work as the Fibonacci differential equation. Under suitable initial conditions inspired from the sequence, the differential equation is solved. At the constant angle steps calculated by employment of the golden ratio, the spiral solution gives the Fibonacci numbers with small deviations. The differential equation considered here is different than the variable coefficient second order differential equations that satisfy Fibonacci polynomials [5, 6]. Fibonacci polynomials are derived from a recurrence relation of a linear polynomial with coefficients being the Fibonacci numbers

$$F_n(x) = xF_{n-1} + F_{n-2}(x), \quad n \geq 3,$$

where

$$F_1(x) = 1, F_2(x) = x.$$

It is shown that these polynomials satisfy a special type of differential equation which is a second order variable-coefficient differential equation

$$(x^2 + 4)y'' + 3xy' - (n^2 - 1)y = 0,$$

while the Fibonacci polynomial differential equation leads to Fibonacci polynomials as solutions, the differential equation presented here leads to Fibonacci numbers at some discrete points.

In contrast to the standard spirals which are formed by quarter circles connected to each other with piecewise constant curvatures, the new spirals are generated from the differential equation which have a continuous change in the radius of curvatures. The angular step size in which the Fibonacci numbers are retrieved is also different than that of the classical spirals. The applications of these new spirals to natural phenomena, living creatures and architecture need further investigation and this study conform a basis of such an analysis.

2 Theoretical Background and Spirals

The difference equation

$$b_{j+2} = b_{j+1} + b_j\ , \quad j = 0, 1, 2 \dots \tag{2.1}$$

produces the Fibonacci sequence with b_0=1 and b_1=1. Table 1 lists the first 16 numbers in the sequence

Table 1: The Fibonacci Sequence

b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}
1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987

One of the most important properties of the sequence is that the ratio of the k+1'th term to the k'th one approaches the well-known Golden Ratio as k tends to infinity

$$\lim_{k\to\infty} \frac{b_{k+1}}{b_k} = \frac{1+\sqrt{5}}{2} \approx 1.618$$

Our aim is to introduce a new Fibonacci spiral which has a continuous change in the radius of curvature. This can be achieved by deriving a differential equation inspired from the difference equation. In contrast, the classical spirals are quarter circles connected to each other where the change in radius of curvature occurs at each 90 degrees of rotation. A sample spiral figure is given for flower heads (Figure 1).

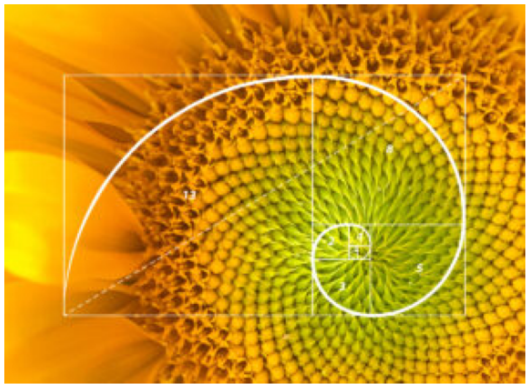

Figure 1: Fibonacci spiral applied to a flower head (Courtesy, Cleveland Design)[4]

The envelope of the spiral is a rectangle with its sides being in golden ratio.

3 The Differential Equation

From the discrete difference relation given in (2.1), a continuous differential equation is derived. The first two difference derivative are defined as [7]

$$D\left(r_k\right) = r_{k+1} - r_k$$

$$D^2\left(r_k\right) = D\left(D\left(r_k\right)\right) = r_{k+2} - r_{k+1} - \left(r_{k+1} - r_k\right) = r_{k+2} - 2r_{k+1} + r_k$$

Hence, the Fibonacci difference equation

$$r_{k+2} - r_{k+1} - r_k = 0 \tag{3.1}$$

can be expressed in terms of difference derivatives as follows

$$D^2\left(r_k\right) + D\left(r_k\right) - r_k = 0. \tag{3.2}$$

Difference equations are discrete equations which do not produce continuous solutions. They produce numbers at each isolated points. In contrast, differential equations produce solutions which are continuous throughout the domain of interest. There is a strong relationship with the discrete differential equations and the continuous differential equations and theoretically speaking, for each differential equation, a discrete differential equation can also be constructed (See some examples in O'Neil, [7]). Most of the numerical methods (Euler, Runge-Kutta, Finite Element etc) in fact search for discrete solutions of continuous differential equations, that is, the solutions are not continuous but valid at each separated steps of integration. Usually, the step size is taken very small to graphically reproduce continuous looking solutions albeit they are discrete. The recurrence relations obtained in series solutions of differential equations (Frobenius method) are also difference equations. Continuous solutions are generally possible if the differential equation has an exact analytical solution or an approximate analytical solution such as the perturbation techniques.

Our aim in this study is to write the difference derivative equation (3.2) associated with the well-known difference equation of the Fibonacci sequence (2.1) and due to the strong connection with the differential equations, switch to the continuous differential equation associated with equation (3.2). Since spirals are continuous curves, a differential equation to describe the curve would become necessary. Polar coordinates instead of Cartesian coordinates are selected since drawing spirals are easier in polar coordinates. The associated differential equation with the difference equation (3.2) in polar coordinates would then be

$$\frac{d^2r}{d\theta^2} + \frac{dr}{d\theta} - r = 0, \tag{3.3}$$

where r is the distance and θ is the angle in polar coordinates. The solution of this constant coefficient second order linear differential equation is

$$r = c_1 exp\left(-\frac{1-\sqrt{5}}{2}\theta\right) + c_2 exp\left(-\frac{1+\sqrt{5}}{2}\theta\right). \tag{3.4}$$

Equation (3.3) is original and named as the Fibonacci differential equation. A spiral solution will be shown in the next section.

4 The Spiral Solution

With inspiration from the first two terms in the Fibonacci sequence (Table 1), one may impose the initial conditions

$$r\,(0) = 1, \quad \frac{dr}{d\theta}\,(0) = 0$$

which yields the solution

$$r = \frac{\sqrt{5}+1}{2\sqrt{5}} exp\left(-\frac{1-\sqrt{5}}{2}\theta\right) + \frac{\sqrt{5}-1}{2\sqrt{5}} exp\left(-\frac{1+\sqrt{5}}{2}\theta\right). \tag{4.1}$$

Equation (4.1) can also be expressed in terms of the golden ration $\varphi = \frac{1+\sqrt{5}}{2}$

$$r = \frac{\varphi}{\sqrt{5}} exp\left(\frac{1}{\varphi}\theta\right) + \frac{1}{\varphi\sqrt{5}} exp\,(-\varphi\theta)\,. \tag{4.2}$$

Assuming $\theta = k\theta_0,\ k = 0, 1, 2, \ldots$ with a constant angle step size θ_0, one may require that the ratio of $r\,[(k+1)\,\theta_0]/r\,[k\theta_0]$ tends to the golden ratio as k tends to infinity, i.e.,

$$\lim_{k\to\infty} \frac{r\,[(k+1)\,\theta_0]}{r\,[k\theta_0]} = \lim_{k\to\infty} \frac{\frac{\varphi}{\sqrt{5}} exp\left(\frac{1}{\varphi}\,(k+1)\,\theta_0\right) + \frac{1}{\varphi\sqrt{5}} exp\,(-\varphi\,(k+1)\,\theta_0)}{\frac{\varphi}{\sqrt{5}} exp\left(\frac{1}{\varphi}k\theta_0\right) + \frac{1}{\varphi\sqrt{5}} exp\,(-\varphi k\theta_0)}\,.$$

As k tends to infinity, the second terms in the numerator and denominator tend to zero and one is left with the ratio of the leading terms

$$\lim_{k\to\infty} \frac{r[(k+1)\,\theta_0]}{r[k\theta_0]} = \exp\left(\frac{1}{\varphi}\theta_0\right) = \varphi$$

from which the step size is

$$\theta_0 = \varphi \ln(\varphi) = 0.778617 \text{ rad} = 44.61^\circ. \tag{4.3}$$

For the spiral curve (4.2), one may expect that for each 44.61 degrees, the distance from the origin would give a Fibonacci number. In Table 2, the approximate Fibonacci numbers from the spiral solution are contrasted with the Fibonacci integers.

Table 2: Comparison of the Fibonacci numbers

k	r_k ($\theta_0 = 0.778617$)	b_k	$e_k = \frac{r_k - b_k}{r_k}\%$
0	1	1	0
1	1.2492	1	24.92
2	1.9167	2	4.17
3	3.0716	3	2.39
4	4.9615	5	0.77
5	8.0254	8	0.32
6	12.9847	13	0.12
7	21.0096	21	0.05
8	33.9941	34	0.02
9	55.0036	55	0.01

As seen from the table, as the Fibonacci numbers become larger, the percentage error drastically reduces, predicting the Fibonacci numbers with precision.

In Figure 2, the Fibonacci spiral is plotted with the dots on it indicating the Fibonacci numbers at each step size. With the reduced errors as the numbers increase, the numbers coincide with the spiral curve.

The calculated spirals are normalized spirals. They can be scaled with an arbitrary r_0 scaling factor for applications. Hence, the scaled spirals are

$$r = r_0 \left(\frac{\varphi}{\sqrt{5}} exp\left(\frac{1}{\varphi}\theta\right) + \frac{1}{\varphi\sqrt{5}} exp\left(-\varphi\theta\right) \right), \tag{4.4}$$

where r_0 may be determined by the specific application.

Note that the continuous spiral drawn here can be expressed as a simple function of $r = r(\theta)$ with reference to a single origin. The piecewise continuous classical spirals have constant radius of curvatures for each 90 degrees but the origin changes at each step and an expression $r = r(\theta)$ with single origin would be much more complicated than the one presented here. Both spirals would not match each other if the classical spirals can also be expressed with reference to a single origin.

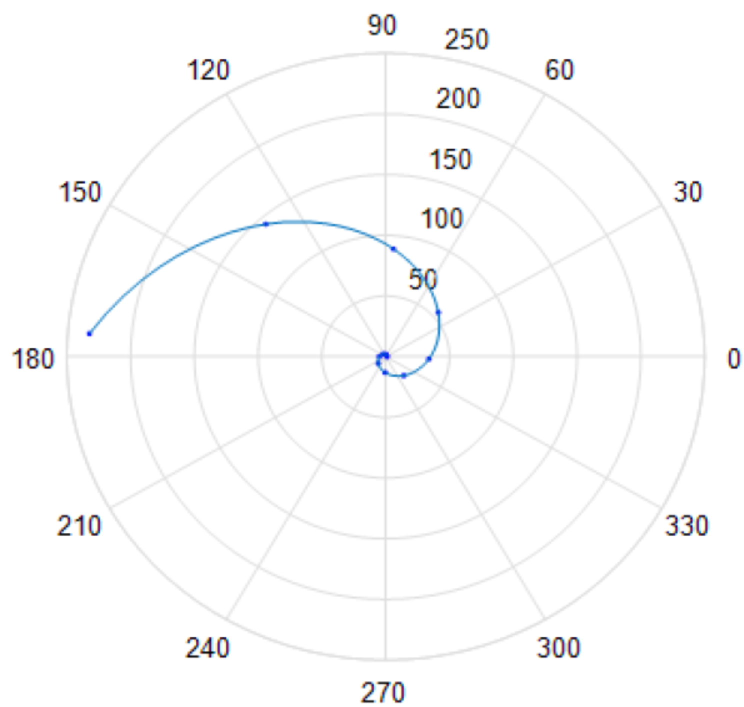

Figure 2: The Fibonacci Spiral

5 Concluding Remarks

The differences in the classical Fibonacci spirals and the new spirals presented in this work are summarized in the below table.

Table 3: Comparisons of Properties of Spirals

	Classical Spirals	New Spirals
Generation	Fibonacci difference equation	Fibonacci differential equation
Radius of curvature	piecewise constant	changing continuously
Step size of angles	90^o	44.61^o
Polar coordinate expression	complex for a single origin reference system	simple for a single origin reference system

It is expected that the new spirals presented in this work may find applications in explaining some geometrical patterns observed in nature. This work constitutes an initial step in accomplishing this effort.

The k-Fibonacci and k-Lucas number sequences are generalizations of Fibonacci number sequences. From the discrete relationship of those numbers, associated differential equations can also be derived in an analogous manner which is a further topic of study.

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