The Potts model and the independence polynomial

Uniqueness of the Gibbs measure and distributions of complex zeros

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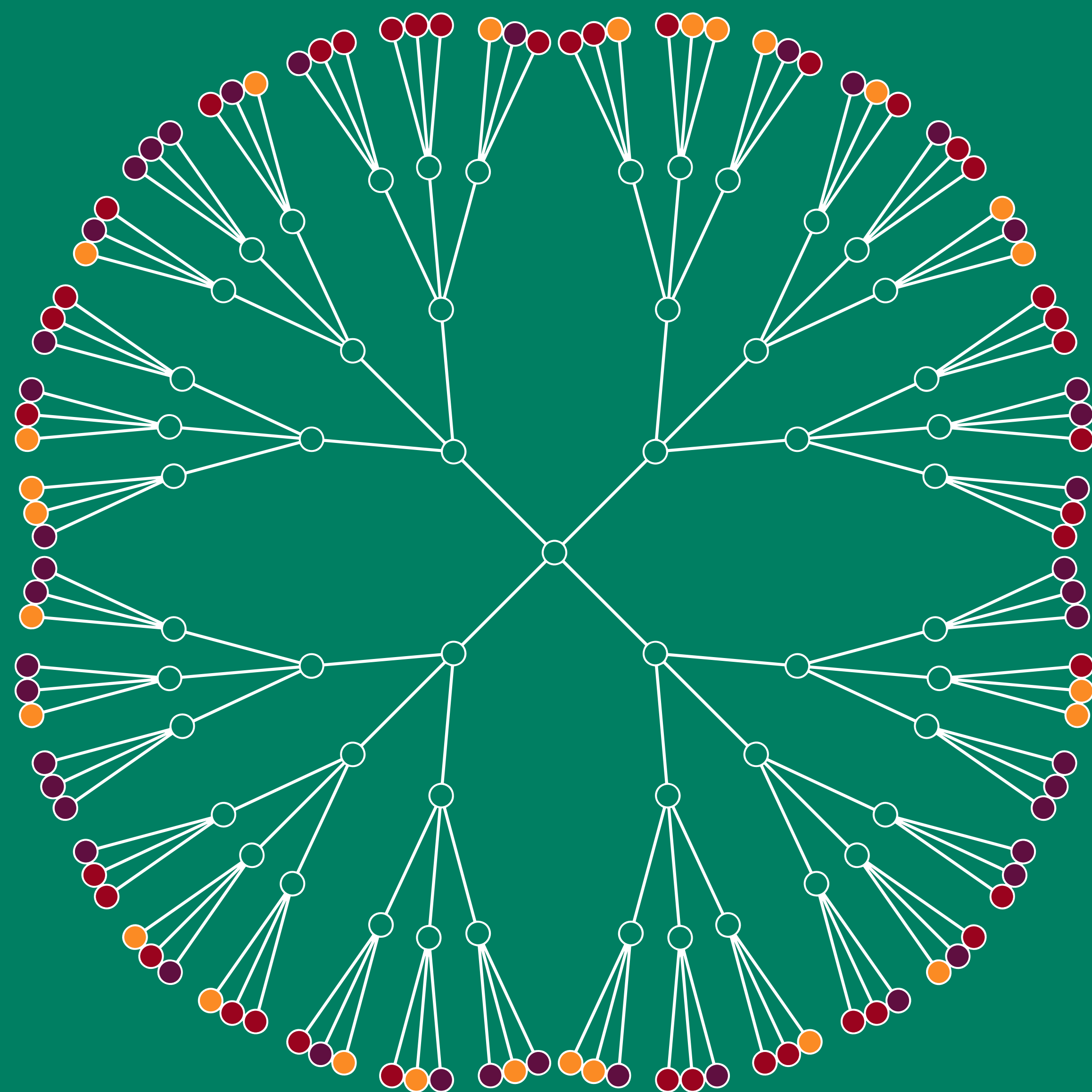

David de Boer

The Potts model and the independence polynomial: uniqueness of the Gibbs measure and distributions of complex zeros

David de Boer

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The Potts model and the independence polynomial
uniqueness of the Gibbs measure and distributions of complex zeros

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CHAPTER

ONE

INTRODUCTION

In this dissertation I study fundamental mathematical properties of the Potts model and the hard-core model. Both models originate in statistical physics. For context I give a short introduction to discrete models from statistical physics and some of the questions that are studied in the literature for these models in Section 1.1. In Section 1.2 I introduce the Potts model and indicate the main results contained in Part I of this dissertation. In Section 1.3 I introduce the hard-core model and indicate the main results contained in Part II of this dissertation. The computational complexity classes $\mathcal{P}$, $\mathcal{NP}$ and $\#\mathcal{P}$ play a role in this dissertation, especially in Chapter 5. Section 1.4 gives a short introduction in computational complexity.

1.1 Discrete models from statistical physics

Statistical mechanical models attempt to capture the macroscopic behavior of materials, such as gasses, fluids or pieces of iron as a function of physical parameters, such as temperature. In particular, statistical physics aims to provide a framework explaining macroscopic behavior of materials from the microscopic states, see the introduction of [FV17]. Drastic changes in macroscopic behavior of the material at a certain value of the parameters are called *phase transitions*. A classical example of a phase transition is the liquid-vapor phase transition of water; water boils at a specific temperature and pressure.

In this dissertation two discrete models that originate in statistical physics play a role, the *Potts model* and the *hard-core model*. The mathematical framework of a discrete model from statistical physics on a finite graph $G = (V, E)$ is as follows. The vertices of G model particles or positions of particles. There is a model dependent finite set of states S. A function $\sigma : V \to S$ is called a *configuration*. Each configuration corresponds to a possible microscopic state of the model. In the hard-core model one takes $S = \{\text{occupied}, \text{unoccupied}\}$ and lets the vertices of G represent potential positions of particles of a gas, see Section 1.3. The Potts model was originally introduced to study ferromagnetism. The graph G models the particles of some material and the set of states is $[q] := \{1, \ldots, q\}$ for an integer $q \geq 2$. The states model the different magnetic spins the particles can take, see Section 1.2. One defines an *energy function* or *Hamiltonian* H that associates an energy to each configuration. The *Gibbs measure* $\Pr_G$ on the set of configurations is the probability measure where each configuration σ has a probability of occurring proportional to $e^{-H(\sigma)/kT}$, where $T \geq 0$ is the temperature and k is the Boltzmann constant. Configurations with large energy are therefore unlikely to occur. Gibbs measures can alternatively be defined by the property that the marginal distributions of the vertices incident to v uniquely determine the marginal distribution at v, i.e. these measures satisfy a Markov property. The introduction of [Geo88] explains the physical relevance of Gibbs measures. In short, a Gibbs measure can be thought of as a macroscopic equilibrium state of the model.

Real world materials contain a large number of particles, a drop of water contains roughly $1.67 \cdot 10^{21}$ molecules. This motivates the idea to model real world materials as an infinite system, i.e. with an infinite graph. The physical idea is to let the volume and the number of particles of the system grow to infinity, while keeping the ratio between volume and number of particles fixed. This process is called the thermodynamic limit, see the introduction of [FV17] for details. Using the Markov property the concept of a Gibbs measure can be successfully generalised to infinite graphs, such as the integer lattice $\mathbb{Z}^d$ or the infinite regular tree $\mathbb{T}_\Delta$[1], see [Geo88] for a general definition. For many models on an infinite graph, including the ones we study, there exists at least one Gibbs measure, which can be proved by a compactness argument cf. [FV17, BW99, Geo88]. On a finite graph the Gibbs measure is clearly unique. However on an infinite graph, the Gibbs measure need no longer be unique. A fundamental question is: at what temperatures is there a unique Gibbs measure and at what temperatures is there a transition from a unique Gibbs measure to multiple Gibbs measures? A temperature T_c at which there is a transition from a unique Gibbs measure to multiple Gibbs measures is called a critical temperature. We say a *uniqueness phase transition* occurs at T_c. There is a surprising connection between the uniqueness phase transition on the infinite regular tree and transitions in the computational complexity of approximately computing partition function of bounded degree graphs for 2-state models, such as the hard-core model. It is conjectured that a similar phenomenon holds for models with a larger number of states, such as the

[1] The countable infinite tree where each vertex is incident to Δ edges.

Potts model. The uniqueness phase transition on the infinite regular tree also plays a role in the design of efficient algorithms to sample from Gibbs measures on finite graphs [BGG$^+$20]. Part I of this dissertation, consisting of Chapters 2, 3 and 4, concerns the uniqueness phase transition of the antiferromagnetic Potts model on the infinite regular tree.

The normalising constant in the Gibbs measure of a finite graph G is $Z_G(T) = \sum_\sigma e^{-H(\sigma)/kT}$ and is called the *partition function* of the model. Consider a sequence of finite graphs G_n converging[2] to an infinite graph G. Define the *free energy per site* as

$$\rho(T) = \lim_{n\to\infty} \frac{1}{|V(G_n)|} \log(Z_{G_n}(T)),$$

which is a function of the temperature T. For many models the free energy per site can be shown to be a well-defined continuous function on positive real parameters. The free energy per site $\rho(T)$ and its derivatives contain interesting information of the system, for example the density of the system. A parameter T at which the free energy per site is non-analytic is called a phase transition of the model. This definition of phase transition goes back to Ehrenfest, see for example [Jae98]. A phase transition in terms of analyticity of the free energy per site need not necessarily be the same as a uniqueness phase transition, though both types of phase transitions mark a change in behavior of the model. Yang and Lee established a connection between analyticity of the free energy per site at a real parameter T_0 and the set of complex zeros of $Z_{G_n}(T)$ accumulating on T_0 for many models, including the hard-core model, when the limit graph is the integer lattice $\mathbb{Z}^d$ [YL52]. This motivates the study of complex zeros of partition functions for sequences of finite graphs.

The study of the complex zeros of partition functions for a class $\mathcal{G}$ of finite graphs also has a motivation from computer science. The computer science framework can be described as follows. Given the model, the temperature T of the model and a class of graphs $\mathcal{G}$, does there exist a polynomial time algorithm that given $G \in \mathcal{G}$ computes $Z_G(T)$? Here polynomial time means the worst-case runtime of the algorithm is bounded by a polynomial in the number of vertices of the graph G. Exact computation is in many cases proven to be computationally hard[3], which implies existence of a polynomial time algorithm is very unlikely. It is therefore natural to relax the computational problem and ask if polynomial time approximate computation is possible: given $G \in \mathcal{G}$ and an $\varepsilon > 0$ can one approximate $Z_G(T)$ in polynomial time with multiplicative error at most ε? Approximate computation turns out to be deeply connected to the location of complex zeros; for many models, including the hard-core model, a zero-free region leads to a polynomial time approximation algorithm by the interpolation

[2]In the sense of van Hove, see Section 3.2.1 in [FV17].

[3]$\mathcal{NP}$-hard or even $\#\mathcal{P}$-hard, see e.g. [Wig19] or Section 1.4 for definitions.

method of Barvinok [Bar16, PR17]. On the other hand, when zeros are known to accumulate at a certain temperature, approximate computation can often be shown to be computationally hard. Part II of this dissertation, consisting of Chapters 5 and 6, concerns the zeros of the partition function of the hard-core model, for bounded degree graphs in Chapter 5 and for tori graphs in Chapter 6.

The next two sections introduce the two specific models and the questions I study for those models in this dissertation.

1.2 Potts model

The Potts model was first introduced by Renfrey Potts on suggestion of his advisor Cyril Domb, to study ferromagnetism [Pot52]. Let a finite graph $G = (V, E)$ be given. For the Potts model the set of *states* or *colors* is $S = [q] = \{1, \ldots, q\}$ for some $q \in \mathbb{Z}_{\geq 2}$. The configurations $\sigma : V \to [q]$ are called *colorings*. For a coloring σ let $m(\sigma)$ denote the number of monochromatic edges in σ, i.e. the number of $\{u, v\} \in E$ for which $\sigma(u) = \sigma(v)$. A *proper coloring* is a coloring for which $m(\sigma) = 0$. The Hamiltonian is defined as $H(\sigma) = -J \cdot m(\sigma)$, where $J \in \mathbb{R} \setminus \{0\}$ is a coupling constant. We define the *weight* $w(\sigma) = e^{\frac{-H(\sigma))}{kT}}$, where $T \in \mathbb{R}_{\geq 0}$ is the temperature and k is the Boltzmann constant. For example, proper colorings receive weight 1. We use the convention to write $w = e^{\frac{J}{kT}} \in \mathbb{R}_{\geq 0}$, so colorings receive weight $w^{m(\sigma)}$. With this convention the Gibbs measure on the graph $G = (V, E)$ is the probability measure that assigns probability

$$\frac{w^{m(\sigma)}}{\sum_{\tau:V\to[q]} w^{m(\tau)}}$$

to each coloring σ. The partition function $Z_{Potts}(G; w) = \sum_{\tau:V\to[q]} w^{m(\tau)}$ of the Potts model is a polynomial in w and can essentially be expressed as an evaluation of the Tutte polynomial, a well known object from combinatorics [FK72, Tut54, Sok05]. When it is clear we are working with the Potts model, we denote the partition function as $Z_G(w)$. For $w \in \mathbb{R}_{>1}$, corresponding to $J > 0$ and $T \geq 0$, colorings σ with larger number of monochromatic edges $m(\sigma)$ receive larger weight. This is referred to as the *ferromagnetic Potts model.* When $w \in [0, 1)$, corresponding to $J < 0$ and $T \geq 0$, colorings with smaller number of monochromatic edges $m(\sigma)$ receive higher weight. This is referred to as the *antiferromagnetic Potts model.*

For the ferromagnetic Potts model on the infinite regular tree $\mathbb{T}_\Delta$ the uniqueness phase transition is completely understood, see Theorem 5 in [GŠVY16]. The proof uses monotonicities in the model which only occur in the ferromagnetic case. For the antiferromagnetic Potts model much less is known. Define

$w_c = \max\{0, 1-\frac{q}{\Delta}\}$. When $w_c > 0$, it is known there are multiple Gibbs measures for $w < w_c$ [PdLM83, PdLM87].

Folklore conjecture. *The q-state antiferromagnetic Potts model on $\mathbb{T}_\Delta$ has a unique Gibbs measure if and only if*

$$\begin{cases} w > w_c & \textit{for } \Delta = q, \\ w \geq w_c & \textit{otherwise.} \end{cases}$$

In other words the folklore conjecture states that w_c is the parameter at which there is a uniqueness phase transition of the antiferromagnetic Potts model on the infinite regular tree for all q and all $\Delta \geq q$. This folklore conjecture was confirmed for $q = 2$ and all Δ by Srivastava, Sinclair and Thurley [SST14] and for $q = 3$ and $\Delta \geq 3$ by Galanis, Goldberg and Yang [GGY18]. In this dissertation, the following theorems are proved.

Main Theorem of Chapter 3. *Let $\Delta \in \mathbb{N}_{\geq 5}$. Then for each $w \in [1 - \frac{4}{\Delta}, 1)$ the 4-state anti-ferromagnetic Potts model with edge interaction parameter w has a unique Gibbs measure on the infinite Δ-regular tree $\mathbb{T}_\Delta$.*

Main Theorem of Chapter 4. *For each integer $q \geq 5$ there exists $\Delta_0 \in \mathbb{N}$ such that for each $\Delta \geq \Delta_0$ and each $w \in [1 - \frac{q}{\Delta}, 1)$ the q-state anti-ferromagnetic Potts model with edge interaction parameter w has a unique Gibbs measure on the infinite Δ-regular tree $\mathbb{T}_\Delta$.*

One of the ingredients in the proof of these theorems is the iteration of a map $F : \mathbb{R}^q \to \mathbb{R}^q$, which depends on Δ and w. We aim to find convex forward invariant sets for F. A natural question that remains is the following.

Question 1.2.1. Can we take $\Delta_0 = q + 1$ in the Main Theorem of Chapter 4?

The main issue is that the proof of the Main Theorem of Chapter 4 uses a compactness argument, which makes the dependency of Δ_0 on q unclear. Additional analysis is needed in order to obtain explicit bounds for Δ_0 in terms of q. Part I, consisting of Chapters 2, 3 and 4, is devoted to the uniqueness phase transition for the antiferromagnetic Potts model and is based on [dBBR22] and [BdBBR22]. Chapter 2 serves as a common introduction to Chapters 3 and 4.

1.3 Hard-core model

The hard-core model is a model for a lattice gas. In this model particles of a gas are assumed to occupy positions of a discrete lattice $\mathcal{L}$. We define the model on any finite graph G. Vertices of G can be occupied or unoccupied by a gas particle. We model this with the set of states $S = \{0,1\}$, by taking 0 for unoccupied and 1 for occupied vertices. The gas particles are assumed to have a hard core, i.e. adjacent vertices of G cannot be both occupied by a gas particle. The Hamiltonian is defined as

$$H(\sigma) = \begin{cases} \infty & \text{if there is an edge } \{u,v\} \text{ with } \sigma(u) = 1 = \sigma(v), \\ \sum_{v \in V} \sigma(v) & \text{otherwise.} \end{cases}$$

We define the weight function by $w(\sigma) = e^{\frac{-H(\sigma)}{kT}}$, where k is the Boltzmann constant and T is the temperature. We write $\lambda = e^{\frac{-1}{kT}}$, so the weight function is

$$w(\sigma) = \begin{cases} 0 & \text{if there is an edge } \{u,v\} \text{ with } \sigma(u) = 1 = \sigma(v), \\ \lambda^{\sum_{v \in V} \sigma(v)} & \text{otherwise.} \end{cases}$$

A set $I \subseteq V$ is called an *independent set* if there is no edge $\{u,v\}$ with both u and v in I. Configurations σ with nonzero weight correspond to independent sets by taking $I = \sigma^{-1}(1)$. Hence the partition function of the hard-core model of a finite graph G is equal to the generating function over all independent sets of G, called the *independence polynomial* in graph theory. The independence polynomial, introduced in [GH83], and its complex zeros for various graph classes have a long history of study, see [LM05] for a survey. We denote the independence polynomial by $Z_{ind}(G;\lambda)$, or often simply by $Z_G(\lambda)$ when it is clear we are working with the independence polynomial.

Evaluations at specific parameters λ carry information on the independent sets of the graph, for example $Z_G(1)$ is the total number of independent sets in G. Computing $Z_G(1)$ exactly is a computationally hard problem[4], already for the class of graphs where each vertex has degree at most 3 [Gre00]. It is therefore natural to relax the problem to approximate computation. For a class of graphs $\mathcal{G}$ and a specific parameter λ we say there exists a polynomial time algorithm approximately computing $|Z_G(\lambda)|$ if given a graph $G \in \mathcal{G}$ and an $\varepsilon > 0$ the algorithm computes a number N in time polynomial in $|V(G)|/\varepsilon$ such that $e^{-\varepsilon} \leq \frac{|Z_G(\lambda)|}{N} \leq e^{\varepsilon}$. When $Z_G(\lambda) = 0$ we allow the algorithm to output any number N. Note for $\lambda \in \mathbb{R}_{\geq 0}$ we have $|Z_G(\lambda)| = Z_G(\lambda)$ as $Z_G(\lambda)$ is a polynomial with positive integer coefficients.

In Chapter 5 the focus is on the class of finite graphs where each vertex is of degree at most Δ, denoted by $\mathcal{G}_\Delta$. Let us denote the set of zeros of independence polynomials of graphs in $\mathcal{G}_\Delta$ by

$$\mathcal{Z}_\Delta := \{\lambda \in \mathbb{C} : \exists G \in \mathcal{G}_\Delta \text{ for which } Z_G(\lambda) = 0\}.$$

[4]In fact $\#\mathcal{P}$-hard, see Section 1.4 for some background on computational complexity, in particular for a definition of $\#\mathcal{P}$-hardness.

We call $\overline{\mathcal{Z}_\Delta}$ the zero-locus. For a vertex v of G we denote by $Z^{\text{in}}_{G,v}(\lambda)$ the independence polynomial where we sum only over independent sets containing v and by $Z^{\text{out}}_{G,v}(\lambda)$ the independence polynomial where we sum only over independent sets not containing v, so that $Z_G(\lambda) = Z^{\text{in}}_{G,v}(\lambda) + Z^{\text{out}}_{G,v}(\lambda)$. The *ratio at* v is the rational function

$$R_{G,v}(\lambda) := \frac{Z^{\text{in}}_{G,v}(\lambda)}{Z^{\text{out}}_{G,v}(\lambda)}.$$

Furthermore, define the family of maps

$$\mathcal{R}_\Delta := \{R_{G,v} : G \in \mathcal{G}_\Delta \text{ and } v \in V(G)\}.$$

Clearly $0 \notin \mathcal{Z}_\Delta$. By a result of Shearer [She85] it turns out that $0 \notin \overline{\mathcal{Z}_\Delta}$. Let $\mathcal{U}_\Delta$ denote the maximal connected and open set containing 0 with $\mathcal{U}_\Delta \subseteq \mathbb{C} \setminus \overline{\mathcal{Z}_\Delta}$. For the class of graphs $\mathcal{G}_\Delta$ and any $\lambda \in \mathcal{U}_\Delta$, there is a polynomial time approximation algorithm to compute $Z_G(\lambda)$, due to the interpolation method by Barvinok [Bar16, PR17]]. One could summarise this by saying that connected and open zero-free regions imply the existence of a polynomial time approximation algorithm. The main result of Chapter 5 establishes a connection in the other direction, building on [BGGŠ20]. It also proves a fundamental connection between the set of zeros and chaotic behavior of the family of ratios.

Main Theorem of Chapter 5. *Parameters λ for which approximately computing the norm of the independence polynomial $|Z_G(\lambda)|$ for graphs $G \in \mathcal{G}_\Delta$ is computationally hard accumulate on any $\mu \in \overline{\mathcal{Z}_\Delta}$. Furthermore $\overline{\mathcal{Z}_\Delta}$ is equal to the set of parameters where the family $\mathcal{R}_\Delta$ behaves chaotically.*

The chaotic behavior is characterised in two different ways that turn out to be equivalent, see Chapter 5 for details. Through the connection between zeros and chaotic behavior of $\overline{\mathcal{R}_\Delta}$ one gets insight in the structure of $\overline{\mathcal{Z}_\Delta}$ and its complement, see for example Proposition 5.4.3 in Chapter 5. We conjecture the following.

Conjecture 1.3.1. *The set $\mathbb{C} \setminus \overline{\mathcal{Z}_\Delta}$ is connected.*

If Conjecture 1.3.1 is true, then $\mathbb{C} \setminus \overline{\mathcal{Z}_\Delta} = \mathcal{U}_\Delta$. The hardness of approximation of the independence polynomial of bounded degree graphs would be essentially understood in terms of the set of zeros of the independence polynomial of bounded degree graphs. In the limit $\Delta \to \infty$ the suitably rescaled complement of the zero-locus converges in the Hausdorff metric to a connected set [BBP21], providing evidence for Conjecture 1.3.1.

From a physics viewpoint it is particularly interesting to consider sequences of graphs G_n that converge to a regular lattice. In Chapter 6 we consider sequences of d-dimensional tori converging to the integer lattice $\mathbb{Z}^d$ for $d \geq 2$, i.e. tori whose minimal cycle lengths tend to infinity. A d-dimensional torus with side lengths $\ell_1, \dots, \ell_d$ is the Cartesian product $\mathbb{Z}_{\ell_1} \times \cdots \times \mathbb{Z}_{\ell_d}$, where we write $\mathbb{Z}_n$ for $\mathbb{Z}/n\mathbb{Z}$. In particular, we study the structure of the set of zeros for various classes of tori, which has motivation from statistical physics and computer science, as discussed in Section 1.1.

Figures 6.1 and 6.2 in Chapter 6 suggest that for classes of tori there is a connection between the set of zeros and the chaotic behavior of the family of ratios of tori graphs. However, establishing such a connection remains open. The main reason is the techniques from Chapter 5 to prove this connection do not apply to the class of tori, as the techniques in Chapter 5 crucially use the class of bounded degree graphs is closed under taking induced subgraphs. The main result in Chapter 6 establishes when the zero set for various classes of tori remains bounded, providing a first result on the structure of the set of zeros for these physically more relevant class of graphs. For technical reasons we only consider tori for which all side lengths are even and call those tori *even.*

Main Theorem of Chapter 6. *Let $\mathcal{F}$ be a family of even d-dimensional tori. If $\mathcal{F}$ is balanced, then the zeros of the independence polynomials $\{Z_{\mathcal{T}} : \mathcal{T} \in \mathcal{F}\}$ are uniformly bounded. If $\mathcal{F}$ is highly unbalanced, then the zeros are not uniformly bounded.*

Here we say that a family of d-dimensional tori $\mathcal{F}$ is *balanced* if there exists a $C > 0$ such that for all $\mathcal{T} \in \mathcal{F}$ we have that $\ell_d \leq \mathrm{Exp}(C \cdot \ell_1)$, where $\ell_1 \leq \cdots \leq \ell_d$ denote the side lengths of $\mathcal{T}$. On the other hand we say that the family is *highly unbalanced* if there is no uniform constant $C > 0$ such that $\ell_d \leq \mathrm{Exp}(C \cdot (\ell_1 \cdots \ell_{d-1})^3)$ for all $\mathcal{T} \in \mathcal{F}$.

Several steps of the proof for boundedness of zeros of balanced tori rely in an essential way on the assumption that the tori are balanced. On the other hand, the highly-unbalanced assumption on the family of tori that guarantees the existence of unbounded zeros seems far from sharp, evidenced for example by the fact that the demonstrated zeros of the tori escape very rapidly in terms of the sizes of the tori. It therefore seems reasonable to expect that the balanced assumption is necessarily, while the highly-unbalanced assumption is not.

Question 1.3.2. Let $\mathcal{F}$ be a family of even d-dimensional tori for which the zeros of the independence polynomials are uniformly bounded. Is $\mathcal{F}$ necessarily balanced?

For further questions, see Section 6.1.6 in Chapter 6. Chapter 5 is based on [dBBG$^+$21] and Chapter 6 is based on [dBBPR23].

1.4 Complexity classes and reductions

The goal of this section is to define complexity classes $\mathcal{NP}$ and $\#\mathcal{P}$ and to define the notion of $\#\mathcal{P}$-hardness, providing background for Chapter 5 of this dissertation. This section is based on the books [Wig19] and [Jer03], though we use different notation to avoid confusion with independent sets. We will use various computational problems as guiding examples in this section.

The first guiding computational problem is the following. Given a graph $G = (V, E)$, an integer $k \geq 0$ and a set $S \subseteq V$, determine if S is an independent set of size k in G. We denote this problem as IsIndset. Computational complexity theory provides a mathematical framework to reason about algorithms for problems such as IsIndset. We wish to define efficient solvable problems in some way. Formally one needs to define what an algorithm is allowed to do and how the input and output of a computational problem are represented. Usually, one defines algorithms using Turing machines and tailor the input and output of the problems to the specific definition of Turing machine chosen. It turns out the standard definition of efficient solvable problem does not depend on the details of the implementation of the Turing machine (e.g. what set of symbols it uses or how many distinct storage tapes it has access to), see for example [AB09]. We represent the finite objects we want to do computations on as finite bit strings. In light of the previous remark, we do not concern ourselves with exactly how we choose such a representation. To have something concrete in mind, one can represent a graph with the adjacency matrix and integers with binary expansions. Denote the set of finite bit strings as $\mathbf{B}$. For $x \in \mathbf{B}$ we denote by $|x|$ the size of x, which is the number of bits in x[5]. Denote $B_n \subset \mathbf{B}$ for the set of bit strings of size n. A computational problem is defined to be a function $f : \mathbf{B} \to \mathbf{B}$. We wish to have an algorithm to compute $f(x)$ given any input $x \in \mathbf{B}$. Note the input and output may have different interpretation. For example, for IsIndset : $\mathbf{B} \to \{0, 1\}$ the input is a triple (G, k, S), where $G = (V, E)$ is a graph, $k \geq 0$ is an integer and $S \subseteq V$ is a set of vertices of G. The output will be 1 if $|S| = k$ and S is an independent set in G and 0 otherwise[6].

Definition 1.4.1 (the class $\mathcal{P}$)**.** A computational problem $f : \mathbf{B} \to \mathbf{B}$ is in the class $\mathcal{P}$ if there is an algorithm computing f and positive constants A, c such that for every n and every $x \in \mathbf{B}_n$ the algorithm computes $f(x)$ in at most An^c steps.

A step of an algorithm is a step of the Turing machine on which the algorithm runs, but as the details of choice of Turing machine does not change the class $\mathcal{P}$ we

[5] For graphs, one defines the size typically to be the number of vertices of G, though again there are many natural definitions of size that will all lead to the same computational complexity of the computational problem.

[6] Formally, f could also receive input strings which are incorrectly formatted, in this case f should output 0 as well.

omit the choice of Turing machine in the definition. Computational problems in $\mathcal{P}$ are considered to be efficiently solvable, see [Wig19] for more background and motivation. The problem IsIndset is in $\mathcal{P}$, as checking for each vertex in S if there is an edge to any other vertex in S is an algorithm computing IsIndset that runs in time polynomial in the size of (G, k, S). Computational problems $f : \mathbf{B} \to \{0,1\}$ are called decision problems. Any computational problem $f : \mathbf{B} \to \mathbf{B}$ can be viewed as a string of decision problems. Hence we sometimes think of $\mathcal{P}$ as consisting of only decision problems. A decision problem f can be represented as a subset $C \subset \mathbf{B}$ such that $x \in C$ if and only if $f(x) = 1$.

The second guiding example is the decision problem Indset : $\mathbf{B} \to \{0,1\}$. The input for Indset is a graph $G = (V, E)$ and an integer $k \geq 0$, the output is 1 if G contains an independent set of size k and 0 otherwise. For this problem, no polynomial time algorithm is known. But given an independent set I in G of size k, one can use IsIndset on input (G, k, I) to check in polynomial time if I is in fact an independent set of size k in G. This is an archetypal example of a problem in $\mathcal{NP}$.

Definition 1.4.2 (the class $\mathcal{NP}$)**.** A decision problem $f : \mathbf{B} \to \{0,1\}$ is in the class $\mathcal{NP}$ if there is a $V_f \in \mathcal{P}$ and a constant c such that

- If $f(x) = 1$, then there exists a witness $y \in \mathbf{B}$ with $|y| \leq c \cdot |x|^c$ and $V_f(x, y) = 1$;
- If $f(x) = 0$, then for all witnesses $y \in \mathbf{B}$ we have $V_f(x, y) = 0$.

The idea behind this definition is that f outputs 0 only on inputs x where the desired structure, such as an independent set of size k, does not exist. If the desired structure does exist the output is 1 and we demand there exist a verifier $V_f \in \mathcal{P}$ with which we can check the correctness of the output. Note that in the definition of the class $\mathcal{NP}$ it is only required that the witness y is of size comparable to the input x; there is no mention of how to obtain a witness y. From the definitions we see each decision problem in $\mathcal{P}$ is in $\mathcal{NP}$, as for any decision problem $f \in \mathcal{P}$ one can take verifier $V_f = f$ and let the witness y be empty.

Let us introduce another famous decision problem in $\mathcal{NP}$. A Boolean formula consists of variables and the logical symbols $\vee, \wedge, \neg$, for example $(x \vee \neg y) \wedge z$ is a Boolean formula. A literal is a Boolean formula of the form x or $\neg x$. A clause is a disjunction of literals, for example $x \vee \neg y$ is a clause. A Boolean formula is in conjunctive normal form if it is a conjunction of clauses, so for example $(x \vee \neg y) \wedge z$ is in conjunctive normal form, while $(x \wedge y) \vee z$ is not. We say a Boolean formula is satisfiable if there is an assignment of true (1) and false (0) to the variables such that the formula is true. For example, $(x \vee \neg y) \wedge z$ is satisfiable, for example take $x = z = y = 1$. The Boolean formula $x \wedge \neg x$ is not satisfiable. Let Sat denote the Boolean satisfiability problem, where the input is a Boolean formula in conjunctive normal form and the output is 1 if the formula is satisfiable and 0 otherwise. Checking an assignment of 0 and 1 to the variables in a Boolean formula B satisfies B can be done in time polynomial in the size of B, hence Sat is in $\mathcal{NP}$.

A crucial tool in the study of complexity of decision problems are so-called Karp reductions. Karp reductions link distinct decision problems, reducing one problem to the other. Let us motivate the definition of a Karp reduction with an example.

The decision problem Sat can be reduced to Indset by the following argument from [Kar72]. Given a Boolean formula B in conjunctive normal form on k variables, let l_i denote the literals and c_j the clauses of B. Form a graph G_B with vertex sets all tuples (l_i, c_j) and an edge between (l_i, c_j) and $(l_{i'}, c_{j'})$ if $j = j'$ or $l_i = \neg l_{i'}$. Then any independent I in G_B of size k corresponds to a satisfying assignment of the variables for B, by letting $l_i = 1$ if $(l_i, c_j) \in I$ and $l_i = 0$ if $(\neg l_i, c_j) \in I$. The construction of G_B takes time polynomial in the size of f. Note also that k is less than the size of f, hence the size of (G_B, k) is polynomial in the size of f. This argument shows, by applying Indset to (G_B, k), that if one has a polynomial time algorithm for Indset, then this would also yield a polynomial time algorithm for Sat.

Definition 1.4.3. Let $f, g : \mathbf{B} \to \{0,1\}$ be decision problems. A function $r : \mathbf{B} \to \mathbf{B}$ is called a *Karp reduction* from f to g if $r \in \mathcal{P}$ and for every $x \in \mathbf{B}$ we have $f(x) = g(r(x))$. We write $f \leq_K g$ if there is a Karp reduction from f to g.

We have shown Sat $\leq_K$ Indset. In general, if $f \leq_K g$ and $g \in \mathcal{P}$ then $f \in \mathcal{P}$ follows. Furthermore, the relation $\leq_K$ is transitive.

Definition 1.4.4. Let $f : \mathbf{B} \to \{0,1\}$ be a decision problem. We say f is *$\mathcal{NP}$-hard* if for all $g \in \mathcal{NP}$ we have $g \leq_K f$. If $f \in \mathcal{NP}$ we say f is *$\mathcal{NP}$-complete.*

If a problem f is $\mathcal{NP}$-complete and $g \leq_K f$, then g is $\mathcal{NP}$-hard. A fundamental result by Cook [Coo71] shows Sat is $\mathcal{NP}$-complete. Together with the reduction Sat $\leq_K$ Indset, this implies Indset is $\mathcal{NP}$-complete. We think of an $\mathcal{NP}$-hard problem as being as hard as all the problems in $\mathcal{NP}$, as finding a polynomial time algorithm for an $\mathcal{NP}$-hard problem would imply finding a polynomial time algorithm for all problems in $\mathcal{NP}$, thus yielding $\mathcal{NP} = \mathcal{P}$, which is widely believed to be false. Hence proving a problem is $\mathcal{NP}$-hard is considered a proof of difficulty of a problem.

The final type of computational problems we will consider are those where we want to count how many solutions exist. The guiding example is the computational problem #Indset : $\mathbf{B} \to \mathbf{B}$. The input to #Indset is a graph $G = (V, E)$ and an integer $k \geq 0$, the output is the number of independent sets of size k in

G. This problem is a counting version of Indset. This is an archetypal example of a problem in $\#\mathcal{P}$, a complexity class introduced by Valiant in [Val79].

Definition 1.4.5 (the class $\#\mathcal{P}$)**.** Let $f \in \mathcal{NP}$. The counting version of f is the problem $\#f : \mathbf{B} \to \mathbf{B}$ defined by $\#f(x) = |\{y \in \mathbf{B} : |y| \leq c \cdot |x|^c \text{ and } V_f(x,y) = 1\}|$, where the constant c and verification algorithm V_f are as in Definition 1.4.2. We define $\#\mathcal{P}$ to be the class of counting versions of problems in $\mathcal{NP}$.

Similarly as in #Indset we can define a counting version of Sat, denoted by #Sat. The reduction from Sat to Indset preserves witnesses, that is each independent set of size k in the graph G_B corresponds to a unique satisfying assignment of the variables occurring in the Boolean formula B, so running #Sat on B yields the same integer as running #Indset on (G_B, k). Hence the problem #Sat also reduces to the problem #Indset. However, in general, one needs a bit more freedom than what is allowed by Karp reductions, as is illustrated in the following example.

Consider the computational problem IndPolyEval, where the input is a pair (G, k), with G a graph and k an integer and the output is the independence polynomial of G evaluated at k, i.e. the integer $Z_{\text{ind}}(G; k)$. Consider also the computational problem IndNumber, with input a graph G and output the size of the largest independent set of G, the so-called *independence number* of G denoted by $\alpha(G)$.

If we assume there is a magical algorithm, often called an oracle, that computes IndPolyEval in constant time, then using this oracle one can also solve IndNumber in polynomial time, by the following argument. Given a graph G, we want to compute $\alpha(G)$ in time polynomial in $|G|$. First, given G we can compute in polynomial time in $|G|$ the integers $0, 1, \ldots, |G|$. Now we use IndPolyEval on (G, i) for all $i \in \{0, 1, \ldots, |G|\}$. As each of the integers in $\{0, 1, \ldots, |G|\}$ is of size bounded by the size of G and we assumed oracle access to IndPolyEval, this all runs in time polynomial in $|G|$. From the integers $Z_{\text{ind}}(G; i)$ for $i \in \{0, 1, \ldots, |G|\}$ we can compute the independence polynomial of G in time polynomial in $|G|$, as the degree of this polynomial is bounded by $|G|$ and we have $|G| + 1$ evaluations. The degree of the independence polynomial is exactly $\alpha(G)$, which is what we wanted to compute. We thus reduced the computational problem IndNumber to the problem IndPolyEval, allowing here to access the oracle for IndPolyEval polynomially many times in the size of the input graph. This type of reduction is called a *Cook reduction* or a *polynomial time Turing reduction.*

Definition 1.4.6. Let $f, g : \mathbf{B} \to \mathbf{B}$ be computational problems. Assume there is an oracle computing g in constant time. There is a *Cook reduction* from f to g, if $f(x)$ is computed by an algorithm in time polynomial in $|x|$, where the algorithm has access to the oracle for g. We write $f \leq_C g$ if there is a Cook reduction from f to g.

Note the oracle is allowed to be used more than one time, and also that after applying the oracle we are allowed to use the outputs. This freedom is exactly what we used when we showed IndNumber $\leq_C$ IndPolyEval. We note for decision problems f, g we have $f \leq_K g$ implies $f \leq_C g$, but the other implication might not hold. Furthermore, if $f \leq_C g$ and $g \in \mathcal{P}$ then $f \in \mathcal{P}$. Using these reductions, we define $\#\mathcal{P}$-hardness and $\#\mathcal{P}$-completeness:

Definition 1.4.7. Let $f : \mathbf{B} \to \mathbf{B}$ be a computational problem. We say f is $\#\mathcal{P}$*-hard* if for all $g \in \#\mathcal{P}$ we have $g \leq_C f$. If $f \in \#\mathcal{P}$ we say f is $\#\mathcal{P}$*-complete.*

From the $\mathcal{NP}$-completeness of Sat and Indset, one can deduce the $\#\mathcal{P}$-completeness of #Indset and #Sat, see for example [Val79, Sim77].

1.5 Publications

This dissertation is primarily based on the work in the following papers.

David de Boer, Pjotr Buys, and Guus Regts, *Uniqueness of the gibbs measure for the 4-state anti-ferromagnetic potts model on the regular tree*, Combinatorics, Probability and Computing **32** (2022), no. 1, 158–182

Ferenc Bencs, David de Boer, Pjotr Buys, and Guus Regts, *Uniqueness of the Gibbs measure for the anti-ferromagnetic Potts model on the infinite* Δ*-regular tree for large* Δ, Preprint, arXiv:2203.15457 (2022), 1–20, to appear in the Journal of Statistical Physics

David de Boer, Pjotr Buys, Lorenzo Guerini, Han Peters, and Guus Regts, *Zeros, chaotic ratios and the computational complexity of approximating the independence polynomial*, Preprint, arXiv:2104.11615, 2021

David de Boer, Pjotr Buys, Han Peters, and Guus Regts, *On boundedness of zeros of the independence polynomial of tori*, Preprint, arXiv:2306.12934, 2023

To each of these papers all authors contributed an equal amount.

Part I:

Uniqueness of the Gibbs measure for the antiferromagnetic Potts model on the regular tree

CHAPTER

TWO

BACKGROUND AND PRELIMINARIES

2.1 Introduction

The Potts model is a model from statistical physics, originally invented to study ferromagnetism [Pot52]; it also plays a central role in probability theory, combinatorics and computer science, see e.g. [Sok05] for background.

Let $G = (V, E)$ be a finite graph. The anti-ferromagnetic Potts model on the graph G has two parameters, a number of *states, or colors,* $q \in \mathbb{Z}_{\geq 2}$ and an edge interaction parameter $w = e^{J/kT}$, with $J < 0$ being a coupling constant, k the Boltzmann constant and T the temperature. The case $q = 2$ is also known as the zero-field Ising model. A *configuration* is a map $\sigma : V \to [q] := \{1, \dots, q\}$. Associated with such a configuration is the *weight* $w^{m(\sigma)}$, where $m(\sigma)$ is the number of edges $e = \{u, v\} \in E$ for which $\sigma(u) = \sigma(v)$. There is a natural probability measure, the *Gibbs measure* $\Pr_{G;q,w}[\cdot]$, on the collection of configurations $\Omega = \{\sigma : V \to [q]\}$ in which a configuration is sampled proportionally to its weight. Formally, for a given configuration $\phi : V \to [q]$ the probability that a random configuration $\mathbf{\Phi}$[1] is equal to ϕ, is given by

$$\Pr_{G;q,w}[\mathbf{\Phi} = \phi] = \frac{w^{m(\phi)}}{\sum_{\sigma:V\to[q]} w^{m(\sigma)}}, \tag{2.1}$$

here the denominator is called *partition function* of the model and we denote it by $Z(G; q, w)$ (or just $Z(G)$ if q and w are clear form the context).

In statistical physics the Potts model is most frequently studied on infinite lattices, such as $\mathbb{Z}^2$. At the cost of introducing some measure theory, the notion

[1] We use the convention to denote random variables with capitals in boldface.

of a Gibbs measure can be extended to such infinite graphs, see e.g. [BW99, BW02, FV17]. For the definition of Gibbs measures, see Section 2.2. While at any temperature the Gibbs measure on a finite graph is unique, this is no longer the case for all infinite lattices. The transition from having a unique Gibbs measure to multiple Gibbs measures in terms of the temperature is referred to as a *uniqueness phase transition* in statistical physics [Geo88, FV17] and it is an important problem to determine the exact temperature, the *critical temperature,* T_c, at which this happens. There exist predictions for the critical temperature on several lattices in the physics literature by Baxter [Bax82, Bax86] (see also [SS97] for more details and further references), but it turns out to be hard to prove these rigorously cf. [SS97].

In Part I of this dissertation we consider the anti-ferromagnetic Potts model on the infinite Δ-regular tree, $\mathbb{T}_\Delta$, also known as the *Bethe lattice*, or *Cayley tree.*

For a number of states $q \geq 3$ define

$$w_c := \max\{0, 1 - \frac{q}{\Delta}\}.$$

The following is a longstanding folklore conjecture (cf.[BGG$^+$20, page 746]).

Folklore conjecture. *The q-state antiferromagnetic Potts model on $\mathbb{T}_\Delta$ has a unique Gibbs measure if and only if*

$$\begin{cases} w > w_c & \text{for } \Delta = q, \\ w \geq w_c & \text{otherwise.} \end{cases}$$

We note that using the well known Dobrushin uniqueness theorem, one obtains uniqueness of the Gibbs measure provided $w > 1 - \frac{q}{2\Delta}$ cf. [BCKL13, SS97], which is still far way from the conjectured threshold. The conjecture was confirmed by Jonasson for the case $w = 0$ [Jon02], by Srivastava, Sinclair and Thurley [SST14] for $q = 2$ (see also [Geo88]; in this case one can map the model to a ferromagnetic model since the tree is bipartite, which is much better understood), by Galanis, Goldberg and Yang for $q = 3$ [GGY18]. Chapter 3 covers the case for $q = 4$ and $\Delta \geq 5$.

Main Theorem of Chapter 3. *Let $\Delta \in \mathbb{N}_{\geq 5}$. Then for each $w \in [1 - \frac{4}{\Delta}, 1)$ the 4-state anti-ferromagnetic Potts model with edge interaction parameter w has a unique Gibbs measure on the infinite Δ-regular tree $\mathbb{T}_\Delta$.*

Our proof of this result follows a different approach than the one taken in [GGY18], which heavily relies on rigorous (but not easily verifiable) computer calculations. In particular, our approach allows us to recover the results from [GGY18], thereby removing the need for these computer calculations. See Theorem 3.4.2 below for the full statement of what we prove with our approach.

In Chapter 4 we confirm the folklore conjecture for all $q \geq 5$ provided the degree of the tree is large enough.

Main Theorem of Chapter 4. *For each integer $q \geq 5$ there exists $\Delta_0 \in \mathbb{N}$ such that for each $\Delta \geq \Delta_0$ and each $w \in [1 - \frac{q}{\Delta}, 1)$ the q-state anti-ferromagnetic Potts model with edge interaction parameter w has a unique Gibbs measure on the infinite Δ-regular tree $\mathbb{T}_\Delta$.*

This result builds on the techniques used in Chapter 3, but instead of fixing q we analyse the uniqueness threshold as $\Delta \to \infty$, see Theorem 4.2.1 for the precise statement.

It has long been known that there are multiple Gibbs measures when $w < w_c$ [PdLM83, PdLM87], see also [GŠV15]) and [BR19, KR17, GRR17, KRK14]. Below Lemma 4.2.2 in Chapter 4 we indicate how one could prove this. So our main results pinpoints the critical temperature for the anti-ferromagnetic Potts model on the infinite regular tree for large enough degree. For later reference we will refer to w_c as the *uniqueness threshold.*

2.1.1 Motivation from computer science

There is a surprising connection between phase transitions on the infinite regular tree and transitions in the computational complexity of approximately computing partition function of 2-state models (not necessarily the Potts model) on bounded degree graphs. For parameters inside the uniqueness region there is an efficient algorithm for this task [Wei06, LLY13, SST14], while for parameters for which there are multiple Gibbs measures on the infinite regular tree, the problem is NP-hard [SS14, GŠV16]. It is conjectured that a similar phenomenon holds for a larger number of states.

While the picture for q-state models for $q \geq 3$ is far from clear, some progress has been made on this problem for the anti-ferromagnetic Potts model. On the hardness side, Galanis, Štefankvovič and Vigoda [GŠV15] showed that for even numbers $\Delta \geq 4$ and any integer $q \geq 3$, approximating the partition function of the Potts model $Z(G; q, w)$ is NP-hard on the family of graphs of maximum degree Δ for any $0 \leq w < q/\Delta = w_c$, which we now know to be the uniqueness threshold (for Δ large enough). On the other side, much less is known about the existence of efficient algorithms for approximating $Z(G; q, w)$ or sampling from the measure $\Pr_{G;q,w}$ for the class of bounded degree graphs when $w > w_c$. Implicit in [BDPR21] there is an efficient algorithm for this problem whenever $1-\alpha q/\Delta < w \leq 1$, with $\alpha = 1/e$, which has been improved to $\alpha = 1/2$ in [LSS22].

For random regular graphs of large enough degree, our main result implies an efficient randomized algorithm to approximately sample from the Gibbs measure $\Pr_{G;q,w}$ for any $w_c < w \leq 1$ by a result of Blanca, Galanis, Goldberg, Štefankovič, Vigoda and Yang [BGG+20, Theorem 2.7]. In [Eft22], Efthymiou proved a similar result for Erdős-Rényi random graphs without the assumption that w_c is equal to the uniqueness threshold on the tree. At the very least this indicates that the uniqueness threshold on the infinite regular tree plays an important role in the study of the complexity of approximating the partition function of and sampling from the Potts model on bounded degree graphs.

2.2 Gibbs measures and uniqueness

We follow Brightwell and Winkler [BW99, BW02] to introduce the notion of Gibbs measures on $\mathbb{T}_\Delta$, see also [Roz13, FV17] for more details and background.

Throughout we fix a degree $\Delta \geq 3$ and an integer $q \geq 2$. We denote the vertex set of $\mathbb{T}_\Delta$ by V_Δ and we denote the space of all configurations $\{\psi : V_\Delta \to [q]\}$ by $\Omega_{q,\Delta}$. For a set $U \subset V_\Delta$ we denote by ∂U the set of vertices in U that are adjacent to some vertex in $V_\Delta \setminus U$. We refer to ∂U as the *boundary* of U. We denote by $U^\circ := U \setminus \partial U$ the *interior* of U. For $\psi \in \Omega_{q,\Delta}$ and $U \subset V_\Delta$ we denote the restriction of ψ to U by $\psi\restriction_U$.

Definition 2.2.1 (Gibbs measure)**.** We equip $\Omega_{q,\Delta}$ with the sigma algebra generated by sets of the form $\{\psi \in \Omega_{q,\Delta} \mid \psi\restriction_U = \phi\}$ where $U \subset V_\Delta$ is a finite set and $\phi : U \to [q]$ a fixed coloring of U. A probability measure μ on $\Omega_{q,\Delta}$ is called a *Gibbs measure* if for any finite set $U \subset V_\Delta$ and μ-almost every $\phi \in \Omega_{q,\Delta}$, we have

$$\Pr_\mu[\mathbf{\Phi}\restriction_{U^\circ} = \phi\restriction_{U^\circ} \mid \mathbf{\Phi}\restriction_{V_\Delta\setminus U^\circ} = \phi\restriction_{V_\Delta\setminus U^\circ}] = \Pr_{U;q,w}[\mathbf{\Phi}\restriction_{U^\circ} = \phi\restriction_{U^\circ} \big| \mathbf{\Phi}\restriction_{\partial U} = \phi\restriction_{\partial U}], \tag{2.2}$$

where the second probability $\Pr_{U;q,w}$ denotes the probability of seeing configuration ϕ on the finite graph $\mathbb{T}_\Delta[U]$ induced by U conditioned on the event of being equal to ϕ on ∂U. This latter probability is obtained by dividing the weight of $\phi\restriction_U$ by the sum of the weights of all colorings of U that agree with ϕ on ∂U, cf. (2.1). To lighten notation, we write $\Pr_U$ instead of $\Pr_{U;q,w}$.

Remark 1. Note that the conditional probability on the left-hand side of (2.2) cannot be computed using the standard formula for conditional probabilities, as we in general condition on an event of measure zero. Therefore the formalism of conditional expectations should be used to evaluate this conditional probability. See [FV17] for more details.

By a compactness argument one can show that there always is at least one Gibbs measure on $\Omega_{q,\Delta}$ cf. [FV17, BW99]. The question of whether there is a unique Gibbs measure can be reformulated in terms of a certain decay of correlations. To do so we require some definitions. Throughout Part I of this dissertation we write $d = \Delta - 1$.

Definition 2.2.2. Let d and n be natural numbers and let $\Delta = d+1$. We define the tree $\mathbb{T}_d^n$ to be the finite tree obtained from $\mathbb{T}_\Delta$ by fixing a root vertex r_d, deleting all vertices at distance more than n from the root, deleting one of the neighbors of r_d and keeping the connected component containing r_d. We denote the set of leaves of $\mathbb{T}_d^n$ by $\Lambda_{n,d}$, except when $n=0$, in which case we let $\Lambda_{0,d} = \{r_d\}$.

Note that in the rooted tree $\mathbb{T}_d^n$ each non leaf vertex has down degree d. We omit the reference to d when this is clear from the context. The next lemma reformulates uniqueness of the Gibbs measure in terms of the dependence on the distribution of the colors of the root vertex on the coloring of the leaves. While this result is well known we will provide a proof for convenience of the reader based on Brightwel and Winkler's proof [BW02, Theorem 3.3] for the case $w = 0$.

Lemma 2.2.3. *The q-state Potts model with parameter $w \geq 0$ on the infinite Δ-regular tree has a* unique *Gibbs measure if and only if for all colors $c \in [q]$ it holds that*

$$\limsup_{n\to\infty} \max_{\tau:\Lambda_{n,d}\to[q]} \left| Pr_{\mathbb{T}_d^n}[\mathbf{\Phi}(r_d) = c \mid \mathbf{\Phi}\restriction_{\Lambda_{n,d}} = \tau] - \frac{1}{q} \right| = 0. \tag{2.3}$$

Proof. We start with the 'if' part. Fix $d = \Delta - 1$ and $w \geq 0$ and let μ be any Gibbs measure on $\mathbb{T} = \mathbb{T}_\Delta$. Let $U \subset V = V(\mathbb{T})$ be a finite set. We aim to show that for any configuration $\psi : U \to [q]$, the probability

$$\Pr_\mu[\mathbf{\Phi}\restriction_U = \psi] \tag{2.4}$$

does not depend on μ.

We may assume that U induces a tree with each vertex of degree Δ or 1 by taking a larger finite set if needed. Suppose that U has ℓ leaves; denote the set of leaves by L. For $n \in \mathbb{Z}_{\geq 1}$ let W_n denote the collection of all vertices of $\mathbb{T}$ at distance at most n from U. The graph induced by $(W_n \setminus U) \cup L$ is the disjoint union of ℓ copies of $\mathbb{T}^n$ each rooted at a leaf of U, we denote the tree rooted at $u \in L$ with T_u. We claim

$$\lim_{n\to\infty} \max_{\rho_n:\partial W_n\to[q]} \left|\Pr_\mu[\mathbf{\Phi}\restriction_U = \psi \,\middle|\, \mathbf{\Phi}\restriction_{\partial W_n} = \rho_n] - \Pr_U[\mathbf{\Psi} = \psi]\right| = 0, \tag{2.5}$$

where $\mathbf{\Psi}$ is drawn from the Potts model distribution on $\mathbb{T}[U]$. This is sufficient because it follows that the difference

$$\begin{aligned}
&|\Pr_\mu[\mathbf{\Phi}\restriction_U = \psi] - \Pr_U[\mathbf{\Psi} = \psi]| = \\
&\left| \sum_{\rho_n:\partial W_n\to[q]} \Pr_\mu[\mathbf{\Phi}\restriction_{\partial W_n} = \rho_n] \cdot \left(\Pr_\mu[\mathbf{\Phi}\restriction_U = \psi \,\middle|\, \mathbf{\Phi}\restriction_{\partial W_n} = \rho_n] - \Pr_U[\mathbf{\Psi} = \psi]\right)\right| \leq \\
&\max_{\rho_n:\partial W_n\to[q]} \left|\Pr_\mu[\mathbf{\Phi}\restriction_U = \psi \,\middle|\, \mathbf{\Phi}\restriction_{\partial W_n} = \rho_n] - \Pr_U[\mathbf{\Psi} = \psi]\right|,
\end{aligned}$$

can be made arbitrarily small, from which we conclude that $\Pr_\mu[\mathbf{\Phi}\restriction_U = \psi] = \Pr_U[\mathbf{\Psi} = \psi]$. As this does not depend on μ it shows that μ is unique.

We now prove the claim. Let $\rho_n : \partial W_n \to [q]$ be arbitrary but fixed. Because μ satisfies the Gibbs property we see

$$\Pr_\mu[\mathbf{\Phi}\restriction_U = \psi \,\big|\, \mathbf{\Phi}\restriction_{\partial W_n} = \rho_n] = \Pr_{W_n}[\mathbf{\Phi}'\restriction_U = \psi \,\big|\, \mathbf{\Phi}'\restriction_{\partial W_n} = \rho_n], \tag{2.6}$$

where $\mathbf{\Phi}'$ is drawn from the Potts model distribution on $\mathbb{T}[W_n]$. We write $\phi \sim \psi$ if two configurations ϕ and ψ are equal where they are both defined. Moreover, we denote the weight of a configuration σ by $\mathrm{wt}(\sigma)$. By definition of the Potts model the right hand side of (2.6) as

$$\begin{aligned}
&\frac{\displaystyle\sum_{\substack{\sigma:W_n\to[q]\\ \sigma\sim\psi,\ \sigma\sim\rho_n}} \mathrm{wt}(\sigma)}{\displaystyle\sum_{\substack{\kappa:W_n\to[q]\\ \kappa\sim\rho_n}} \mathrm{wt}(\kappa)} = \frac{\mathrm{wt}(\psi)\displaystyle\sum_{\substack{(\sigma_u)_{u\in L},\,\sigma_u:T_u\to[q]\\ \sigma_u\sim\rho_n,\ \sigma_u\sim\psi}}\prod_{u\in L}\mathrm{wt}(\sigma_u)}{\displaystyle\sum_{\kappa:U\to[q]}\mathrm{wt}(\kappa)\sum_{\substack{(\gamma_u)_{u\in L},\,\gamma_u:T_u\to[q]\\ \gamma_u\sim\rho_n,\ \gamma_u\sim\kappa}}\prod_{u\in L}\mathrm{wt}(\gamma_u)} = \\
&\frac{\mathrm{wt}(\psi)\displaystyle\prod_{u\in L}\sum_{\substack{\sigma_u:T_u\to[q]\\ \sigma_u\sim\rho_n,\,\sigma_u\sim\psi}}\mathrm{wt}(\sigma_u)}{\displaystyle\sum_{\kappa:U\to[q]}\mathrm{wt}(\kappa)\prod_{u\in L}\sum_{\substack{\gamma_u:T_u\to[q]\\ \gamma_u\sim\rho_n,\,\gamma_u\sim\kappa}}\mathrm{wt}(\gamma_u)} = \\
&\frac{\mathrm{wt}\,(\psi)}{\sum_{\kappa:U\to[q]}\mathrm{wt}(\kappa)\displaystyle\prod_{u\in L}\frac{\Pr_{T_u}[\mathbf{\Phi}'_{\mathbf{u}}(u)=\kappa(u)\,\big|\,\mathbf{\Phi}'_{\mathbf{u}}\restriction_{\partial T_u}=\rho_n\restriction_{\partial T_u}]}{\Pr_{T_u}[\mathbf{\Phi}'_{\mathbf{u}}(u)=\psi(u)\,\big|\,\mathbf{\Phi}'_{\mathbf{u}}\restriction_{\partial T_u}=\rho_n\restriction_{\partial T_u}]}},
\end{aligned}$$

where $\partial T_u = T_u \cap \partial W_n$ and $\mathbf{\Phi}'_{\mathbf{u}}$ is drawn from the the Potts model distribution on $\mathbb{T}[T_u]$. As n goes to infinity the distance between the root u of T_u and its leaves becomes arbitrarily large. It therefore follows from equation (2.3) that the expression inside the final product gets arbitrarily close to 1 uniformly over all ρ_n. We can thus conclude that $\Pr_\mu[\mathbf{\Phi}\restriction_U = \psi \,\big|\, \mathbf{\Phi}\restriction_{\partial W_n} = \rho_n]$ converges to

$$\frac{\mathrm{wt}(\psi)}{\sum_{\kappa:U\to[q]}\mathrm{wt}(\kappa)} = \Pr_U[\mathbf{\Psi} = \psi]$$

uniformly, which was our claim.

For the 'only if' part we merely sketch the argument. Suppose the limsup is not equal to 0 for some color $c \in [q]$. Then there must be distinct colors c and c', a number $\varepsilon > 0$, a sequence $\{n_i\}$ of natural numbers and boundary conditions τ_i on the leaves of $\mathbb{T}_d^{n_i}$ such that the associated probabilities of the roots getting color c (resp. c') are at least $1/q + \varepsilon$ (resp. at most $1/q - \varepsilon$). Let τ_i' be the boundary condition on the leaves of $\mathbb{T}_d^{n_i}$ obtained from τ_i by flipping the colors c and c'. By symmetry, these respective probabilities are then reversed. We can then create two distinct Gibbs measures with a limiting process using the boundary conditions τ_i and τ_i' respectively. $\square$

We note that (2.3) is the property of uniqueness used in algorithmic applications [BGG$^+$20].

CHAPTER
THREE

UNIQUENESS OF THE GIBBS MEASURE FOR THE 4-STATE ANTI-FERROMAGNETIC POTTS MODEL ON THE REGULAR TREE

3.1 Organization

In this chapter we prove the following theorem.

Main Theorem of Chapter 3. *Let $\Delta \in \mathbb{N}_{\geq 5}$. Then for each $w \in [1 - \frac{4}{\Delta}, 1)$ the 4-state anti-ferromagnetic Potts model with edge interaction parameter w has a unique Gibbs measure on the infinite Δ-regular tree $\mathbb{T}_\Delta$.*

In the next section we discuss our approach towards proving the main theorem arriving at a geometric condition for uniqueness that we check in Section sec:proof main, deferring the verification of a crucial inequality to Section sec: Inequality section. Finally, in Section sec:conclude we finish with some concluding remarks and open questions.

3.2 Approach and preliminaries

Our main goal in this section is to derive a geometric condition for ratios of probabilities that implies uniqueness of the Gibbs measure on the Δ-regular tree. This condition will then be verified in the following sections. Along the way we will comment on how our approach relates to the approach of Galanis, Goldberg and Yang [GGY18].

3.2.1 Ratios of probabilities and the tree recursion

Instead of working directly with the probabilities we work with ratios of probabilities just as in [GGY18].

Let us introduce a few concepts to facilitate the discussion. Fix $n, d \in \mathbb{N}$, write $\mathbb{T}_d^n = (V, E)$ and let $\tau : \Lambda_{n,d} \to [q]$, see Definition 2.2.2. The map τ is called a *boundary condition on* $\Lambda_{n,d}$. We denote by

$$Z_\tau(\mathbb{T}_d^n) = \sum_{\substack{\sigma: V \to [q] \\ \sigma\restriction_{\Lambda_{n,d}} = \tau}} w^{m(\sigma)}, \tag{3.1}$$

the restricted partition function. For $i \in [q]$ we denote by $Z_{i,\tau}(\mathbb{T}_d^n)$ the sum (3.1) restricted to those σ that associate color i to the root vertex. We define the *ratio*

$$R_{i,\tau}(\mathbb{T}_d^n) = \frac{Z_{i,\tau}(\mathbb{T}_d^n)}{Z_{q,\tau}(\mathbb{T}_d^n)}. \tag{3.2}$$

Note that $R_{q,\tau}(\mathbb{T}_d^n) = 1$. We moreover remark that $R_{i,\tau}(\mathbb{T}_d^n)$ can be interpreted as the ratio of the probabilities that the root gets color i (resp. q) given the boundary condition τ on $\Lambda_{n,d}$.

We define for $n \geq 0$, $\widehat{\mathbb{T}}_d^n$ to be the rooted tree obtained from $\mathbb{T}_d^n$ by adding a new root $\hat{r}_d$ connecting it to the original root r_d with a single edge. Note that the set of non-root leaves of $\widehat{\mathbb{T}}_d^n$ is just $\Lambda_{n,d}$. For any boundary condition $\tau : \Lambda_{n,d} \to [q]$ on $\Lambda_{n,d}$ we define the restricted partition function, $Z_{i,\tau}(\widehat{\mathbb{T}}_d^n)$ and ratio $R_{i,\tau}(\widehat{\mathbb{T}}_d^n)$ analogously as for $\mathbb{T}_d^n$.

The next lemma provides a sufficient condition for $\mathbb{T}_\Delta$ to have a unique Gibbs measure in terms of these ratios, which we prove at the end of this section.

Lemma 3.2.1. *Let $q, d \in \mathbb{N}$ and $w \in (0,1)$. Suppose that for all $i \in [q-1]$ and for all $\delta > 0$ there exists $N > 0$ such that for all $n \geq N$ and for all boundary conditions $\tau : \Lambda_{n,d} \to [q]$ on $\Lambda_{n,d}$ we have*

$$|R_{i,\tau}(\widehat{\mathbb{T}}_d^n) - 1| < \delta,$$

then the tree $\mathbb{T}_\Delta$ with $\Delta = d+1$ has a unique Gibbs measure.

An advantage of working with the ratios of probabilities is that the well known tree recursion for the Potts model takes a convenient form.

Lemma 3.2.2. *Let $n, d \in \mathbb{N}$ and let $\tau : \Lambda_{n,d} \to [q]$ be a boundary condition on $\Lambda_{n,d}$. Let for $i = 1, \ldots, d$, $T_i = \widehat{\mathbb{T}}_d^{n-1}$ be the components of $\mathbb{T}_d^n - r_d$ where we attach a new root vertex to r_{d-1}. Let τ_i be the restriction of τ to $\Lambda_{n-1,d} \to [q]$ viewed as a subset of the vertices of T_i. Then we have for each $i \in [q-1]$,*

$$R_{i,\tau}(\widehat{\mathbb{T}}_d^n) = \frac{1 + w\prod_{s=1}^d R_{i,\tau_s}(\widehat{\mathbb{T}}_d^{n-1}) + \sum_{l \in [q-1]\setminus\{i\}} \prod_{s=1}^d R_{l,\tau_s}(\widehat{\mathbb{T}}_d^{n-1})}{w + \sum_{l \in [q-1]} \prod_{s=1}^d R_{l,\tau_s}(\widehat{\mathbb{T}}_d^{n-1})}. \tag{3.3}$$

For completeness we provide a proof for this lemma at the end of this section.

A direct analysis of the recursion in Lemma 3.2.2 is not straightforward, as it does not contract uniformly on a symmetric domain. In [GGY18] this is remedied by looking at the two-step recursion, that is they analyze the behaviour of the ratio at depth n as a function of the ratios at depth $n+2$. They show with substantial, yet rigorous, aid of a computer algebra package that this two-step recursion does contract on a symmetric domain (when $q = 3$ and w and d are as they should be). We however take a different, more geometric approach and work instead with the one-step recursion, as described in the next subsection.

3.2.2 A geometric condition for uniqueness

To state a geometric condition, we first introduce some functions that allow us to treat the tree recursion from Lemma 3.2.2 more concisely. Let $q \in \mathbb{Z}_{\geq 2}, d \in \mathbb{Z}_{\geq 1}$ and $w \in [0, 1)$. For $i \in [q]$ let μ_i be the map from $\mathbb{R}^q_{>0}$ to $\mathbb{R}_{>0}$ given by

$$\mu_i(x_1, \dots, x_q) = (w-1)x_i + \sum_{j=1}^{q} x_j.$$

Furthermore, we define

$$\tilde{G}(x_1, \dots, x_q) = (\mu_1(x_1, \dots, x_q), \dots, \mu_q(x_1, \dots, x_q))$$

and

$$\tilde{F}(x_1, \dots, x_q) = \tilde{G}(x_1^d, \dots, x_q^d).$$

Both $\tilde{F}$ and $\tilde{G}$ are homogeneous maps from $\mathbb{R}^q_{>0}$ to itself. For $x, y \in \mathbb{R}^q_{>0}$ we define an equivalence relation $x \sim y$ if and only if $x = \lambda y$ for some $\lambda > 0$. We define $\mathbb{P}^{q-1}_{>0} = \mathbb{R}^q_{>0}/\sim$ and denote elements of $\mathbb{P}^{q-1}_{>0}$ as $[x_1 : \cdots : x_q]$. We note that since $\tilde{F}$ and $\tilde{G}$ are homogeneous they are also well-defined as maps from $\mathbb{P}^{q-1}_{>0}$ to itself and from now on we consider them as such.

Let $\pi : \mathbb{P}^{q-1}_{>0} \to \mathbb{R}^{q-1}_{>0}$ be the projection map defined by $\pi([x_1 : \cdots : x_q]) = (x_1/x_q, \dots, x_{q-1}/x_q)$ with inverse $\iota : \mathbb{R}^{q-1}_{>0} \to \mathbb{P}^{q-1}_{>0}$ defined by $\iota(x_1, \dots, x_{q-1}) = [x_1 : \cdots : x_{q-1} : 1]$. Note that π and ι are continuous. We define the maps G, F from $\mathbb{R}^{q-1}_{>0}$ to itself by $\pi \circ \tilde{G} \circ \iota$ and $\pi \circ \tilde{F} \circ \iota$. Explicitly we have

$$G(x_1, x_2) = \left(\frac{wx_1 + x_2 + 1}{x_1 + x_2 + w}, \frac{x_1 + wx_2 + 1}{x_1 + x_2 + w} \right) \quad \text{and} \quad F(x_1, x_2) = G(x_1^d, x_2^d)$$

for $q = 3$. For $q = 4$ we have

$$G(x_1, x_2, x_3) = \left(\frac{wx_1 + x_2 + x_3 + 1}{x_1 + x_2 + x_3 + w}, \frac{x_1 + wx_2 + x_3 + 1}{x_1 + x_2 + x_3 + w}, \frac{x_1 + x_2 + wx_3 + 1}{x_1 + x_2 + x_3 + w} \right)$$

and $F(x_1, x_2, x_3) = G(x_1^d, x_2^d, x_3^d)$.

With this definition the recursion from Lemma 3.2.2 can now be stated as follows. Following the notation of the lemma, denote by x_l the following log convex combination of the ratios $R_{l,\tau_s}(\widehat{\mathbb{T}}_d^{n-1})$,

$$x_l = \left(\prod_{s=1}^{d} R_{l,\tau_s}(\widehat{\mathbb{T}}_d^{n-1})\right)^{1/d}. \tag{3.4}$$

Then

$$\left(R_{1,\tau}(\widehat{\mathbb{T}}_d^{n}), \ldots, R_{q-1,\tau}(\widehat{\mathbb{T}}_d^{n})\right) = F(x_1, \ldots, x_{q-1}). \tag{3.5}$$

We say that a subset $\mathcal{T} \subseteq \mathbb{R}_{>0}^n$ is *log convex* if $\log(\mathcal{T})$ is a convex subset of $\mathbb{R}^n$, where $\log(\mathcal{T})$ denotes the set consisting of elements of $\mathcal{T}$ with the logarithm applied to their individual entries. The next lemma gives sufficient conditions for uniqueness on the infinite regular tree.

Lemma 3.2.3. *Suppose that $q \geq 2$, $d \geq 2$ and $w > 0$ are such that there exists a sequence $\{\mathcal{T}_n\}_{n\geq 0}$ of log convex subsets of $\mathbb{R}_{>0}^{q-1}$ with the following properties.*

1. *Both the vector with every entry equal to $1/w$ and the vectors obtained from the all-ones vector with a single entry changed to w are elements of $\mathcal{T}_0$.*
2. *For every m we have $F(\mathcal{T}_m) \subseteq \mathcal{T}_{m+1}$.*
3. *For every $\varepsilon > 0$ there is an M such that for all $m \geq M$ every element of $\mathcal{T}_m$ has at most distance ε to the all-ones vector.*

Then the anti-ferromagnetic Potts model with parameter w has has a unique Gibbs measure on $\mathbb{T}_d$.

Proof. By Lemma 3.2.1, it suffices to show that regardless of the boundary condition τ on $\Lambda_{n,d}$, $R_{i,\tau}(\widehat{\mathbb{T}}_d^n) \to 1$ as $n \to \infty$.

First of all we claim that for all $n \geq 0$ and all boundary conditions $\tau : \Lambda_{n,d} \to [q]$ on $\Lambda_{n,d}$ we have $(R_{1,\tau}(\widehat{\mathbb{T}}_d^n), \ldots, R_{q-1,\tau}(\widehat{\mathbb{T}}_d^n)) \in \mathcal{T}_n$. We prove this by induction on n. For the base case, $n = 0$, we note that $\widehat{\mathbb{T}}_d^0$ consists of one free root $\hat{r}_d$, connected to a colored vertex v. If v is colored $i \in [q-1]$, then $(R_{1,\tau}(\widehat{\mathbb{T}}_d^0), \ldots, R_{q-1,\tau}(\widehat{\mathbb{T}}_d^0))$ consists of a w on position i and ones everywhere else. If v is colored q, then $(R_{1,\tau}(\widehat{\mathbb{T}}_d^0), \ldots, R_{q-1,\tau}(\widehat{\mathbb{T}}_d^0)) = (1/w, \ldots, 1/w)$. So the base case follows from item (1).

Suppose next that for some $n \geq 0$ the claim holds. Let $\tau : \Lambda_{n+1,d} \to [q]$ be any boundary condition on $\Lambda_{n+1,d}$. It then follows from (3.4), (3.5) and the assumptions that $\mathcal{T}_n$ is log convex and $F(\mathcal{T}_n) \subseteq (\mathcal{T}_{n+1})$ that

$$\left(R_{1,\tau}(\widehat{\mathbb{T}}_d^{n}), \ldots, R_{q-1,\tau}(\widehat{\mathbb{T}}_d^{n})\right) \in \mathcal{T}_{n+1},$$

completing the induction.

From the claim we just proved and item (3) it then follows that given $\varepsilon > 0$ there exists $N > 0$ such that for all $n \geq N$, any boundary condition τ on $\Lambda_{n,d}$ and color i, $|R_{i,\tau}(\widehat{\mathbb{T}}_d^n) - 1| < \varepsilon$. This concludes the proof. $\square$

In the next section we will construct a sequence of regions $\{\mathcal{T}_n\}_{n\geq 0}$ satisfying the conditions of the lemma. In Subsection sec: symmetry of F we describe a certain symmetry that the map F exhibits, corresponding to the symmetry of the colors in the Potts model. When a region $\mathcal{T}$ has a corresponding symmetry it is easier to understand the image $F(\mathcal{T})$. This is explained in Lemma 3.3.2. In Subsection sec: regions Tab we define a two parameter family of sets $\mathcal{T}_{a,b}$ that display the required symmetry. In Lemma 3.3.4 we prove that if simple analytic conditions in a and b are satisfied the sets $\mathcal{T}_{a,b}$ are log-convex. In Lemma 3.3.6 we give inner and outer approximations of the sets $\mathcal{T}_{a,b}$ with simple polytopes. This is convenient since the map G is a fractional linear transformation and therefore preserves convex sets. These are used in Lemma 3.4.1 in Subsection sec: main thm proof where we show that if more involved analytical conditions are satisfied $\mathcal{T}_{a,b}$ gets mapped strictly inside itself by F. We then combine all ingredients to prove Theorem 3.4.2. In the proof of Theorem 3.4.2 we show that we can construct a sequence $\mathcal{T}_n = \mathcal{T}_{a_n,b_n}$ that satisfies the conditions of Lemma 3.2.3 using the fact that we can keep satisfying the analytic conditions on a_n and b_n. This uses a number of technical inequalities whose verification we have moved to Section sec: Inequality section to preserve the flow of the text.

We finish this section be providing proofs of Lemma 3.2.1 and Lemma 3.2.2.

3.2.3 Proofs of Lemma 3.2.1 and Lemma 3.2.2

Proof of Lemma 3.2.1. The ratios for $\widehat{\mathbb{T}}_d^n$ and $\mathbb{T}_d^n$ can easily be expressed in terms of each other. Fix any $\tau : \Lambda_{n,d} \to [q]$. Then for any $i = 1, \dots, q-1$,

$$R_{i,\tau}(\widehat{\mathbb{T}}_d^n) = \frac{(w-1)R_{i,\tau}(\mathbb{T}_d^n) + \sum_{j=1}^{q-1} R_{j,\tau}(\mathbb{T}_d^n) + 1}{\sum_{j=1}^{q-1} R_{j,\tau}(\mathbb{T}_d^n) + w} \tag{3.6}$$

and

$$R_{i,\tau}(\mathbb{T}_d^n) = (R_{i,\tau}(\widehat{\mathbb{T}}_d^{n-1}))^d. \tag{3.7}$$

We may thus assume that for each $\delta > 0$ there exist N' such that for all $n \geq N'$,$|R_{i,\tau}(\mathbb{T}_d^n) - 1| < \delta$ for all $\tau : \Lambda_{n,d} \to [q]$.

For readability, we omit the reference to the subscript d in what follows. For any $i = 1, \dots, q$ we have

$$\Pr_{\mathbb{T}^n}[\mathbf{\Phi}(r) = i \mid \mathbf{\Phi}\restriction_{\Lambda_n} = \tau] = \frac{Z_{i,\tau}(\mathbb{T}^n)}{Z_\tau(\mathbb{T}^n)} = \frac{Z_{i,\tau}(\mathbb{T}^n)}{\sum_{j=1}^q Z_{j,\tau}(\mathbb{T}^n)}.$$

Hence for any $i \in [q]$, upon dividing both the numerator and denominator by $Z_{q,\tau}(\mathbb{T}^n)$, we obtain

$$\Pr_{\mathbb{T}^n}[\mathbf{\Phi}(r) = i \mid \mathbf{\Phi}\restriction_{\Lambda_n} = \tau] = \frac{R_{i,\tau}(\mathbb{T}^n)}{\sum_{i=1}^{q-1} R_{i,\tau}(\mathbb{T}^n) + 1}.$$

Now since the map

$$(x_1, \ldots, x_{q-1}) \mapsto \max_{i \in [q-1]} \left(\left| \frac{x_i}{\sum_{j=1}^{q-1} x_j + 1} - \frac{1}{q} \right|, \left| \frac{1}{\sum_{j=1}^{q-1} x_j + 1} - \frac{1}{q} \right| \right)$$

is continuous and maps $(1, \ldots, 1)$ to 0, it follows that for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|R_{i,\tau}(\mathbb{T}_n) - 1| < \delta$ for all boundary conditions τ on Λ_n and $i = 1. \ldots, q-1$, implies that

$$\max_{\tau:\Lambda_n \to [q]} \left| \Pr_{\mathbb{T}^n}[\mathbf{\Phi}(r) = i \mid \mathbf{\Phi}\restriction_{\Lambda_n} = \tau] - \frac{1}{q} \right| < \varepsilon.$$

We conclude that the conditions of Lemma 2.2.3 are satisfied and hence $\mathbb{T}_\Delta$ has a unique Gibbs measure. □

We next provide a proof for the tree recursion.

Proof of Lemma 3.2.2. For readability we omit d from the notation. We have

$$R_{i,\tau}(\widehat{\mathbb{T}}^n) = \frac{Z_{i,\tau}(\widehat{\mathbb{T}}^n)}{Z_{q,\tau}(\widehat{\mathbb{T}}^n)} = \frac{\sum_{l \in [q] \setminus \{i\}} Z_{l,\tau}(\mathbb{T}^n) + w Z_{i,\tau}(\mathbb{T}^n)}{\sum_{l \in [q-1]} Z_{l,\tau}(\mathbb{T}^n) + w Z_{q,\tau}(\mathbb{T}^n)}, \tag{3.8}$$

as a factor w is picked up when the unique neighbour of the root vertex is assigned the same color as the root vertex, $\hat{r}_d$, of $\widehat{\mathbb{T}}^n$. Note that for any color $c \in [q]$ we have $Z_{c,\tau}(\mathbb{T}^n) = \prod_{s=1}^{d} Z_{c,\tau_s}(\widehat{\mathbb{T}}^{n-1})$. Plugging this in into (3.8) and dividing the numerator and denominator by $\prod_{s=1}^{d} Z_{q,\tau_s}(\widehat{\mathbb{T}}^{n-1})$, we arrive at the desired expression. □

3.3 Ingredients for the proof of the main theorem

In this section we collect more properties of the recursion which together with Lemma 3.2.3 prove the main theorem.

3.3.1 Symmetry of the map F

In order to find suitable sets $\mathcal{T}$ such that $F(\mathcal{T}) \subseteq \mathcal{T}$ we will exploit a symmetry that the map F exhibits, due to the inherent symmetry of permuting the colors $[q]$ in the Potts model. To make this formal we will define a few self-maps on, and regions of, the spaces $\mathbb{P}^{q-1}_{>0}, \mathbb{R}^{q-1}_{>0}$ and $\mathbb{R}^{q-1}$. To avoid confusion, we will denote self-maps on and subsets of $\mathbb{P}^{q-1}_{>0}$ with a tilde, self-maps on and subsets of $\mathbb{R}^{q-1}_{>0}$ without additional notation and self-maps on and subsets of $\mathbb{R}^{q-1}$ with a hat. When a self-map or subset is used as an index, we will drop the hat or tilde in the index.

The three spaces $\mathbb{P}^{q-1}_{>0}, \mathbb{R}^{q-1}_{>0}$ and $\mathbb{R}^{q-1}$ are homeomorphic, with homeomorphisms $\pi : \mathbb{P}^{q-1}_{>0} \to \mathbb{R}^{q-1}_{>0}$ with inverse ι and $\log : \mathbb{R}^{q-1}_{>0} \to \mathbb{R}^{q-1}$ with inverse exp. We define the self-maps $\hat{G}, \hat{F}$ on $\mathbb{R}^{q-1}$ by $\hat{G} = \log \circ G \circ \exp$ and $\hat{F} = \log \circ F \circ \exp$. To summarize, we have the following diagram of continuous maps

$$\begin{tikzcd}
\mathbb{P}^{q-1}_{>0} \arrow[r, shift left, "\pi"] \arrow[d, "\tilde{G},\tilde{F}"] & \mathbb{R}^{q-1}_{>0} \arrow[l, shift left, "\iota"] \arrow[r, shift left, "\log"] \arrow[d, "G,F"] & \mathbb{R}^{q-1} \arrow[l, shift left, "\exp"] \arrow[d, "\hat{G},\hat{F}"] \\
\mathbb{P}^{q-1}_{>0} \arrow[r, shift left, "\pi"] & \mathbb{R}^{q-1}_{>0} \arrow[l, shift left, "\iota"] \arrow[r, shift left, "\log"] & \mathbb{R}^{q-1} \arrow[l, shift left, "\exp"] .
\end{tikzcd}$$

Let S_q denote the symmetric group on q elements. This group acts on $\mathbb{P}^{q-1}_{>0}$ be permuting the entries, which corresponds to permuting the colors in the Potts model. For $\sigma \in S_q$ we denote the map from $\mathbb{P}^{q-1}_{>0}$ to itself corresponding to this action by $\tilde{M}_\sigma$. We use this action to also define an action on $\mathbb{R}^{q-1}_{>0}$ by letting $M_\sigma(x) = (\pi \circ \tilde{M}_\sigma \circ \iota)(x)$. It is easy to see that the action of S_q on $\mathbb{P}^{q-1}_{>0}$ commutes with $\tilde{G}$ and $\tilde{F}$. It follows that the action of S_q on $\mathbb{R}^{q-1}_{>0}$ also commutes with F and G. Similarly, we define the map $\hat{M}_\sigma$ on $x \in \mathbb{R}^{q-1}$ by $\hat{M}_\sigma(x) = (\log \circ\, M_\sigma \circ \exp)(x)$ and we note that this action commutes with $\hat{G}$ and $\hat{F}$.

Example 3.3.1. As an example we present the table of the action of S_q for $q = 3$ on a point in all the three coordinates.

σ	id	(12)	(13)	(23)	(123)	(132)
$\tilde{M}_\sigma([x:y:z])$	$[x:y:z]$	$[y:x:z]$	$[z:y:x]$	$[x:z:y]$	$[z:x:y]$	$[y:z:x]$
$M_\sigma(x,y)$	(x,y)	(y,x)	$(1/x, y/x)$	$(x/y, 1/y)$	$(1/y, x/y)$	$(y/x, 1/x)$
$\hat{M}_\sigma(x,y)$	(x,y)	(y,x)	$(-x, y-x)$	$(x-y, -y)$	$(-y, x-y)$	$(y-x, -x)$

Note that in general $\hat{M}_\sigma$ is a linear map for all $\sigma \in S_q$. In fact, the map $\sigma \mapsto \hat{M}_\sigma$ is an irreducible representation of S_q called the standard representation, but we will not use this.

For any permutation $\tau \in S_q$ we define the following subset of $\mathbb{P}^{q-1}_{>0}$

$$\tilde{\mathcal{R}}_\tau = \left\{ [x_1 : \cdots : x_q] \in \mathbb{P}^{q-1}_{>0} : x_{\tau(1)} \leq x_{\tau(2)} \leq \cdots \leq x_{\tau(q)} \right\}.$$

Furthermore, we let $\mathcal{R}_\tau = \pi(\tilde{\mathcal{R}}_\tau)$ and $\hat{\mathcal{R}}_\tau = \log(\mathcal{R}_\tau)$. Note that if $x \in \mathbb{P}^{q-1}_{>0}$ has the property that $x_i \geq x_j$ then $\mu_i(x) \leq \mu_j(x)$, recalling that

$$\mu_i(x_1, \ldots, x_q) = (w-1)x_i + \sum_{j=1}^{q} x_j.$$

It follows that the map $\tilde{G}$ maps $\tilde{\mathcal{R}}_\tau$ into $\tilde{\mathcal{R}}_{\tau \circ m}$, where $m \in S_q$ denotes the permutation with $m(l) = q+1-l$ for $l \in [q]$. The same is true for $\tilde{F}$ because $x \mapsto x^d$ maps any $\tilde{\mathcal{R}}_\tau$ to itself. It follows that G and F map $\mathcal{R}_\tau$ into $\mathcal{R}_{\tau \circ m}$ and that $\hat{G}$ and $\hat{F}$ map $\hat{\mathcal{R}}_\tau$ into $\hat{\mathcal{R}}_{\tau \circ m}$. In Figure 3.1 the regions $\mathcal{R}_\tau$ and $\hat{\mathcal{R}}_\tau$ are depicted when $q = 3$.

The main purpose of the considerations of this section up until this point is to state and prove the following simple lemma.

Lemma 3.3.2. *Suppose $\mathcal{T} \subseteq \mathbb{R}^{q-1}_{>0}$ is a set such that $M_\sigma(\mathcal{T}) = \mathcal{T}$ for all $\sigma \in S_q$. Suppose also that there is a permutation $\tau \in S_q$ such that*

$$F(\mathcal{T} \cap \mathcal{R}_\tau) \subseteq \operatorname{int}(\mathcal{T}).$$

Then $F(\mathcal{T}) \subseteq \operatorname{int}(\mathcal{T})$.

Proof. Let $x \in \mathcal{T}$. There is a $\sigma \in S_q$ such that $M_\sigma(x) \in \mathcal{R}_\tau$ and thus $M_\sigma(x) \in \mathcal{T} \cap \mathcal{R}_\tau$. It follows from the assumption that $(F \circ M_\sigma)(x) \in \operatorname{int}(\mathcal{T})$. Because M_σ commutes with F we find that $(M_\sigma \circ F)(x) \in \operatorname{int}(\mathcal{T})$. We conclude that $F(x) \in M_\sigma^{-1}(\operatorname{int}(\mathcal{T})) = M_{\sigma^{-1}}(\operatorname{int}(\mathcal{T})) \subseteq \mathcal{T}$. Because M_σ is continuous it follows that $M_\sigma^{-1}(\operatorname{int}(\mathcal{T}))$ is an open subset of $\mathcal{T}$ and hence $F(x) \in \operatorname{int}(\mathcal{T})$. $\square$

In the next section we will define a family of regions $\mathcal{T}_{a,b}$ for $a, b > 1$ with the property $M_\sigma(\mathcal{T}_{a,b}) = \mathcal{T}_{a,b}$ for all $\sigma \in S_q$. Our goal will be to show that for certain choices of parameters (a, b) we have $F(\mathcal{T}_{a,b}) \subseteq \operatorname{int}(\mathcal{T}_{a,b})$. Because of Lemma 3.3.2 it will be enough to restrict ourselves to one well chosen region $\mathcal{R}_\tau$.

3.3.2 Definition and properties of the sets $\mathcal{T}_{a,b}$

For $q = 3$ and $q = 4$ we will define a family of log convex sets $\mathcal{T}_{a,b} \subseteq \mathbb{R}^{q-1}_{>0}$ with the property that $M_\sigma(\mathcal{T}_{a,b}) = \mathcal{T}_{a,b}$ for all $\sigma \in S_q$. We will do this by defining the convex sets $\hat{\mathcal{T}}_{a,b} \subseteq \mathbb{R}^{q-1}$ and then letting $\mathcal{T}_{a,b} = \exp(\hat{\mathcal{T}}_{a,b})$.

Let $a, b > 1$. To avoid having to write too many logarithms we let $\hat{a} = \log(a)$ and $\hat{b} = \log(b)$. For $q = 3$ we define the following half-space of $\mathbb{R}^2$

$$\hat{H}_{a,b} = \{(x, y) \in \mathbb{R}^2 : -\hat{b} \cdot x + \hat{a} \cdot y \leq \hat{a}\hat{b}\}.$$

Subsequently, we define

$$\hat{\mathcal{T}}_{a,b} = \bigcup_{\sigma \in S_3} \hat{M}_\sigma \left(\hat{\mathcal{R}}_{(23)} \cap \hat{H}_{a,b} \right).$$

Similarly, for $q = 4$, we define the half-space

$$\hat{H}_{a,b} = \{(x, y, z) \in \mathbb{R}^3 : -\hat{b} \cdot x + \hat{a} \cdot z \leq \hat{a}\hat{b}\}$$

and the region

$$\hat{\mathcal{T}}_{a,b} = \bigcup_{\sigma \in S_4} \hat{M}_\sigma \left(\hat{\mathcal{R}}_{(243)} \cap \hat{H}_{a,b} \right).$$

For both $q = 3$ and $q = 4$ we let $\mathcal{T}_{a,b} = \exp(\hat{\mathcal{T}}_{a,b})$. Figure 3.1 contains an image of $\hat{\mathcal{T}}_{a,b}$ and $\mathcal{T}_{a,b}$ for $q = 3$. Figure 3.2 contains an image of $\hat{\mathcal{T}}_{a,b}$ for $q = 4$; we highlighted the region $\hat{\mathcal{R}}_{(243)} \cap \hat{H}_{a,b}$ in orange. We have chosen to give the sets $\hat{\mathcal{T}}_{a,b}$ the same name for $q = 3$ and $q = 4$. This is because many of the properties of $\hat{\mathcal{T}}_{a,b}$ that we will prove hold for both cases and are proved in a similar way. Unless otherwise stated one should assume that any statement involving $\hat{\mathcal{T}}_{a,b}$ refers to the corresponding statement for both $q = 3$ and $q = 4$.

We first state a basic lemma relating the half-space representation and the vertex representation of a polytope. This lemma will be used a number of times in the remainder of the section to derive useful properties of the sets $\mathcal{T}_{a,b}$.

Lemma 3.3.3. *Let $H_1, \ldots, H_n$ be closed half-spaces in $\mathbb{R}^{n-1}$. Furthermore, let $p_1, \ldots, p_n \in \mathbb{R}^{n-1}$ with the property that for all $i \in [n]$ we have $p_i \in \operatorname{int}(H_i)$ and $p_i \in \partial H_j$ for $j \neq i$. Then*

$$\bigcap_{i=1}^{n} H_i = \operatorname{Conv}\left(\{p_1, \ldots, p_n\}\right),$$

where $\operatorname{Conv}(S)$ *denotes the convex hull of the set S.*

Proof. We give a sketch of the proof. The conditions on the p_i imply that the set $\{p_1, \ldots, p_n\}$ is affinely independent, i.e. the set $\{p_1 - p_n, \ldots, p_{n-1} - p_n\}$ is linearly independent. Therefore there exists an invertible affine transformation T with $T(v) = M(v - p_n)$ for some invertible linear transformation M, such that $T(p_i) = e_i$ for $i \in [n-1]$, where e_i denotes a standard basis vector. From the conditions on the p_i it follows that $T(H_i) = \{x \in \mathbb{R}^{n-1} : x_i \geq 0\}$ for $i \in [n-1]$ and $T(H_n) = \{x \in \mathbb{R}^{n-1} : \sum_{i=1}^{n-1} x_i \leq 1\}$. As affine transformations preserve convexity, we see

$$\begin{aligned}T\left(\bigcap_{i=1}^{n} H_i\right) &= \bigcap_{i=1}^{n} T(H_i) = \bigcap_{i=1}^{n-1}\{x\in\mathbb{R}^{n-1} : x_i \geq 0\} \cap \{x\in\mathbb{R}^{n-1} : \sum_{i=1}^{n-1} x_i \leq 1\}\\ &= \mathrm{Conv}\left(\{e_1,\dots,e_{n-1},0\}\right) = \mathrm{Conv}\left(\{T(p_1),\dots,T(p_n)\}\right)\\ &= T\left(\mathrm{Conv}\left(\{p_1,\dots,p_n\}\right)\right).\end{aligned}$$

The lemma now follows from the fact that T is invertible. □

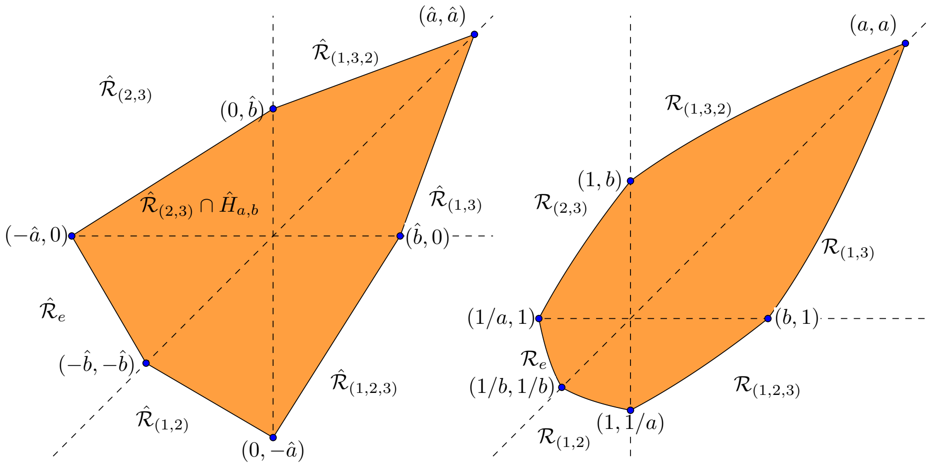


Figure 3.1: Images for $q=3$ of $\hat{\mathcal{T}}_{a,b}$ on the left and $\mathcal{T}_{a,b}$ on the right. The boundaries of the regions $\hat{\mathcal{R}}_\tau$ and $\mathcal{R}_\tau$ are drawn with dashed lines.

Lemma 3.3.4. *For $a,b\in\mathbb{R}_{>1}$ with $b\leq a\leq b^2$ we have that $\hat{\mathcal{T}}_{a,b}$ is convex, or equivalently, that $\mathcal{T}_{a,b}$ is log convex.*

Proof. Recall that we let $\hat a=\log(a)$ and $\hat b=\log(b)$ and observe that these are two positive real numbers. Also recall that the action of S_q on $\mathbb{R}^{q-1}$ is given by linear maps. It follows that the half-space $\hat H_{a,b}$ gets mapped to a half-space by $\hat M_\sigma$ for any $\sigma\in S_q$. We will show that for the choices of parameters stated in the lemma we have

$$\hat{\mathcal{T}}_{a,b} = \bigcap_{\sigma\in S_q}\hat M_\sigma\left(\hat H_{a,b}\right) \tag{3.9}$$

for both $q=3$ and $q=4$. This equality implies that $\hat{\mathcal{T}}_{a,b}$ is convex because an intersection of half-spaces is convex. In fact, it implies that $\hat{\mathcal{T}}_{a,b}$ is a convex polytope.

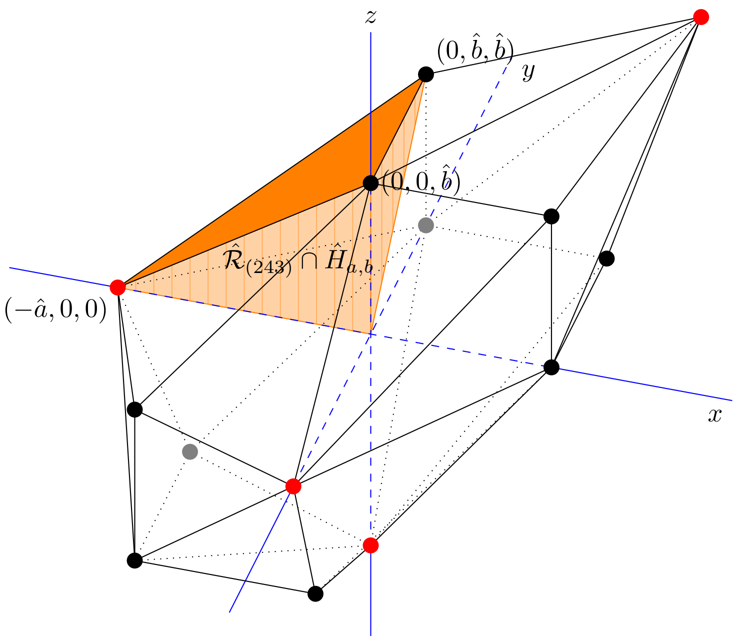


Figure 3.2: Image for $q = 4$ of $\hat{\mathcal{T}}_{a,b}$. The red dots depend on $\hat{a}$, the black dots depend on $\hat{b}$. In orange the region $\hat{\mathcal{R}}_{(243)} \cap \hat{H}_{a,b}$ is depicted.

We will first prove that the right-hand side of (3.9) is contained in the left-hand side. To that effect take an element $x \in \bigcap_{\sigma \in S_q} \hat{M}_\sigma \left(\hat{H}_{a,b}\right)$. Because the collection $\{\hat{\mathcal{R}}_\sigma\}_{\sigma \in S_q}$ covers $\mathbb{R}^{q-1}$, there is a $\tau \in S_q$ such that $x \in \hat{\mathcal{R}}_\tau$. For $q = 3$ let $\sigma \in S_3$ such that $\sigma \cdot (23) = \tau$. We see that $x \in \hat{\mathcal{R}}_\tau \cap \hat{M}_\sigma(\hat{H}_{a,b})$ and thus $x \in \hat{M}_\sigma \left(\hat{\mathcal{R}}_{(23)} \cap \hat{H}_{a,b}\right)$, from which it follows $x \in \hat{\mathcal{T}}_{a,b}$. Similarly, for $q = 4$ we let $\sigma \in S_4$ such that $\sigma \cdot (243) = \tau$. It follows in exactly the same way that $x \in \hat{\mathcal{T}}_{a,b}$.

The proof that the left-hand side of (3.9) is contained in the right-hand side is slightly more involved. Assume that $q = 3$. We first show that

$$\hat{\mathcal{R}}_{(23)} \cap \hat{H}_{a,b} = \operatorname{Conv}\left(\{(0,0), (-\hat{a}, 0), (0, \hat{b})\}\right), \tag{3.10}$$

While this is easily seen to be true from Figure 3.1, we provide a formal proof. Note that $\hat{\mathcal{R}}_{(23)}$ is the intersection of $\hat{H}_{x \leq 0} = \{(x, y) \in \mathbb{R}^2 : x \leq 0\}$ and $\hat{H}_{y \geq 0} = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$. One can check that $(0,0) \in \partial \hat{H}_{x \leq 0} \cap \partial \hat{H}_{y \geq 0} \cap \operatorname{int}(\hat{H}_{a,b})$, $(-\hat{a}, 0) \in \partial \hat{H}_{a,b} \cap \partial \hat{H}_{y \geq 0} \cap \operatorname{int}(\hat{H}_{x \leq 0})$ and $(0, \hat{b}) \in \partial \hat{H}_{a,b} \cap \partial \hat{H}_{x \leq 0} \cap \operatorname{int}(\hat{H}_{y \geq 0})$. Equation (3.10) then follows from Lemma 3.3.3.

We obtain

$$\begin{aligned}\hat{\mathcal{T}}_{a,b} &= \bigcup_{\sigma\in S_3} \hat{M}_\sigma\left(\mathrm{Conv}\left(\{(0,0),(-\hat{a},0),(0,\hat{b})\}\right)\right)\\ &= \bigcup_{\sigma\in S_3} \mathrm{Conv}\left(\{(0,0),\hat{M}_\sigma(-\hat{a},0),\hat{M}_\sigma(0,\hat{b})\}\right).\end{aligned}$$

We want to show that this is a subset of $\bigcap_{\sigma\in S_3}\hat{M}_\sigma\left(\hat{H}_{a,b}\right)$. Because all these half-spaces are convex, it is enough to show that the set

$$P = \{(0,0)\} \cup \bigcup_{\sigma\in S_3}\{\hat{M}_\sigma(-\hat{a},0),\hat{M}_\sigma(0,\hat{b})\}$$

is a subset of $\hat{M}_\tau(H_{a,b})$ for all $\tau\in S_3$. Because the set P is invariant under the action of S_3 it is sufficient to show that $P\subseteq \hat{H}_{a,b}$. We can calculate P explicitly to obtain

$$P = \{(0,0),(0,-\hat{a}),(-\hat{a},0),(\hat{a},\hat{a}),(0,\hat{b}),(\hat{b},0),(-\hat{b},-\hat{b})\}.$$

To check that these points lie in $\hat{H}_{a,b}$ we have to check that for each $(x,y)\in P$ we have $-\hat{b}\cdot x+\hat{a}\cdot y\le \hat{a}\hat{b}$. The inequality is trivially true for all but the points $(\hat{a},\hat{a})$ and $(-\hat{b},-\hat{b})$. One can confirm that the inequalities obtained by filling in these two points are simultaneously satisfied if and only if $\hat{b}/2\le\hat{a}\le 2\hat{b}$. Because $\hat{a}=\log(a)$ and $\hat{b}=\log(b)$ this is equivalent to $\sqrt{b}\le a\le b^2$. This shows that for these choices of a and b the left-hand side of (3.9) is contained in the right-hand side, which concludes the proof for $q=3$.

The proof for $q=4$ follows the same path. One can show in very similar way to the $q=3$ case that

$$\hat{\mathcal{R}}_{(243)}\cap\hat{H}_{a,b} = \mathrm{Conv}\left(\{(0,0,0),(-\hat{a},0,0),(0,0,\hat{b}),(0,\hat{b},\hat{b})\}\right)$$

and thus that

$$\hat{\mathcal{T}}_{a,b} = \bigcup_{\sigma\in S_4}\mathrm{Conv}\left(\{(0,0,0),\hat{M}_\sigma(-\hat{a},0,0),\hat{M}_\sigma(0,0,\hat{b}),\hat{M}_\sigma(0,\hat{b},\hat{b})\}\right).$$

It is again sufficient to show that

$$P = \{(0,0,0)\}\cup\bigcup_{\sigma\in S_4}\{\hat{M}_\sigma(-\hat{a},0,0),\hat{M}_\sigma(0,0,\hat{b}),\hat{M}_\sigma(0,\hat{b},\hat{b})\}$$

is a subset of $\hat{H}_{a,b}$. Explicitly we have

$$\begin{aligned}P = \{&(0,0,0),(-\hat{a},0,0),(0,-\hat{a},0),(0,0,-\hat{a}),(\hat{a},\hat{a},\hat{a}),(\hat{b},0,0),(0,\hat{b},0),(0,0,\hat{b}),\\ &(0,\hat{b},\hat{b}),(\hat{b},0,\hat{b}),(\hat{b},\hat{b},0),(0,-\hat{b},-\hat{b}),(-\hat{b},0,-\hat{b}),(-\hat{b},-\hat{b},0),(-\hat{b},-\hat{b},-\hat{b})\}.\end{aligned}$$

We need to check that for $(x, y, z) \in P$ we have $(x, y, z) \in \hat{H}_{a,b}$, that is, we have $-\hat{b} \cdot x + \hat{a} \cdot z \leq \hat{a}\hat{b}$. This inequality holds simultaneously for the points $(-\hat{b}, -\hat{b}, 0)$ and $(\hat{a}, \hat{a}, \hat{a})$ if and only $\hat{b} \leq \hat{a} \leq 2\hat{b}$. It can be seen that the inequalities obtained from the other points also hold for this regime of parameters. This concludes the proof that the left-hand side of (3.9) is contained in the right-hand side for $q = 4$, which is the final thing that we needed to show to prove the lemma. □

Lemma 3.3.2 states that it is enough to understand $F(\mathcal{T}_{a,b} \cap \mathcal{R}_\sigma)$ for a specific σ to understand the whole image $F(\mathcal{T}_{a,b})$. In the following two lemmas we calculate $\mathcal{T}_{a,b} \cap \mathcal{R}_\sigma$ more explicitly for $\sigma = (123)$ and $\sigma = (134)$ for $q = 3$ and $q = 4$ respectively. Because F maps $\mathcal{R}_{(123)}$ into $\mathcal{R}_{(23)}$ for $q = 3$ and $\mathcal{R}_{(134)}$ into $\mathcal{R}_{(243)}$ for $q = 4$, we describe $\mathcal{T}_{a,b} \cap \mathcal{R}_\sigma$ for these instances of σ too. The choice for these specific permutations σ is arbitrary, but does seem to make the upcoming analysis more pleasant than for some other choices.

Lemma 3.3.5. *Let $a, b \in \mathbb{R}_{>1}$ and define*

$$l_{a,b}(x) = b \cdot x^{\log(b)/\log(a)}$$

For $q = 3$ we have

$$\mathcal{T}_{a,b} \cap \mathcal{R}_{(23)} = \{(x, y) \in \mathbb{R}^2_{>0} : x \leq 1 \leq y \leq l_{a,b}(x)\}$$

and

$$\mathcal{T}_{a,b} \cap \mathcal{R}_{(123)} = \{(x, y) \in \mathbb{R}^2_{>0} : y \leq 1 \leq x \leq l_{a,b}(y)\}.$$

For $q = 4$ we have

$$\mathcal{T}_{a,b} \cap \mathcal{R}_{(243)} = \{(x, y, z) \in \mathbb{R}^3_{>0} : x \leq 1 \leq y \leq z \leq l_{a,b}(x)\}.$$

and

$$\mathcal{T}_{a,b} \cap \mathcal{R}_{(134)} = \{(x, y, z) \in \mathbb{R}^3_{>0} : z \leq y \leq 1 \leq x \leq y \cdot l_{a,b}(z/y)\}.$$

Proof. We will first prove the statement for $q = 3$. Recall that $\hat{\mathcal{T}}_{a,b} \cap \hat{\mathcal{R}}_{(23)} = \hat{\mathcal{H}}_{a,b} \cap \hat{\mathcal{R}}_{(23)}$, where $\hat{\mathcal{H}}_{a,b} = \{(x, y) \in \mathbb{R}^2 : -\hat{b} \cdot x + \hat{a} \cdot y \leq \hat{a}\hat{b}\}$ and $\hat{\mathcal{R}}_{(23)} = \{(x, y) \in \mathbb{R}^2 : x \leq 0 \leq y\}$. Therefore we can write

$$\hat{\mathcal{H}}_{a,b} \cap \hat{\mathcal{R}}_{(23)} = \{(\hat{x}, \hat{y}) \in \mathbb{R}^2 : \hat{x} \leq 0 \leq \hat{y} \leq \frac{\hat{b}}{\hat{a}} \cdot \hat{x} + \hat{b}\}.$$

If we replace $\hat{a}$ by $\log(a)$ and $\hat{b}$ by $\log(b)$ we find that

$$\mathcal{T}_{a,b} \cap \mathcal{R}_{(23)} = \exp(\hat{\mathcal{H}}_{a,b} \cap \hat{\mathcal{R}}_{(23)}) =$$
$$\{(e^{\hat{x}}, e^{\hat{y}}) \in \mathbb{R}^2_{>0} : \hat{x} \leq 0 \leq \hat{y} \leq \frac{\log(b)}{\log(a)} \cdot \hat{x} + \log(b)\}.$$

By applying exp to the individual components of the inequalities and replacing $x = e^{\hat{x}}$ and $y = e^{\hat{y}}$, we obtain the equality stated in the lemma. To prove the other equality for $q = 3$ we note that for $\sigma = (12)$ we have $\sigma(23) = (123)$ and thus $M_\sigma(\mathcal{T}_{a,b} \cap \mathcal{R}_{(23)}) = \mathcal{T}_{a,b} \cap \mathcal{R}_{(123)}$. For $(x,y) \in \mathbb{R}^2_{>0}$ we have $M_{(12)}(x,y) = (y,x)$ and thus $\mathcal{T}_{a,b} \cap \mathcal{R}_{(123)}$ equals

$$\{(y,x) \in \mathbb{R}^2_{>0} : x \leq 1 \leq y \leq l_{a,b}(x)\} = \{(x,y) \in \mathbb{R}^2_{>0} : y \leq 1 \leq x \leq l_{a,b}(y)\}.$$

To prove the statements given for $q = 4$ we recall that in that case $\hat{H}_{a,b} = \{(x,y,z) \in \mathbb{R}^3 : -\hat{b} \cdot x + \hat{a} \cdot z \leq \hat{a}\hat{b}\}$ and $\hat{\mathcal{R}}_{(243)} = \{(x,y,z) \in \mathcal{R}^3 : x \leq 0 \leq y \leq z\}$. Therefore

$$\hat{\mathcal{T}}_{a,b} \cap \hat{\mathcal{R}}_{(243)} = \hat{\mathcal{H}}_{a,b} \cap \hat{\mathcal{R}}_{(243)} = \{(\hat{x},\hat{y},\hat{z}) \in \mathbb{R}^3 : \hat{x} \leq 0 \leq \hat{y} \leq \hat{z} \leq \frac{\hat{b}}{\hat{a}} \cdot \hat{x} + \hat{b}\}.$$

Similarly, as in the $q = 3$ case, it follows that

$$\mathcal{T}_{a,b} \cap \mathcal{R}_{(243)} = \exp(\hat{\mathcal{T}}_{a,b} \cap \hat{\mathcal{R}}_{(243)}) = \{(x,y,z) \in \mathbb{R}^3_{>0} : x \leq 1 \leq y \leq z \leq l_{a,b}(x)\}.$$

If we let $\sigma = (13)(24)$ we have $\sigma \cdot (243) = (134)$. For this σ and $(x,y,z) \in \mathbb{R}^3_{>0}$ we have $M_\sigma(x,y,z) = (z/y, 1/y, x/y)$. We find that

$$\begin{aligned}
\mathcal{T}_{a,b} \cap \mathcal{R}_{(134)} &= M_\sigma(\mathcal{T}_{a,b} \cap \mathcal{R}_{(243)}) \\
&= \{(z/y, 1/y, x/y) \in \mathbb{R}^3_{>0} : x \leq 1 \leq y \leq z \leq l_{a,b}(x)\} \\
&= \{(x,y,z) \in \mathbb{R}^3_{>0} : z/y \leq 1 \leq 1/y \leq x/y \leq l_{a,b}(z/y)\} \\
&= \{(x,y,z) \in \mathbb{R}^3_{>0} : z \leq y \leq 1 \leq x \leq y \cdot l_{a,b}(z/y)\}.
\end{aligned}$$

□

The next lemma provides inner and outer approximations of the sets $\mathcal{T}_{a,b}$ with simple polytopes.

Lemma 3.3.6. *Let $a, b \in \mathbb{R}_{>1}$ with $a \geq b$. Then for q=3 we have*

$$\mathrm{Conv}(\{(1,1), (1/a, 1), (1, b)\}) \subseteq \mathcal{T}_{a,b} \cap \mathcal{R}_{(23)}$$

and

$$\mathcal{T}_{a,b} \cap \mathcal{R}_{(123)} \subseteq \mathrm{Conv}(\{(1,1), (b,1), (1, 1 - \frac{(b-1)\log(a)}{b\log(b)})\}).$$

For q=4 we have

$$\mathrm{Conv}(\{(1,1,1), (1/a,1,1), (1,b,b), (1,1,b)\}) \subseteq \mathcal{T}_{a,b} \cap \mathcal{R}_{(243)}$$

and

$$\mathcal{T}_{a,b} \cap \mathcal{R}_{(134)} \subseteq \mathrm{Conv}(\{(1,1,1), (b,1,1), (1,1/b,1/b), (1,1,1 - \frac{(b-1)\log(a)}{b\log(b)})\}).$$

Proof. Let $l_{a,b}$ be as in Lemma 3.3.5. We define $c = \log(b)/\log(a)$ so that we can write $l_{a,b}(x) = b \cdot x^c$. By assumption $c \leq 1$, therefore the function $l_{a,b}$ is concave and thus the sets

$$\{(x,y) \in \mathbb{R}^2_{>0} : y \leq l_{a,b}(x)\} \quad \text{and} \quad \{(x,y,z) \in \mathbb{R}^3_{>0} : z \leq l_{a,b}(x)\}$$

are convex. It follows now from Lemma 3.3.5 that the sets $\mathcal{T}_{a,b} \cap \mathcal{R}_{(23)}$ for $q = 3$ and $\mathcal{T}_{a,b} \cap \mathcal{R}_{(243)}$ for $q = 4$ are convex. It is easy to see that the former set contains the points $(1,1), (1/a, 1)$ and $(1, b)$ and that the latter set contains the points $(1,1,1), (1/a,1,1), (1,b,b)$ and $(1,1,b)$. This is enough to conclude that the first stated inclusions for $q = 3$ and $q = 4$ hold.

Because $l_{a,b}$ is concave we find that for all $x > 0$

$$l_{a,b}(x) \leq l'_{a,b}(1)(x-1) + l_{a,b}(1) = bc(x-1) + b. \tag{3.11}$$

Therefore, using Lemma 3.3.5, we have the following inclusion for $q = 3$

$$\begin{aligned} \mathcal{T}_{a,b} \cap \mathcal{R}_{(123)} &= \{(x,y) \in \mathbb{R}^2_{>0} : y \leq 1 \leq x \leq l_{a,b}(y)\} \\ &\subseteq \{(x,y) \in \mathbb{R}^2 : y \leq 1 \leq x \leq bc(y-1) + b\}. \end{aligned}$$

Note that in the latter set we do not require x and y to be positive. This set can also be written as the intersection of the following three half-spaces

$$\begin{aligned} &H_1 = \{(x,y) \in \mathbb{R}^2 : y \leq 1\},\ H_2 = \{(x,y) \in \mathbb{R}^2 : 1 \leq x\} \text{ and} \\ &H_3 = \{(x,y) \in \mathbb{R}^2 : x \leq bc(y-1) + b\}. \end{aligned}$$

Note that $(1,1) \in \partial H_1 \cap \partial H_2 \cap \operatorname{int}(H_3)$, $(b,1) \in \partial H_1 \cap \operatorname{int}(H_2) \cap \partial H_3$ and $(1, 1 - \frac{b-1}{bc}) \in \operatorname{int}(H_1) \cap \partial H_2 \cap \partial H_3$. The second inclusion for $q = 3$ stated in the lemma follows from Lemma 3.3.3.

From Lemma 3.3.5 and equation (3.11) we deduce that for q=4

$$\mathcal{T}_{a,b} \cap \mathcal{R}_{(134)} \subseteq \{(x,y,z) \in \mathbb{R}^3 : z \leq y \leq 1 \leq x \leq y \cdot (bc(z/y - 1) + b)\}.$$

So $\mathcal{T}_{a,b} \cap \mathcal{R}_{(134)}$ is contained in the intersection of the following half-spaces

$$\begin{aligned} H_1 &= \{(x,y,z) \in \mathbb{R}^3 : z \leq y\}, \quad H_2 = \{(x,y,z) \in \mathbb{R}^3 : y \leq 1\}, \\ H_3 &= \{(x,y,z) \in \mathbb{R}^3 : 1 \leq x\}, \quad H_4 = \{(x,y,z) \in \mathbb{R}^3 : x \leq bc(z-y) + by\}. \end{aligned}$$

We see that $(1,1,1) \in \partial H_1 \cap \partial H_2 \cap \partial H_3 \cap \operatorname{int}(H_4)$, $(b,1,1) \in \partial H_1 \cap \partial H_2 \cap \operatorname{int}(H_3) \cap \partial H_4$, $(1, 1/b, 1/b) \in \partial H_1 \cap \operatorname{int}(H_2) \cap \partial H_3 \cap \partial H_4$ and $(1,1,1 - \frac{b-1}{bc}) \in \operatorname{int}(H_1) \cap \partial H_2 \cap \partial H_3 \cap \partial H_4$. Using Lemma 3.3.3 we can conclude that the last stated inclusion in the lemma indeed holds. $\square$

3.4 Proof of the main theorem

In this section we prove the main theorem. We utilize a number of inequalities for which the proofs can be found in the next section.

Lemma 3.4.1. *Let $q \in \{3, 4\}$, $d \in \mathbb{Z}_{\geq 2}$ for $q = 3$ and $d \in \mathbb{Z}_{\geq 4}$ for $q = 4$ and let $1 - q/\Delta \leq w < 1$ where $\Delta = d + 1$. Let $a, b \in \mathbb{R}_{>1}$ such that*

$$\max\left\{b, \frac{b^d + w + q - 2}{wb^d + (q-1)}\right\} < a < b^{\frac{b^d(q-1+w)(b-1)}{(b^d-1)(b-w)}}. \tag{3.12}$$

Then $F(\mathcal{T}_{a,b}) \subseteq \mathrm{int}(\mathcal{T}_{a,b})$.

Proof. Recall from Section sec: Intro functions that we can write F as the composition $G \circ P$, where $P(x_1, \ldots, x_{q-1}) = (x_1^d, \ldots, x_{q-1}^d)$. In logarithmic coordinates the map $\hat{P} = \log \circ P \circ \exp$ acts as multiplication by d. In the proof of Lemma 3.3.4 we showed that $\hat{\mathcal{T}}_{a,b}$ is a polytope whose vertices have entries 0, $\pm\hat{a}$ or $\pm\hat{b}$. It follows that $\hat{P}(\hat{\mathcal{T}}_{a,b})$ is the same polytope where $\hat{a}$ and $\hat{b}$ are replaced by $d \cdot \hat{a}$ and $d \cdot \hat{b}$ respectively. Because $\hat{a} = \log(a)$ and $\hat{b} = \log(b)$, we can conclude that $P(\mathcal{T}_{a,b}) = \mathcal{T}_{a^d,b^d}$. It follows from Lemma 3.3.2 that it is enough to show that $G(\mathcal{T}_{a^d,b^d} \cap \mathcal{R}_{(123)}) = F(\mathcal{T}_{a,b} \cap \mathcal{R}_{(123)}) \subseteq \mathrm{int}(\mathcal{T}_{a,b})$ for $q = 3$ and $G(\mathcal{T}_{a^d,b^d} \cap \mathcal{R}_{(134)}) = F(\mathcal{T}_{a,b} \cap \mathcal{R}_{(134)}) \subseteq \mathrm{int}(\mathcal{T}_{a,b})$ for $q = 4$.

We use Lemma 3.3.6 to conclude that it is enough to show that

$$G\big(\,\mathrm{Conv}(\{(1,1), (b^d, 1), (1, 1 - \frac{(b^d-1)\log(a)}{b^d \log(b)})\})\big) \subseteq \mathrm{int}(\mathcal{T}_{a,b}) \tag{3.13}$$

for $q = 3$ and

$$\begin{aligned} G\big(\,\mathrm{Conv}(\{(1,1,1), (b^d,1,1), (1, 1/b^d, 1/b^d), (1,1,1 - \frac{(b^d-1)\log(a)}{b^d\log(b)})\})\big) \\ \subseteq \mathrm{int}(\mathcal{T}_{a,b}) \end{aligned} \tag{3.14}$$

for $q = 4$. We have to be careful here because initially we defined G as a map on $\mathbb{R}_{>0}^{q-1}$. We can extend G to the half-space $H = \{(x_1, \ldots, x_{q-1}) : x_1 + \cdots + x_{q-1} + w > 0\}$. To show that the sets in equations (3.13) and (3.14) are contained in H it is enough to show that the vertices of these convex hulls are contained in H. This is clear for all but the last written vertex in either case. We will show that the equation $x_1 + \cdots + x_{q-1} + w > 0$ does indeed hold for these two points.

Namely, by (3.12) we have

$$\begin{aligned} x_1 + \cdots + x_{q-1} + w &= q - 1 - \frac{(b^d - 1)\log(a)}{b^d \log(b)} + w \\ &> q - 1 - \frac{(b^d-1)}{b^d} \cdot \frac{b^d(q-1+w)(b-1)}{(b^d-1)(b-w)} + w \\ &= \frac{(1-w)(q-1+w)}{b-w} \geq 0, \end{aligned}$$

as desired.

The map G is a linear-fractional function, which means that G sends line segments to line segments (see e.g. Section 2.3.3 of [BV04]). Thus, for any set of points $p_1, \ldots, p_n$ we have $G(\mathrm{Conv}(\{p_1, \ldots, p_n\})) = \mathrm{Conv}\left(\{G(p_1), \ldots, G(p_n)\}\right)$. Let

$$f_q(x) = \frac{wx + q - 1}{x + q - 2 + w} \quad \text{and} \quad g(x) = \frac{(1+w)x + 2}{2x + 1 + w}.$$

The left-hand side of (3.13) is equal to

$$\mathrm{Conv}\left(\{(1,1), (f_3(b^d), 1), (1, f_3\left(1 - \frac{(b^d-1)\log(a)}{b^d \log(b)}\right))\}\right)$$

and the left-hand side of (3.14) is equal to

$$\begin{gathered} \mathrm{Conv}\Bigg(\{(1,1,1), (f_4(b^d),1,1), (1, g(1/b^d), g(1/b^d)), \\ (1, 1, f_4\left(1 - \frac{(b^d-1)\log(a)}{b^d \log(b)}\right)\}\Bigg). \end{gathered}$$

We can use Lemma 3.3.6 to see that it is enough to show that

$$f_q(b^d) > 1/a, \quad g(1/b^d) < b \quad \text{and} \quad f_q\left(1 - \frac{(b^d-1)\log(a)}{b^d \log(b)}\right) < b \tag{3.15}$$

to conclude that these sets are contained in $\mathcal{T}_{a,b} \cap \mathcal{R}_{(23)}$ and $\mathcal{T}_{a,b} \cap \mathcal{R}_{(243)}$ respectively. The first inequality follows directly from the assumptions. The second inequality follows from item (4) of Theorem 3.5.1 below. For the last inequality we note that f_q is strictly decreasing and $1 - \frac{(b^d-1)\log(a)}{b^d \log(b)}$ is also strictly decreasing in a. Therefore, it is enough to show the inequality for $a = b^{\frac{b^d(q-1+w)(b-1)}{(b^d-1)(b-w)}}$. We obtain

$$f_q\left(1 - \frac{(b^d-1)\log(a)}{b^d \log(b)}\right) < f_q\left(1 - \frac{(q-1+w)(b-1)}{b-w}\right) = b.$$

Because the inequalities in (3.15) are strict we can even conclude that $\mathcal{T}_{a,b}$ gets mapped strictly inside itself by F, i.e. $F(\mathcal{T}_{a,b}) \subseteq \text{int}(\mathcal{T}_{a,b})$. $\square$

Theorem 3.4.2. *Let $q \in \{3,4\}$, $d \in \mathbb{Z}_{\geq 2}$ for $q = 3$ and $d \in \mathbb{Z}_{\geq 4}$ for $q = 4$ and let $1 - q/\Delta \leq w < 1$ with $w > 0$ and $\Delta = d + 1$. Then the q-state Potts model with weight w on the infinite Δ-regular tree, $\mathbb{T}_\Delta$, has a unique Gibbs measure.*

Proof. We will construct a sequence of subsets $\{\mathcal{T}_n\}_{n\geq 0}$ as is described in Lemma 3.2.3. Define the functions

$$L(b) = \max\left\{b, \frac{b^d + w + q - 2}{wb^d + (q-1)}\right\} \quad \text{and} \quad U(b) = \min\left\{b^2, b^{\frac{b^d(q-1+w)(b-1)}{(b^d-1)(b-w)}}\right\}.$$

It follows from items (1), (2) and (3) of Theorem 3.5.1 that $L(b) < U(b)$ for $b > 1$. We define $M(b) = (L(b) + U(b))/2$ and note that $L(b) < M(b) < U(b)$ for $b > 1$. Any element of $\mathbb{R}^{q-1}_{>0}$ is contained in $\mathcal{T}_{b,b}$ for a large enough value of b. It follows that we can choose $b_0 > 1$ such that $\mathcal{T}_{b_0,b_0}$ contains both the vector with every entry equal to $1/w$ and the vectors obtained from the all-ones vector with a single entry changed to w. Because $M(b_0) > b_0$ we have $\mathcal{T}_{b_0,b_0} \subset \mathcal{T}_{M(b_0),b_0}$ and thus $\mathcal{T}_{M(b_0),b_0}$ contains these vectors too. Inductively we now define b_n for $n \geq 1$ by

$$b_n = \inf\left\{b : F(\mathcal{T}_{M(b_{n-1}),b_{n-1}}) \subseteq \mathcal{T}_{M(b),b}\right\}.$$

Because $\mathcal{T}_{M(b),b}$ moves continuously with b it follows from Lemma 3.4.1 that $\{b_n\}_{n\geq 1}$ is a strictly decreasing sequence. The sequence is clearly bounded below by 1 and thus it must have a limit. We claim that this limit is 1. For the sake of contradiction assume that it has a limit $b_\infty > 1$. The set $\mathcal{T}_{M(b_\infty),b_\infty}$ gets mapped strictly inside itself by F and thus there is a $b' < b_\infty$ such that $\mathcal{T}_{M(b_\infty),b_\infty}$ also gets mapped strictly inside $\mathcal{T}_{M(b'),b'}$. This is an open condition, so there is an $\varepsilon > 0$ such that $\mathcal{T}_{M(b),b}$ gets mapped strictly inside $\mathcal{T}_{M(b'),b'}$ for all $b \in [b_\infty, b_\infty + \varepsilon)$. There must be an integer N such that $b_N \in [b_\infty, b_\infty + \varepsilon)$, but then $b_{N+1} < b' < b_\infty$, so b_∞ cannot be the limit of the decreasing sequence $\{b_n\}_{n\geq 0}$.

We define $\mathcal{T}_n = \mathcal{T}_{M(b_n),b_n}$. We have $b < M(b) < b^2$, so it follows from Lemma 3.3.4 that every $\mathcal{T}_n$ is log-convex. We have chosen $\mathcal{T}_0$ such that condition (1) of Lemma 3.2.3 is satisfied. By construction $F(\mathcal{T}_m) \subseteq \mathcal{T}_{m+1}$ for all m and thus condition (2) of Lemma 3.2.3 is satisfied. Finally, because both b_n and $M(b_n)$ converge to 1, it follows that the sequence of sets $\mathcal{T}_n$ converges to the set consisting of just the all-ones vector. This means that condition (3) of Lemma 3.2.3 is satisfied. We can conclude that $\mathbb{T}_\Delta$ has a unique Gibbs measure. $\square$

Remark 2. The assumption $w > 0$ is critical in the case $q = 3$ and $d = 2$, as it is well known there are multiple Gibbs measures at $w_c = 0$ when $q = \Delta$. One sees this in our argument as well. For the base case of the induction, condition (1) of

Lemma 3.2.3, we need $\mathcal{T}_0$ to contain the vectors $(1,w)=(1,0)$, $(w,1)=(0,1)$ and $(1/w,1/w)=(\infty,\infty)$. If we take the log convex hull of these vectors and apply F, we obtain a region that again contains the vectors $(1,w)=(1,0)$, $(w,1)=(0,1)$ and $(1/w,1/w)=(\infty,\infty)$. It is thus possible to choose boundary conditions that yield unbounded ratios at an arbitrary distance from the leaves. This observation is closely related to the existence of so-called frozen colorings [BW00]. These give distinct *trivial* Gibbs measures, each supported on a single coloring of $\mathbb{T}_2$.

3.5 Proof of the inequalities

This section is dedicated to showing all the inequalities from the previous section are satisfied. We define the following functions

$$l(q,d,w,b)=\frac{b^d+q-2+w}{wb^d+q-1}, \qquad g(d,w,b)=\frac{2b^\Delta+w}{(1+w)b^d+2},$$
$$h(q,d,w,b)=\frac{b^d(b-1)(q-1+w)}{(b^d-1)\,(b-w)}, \qquad u(q,d,w,b)=b^{h(q,d,w,b)}.$$

We mostly consider these as functions in b and consider only $b\geq 1$. Note that $h(q,d,w,b)$ has a removable singularity in $b=1$ with $h(q,d,w,1)=\frac{q-1+w}{d(1-w)}$. The theorem we prove in this section is the following.

Theorem 3.5.1. *For $q=3, d\geq 2$ and $w\in[1-\frac{3}{\Delta},1)$ or for $q=4, d\geq 4$ and $w\in[1-\frac{4}{\Delta},1)$ with $\Delta=d+1$ we have for each $b>1$*

1. *$u(q,d,w,b)>l(q,d,w,b)$,*
2. *$u(q,d,w,b)>b$,*
3. *$b^2>l(q,d,w,b)$.*

And for all $b>1$ and $d\geq 3$ and $w\in[1-\frac{4}{\Delta},1)$ we have

(4) $g(d,w,b)<b$.

In the next section we show it is enough to prove Theorem 3.5.1 holds for $w=w_c=1-\frac{q}{\Delta}$ where we take $q=4$ in inequality (4). Subsequently, inequality (2) is proved in Corollary 3.5.5, inequality (3) is proved in Lemma 3.5.6 and inequality (4) is proved in Lemma 3.5.3. The proof of inequality (1) is the most involved and is the result of Lemma 3.5.7 and Lemma 3.5.8.

3.5.1 Reduction to $w = w_c$

Lemma 3.5.2. *Let $q \geq 2, d \geq 1$ and $w \in [0,1)$. For $b > 1$ we have $l(q,d,w,b)$ and $g(d,w,b)$ are decreasing in w, while $u(q,d,w,b)$ is increasing in w.*

Proof. We compute

$$\begin{aligned}
\frac{\partial}{\partial w} l(q,d,w,b) &= -\frac{(b^d-1)(b^d+q-1)}{(wb^d+q-1)^2}, \\
\frac{\partial}{\partial w} g(d,w,b) &= -\frac{2(b^{2d}-1)}{((1+w)b^d+2)^2}, \\
\frac{\partial}{\partial w} u(q,d,w,b) &= u(q,d,w,b) \cdot \frac{b^d(b-1)(b+q-1)\log b}{(b^d-1)(b-w)^2}.
\end{aligned}$$

We see that for $b > 1$ we have

$$\frac{\partial}{\partial w} l(q,d,w,b) < 0, \; \frac{\partial}{\partial w} g(d,w,b) < 0 \quad \text{and} \quad \frac{\partial}{\partial w} u(q,d,w,b) > 0,$$

so the lemma follows. □

From Lemma 3.5.2 it follows that if we can show Theorem 3.5.1 holds for $w = w_c$, then it also holds for all $w \in [w_c, 1)$. So from now on we will work with $l(b, w_c, d, q)$, $u(b, w_c, d, q)$ and $h(b, w_c, d)$. To shorten notation we write

$$\begin{aligned}
l(b) &= \frac{\Delta b^d + d(q-1) - 1}{(d-q+1)b^d + \Delta(q-1)}, \quad & g(b) &= \frac{\Delta b^d + d - 1}{(d-1)b^d + \Delta}, \\
h(b) &= \frac{dqb^d(b-1)}{(b^d-1)\left(\Delta(b-1)+q\right)}, \quad & u(b) &= b^{h(b)}.
\end{aligned}$$

We note that the function h has a removable singularity in 1 with $h(1) = 1$.

3.5.2 Inequalities $g(b) < b$, $u(b) > b$ and $b^2 > l(b)$

We will start by showing $g(b) < b$ holds for $b > 1$ and $d \geq 2$.

Lemma 3.5.3. *Let $d \geq 2$ and $b > 1$. Then we have $g(b) < b$.*

Proof. We have $g(1) = 1$ and $g'(1) = 1$. Furthermore, one can see

$$g''(b) = -\frac{4d^2\left(d^2-1\right)\left(b^d-1\right)b^{d-2}}{\left((d-1)b^d+\Delta\right)^3} < 0$$

for $d \geq 2$ and $b > 1$. This implies $g(b) < b$ for $d \geq 2$ and $b > 1$. □

Next we show that h is increasing in b. This fact will immediately give us inequality (2). Furthermore, it is also helpful in proving a sufficient condition for inequality (1) to hold, see Lemma 3.5.7 below.

Lemma 3.5.4. *For all $b > 1, d \geq 2$ and $q \geq 2$ we have $h'(b) > 0$.*

Proof. We compute

$$h'(b) = \frac{kdb^{d-1}\left(qb^{\Delta} - d\left(\Delta\right)b^2 + \left(2d^2 - dq + 2d - q\right)b - d^2 + dq - d\right)}{\left(b^d - 1\right)^2\left(\Delta(b-1) + q\right)^2}.$$

It suffices to show that

$$m(b) := qb^{\Delta} - d\left(\Delta\right)b^2 + \left(2d^2 - dq + 2d - q\right)b - d^2 + dq - d$$

is positive for $b > 1$. We compute

$$\begin{aligned} m'(b) &= \Delta(k(b^d - 1) - 2d(b-1)), \\ m''(b) &= d\Delta(kb^{d-1} - 2). \end{aligned}$$

We see $m''(b) > 0$ for $b > 1, d \geq 2$ and $q \geq 2$. Noting that $m'(1) = 0$ and $m(1) = 0$, it follows that $m'(b)$ and $m(b)$ are strictly positive for $b > 1$. $\square$

This immediately implies inequality (2).

Corollary 3.5.5. *For $b > 1$ we have $u(b) > b$.*

Proof. Recall $u(b) = b^{h(b)}$. As $h(1) = 1$ and $h'(b) > 0$ for $b > 1$ by Lemma 3.5.4, we see $u(b) > b$ for $b > 1$ follows. $\square$

Until this point, we did not need to assume $q = 3$ or $q = 4$ for the computations to work, but for inequality (3) to hold we do need some restrictions on q and d.

Lemma 3.5.6. *For $q = 3$ and $d \geq 2$ and for $q = 4$ and $d \geq 4$ we have $l(b) < b^2$, for all $b > 1$.*

Proof. Multiplying both sides of the inequality with the positive factor $(d - q + 1)b^d + \Delta(q-1)$ we obtain the equivalent inequality

$$\Delta b^d + d(q-1) - 1 < (d - q + 1)b^{d+2} + \Delta(q-1)b^2.$$

To show that this inequality holds we show that the polynomial

$$Q(b) = (d - q + 1)b^{d+2} - \Delta b^d + \Delta(q-1)b^2 - d(q-1) + 1$$

is strictly positive for $b > 1$. For $d \geq 2$ we compute

$$\begin{aligned} Q'(b) &= (d+2)(d-q+1)b^{\Delta} - \Delta d b^{d-1} + 2\Delta(q-1)b, \\ Q''(b) &= (d+2)\Delta(d-q+1)b^{d} - \Delta d(d-1)b^{d-2} + 2\Delta(q-1), \\ Q'''(b) &= d\Delta b^{d-3}\left((d+2)(d-q+1)b^2 - (d-1)(d-2)\right), \end{aligned}$$

Because $\Delta \geq k$ we find that for all $b \geq 1$

$$(d+2)(d-q+1)b^2 - (d-1)(d-2) \geq (d+2)(d-q+1) - (d-1)(d-2) = (6-q)d - 2q.$$

For $q = 3$ this quantity is nonnegative for $d \geq 2$ and for $q = 4$ this quantity is nonnegative for $d \geq 4$. So in our case we can conclude that $Q'''(b) \geq 0$ for all $b \geq 1$. As we have $Q''(1) = d\Delta > 0$, $Q'(1) = 3d > 0$ and $Q(1) = 0$, it follows that $Q''(b), Q'(b)$ and $Q(b)$ are strictly positive for $b > 1$. □

3.5.3 The inequality $u(b) > l(b)$

The following lemma contains a sufficient condition to prove this inequality. In the remainder of the section we prove that this condition is satisfied.

Lemma 3.5.7. *Suppose for all $b > 1$ we have*

$$\frac{l'(b)}{l(b)} < \frac{h(b)}{b} + 2\frac{b-1}{b+1}g'(b). \tag{3.16}$$

Then $u(b) > l(b)$ for all $b > 1$.

Proof. As $l(b)$ and $u(b)$ are strictly positive for $b \geq 1$, we can define

$$F(b) = \log(u(b)) - \log(l(b))$$

for $b \geq 1$. Then we have

$$F'(b) = \frac{u'(b)}{u(b)} - \frac{l'(b)}{l(b)} = \frac{h(b)}{b} + \log(b)h'(b) - \frac{l'(b)}{l(b)}.$$

For $b > 1$ we have that $h'(b) > 0$ by Lemma 3.5.4 and $\log(b) > 2(b-1)/(b+1)$, therefore

$$F'(b) > \frac{h(b)}{b} + 2\frac{b-1}{b+1}h'(b) - \frac{l'(b)}{l(b)},$$

which is positive by (3.16). It is easy to see $F'(1) = 0$. Hence F has a global minimum in $b = 1$. As $F(1) = 0$, it follows that $u(b) > l(b)$ for all $b > 1$, which is what we wanted to show. □

This lemma is useful because proving the inequality $u(b) > l(b)$ for all $b > 1$ can now be reduced to proving inequalities involving rational functions and with some work to inequalities involving only polynomials. The next lemma shows that (3.16) holds. For this to work we do need to restrict to $q = 3$ and $d \geq 2$ or $q = 4$ and $d \geq 4$.

Lemma 3.5.8. *For $q = 3$ and $d \geq 2$ and for $q = 4$ and $d \geq 4$ and any $b > 1$ we have*

$$\frac{l'(b)}{l(b)} < \frac{h(b)}{b} + 2\frac{b-1}{b+1}h'(b).$$

Proof. We introduce the following polynomials

$$\begin{aligned} p(b) &= \Delta b^d + d(q-1) - 1, & q(b) &= (d-q+1)b^d + \Delta(q-1), \\ s(b) &= dqb^d(b-1), & t(b) &= \left(b^d - 1\right)(\Delta(b-1) + q). \end{aligned}$$

Thus $l(b) = p(b)/q(b)$ and $h(b) = s(b)/t(b)$. Furthermore, we define $r(b) = q(b)p'(b) - p(b)q'(b)$ and $v(b) = t(b)s'(b) - s(b)t'(b)$. It is worth noting that $r(b)$ simplifies to $q^2d^2b^{d-1}$. The inequality we want to prove can now be written as

$$\frac{r(b)}{p(b)q(b)} < \frac{s(b)}{b \cdot t(b)} + 2\frac{(b-1)v(b)}{(b+1)t(b)^2}.$$

For $b > 1$ the quantity $b(b+1)p(b)q(b)t(b)^2$ is strictly positive and thus it is equivalent to prove the inequality, where we have multiplied both sides by this term. We see that it is enough to prove that the following polynomial is strictly positive for all $b > 1$

$$P(b) = (b+1)s(b)p(b)q(b)t(b) + 2b(b-1)v(b)p(b)q(b) - b(b+1)r(b)t(b)^2. \quad (3.17)$$

It can be checked that the terms $s(b)$, $b \cdot v(b)$ and $b \cdot r(b)$ all contain a factor qdb^d and thus $P_0(b) = P(b)/(kdb^d)$ is a polynomial in b whose coefficients are polynomials in d. The remainder of the proof will be dedicated to showing that $P_0(b)$ is strictly positive for $b > 1$.

To avoid ambiguity later, we prove this for $q = 3$ in the two cases $d = 2$ and $d = 3$ separately. For $d = 2$ we have

$$P_0(b) = 54(b-1)^6 + 54(b-1)^5$$

and for $d = 3$ we have

$$\begin{aligned} P_0(b) =& 16(b-1)^{12} + 212(b-1)^{11} + 1236(b-1)^{10} + 4116(b-1)^9 + 8793(b-1)^8 \\ &+ 12789(b-1)^7 + 12123(b-1)^6 + 6318(b-1)^5 + 1458(b-1)^4. \end{aligned}$$

In both cases all the coefficients of $P_0(b)$ are strictly positive when written as a polynomial in $b-1$ and thus the polynomials are strictly positive for $b > 1$.

We will now assume that $d \geq 4$. It can be seen by cross-multiplying the terms in the individual polynomials in (3.17) that the only coefficients of $P_0(b)$ that can be non-zero appear in the $b^{i \cdot d+j}$ terms where $i, j \in \{0,1,2,3\}$. The exact coefficients are recorded in Table 3.1. For $n \in \{1,2,3\}$ we inductively define the polynomials $P_n(b) = P_{n-1}^{(4)}(b)/b^{d-4}$. Note that in this way $P_n(b)$ is a polynomial whose only non-zero coefficients appear in the $b^{i \cdot d+j}$ term, where $0 \leq i \leq 3-n$ and $j \in \{0,1,2,3\}$.

Table 3.1: The coefficients of $P_0(b)$ for $d \geq 4$.

Term of $P_0(b)$	Coefficient $q=3$	Coefficient $q=4$
b^0	$(d-2)\left(8d^3-3d^2+2\right)$	$(d-3)\left(18d^3-d^2+3\right)$
b^1	$-24d^4+19d^3+60d^2-24d-14$	$-54d^4+67d^3+173d^2-39d-27$
b^2	$\Delta(3d-1)\left(8d^2+d-16\right)$	$\Delta\left(54d^3-23d^2-124d+33\right)$
b^3	$-\Delta^2\left(8d^2+3d-2\right)$	$-\Delta^2\left(18d^2+7d-3\right)$
b^d	$(d-2)\left(8d^3+4d^2-d-6\right)$	$(d-3)\left(12d^3+3d^2-2d-9\right)$
b^Δ	$-3\left(8d^4-8d^3+15d^2-19d-14\right)$	$-36d^4+73d^3-129d^2+99d+81$
b^{d+2}	$3\left(8d^4-4d^3+15d^2+20d-16\right)$	$36d^4-47d^3+115d^2+163d-99$
b^{d+3}	$-\Delta\left(8d^3-8d^2-d+6\right)$	$-\Delta\left(12d^3-19d^2-6d+9\right)$
b^{2d}	$2(d-2)\Delta\left(d^2-2d+3\right)$	$(d-3)\Delta\left(2d^2-5d+9\right)$
$b^{2\Delta}$	$-3\left(2d^4-4d^3+3d^2+14\Delta 4\right)$	$-6d^4+21d^3-37d^2-81d-81$
b^{2d+2}	$3\left(2d^4-2d^3-7\Delta 6\right)$	$6d^4-15d^3+19d^2-53d+99$
b^{2d+3}	$-(d-2)\Delta\left(2d^2+2d+3\right)$	$-(d-3)\Delta\left(2d^2+d+3\right)$
b^{3d}	$(d-2)^2\Delta$	$(d-3)^2\Delta$
$b^{3\Delta}$	$-(d-2)\Delta(d+7)$	$-(d-3)\Delta(d+9)$
b^{3d+2}	$-(d-8)(d-2)\Delta$	$-(d-11)(d-3)\Delta$
b^{3d+3}	$(d-2)\Delta^2$	$(d-3)\Delta^2$

The values of $P_j^{(i)}(1)$ as a polynomial in $x = d-4$, up to a common positive multiplicative factor, for $q=3$ and $q=4$ are contained in tables 3.2 and 3.3 respectively. These polynomials have only nonnegative coefficients, from which it follows that their values are nonnegative for all $d \geq 4$.

The polynomial $P_3(b)$ is a cubic polynomial and thus its third derivative $P_3^{(3)}(b)$ is constant. Its exact value, which is recorded in Table 3.2 for $q=3$ and in Table 3.3 for $q=4$, is strictly positive for all $x \geq 0$, i.e. for all $d \geq 4$. We claim that it now follows inductively that $P_j^{(i)}(b)$ is strictly positive for all $b > 1$. Namely, suppose that for $i \in \{0,1,2,3\}$ we have shown that $P_j^{(i+1)}(b)$ is strictly positive for $b > 1$. Then it follows that $P_j^{(i)}(b)$ is strictly increasing. Because $P_j^{(i)}(1) \geq 0$ (cf. Table 3.2 and Table 3.3), we can conclude from this that $P_j^{(i)}(b)$ is also strictly positive for $b > 1$. Furthermore, if $P_{j+1}(b) > 0$ for $b > 1$ then the same follows for $P_j^{(4)}$ because $b^{d-4} \cdot P_{j+1}(b) = P_j^{(4)}(b)$. In conclusion, it follows that $P_0(b) > 0$ for $b > 1$, which is what we set out to prove.

$\square$

Table 3.2: The values of $P_j^{(i)}(1)$ for $q = 3$ in the variable $x = d - 4$ divided by $6(x+4)^3(x+5)$ for $i, j \in \{0, 1, 2, 3\}$.

$P_0(1)$	0
$P_0^{(1)}(1)$	0
$P_0^{(2)}(1)$	0
$P_0^{(3)}(1)$	0
$P_1(1)$	$54(x+2)$
$P_1^{(1)}(1)$	$3\left(122x^2+759x+1045\right)$
$P_1^{(2)}(1)$	$3\left(478x^3+5019x^2+16831x+17560\right)$
$P_1^{(3)}(1)$	$4276x^4+61731x^3+328134x^2+754415x+623616$
$P_2(1)$	$4\left(2864x^5+51218x^4+363231x^3+1272211x^2+2188942x+1467858\right)$
$P_2^{(1)}(1)$	$2\left(8800x^6+200624x^5+1895748x^4+9479789x^3+26371144x^2+38515725x+22913226\right)$
$P_2^{(2)}(1)$	$4\left(6100x^7+166078x^6+1935943x^5+12502085x^4+48198140x^3+110605547x^2+139341417x+73916010\right)$
$P_2^{(3)}(1)$	$4(x+2)\left(7948x^7+229772x^6+2871108x^5+20093453x^4+85033465x^3+217534941x^2+311415975x+192411450\right)$
$P_3(1)$	$12(x+2)\left(3324x^8+105498x^7+1478477x^6+11945536x^5+60841362x^4+199973638x^3+414113609x^2+493884000x+259667100\right)$
$P_3^{(1)}(1)$	$4(x+2)(x+5)(2x+9)(3x+13)\left(372x^6+10607x^5+124569x^4+775749x^3+2712487x^2+5063412x+3950100\right)$
$P_3^{(2)}(1)$	$4(x+2)(x+5)^2(x+6)(2x+9)(3x+13)(3x+14)\left(12x^4+368x^3+3431x^2+13148x+18249\right)$
$P_3^{(3)}(1)$	$36(x+2)(x+5)^4(x+6)(x+7)(2x+9)(2x+11)(3x+13)(3x+14)$

Table 3.3: The values of $P_j^{(i)}(1)$ for $q = 4$ in the variable $x = d - 4$ divided by $8(x+4)^3(x+5)$ for $i, j \in \{0, 1, 2, 3\}$.

$P_0(1)$	0
$P_0^{(1)}(1)$	0
$P_0^{(2)}(1)$	0
$P_0^{(3)}(1)$	0
$P_1(1)$	$48x$
$P_1^{(1)}(1)$	$8\left(44x^2+215x+135\right)$
$P_1^{(2)}(1)$	$2\left(709x^3+6684x^2+18585x+12690\right)$
$P_1^{(3)}(1)$	$4134x^4+55427x^3+265045x^2+515121x+308889$
$P_2(1)$	$4\left(2699x^5+45392x^4+297314x^3+933894x^2+1366575x+695142\right)$
$P_2^{(1)}(1)$	$2\left(8020x^6+174193x^5+1549849x^4+7170108x^3+17941968x^2+22435155x+10317699\right)$
$P_2^{(2)}(1)$	$2\left(10878x^7+284024x^6+3149973x^5+19136364x^4+68251666x^3+141154110x^2+153239211x+64122030\right)$
$P_2^{(3)}(1)$	$3(x+1)\left(9316x^7+267882x^6+3329185x^5+23169850x^4+97489094x^3+247912018x^2+352706325x+216527850\right)$
$P_3(1)$	$12(x+1)\left(2889x^8+91143x^7+1269517x^6+10192836x^5+51576597x^4+168375593x^3+346232169x^2+409934700x+213929100\right)$
$P_3^{(1)}(1)$	$3(x+1)(x+5)(2x+9)(3x+13)\left(408x^6+11669x^5+136865x^4+848735x^3+2949267x^2+5463996x+4227300\right)$
$P_3^{(2)}(1)$	$6(x+1)(x+5)^2(x+6)(2x+9)(3x+13)(3x+14)\left(6x^4+193x^3+1807x^2+6883x+9471\right)$
$P_3^{(3)}(1)$	$27(x+1)(x+5)^4(x+6)(x+7)(2x+9)(2x+11)(3x+13)(3x+14)$

We have shown the inequalities (1), (2), (3) all hold, thus the conditions in Lemma 3.4.1 and Lemma 3.3.4 are satisfied.

3.6 Concluding remarks

We conclude with some remarks concerning the possibility of expanding our approach and with some questions.

Generalisation The biggest challenge to generalizing our method to other values of (q, Δ) comes from the fact that inequality (3) from Theorem 3.5.1 is not necessarily true for all $b > 1$. This suggests that it might not be possible in all cases to find arbitrarily large log convex regions that get mapped into themselves. We suspect that in general this is indeed impossible when one requires the regions to have the symmetry that we use in this chapter, that is regions $\mathcal{T} \subset \mathbb{R}_{>0}^{q-1}$ with $M_\sigma(\mathcal{T}) = \mathcal{T}$ for all $\sigma \in S_q$. A consequence is that in some cases we cannot make the region large enough to start the induction laid out in Lemma 3.2.3. Fortunately, inequality (3) does hold near $b > 1$ for all (q, Δ) with $\Delta \geq q$ and $w \geq w_c$. This suggests that, at least when w_c is close to 1, i.e. when Δ is large enough compared to q, our methods could still be applied. Moreover, it might be possible to find a separate argument to show that the ratios of $\hat{\mathbb{T}}_d^n$ get at least moderately close to 1 for some n. This could then be used to bootstrap the induction in Lemma 3.2.3.

There are two more complications that prevent us from applying our method directly to other values of (q, Δ). We suspect that these can be overcome with more thorough analysis. The first one comes from the fact that inequality (1) from Theorem 3.5.1 is no longer satisfied for most values of (q, Δ) and w_c. The precise form of this inequality highly depends on our method of proof and specifically on our choice of upper bound for $l_{a,b}$ in equation (3.11). Computer analysis suggests that by taking different upper bounds for $l_{a,b}$, specifically taking tangent lines at different points, the proof that $\mathcal{T}_{a,b}$ gets mapped into itself for a and b near 1 can be salvaged. The other complication appears when $q \geq 5$. In this case one obtains more inequalities analogous to inequality (4) from Theorem 3.5.1. These are not all satisfied when we take a naive generalization of the region $\mathcal{T}_{a,b}$. We suspect that this can be remedied by letting the regions depend on more than just two parameters. This leads to the analysis specifically of the log convexity of the regions becoming more involved. Of course, it is also possible to work with different log convex regions, for example the regions we work with in Chapter 4.

The case $(q, \Delta) = (4, 4)$ Unfortunately, our approach does not allows us to handle the case $(q, \Delta) = (4, 4)$. We briefly explain the complications. In inequality (3.11) we use the tangent line of $l_{a,b}(x)$ at $x = 1$ to upper bound $l_{a,b}(x)$; this makes the calculus easier and this choice works for $q = 4$ and $\Delta \geq 5$. We have evidence that by using the tangent line at a different point in inequality (3.11) the calculations that follow from this upper bound also work for the case $q = 4$ and $\Delta = 4$. However, we can show that inequality (3) in Theorem 3.5.1 fails when $(q, \Delta) = (4, 4)$ and $b > 1$ is large enough, meaning that in that case the set $\mathcal{T}_{a,b}$ cannot both be log convex and satisfy $F(\mathcal{T}_{a,b}) \subset \mathcal{T}_{a,b}$. For $b > 1$ close enough to 1 inequality (3) in Theorem 3.5.1 does hold. We suspect that our approach can be tweaked to show uniqueness for all $w \in (0, 1)$ when $(q, \Delta) = (4, 4)$, possibly by finding a separate argument to show that the ratios of $\hat{\mathbb{T}}_3^n$ get at least moderately close to 1 for some n, bootstrapping the induction in Lemma 3.2.3.

Zero-free region Our final comment is related to the following question. Given $q \in \mathbb{N}$ and $\Delta \geq q$. Does there exist a region U in $\mathbb{C}$ containing the interval $(1 - \frac{q}{\Delta}, 1]$ such that for any $w \in U$ and any graph G of maximum degree Δ the partition function $Z(G, q, w) \neq 0$? (If so this would yield an efficient algorithm for approximately computing $Z(G, q, w)$ in this region by Barvinok's method [Bar16] combined with [PR17].)

Following [BDPR21], to prove this, for say $q = 3$, we would essentially need to find a log convex set $S \subsetneq \mathbb{C}^2$ such that the map F maps S into S and such that S satisfies some additional properties that we will not discuss here. We suspect that the sets $\mathcal{T}_{a,b}$ we have constructed may be helpful in determining whether such a set S can be constructed.

CHAPTER
FOUR

UNIQUENESS OF THE GIBBS MEASURE FOR THE ANTI-FERROMAGNETIC POTTS MODEL ON THE INFINITE Δ-REGULAR TREE FOR LARGE Δ

4.1 Organization

In this chapter we prove the following theorem.

Main Theorem of Chapter 4. *For each integer $q \geq 5$ there exists $\Delta_0 \in \mathbb{N}$ such that for each $\Delta \geq \Delta_0$ and each $w \in [1-\frac{q}{\Delta}, 1)$ the q-state anti-ferromagnetic Potts model with edge interaction parameter w has a unique Gibbs measure on the infinite Δ-regular tree $\mathbb{T}_\Delta$.*

In the next section we give a more technical overview of our approach. In particular we recall some results from Chapter 3 that we will use and set up some terminology. We also gather two results that will be used to prove the main theorem, leaving the proofs of these results to Section 4.3 and Section 4.4 respectively. Assuming these results, the main theorem will be proved in Section 4.2.4.

4.2 Approach, preliminaries and proof outline

Our approach to prove the main theorem is based on the approach from Chapter 3 for the cases $q = 3, 4$. As is well known, to prove uniqueness it suffices to show that for a given root vertex, say v, the probability that v receives a color $i \in [q]$, conditioned on the event that the vertices at distance n from v receive a fixed coloring, converges to $1/q$ as $n \to \infty$ regardless of the fixed coloring of the vertices at distance n, see Lemma 2.2.3. Instead of studying these probabilities, we study ratios of these probabilities. It then suffices to show that these converge to 1. The ratios at the root vertex v can be expressed as a rational function of the ratios at the neighbors of v. See Lemma 4.2.2 below. This function is rather difficult to analyze directly and as in Chapter 3 we analyze a simpler function coupled with a geometric approach. A key new ingredient of our approach in this chapter is to take the limit of Δ, the degree of the tree, to infinity and analyze the resulting function. This function turns out be even simpler and behaves much better in a geometric sense. With some work we translate the results for the limit case back to the finite case and therefore obtain results for Δ large enough. This is inspired by a recent paper [BBP21] in which this idea was used to give a precise description of the location of the zeros of the independence polynomial for bounded degree graphs of large degree.

4.2.1 Reformulation of the main result

We will reformulate the main theorem here in terms of the conditional distribution of the color of the root vertex of $\mathbb{T}_\Delta$ conditioned on a fixed coloring of the vertices at a certain distance from the root.

Let $\Delta \geq 2$ be an integer. In what follows it will be convenient to write $d = \Delta - 1$ and let $n \in \mathbb{Z}_{\geq 0}$. We use the notations $\mathbb{T}_d^n$ and $\Lambda_{n,d}$ introduced in Definition 2.2.2. For a positive integer q we call a map $\tau : \Lambda_{n,d} \to [q]$ a *boundary condition on* $\Lambda_{n,d}$.

The following theorem may be seen as a more precise form of our main result.

Theorem 4.2.1. *Let $q \geq 3$ be a positive integer. There exist constants $C > 0$ and $d_0 > 0$ such that for all integers $d \geq d_0$ and all $\alpha \in (0,1)$ the following holds for any $i \in \{1, \ldots, q\}$:*

$$\lim_{n\to\infty} \max_{\tau:\Lambda_{n,d}\to[q]} \left| Pr_{\mathbb{T}_d^n,q,w_c}[\mathbf{\Phi}(r) = i \mid \mathbf{\Phi}\restriction_{\Lambda_{n,d}} = \tau] - \frac{1}{q} \right| = 0, \tag{4.1}$$

for any boundary condition τ on $\Lambda_{n,d}$ and edge interaction $w(\alpha) = 1 - \frac{\alpha q}{d+1}$,

$$\left| Pr_{\mathbb{T}_d^n,q,w(\alpha)}[\mathbf{\Phi}(r) = i \mid \mathbf{\Phi}\restriction_{\Lambda_{n,d}} = \tau] - \frac{1}{q} \right| \leq C\alpha^{n/2}. \tag{4.2}$$

Remark 3. We can in fact strengthen (4.2) in two ways. First of all, for any $\alpha < \hat{\alpha} < 1$ there exists a constant $C_{\hat{\alpha}} > 0$ such that the right-hand side of (4.2) can be replaced by $C_{\hat{\alpha}}\hat{\alpha}^n$. Secondly, for any fixed $d \geq d_0$ there exist a constant $C_d > 0$ such that the right-hand side of (4.2) can be replaced by $C_d\alpha^n$.

As is well known (we provide a proof in Lemma 3.2.1 of Chapter 3) Theorem 4.2.1 directly implies the main theorem. Therefore the remainder of the chapter is devoted to proving Theorem 4.2.1.

4.2.2 Log-ratios of probabilities

Theorem 4.2.1 is formulated in terms of certain conditional probabilities. For our purposes it turns out to be convenient to reformulate this into *log-ratios* of these probabilities. To introduce these, we recall some relevant definitions from Chapter 3. Throughout we fix an integer $q \geq 3$.

Given a (finite) graph $G = (V, E)$ and a subset $U \subseteq V$ of vertices, we call $\tau : U \to [q]$ a *boundary condition on* U. We say vertices in U are *fixed* and vertices in $V \setminus U$ are *free*. The *partition function restricted to* τ is defined as

$$Z_{U,\tau}(G; q, w) = \sum_{\substack{\sigma: V \to [q] \\ \sigma \restriction_U = \tau}} w^{m(\sigma)}.$$

We just write $Z(G)$ if U, τ and q, w are clear from the context. Given a boundary condition $\tau : U \to [q]$ on U, a free vertex $v \in V \setminus U$ and a state $i \in [q]$ we define $\tau_{v,i}$ as the unique boundary condition on $U \cup \{v\}$ that extends τ and associates i to v. When U and τ are clear from the context, we will denote $Z_{U \cup \{v\}, \tau_{v,i}}(G)$ as $Z_i^v(G)$. Let $\tau : U \to [q]$ be a boundary condition on U and $v \in V$ be a free vertex. For any $i \in [q]$ we define the *log-ratio* $\tilde{R}_{G,v,i}$ as

$$\tilde{R}_{G,v,i} := \log(Z_i^v(G)) - \log(Z_q^v(G)),$$

where log denotes the natural logarithm. Note that $\tilde{R}_{G,v,q} = 0$. We moreover remark that $\tilde{R}_{G,v,i}$ can be interpreted as the logarithm of the ratio of the probabilities that the root gets color i (resp. q) conditioned on the event that U is colored according to τ.

For trees the log-ratios at the root vertex can be recursively computed from the log-ratios of its neighbors. To describe this compactly we introduce some notation that will be used extensively throughout the chapter. Fix $d \in \mathbb{R}_{>1}$ and let $\alpha \in (0, 1]$. Define the maps $G_{d,\alpha;i}, F_{d,\alpha;i} : \mathbb{R}^{q-1} \to \mathbb{R}$ for $i \in \{1, \ldots, q-1\}$ as

$$G_{d,\alpha;i}(x_1, \ldots, x_{q-1}) = \frac{1 - x_i}{\sum_{j=1}^{q-1} x_j + 1 - \frac{\alpha \cdot q}{d+1}} \tag{4.3}$$

and

$$F_{d,\alpha;i}(x_1, \ldots, x_q) = d \log\left(1 + \frac{\alpha \cdot q}{d+1} \cdot G_{d,\alpha;i}(\exp(x_1), \ldots, \exp(x_{q-1}))\right). \tag{4.4}$$

Define the map $F_{d,\alpha} : \mathbb{R}^{q-1} \to \mathbb{R}^{q-1}$ whose ith coordinate function is given by $F_{d,\alpha;i}(x_1, \ldots, x_{q-1})$ and define $G_{d,\alpha}$ similarly. To suppress notation we write $F_d = F_{d,1}$ and $G_d = G_{d,1}$. We also define $\exp(x_1, \ldots, x_{q-1}) = (\exp(x_1), \ldots, \exp(x_{q-1}))$ and $\log(x_1, \ldots, x_{q-1}) = (\log(x_1), \ldots, \log(x_{q-1}))$. We note that $G_{d,\alpha}$ and $F_{d,\alpha}$ are analytic in $1/d$ near 0 when viewing d as a variable. We will now use the map $F_{d,\alpha}$ to give a compact description of the tree recurrence for log-ratios.

Lemma 4.2.2. *Let $T = (V, E)$ be a tree, $\tau : U \to [q]$ a boundary condition on $U \subsetneq V$. Let v be a free vertex of degree $d \geq 1$ with neighbors $v_1, \ldots, v_d$. Denote T_i for the tree that is the connected component of $T - v$ containing v_i. Restrict τ to each T_i in the natural way. Write $\tilde{R}_{i,j}$ for the log-ratio $\tilde{R}_{T_i,v_i,j}$. Then for α such that $w = 1 - \frac{\alpha \cdot q}{d+1}$,*

$$(\tilde{R}_{T,v,1}, \ldots, \tilde{R}_{T,v,q-1}) = \sum_{i=1}^{d} \frac{1}{d} F_{d,\alpha}(\tilde{R}_{i,1}, \ldots, \tilde{R}_{i,q-1}), \tag{4.5}$$

a convex combination of the images of the map $F_{d,\alpha}$.

Proof. By focusing on the jth entry of the left-hand side and substituting $R_{T,v,j} = \exp(\tilde{R}_{T,v,j})$, we see that (4.5) follows from the well known recursion for ratios

$$R_{T,v,i} = \prod_{s=1}^{d} \frac{\sum_{l \in [q-1] \setminus \{i\}} R_{T_s,v_s,l} + w R_{T_s,v_s,i} + 1}{\sum_{l \in [q-1]} R_{T_s,u_s,l} + w}. \tag{4.6}$$

This recursion follows from Lemma 3.2.2 and Equation 3.7 in Chapter 3. $\square$

We note that if the boundary condition τ is constant on the leaves of the tree $\mathbb{T}^n_{d+1}$, then the log-ratios at the root can be obtained by iterating the univariate function f given by $f(x) = F_{d,\alpha}(x, \ldots, x)$ at $w = w(\alpha)$. The point $x = 0$ is a fixed point of f; it satisfies $|f'(0)| \leq 1$ if and only if $w \geq w_c$. From this it is not difficult to extract that there exist multiple Gibbs measures when $w < w_c$.

Denote $\vec{0}$ for the zero vector in $\mathbb{R}^{q-1}$. (Throughout we will denote vectors in boldface.) We define for any $n \geq 1$ the set of possible log-ratio vectors

$$\mathcal{R}_n := \{(\tilde{R}_{\mathbb{T}^n_{d+1},r,1}, \ldots, \tilde{R}_{\mathbb{T}^n_{d+1},r,q-1}) \in \mathbb{R}^{q-1} | \tau : \Lambda_n \to [q]\}.$$

Here the ratios $\tilde{R}_{\mathbb{T}^n_{d+1},r,1}$ depend on τ but this is not visible in the notation. The following lemma shows how the recursion from Lemma 4.2.2 will be used.

Lemma 4.2.3. *Let $q \geq 3$ and $d \geq 2$ be integers. If there exists a sequence $\{\mathcal{T}_n\}_{n \geq 1}$ of convex subsets of $\mathbb{R}^{q-1}$ with the following properties:*

1. *$\mathcal{R}_1 \subseteq \mathcal{T}_1$,*

2. *for every* $n \geq 1$, $F_d(\mathcal{T}_n) \subseteq \mathcal{T}_{n+1}$,
3. *for every* $\varepsilon > 0$ *there is an* $N \geq 1$ *such that for all* $n \geq N$ *we have* $\sup_{\mathbf{r} \in \mathcal{T}_\mathbf{n}} \|\mathbf{r}\|_1 \leq \varepsilon$,

then

$$\lim_{n \to \infty} \max_{\tau : \Lambda_{n,d+1} \to [q]} \left| Pr_{\mathbb{T}_d^n, q, w_c}[\mathbf{\Phi}(r) = i \mid \mathbf{\Phi}\restriction_{\Lambda_{n,d}} = \tau] - \frac{1}{q} \right| = 0. \tag{4.7}$$

Proof. The proof is analogous to the proof of Lemma 3.2.3 and we therefore omit it. □

We note that the lemma is only stated for $\alpha = 1$. An analogues statement for $\alpha \in (0,1)$ and F_d replaced by $F_{d,\alpha}$ with a more accurate dependence of N on ε follows from a certain monotonicity of $F_{d,\alpha}$, as will be explained in the proof of the main theorem below.

In the next section we construct a family of convex sets that allows us to form a sequence $\{\mathcal{T}_n\}_{n \geq 1}$ with the properties required by the lemma.

4.2.3 Construction of suitable convex sets

We need the standard $q-2$-simplex, which we denote as

$$\text{Simp} = \left\{ (t_1, \ldots, t_{q-2}, 1 - \sum_{i=1}^{q-2} t_i) \mid t_i \geq 0 \text{ for all } i, \ \sum_{i=1}^{q-2} t_i \leq 1 \right\},$$

to avoid confusion with the degree Δ of the infinite regular tree.

The symmetric group S_q acts on $\mathbb{R}^q$ by permuting entries of vectors. Consider $\mathbb{R}^{q-1} \subset \mathbb{R}^q$ as the subspace spanned by $\{\vec{e}_1 - \vec{e}_q, \ldots, \vec{e}_{q-1} - \vec{e}_q\}$, where $\vec{e}_i$ denotes the ith standard base vector in $\mathbb{R}^q$. This induces a linear action of S_q on $\mathbb{R}^{q-1}$, also known as the the standard representation of S_q and denoted by $\vec{x} \mapsto \pi \cdot \vec{x}$ for $\vec{x} \in \mathbb{R}^{q-1}$ and $\pi \in S_q$. The following lemma shows that the map $F_{d,\alpha}$ is S_q-equivariant for any $\alpha \in (0,1]$, essentially because the action permutes the q colors of the Potts model and no color plays a special role.

Lemma 4.2.4. *For any* $\pi \in S_q$, *any* $\alpha \in (0,1]$, *any* $\vec{x} \in \mathbb{R}^{q-1}$ *and any* d *we have*

$$\pi \cdot F_{d,\alpha}(\vec{x}) = F_{d,\alpha}(\pi \cdot \vec{x}).$$

Proof. This follows as in Section 3.3.1. □

Define for $c \geq 0$ the half space

$$H_{\geq -c} := \left\{ \vec{x} \in \mathbb{R}^{q-1} \mid \sum_{i=1}^{q-1} x_i \geq -c \right\}. \tag{4.8}$$

Define the set

$$P_c = \bigcap_{\pi \in S_q} \pi \cdot H_{\geq -c}. \tag{4.9}$$

Note that for each $c \geq 0$ the set P_c equals the convex polytope

$$\mathrm{conv}\big(\{(-c,0,\dots,0),\dots(0,\dots,0,-c),(c,\dots,c)\}\big).$$

Denote $D_c := \mathrm{conv}\big(\{(-c,0,\dots,0),\dots(0,\dots,0,-c),(0,\dots,0)\}\big)$. Then we have

$$P_c = \bigcup_{\pi \in S_q} \pi \cdot D_c. \tag{4.10}$$

We refer to D_c as the fundamental domain of the action of S_q on $\mathbb{R}^{q-1}$.

The following two propositions capture the image of P_c under applications of the map F_d.

Proposition 4.2.5. *Let $q \geq 3$ be an integer. Then there exists $d_1 > 0$ such that for all $d \geq d_1$ and $c \in [0, q+1]$, $F_d(P_c)$ is convex.*

Proposition 4.2.6. *Let $q \geq 3$ be an integer. There exists $d_2 > 0$ such that for all $d \geq d_2$ the following holds: for any $c \in (0, q+1]$ there exists $0 < c' < c$ such that*

$$F_d^{\circ 2}(P_c) \subseteq P_{c'}.$$

An intuitive explanation for why we need $F_d^{\circ 2}$ and cannot work with F_d directly is that the derivative of F_d at $\mathbf{0}$ is equal to $-\mathrm{Id}$, which reflects the fact that we are dealing with an anti-ferromagnetic model, while the derivative of $F_d^{\circ 2}$ at $\mathbf{0}$ is equal to Id.

We postpone the proofs of the two results above to the subsequent sections. A crucial ingredient in both proofs will be to analyze the limit $\lim_{d\to\infty} F_d$. We first utilize the two propositions to give a proof of Theorem 4.2.1.

4.2.4 A proof of Theorem 4.2.1

Fix an integer $q \geq 3$. Let d_1, d_2 be the constants from Proposition 4.2.5 and 4.2.6 respectively. Let $d_0 \geq \max\{d_1, d_2\}$ large enough to be determined below. Note that the log-ratios at depth 0 are of the form $\infty \cdot \vec{e}_i$ and $-\infty \cdot \vec{1}$, where $\vec{1}$ denotes the all ones vector. This comes from the fact that the probabilities at level 0 are either 1 or 0 and so the ratios are of the form $\vec{1} + \infty\vec{e}_i$ or $\vec{0}$. This implies that the log-ratios at depth 1 are convex combinations of $F_d(\infty \cdot \vec{e}_i) = d\log(1 + \frac{-q}{d+1})\vec{e}_i$ and $F_d(-\infty \cdot \vec{1}) = d\log(1 + \frac{q}{d+1-q})\vec{1}$. So for $d \geq d_0$ and d_0 large enough they are certainly contained in P_{q+1}.

We start with the proof of (4.1). We construct a decreasing sequence $\{c_n\}_{n\in\mathbb{N}}$ and let $\mathcal{T}_{2n-1} = P_{c_n}$. For even $n > 0$ we set $\mathcal{T}_n = F_d(P_{c_{n-1}})$, which is convex by Proposition 4.2.5. We set $c_1 = q+1$ and for $n \geq 1$, given c_n, we can choose, by Proposition 4.2.6, $c_{n+1} < c_n$ so that $F_d^{\circ 2}(P_{c_n}) \subseteq P_{c_{n+1}}$. Choose such a c_{n+1} as small as possible. We claim that the sequence $\{c_n\}_{n\in\mathbb{N}}$ converges to 0. Suppose not then it must have a limit $c > 0$. Choose $c' < c$ such that $F_d^{\circ 2}(P_c) \subseteq P_{c'}$. Then for n large enough we must have $F_d^{\circ 2}(P_{c_n}) \subseteq P_{c/2+c'/2}$, contradicting the choice of c_{n+1}.

Since $\{c_n\}_{n\in\mathbb{N}}$ converges to 0, it follows that the sequence $\mathcal{T}_n$ converges to $\{0\}$. With Lemma 4.2.3 this implies (4.1).

To prove the second part let $\alpha \in (0,1)$. Consider the decreasing sequence $\{c_n\}_{n\in\mathbb{N}}$ with $c_n = (q+1)\alpha^{n-1}$. Set $\mathcal{T}_{2n-1} = P_{c_n}$ and $\mathcal{T}_{2n} = F_{d,\alpha}(P_{c_{n-1}})$. We use the following observation.

Lemma 4.2.7. *For any $\alpha \in (0,1]$, any $\vec{x} \in \mathbb{R}^{q-1}$ and any integer d there is $d' \geq d$ such that $F_{d,\alpha}(\vec{x}) = \frac{d}{d'} \cdot F_{d'}(\vec{x})$. Moreover, $\frac{d}{d'} \leq \alpha$.*

Proof. When viewing α and d as variables, $\frac{1}{d}F_{d,\alpha;i}$ only depends on the ratio $\frac{\alpha}{d+1}$. Therefore the first statement of the lemma holds with d' defined by $\frac{\alpha}{d+1} = \frac{1}{d'+1}$. Since $\frac{d}{d'} = \frac{\alpha d}{d+1-\alpha}$, the second statement also holds. □

The lemma above implies that $F_{d,\alpha}(P_{c_n}) = \frac{d}{d'} \cdot F_{d'}(P_{c_n})$ and hence is convex for each c_n. It moreover implies that

$$F_{d,\alpha}^{\circ 2}(P_{c_n}) \subset \alpha F_{d'}(\alpha F_{d'}(P_{c_n}))) \subset \alpha P_{c_n} = P_{c_{n+1}}.$$

By basic properties of the logarithm, (4.2) now quickly follows. This finishes the proof of Theorem 4.2.1.

The strengthening mentioned in Remark 3 can be derived from the fact that the derivative of $F_{d,\alpha}$ at $\vec{0}$ is equal to $\frac{-\alpha d}{d+1-\alpha}\mathrm{Id}$. Note that $\frac{\alpha d}{d+1-\alpha} < \alpha$ for all $\alpha \in (0,1)$ and d. Therefore on a small enough open ball B around $\vec{0}$ the operator norm of the derivative of $F_{d,\alpha}$ can be bounded by $\hat{\alpha}$ for all $d \geq d_0$ (and by α for fixed $d \geq d_0$). Then for any integer $n \geq 0$, $F_{d,\alpha}^{\circ n}(B) \subset \hat{\alpha}^n B$ ($\alpha^n B$ respectively). For n_0 large enough $P_{c_{n_0}}$ is contained in this ball B. For $n > 2n_0$ we then set $\mathcal{T}_n = \hat{\alpha}^{n-2n_0} B$ ($\alpha^{n-2n_0} B$ respectively). The statements in the remark now follow quickly.

4.2.5 The $d \to \infty$ limit map

As mentioned above, an important tool in our approach is to analyze the maps F_d as $d \to \infty$. Since $F_d(\mathbb{R}^{q-1})$ is bounded, it follows that as $d \to \infty$, $F_d(x_1, \ldots, x_{q-1})$ converges uniformly to the limit map

$$F_\infty(x_1, \ldots, x_{q-1}), \tag{4.11}$$

with coordinate functions

$$F_{\infty;i}(x_1,\dots,x_{q-1}) := q\frac{1-e^{x_i}}{\sum_{j=1}^{q-1} e^{x_i}+1}. \tag{4.12}$$

We write $G_{\infty;i}(x_1,\dots,x_{q-1}) = q\frac{1-x_i}{\sum_{j=1}^{q-1} x_j+1}$ for the ith coordinate function of the fractional linear map G_∞. Note that $F_\infty = G_\infty \circ \exp$.

By Lemma 4.2.4 for any $\pi \in S_q$, any $\vec{x} \in \mathbb{R}^{q-1}$ and any d we have $\pi \cdot F_d(\vec{x}) = F_d(\pi \cdot \vec{x})$. As the action of π on $\mathbb{R}^{q-1}$ does not depend on d, we immediately see $\pi \cdot F_\infty(\vec{x}) = F_\infty(\pi \cdot \vec{x})$ follows.

In the next two sections we will prove Propositions 4.2.5 and 4.2.6. The idea is to first prove a variant of these propositions for the map F_∞ and then use that $F_d \to F_\infty$ uniformly to finally prove the actual statements. We use the description of P_c as intersection of half spaces $\pi \cdot H_{\geq -c}$ in Section 4.3 and the description as the union of the $\pi \cdot D_c$ in Section 4.4.

4.3 Convexity of the forward image of P_c

This section is dedicated to proving Proposition 4.2.5.

For $\mu \in \mathbb{R}$ we define the half space $H_{\geq\mu}$ as in (4.8). The half space $H_{\leq\mu}$ is defined similarly. We denote by H_μ the affine space which is the boundary of $H_{\leq\mu}$.

In what follows we will often use that the map G_∞ is a fractional linear transformation and thus preserves lines and hence maps convex sets to convex sets, see e.g. [BV04, Section 2.3].

Lemma 4.3.1. *For all $c > 0$, the set $\exp(H_{\geq -c}) := \{\exp(\vec{x}) \mid \vec{x} \in H_{\geq -c}\}$ is strictly convex, consequently*

$$G_\infty(\exp(H_{\geq -c}))$$

is strictly convex.

Proof. Since G_∞ is a fractional linear transformation, it preserves convex sets. It therefore suffices to show that $\exp(H_{\geq -c})$ is strictly convex.

To this end take any $\vec{x}, \vec{y} \in \exp(H_{\geq -c})$ and let $\lambda \in (0,1)$. We need to show that $\lambda\vec{x} + (1-\lambda)\vec{y} \in \exp(H_{\geq -c})$. By strict concavity of the logarithm we have

$$\sum_{i=1}^{q-1} \log(\lambda x_i + (1-\lambda)y_i) \geq \sum_{i=1}^{q-1} \lambda \log(x_i) + (1-\lambda)\log(y_i) > -c,$$

we conclude that $\exp(H_{\geq -c})$ is strictly convex. $\square$

In what follows we need the *angle* between the tangent space of $G_\infty(\exp(H_{-c}))$ for $c > 0$ at $G_\infty(\vec{x})$ for any $\vec{x} \in \exp(H_{-c})$ and the space H_0. This angle is defined as the angle of a normal vector of the tangent space pointing towards the interior of $G_\infty(\exp(H_{\geq -c}))$ and the vector $-\vec{1}$ (which is a normal vector of H_0).

Lemma 4.3.2. *For any $c \in [0, q+1]$ and any $\vec{x} \in \exp(H_{-c})$ the angle between the tangent space of $G_\infty(\exp(H_{-c}))$ at $G_\infty(\vec{x})$ and H_0 is strictly less than $\pi/2$.*

Proof. We will first show that the tangent space cannot be orthogonal to H_0.

The map G_∞ is invertible (when restricted to $\mathbb{R}^{q-1}_{>0}$) with inverse G_∞^{-1} whose coordinate functions are given by

$$G^{-1}_{\infty,i}(y_1, \ldots, y_{q-1}) = \frac{-qy_i}{\sum_{i=1}^{q-1} y_i + q} + 1.$$

Define $g : \mathbb{R}^{q-1} \setminus H_{-q} \to \mathbb{R}$ by $g(\vec{y}) = \prod_{i=1}^{q-1} G^{-1}_{\infty,i}(\vec{y})$. Then the image of $\exp(H_{-c})$ under G_∞ is contained in the hypersurface $\{\vec{y} \in \mathbb{R}^{q-1} \mid g(\vec{y}) = \exp(-c)\}$. Therefore a normal vector of the tangent space of $G_\infty(\exp(H_{-c}))$ at $\vec{y} = G_\infty(\vec{x})$ is given by the gradient of the function g. Thus to show that this tangent space is not orthogonal to H_0, we need to show that

$$\sum_{i=1}^{q-1} \tfrac{\partial}{\partial y_i} g(\vec{y}) \neq 0. \tag{4.13}$$

We have

$$\begin{aligned}
\sum_{i=1}^{q-1} \tfrac{\partial}{\partial y_i} g(\vec{y}) &= \sum_{i=1}^{q-1}\sum_{j=1}^{q-1} \frac{\prod_{k=1}^{q-1} G^{-1}_{\infty,k}(\vec{y})}{G^{-1}_{\infty,j}(\vec{y})} \tfrac{\partial}{\partial y_i} G^{-1}_{\infty,j}(\vec{y}) \\
&= \sum_{j=1}^{q-1} \frac{\prod_{k=1}^{q-1} G^{-1}_{\infty,k}(\vec{y})}{G^{-1}_{\infty,j}(\vec{y})} \sum_{i=1}^{q-1} \tfrac{\partial}{\partial y_i} G^{-1}_{\infty,j}(\vec{y}) \\
&= \sum_{j=1}^{q-1} \frac{\prod_{k=1}^{q-1} G^{-1}_{\infty,k}(\vec{y})}{G^{-1}_{\infty,j}(\vec{y})} \cdot \frac{-q(\sum_{i=1}^{q-1} y_i + q) + q(q-1)y_j}{(\sum_{i=1}^{q-1} y_i + q)^2} \\
&= \sum_{j=1}^{q-1} \frac{\prod_{k=1}^{q-1} G^{-1}_{\infty,k}(\vec{y})}{G^{-1}_{\infty,j}(\vec{y})} \cdot \frac{-(q-1)G^{-1}_{\infty,j}(\vec{y}) - 1}{\sum_{i=1}^{q-1} y_i + q}.
\end{aligned}$$

Since $G^{-1}_{\infty,k}(\vec{y}) > 0$ for each k, all terms in the final sum are nonzero and have the same sign. This proves (4.13).

Since the angle between the tangent space of $G_\infty(\exp(H_{-c}))$ at $G_\infty(\vec{x})$ and H_0 depends continuously on $\vec{x}$ this angle should either be always less than $\pi/2$ or always be bigger. Since by the previous lemma the set $G_\infty(\exp(H_{\geq -c}))$ is convex, it is the former. $\square$

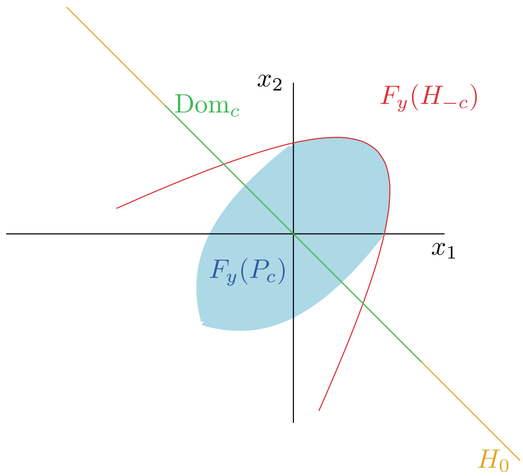


Figure 4.1: Depicting the situation in Lemma 4.3.3, for $q = 3, c = 2$ and $y = \frac{1}{20}$. The domain Dom_c of the function $h_{y,c}$ which we define in the proof of Lemma 4.3.3 is made by choosing $a' = -3$.

We next continue with the finite case. We will need the following definition. The *hypograph* of a function $f : D \to \mathbb{R}$ is the region $\{(x, y) \mid x \in D, y \leq f(x)\}$. Below we will consider a hypersurface contained in $\mathbb{R}^{q-1}$ that we view as the graph of a function with domain contained in H_0. In this context the hypograph of such a function is again contained in $\mathbb{R}^{q-1}$, but the 'positive y-axis' points in the direction of $\vec{1}$ as seen from $\vec{0} \in H_0$.

Lemma 4.3.3. *There exists $y_1 > 0$ such that for all $y \in [0, y_1)$ and $c \in [0, q+1]$ the set $F_y(P_c)$ is contained in the hypograph of a concave function, $h_{y,c}$, with a convex compact domain in H_0.*

Proof. We first prove that for any $\vec{x} \in H_0$ and $c \in [0, q + 1]$ there exists an open neighborhood $W_{c,\vec{x}} = Y_{c,\vec{x}} \times C_{c,\vec{x}} \times X_{c,\vec{x}}$ of $(0, c, \vec{x}) \in [0, 1] \times [0, q+1] \times \mathbb{R}^{q-1}$ such that the following holds for any $(y', c', \vec{x}') \in W_{c,\vec{x}}$:

$$\text{the angle between the tangent space of } F_{1/y'}(H_{-c'}) \text{ at } F_{1/y'}(\vec{x}'_{c'}) \text{ and } H_0 \text{ is strictly less than } \pi/2, \tag{4.14}$$

where we denote $\vec{x}_c := \vec{x} - \frac{c}{q-1}\vec{1} \in H_{-c}$. To see this note that by the previous lemma we have that the tangent space of $F_\infty(H_{-c})$ at $F_\infty(\vec{x}_c)$ is not orthogonal to H_0 and in fact makes an angle of less than $\pi/2$ with H_0. Say it has angle $\pi/2 - \gamma$. Since $(y, c, \vec{x}) \mapsto F_{1/y}(\vec{x}_c)$ is analytic, there exists an open neighborhood

W_0 of $(0, c, \vec{x})$ such that for any $(y', \vec{x'}, c') \in W_0$ the angle between the tangent space of $F_{1/y'}(H_{-c'})$ at $F_{1/y'}(\vec{x'}_{c'})$ and H_0 is at most $\pi/2 - \gamma/2$. Clearly, W_0 contains an open neighborhood of $(0, c, \vec{x})$ of the form $Y \times C \times X$ proving (4.14).

Next fix $c \in [0, q+1]$ and $\vec{x} \in H_0$ and write $W_{c,\vec{x}} = Y \times C \times X$. Together with the implicit function theorem, (4.14) now implies that for each $y' \in Y$ and any $c' \in C$, that locally at $\vec{x}_{c'}$, $F_{1/y'}(H_{-c})$ is the graph of an analytic function $f_{y',c',\vec{x}}$ on an open domain contained in H_0. Here we use that $F_{1/y}$ is invertible with analytic inverse. By choosing Y and C small enough, we may by continuity assume that we have a common open domain, $D_{c,\vec{x}}$, for these functions for all $c' \in C$ and $y' \in Y$, where we may moreover assume that these functions are all defined on the closure of $D_{c,\vec{x}}$.

We next claim, provided the neighbourhood $W = Y_{c,\vec{x}} \times C_{c,\vec{x}}$ is chosen small enough, that for each $y' \in Y$ and $c' \in C$,

$$\text{the largest eigenvalue of the Hessian } f_{y',c',\vec{x}} \text{ on } D_{c,\vec{x}} \text{ is strictly less than } 0. \tag{4.15}$$

To see this we note that by the previous lemma we know that $F_\infty(H_{\geq -c})$ is strictly convex. Therefore the Hessian[1] of $f_{0,c,\vec{x}}$ on $D_{c,\vec{x}}$ is negative definite, say its largest eigenvalue is $\delta < 0$. Similarly as before, there exists an open neighborhood $W' \subseteq W$ of $(0, c)$ of the form $W' = Y' \times C'$ such that for each $y' \in Y'$ and $c' \in C'$, the function $f_{y',c',\vec{x}}$ has a negative definite Hessian with largest eigenvalue at most $\delta/2 < 0$ for each $\vec{z} \in D_{c,\vec{x}}$ (by compactness of the closure of $D_{c,\vec{x}}$). We now want to patch all these function to form a global function on a compact and convex domain. We first collect some properties of $F_{1/y}$ that will allow us to define the domain.

First of all note that by compactness there exists $a > 0$ such that for each $c \in [0, q+1]$, $\exp(P_c) \subset H_{\leq a}$ (where the inclusion is strict). We now fix such a value of a. Since G_∞ is S_q-equivariant, we know that $G_\infty(H_{\leq a}) = H_{\geq a'}$ for some $a' \in \mathbb{R}$. We now choose $y^* > 0$ small enough such that the following two inclusions hold for all $y \in [0, y^*]$ and $c \in [0, q+1]$

$$F_{1/y}(P_c) \subset H_{\geq a'}, \tag{4.16}$$

$$\mathrm{proj}_{H_0}(F_\infty(H_{-c}) \cap H_{\geq a'}) \subset \mathrm{proj}_{H_0}(F_{1/y}(H_{-c})), \tag{4.17}$$

where proj_{H_0} denotes the orthogonal projection onto the space H_0. The first inclusion holds since $F_{1/y}$ converges uniformly to F_∞ as $y \to 0$. For the second inclusion note that

$$F_\infty(H_{-c}) \cap H_{\geq a'} = G_\infty(\exp(H_{-c}) \cap H_{\leq a}) \subset F_\infty(H_{-c}).$$

[1]Recall that the *Hessian* of a function $f : U \to \mathbb{R}$ for an open set $U \subseteq \mathbb{R}^n$ at a point $u \in U$ is defined as the $n \times n$ matrix $H_f(u)$ with $(H_f(u))_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j}(u)$. When these partial derivatives are continuous and the domain U is convex, f is concave if and only if its Hessian is negative definite at each point of the domain U [BV04].

Because $\exp(H_{-c}) \cap H_{\leq a}$ is compact, the desired conclusion follows since $F_{1/y} \to F_\infty$ uniformly as $y \to 0$.

Let us now consider for $c \in [0, q+1]$ the projection

$$\mathrm{Dom}_c := \mathrm{proj}_{H_0}(F_\infty(H_{-c}) \cap H_{\geq a'}),$$

see Figure 4.1. Since $F_\infty(H_{-c}) \cap H_{\geq a'}$ is convex by Lemma 4.3.1 and compact, it follows that Dom_c is compact and convex for each $c \in [0, q+1]$. Moreover, we claim that

$$\bigcup_{c \in [0,q+1]} (\{c\} \times \mathrm{Dom}_c) \subseteq [0, q+1] \times H_0 \text{ is compact.} \tag{4.18}$$

Indeed, it is the continuous image of the compact set $\exp(H_{\geq -q-1}) \cap H_{\leq a}$ under the map

$$\exp(H_{\geq -q-1}) \cap H_{\leq a} \to [0, q+1] \times H_0$$

defined by

$$\vec{x} \mapsto \left(\sum_{i=1}^{q-1} x_i, \mathrm{proj}_{H_0}(G_\infty(\vec{x})) \right).$$

By (4.17) Dom_c is contained in $\mathrm{proj}_{H_0}(F_{1/y}(H_{-c}))$ for all $y \in [0, y^*]$ and $c \in [0, q+1]$. It follows that the sets $Y_{c,\vec{x}} \times C_{c,\vec{x}} \times D_{c,\vec{x}}$, where $\vec{x}$ ranges over H_0 and c over $[0, q+1]$, form an open cover of $\{0\} \times \cup_{c \in [0,q+1]} (\{c\} \times \mathrm{Dom}_c)$. Since the latter set is compact by (4.18), we can take a finite sub cover. Therefore there exists $y_1 > 0$ such that for each $y \in [0, y_1)$ and each $c \in [0, q+1]$ we obtain a unique global function $h_{y,c}$ on the union of these finitely many domains, which by (4.15) has a strictly negative definite Hessian. By construction the union of these domains contains Dom_c for each $c \in [0, q+1]$. Consequently, restricted to Dom_c, $h_{y,c}$ is a concave function for each $y \in [0, y_1)$ and $c \in [0, q+1]$. By (4.16), it follows that $F_{1/y}(P_c)$ is contained in the hypograph of $h_{y,c}$, as desired. $\square$

We can now finally prove Proposition 4.2.5, which we restate here for convenience.

Proposition 4.2.5. *Let $q \geq 3$ be an integer. Then there exists $d_1 > 0$ such that for all $d \geq d_1$ and $c \in [0, q+1]$, $F_d(P_c)$ is convex.*

Proof. By the previous lemma we conclude that for d larger than $1/y_1$, $F_d(P_c)$ is contained in the hypograph of the function $h_{1/d,c}$, denoted by $\mathrm{hypo}(h_{c,1/d})$ and moreover that this hypograph is convex, as the function $h_{1/d,c}$ is concave on a convex domain.

Since P_c is invariant under the S_q-action, it follows that

$$\exp(P_c) = \bigcap_{\pi \in S_q} \pi \cdot (\exp(H_{\geq -c}) \cap H_{\leq a})$$

and therefore by Lemma 4.2.4,

$$F_d(P_c) = \bigcap_{\pi \in S_q} \pi \cdot (F_d(P_c)) \subseteq \bigcap_{\pi \in S_q} \pi \cdot \mathrm{hypo}(h_{1/d,c}). \tag{4.19}$$

We now claim that the final inclusion in (4.19) is in fact an equality. To see the other inclusion, take some $\vec{z} \in \cap_{\pi \in S_q} \pi \cdot \mathrm{hypo}(h_{1/d,c})$. By symmetry, we may assume that $\vec{z}$ is contained in $\mathbb{R}^{q-1}_{\geq 0}$. Then $\vec{z}$ is equal to $F_d(\vec{x})$ for some $\vec{x} \in H_{\geq -c} \cap \mathbb{R}^{q-1}_{\leq 0}$, implying that $\vec{z}$ is indeed contained in $F_d(P_c)$.

This then implies that $F_d(P_c)$ is indeed convex being equal to the intersection of the convex sets $\pi \cdot \mathrm{hypo}(h_{1/d,c})$. $\square$

4.4 Forward invariance of P_c in two iterations

This section is dedicated to proving Proposition 4.2.6. We start with a version of the proposition for $d = \infty$ and after that consider finite d.

4.4.1 Two iterations of F_∞

Let $\Phi : \mathbb{R}^{q-1} \to \mathbb{R}^{q-1}$ be defined by

$$\Phi(x_1, \ldots, x_{q-1}) = F_\infty^{\circ 2}(x_1, \ldots, x_{q-1})$$

and its 'restriction' to the half line $\mathbb{R}_{\leq/0} \cdot \vec{1}$, $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, by

$$\phi(t) = -\langle \Phi(-t/(q-1) \cdot \vec{1}), \vec{1} \rangle,$$

where we use $\langle \cdot, \cdot \rangle$ to denote the standard inner product on $\mathbb{R}^{q-1}$.

This subsection is devoted to proving the following result.

Proposition 4.4.1. *For any $c \geq 0$ we have*

$$\Phi(P_c) \subseteq P_{\phi(c)} \subsetneq P_c.$$

By the definition of P_c in terms of D_c, (4.10), and the S_q-equivariance of the map F_∞ and hence of the map Φ, it suffices to prove this for P_c replaced by D_c. This can be derived from the following two statements:

(i) For any $c \geq 0$ the minimum of $\langle \Phi(\vec{x}), \vec{1} \rangle$ on $-c\,\mathrm{Simp}$ is attained at $-c/(q-1) \cdot \vec{1}$.

(ii) For any $c > 0$ we have $\phi(c) < c$.

Indeed, these statements imply that for any $c > 0$ we have that $\Phi(-c\,\mathrm{Simp}) \subseteq D_{\phi(c)} \subsetneq D_c$. Clearly this is sufficient, since $D_c = \cup_{0 \leq c' \leq c} - c'\,\mathrm{Simp}$ and therefore

$$\Phi(D_c) = \cup_{0 \leq c' \leq c} \Phi(-c'\,\mathrm{Simp}) \subseteq \cup_{0 \leq c' \leq c} D_{\phi(c')} \subseteq D_{\phi(c)} \subsetneq D_c.$$

We next prove both statements, starting with the first one.

Statement (i)

Proposition 4.4.2. *Let $c \geq 0$. Then for any $\vec{x} \in -c\,\mathrm{Simp}$ we have that*

$$\langle \Phi(\vec{x}), \vec{1} \rangle \geq \left\langle \Phi\left(\frac{-c}{q-1}\vec{1}\right), \vec{1} \right\rangle.$$

Moreover, equality happens only at $\vec{x} = \frac{-c}{q-1}\vec{1}$.

Before giving a proof, let us fix some further notation. By definition we have

$$\langle \Phi(\vec{x}), \vec{1} \rangle = \sum_{i=1}^{q-1} q \frac{1 - e^{F_{\infty;i}(\vec{x})}}{\sum_{j=1}^{q-1} e^{F_{\infty;j}(\vec{x})} + 1} = \frac{q^2}{\sum_{j=1}^{q-1} e^{F_{\infty;j}(\vec{x})} + 1} - q,$$

where we recall that $F_{\infty;j}$ denotes the jth coordinate function of F_∞. Thus the ith coordinate of the gradient of $\langle \Phi(\vec{x}), \vec{1} \rangle$ is given by

$$\begin{aligned}
\psi_i(\vec{x}) &:= \frac{-q^2}{\left(\sum_{j=1}^{q-1} e^{F_{\infty;j}(\vec{x})} + 1\right)^2} \left(\sum_{j=1}^{q-1} e^{F_{\infty;j}(\vec{x})} \cdot \frac{\partial F_{\infty;j}}{\partial x_i}(\vec{x}) \right) \\
&= \frac{q^3 e^{x_i} \left(e^{F_{\infty;i}(\vec{x})}(1 + \sum_{j=1}^{q-1} e^{x_j}) + \sum_{j=1}^{q-1} e^{F_{\infty;j}(\vec{x})}(1 - e^{x_j}) \right)}{\left(\sum_{j=1}^{q-1} e^{F_{\infty;j}(\vec{x})} + 1\right)^2 \left(\sum_{j=1}^{q-1} e^{x_j} + 1\right)^2}.
\end{aligned}$$

Let us define the following functions $v_i : \mathbb{R}^{q-1} \to \mathbb{R}$ for $i = 1, \ldots, q-1$ as

$$v_i(\vec{x}) := x_i \left(e^{G_i}(1 + \sum_{j=1}^{q-1} x_j) + \sum_{j=1}^{q-1} e^{G_j}(1 - x_j) \right),$$

where we write

$$G_i := G_{\infty;i}(\vec{x}) = \frac{q(1 - x_i)}{1 + x_1 + \cdots + x_{q-1}}.$$

Then we see that

$$\psi_i(\vec{x}) = \frac{q^3}{\left(\sum_{j=1}^{q-1} e^{F_{\infty;j}(\vec{x})} + 1\right)^2 \left(\sum_{j=1}^{q-1} e^{x_j} + 1\right)^2} \cdot v_i(e^{x_1}, \dots, e^{x_{q-1}}),$$

and $\psi_1(\vec{x}) = \cdots = \psi_{q-1}(\vec{x})$ if and only if $v_1(\exp(\vec{x})) = \cdots = v_{q-1}(\exp(\vec{x}))$.

Proof of Proposition 4.4.2. First of all observe that the function $\langle \Phi(\vec{x}), \vec{1} \rangle$ is invariant under the permutation of the coordinates of $\vec{x}$. Thus we can assume that

$$\vec{x} \in U := \{\vec{y} \in \mathbb{R}^{q-1} \mid 0 \geq y_1 \cdots \geq y_{q-1}\}$$

and not all the coordinates of $\vec{x}$ are equal.

Now it is enough to show that there exists a vector $\vec{0} \neq \vec{w} \in \mathbb{R}^{q-1}$ such that in the direction of $\vec{w}$ the function is (strictly) decreasing, $\langle \vec{w}, \vec{1} \rangle = 0$ and $\vec{x} + t_0 \vec{w} \in U$ for some small $t_0 > 0$. Let

$$\ell = \min\{1 \leq i \leq q-2 \mid x_i > x_{i+1}\},$$

which is finite, since not all of the coordinates of $\vec{x}$ are equal.

We claim that $\vec{w} = -\frac{\vec{e}_1 + \cdots + \vec{e}_\ell}{\ell} + \vec{e}_{\ell+1}$ satisfies the desired conditions. Clearly, $\vec{w}$ is perpendicular to $\vec{1}$ and $\vec{x} + t\vec{w} \in U$ for t small enough. Now let us calculate the derivative of

$$g(t) := \langle \Phi(\vec{x} + t\vec{w}), \vec{1} \rangle.$$

Using the notation defined above, we obtain

$$\begin{aligned} g'(0) &= -\frac{\psi_1(\vec{x}) + \cdots + \psi_\ell(\vec{x})}{\ell} + \psi_{\ell+1}(\vec{x}) \\ &= -\psi_\ell(\vec{x}) + \psi_{\ell+1}(\vec{x}) \\ &= -C \cdot (v_\ell(\exp(\vec{x})) - v_{\ell+1}(\exp(\vec{x}))) \\ &= -C \cdot (v_\ell(\vec{y}) - v_{\ell+1}(\vec{y})), \end{aligned}$$

where $C > 0$ and $\vec{y} = \exp(\vec{x})$. In particular,

$$1 \geq y_1 = y_2 = \ldots = y_\ell > y_{\ell+1} \geq \ldots \geq y_{q-1} \geq 0.$$

So to conclude that $g'(0) < 0$ and finish the proof, we need to show that

$$v_\ell(\vec{y}) - v_{\ell+1}(\vec{y}) > 0. \tag{4.20}$$

Lemma 4.4.3 shows that we may assume $\vec{y}$ satisfies $1 \geq y_1 = y_2 = \ldots = y_\ell > y_{\ell+1} \geq y_{\ell+2} = \ldots = y_{q-1} \geq 0$. Lemma 4.4.4 below shows that for those vectors $\vec{y}$ (4.20) is indeed true. So by combining Lemma 4.4.3 and Lemma 4.4.4 below we obtain (4.20) and finish the proof. $\square$

Lemma 4.4.3. *If* $1 \geq y_1 = y_2 \ldots = y_\ell > y_{\ell+1} \geq \ldots \geq y_{q-1} \geq 0$ *for some* $1 \leq \ell \leq q-2$*, then*

$$v_\ell(\vec{y}) - v_{\ell+1}(\vec{y}) \geq v_\ell(\vec{x}) - v_{\ell+1}(\vec{x}),$$

where $\vec{x} \in \mathbb{R}^{q-1}$ *is defined as*

$$x_j = \begin{cases} y_j & \text{if } j \leq \ell+1 \\ \frac{y_{\ell+2}+\cdots+y_{q-1}}{q-\ell-2} & \text{if } j > \ell+1 \end{cases}$$

for $1 \leq j \leq q-1$.

Proof. By continuity, it suffices to show

$$v_\ell(\vec{y}) - v_{\ell+1}(\vec{y}) \geq v_\ell(\vec{x}) - v_{\ell+1}(\vec{x}), \tag{4.21}$$

where $\vec{x} \in \mathbb{R}^{q-1}$ is defined as

$$x_j = \begin{cases} y_j & \text{if } j \neq i, i+1 \\ \frac{y_i+y_{i+1}}{2} & \text{if } j = i \text{ or } j = i+1 \end{cases}$$

for $1 \leq j \leq q-1$ and any $i \geq \ell+2$.

For $t \in \mathbb{R}$ we define $\vec{y}(t)$ by

$$y_j(t) := \begin{cases} y_j & \text{if } j \neq i, i+1 \\ y_i - t & \text{if } j = i \\ y_{i+1} + t & \text{if } j = i+1 \end{cases}$$

for $j = 1, \ldots, q-1$. Note that $\vec{y}(0) = \vec{y}$ and $\vec{y}(y_i/2 - y_{i+1}/2) = \vec{x}$. We further define

$$\xi(t) := v_\ell(\vec{y}(t)) - v_{\ell+1}(\vec{y}(t)).$$

After a straightforward calculation we can express $\xi(t)$ as

$$\begin{aligned} \xi(t) = {} & y_\ell e^{G_\ell}(1 + \sum_{j\geq 1}^{q-1} y_j) - y_{\ell+1} e^{G_{\ell+1}}(1 + \sum_{j\geq 1}^{q-1} y_j) \\ & + y_\ell \sum_{j\neq i,i+1} e^{G_j}(1-y_j) - y_{\ell+1} \sum_{j \neq i,i+1} e^{G_j}(1-y_j) \\ & + (y_\ell - y_{\ell+1}) \left(e^{G_i(t)}(1 - y_i + t) + e^{G_{i+1}(t)}(1 - y_{i+1} - t) \right), \end{aligned}$$

where we write $G_\ell := G_{\infty;\ell}(\vec{y}(t)) = \frac{q(1-y_\ell)}{1+y_1+\cdots+y_{q-1}}$, for $\ell \notin \{i, i+1\}$ and we write $G_\ell(t) = G_{\infty;\ell}(\vec{y}(t))$ when $\ell \in \{i, i+1\}$. This notation indicates that G_ℓ

is a constant function of t when $\ell \notin \{i, i+1\}$. Now observe that the function appearing in the last row,

$$g(t) := e^{G_i(t)}(1 - y_i + t) + e^{G_{i+1}(t)}(1 - y_{i+1} - t),$$

is convex on $t \in [0, y_i - y_{i+1}]$, since its second derivative is given by

$$\begin{aligned} g''(t) = e^{G_i(t)} \frac{(1 - y_i + t)q^2}{(1 + y_1 + \cdots + y_{q-1})^2} + 2e^{G_i(t)} \frac{q}{1 + y_1 + \cdots + y_{q-1}} \\ + e^{G_{i+1}(t)} \frac{(1 - y_{i+1} - t)q^2}{(1 + y_1 + \cdots + y_{q-1})^2} + 2e^{G_{i+1}(t)} \frac{q}{1 + y_1 + \cdots + y_{q-1}} > 0. \end{aligned}$$

As $g(t) = g(y_i - y_{i+1} - t)$, we obtain that $g(t)$ has a unique minimizer in $[0, y_i - y_{i+1}]$ exactly at t such that $= y_i - y_{i+1} - t$. In other words,

$$t = \frac{y_i - x_{i+1}}{2}$$

is the unique minimizer of $g(t)$ on this interval and thus for $\xi(t)$. This implies (4.21) and hence the lemma. ☐

Lemma 4.4.4. *Let* $1 \geq x_1 > x_2 \geq x_3 \geq 0$ *and* $q - 2 \geq l \geq 1$. *Then*

$$v_l(\underbrace{x_1, \cdots, x_1}_{l}, x_2, \underbrace{x_3, \cdots, x_3}_{q-l-2}) > v_{l+1}(\underbrace{x_1, \cdots, x_1}_{l}, x_2, \underbrace{x_3, \cdots, x_3}_{q-l-2}).$$

Proof. The algebraic manipulations that are done in this proof, while elementary, involve quite large expressions. Therefore we have supplied additional Mathematica code in Appendix 4.6 that can be used to verify the computations. We define

$$\begin{aligned} \xi(y_1, y_2, y_3; t) := &(y_1 y_3(t - l - 1) + (l + 1)y_1 + (l + 1)y_1 y_2 - l y_2)\, e^{A_1(y_1, y_2, y_3; t)} + \\ &(-y_2 y_3(t - l - 1) - (l + 1)y_1 y_2 + y_1 - 2y_2)\, e^{A_2(y_1, y_2, y_3; t)} + \\ &(y_1 - y_2)\,(1 - y_3)\,(t - l - 1)e^{A_3(y_1, y_2, y_3; t)}, \end{aligned}$$

where

$$A_i(y_1, y_2, y_3; t) := \frac{(t + 1)(1 - y_i)}{1 + l y_1 + y_2 + (t - (l + 1))y_3}$$

for $i = 1, 2, 3$ (see Listing 4.1). One can check that $\xi(x_1, x_2, x_3; q - 1)$ is equal to

$$v_l(x_1, \cdots, x_1, x_2, x_3, \cdots, x_3) - v_{l+1}(x_1, \cdots, x_1, x_2, x_3, \cdots, x_3).$$

We will treat t as a variable and vary it while keeping the values that appear in the exponents constant. To that effect let $C_i = A_i(x_1, x_2, x_3; q-1)$ and define

$$\begin{aligned}
y_1(t) &= \frac{C_1(l-t-1) + C_3(t-l-1) + C_2 + t + 1}{C_3(t-l-1) + C_1 l + C_2 + t + 1}, \\
y_2(t) &= \frac{C_3(t-l-1) + C_1 l - C_2 t + t + 1}{C_3(t-l-1) + C_1 l + C_2 + t + 1}, \\
y_3(t) &= \frac{C_1 l - C_3(l+2) + C_2 + t + 1}{C_3(t-l-1) + C_1 l + C_2 + t + 1}.
\end{aligned}$$

These values are chosen such that for $t_0 = q-1$ we have $y_i(t_0) = x_i$ and $A_i(y_1(t), y_2(t), y_3(t); t) = C_i$ independently of t for $i = 1, 2, 3$ (see Listings 4.2 and 4.3). Therefore $\xi(y_1(t), y_2(t), y_3(t); t)$ is a rational function of t and we want to show that it is positive at $t = q-1$. We can explicitly calculate that

$$\xi(y_1(t), y_2(t), y_3(t); t) = \left(\frac{1+t}{C_3(t-l-1) + C_1 l + C_2 + t + 1}\right)^2 \cdot r(t),$$

where r is a linear function (see Listing 4.4). It is thus enough to show that $r(q-1) > 0$. We will do this by showing that $r(l+1) > 0$ and that the slope of r is positive. We find that $r(l+1)$ is equal to

$$r(l+1) = u_1 \cdot e^{C_1} + u_2 \cdot e^{C_2},$$

where

$$\begin{aligned}
u_1 &= 2 + l + C_2 - 2C_1 + lC_1C_2 - lC_1^2 \\
u_2 &= -\left(2 + l + lC_1 - (l+1)C_2 + C_1C_2 - C_2^2\right).
\end{aligned}$$

This is part of the output of Listing 4.5. Note that by construction, since $1 \geq x_1 > x_2 \geq x_3$, we have $0 \leq C_1 < C_2 \leq C_3$. Therefore the sum of the coefficients of e^{C_1} and e^{C_2} satisfies

$$\begin{aligned}
u_1 + u_2 &= (l+2)(C_2 - C_1) + (l-1)C_1C_2 - lC_1^2 + C_2^2 \\
&= (l + 2 + C_2 + lC_1)(C_2 - C_1) > 0.
\end{aligned}$$

Now we will separate two cases depending on the sign of the coefficient of u_2. If u_2 is non-negative, then

$$r(l+1) = u_1 e^{C_1} + u_2 e^{C_2} \geq u_1 e^{C_1} + u_2 e^{C_1} = (u_1 + u_2) e^{C_1} > 0.$$

If u_2 is negative, then

$$2 + (1 + C_1 - C_2)l > C_2 - C_1C_2 + C_2^2 = (1 + C_2 - C_1)C_2.$$

In particular $2 + (1 + C_1 - C_2)l > 0$. Thus

$$\begin{aligned} r(l+1) &= e^{C_2}(u_1 e^{C_1 - C_2} - u_2) \\ &\geq (1 + C_1 - C_2)u_1 - u_2 = C_1(C_2 - C_1)(2 + (1 + C_1 - C_2)l) > 0. \end{aligned}$$

The slope of r is given by

$$s := (1 + C_3 - C_1)\, e^{C_1} - (1 + C_3 - C_2)e^{C_2} + (C_2 - C_1)C_3 e^{C_3}.$$

This is part of the output of Listing 4.5. To show that this is positive we show that $s \cdot e^{-C_2}$ is positive. Because both $1 + C_3 - C_1$ and $C_2 - C_1$ are positive we find

$$\begin{aligned} &s \cdot e^{-C_2} = (1 + C_3 - C_1)\, e^{C_1 - C_2} - (1 + C_3 - C_2) + (C_2 - C_1)C_3 e^{C_3 - C_2} \geq \\ &(1 + C_3 - C_1)(1 + C_1 - C_2) - (1 + C_3 - C_2) + (C_2 - C_1)C_3(1 + C_3 - C_2) = \\ &(C_2 - C_1)(C_1 + C_3(C_3 - C_2)), \end{aligned}$$

which is positive because $0 \leq C_1 < C_2 \leq C_3$. This concludes the proof. $\square$

We now continue with the second statement.

Statement (ii)

Proposition 4.4.5. *For any $x > 0$ we have that*

$$\left\langle \Phi\left(\frac{-x}{q-1}\vec{1}\right), \vec{1} \right\rangle > -x.$$

Proof. The statement is equivalent to

$$\phi(x) < x.$$

for $x > 0$. By definition we know that

$$\phi(x) = (q-1)\frac{q(e^{f(x)} - 1)}{(q-1)e^{f(x)} + 1},$$

where

$$f(x) = -q\frac{e^{-x/(q-1)} - 1}{(q-1)e^{-x/(q-1)} + 1}.$$

First note that $\phi(\mathbb{R}_{>0}) \subseteq (0, q)$. This means that if $x \geq q$, the statement holds. Thus we can assume that $0 < x < q$. Now, the inequality $\phi(x) < x$ can be written as

$$e^{f(x)} < \frac{x + q(q-1)}{(q-1)(q-x)},$$

because $q-x>0$. By taking logarithm of both sides, we see that $\phi(x)<x$ is equivalent to

$$-q\frac{e^{-x/(q-1)}-1}{(q-1)e^{-x/(q-1)}+1}<\log\left(\frac{x+q(q-1)}{(q-1)(q-x)}\right).$$

Since $\frac{x+q(q-1)}{(q-1)(q-x)}>\frac{0+q(q-1)}{(q-1)q}\geq 1$, we can use the inequality $\log(b)>2\frac{b-1}{b+1}$ for $b=\frac{x+q(q-1)}{(q-1)(q-x)}$. Therefore, to show $\phi(x)<x$, it is sufficient to prove that

$$-q\frac{e^{-x/(q-1)}-1}{(q-1)e^{-x/(q-1)}+1}\leq\frac{-2qx}{(q-2)x-2q(q-1)},$$

or, equivalently

$$(2q-2-x)\leq(x+2q-2)e^{-x/(q-1)}.$$

This follows from the fact that $g(t)=(t+2q-2)e^{-t/(q-1)}-(2q-2-t)$ is a convex function on $\mathbb{R}_{\geq 0}$, its derivative satisfies $g'(0)=0$ and $g(0)=0$. This concludes the proof. $\square$

4.4.2 Two iterations of F_d

As before, we view $y=1/d$ as a continuous variable. Let us define $\Phi:\mathbb{R}^{q-1}\times[0,\frac{1}{2}]\to\mathbb{R}^{q-1}$ by

$$\Phi(x_1,\ldots,x_{q-1},y)=F_{1/y}^{\circ 2}(x_1,\ldots,x_{q-1}).$$

Note that this map is analytic in all its variables. For simplicity, if y^* is fixed, then we use the notation $\Phi_{y^*}(x_1,\ldots,x_{q-1})$ for $\Phi(x_1,\ldots,x_{q-1},y)|_{y=y^*}$, and if $y=0$, then $\Phi(x_1,\ldots,x_{q-1}):=\Phi_0(x_1,\ldots,x_{q-1})$.

Lemma 4.4.6. *There exist positive constants $A>0$ and $c_0>0$, such that for any $0<c\leq c_0$ we have*

$$c-\phi(c)\geq Ac^3.$$

Proof. By definition we know that

$$\phi(x)=(g\circ f)(x)=(q-1)\frac{q(e^{f(x)}-1)}{(q-1)e^{f(x)}+1},$$

where

$$f(x)=-q\frac{e^{-x/(q-1)}-1}{(q-1)e^{-x/(q-1)}+1},$$
$$g(x)=(q-1)q\frac{e^{x}-1}{(q-1)e^{x}+1}.$$

Let us calculate the Taylor expansion of $f(x)$ and $g(x)$ around 0:

$$f(x) = \frac{1}{q-1}x + \frac{q-2}{2(q-1)^2 q}x^2 + \frac{(q^2-6q+6)}{6(q-1)^3q^2}x^3 + O(x^4),$$
$$g(x) = (q-1)x - \frac{(q-1)(q-2)}{2q}x^2 + \frac{(q-1)(q^2-6q+6)}{6q^2}x^3 + O(x^4).$$

Thus their composition has the following Taylor expansion around 0:

$$(g \circ f)(x) = x - \frac{1}{6(q-1)^2}x^3 + O(x^4).$$

This implies that there exists $c_0 > 0$ and $A > 0$, such that for any $c_0 \geq x \geq 0$ we have

$$x - \phi(x) \geq Ax^3,$$

as desired. $\square$

The next proposition implies forward invariance of P_c under $F_d^{\circ 2}$ for c small enough and d large enough.

Proposition 4.4.7. *There exists $c_0 > 0$ and $d_0 > 0$. Such that for all $c \in (0, c_0]$ and integers $d \geq d_0$ there exists $0 < c' < c$ such that*

$$F_d^{\circ 2}(D_c) \subset D_{c'}.$$

Proof. By the previous lemma we know that there is a $c_0' > 0$ and an $A > 0$, such that for any $c \leq c_0'$ we have

$$\|\Phi(-c/(q-1) \cdot \vec{1}) + c/(q-1) \cdot \vec{1}\| \geq Ac^3.$$

Here we denote by $\|\vec{x}\| = \left(\sum_{i=1}^{q-1} x_i^2\right)^{1/2}$, the standard 2-norm on $\mathbb{R}^{q-1}$. By Proposition 4.4.2, we have that for any $\vec{x} \in D_c$, $\Phi(\vec{x})$ is contained in $D_{\phi(c)}$. Therefore, denoting by $B_r(y)$ the ball of radius r around y,

$$B_{Ac^3/2}(\Phi(\vec{x})) \cap (-\infty, 0]^{q-1} \subseteq D_{\phi(c)+Ac^3/2} \subsetneq D_c. \tag{4.22}$$

Now let us consider the Taylor approximation of $\Phi_y(x_1, \ldots, x_{q-1})$ at $\vec{0} = (0, \ldots, 0)$. Since for any $y^* \in [0,1]$ the map $F_{1/y^*}(x_1, \ldots, x_{q-1})$ has $\vec{0}$ as a fixed point of derivative $-\mathrm{Id}$, there exists constants $c_1, C_1 \geq 0$ such that for any $y \in [0,1]$ and $\vec{x} = (x_1, \ldots, x_{q-1}) \in [-c_1, 0]^{q-1}$ we have

$$\|\Phi_y(\vec{x}) - \mathrm{Id}(\vec{x}) - T_{3,y}(\vec{x})\| \leq C_1 \cdot \|\vec{x}\|^4,$$

where $\mathrm{Id}(\vec{x}) + T_{3,y}(\vec{x})$ is the 3rd order Taylor approximation of $\Phi_y(\vec{x})$ at $\vec{0}$. Note that the second order term is equal to 0 because the derivative of $F_{1/y^*}(\vec{x})$ at $\vec{x} = \vec{0}$ equals $-\mathrm{Id}$. In particular, $T_{3,y}(\vec{x}) = T_y((\vec{x}), (\vec{x}), (\vec{x}))$ for some multi-linear map $T_y \in \mathrm{Mult}((\mathbb{R}^{q-1})^3, \mathbb{R}^{q-1})$, and as $y \to 0$ the map $T_{3,y}$ converges uniformly on $[-q, 0]^{q-1}$ to $T_{3,0}$. Specifically, for any $\vec{x} = (x_1, \ldots, x_{q-1}) \in [-c_1, 0]^{q-1}$

$$\|T_{3,y}(\vec{x}) - T_{3,0}(\vec{x})\| \leq A_3(y)\|\vec{x}\|^3$$

for some function A_3 that satisfies $\lim_{y\to 0} A_3(y) = 0$.

Putting this together and making use of the triangle inequality, we obtain that for any $0 < c \leq \min\{c_1, c_0'\}$ and any $\vec{x} = (x_1, \ldots, x_{q-1}) \in D_c$

$$\begin{aligned}\|\Phi_y(\vec{x}) - \Phi(\vec{x})\| &\leq \|\Phi_y(\vec{x}) - \mathrm{Id}(\vec{x}) - T_{3,y}(\vec{x})\| \\ &\quad + \|\mathrm{Id}(\vec{x}) + T_{3,y}(\vec{x}) - \mathrm{Id}(\vec{x}) - T_{3,0}(\vec{x})\| \\ &\quad + \|\mathrm{Id}(\vec{x}) + T_{3,0}(\vec{x}) - \Phi(\vec{x})\| \\ &\leq 2C_1\|x\|^4 + A_3(y))\|x\|^3 \leq K(2C_1 c + A_3(y))c^3,\end{aligned}$$

for some constant $K > 0$ (using that the 2-norm and the 1-norm are equivalent on $\mathbb{R}^{q-1}$.) Now let us fix $0 < c_0 \leq \min\{c_1, c_0'\}$ small enough such that $K2C_1c_0 < A/4$ and fix a $y_0 > 0$ such that for any any $0 \leq y \leq y_0$ we have $KA_3(y) \leq A/4$.

Then by (4.22), for any $0 \leq y \leq y_0$, $0 \leq c \leq c_0$ and $\vec{x} = (x_1, \ldots, x_{q-1}) \in D_c$,

$$\Phi_y(D_c) \subseteq B_{Ac^3/2}(\Phi(D_c)) \cap (-\infty, 0]^{q-1} \subseteq D_{\phi(c)+Ac^3/2} \subsetneq D_c.$$

So we can take $c' = \phi(c) + Ac^3/2$. □

4.4.3 Proof of Proposition 4.2.6

We are now ready to prove Proposition 4.2.6, which we restate here for convenience.

Proposition 4.2.6. *Let $q \geq 3$ be an integer. There exists $d_2 > 0$ such that for all $d \geq d_2$ the following holds: for any $c \in (0, q+1]$ there exists $0 < c' < c$ such that*

$$F_d^{\circ 2}(P_c) \subseteq P_{c'}.$$

Proof. We know by Proposition 4.4.7 there is a $d_0 > 0$ and a $c_0 > 0$ such that for $d \geq d_0$ and $c \in (0, c_0)$ there exist $c' < c$ such that $F_d^{\circ 2}(D_c) \subset D_{c'}$. As $P_c = \cup_{\pi\in S_q} \pi \cdot D_c$, we see by Lemma 4.2.4 that for $d \geq d_0$ and $c \in (0, c_0)$ we have $F_d^{\circ 2}(P_c) \subset P_{c'}$.

Next we consider $c \in [c_0, q+1]$. By Proposition 4.4.1 we know $F_\infty^{\circ 2}(P_c) \subset P_{\phi(c)}$ and $\phi(c) < c$ for any $c > 0$. As F_d converges to F_∞ uniformly, we see for each $c \in [c_0, q+1]$ there is a $d_c > 0$ large enough such that for $d \geq d_c$ and

$c' = c/2 + \phi(c)/2$ we have $F_d^{\circ 2}(P_{\hat{c}}) \subsetneq P_{c'}$ for all $\hat{c}$ sufficiently close to c. By compactness of $[c_0, q+1]$, we obtain that there is a $d_{\max} > 0$ such that for any $d > d_{\max}$ and any $c \in [c_0, q+1]$ there exists $c' < c$ such that $F_d^{\circ 2}(P_c) \subsetneq P_{c'}$. The proposition now follows by taking $d_2 = \max(d_0, d_{\max})$. □

4.5 Concluding remarks

Although we have only proved uniqueness of the Gibbs measure on the infinite regular tree for a sufficiently large degree d, our method could conceivably be extended to smaller values of d. With the aid of a computer we managed to check that for $q = 3$ and $q = 4$ and all $d \geq 2$ the map $F_d^{\circ 2}$ maps P_c into $P_{\phi_d(-c)}$, where ϕ_d is the restriction of $-F_d^{\circ 2}$ to the line $\mathbb{R} \cdot \vec{1}$. It seems reasonable to expect that for other small values of q a similar statement could be proved. A general approach is elusive so far. It is moreover also not clear that $F_d(P_c)$ is convex, not even for $q = 3$. In fact, for $q = 3$ and c large enough $F_3(P_c)$ is **not** convex. But for reasonable values of c it does appear to be convex. For larger values of q this is even less clear.

Knowing that there is a unique Gibbs measure on the infinite regular tree is by itself not sufficient to design efficient algorithms to approximately compute the partition function/sample from the associated distribution on all bounded degree graphs. One needs a stronger notion of decay of correlations, often called *strong spatial mixing* [Wei06, GK12, GKM15, LY13] or absence of complex zeros for the partition function near the real interval $[w, 1]$ [Bar16, PR17, BDPR21, LSS22]. It is not clear whether our current approach is capable of proving such statements (these certainly do not follow automatically), but we hope that it may serve as a building block in determining the threshold(s) for strong spatial mixing and absence of complex zeros. While writing this dissertation, for the case $w = 0$ corresponding to proper colorings, strong spatial mixing on the infinite tree was proved for $q \geq \Delta + 3$ in [CLMM23], very close to the uniqueness threshold $q \geq \Delta + 1$. in the same paper uniqueness of the Gibbs measure on $\mathbb{T}_\Delta$ was confirmed for all q, Δ and $w \geq \max(1 - \frac{q-1}{\Delta+4}, 0) \geq \max(1 - \frac{q}{\Delta}, 0) = w_c$ in [CLMM23], providing further partial confirmation of the folklore conjecture.

4.6 Supplementary Mathematica code to Lemma 4.4.4

The functions A_i for $i = 1, 2, 3$ and ξ are defined as follows.

Listing 4.1: The functions A_i and ξ

```
A1[y1_, y2_, y3_, m_] := (m + 1) (1 − y1)/(1 + l y1 + y2 + (m − (l + 1)) y3)
A2[y1_, y2_, y3_, m_] := (m + 1) (1 − y2)/(1 + l y1 + y2 + (m − (l + 1)) y3)
A3[y1_, y2_, y3_, m_] := (m + 1) (1 − y3)/(1 + l y1 + y2 + (m − (l + 1)) y3)

Xi[y1_, y2_, y3_, m_] := (y1 y3 (m − l − 1) + (l + 1) y1 + (l + 1) y1 y2 − l y2) Exp[A1[y1,
      y2, y3, m]]
+ (−y2 y3 (m − l − 1) − (l + 1) y1 y2 + y1 − 2 y2) Exp[A2[y1, y2, y3, m]]
+ (y1 − y2) (1 − y3) (m − l − 1) Exp[A3[y1, y2, y3, m]]
```

The functions $y_i(t)$ are defined as follows.

Listing 4.2: The functions y_i

```
{y1[t_], y2[t_], y3[t_]} = {y1, y2, y3} /. Solve[A1[y1, y2, y3, t] == C1 && A2[y1, y2, y3,
      t] == C2 && A3[y1, y2, y3, t] == C3, {y1, y2, y3}][[1]]
```

Listing 4.3: Verification that $y_i(q-1) = x_i$. This expression yields $\{x_1, x_2, x_3\}$

```
Simplify[{y1[q − 1], y2[q − 1], y3[q − 1]} /. {Rule[C1, A1[x1, x2, x3, q − 1]], Rule[C2, A2
      [x1, x2, x3, q − 1]], Rule[C3, A3[x1, x2, x3, q − 1]]}]
```

The function $r(t)$ can subsequently be found with the following code.

Listing 4.4: The function r

```
r[t_] = Simplify[Xi[y1[t], y2[t], y3[t], t] ((1 + t)/(1 + C2 − C3 + C1 l − C3 l + t + C3
      t))^(−2)]
```

It can be observed that r is indeed linear in t. To calculate $r(l+1)$ and the slope of r we use the following piece of code.

Listing 4.5: The values of $r(l+1)$ and the slope of r

```
Simplify[{r[l + 1], Coefficient[r[t], t]}]
```

PART II:

ZEROS OF THE INDEPENDENCE POLYNOMIAL ON BOUNDED DEGREE GRAPHS AND TORI

CHAPTER

FIVE

ZEROS, CHAOTIC RATIOS AND THE COMPUTATIONAL COMPLEXITY OF APPROXIMATING THE INDEPENDENCE POLYNOMIAL FOR BOUNDED DEGREE GRAPHS

5.1 Introduction

The independence polynomial of a graph $G = (V, E)$ is defined by

$$Z_G(\lambda) = \sum_{I \subseteq V} \lambda^{|I|},$$

where the sum is taken over all *independent* subsets I of the vertex set V. Recall that I is said to be independent if no two vertices in I are connected by an edge. Note that $Z_G(1)$ equals the number of independent subsets of V.

In statistical physics the independence polynomial is known as the partition function of the hard-core model. Of particular interest from a physics perspective is the location of the zeros of the partition function for certain classes of graphs. Away from these zeros the free energy is analytic, i.e. there are no phase transitions in the Lee-Yang sense cf. [YL52].

It turns out that exact computation of the independence polynomial for large graphs is not feasible for most values of λ; it is a $\#\mathcal{P}$-Hard problem[1], cf. [Rot96,

[1] The complexity class $\#\mathcal{P}$ may be seen as the counting version of the complexity class $\mathcal{NP}$. For example, the problem of deciding whether a graph on n vertices contains an independent set of size k is a problem in $\mathcal{NP}$, while the problem of determining the number of independent sets of size k is in $\#\mathcal{P}$. See [Val79, AB09] for further background.

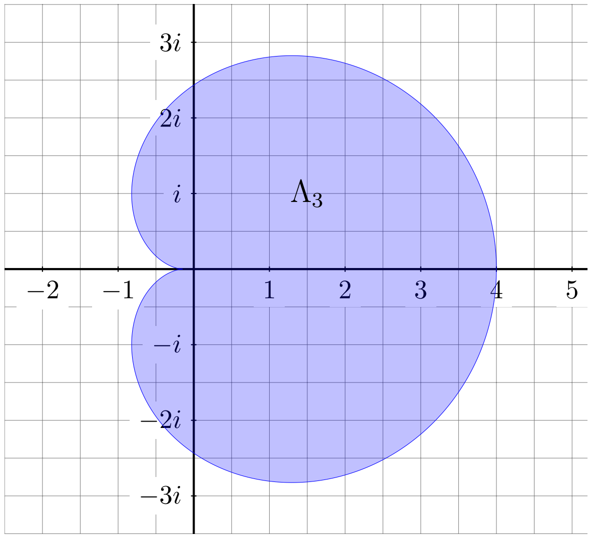


Figure 5.1: The cardioid Λ_3.

Gre00, Vad01]. A question that has received significant interest is for which $\lambda \in \mathbb{C}$ there exist polynomial time algorithms that approximate $Z_G(\lambda)$, up to small multiplicative constants. See e.g. [Wei06, SS14, Bar16, PR17, BGGŠ20, ALOG20] and the references therein.

Surprisingly, much like absence of zeros implies absence of phase transitions (in the Lee-Yang sense), absence of zeros implies the existence of efficient algorithms for this computational problem. More formally, on any connected open set containing the origin on which the independence polynomial does not vanish for all graphs of a given maximum degree Δ there exists an efficient algorithm for approximating the independence polynomial [Bar16, PR17]. Let us denote the maximal connected 'zero-free' set by $\mathcal{U}_\Delta$. For real values of λ in the complement of the closure of $\mathcal{U}_\Delta$, approximating the partition function is computationally hard [SS05, SS14, PR19, BGGŠ20]. In other words, the absence/presence of complex zeros near the real axis marks a transition in the computational complexity of approximating the independence polynomial of graphs of bounded degree Δ for real values of λ. The transition point for positive λ coincides with the phase transition for the hard-core model on the Cayley tree of degree Δ.

A natural question is whether a similar phenomenon manifests itself for non-real λ. Bezaková, Galanis, Goldberg and Štefankovič [BGGŠ20] made an important contribution towards solving this question, by showing that for any integer

$\Delta \geq 3$ and non-real λ outside a certain cardioid[2], Λ_Δ, approximation of the independence polynomial for graphs of bounded degree at most Δ is computationally hard. (In fact $\#\mathcal{P}$-hard.) See Figure 5.1 for a picture of Λ_3 and Definition 5.2.5 for the definition of Λ_Δ. Earlier it was shown by Peters and Regts [PR19] that zeros of the independence polynomial of graphs of maximum degree at most Δ accumulate on the entire boundary of Λ_Δ. In particular the maximal connected 'zero-free' set containing 0, denoted by $\mathcal{U}_\Delta$, is contained in the cardioid; their intersections with the real axis in fact coincide [SS05, PR19]. See [BC18] and [BCSV23] for more results on $\mathcal{U}_\Delta$. Buys [Buy21] however showed that Λ_Δ does contain zeros of the independence polynomial of graphs of bounded degree Δ. This in particular indicates that the result of [BGGŠ20] does not fully answer the question how zeros relate to computational hardness for non-real λ.

The goal of this chapter is to solve this question by directly relating, for any fixed integer $\Delta \geq 3$, the zeros for the family of graphs of maximum degree at most Δ to the parameters where approximating evaluations of the independence polynomial is computationally hard. Our result is obtained by studying a natural family of rational maps associated to this family of graphs, using techniques and ideas from complex dynamics. We show that 'chaotic behaviour' of this family is equivalent to the presence of zeros, and implies computational hardness.

5.1.1 Occupation ratios

Given $\Delta \in \mathbb{Z}_{\geq 2}$, we define $\mathcal{G}_\Delta$ as the collection of finite simple rooted graphs (G, v) such that the maximum degree of G is at most Δ. For $i \in \{1, \ldots, \Delta\}$ we define $\mathcal{G}_\Delta^i = \{(G, v) \in \mathcal{G}_\Delta : \deg(v) \leq i\}$. The *occupation ratio*, or *ratio* for short, of a rooted graph (G, v) is defined by the rational function

$$R_{G,v}(\lambda) := \frac{Z_G^{\text{in}}(\lambda)}{Z_G^{\text{out}}(\lambda)},$$

where "in" means that in the definition of $Z_G(\lambda)$ the sum is taken only over independent sets I that contain the marked point v, while "out" means that the independent sets do not contain v. The ratio is a very useful tool in studying the zeros of the independence polynomial, see Lemma 5.2.1 below, and has been key in several of the aforementioned works. The ratio is also relevant from a statistical physics perspective as it is closely related to the *free energy*.

When (G, v) is a rooted Cayley tree of depth $n-1$ and down-degree $d = \Delta - 1$, the ratio satisfies

$$R_{G,v}(\lambda) = f_{\lambda,d}^n(0),$$

[2] Although the domain Λ_Δ resembles a cardioid, it is formally not a cardioid. However, as discussed in Section 5.7, it plays an analogous role as the Main Cardioid of the Mandelbrot set, justifying our use of the term cardioid.

where

$$f_{\lambda,d}(z) := \frac{\lambda}{(1+z)^d}$$

and throughout the chapter we write f^n for the n-th iterate of the map f.

In this context it is therefore natural to consider λ as the parameter which determines the orbit of the marked point 0. This type of setting is often studied in complex dynamical systems, where one is interested in the sets where the parameter λ is *active* or *passive*. A parameter λ_0 is said to be passive if the family of rational functions $\{\lambda \mapsto f_{\lambda,d}^{\circ n}(0)\}$ is normal at λ_0, i.e. there exists a neighborhood such that every sequence in this family has a subsequence that converges uniformly. A parameter is active if it is not passive. The most well-known activity-locus is undoubtedly the boundary of the Mandelbrot set, where the iterates of the functions $z^2 + c$ are considered. Following this terminology we define the *activity-locus*, $\mathcal{A}_\Delta$, by

$$\mathcal{A}_\Delta := \{\lambda_0 \in \mathbb{C} \mid \{\lambda \mapsto R_{G,v}(\lambda) \mid (G,v) \in \mathcal{G}_\Delta\} \text{ is not locally normal at } \lambda_0\}.$$

Another notion of chaotic behaviour of the ratios appears in the proof of the result of Bezaková, Galanis, Goldberg and Šefankovič [BGGŠ20]. An important step towards proving $\#\mathcal{P}$-hardness is showing that for every non-real λ outside of the closed cardioid $\overline{\Lambda_\Delta}$ the set of values $\{R_{G,v}(\lambda) \mid (G,v) \in \mathcal{G}^1_\Delta\}$ is dense in $\hat{\mathbb{C}}$. Motivated by this we define

$$\mathcal{D}_\Delta := \{\lambda \in \mathbb{C} \mid \{R_{G,v}(\lambda) \mid (G,v) \in \mathcal{G}^1_\Delta\} \text{ is dense in } \hat{\mathbb{C}}\}$$

and refer to the closure of $\mathcal{D}_\Delta$ as the *density-locus*. We will prove it is equal to the activity-locus, thereby showing that these two notions of chaotic behaviour of the ratios are essentially equivalent.

5.1.2 Main result

To state our main result connecting the presence of zeros to computational hardness, we define the *zero-locus* as the closure of

$$\mathcal{Z}_\Delta = \{\lambda \in \mathbb{C} : Z_G(\lambda) = 0 \text{ for some } G \in \mathcal{G}_\Delta\}.$$

We informally define the *$\#\mathcal{P}$-locus* as the closure of the collection of λ for which approximating $Z_G(\lambda)$ is $\#\mathcal{P}$-hard for $G \in \mathcal{G}_\Delta$. See Section 5.1.3 below for a formal definition.

The main results of this chapter can now be stated succinctly as follows.

Main Theorem. *For any integer $\Delta \geq 3$ the zero-locus, the activity-locus and the density-locus are equal and contained in the $\#\mathcal{P}$-locus. In other words:*

$$\overline{\mathcal{Z}_\Delta} = \mathcal{A}_\Delta = \overline{\mathcal{D}_\Delta} \subseteq \overline{\#\mathcal{P}_\Delta}.$$

We remark that the topological structure of the complement of the zero-locus is not yet understood. We have the following conjecture.

Conjecture 5.1.1. *For each integer $\Delta \geq 3$, the set $\mathbb{C} \setminus \overline{\mathcal{Z}_\Delta}$ is connected.*

Should this conjecture be true, then by Proposition 5.4.3 below, we know that the maximal 'zero-free' set containing 0, $\mathcal{U}_\Delta$, equals the complement of the zero-locus. Since there exists a polynomial time algorithm [Bar16, PR17] for approximating the independence polynomial on $\mathcal{U}_\Delta$, this would imply with our main theorem a complete understanding of the computational complexity of approximating the independence polynomial in terms of the location of the zeros as well as in terms of chaotic behaviour of the ratios.

Remark 4. We note that [PR19] and [BGPR22] combined implicitly contain similar equivalent characterizations for the Lee-Yang zeros of the partition function of the ferromagnetic Ising model on bounded degree graphs. In that setting the complement of the zero-locus is in fact connected when the edge interaction parameter is sub-critical.

5.1.3 Computational complexity

We formally state here the computational problems we are interested in. We denote by $\mathbb{Q}[i]$ the collection of complex numbers with rational real and imaginary part. Let $\lambda \in \mathbb{Q}[i]$, $\Delta \in \mathbb{N}$ and consider the following computational problems.

Name #Hard-CoreNorm(λ, Δ)

Input A graph G of maximum degree at most Δ.

Output If $Z_G(\lambda) \neq 0$ the algorithm must output a rational number N such that $N/1.001 \leq |Z_G(\lambda)| \leq 1.001N$. If $Z_G(\lambda) = 0$ the algorithm may output any rational number.

Name #Hard-CoreArg(λ, Δ)

Input A graph G of maximum degree at most Δ.

Output If $Z_G(\lambda) \neq 0$ the algorithm must output a rational number A such that $|A - a| \leq \pi/3$ for some $a \in \arg(Z_G(\lambda))$. If $Z_G(\lambda) = 0$ the algorithm may output any rational number.

We can now formally define the $\#\mathcal{P}$-locus, as the closure of the set,

$$\#\mathcal{P}_\Delta := \{\lambda \in \mathbb{Q}[i] : \text{ the problem } \#\text{Hard-CoreNorm}(\lambda, \Delta) \text{ is } \#\mathcal{P}\text{-hard}\}.$$

We remark that in the definition of $\#\mathcal{P}_\Delta$ replacing #Hard-CoreNorm(λ, Δ) by #Hard-CoreArg(λ, Δ) does not alter the validity of the main theorem.

We moreover note that the constant 1.001 is rather arbitrary. It originates from [BGGŠ20]. As remarked there the constant can be replaced by any other constant. Let us quickly explain the idea. If say #Hard-CoreNorm(λ, Δ) is $\#\mathcal{P}$-hard, but there would be a polynomial time algorithm for the problem with 1.001 replaced by 1.001^2, then we could run this algorithm on the disjoint union of two copies of the same graph G obtaining an a 1.001^2 approximation to the norm of $Z_{G\cup G}(\lambda) = Z_G(\lambda)^2$. This would immediately gives us a 1.001-approximation to the norm of $Z_G(\lambda)$. Since the number of vertices of $G \cup G$ is polynomial in the number of vertices of G, we would thus also get a polynomial time algorithm for the problem with constant 1.001.

Organization. After introducing preliminary definitions and results in section 2, we treat the degree $\Delta = 2$ case in section 3. While the equalities between different loci are different when $\Delta = 2$, the explicit descriptions of the zero- and activity-locus will be used in the higher degree cases.

In section 4 we prove the equality of the zero-locus and the activity-locus. The inclusion of the latter in the former is actually an immediate consequence of Montel's Theorem, and proved earlier in Corollary 5.2.10. We end that section by showing that connected components of the complement of the zero-locus are simply connected.

In section 5 we prove the equality of the activity- and the density-locus, and in section 6 we prove that the density-locus is contained in the $\#\mathcal{P}$-locus.

We end this chapter by discussing a special subclass of graphs: the finite Cayley trees of fixed down-degree $\Delta - 1$. In this setting classical results from complex dynamical systems can be used to obtain a precise description of the zero- and activity-locus. While there zeros do not lie in the activity-locus, the activity-locus equals the accumulation set of the zeros.

5.2 Preliminaries

In this section we collect some preliminary results and conventions that will be used in the remainder of the chapter. The results in this section are not necessarily new, but often cannot be found in the literature in the exact way they are stated here. For convenience of the reader we include proofs, especially when the methods are similar to those used later in the chapter.

5.2.1 Ratios of graphs and trees.

Recall that for a rooted graph (G, v) the occupation ratio is defined as the following rational function in λ

$$R_{G,v}(\lambda) = \frac{Z_G^{\text{in}}(\lambda)}{Z_G^{\text{out}}(\lambda)}.$$

We note that $Z_G(\lambda) = Z_G^{\text{in}}(\lambda) + Z_G^{\text{out}}(\lambda)$, which implies that $Z_G(\lambda) = 0$ if and only if $R_{G,v}(\lambda) = -1$, unless $Z_{G,v}^{\text{in}}(\lambda)$ and $Z_{G,v}^{\text{out}}(\lambda)$ both vanish, in which case the value of the rational function $R_{G,v}(\lambda)$ may not equal -1. The next lemma will show that we can often ignore this difficulty.

We will write $G - v$ for the graph G with vertex v removed, and $G - N[v]$ for the graph with $N[v]$ removed, where $N[v] = \{u \in V(G) : \{u, v\} \in E(G)\} \cup \{v\}$ is the closed neighborhood of v. We observe that $Z_{G,v}^{out}(\lambda) = Z_{G-v}(\lambda)$, and similarly $Z_{G,v}^{in}(\lambda) = \lambda \cdot Z_{G-N[v]}(\lambda)$.

Lemma 5.2.1. *Let $\lambda \in \mathbb{C}^*$. The following three statements are equivalent.*

1. *There exists a graph G of maximum degree at most Δ for which $Z_G(\lambda) = 0$.*
2. *There exists a rooted graph $(G, v) \in \mathcal{G}_\Delta$ for which $R_{G,v}(\lambda) = -1$.*
3. *There exists a rooted graph $(G, v) \in \mathcal{G}_\Delta$ for which $R_{G,v}(\lambda) \in \{-1, 0, \infty\}$.*

Proof. Assume that (1) holds, then there is a graph G of maximum degree at most Δ for which $Z_G(\lambda) = 0$. Without loss of generality we can assume $G \in \mathcal{G}_\Delta$ satisfies $Z_G(\lambda) = 0$ and has a minimal number of vertices, i.e. for any graph $H \in \mathcal{G}_\Delta$ with $Z_H(\lambda) = 0$ we have $|V(G)| \leq |V(H)|$. For any vertex $v \in V(G)$ we have

$$0 = Z_G(\lambda) = Z_{G,v}^{in}(\lambda) + Z_{G,v}^{out}(\lambda).$$

As $|V(G-v)| < V(G)$ we have $Z_{G,v}^{out}(\lambda) = Z_{G-v}(\lambda) \neq 0$, which implies $R_{G,v}(\lambda) = -1$. Thus (2) holds. Trivially, if (2) holds then also (3) holds. To complete the proof we will assume (3) holds and show that (1) follows.

Assume there is a rooted graph $(G, v) \in \mathcal{G}_\Delta$ for which $R_{G,v}(\lambda) \in \{-1, 0, \infty\}$. If $R_{G,v}(\lambda) = -1$, we either have $Z_{G,v}^{out}(\lambda) = 0$, in which case (1) follows, or $Z_{G,v}^{out}(\lambda) \neq 0$, in which case $Z_{G,v}^{in}(\lambda) = -Z_{G,v}^{out}(\lambda)$ and (1) follows as well. If $R_{G,v}(\lambda) = \infty$ we have $Z_{G,v}^{out}(\lambda) = 0$. As $Z_{G,v}^{out}(\lambda) = Z_{G-v}(\lambda)$ we see (1) holds. The final case is $R_{G,v}(\lambda) = 0$, in which case we have $0 = Z_{G,v}^{in}(\lambda) = \lambda \cdot Z_{G-N[v]}(\lambda)$. Now as $\lambda \neq 0$, we must have $Z_{G-N[v]}(\lambda) = 0$, which concludes the proof. $\square$

Note that for $\lambda = 0$ we have $R_{G,v}(\lambda) = 0$ and $Z_G(\lambda) = 1$ for any graph G and any vertex $v \in V(G)$. Hence for $\lambda = 0$, statements (1) and (2) in Lemma 5.2.1 are still equivalent, while statement (3) is not equivalent to (1) or (2).

The following result due to Bencs [Ben18] will play an important role in this chapter.

Theorem 5.2.2. *Let $(G, v) \in \mathcal{G}^i_\Delta$ be a rooted connected graph. Then there is a rooted tree $(T, u) \in \mathcal{G}^i_\Delta$ and induced graphs $G_1, \ldots, G_k$ of G such that*

(i) $Z_T = Z_G \prod_{i=1}^k Z_{G_i}$,

(ii) $R_{G,v} = R_{T,u}$.

The following result is a consequence.

Lemma 5.2.3. *Let $\lambda \in \mathbb{C}$ and $(G, v) \in \mathcal{G}_\Delta$ with $Z_G(\lambda) = 0$. Then there is a rooted tree $(T, u) \in \mathcal{G}^1_\Delta$ such that $Z_T(\lambda) = 0$ and $R_{T,u}(\lambda) = -1$.*

Proof. Note that for any graph G we have $Z_G(0) = 1$, so we can assume $\lambda \neq 0$. By Lemma 5.2.1 there exists a rooted graph $(G, v) \in \mathcal{G}_\Delta$ such that $R_{G,v}(\lambda) = -1$. By Theorem 5.2.2(i) we see there is a rooted tree $(T, u) \in \mathcal{G}_\Delta$ with $Z_T(\lambda) = 0$. It follows there is a tree $\tilde{T}$ of maximum degree Δ with a minimal number of vertices such that $Z_{\tilde{T}}(\lambda) = 0$. For $\tilde{T}$ and any vertex $v \in V(\tilde{T})$ we have $R_{\tilde{T},v}(\lambda) = -1$. The lemma follows by choosing v a leaf of $\tilde{T}$. □

At a later stage we will need to worry about the degree of the root vertex in our definition of the activity- and density-locus. We therefore introduce some definitions to facilitate their discussion.

Fix an integer $\Delta \geq 2$ throughout. For $i = 1, \ldots, \Delta$ we denote the family of ratios with root degree at most i by

$$\mathcal{R}^i_\Delta := \{R_{G,v)} \mid (G, v) \in \mathcal{G}^i_\Delta\}.$$

We just write $\mathcal{R}_\Delta$ instead of $\mathcal{R}^\Delta_\Delta$. For a given $\lambda \in \mathbb{C}$, we denote the set of values of these ratios by

$$\mathcal{R}^i_\Delta(\lambda) := \{R_{G,v)}(\lambda) \mid (G, v) \in \mathcal{G}^i_\Delta\}.$$

Then we define $\mathcal{A}^i_\Delta$ to be the collection of λ_0 at which the family $\mathcal{R}^i_\Delta$ is not normal. We just write $\mathcal{A}_\Delta$ instead of $\mathcal{A}^\Delta_\Delta$. Finally, we introduce $\mathcal{D}^i_\Delta$ to be the collection of λ for which the set $\mathcal{R}^i_\Delta(\lambda)$ is dense in $\mathbb{C}$. Note that we denote $\mathcal{D}^1_\Delta$ by $\mathcal{D}_\Delta$ (as opposed to the above convention).

5.2.2 Graph manipulations and definition of the cardioid

The recursion formula given in the following lemma is well known.

Lemma 5.2.4. *Let $T = (V, E)$ be a tree and v a vertex of T. Suppose v is connected to $d \geq 1$ other vertices $u_1, \ldots, u_d$. Denote T_s for the tree that is the connected component of $T - v$ containing u_s. Then we have*

$$R_{T,v}(\lambda) = \frac{\lambda}{\prod_{s=1}^{d}(1 + R_{T_s,u_s}(\lambda))}. \tag{5.1}$$

Proof. We have

$$\frac{Z^{in}_{T,v}(\lambda)}{Z^{out}_{T,v}(\lambda)} = \lambda \frac{Z_{T-N[v]}(\lambda)}{Z_{T-v}(\lambda)} = \lambda \prod_{s=1}^{d} \frac{Z^{out}_{T_s,u_s}(\lambda)}{Z_{T_s}(\lambda)} = \lambda \prod_{s=1}^{d} \frac{Z^{out}_{T_s,u_s}(\lambda)}{Z^{out}_{T_s,u_s}(\lambda) + Z^{in}_{T_s,u_s}(\lambda)}, \tag{5.2}$$

where in the second equality we use that the partition function of a graph factors into the partition functions of its connected components. By dividing for each $s \in \{1, \ldots, d\}$ the denominator and enumerator of the right hand side of equation (5.2) by $Z^{out}_{T_s,u_s}(\lambda)$ we obtain the desired formula. □

This lemma implies the claim from the introduction that the ratios of Cayley trees are given by iterating $f_{\lambda,d}(z) = \frac{\lambda}{(1+z)^d}$. We refer to Section 5.7 for an in-depth discussion of Cayley trees and their associated dynamics.

Definition 5.2.5. Define the cardioid Λ_Δ as the closure of the set of parameters λ for which $f_{\lambda,d}$ has an attracting fixed point.

Note that $0 \in \Lambda_\Delta$. One can show, see Section 2.1 in [PR19], that

$$\Lambda_\Delta = \left\{ \frac{z}{(1-z)^\Delta} \mid |z| \leq \frac{1}{\Delta - 1} \right\}.$$

Taking $z = \frac{-1}{\Delta-1}$, we observe that

$$\lambda^*(\Delta) = \frac{-(\Delta-1)^{\Delta-1}}{\Delta^\Delta}$$

is the intersection point of Λ_Δ with the negative real line.

Let $G = (V, E)$ be a graph and let (G_i, v_i) be rooted graphs for $i \in V$. We refer to the graph obtained from G and the G_i by identifying each vertex $i \in V$ with v_i as *implementing* the G_i in G, see Figure 5.2. The next lemmas describe the effect on the ratios for various choices of G and G_i.

Lemma 5.2.6. *Let P_n denote the path on n vertices. Let (G_i, v_i) be rooted graphs for $i \in \{1, \ldots, n\}$ and denote $\mu_i(\lambda) = R_{G_i,v_i}(\lambda)$. Let $\tilde{P}_n$ be the graph obtained by implementing the G_i in P_n. Then*

$$R_{\tilde{P}_n,v_n}(\lambda) = (f_{\mu_n(\lambda)} \circ \cdots \circ f_{\mu_1(\lambda)})(0),$$

where $f_\mu(z) = \frac{\mu}{1+z}$.

Proof. We use induction on n. For $n = 1$, by definition we have $R_{G_1,v_1}(\lambda) = \mu_1(\lambda) = f_{\mu_1(\lambda)}(0)$. As $\tilde{P}_1 = G_1$, we have $R_{\tilde{P}_1,v_1}(\lambda) = R_{G_1,v_1}(\lambda)$. The base case follows.

Suppose the statement holds for some $n \geq 1$. The vertex v_{n+1} has 1 neighbor that is part of the path P_n. Let us denote that neighbor as v_n. It follows that

$$\begin{aligned} R_{\tilde{P}_{n+1},v_{n+1}}(\lambda) &= \frac{Z^{in}_{\tilde{P}_{n+1},v_{n+1}}(\lambda)}{Z^{out}_{\tilde{P}_{n+1},v_{n+1}}(\lambda)} = \frac{Z^{in}_{G_{n+1},v_{n+1}}(\lambda)}{Z^{out}_{G_{n+1},v_{n+1}}(\lambda)} \cdot \frac{Z^{out}_{\tilde{P}_n,v_n}(\lambda)}{Z_{\tilde{P}_n,v_n}(\lambda)} \\ &= R_{G_{n+1},v_{n+1}}(\lambda) \cdot \frac{Z^{out}_{\tilde{P}_n,v_n}(\lambda)}{Z^{out}_{\tilde{P}_n,v_n}(\lambda) + Z^{in}_{\tilde{P}_n,v_n}(\lambda)} = \frac{R_{G_{n+1},v_{n+1}}(\lambda)}{1 + R_{\tilde{P}_n,v_n}(\lambda)} \\ &= f_{\mu_{n+1}(\lambda)}(R_{\tilde{P}_n,v_n}(\lambda)), \end{aligned}$$

where in the second equality we use that the partition function of a graph factors into the partition functions of its connected components.

By the induction hypothesis we have

$$R_{\tilde{P}_n,v_n}(\lambda) = (f_{\mu_n(\lambda)} \circ \cdots \circ f_{\mu_1(\lambda)})(0),$$

from which it follows that

$$R_{\tilde{P}_{n+1},v_{n+1}}(\lambda) = (f_{\mu_{n+1}(\lambda)} \circ f_{\mu_n(\lambda)} \circ \cdots \circ f_{\mu_1(\lambda)})(0),$$

completing the proof. □

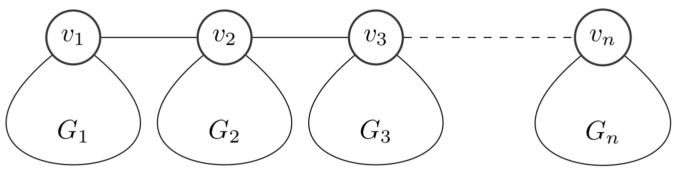


Figure 5.2: The graph $\tilde{P}_n$ in Lemma 5.2.6

Remark 5. Note that if the graphs G_i in Lemma 5.2.6 are all of maximum degree Δ and the roots v_i have degree at most $\Delta - 2$ for $i \in \{2, \ldots, n-1\}$ and at most degree $\Delta - 1$ for $i \in \{1, n\}$, then the graph $\tilde{P}_n$ is also of maximum degree Δ.

Lemma 5.2.7. *Let $G = (V, E)$ be a graph and denote $n = |V|$. Let (H, v) be a rooted graph. Let $\tilde{G} = (\tilde{V}, \tilde{E})$ be obtained from G by implementing n copies of (H, v) in G. Then for any $w \in V$ we have*

$$\frac{Z_{\tilde{G},w}(\lambda)}{(Z^{out}_{H,v}(\lambda))^n} = Z_{G,w}(R_{H,u}(\lambda)) \tag{5.3}$$

and

$$R_{\tilde{G},w}(\lambda) = R_{G,w}(R_{H,v}(\lambda)). \tag{5.4}$$

Proof. We have

$$\frac{Z^{in}_{\tilde{G},w}(\lambda)}{(Z^{out}_{H,v}(\lambda))^n} = \frac{\sum_{\substack{I \in \mathcal{I}(G)\\ w \in I}} Z^{in}_{H,v}(\lambda)^{|I|} Z^{out}_{H,v}(\lambda)^{n-|I|}}{(Z^{out}_{H,v}(\lambda))^n} = Z^{in}_{G,w}(R_{H,v}(\lambda)) \tag{5.5}$$

and

$$\frac{Z^{out}_{\tilde{G},w}(\lambda)}{(Z^{out}_{H,v}(\lambda))^n} = \frac{\sum_{\substack{I \in \mathcal{I}(G)\\ w \notin I}} Z^{in}_{H,v}(\lambda)^{|I|} Z^{out}_{H,v}(\lambda)^{n-|I|}}{(Z^{out}_{H,v}(\lambda))^n} = Z^{out}_{G,w}(R_{H,v}(\lambda)). \tag{5.6}$$

Equality (5.3) follows from equalities (5.5) and (5.6) noting that for any graph W and any vertex u of W we have $Z_W(\lambda) = Z^{in}_{W,u}(\lambda) + Z^{out}_{W,u}(\lambda)$. Equality (5.4) follows from equalities (5.5) and (5.6) and the definition of the ratio. □

We will also need the following slight variation on Lemma 5.2.4.

Lemma 5.2.8. *Let (G_1, v_1) and (G_2, v_2) be rooted graphs, and define the rooted graph $(\tilde{G}, \tilde{v})$ by identifying the roots v_1 and v_2. Then*

$$R_{\tilde{G},\tilde{v}}(\lambda) = \lambda^{-1} \cdot R_{G_1,v_1}(\lambda) \cdot R_{G_2,v_2}(\lambda).$$

Proof. We compute

$$R_{\tilde{G},\tilde{v}}(\lambda) = \frac{Z^{in}_{\tilde{G},\tilde{v}}(\lambda)}{Z^{out}_{\tilde{G},\tilde{v}}(\lambda)} = \frac{Z^{in}_{G_1,v_1}(\lambda) \cdot Z^{in}_{G_2,v_2}(\lambda) \cdot \lambda^{-1}}{Z^{out}_{G_1,v_1}(\lambda) \cdot Z^{out}_{G_2,v_2}(\lambda)} = \lambda^{-1} \cdot R_{G_1,v_1}(\lambda) \cdot R_{G_2,v_2}(\lambda).$$

□

5.2.3 The Shearer region

Denote the open disk around 0 with radius $\frac{(\Delta-1)^{\Delta-1}}{\Delta^\Delta}$ by B_Δ. This region, also known as the Shearer region, is the maximal open disk centered around 0 that is zero free for the independence polynomial of graphs of maximum degree Δ [SS05, She85]. We will show the Shearer region is also disjoint from the activity-locus and the density-locus, which will later be used to deal with the $\lambda = 0$ case in the proof of our main theorem.

Lemma 5.2.9. *Let $\Delta \geq 2$ be an integer. Then B_Δ is disjoint from the activity-locus, the zero-locus and the density-locus, i.e., we have $B_\Delta \cap \overline{\mathcal{D}_\Delta} = B_\Delta \cap \mathcal{A}_\Delta = B_\Delta \cap \overline{\mathcal{Z}_\Delta} = \emptyset$.*

Proof. We claim that for any rooted graph $(G, v) \in \mathcal{G}_\Delta$ and any $\lambda \in B_\Delta$ we have

$$|R_{G,v}(\lambda)| < \begin{cases} \frac{1}{\Delta} & \text{if } \deg(v) \leq \Delta - 1, \\ \frac{1}{\Delta-1} & \text{otherwise.} \end{cases}$$

By Theorem 5.2.2 we can equivalently work with rooted trees $(T, v) \in \mathcal{G}_\Delta$ instead of rooted graphs.

We will prove the claim by induction on the number of vertices of T. If $|V(T)| = 1$, we have $\deg(v) = 0$ and therefore $R_{T,v}(\lambda) = \lambda$. The claim then follows as $\frac{(\Delta-1)^{(\Delta-1)}}{\Delta^\Delta} < \frac{1}{\Delta}$ for all $\Delta \geq 2$. Suppose the claim holds for all rooted trees $(T, v) \in \mathcal{G}_\Delta$ with $|V(T)| \leq n$ for some $n \geq 1$. Let $(\tilde{T}, \tilde{v}) \in \mathcal{G}_\Delta$ be a rooted tree with $n + 1$ vertices. Denote the d children of $\tilde{v}$ as $u_1, \ldots, u_d$ and denote (T_i, u_i) for the rooted subtree of $\tilde{T}$ with root u_i. By Lemma 5.2.4 we have

$$R_{\tilde{T},\tilde{v}}(\lambda) = \frac{\lambda}{\prod_{i=1}^d (1 + R_{T_i,u_i}(\lambda))}.$$

We note that each (T_i, u_i) has at most n vertices, hence the induction hypotheses applies. Furthermore in T_i we have $\deg(u_i) \leq \Delta - 1$ as $\tilde{T}$ has maximum degree at most Δ. Thus we see

$$\begin{aligned} |R_{\tilde{T},\tilde{v}}(\lambda)| &= \frac{|\lambda|}{\prod_{i=1}^d |1 + R_{T_i,u_i}(\lambda)|} \leq \frac{|\lambda|}{\prod_{i=1}^d (1 - |R_{T_i,u_i}(\lambda)|)} \\ &< \frac{|\lambda|}{(1 - \frac{1}{\Delta})^d} = \frac{\Delta^d |\lambda|}{(1 - \Delta)^d} < \frac{(\Delta-1)^{\Delta-1-d}}{\Delta^{\Delta-d}}. \end{aligned}$$

Now if $d \leq \Delta - 1$, we see $\frac{(\Delta-1)^{\Delta-1-d}}{\Delta^{\Delta-d}} < \frac{1}{\Delta}$ hence the claim follows for that case. If $d = \Delta$ we have $\frac{(\Delta-1)^{\Delta-1-d}}{\Delta^{\Delta-d}} = \frac{1}{\Delta-1}$, which proves the claim.

It follows from the claim above that the family of ratios $\mathcal{R}_\Delta$ maps B_Δ into the open unit disk, for all $\Delta \geq 2$. So clearly $B_\Delta \cap \mathcal{D}_\Delta = \emptyset$. As B_Δ is open, we have $B_\Delta \cap \overline{\mathcal{D}_\Delta} = \emptyset$.

By Montel's Theorem the family $\mathcal{R}_\Delta$ is normal on B_Δ, so $B_\Delta \cap \mathcal{A}_\Delta = \emptyset$. We showed for all rooted graphs $(G, v) \in \mathcal{G}_\Delta$ that $|R_{G,v}(\lambda)| < \frac{1}{\Delta-1} \leq 1$, hence the ratio will never equal -1. For $\lambda \neq 0$, we see by Lemma 5.2.1 that $\lambda \notin \mathcal{Z}_\Delta$. For $\lambda = 0$ we note that $Z_G(0) = 1$ for any graph $G \in \mathcal{G}_\Delta$. It follows that $B_\Delta \cap \mathcal{Z}_\Delta = \emptyset$. Again, as B_Δ is open, we have $B_\Delta \cap \overline{\mathcal{Z}_\Delta} = \emptyset$. This completes the proof. $\square$

Remark 6. We note that on the negative real line the Shearer region agrees with the interior of the cardioid, i.e we have $\mathbb{R}_{\leq 0} \cap B_\Delta = \mathbb{R}_{\leq 0} \cap \operatorname{int} \Lambda_\Delta$ for all integers $\Delta \geq 3$.

Lemmas 5.2.9 and 5.2.1 together imply one of the inclusions in our main result.

Corollary 5.2.10. *For all $\Delta \geq 2$ the activity-locus is contained in the zero-locus, i.e. $\mathcal{A}_\Delta \subseteq \overline{\mathcal{Z}_\Delta}$.*

Proof. Equivalently, we want to show that for any $\lambda \in \mathbb{C} \setminus \overline{\mathcal{Z}_\Delta}$ the family $\mathcal{R}_\Delta$ is normal at λ. By Lemma 5.2.9 this is the case for $\lambda = 0$ and thus we assume that $\lambda \neq 0$. Take a sufficiently small neighborhood U around λ such that $0 \notin U$ and $U \cap \overline{\mathcal{Z}_\Delta} = \emptyset$. It follows from Lemma 5.2.1 that the family $\mathcal{R}_\Delta$ avoids $\{-1, 0, \infty\}$ for all $\lambda' \in U$. Hence by Montel's Theorem the family is normal on U. □

5.3 Graphs with maximum degree at most two

In this section we will deal with graphs of maximum degree at most two, in other words graphs for which each component is a path or a cycle. We will show that

$$\overline{\mathcal{Z}_2} = \mathcal{A}_2 = (-\infty, -1/4] \quad \text{and} \quad \mathcal{D}_2^1 = \mathcal{D}_2^2 = \emptyset.$$

An explicit description of $\mathcal{Z}_2$ was already known [HL72, SS05]; we provide a new proof for the sake of completeness.

Note that this is in contrast to the situation for $\Delta \geq 3$ as stated in the main theorem. It follows from Lemma 5.2.3 that $\mathcal{Z}_2$ is equal to the set of λ for which there is a $(T, v) \in \mathcal{G}_2^1$, with T a tree, such that $R_{T,v}(\lambda) = -1$. The collection $\mathcal{G}_2^1$ consists of rooted graphs where the component containing the root is a path rooted at an endpoint. Let (P_n, v_n) denote a path on n vertices rooted at an endpoint v_n. If we let $f_\lambda(z) = \lambda/(1+z)$ then it follows from Lemma 5.2.6 that $R_{P_n,v_n}(\lambda) = f_\lambda^n(0)$. For fixed λ the map f_λ is a Möbius transformation and therefore we first review some properties of Möbius transformation.

5.3.1 Möbius transformations

Everything that is done in this section can for example be found in [Bea95, Section 4.3]. Let $\mathcal{M}$ denote the group of Möbius transformations with composition as group operation and let $\mathrm{GL}_2(\mathbb{C})$ denote the group of 2×2 invertible matrices with complex entries. The following map is a surjective group homomorphism.

$$\Phi : \mathrm{GL}_2(\mathbb{C}) \to \mathcal{M}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(z \mapsto \frac{az+b}{cz+d} \right).$$

For any $g \in \mathcal{M}$ take an element $A \in \Phi^{-1}(\{g\})$ and define $\operatorname{tr}^2(g) = \operatorname{tr}(A)^2 / \det(A)$. This quantity does not depend on the choice of A and thus tr^2 is a well-defined function on $\mathcal{M}$. We say that elements $f, g \in \mathcal{M}$ are conjugate if there exists an $h \in \mathcal{M}$ such that $f = h \circ g \circ h^{-1}$.

Lemma 5.3.1 ([Bea95, Theorem 4.3.4])**.** *Let $f, g \in \mathcal{M}$ not equal to the identity. The maps f, g are conjugate if and only if $\operatorname{tr}^2(f) = \operatorname{tr}^2(g)$. It follows that g is conjugate to*

- *a rotation $z \mapsto e^{i\theta} \cdot z$ for some $\theta \in (0, \pi]$ if and only if $\operatorname{tr}^2(g) \in [0, 4)$;*
- *the translation $z \mapsto z + 1$ if and only if $\operatorname{tr}^2(g) = 4$;*
- *a multiplication $z \mapsto \xi \cdot z$ for some $\xi \in \mathbb{C}^*$ with $|\xi| < 1$ if and only if $\operatorname{tr}^2(g) \in \mathbb{C} \setminus [0, 4]$.*

The map g is said to be elliptic, parabolic or loxodromic in these three cases respectively.

Observe that if $f = h \circ g \circ h^{-1}$ then $f^n = h \circ g^n \circ h^{-1}$. It follows that, to understand the dynamical behaviour of a Möbius transformation g, it is enough to understand the dynamical behaviour of any element in the conjugacy class of g. If g is loxodromic then it has two distinct fixed points in $\widehat{\mathbb{C}}$, one of which is attracting and the other is repelling. Under iteration of g the orbit of every initial value except for the repelling fixed point converges to the attracting fixed point. If g is parabolic then g has a unique fixed point, and under iteration of g all orbits converge to this fixed point. If g is elliptic then g is conjugate to a rotation $z \mapsto e^{i\theta} \cdot z$. We say that g is conjugate to a *rational rotation* if θ is a rational multiple of π and otherwise we say that g is conjugate to an *irrational rotation*. If g is conjugate to a rational rotation there is a positive integer n such that g^n is equal to the identity. If g is conjugate to an irrational rotation it has two fixed points, say p, q, and $\widehat{\mathbb{C}} \setminus \{p, q\}$ is foliated by generalized circles on which g acts conjugately to an irrational rotation.

We end this subsection by classifying f_λ in terms of its parameter.

Lemma 5.3.2. *The Möbius transformation f_λ is*

- *elliptic if $\lambda \in (-\infty, -1/4)$;*
- *parabolic if $\lambda = -1/4$;*
- *loxodromic if $\lambda \in \mathbb{C}^* \setminus (-\infty, -1/4]$.*

Proof. This follows from Lemma 5.3.1 and the fact that $\operatorname{tr}^2(f_\lambda) = -1/\lambda$. $\square$

5.3.2 Determining the zero and activity-locus

In this subsection we we will show that both $\overline{\mathcal{Z}_2}$ and $\mathcal{A}_2^1$ are equal to $(-\infty, -1/4]$. By definition we have $\mathcal{A}_2^1 \subseteq \mathcal{A}_2$ and by Corollary 5.2.10 we have $\mathcal{A}_2 \subseteq \overline{\mathcal{Z}_2}$, hence it will follow that $\mathcal{A}_2$ is equal to $(-\infty, -1/4]$ as well.

Lemma 5.3.3. *Zeros of Z_G for graphs $G \in \mathcal{G}_2$ form a dense subset of the interval $(-\infty, -1/4)$, hence $\overline{\mathcal{Z}_2} = (-\infty, -1/4]$.*

Proof. We claim that $\lambda \in \mathcal{Z}_2$ if and only if f_λ is conjugate to a rational rotation.

First suppose that $\lambda \in \mathcal{Z}_2$. Then, by Lemma 5.2.3, there is an $n \geq 1$ such that for the path on n vertices P_n rooted at the endpoint v_n we have $R_{P_n,v_n}(\lambda) = -1$ and thus $f_\lambda^n(0) = -1$. Because $f_\lambda^2(-1) = 0$ regardless of the value of λ we obtain that $f_\lambda^{n+2}(0) = 0$. This means that 0 is a periodic point of f_λ of period strictly larger than 1. This can only occur if f_λ is conjugate to a rational rotation, as is explained in Section 5.3.1.

Suppose that f_λ is conjugate to a rational rotation. Note that this implies that λ is not equal to zero. Take the smallest positive integer n such that f_λ^n is equal to the identity and thus specifically $f_\lambda^n(0) = 0$. Note that $f_\lambda(0) = \lambda$ and $f_\lambda^2(0) = \lambda/(1+\lambda)$ and thus $n \geq 3$. Since $f_\lambda^{-2}(0) = -1$ we obtain that $R_{P_{n-2},v_{n-2}}(\lambda) = f_\lambda^{n-2}(0) = -1$. It follows from the proof of Lemma 5.2.1 that λ is a root of $Z_{P_{n-2}}$.

Parameters λ for which f_λ is conjugate to a rational rotation lie dense in the set of parameters for which f_λ is conjugate to any rotation. It follows from Lemma 5.3.2 that $\overline{\mathcal{Z}_2} = (-\infty, -1/4]$. □

We remark that tr^2 of the map that sends z to $e^{i\theta} \cdot z$ is equal to $2(1+\cos(\theta))$. By comparing this to the value of $\mathrm{tr}^2(f_\lambda)$ it follows from the previous proof that

$$\mathcal{Z}_2 = \left\{ \frac{-1}{2(1+\cos(t\pi))} : t \in (0,1) \cap \mathbb{Q} \right\}.$$

We will now prove the final lemma needed to determine $\mathcal{A}_2$.

Lemma 5.3.4. *The family $\mathcal{R}_2^1$ is not normal around any $\lambda \in (-\infty, -1/4]$, i.e. $(-\infty, -1/4] \subseteq \mathcal{A}_2^1$.*

Proof. Recall that

$$\mathcal{R}_2^1 = \{R_{P_n,v_n} : n \geq 1\} = \{\lambda \mapsto f_\lambda^n(0) : n \geq 1\}.$$

Take a $\lambda_0 \in (-\infty, -1/4]$ and suppose for the sake of contradiction that there exists a neighborhood U of λ_0 on which $\mathcal{R}_2^1$ is normal. We take U connected and sufficiently small so that it does not contain 0, and 0 is not a fixed point of f_λ for any $\lambda \in U$. Because $\mathcal{R}_2^1$ is normal on U there exists a subsequence of $\{R_{P_n,v_n}\}_{n\geq 1}$ that converges locally uniformly to a holomorphic function $F : U \to \widehat{\mathbb{C}}$. For $\lambda \in U \setminus (-\infty, -1/4]$ the map f_λ is loxodromic, hence $f_\lambda^n(0)$ converges to the attracting fixed point of f_λ as n goes to infinity. This means that for $\lambda \in U \setminus (-\infty, -1/4]$ we have $f_\lambda(F(\lambda)) = F(\lambda)$. Because $U \setminus (-\infty, -1/4]$ is non-empty and open in U it follows from the identity theorem for holomorphic functions that the equality $f_\lambda(F(\lambda)) = F(\lambda)$ must hold for all $\lambda \in U$. The set U contains a parameter λ_1 for which f_{λ_1} is elliptic. The value 0 is not a fixed point of f_{λ_1} and thus the distance of $f_{\lambda_1}^n(0)$ to both of the fixed points of f_{λ_1} is uniformly bounded below for all n by a positive constant. This means that no subsequence of $\{R_{P_n,v_n}(\lambda_1)\}_{n\geq 1}$ can converge to the fixed point $F(\lambda_1)$. We conclude that $\mathcal{R}_2^1$ is not normal at λ_0. $\square$

It follows from the previous two lemmas and Corollary 5.2.10 that

$$(-\infty, -1/4] \subseteq \mathcal{A}_2^1 \subseteq \mathcal{A}_2^2 \subseteq \overline{\mathcal{Z}_2} = (-\infty, -1/4].$$

Therefore we can conclude that both $\mathcal{A}_2$ and $\overline{\mathcal{Z}_2}$ are equal to $(-\infty, -1/4]$.

5.3.3 Determining the density-locus.

Recall that for $\lambda \in \mathbb{C}$ we defined $\mathcal{R}_\Delta^i(\lambda) = \{R_{G,v}(\lambda) : (G, v) \in \mathcal{G}_\Delta^i\}$. Subsequently we defined $\mathcal{D}_\Delta^i$ as the set consisting of those λ for which $\mathcal{R}_\Delta^i(\lambda)$ is dense in $\widehat{\mathbb{C}}$. It is thus clear that $\mathcal{D}_2^1 \subseteq \mathcal{D}_2^2$. To conclude the section we show the following.

Lemma 5.3.5. *There is no $\lambda_0 \in \mathbb{C}$ for which $\mathcal{R}_2^2(\lambda_0)$ is dense in $\widehat{\mathbb{C}}$, i.e. $\mathcal{D}_2^2 = \emptyset$.*

Proof. It follows from Theorem 5.2.2 that

$$\mathcal{R}_2^2(\lambda) = \{R_{T,v}(\lambda) : (T, v) \in \mathcal{G}_2^2 \text{ with } T \text{ a tree}\}.$$

A rooted tree $(T, v) \in \mathcal{G}_2^2$ can be viewed as a vertex v onto which two rooted paths (P_n, v_n) and (P_m, v_m) are attached for $n, m \geq 0$. It follows from Lemma 5.2.4 and Lemma 5.2.6 that

$$R_{T,v}(\lambda) = \lambda \cdot \frac{1}{1 + f_\lambda^n(0)} \cdot \frac{1}{1 + f_\lambda^m(0)} = \frac{1}{\lambda} \cdot f_\lambda^{n+1}(0) \cdot f_\lambda^{m+1}(0).$$

For a specific λ_0 the right-hand side of this equality is not defined if $f_{\lambda_0}^{n+1}(0)$ and $f_{\lambda_0}^{m+1}(0)$ take on the values 0 and ∞. Recall that if $f_{\lambda_0}^{n+1}(0) = \infty$, then $f_{\lambda_0}^n(0) = -1$, which implies that $\lambda_0 \in \mathcal{Z}_2$. If this is the case then Lemma 5.3.3 implies that λ_0 is real, and thus $\mathcal{R}_2^2(\lambda_0)$ is contained in $\mathbb{R} \cup \{\infty\}$ and is not dense in $\widehat{\mathbb{C}}$.

Assume that λ_0 is not real. In this case we have the equality

$$\mathcal{R}_2^2(\lambda_0) = \left\{ \frac{1}{\lambda_0} \cdot f_{\lambda_0}^{n+1}(0) \cdot f_{\lambda_0}^{m+1}(0) : n, m \geq 0 \right\}.$$

The map f_{λ_0} is loxodromic, hence the orbit of 0 converges to an attracting fixed point without passing through ∞. Note that $f_{\lambda_0}(\infty) = 0$, therefore ∞ is not the attracting fixed point, and thus there is a positive bound $B \in \mathbb{R}_{>0}$ such that $|f_{\lambda_0}^n(0)| < B$ for all n. It follows that

$$\left| \frac{1}{\lambda_0} \cdot f_{\lambda_0}^{n+1}(0) \cdot f_{\lambda_0}^{m+1}(0) \right| < \frac{B^2}{|\lambda_0|}$$

for all n, m, and thus $\mathcal{R}_2^2(\lambda_0)$ is bounded and in particular not dense in $\widehat{\mathbb{C}}$. $\square$

5.4 Equality of the zero-locus and the activity-locus for $\Delta \geq 3$

In this section we prove the equalities $\mathcal{A}_\Delta^1 = \mathcal{A}_\Delta^2 = \cdots = \mathcal{A}_\Delta^\Delta = \overline{\mathcal{Z}_\Delta}$ for $\Delta \geq 3$, thereby proving that the activity-locus is equal to the zero-locus. Our strategy is similar to the $\Delta = 2$ case. By definition we have $\mathcal{A}_\Delta^1 \subseteq \mathcal{A}_\Delta^2 \subseteq \cdots \subseteq \mathcal{A}_\Delta^\Delta$. We will first show that $\mathcal{A}_\Delta^1 = \mathcal{A}_\Delta^2 = \cdots = \mathcal{A}_\Delta^{\Delta-1}$ and subsequently we will show that $\overline{\mathcal{Z}_\Delta} \subseteq \mathcal{A}_\Delta^{\Delta-1}$. Then Corollary 5.2.10, which states that $\mathcal{A}_\Delta^\Delta \subseteq \overline{\mathcal{Z}_\Delta}$, is enough to arrive at our desired conclusion.

Lemma 5.4.1. *The family* $\mathcal{R}_\Delta^1$ *is normal at* $\lambda_0 \in \mathbb{C}$ *if and only if* $\mathcal{R}_\Delta^{\Delta-1}$ *is normal at* λ_0*, and hence* $\mathcal{A}_\Delta^1 = \mathcal{A}_\Delta^{\Delta-1}$*.*

Proof. Recall that

$$\mathcal{R}_\Delta^i := \{R_{G,v} : (G, v) \in \mathcal{G}_\Delta^i\}$$

and thus $\mathcal{R}_\Delta^1 \subseteq \mathcal{R}_\Delta^{\Delta-1}$. It follows that if $\mathcal{R}_\Delta^{\Delta-1}$ is normal at λ_0 then the same holds for $\mathcal{R}_\Delta^1$.

To show the other direction, assume that $\mathcal{R}_\Delta^1$ is normal at λ_0. Note that the family $\mathcal{R}_\Delta^\Delta$ is normal at 0 by Lemma 5.2.9, hence we can assume $\lambda_0 \neq 0$. As $\mathcal{R}_\Delta^1$ is normal at λ_0, there is a neighborhood U of λ_0 on which $\mathcal{R}_\Delta^1$ is a normal family. We can take U such that $0 \notin U$. We will show that $\mathcal{R}_\Delta^{\Delta-1}$ is also a normal family on U. To that effect take a sequence of rooted graphs $\{(G_n, v_n)\}_{n\geq 1} \subseteq \mathcal{G}_\Delta^{\Delta-1}$. Construct the rooted graphs $(\hat{G}_n, \hat{v}_n) \in \mathcal{G}_\Delta^1$ by attaching a root $\hat{v}_n$ to the root v_n of G_n by a single edge. By assumption the sequence $\{R_{\hat{G}_n, \hat{v}_n}\}_{n\geq 1}$ has a subsequence that converges locally uniformly to a function $H : U \to \widehat{\mathbb{C}}$. Let $I \subseteq \mathbb{N}$ be the indices belonging to this subsequence. By Lemma 5.2.6 we have

$R_{\hat{G}_n,\hat{v}_n}(\lambda) = f_\lambda(R_{G_n,v_n}(\lambda))$ for every $\lambda \in U$. Because U does not contain 0, the Möbius transformation f_λ is invertible for every $\lambda \in U$. Therefore for these λ we have

$$\begin{aligned}\lim_{\substack{n\to\infty\\ n\in I}} R_{G_n,v_n}(\lambda) &= \lim_{\substack{n\to\infty\\ n\in I}} f_\lambda^{-1}(f_\lambda(R_{G_n,v_n}(\lambda)))\\ &= \lim_{\substack{n\to\infty\\ n\in I}} f_\lambda^{-1}(R_{\hat{G}_n,\hat{v}_n}(\lambda)) = f_\lambda^{-1}(H(\lambda)).\end{aligned}$$

Because the map $U \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ that sends (λ, z) to $f_\lambda^{-1}(z)$ is continuous, we can conclude that this limit converges locally uniformly on U. Therefore we have shown that the sequence $\{R_{G_n,v_n}\}_{n\geq 1}$ has a subsequence that converges locally uniformly to the holomorphic function $\lambda \mapsto f_\lambda^{-1}(H(\lambda))$, and thus $\mathcal{R}_\Delta^{\Delta-1}$ is normal at λ_0. $\square$

Proposition 5.4.2. *Let $\Delta \geq 3$. Then $\overline{\mathcal{Z}_\Delta} \subseteq \mathcal{A}_\Delta^{\Delta-1}$, and hence the zero-locus is contained in the activity-locus.*

Proof. Let us assume $\lambda \in \overline{\mathcal{Z}_\Delta}$. Then for any open neighborhood V of λ there is a $\lambda_0 \in V$ for which $Z_G(\lambda_0) = 0$ for some $G \in \mathcal{G}_\Delta$. We will prove that the family $\mathcal{R}_\Delta^{\Delta-1}$ cannot be normal on V.

By Lemma 5.2.3 there is a a rooted tree $(T, u) \in \mathcal{G}_\Delta^1$ for which $R_{T,u}(\lambda_0) = -1$. Consider the rooted trees (T_n, v) obtained by implementing a copy of (T, u) in every vertex of the rooted paths (P_n, v). It follows from Lemma 5.2.7 that

$$R_{T_n,v} = R_{P_n,v} \circ R_{T,u}.$$

We note that in T_n the root v has degree $2 \leq \Delta - 1$. Furthermore $R_{T,u}$ maps a neighborhood of λ_0 holomorphically to a neighborhood of -1, since $R_{T,u}$ is not constantly equal to -1. Lemma 5.3.4 states that the family $\{R_{P_n,v}\}_{n>0}$ is not normal at -1 and thus it follows that $\{R_{T_n,v}\}_{n>0}$ is not normal at λ_0. $\square$

Summarising we have the following relations between sets

$$\mathcal{A}_\Delta^1 \overset{(1)}{=} \mathcal{A}_\Delta^{\Delta-1} \subseteq \mathcal{A}_\Delta^\Delta \overset{(2)}{\subseteq} \overline{\mathcal{Z}_\Delta} \overset{(3)}{\subseteq} \mathcal{A}_\Delta^{\Delta-1},$$

where equality (1) is due to Lemma 5.4.1, inclusion (2) is due to Corollary 5.2.10 and inclusion (3) is due to Proposition 5.4.2. It follows that $\mathcal{A}_\Delta^1 = \ldots = \mathcal{A}_\Delta^\Delta$, and $\mathcal{A}_\Delta = \overline{\mathcal{Z}_\Delta}$ for all $\Delta \geq 2$.

5.4.1 The complement of the zero-locus

As an application of the equality of the zero-locus and the activity-locus, we show here that each component of the complement of the zero-locus is simply connected.

Proposition 5.4.3. *Let* $\Delta \geq 2$ *be an integer. Any connected component of the complement of the zero-locus,* $\mathbb{C} \setminus \overline{\mathcal{Z}_\Delta}$*, is simply connected.*

Proof. For $\Delta = 2$ the statement follows directly by the exact characterization of the closure of the zero-locus. We will therefore assume that $\Delta \geq 3$.

Let γ be a simple closed curve contained in the complement of the zero-locus, $\mathbb{C} \setminus \overline{\mathcal{Z}_\Delta}$. It is sufficient to prove that the interior of γ, which we will denote by V, is zero free. Let us suppose for the sake of a contradiction that this is not the case.

Let T be a minimal tree for which $Z_T(\lambda_0) = 0$ for some $\lambda_0 \in V$. Let v be a leaf of T. Since $|T|$ is chosen minimal it follows that $R_{T,v}(\lambda_0) = -1$. Denote the neighbor of v in T by w. By minimality of T it also follows that $R_{T-v,w}(\lambda) \neq -1$ for any $\lambda \in \overline{V}$.

Note that V is necessarily bounded, as it is a subset of the cardioid, Λ_Δ. Hence by compactness of $\overline{V}$ it follows that $R_{T-v,w}$ is bounded away from -1 on V. Since

$$R_{T,v}(\lambda) = \frac{\lambda}{1 + R_{T-v,w}(\lambda)}$$

it follows that $R_{T,v}$ is bounded on $\overline{V}$. By the Open Mapping Theorem for holomorphic functions it follows that there must be a $\lambda_1 \in \partial V = \gamma$ for which

$$R_{T,v}(\lambda_1) \in (-\infty, -1).$$

Use Lemma 5.2.6 to implement the rooted tree (T, v) in the paths P_n to obtain a sequence of rooted graphs $\{(G_n, u_n)\}_{n \geq 1}$ with

$$R_{G_n,u_n} = R_{P_n,u_n} \circ R_{T,v}.$$

Since $R_{T,v}(\lambda_1) \in (-\infty, -1)$, which is contained in the half-line where the family $\{\lambda \mapsto R_{P_n,u}(\lambda)\}_{n \in \mathbb{N}}$ is not normal, it follows that the family of ratios $\{R_{G_n,u_n}\}$ is not normal at λ_1. This contradicts the assumption that γ is contained in $\mathbb{C} \setminus \overline{\mathcal{Z}_\Delta}$, by the equivalence of the activity-locus and the zero-locus. $\square$

5.5 Equality of the density-locus and the activity-locus for $\Delta \geq 3$

We first show the inclusion $\overline{\mathcal{D}_\Delta} \subseteq \mathcal{A}_\Delta$ holds. Note that as $\mathcal{A}_\Delta$ is closed, it suffices to show $\mathcal{D}_\Delta \subseteq \mathcal{A}_\Delta$.

Theorem 5.5.1. *The density-locus is contained in the activity-locus. More precisely, we have* $\mathcal{D}_\Delta \subseteq \mathcal{A}_\Delta$ *for all* $\Delta \geq 3$.

Remark 7. Recall the remarkable Proposition 6 in [BGGŠ20], in which it is shown that non-real $\lambda \in \mathbb{Q}[i]$ outside the cardioid Λ_Δ are contained in the density-locus. As a consequence Theorem 5.5.1 implies that $\mathcal{Z}_\Delta$ is dense in the complement of the cardioid.

The proof of Theorem 5.5.1 is by contradiction. So we will assume that there is a $\lambda_0 \in \mathcal{D}_\Delta$ with $\lambda_0 \notin \mathcal{A}_\Delta$ and arrive at a contradiction. In order to do this, we state and prove three helpful lemmas.

Lemma 5.5.2. *Let $\lambda_0 \in \mathbb{C}\setminus\mathcal{A}_\Delta$. Assume the family $\mathcal{R}_\Delta$ is normal on some open neighborhood U of λ_0 and that $\{R_{G_n,v_n}(\lambda_0)\}_{n\geq 1}$ converges to -1 for a sequence $\{(G_n, v_n)\}_{n\geq 1}$ of rooted graphs from $\mathcal{G}_\Delta$. Then $\{R_{G_n,v_n}\}_{n\geq 1}$ converges to -1 locally uniformly on U.*

Proof. It follows from the conclusion of Section 5.4, i.e. $\mathcal{A}_\Delta = \overline{\mathcal{Z}_\Delta}$, that $Z_G(\lambda) \neq 0$ for all $\lambda \in U$ and $G \in \mathcal{G}_\Delta$. Suppose $\{R_{G_n,v_n}\}_{n\geq 1}$ does not converge to -1 locally uniformly on U. Then, after taking a subsequence if necessary, we may assume that $\{R_{G_n,v_n}\}_{n\geq 1}$ converges locally uniformly on U to a non-constant holomorphic function f. Clearly $f(\lambda_0) = -1$. Since zeros of holomorphic functions are isolated there exists $\varepsilon > 0$ so that $\overline{B(\lambda_0, \varepsilon)} \subset U$ and such that

$$\delta := \inf_{\lambda\in\partial B(\lambda_0,\varepsilon)} |f(\lambda) + 1| > 0.$$

Let n be sufficiently large so that $|R_{G_n,v_n} - f| < \delta$ uniformly on $\overline{B(\lambda_0, \varepsilon)}$. Then

$$|(R_{G_n,v_n}(\lambda) + 1) - (f(\lambda) + 1)| < \delta < |f(\lambda) + 1| + |R_{G_n,v_n}(\lambda) + 1|$$

for all $\lambda \in \partial B(\lambda_0, \varepsilon)$. By Rouché's theorem there exists $\lambda_1 \in B(\lambda_0, \varepsilon)$ for which $R_{G_n,v_n}(\lambda_1) = -1$. By Lemma 5.2.1 it follows λ_1 is a zero of the independence polynomial Z_G for some graph G of maximum degree at most Δ, which is a contradiction as we assumed $\lambda_0 \in \mathbb{C} \setminus \mathcal{A}_\Delta = \mathbb{C} \setminus \overline{\mathcal{Z}_\Delta}$. □

Lemma 5.5.3. *Let $\lambda_0 \in \mathbb{C} \setminus \mathcal{A}_\Delta$. Assume the family $\mathcal{R}_\Delta$ is normal on some open neighborhood U of λ_0 and that $\{R_{G_n,v_n}(\lambda_0)\}_{n\geq 1}$ converges to $\mu \leq -\frac{1}{4}$ for a sequence of rooted graphs $\{(G_n, v_n)\}_{n\geq 1}$ in $\mathcal{G}^1_\Delta$. Then $\{R_{G_n,v_n}\}_{n\geq 1}$ converges to μ locally uniformly on U.*

Proof. If this is not the case then, as in the previous lemma, we may assume that $\{R_{G_n,v_n}\}_{n\geq 1}$ converges locally uniformly to a non-constant holomorphic function f with $f(\lambda_0) = \mu$. By Rouché's theorem we can find $\lambda_1 \in U$ and n sufficiently large so that $R_{G_n,v_n}(\lambda_1) = \mu$, by the same argument as in the previous lemma.

Consider the family of rooted graphs $\{(\widetilde{G}_k, w_k)\}$ obtained by implementing (G_n, v_n) in every vertex of the rooted paths (P_k, w_k), where P_k is the path with k

vertices and w_k is one of its extreme vertices. Since v_n has degree 1, the graph $\widetilde{G}_k$ has maximum degree at most Δ. Hence by Lemma 5.2.6 we have

$$R_{\widetilde{G}_k,w_k}(\lambda) = f^k_{R_{G_n,v_n}(\lambda)}(0).$$

By Lemma 5.3.4 the family $\{R_{P_k,w_k}\} = \{\lambda \mapsto f^k_\lambda(0)\}$ is non-normal at $\lambda = \mu$, and therefore the family $\{R_{\widetilde{G}_k,w_k}\}$ is non-normal at λ_1, contradicting the fact that the family $\mathcal{R}_\Delta$ is normal on U. □

Lemma 5.5.4. *Assume there is a* $\lambda_0 \in \mathcal{D}_\Delta$ *with* $\lambda_0 \notin \mathcal{A}_\Delta$*. Denote* U *for an open neighborhood of* λ_0 *on which the family* $\mathcal{R}_\Delta$ *is normal. Assume furthermore that* $\{R_{G_n,v_n}(\lambda_0)\}_{n\geq 1}$ *converges to* $\mu \in \mathbb{R}$ *for a sequence of rooted graphs* $\{(G_n, v_n)\}_{n\geq 1}$ *in* $\mathcal{G}^{\Delta-1}_\Delta$*. Then* $\{R_{G_n,v_n}\}_{n\geq 1}$ *converges to* μ *locally uniformly on* U*.*

Proof. If $\mu = -1$ the result follows by Lemma 5.5.2. We may therefore assume that $\mu \neq -1$. Recall that we denote $f_\lambda(z) = \lambda/(1 + z)$. We will show for each $\mu \in \mathbb{R}$ there exists $\mu_1, \mu_2, \mu_3 \leq -\frac{1}{4}$ so that

$$f_{\mu_m} \circ \cdots \circ f_{\mu_1}(\mu) = -1,$$

for some $m \leq 3$. We distinguish between different cases

1. $\mu \geq -3/4$. Take $\mu_1 = -1 - \mu \leq -1/4$, one can check $f_{\mu_1}(\mu) = -1$.
2. $\mu < -1$. Take $\mu_1 = -1/4$ and $\mu_2 = 1 - f_{\mu_1}(\mu)$, then $f_{\mu_1}(\mu) > 0 > -3/4$ and so $\mu_2 \leq -1/4$. One can check $f_{\mu_2} \circ f_{\mu_1}(\mu) = -1$.
3. $-1 < \mu < -3/4$. Take $\mu_1 = \mu_2 = -1/4$ and $\mu_3 = 1 - f_{\mu_2}(f_{\mu_1}(\mu))$, then $f_{\mu_1}(\mu) < -1$ so we see $\mu_3 \leq -1/4$. One can check $f_{\mu_3} \circ f_{\mu_2} \circ f_{\mu_1}(\mu) = -1$.

We may assume that $\{R_{G_n,v_n}\}_{n\geq 1}$ converges locally uniformly on U to a holomorphic function f with $f(\lambda_0) = \mu$. We want to show f is constant on U. Since the set $\{R_{G,v}(\lambda_0) : (G, v) \in \mathcal{G}^1_\Delta\}$ is dense in $\hat{\mathbb{C}}$ by assumption, we can choose sequences of rooted graphs $\{(G^i_n, v^i_n)\}_{n\geq 1}$ in $\mathcal{G}^1_\Delta$ so that $\{R_{G^i_n,v^i_n}(\lambda_0)\}_{n\geq 1}$ converges to μ_i for each $i = 1, \ldots, m$. By Lemma 5.5.3 every sequence $\{R_{G^i_n,v^i_n}\}_{n\geq 1}$ converges locally uniformly on U to the constant function μ_i for each i.

Consider for each $n \geq 1$, the rooted graph $(\widetilde{G}_n, v^m_n)$ obtained by implementing the rooted graphs (G_n, v_n), $(G^1_n, v^1_n), \ldots, (G^m_n, v^m_n)$ on the vertices of the path P_{m+1} of length m. Note that $\widetilde{G}_n$ has maximum degree at most Δ.

It follows from Lemma 5.2.6 that

$$R_{\widetilde{G}_n,v^m_n}(\lambda) = f_{R_{G^m_n,v^m_n}(\lambda)} \circ \cdots \circ f_{R_{G^1_n,v^1_n}(\lambda)} \circ R_{G_n,v_n}(\lambda).$$

By our choice of the μ_i the sequence of ratios $\{R_{\widetilde{G}_n,v_n^m}(\lambda_0)\}_{n\geq 1}$ converges to $f_{\mu_m}\circ\cdots\circ f_{\mu_1}(\mu) = -1$. Hence by Lemma 5.5.2 the sequence of ratios $\{R_{\widetilde{G}_n,v_n^m}\}_{n\geq 1}$ converges locally uniformly to the constant function -1. Furthermore the sequence of ratios $\{R_{\widetilde{G}_n,v_n^m}\}_{n\geq 1}$ converges to the function $F := f_{\mu_m}\circ\cdots\circ f_{\mu_1}\circ f$. As $f_{\mu_m}\circ\cdots\circ f_{\mu_1}(z)$ is a non-constant holomorphic function and $F=-1$ on U, it follows that f is constant on U, as desired. $\square$

We are now ready to prove Theorem 5.5.1.

Proof of Theorem 5.5.1. Assume for the purpose of a contradiction that there exists $\lambda_0 \in \mathcal{D}^1_\Delta$ with $\lambda_0 \notin \mathcal{A}_\Delta$. We note that by Lemma 5.2.9 we know $\lambda_0 \neq 0$. Throughout the proof denote U for an open neighborhood of λ_0 on which the family $\mathcal{R}_\Delta$ is normal; we may assume $0 \notin U$ by taking U small enough. Assume first that λ_0 is not purely imaginary. Consider the real number $c = \frac{-|\lambda_0|^2}{2\mathrm{Re}\,\lambda_0}$ and notice that

$$\frac{\lambda_0^2}{\lambda_0 + c} = 2\mathrm{Re}\,\lambda_0 \in \mathbb{R}.$$

Choose two sequences of rooted graphs $\{G_n, v_n\}_{n\geq 1}$, $\{(H_n, w_n)\}_{n\geq 1}$ in $\mathcal{G}^1_\Delta$ so that $\{R_{G_n,v_n}(\lambda_0)\}_{n\geq 1}$ and $\{R_{H_n,w_n}(\lambda_0)\}_{n\geq 1}$ converge to respectively 1 and c. By Lemma 5.5.4 we must have that these sequence of ratios converge locally uniformly on U to the respective constants 1 and c.

Consider the sequence of graphs $\widetilde{G}_{n\geq 1}$ constructed by merging v_n and w_n and by then connecting this vertex to a vertex $\widetilde{v}_n$.

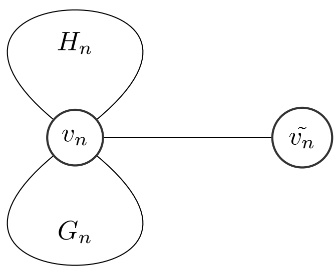


Figure 5.3: The rooted graph $\widetilde{G}_n$ in the proof of Theorem 5.5.1

It follows from Lemma 5.2.6 and Lemma 5.2.8 for all $\lambda \in U$ that

$$R_{\widetilde{G}_n,\widetilde{v}_n}(\lambda) = \frac{\lambda}{1 + \lambda^{-1} R_{G_n,v_n}(\lambda)\, R_{H_n,w_n}(\lambda)},$$

where we use $0 \notin U$. Hence the sequence of holomorphic functions $\{R_{\widetilde{G}_n,\widetilde{v}_n}\}_{n\geq 1}$ converges locally uniformly on U to the function $f(\lambda) = \frac{\lambda^2}{\lambda+c}$ as $n \to \infty$. Note that f is not a constant function, and that $f(\lambda_0) \in \mathbb{R}$, contradicting Lemma 5.5.4. This contradiction completes the proof for λ_0 not purely imaginary.

Assume instead that λ_0 is purely imaginary and let $(G, v) \in \mathcal{G}^1_\Delta$ so that $R_{G,v}(\lambda_0)$ is not purely imaginary. For $c \in \mathbb{R}$ to be determined later choose again two sequences of rooted graphs $\{G_n, v_n\}_{n\geq 1}$, $\{(H_n, w_n)\}_{n\geq 1}$ in $\mathcal{G}^1_\Delta$ such that sequences $\{R_{G_n,v_n}(\lambda_0)\}_{n\geq 1}$ and $\{R_{H_n,w_n}(\lambda_0)\}_{n\geq 1}$ converge to 1 and c respectively. Define for each $n \geq 1$, $(\tilde{G_n}, \widetilde{v_n})$ as above and let (K_n, v_n) be the rooted graph obtained from the disjoint union of $(\tilde{G_n}, \widetilde{v_n})$ and (G, v) by identifying the vertex $\widetilde{v}_n$ with v. It follows from Lemma 5.2.6 and Lemma 5.2.8 for $\lambda \in U$ that

$$R_{K_n,v_n}(\lambda) = \frac{R_{G,v}(\lambda)}{1 + \lambda^{-1} R_{G_n,v_n}(\lambda)\, R_{H_n,w_n}(\lambda)},$$

where we use $0 \notin U$. Thus in order to follow the same argument as before we require $c \in \mathbb{R}$ for which

$$\frac{\lambda_0 \cdot R_{G,v}(\lambda_0)}{\lambda_0 + c} \in \mathbb{R}.$$

It is clear that such real number c exists, hence the identical argument leads to the desired contradiction. □

We will now show the other inclusion $\mathcal{A}_\Delta \subseteq \overline{D_\Delta}$ also holds for all $\Delta \geq 3$. We first show the inclusion holds for non-real parameters $\lambda \in \mathcal{A}_\Delta$.

Theorem 5.5.5. *Let $\Delta \geq 3$ and suppose that the family $\mathcal{R}_\Delta$ is not normal in any neighborhood of $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. Then there exists λ_1 arbitrarily close to λ_0 for which the set $\{R_{G,v}(\lambda_1) : (G, v) \in \mathcal{G}^1_\Delta\}$ is dense in $\hat{\mathbb{C}}$.*

Proof. Because $\overline{\mathcal{Z}_\Delta} = \mathcal{A}_\Delta$ there exists λ_2 arbitrarily close to λ_0 for which there is a graph G of maximum degree at most Δ such that $Z_G(\lambda_2) = 0$. We claim that we can assume $\lambda_2 \notin \mathbb{R}$. This is clear if $\lambda_0 \notin \mathbb{R}$. Moreover, if $\lambda_0 \in \mathbb{R}$ then λ_0 is a strictly positive real number. Because $Z_G(x) > 0$ for any positive real number x, it follows that λ_2 is necessarily not real as long as it is sufficiently close to λ_0.

By Lemma 5.2.3 there is a rooted tree $(T, v) \in \mathcal{G}^1_\Delta$ such that $R_{T,v}(\lambda_2) = -1$. Since the rational function $\lambda \mapsto R_{T,v}(\lambda)$ is non-constant, it is an open map. The image of a neighborhood of λ_2 therefore contains a small open real interval around -1. Recall that Lemma 5.3.2 states that for $\mu \in (-\infty, -1/4)$ the map $f_\mu : z \mapsto \mu/(1 + z)$ is conjugate to a rotation $w \mapsto e^{i\theta} \cdot w$. Furthermore, by comparing tr^2 of both maps, it is not hard to see that those parameters μ for which f_μ is conjugate to an irrational rotation lie dense in $(-\infty, -1/4)$. Therefore we can choose a $\lambda_1 \in \mathbb{C} \setminus \mathbb{R}$ arbitrarily close to λ_2 such that for $\mu := R_{T,v}(\lambda_1)$

the map f_μ is conjugate to an irrational rotation. From now on μ is fixed to be this value.

Let p, q be the two fixed points of the transformation f_μ. In Section 5.3.1 we explained that $\hat{\mathbb{C}} \setminus \{p, q\}$ is foliated by generalized circles invariant under f_μ, and on which f_μ acts conjugate to an irrational rotation. We denote the generalized circle through z by C_z, and write C_q and C_p for $\{q\}$ and $\{p\}$ respectively. The map $z \mapsto C_z$ is continuous as a map from $\hat{\mathbb{C}}$ to the space $\{K \subseteq \hat{\mathbb{C}} : K \text{ compact}\}$ equipped with the Hausdorff metric.

Our goal is to show that $\mathcal{R}^1_\Delta(\lambda_1)$ is dense in $\hat{\mathbb{C}}$. We first claim that if $w \in \mathcal{R}^1_\Delta(\lambda_1)$, then $\mathcal{R}^2_\Delta(\lambda_1) \cap C_w$ is dense in C_w.

To prove the claim, let $(H, u) \in \mathcal{G}^1_\Delta$ be a rooted graph such that $R_{H,u}(\lambda_1) = w$. Let $\tilde{G}_n$ as follows be obtained from the path P_{n+1} on $n+1$ vertices, labeled v_0 up to v_n, by implementing (H, u) at v_0 and the rooted tree (T, v) at the remaining n vertices of P_{n+1}, see Figure 5.4. Now by Lemma 5.2.6 we have

$$R_{\tilde{G}_n, v_n}(\lambda_1) = f_\mu^n(R_{H,u}(\lambda_1)) = f_\mu^n(w).$$

Observe that for each $n \geq 1$ we have $(\tilde{G}_n, v_n) \in \mathcal{G}^2_\Delta$. Because f_μ acts conjugately to an irrational rotation on C_w it follows that $\mathcal{R}^2_\Delta(\lambda_1) \cap C_w$ is dense in C_w.

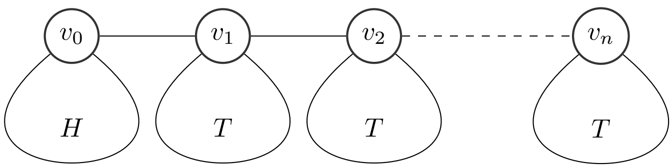


Figure 5.4: The graph $(\tilde{G}_n, v_n)$ in the proof of the claim

Because $\mu \in \mathcal{R}^1_\Delta(\lambda_1)$ and $C_\mu = \hat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ it follows from the claim that $\mathcal{R}^2_\Delta(\lambda_1) \cap \hat{\mathbb{R}}$ is dense in $\hat{\mathbb{R}}$. Observe that $f_{\lambda_1}(\hat{\mathbb{R}}) = \lambda_1 \cdot \hat{\mathbb{R}}$. So by attaching a vertex at the root with an edge, we obtain that $\mathcal{R}^1_\Delta(\lambda_1) \cap \lambda_1 \cdot \hat{\mathbb{R}}$ is dense in $\lambda_1 \cdot \hat{\mathbb{R}}$.

The set

$$U = \{z \in \hat{\mathbb{C}} : C_z \text{ intersects } \lambda_1 \cdot \hat{\mathbb{R}} \text{ transversely}\}$$

is an open set in $\hat{\mathbb{C}}$, see Figure 5.5. Because $\lambda_1 \notin \mathbb{R}$ we see that $C_{-1} = \hat{\mathbb{R}}$ intersects $\lambda_1 \cdot \hat{\mathbb{R}}$ transversely, and thus $-1 \in U$. The set U is contained in $\cup_{w \in \lambda_1 \cdot \hat{\mathbb{R}}} C_w$. Because $\mathcal{R}^1_\Delta(\lambda_1) \cap \lambda_1 \cdot \hat{\mathbb{R}}$ is dense in $\lambda_1 \cdot \hat{\mathbb{R}}$, it follows that $\cup_{w \in \mathcal{R}^1_\Delta(\lambda_1)} C_w$ is dense in U. From the claim we proved earlier, it follows $\mathcal{R}^2_\Delta(\lambda_1)$ is dense in U. Attaching a vertex to the root of a tree in $\mathcal{G}^2_\Delta$ with ratio r yields a rooted tree in $\mathcal{G}^1_\Delta$ with ratio $f_{\lambda_1}(r)$, and thus $\mathcal{R}^1_\Delta(\lambda_1)$ is dense in the neighborhood $U_\infty := f_{\lambda_1}(U)$ of ∞.

For two rooted trees $(T_1, v_1) \in \mathcal{G}^1_\Delta$ and $(T_2, v_2) \in \mathcal{G}^2_\Delta$ with ratios r_1 and r_2 respectively we can define the rooted tree $(T_3, v_1) \in \mathcal{G}^2_\Delta$ by adding an edge between the roots of T_1 and T_2 and considering v_1 the root of the obtained tree. By Lemma 5.2.6 the ratio of (T_3, v_1) is given by

$$F(r_1, r_2) := f_{r_1}(r_2) = \frac{r_1}{1 + r_2},$$

under the assumption $(r_1, r_2) \notin \{(0, -1), (\infty, \infty)\}$. It is not hard to see that

$$F(U_\infty \times \hat{\mathbb{R}} \setminus \{(0, -1), (\infty, \infty)\}) = \hat{\mathbb{C}}.$$

Because $\mathcal{R}^1_\Delta(\lambda_1)$ is dense in U_∞ and $\mathcal{R}^2_\Delta(\lambda_1)$ is dense in $\hat{\mathbb{R}}$ it follows that $\mathcal{R}^2_\Delta(\lambda_1)$ is dense in $\hat{\mathbb{C}}$. We finally conclude that $\mathcal{R}^1_\Delta(\lambda_1)$ is dense in $f_{\lambda_1}(\hat{\mathbb{C}}) = \hat{\mathbb{C}}$. $\square$

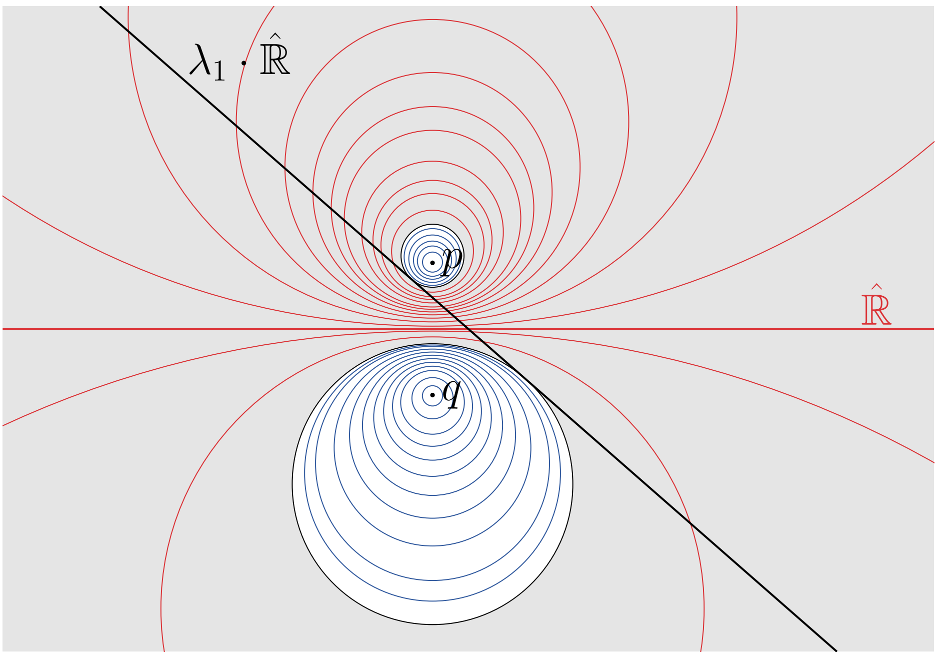


Figure 5.5: The generalized circles $\lambda_1 \cdot \hat{\mathbb{R}}$ and $\hat{\mathbb{R}}$ intersect in the points 0 and ∞. A region around 0 is drawn. The open set U is shaded in gray. Examples of generalized circles C_w that intersect $\lambda_1 \cdot \hat{\mathbb{R}}$ transversely are drawn in red, while examples of circles that do not intersect $\lambda_1 \cdot \hat{\mathbb{R}}$ are drawn in blue.

We can now finally prove the inclusion $\mathcal{A}_\Delta \subseteq \overline{\mathcal{D}_\Delta}$ building on Proposition 6 of [BGGŠ20] to deal with the real parameters $\lambda \in \mathcal{A}_\Delta$.

Theorem 5.5.6. *Let $\Delta \geq 3$. Then the activity locus is contained in the density locus, i.e. $\mathcal{A}_\Delta \subseteq \overline{\mathcal{D}_\Delta}$.*

Proof. Let $\lambda_0 \in \mathcal{A}_\Delta$. If $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, then $\lambda_0 \in \overline{\mathcal{D}_\Delta}$ follows from Theorem 5.5.5.

We know $\overline{\mathcal{Z}_\Delta} = \mathcal{A}_\Delta$. By Remark 6 we know that

$$\overline{\mathcal{Z}_\Delta} \cap \mathbb{R}_{\leq 0} = \mathbb{R}_{\leq 0} \setminus \mathrm{int}(\Lambda_\Delta)$$

for all $\Delta \geq 3$. Proposition 6 of [BGGŠ20] implies that $\mathbb{C} \setminus (\mathbb{R} \cup \Lambda_\Delta) \subseteq \overline{\mathcal{D}_\Delta}$ for $\Delta \geq 3$. Hence it follows that $\mathbb{R}_{\leq 0} \setminus \mathrm{int}(\Lambda_\Delta) \subseteq \overline{\mathcal{D}_\Delta}$, which completes the proof. $\square$

5.6 Density implies $\#\mathcal{P}$-hardness

In this section we will show that the density-locus is contained in the $\#\mathcal{P}$-locus. To prove our result we will need to to show 'exponential' density for ratios of a specific family of trees: we need to get ε-close to a given point $P \in \mathbb{Q}[i]$ with ratios of trees of size at most $O(\log(1/\varepsilon) + \mathrm{size}(P))$. Here $\mathrm{size}(P)$ denotes the sum of the bit sizes of the real and imaginary part of P. Moreover, we denote for rational $\varepsilon > 0$ by $\mathrm{size}(\varepsilon, P)$ the sum of the the bit size of ε and $\mathrm{size}(P)$.

Let $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$. Then the Möbius transformation f_{λ_0} is loxodromic (cf. Section 5.3.1) and hence has a repelling fixed point, which we denote by z_0.

Let

$$A := \{z \in \mathbb{C} : 2\pi/3 - 0.01 < \arg z < 2\pi/3, 1/17 < |z| < 1/16\}. \tag{5.7}$$

Let $\mathcal{T} = \{(G_1, v_1), \ldots, (G_m, v_m), (\overline{G}_1, \overline{v}_1), \ldots, (\overline{G}_M, \overline{v}_M\}$ be a family of rooted trees and U an open disk containing z_0. The pair $(\mathcal{T}, U)$ is called a *fast implementer for* λ_0 if the ratios $\mu_i := R_{G_i,v_i}$ and $\chi_i := R_{\overline{G}_i,\overline{v}_i}$ are such that the maps $g_i := f_{\mu_i} \circ f_{\chi_i}$ are loxodromic and satisfy

1. the attracting fixed point z_i of g_i lies in U for all i,
2. $\overline{U} \subseteq \cup_{i=1}^{M} g_i(U)$,
3. $g_i'(z) \in A$ for all i and all $z \in \overline{U}$,

and the disk U is such that

1. $\overline{U} \subset f_{\lambda_0}(U)$,
2. $\overline{U}$ does not contain the attracting fixed point of f_{λ_0},
3. U has three rational points on its boundary.

We have the following results concerning fast implementers.

Lemma 5.6.1. *Let $\Delta \geq 3$ be an integer. Let $\lambda_0 \in \mathcal{D}_\Delta$. Then there exists a fast implementer $(\mathcal{T}, U)$ for λ_0.*

Lemma 5.6.2. *Let $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$ and assume that there exists a fast implementer $(\{(G_1, v_1), \ldots, (G_m, v_m), (\overline{G}_1, \overline{v_1}), \ldots, (\overline{G}_M, \overline{v}_M)\}, U)$ for λ_0. Then, given $P \in \mathbb{C}$ and $\varepsilon > 0$ there exists an algorithm that yields a sequence of ratios*

$$w_1, \ldots, w_K \in \{\lambda_0\} \cup \bigcup_{i=1}^{M} \{\mu_i := R_{G_i, v_i}, \chi_i := R_{\overline{G}_i, \overline{v_i}}\}$$

such that

- $|(f_{w_K} \circ \cdots \circ f_{w_1})(0) - P| < \varepsilon$,
- $w_K = \lambda_0$ *and*
- $K = \mathcal{O}(\max(\log(1/\varepsilon), \log(|P|/\varepsilon)))$.

If $\lambda_0 \in \mathbb{Q}[i]$ and the input parameters P, ε are also in $\mathbb{Q}[i]$ then the algorithm runs in poly$(\mathrm{size}(P, \varepsilon))$ *time.*

We provide proofs for these lemmas in the next subsection, but first we collect some consequences.

Corollary 5.6.3. *Let $\Delta \geq 3$ be an integer. The set $\mathcal{D}_\Delta$ is an open set.*

Proof. Let $\lambda_0 \in \mathcal{D}_\Delta$. Let $(\mathcal{T}, U)$ be fast implementer, as guaranteed to exist by Lemma 5.6.1. For λ nearby λ_0 we still have that the repelling fixed point of f_λ is contained in U, its attracting fixed point does not lie in $\overline{U}$ and $\overline{U} \subset f_\lambda(U)$. In other words $(\mathcal{T}, U)$ is a fast implementer for λ. Therefore applying the algorithm of Lemma 5.6.2 to λ we obtain that the collection of values $\{R_{G,v}(\lambda) \mid (G, v) \in \mathcal{G}^1_\Delta\}$ is dense in $\hat{\mathbb{C}}$ and hence $\lambda \in \mathcal{D}_\Delta$. $\square$

For our next corollary we first need a result about the set

$$\mathcal{E}_\Delta := \{\lambda \in \mathbb{Q}[i] \mid Z_G(\lambda) = 0 \text{ for some } G \in \mathcal{G}_\Delta\}.$$

Lemma 5.6.4. *Let $\Delta \geq 3$ be an integer. Then the collection $\mathcal{E}_\Delta$ is contained in the set*

$$\Big\{(a + ib)^{-1} \mid a, b \in \mathbb{Z},\ 0 < \sqrt{a^2 + b^2} \leq \frac{\Delta^\Delta}{(\Delta - 1)^{\Delta - 1}}\Big\}.$$

Proof. Let $\lambda \in \mathcal{E}_\Delta$. Then there exists a graph $G = (V, E)$ such that $1/\lambda$ is a root of $P(z) := z^{|V|} Z_G(1/z)$. Now P is a monic polynomial and therefore $1/\lambda \in \mathbb{Z}[i]$ (since $\mathbb{Z}[i]$ is integrally closed by Gauss's lemma). We also know that $|1/\lambda| \leq \frac{\Delta^\Delta}{(\Delta-1)^{\Delta-1}}$ by Lemma 5.2.9. This proves the lemma. $\square$

Corollary 5.6.5. *Let $\Delta \geq 3$ be an integer. Let $\lambda_0 \in (\mathcal{D}_\Delta \cap \mathbb{Q}[i]) \setminus \mathcal{E}_\Delta$. Then given $P \in \mathbb{Q}[i]$ and rational $\varepsilon > 0$ there exists an algorithm that generates a rooted tree (T, v) such that $|R_{T,v}(\lambda_0) - P| \leq \varepsilon$ and $Z^{out}_{T,v}(\lambda_0) \neq 0$, and outputs $Z^{in}_{T,v}(\lambda_0)$ and $Z^{out}_{T,v}(\lambda_0)$ in time bounded by $poly(\mathrm{size}(\varepsilon, P))$.*

Proof. We first perform a brute force, but constant time, computation to obtain a fast implementer $(\{(G_1, v_1), \ldots, (G_m, v_m), (\overline{G}_1, \overline{v_1}), \ldots, (\overline{G}_M, \overline{v}_M\}, U)$ for λ_0. Denote for $i = 1, \ldots, M$, $\mu_i := R_{G_i, v_i}$ and $\chi_i := R_{\overline{G}_i, \overline{v_i}}$ and $g_i := f_{\mu_i} \circ f_{\chi_i}$.

The algorithm of Lemma 5.6.2 applied to P returns in time $poly(\mathrm{size}(\varepsilon, P))$ a sequence of ratios $\omega_1 \ldots, \omega_K \in \{\lambda_0\} \cup \bigcup_{i=1}^M \{\mu_i, \chi_i\}$ that, by Lemma 5.2.6, correspond to the implementation of the trees G_i and $\overline{G}_i$ on a path with $K = \mathcal{O}(\max(\log(1/\varepsilon), \log(|P|/\varepsilon))$ vertices. The resulting rooted tree (T, v) has maximum degree at most Δ and root degree 1 and satisfies $|R_{T,v}(\lambda_0) - P| \leq \varepsilon$. Denote the rooted tree corresponding to the sequence $\omega_1, \ldots \omega_i$ by (T_i, u_i). Then (T_{i+1}, u_{i+1}) is obtained from (T_i, u_i) by adding the edge $\{v_{i+1}, u_i\}$ to T_i and gluing a rooted tree $(H, v) \in \{K_1, (G_j, v_j), (\overline{G_j}, v_j) \mid j = 1, \ldots, M\}$ to u_{i+1} (here K_1 denotes a single vertex.) We then have

$$\left(Z^{\mathrm{in}}_{T_{i+1}, u_{i+1}}(\lambda_0), Z^{\mathrm{out}}_{T_{i+1}, u_{i+1}}(\lambda_0)\right) = \left(Z^{\mathrm{in}}_{H,v}(\lambda_0) Z^{\mathrm{out}}_{T_i, u_i}(\lambda_0), Z^{\mathrm{out}}_{H,v}(\lambda_0) Z_{T_i}(\lambda_0)\right). \tag{5.8}$$

Note that (5.8) describes a simple recurrence to compute $Z^{\mathrm{in}}_{T,v}(\lambda_0)$ and $Z^{\mathrm{out}}_{T,v}(\lambda_0)$ in time linear in the number of vertices of T.

Finally, we remark that $Z^{\mathrm{out}}_{T,v}(\lambda_0) \neq 0$ since $\lambda_0 \notin \mathcal{E}_\Delta$ by assumption. □

We can now prove the desired inclusion of the density-locus in the #$\mathcal{P}$-locus.

Theorem 5.6.6. *For any integer $\Delta \geq 3$ the density-locus $\overline{\mathcal{D}_\Delta}$ is contained in the #$\mathcal{P}$-locus $\overline{\#\mathcal{P}_\Delta}$.*

Proof. We will show that for any $\lambda_0 \in (\mathcal{D}_\Delta \cap \mathbb{Q}[i]) \setminus \mathcal{E}_\Delta$ the computational problem #Hard-CoreNorm(λ_0, Δ) is #$\mathcal{P}$-hard. Since $\mathcal{D}^1_\Delta$ is an open set and $\mathcal{E}_\Delta$ is finite, this implies the theorem.

This in fact follows directly from the work of [BGGŠ20]. Let us briefly indicate why. In [BGGŠ20, Section 6] the authors show that a polynomial time algorithm for #Hard-CoreNorm(λ_0, Δ) combined with the statement of Corollary 5.6.5 for λ_0 yields an algorithm that on input of a graph G of maximum degree at most Δ exactly computes $Z_G(1)$, the number of independent sets of G, in polynomial time in the number of vertices of G. (The algorithm is obtained by cleverly utilizing Corollary 5.6.5 for suitable choices of P and gluing combinations of the obtained trees to G and applying the assumed algorithm for #Hard-CoreNorm(λ_0, Δ) to the resulting graph.) Since determining $Z_G(1)$ is a known #$\mathcal{P}$-complete problem, this implies that #Hard-CoreNorm(λ_0, Δ) is #$\mathcal{P}$-hard. □

We note that our result does not allow us to say anything about the complexity of #Hard-CoreNorm(λ_0, Δ) for $\lambda_0 \in \partial(\mathcal{D}_\Delta) \cap \mathbb{Q}[i]$. For example, for $\lambda_0 \in \partial(\mathcal{D}_\Delta) \cap \mathbb{Q}_{\leq 0}$ it follows from [BGGŠ20] that the problem #Hard-CoreNorm(λ, Δ) is #$\mathcal{P}$-hard. For $\lambda \in \mathbb{Q}$ such that $\lambda \geq \lambda_c(\Delta) := \frac{(\Delta-1)^{\Delta-1)}}{(\Delta-2)^\Delta}$ we know from [BGGŠ20] that $\lambda \in \partial(\mathcal{D}_\Delta)$, while the complexity of #Hard-CoreNorm$(\lambda_c(\Delta), \Delta)$ is unknown. For $\lambda > \lambda_c(\Delta)$ the problem Hard-CoreNorm(λ, Δ) is only known to be $\mathcal{NP}$-hard [SS14], and unlikely to be #$\mathcal{P}$-hard cf.[BGGŠ20].

5.6.1 Proofs of Lemma 5.6.1 and Lemma 5.6.2

The next lemma directly implies Lemma 5.6.1.

Lemma 5.6.7. *Given $z_0 \in \mathbb{C} \setminus \{-1, 0\}$, a dense subset D of $\mathbb{C}^*$ and a non-empty open subset A of the unit disk $\mathbb{D}$ then there exists a finite set of tuples $\{(\mu_i, \chi_i)\}_{i=1}^M \subset D \times D$ and an arbitrarily small open disk $U \subseteq \mathbb{C}$ containing z_0 such that the maps $g_i := f_{\mu_i} \circ f_{\chi_i}$ are loxodromic Möbius transformations and*

1. *the attracting fixed point z_i of g_i lies in U for all i,*
2. *$\overline{U} \subseteq \cup_{i=1}^M g_i(U)$,*
3. *$g_i'(z) \in A$ for all i and all $z \in \overline{U}$.*

Proof. We denote $g_{\mu,\chi} = f_\mu \circ f_\chi$ throughout this proof. Note $g_{\mu,\chi}$ is a Möbius transformation for $\mu, \chi \neq 0$. Without loss of generality assume that A is bounded away from 0. Take $\alpha \in A$ such that $\alpha \neq \frac{z_0}{z_0+1}$. Note that $\chi_0 = \frac{(z_0+1)^2\alpha}{z_0-(z_0+1)\alpha}$ and $\mu_0 = \frac{z_0(z_0+\chi_0+1)}{z_0+1}$ are nonzero and well-defined as $z_0 \neq -1, 0$ and $\alpha \neq \frac{z_0}{z_0+1}, 0$. Furthermore we have $g_{\mu_0,\chi_0}(z_0) = z_0$ and $g'_{\mu_0,\chi_0}(z_0) = \alpha$.

Define $F : (\mathbb{C}^*)^2 \times \mathbb{C} \to \hat{\mathbb{C}}$ as $F(\mu, \chi, z) = g_{\mu,\chi}(z) - z$. Now as $\frac{\partial F}{\partial z}(\mu_0, \chi_0, z_0) = \alpha - 1 \neq 0$, the implicit function theorem gives an open neighborhood W of (μ_0, χ_0) and a holomorphic function $h : W \to \mathbb{C}$ with $h(\mu_0, \chi_0) = z_0$ and $F(\mu, \chi, h(\mu, \chi)) = 0$ for all $(\mu, \chi) \in W$. As h is a non-constant holomorphic map, it is an open map and so $h(W)$ is an open neighborhood of z_0.

Let $B \subseteq A$ be an open set in $\mathbb{C}$ with $\alpha \in B$ and $\overline{B} \subseteq A$. Denote $H(\mu, \chi, z) = \frac{\partial g_{\mu,\chi}}{\partial z}(z) = \frac{\mu\chi}{(1+z+\chi)^2}$, note that H is continuous as a function on $\mathbb{C}^3 \setminus \{(\mu, \chi, z) : \chi + z + 1 = 0\}$. It follows there is an open neighborhood C of z_0 such that we have $H(\mu_0, \xi_0, z) \in B$ for all $z \in C$. We have $\{(\mu_0, \chi_0)\} \times \overline{C} \subseteq H^{-1}(\overline{B}) \subseteq H^{-1}(A)$. As $H^{-1}(A)$ is an open subset of $\mathbb{C}^3 \setminus \{(\mu, \chi, z) : \chi + z + 1 = 0\}$ containing the set $\{(\mu_0, \chi_0)\} \times \overline{C}$, by a compactness argument it follows that $H^{-1}(A)$ contains a set of the form $L \times C$, for some open neighborhood L of the point (μ_0, χ_0). Hence the set $Y := L \cap W \cap h^{-1}(C)$ is an open neighborhood of (μ_0, χ_0) and so $h(Y)$ is an open neighborhood of z_0.

Take $U \subset h(Y)$ an open disk containing z_0, such that $\overline{U} \subset h(Y)$. Note that we can take U arbitrarily small. By construction, we have for all $(\mu, \chi) \in Y$ that $g'_{\mu,\chi}(z) \in A$ for all $z \in \overline{U}$. Furthermore, we have $F(\mu, \chi, h(\mu, \chi)) = 0$, so $h(\mu, \chi)$ is the attracting fixed point of $g_{\mu,\chi}$. Note $D \times D$ is dense in $h^{-1}(U)$, hence the fixed points of $g_{\mu,\chi}$ for $(\mu, \chi) \in h^{-1}(U) \cap (D \times D)$ lie dense in U. There is a uniform lower bound on the diameters of the disks $g_{\mu,\chi}(U)$ for $(\mu, \chi) \in h^{-1}(U)$, because $g'_{\mu,\chi}(z) \in A$ for all $z \in U$ and A is bounded away from 0. Therefore

$$\left\{g_{\mu,\chi}(U) : (\mu, \chi) \in h^{-1}(U) \cap (D \times D)\right\}$$

is an open cover of $\overline{U}$. There is a finite set of tuples $\{(\mu_i, \chi_i)\}_{i=1}^{M} \subseteq h^{-1}(U) \cap (D \times D)$ such that $\overline{U} \subseteq \cup_{i=1}^{M} g_{\mu_i,\chi_i}(U)$ by compactness of $\overline{U}$. We thus found the desired set of tuples in $D \times D$ and the open disk U containing z_0. $\square$

We next focus on proving Lemma 5.6.2. To this end let $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$ and let $(\{(G_1, v_1), \ldots, (G_m, v_m), (\overline{G}_1, \overline{v_1}), \ldots, (\overline{G}_M, \overline{v}_M)\}, U)$ be a fast implementer for λ_0. We fix these throughout this section. We denote the repelling fixed point of f_{λ_0} by z_0 and we denote for $i = 1, \ldots, M$, $\mu_i := R_{G_i,v_i}$, $\chi_i := R_{\overline{G_i},\overline{v_i}}$ and $g_i := f_{\mu_i} \circ f_{\chi_i}$. We distinguish between the case that P is close to the attracting fixed point of f_{λ_0} and the case that it is not. In the first case the algorithm is much simpler.

Let a be the attracting fixed point of f_{λ_0}. Because $f_{\lambda_0}(\infty) = 0$ we observe that ∞ is not a fixed point and thus $a \in \mathbb{C}$. Suppose that $|P - a| \leq \varepsilon/2$. Choose $\delta > 0$ for which there is a constant $\eta < 1$ such that $|f'_{\lambda_0}(z)| < \eta$ for all $z \in B(a, \delta)$. The point 0 is not a fixed point of f_{λ_0} because $f_{\lambda_0}(0) = \lambda_0 \neq 0$ and thus $f^n_{\lambda_0}(0)$ converges to a as $n \to \infty$. It follows that there is a constant N_0 such that $f^{N_0}_{\lambda_0}(0) \in B(a, \delta)$. Note that the value of N_0 does not depend on the input parameters. Now let $N_\varepsilon = \max\{\lceil \log_\eta(\frac{\varepsilon}{2\delta})\rceil, 0\} + 1$. Then for any $w \in B(a, \delta)$ we have

$$|f^{N_\varepsilon}_{\lambda_0}(w) - a| < \eta^{N_\varepsilon}|w - a| < \varepsilon/2$$

and thus for $K = N_0 + N_\varepsilon$ we have

$$|f^K_{\lambda_0}(0) - P| \leq |f^{N_\varepsilon}_{\lambda_0}(f^N_{\lambda_0}(0)) - a| + |a - P| < \varepsilon/2 + \varepsilon/2 < \varepsilon.$$

Because $K = \mathcal{O}(\log(1/\varepsilon))$ this describes the algorithm when $|P - a| \leq \varepsilon/2$.

The case that $|P-a| > \varepsilon/2$ is more involved and we will describe the algorithm as a sequence of simpler subroutines. Just as in Lemma 5.6.7 let z_i denote the attracting fixed point of g_i. We will show first show that, given a parameter Q that is at most distance ε away from some z_i, we only have to apply g_i to the starting value 0 an $\mathcal{O}(\log(1/\varepsilon))$ number of times to get ε close to Q. Morally, this should be true because after a fixed number of steps the orbit of 0 converges exponentially quickly to z_i and because z_i is close to Q the orbit should also get close to Q. The only way that this reasoning could be incorrect is if z_i and Q are almost ε apart and the orbit of 0 converges to z_i from the wrong direction. An example of this is given by the red orbit in Figure 5.6. This is the reason that we required $g_i'(z_i)$ to have an argument close to $2\pi/3$ in which case the above reasoning is correct as the green orbit in Figure 5.6 demonstrates. In the following proof most time is spent on making this precise.

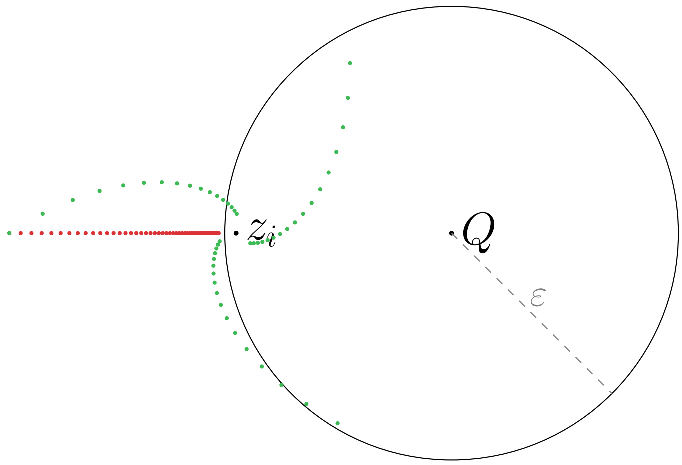


Figure 5.6: An example of two orbits with the same initial value converging to z_i under iteration of two different maps. For the red orbit the derivative at z_i is real. For the green orbit the derivative at z_i has the same magnitude, while its argument is a little less than $2\pi/3$.

Lemma 5.6.8. *There exists an algorithm that, given $\varepsilon > 0$, $Q \in \mathbb{C}$ and $i \in \{1, \dots, M\}$ such that $|Q - z_i| < \varepsilon$, yields an integer K such that $|g_i^K(0) - Q| < \varepsilon$, where $K = \mathcal{O}(\log(1/\varepsilon))$. If $\lambda_0 \in \mathbb{Q}[i]$ and the input parameters Q, ε lie in $\mathbb{Q}[i]$ then the algorithm runs in poly*$(\mathrm{size}(Q, \varepsilon))$ *time.*

Proof. Let δ be such that $B(z_i, \delta) \subseteq U$ and let $\varepsilon' = \min\{\varepsilon/2, \delta\}$. Note that $g_i(0) = \frac{\mu_i}{1+\chi_i} \neq 0$ and thus 0 is not a fixed point of g_i. Because z_i is the attracting fixed point of g_i we can find (in a similar way as described above) a positive integer $\tilde{K}$ that is $\mathcal{O}(\log(1/\varepsilon')) = \mathcal{O}(\log(1/\varepsilon))$ such that $|g_i^{\tilde{K}}(0) - z_i| < \varepsilon'$. If $|Q - z_i| \leq \varepsilon/2$ we are done because then

$$|g_i^{\tilde{K}}(0) - Q| \leq |g_i^{\tilde{K}}(0) - z_i| + |Q - z_i| < \varepsilon' + \varepsilon/2 \leq \varepsilon.$$

So from now on we assume that $|Q - z_i| > \varepsilon/2$. Define the following sector S of

$B(z_i, \varepsilon')$

$$S = \{z_i + \xi \cdot \left(\frac{Q - z_i}{|Q - z_i|}\right) : |\xi| < \varepsilon', -\pi/3 \leq \arg(\xi) \leq \pi/3\}.$$

We claim that $S \subseteq B(Q, \varepsilon)$. To show this note that

$$\left| Q - \left[z_i + \xi \cdot \left(\frac{Q - z_i}{|Q - z_i|} \right) \right] \right| = |Q - z_i| \cdot \left| 1 - \frac{\xi}{|Q - z_i|} \right| < \varepsilon \cdot \left| 1 - \frac{\xi}{|Q - z_i|} \right|.$$

If ξ is as in the definition of S the complex number $\xi/|Q - z_i|$ has its argument between $-\pi/3$ and $\pi/3$. Furthermore, because $|\xi| < \varepsilon/2$ and $|Q - z_i| > \varepsilon/2$, its norm is bounded above by 1. It follows that the norm of $1 - \xi/|Q - z_i|$ is at most 1. Indeed, because $|1 - re^{i\phi}|^2 = 1 + r^2 - 2r\cos(\phi)$, the statement $|1 - re^{i\phi}| \leq 1$ is equivalent to $r = 0$ or $r \leq 2\cos(\phi)$, and the latter is satisfied for all $0 \leq r \leq 1$ and $-\pi/3 \leq \phi \leq \pi/3$. The claim follows.

We claim that for $w \in B(z_i, \varepsilon')$ the intersection of $\{w, g_i(w), g_i^2(w), g_i^3(w)\}$ with S is not empty. Note that because $\varepsilon' \leq \delta$ we have that $B(z_i, \varepsilon') \subseteq U$ and thus $g'(w) \in A$ for every $w \in B(z_i, \varepsilon')$. It follows that applying g_i to w has the effect of rotating around z_i with an angle strictly between $2\pi/3 - 0.01$ and $2\pi/3$ and contracting towards z_i. Therefore applying g_i to w three times has the effect of rotating w a little less than a full circle around z_i, with steps that are strictly less than $2\pi/3$ radians. Because the internal angle of the sector S is $2\pi/3$ the orbit $w, g_i(w), g_i^2(w), g_i^3(w)$ cannot miss S.

To summarize the algorithm, define ε' and determine $\tilde{K}$ such that $g_i^{\tilde{K}}(0) \in B(z_i, \varepsilon')$. Then determine a $j \in \{0, 1, 2, 3\}$ such that $|g_i^{\tilde{K}+j}(0) - Q| < \varepsilon$. We have shown that there exists at least one such j. The output of the algorithm is $K = \tilde{K} + j$.

□

We shall now describe an algorithm that does the following. Given a disk D of radius ε inside U, it returns an index i, a disk $\tilde{D}$ of radius at least ε containing z_i and a sequence of indices $j_1, \ldots, j_K$ such that $(g_{j_1} \circ \cdots \circ g_{j_{K-1}})(\tilde{D}) \subseteq D$. To describe the computational complexity of this algorithm, we need a finite way to represent disks in the complex plane. A pleasant way for our purposes is to represent an open disk D by three distinct points P_1, P_2, P_3 on its boundary. This is an unambiguous way to represent a disk because three different points on a circle uniquely determine that circle. If $P_1, P_2, P_3 \in \mathbb{Q}[i]$ we say that the disk D is rational and that $\text{size}(D) = \text{size}(P_1) + \text{size}(P_2) + \text{size}(P_3)$.

Recall that a Möbius transformation maps generalized circles (circles and straight lines) to generalized circles. In what follows, we will apply Möbius transformations to disks in the complex plane. We shall make sure that the image of the disks involved is always again a disk in the complex plane and not the complement of a disk or a half-plane as it could in general be. Therefore, if D is a disk represented by P_1, P_2 and P_3 and g is one of the Möbius transformations, then $g(D)$ will be a disk represented by $g(P_1), g(P_2)$ and $g(P_3)$. Note that if D is rational and g has rational coefficients then $g(D)$ is a again rational. The Möbius transformations that we will apply come from a fixed finite set and thus there is a fixed constant C for which $\text{size}(g(D)) \leq C \cdot \text{size}(D)$.

Let us denote $P_j = x_j + iy_j$ for $j \in \{1, 2, 3\}$. The center $c_D = x + yi$ of the disk D is known as the circumcenter of the triangle with vertices P_1, P_2 and P_3. The coordinates of c_D can be calculated using the well known and easy to derive formulas

$$x = \frac{(x_1^2 + y_1^2)(y_2 - y_3) + (x_2^2 + y_2^2)(y_3 - y_1) + (x_3^2 + y_3^2)(y_1 - y_2)}{2(x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2))},$$
$$y = \frac{(x_1^2 + y_1^2)(x_3 - x_2) + (x_2^2 + y_2^2)(x_1 - x_3) + (x_3^2 + y_3^2)(x_2 - x_1)}{2(x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2))}.$$

We note that if D is rational, then c_D is rational and can be computed in time linear in $\text{size}(D)$. We can also decide whether a given point $Q \in \mathbb{Q}[i]$ lies in a given rational disk D in time linear in $\text{size}(Q)$ and $\text{size}(D)$.

We next need a lemma concerning a geometric construction involving disks.

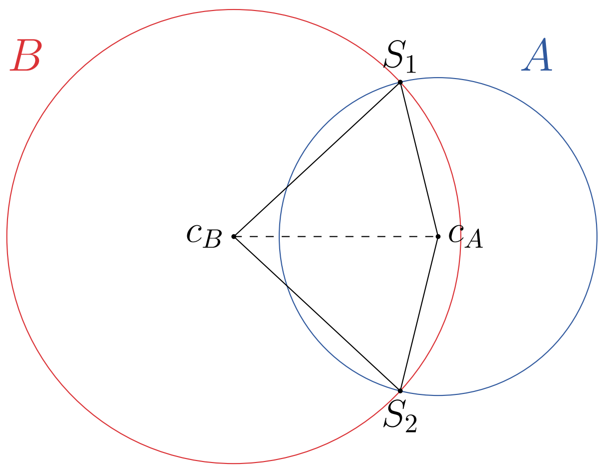


Figure 5.7: -

Lemma 5.6.9. *There exists an algorithm that, given two disks A, B in the complex plane for which the center of A is contained in B and B is not contained*

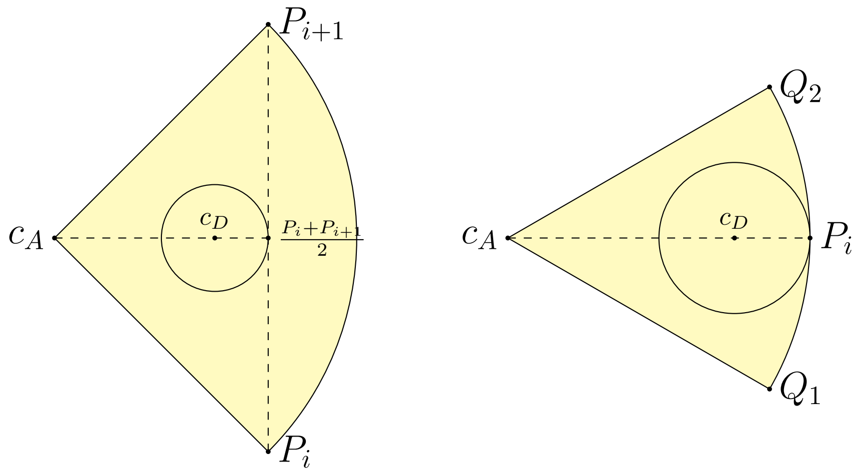


Figure 5.8: -

in A, returns a disk D contained in both A and B, such that the area of D is at least $1/128$ times that of A. Furthermore, if A and B are rational then D is rational and both the running time of the algorithm and $\mathrm{size}(D)$ *are bounded by a fixed constant times* $\mathrm{size}(A, B)$.

Proof. For a disk D we denote its center by c_D and its radius by r_D and recall that if D is rational then c_D is rational and can be computed efficiently. For two distinct points P, Q on the boundary of D we denote the closed counterclockwise arc from P to Q by $\mathrm{Arc}_D(P, Q)$ and we denote the sector given by the convex hull of $\mathrm{Arc}_D(P, Q)$ and c_D by $\mathrm{Sec}_D(P, Q)$. We note that the internal angle of both $\mathrm{Arc}_D(P, Q)$ and $\mathrm{Sec}_D(P, Q)$ is given by the arclength of $\mathrm{Arc}_D(P, Q)$ divided by r_D. We claim that either a sector of A whose internal angle is greater than $2\pi/3$ is contained in the closure of B or a sector of B whose internal angle is greater than $2\pi/3$ is contained in the closure of A.

If the boundaries of A and B either do not intersect or intersect in one point then A is contained in B and the claim is obvious. Otherwise let S_1, S_2 be the two intersection points such that $\mathrm{Arc}_A(S_1, S_2)$ is contained in B and thus $\mathrm{Arc}_B(S_2, S_1)$ is contained in A, see Figure 5.7. Consider the quadrilateral $\Box c_B S_1 c_A S_2$ and suppose towards contradiction that the internal angles at both c_A and c_B are at most $2\pi/3$, then the sum of the internal angles at S_1 and S_2 is at least $2\pi/3$ and since they are equal by symmetry the internal angle at S_1 is at least $\pi/3$. By then considering the triangle $\triangle c_B S_1 c_A$ it should follow that $|c_B - c_A| \geq |c_B - S_1| = r_B$, which contradicts the assumption that c_A is contained in B. We therefore find that either the angle $\angle S_2 c_B S_1$ or $\angle S_2 c_A S_1$ is at least $2\pi/3$. If the latter is the case then both $\mathrm{Arc}_A(S_1, S_2)$ and c_A are contained in the closure of B and thus the same is true for $\mathrm{Sec}_A(S_1, S_2)$. If the angle $\angle S_2 c_A S_1$ is less than $2\pi/3$, then $\angle S_2 c_B S_1$ is at least $2\pi/3$. It follows that $\angle c_A c_B S_1$ is the largest internal angle of the triangle $\triangle c_B S_1 c_A$ and thus $|c_A - c_B| < |c_A - S_1| = r_A$, from which it follows that c_B is contained in A. So in this case $\mathrm{Sec}_B(S_2, S_1)$ is contained in A.

For the algorithm we do not need to know whether a large sector of B is contained in A or vice versa. Assume for simplicity that a sector S of A with internal angle at least $2\pi/3$ is contained in B. Take S to be as large as possible. In the case that A is contained in B we let $S = A$. Let P_0 denote one of the given (rational) points on the boundary of A. Now for $i = 0, 1, 2$ inductively define P_{i+1} as P_i rotated around c_A with an angle of $\pi/2$. Calculating these points is computationally easy because $P_{i+1} = c_A + i(P_i - c_A)$. Now one of the following is guaranteed to be the case.

1. Two consecutive points P_i and P_{i+1} are contained in S.

2. There is a unique index i such that $P_i \in S$.

Determining which of the two cases is true is easy since checking membership of S is equivalent to checking membership of B. In the first case we note that $\mathrm{Sec}_A(P_i, P_{i+1})$ is contained in both the closures of A and B. Now let $R = (P_i + P_{i+1})/2$ and $c_D = (c_A + 3R)/4$ and let D be the disk with center c_D and the point R on the boundary, see Figure 5.8. It can be checked that D is now contained in $\mathrm{Sec}_A(P_i, P_{i+1})$ and its area is $1/32$ that of A.

In the second case note that the arc $\mathrm{Arc}_A(Q_1, Q_2)$ containing P_i such that the internal angle of both $\mathrm{Arc}_A(Q_1, P_i)$ and $\mathrm{Arc}_A(P_i, Q_2)$ is $\pi/6$ contained in S, otherwise, since the internal angle of S is at least $2\pi/3$, S has to contain two consecutive points P_i and P_{i+1}. Now let D be the disk with center $(c_A + 3P_i)/4$ containing P_i on its boundary, see Figure 5.8. It can be checked that D is contained in $\mathrm{Sec}_A(Q_1, Q_2)$ and its area is $1/16$ that of A.

The algorithm above is only guaranteed to successfully return a disk contained in both A and B if a large sector of A is contained in B. Therefore we have to run the algorithm described above (and let it fail if neither of the two described cases is true) and run the same algorithm with the roles of A and B reversed. If both instances of the algorithm return a disk, say D_1 and D_2, then at least one of them is contained in both A and B but the other one might not be. So in this case we have to run one final check to see which one of the two disks is indeed contained in both A and B (which is computationally easy). If they both are we can return either D_1 or D_2.

In conclusion we obtain a disk D contained in both A and B that is either at least $1/32$ of the area of A or $1/32$ of the area of B. Because the area of B is at least $1/4$ that of A (otherwise $r_B < r_A/2$ and $c_A \in B$ would imply $B \subseteq A$), we can conclude that the area of D is at least $(1/4) \cdot (1/32) = 1/128$ that of A.

$\square$

Lemma 5.6.10. *There exists an algorithm that, given a disk $D_1 \subseteq U$ with radius ε, returns an index $i \in \{1, \dots, M\}$, a sequence of indices $j_1, \dots, j_K \in \{1, \dots, M\}$ and a disk D_K such that $z_i \in D_K$, the radius of D_K is at least ε and $(g_{j_1} \circ \cdots \circ g_{j_{K-1}})(D_K) \subseteq D_1$. Furthermore, $K = \mathcal{O}(\log(1/\varepsilon))$. If $\lambda_0 \in \mathbb{Q}[i]$ and D_1 and ε are both rational then D_K is rational and the algorithm runs in poly$(\mathrm{size}(D_1, \varepsilon))$ time.*

Proof. For every index i let $U_i = g_i(U)$. Recall that we took U as a rational disk. Because the derivative of g_i is bounded on U, the image U_i is again a disk in the complex plane. If λ_0 is rational, the coefficients of g_i are rational and then U_i is also rational. The point z_i is fixed for g_i and contained in U, therefore, also contained in U_i. We describe a procedure to generate a sequence of disks $\{D_n\}_{n \geq 1}$, starting with the given disk D_1. The sequence is defined in such a way such that $D_n \subseteq U$ for all n, which is, by assumption, the case for D_1.

Suppose we have arrived at disk $D_n \subseteq U$. Check if there is any index $i \in \{1, \dots, M\}$ such that $z_i \in D_n$, if there is stop the procedure and let $K = n$. Otherwise, let m_n be the center of D_n and determine an index $j_n \in \{1, \dots, M\}$ such that $m_n \in U_{j_n}$. Such an index must exist because $m_n \in U$ and the disks $U_1, \dots, U_M$ cover U. Because the center of D_n lies in U_{j_n} but U_{j_n} is not contained in D_n (z_{j_n} does not lie in D_n) we can use Lemma 5.6.9 to generate a disk $\tilde{D}_n$ that is contained in both D_n and U_{j_n} whose area is at least $1/128$ times that of D_n and which can be assumed to be rational if D_n is. Now we define $D_{n+1} = g_{j_n}^{-1}(\tilde{D}_n)$. Because $\tilde{D}_n \subseteq U_{j_n}$ the disk D_{n+1} lies in U and because $\tilde{D}_n \subseteq D_n$ the disk $g_{j_n}(D_{n+1})$ lies in D_n. Furthermore, by the properties of the fast implementer, $g_{j_n}^{-1}$ is expanding the norm on U_{j_n} with a factor at least 16, the area of D_{n+1} is at least $16^2 \cdot (1/128) = 2$ times that of D_n. This means that the area of D_n grows exponentially with n and thus, because the area of U is fixed, the procedure will terminate after $K = \mathcal{O}(\log(1/\varepsilon))$ steps. Note that indeed $z_i \in D_K$ for some i, the radius of D_K is at least that of D_1 and $(g_{j_1} \circ \cdots \circ j_{K-1})(D_K) \subseteq D_1$. $\square$

Recall that we had defined a and z_0 to be the attracting and repelling fixed point of f_{λ_0} respectively. We have already described the algorithm in Lemma 5.6.2 when P is near a. What follows is the final lemma needed to describe the algorithm when P is not near a.

Lemma 5.6.11. *There exists a fixed positive constant c and an algorithm that, given $P \in \mathbb{C}$ and $\varepsilon > 0$ such that $|P - a| \geq \varepsilon/2$, yields a disk $D \subseteq U$ and a positive integer K with $f_{\lambda_0}^K(D) \subseteq B(P, \varepsilon)$, such that the radius of D is at least*

$c \cdot \min(\varepsilon, \varepsilon/|P|^2)$ and $K = \mathcal{O}(\log(1/\varepsilon))$. If both λ_0 and the input parameters are in $\mathbb{Q}[i]$ then D is also rational and both size(D) *and the running time of the algorithm is polynomial in* size(P, ε).

Proof. Let V be a compact neighborhood of a such that $|f'_{\lambda_0}(z)| < \xi < 1$ for some constant ξ for all $z \in V$. We first claim that there is an integer N such that the complement of $f^N_{\lambda_0}(U)$ is contained in V. To show this let $U_n = f^n_{\lambda_0}(U)$. Recall that we assumed that $\overline{U} = \overline{U_0} \subseteq U_1$ and thus inductively $\overline{U_n} \subseteq U_{n+1}$. Under iteration of $f^{-1}_{\lambda_0}$ every initial point that is not a converges to z_0 and thus eventually lands in U. Therefore,

$$\bigcup_{n=1}^{\infty} U_n = \widehat{\mathbb{C}} \setminus \{a\}.$$

For n large enough the point ∞ is contained in U_n and from then on the sequence $(U_n)^c$ consists of nested disks, containing a, whose radii must necessarily converge to 0, proving that there is an N such that $(U_N)^c$ is contained in V. Note that N does not depend on the input parameters. Let D_0 be the interior of $(U_N)^c$, this is a rational disk whose size also does not depend on the input, and let $D_i = f^i_{\lambda_0}(D_0)$ for $i \in \{1, 2, 3\}$. Let $\delta > 0$ be a constant smaller than the minimum distance between points on the boundary of D_i and D_{i-1}. From now on we will assume that $\varepsilon < \delta$. Finally let $h = f^{-(N+3)}_{\lambda_0}$.

If P lies outside D_2, then let $\tilde{D}$ be the disk of radius ε represented by $P + \varepsilon, P + i\varepsilon$ and $P - \varepsilon$. Note that $\tilde{D}$ lies outside D_3 and thus $D = h(\tilde{D}) \subset U$. Because the derivative of a Möbius transformation of the form $z \mapsto (az + b)/(cz + d)$ is $z \mapsto \frac{ad-bc}{(cz+d)^2}$ there is a constant c_1 such that the radius of D is at least $c_1 \cdot \min(\varepsilon, \varepsilon/|P|^2)$. In this case D and $K = N + 3$ are the output of the algorithm.

If P lies inside D_2 we determine N_0 such that for $P_{N_0} := f^{-N_0}_{\lambda}(P)$ we have $P_{N_0} \in D_1$ and $P_{N_0} \notin D_2$. Because $|P - a| \geq \varepsilon/2$ and $D_1 \subset V$ we find that $N_0 = \mathcal{O}(\log(1/\varepsilon))$. Let $\tilde{D}$ be the disk of radius ε represented by $P_{N_0} + \varepsilon, P_{N_0} + i\varepsilon$ and $P_{N_0} - \varepsilon$. Note that again $\tilde{D}$ lies outside D_3 and thus $D = h(\tilde{D}) \subseteq U$. Furthermore, because $\tilde{D} \subset D_0 \subset V$ and f_{λ_0} is attracting on V it follows that $f^{N_0}_{\lambda_0}(\tilde{D}) \subseteq B(P, \varepsilon)$. Finally, if we let c_2 be the minimum of $|h'(z)|$ for $z \in D_0$, we find that the radius of D is at least $c_2 \cdot \varepsilon$. So in this case the output is the disk D together with $K = N_0 + N + 3$.

□

We are now ready to complete the proof of Lemma 5.6.2.

Proof of Lemma 5.6.2. Recall we had defined a to be the attracting fixed point of f_{λ_0} and that we already described the algorithm in the case that $|P - a| < \varepsilon/2$, therefore we assume that $|P - a| \geq \varepsilon/2$.

It follows from Lemma 5.6.11 that we can generate a disk $D_1 \subseteq U$ of radius $r = \mathcal{O}(\min\{\varepsilon, \varepsilon/|P|^2\})$ and whose size is polynomial in size(P,ε) together with a positive integer K_1 that is $\mathcal{O}(\log(1/\varepsilon))$ such that $f_{\lambda_0}^{K_1}(D_1)$ is contained in $B(P,\varepsilon)$. From Lemma 5.6.10 it follows that we can find an index $i \in \{1,\ldots,M\}$, a sequence of indices $j_1,\ldots,j_{K_2}$ and a disk D_2 such that $z_i \in D_2$, the radius of D_2 is at least r, its size is polynomial in size(r,D_1), which is again polynomial in size(ε,P), and such that

$$(g_{j_1} \circ \cdots \circ g_{j_{K_2}})(D_2) \subseteq D_1.$$

Furthermore $K_2 = \mathcal{O}(\log(1/r)) = \mathcal{O}(\max(\log(1/\varepsilon), \log(|P|/\varepsilon)))$. Finally let Q be the center of D_2 and note that size(Q) is polynomial in size(D_2). Then, because $|Q - z_i| < r$, it follows from Lemma 5.6.8 that we can generate a K_3 such that $g_i^{K_3}(0) \in D_2$, where $K_3 = \mathcal{O}(\log(1/r)) = \mathcal{O}(\max(\log(1/\varepsilon), \log(|P|/\varepsilon)))$. Concluding, we find that

$$(f_{\lambda_0}^{K_1} \circ g_{j_1} \circ \cdots \circ g_{j_{K_2}} \circ \circ g_i^{K_3})(0) \in B(P,\varepsilon).$$

Furthermore, adding the running times of the individual algorithms, we find that the final algorithm runs in *poly*(size(P,ε)) time. □

5.7 Activity and zeros for Cayley trees

For fixed $\Delta \geq 2$ notions such as the activity-locus and the zero sets can be considered for subcollections of $\mathcal{G}_\Delta$. Particularly interesting subcollections from a physical viewpoint are given by subgraphs of regular lattices. However, it is notoriously difficult to rigorously deduce the properties for such collections.

A much simpler collection of rooted graphs in $\mathcal{G}_\Delta$ is given by finite Cayley trees, and we will describe the properties of those in this section. The trees are uniquely determined by the conditions that every leaf has fixed distance n to the root vertex v, and every non-leaf has down-degree $d = \Delta - 1$. The root vertex therefore has degree d, while every other non-leaf has degree Δ. We denote the Cayley tree of depth n by T_n, and its root by v_n.

As an immediate consequence of Lemma 5.2.4 we obtain

$$R_{T_n,v_n}(\lambda) = f_{\lambda,d}(R_{T_{n-1},v_{n-1}}(\lambda)),$$

where $f_{\lambda,d}(z) = \lambda/(1+z)^d$. Since the ratio of a single point is given by $\lambda = f_{\lambda,d}(0)$, it follows inductively that

$$R_{T_n,v_n}(\lambda) = f_{\lambda,d}^{n+1}(0).$$

In fact, since

$$Z_{T_n,v_n}^{out}(\lambda) = \left(Z_{T_{n-1},v_{n-1}}(\lambda)\right)^d$$

and

$$Z^{in}_{T_n,v_n}(\lambda) = \lambda \left(Z^{out}_{T_{n-1},v_{n-1}}(\lambda) \right)^d$$

it follows by induction on n that for $\lambda \in \mathbb{C} \setminus \{0\}$ the polynomials $Z^{in}_{T_n,v_n}(\lambda)$ and $Z^{out}_{T_n,v_n}(\lambda)$ cannot vanish simultaneously. For $\lambda \in \mathbb{C} \setminus \{0\}$ it follows that $Z_{T_n}(\lambda) = 0$ if and only if $R_{T_n,v_n}(\lambda) = f^n_{\lambda,d}(0) = -1$.

In what follows we deduce properties of the zeros of $Z_{T_n}(\lambda)$ and the activity-locus of $f^n_{\lambda,d}(0)$ from well known results in the field of holomorphic dynamical systems, occasionally adapting the proofs to our setting. We refer the reader to the standard references [Mil06, CG93].

Observe that $f_{\lambda,d}(-1) = \infty$ and $f_{\lambda,d}(\infty) = 0$, and $f'_{\lambda,d}(-1) = f'_{\lambda,d}(\infty) = 0$. Thus if $f^n_{\lambda_0,d}(0) = -1$ for some λ_0 and n, then 0 is an attracting periodic cycle of period $n+2$. This cycle is stable under perturbations of λ_0, i.e. the attracting cycle persists and in fact varies holomorphically for nearby parameters $\lambda \sim \lambda_0$ by the implicit function theorem.

Recall that every attracting cycle attracts the orbit of a critical point. But $f_{\lambda,d}$ has only one critical orbit: the orbit of -1, ∞ and 0. Thus whenever $f_{\lambda,d}$ has an attracting cycle, the orbit $f^n_{\lambda,d}(0)$ converges to the attracting cycle. In fact, the convergence is uniform in a neighborhood of the parameter λ_0, hence λ_0 cannot lie in the activity-locus. The situation is therefore fundamentally different from the setting where the whole family of graphs $\mathcal{G}_\Delta$ is considered, as there λ_0 must lie in the activity-locus. The following however does hold:

Proposition 5.7.1. *The activity-locus of the family $\{T_n, v_n\}$ equals the collection of accumulation points of the zeros of the collection $\{Z_{T_n}\}$.*

Proof. If there are no zeros in a neighborhood of some λ_0, then the family $\{R_{T_n,v_n}\}$ avoids the values $0, -1$ and ∞, and is normal by Montel's Theorem.

Suppose on the other hand that λ_0 is an accumulation point of zeros $\lambda_1, \lambda_2, \ldots$. Let $n_1, n_2, \ldots$ be the minimal integers for which $f^{n_i}(\lambda_i) = -1$. Since for fixed n the zeros of Z_{T_n} are isolated, we may assume that $n_i \to \infty$ and (n_i) is strictly increasing.

When for a parameter λ the rational function f has an attracting periodic cycle, the unique critical orbit $\{f^n(0)\}_{n \geq 1}$ must converge to this periodic orbit. Since attracting periodic cycles are stable, i.e. they persist under small changes of the parameter λ, such parameters lie in a passivity component, i.e., a maximal connected open subset where the family $\{\lambda \mapsto f^n(0)\}$ is normal. The passivity component agrees exactly with the connected component where the attracting periodic cycle persists, since by [MnSS83] the parameter must become active when the periodic cycle becomes neutral.

Thus, λ_i lies in a connected component of the open set where the family $\{\lambda \mapsto f^n(0)\}$ is normal, and associated to this component is the unique period

$n_i + 2$. Since the sequence $\{n_i\}_{i \geq 1}$ is strictly increasing, the parameters λ_i must all lie in distinct connected components. It follows that the limit parameter λ_0 cannot lie in an open component where the family is normal, and therefore λ_0 must be an active parameter. □

The activity-locus for Cayley trees of down degree $d = 2, 3$ and 4 is illustrated in Figure 5.9. Each of these diagrams represents the spherical derivative of the function $\lambda \mapsto f^{120}_{\lambda,d}(0)$.

It follows from Proposition 5.7.1 above, plus the observation that zeros do not lie in the activity-locus for the Cayley tree setting, that the Cayley tree activity-locus never has interior. On the other hand, it follows from the universality of the Mandelbrot set, a result due to McMullen [McM00], that the activity-locus must contain a quasiconformal image of the Mandelbrot set of some degree. Therefore by Shishikura's result [Shi98] the Hausdorff dimension of the activity-locus is equal to 2 for any $d \geq 2$.

It follows from the proof of Proposition 5.7.1 that the complement of the activity-locus consists of infinitely many connected components. Each λ for which $f_{\lambda,d}$ has an attracting periodic cycle lies in such a passive component, a so-called *hyperbolic* component associated to the period k. Whether all connected components are hyperbolic is an open question, which is conjectured to hold for quadratic polynomials.

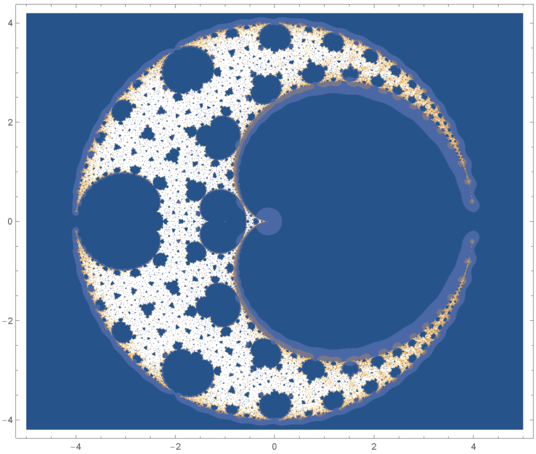
For any down-degree d there are two special connected components that can easily be identified. The unbounded component is always a hyperbolic component of period 2. For degree 2 this is the complement of the closed disk of radius 4. For down-degrees 3 and 4 the boundary has respectively 1 and 2 singular points.

For each down-degree $d = \Delta - 1$ there is a single hyperbolic component of period 1, which contains of course the parameter $\lambda = 0$ and equals the cardioid Λ_Δ.

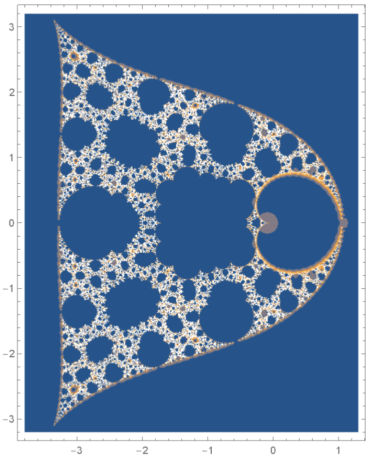
Apart from these two special hyperbolic components, any hyperbolic component contains a unique zero of the partition function, i.e. a unique parameter λ for which $f^n_{\lambda,d}(\lambda) = -1$ for some $n \in \mathbb{N}$. Since $f^2_{\lambda,d}(-1) = 0$ and $f^{-2}_{\lambda,d}(0) = \{-1\}$, these are exactly the parameters $\lambda \in \mathbb{C} \setminus \{0\}$ for which the unique critical orbit $\{f^i_{\lambda,d}(0)\}_{i \in \mathbb{N}}$ is periodic, i.e. for which $f_{\lambda,d}$ is super-attracting.

For the family $p_c(z) = z^2 + c$ the fact that every hyperbolic component of the Mandelbrot set contains a unique super-attracting parameter is a consequence of the Multiplier Theorem, due to Douady-Hubbard and Sullivan, see [Dou83].

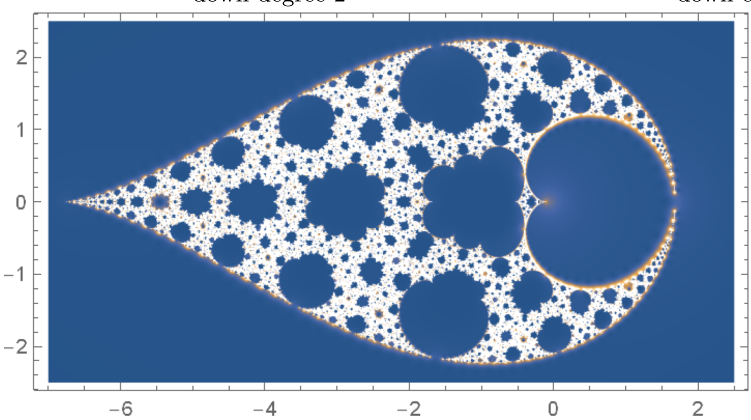
Let us recall this fundamental result in the field. Let H be a hyperbolic component of the Mandelbrot set, say of period n. For every parameter $c \in H$ there exist an attracting periodic cycle $a_0, a_1, \ldots, a_n = a_0$. The multiplier $h(c) = (f^n_c)'(a_0)$ is independent from the choice of a_n, and gives a holomorphic map from H to the unit disk.



down-degree 2



down-degree 4



down-degree 3

Figure 5.9: The activity-locus of Cayley trees for down-degrees $2, 3$ and 4. For each pixel the spherical derivative of the occupation ratio is computed for the Cayley tree of depth 120. Pixels for which this derivative is sufficiently large are depicted in white, suggesting that the corresponding parameter λ lies approximately on the activity locus.

Theorem 5.7.2 (Multiplier Theorem). *For every hyperbolic component H the map $c \mapsto h(c)$ gives a conformal bijection from H to the unit disk.*

The proof of the Multiplier Theorem can be found in [CG93], Theorem 2.1 on page 133, and can be applied almost directly to our setting. We present a high-level discussion to outline how the proof adapts to our setting.

Let H be a hyperbolic component of period at least 3. One easily sees that $h(\lambda)$, the multiplier of the attracting periodic cycle of $f_{\lambda,d}$ is a holomorphic and surjective map from the hyperbolic component H to the unit disk $\mathbb{D}$, hence is a branched covering. Let Z be the set of super-attracting parameters in H, i.e. $Z = h^{-1}(0)$. If it can be shown that $h : H \setminus Z \to \mathbb{D} \setminus \{0\}$ is a covering map, it follows from the Riemann-Hurwitz Theorem that $\mathrm{card}(Z) = 1$.

Thus, it needs to be shown that h is locally invertible near parameters $\lambda_0 \in H \setminus Z$. Write $\eta_0 = h(\lambda_0) \in \mathbb{D}$, and consider values of η near η_0. Following the proof of the Multiplier Theorem one applies quasiconformal surgery by modifying the ellipse field near the attracting periodic cycle in order to obtain attracting periodic cycles with multipliers η. Using the dynamics the ellipse field can be extended to the full basin of the attracting cycle, obtaining an invariant ellipse field that is invariant under the map $f_{\lambda_0,d}$. The ellipse field corresponds to a Beltrami coefficient, which can be extended to the entire Riemann sphere by setting it equal to 0 outside of the basin of attraction. The Measurable Riemann Mapping Theorem gives a holomorphic family of quasiconformal maps φ_η, with φ_{η_0} the identity. By composing with suitable Möbius transformations we can guarantee that the points $-1, \infty$ and 0 are fixed under all φ_η.

Since each ellipse field is invariant under $f_{\lambda_0,d}$, conjugating $f_{\lambda,d}$ by φ_η yields a holomorphic family of self-maps of the Riemann sphere $g_{\eta,d}$, which are necessarily rational functions of the same degree d. In fact, since each φ_η fixes the points $-1, \infty$ and 0, each rational function $f_{\eta,d}$ must send -1 to ∞ and ∞ to 0, each with local degree d. It follows that the rational function $g_{\eta,d}$ must be of the form

$$g_{\eta,d}(z) = \frac{\lambda(\eta)}{(1+z)^d}.$$

It follows that $\lambda(\eta)$ gives a local inverse of the multiplier function h, completing this step of the proof. This step guarantees that there exists a unique zero in each hyperbolic component of period at least 3, which equals the super-attracting *center* of the hyperbolic component. The proof of the Multiplier Theorem in our setting can be concluded by analyzing the local degree near the center. We have therefore obtained the following description of the zeros of the Cayley trees:

Corollary 5.7.3. *Every $\lambda \in \mathbb{C}$ for which $Z_{T_n}(\lambda) = 0$ for some $n \in \mathbb{N}$ is the center of a hyperbolic component of the complement of the activity locus. On the other hand: apart from the two special hyperbolic components, the unbounded* *component and the component containing* 0*, for each center* λ *of a hyperbolic component there exists an* $n \in \mathbb{N}$ *for which* $Z_{T_n}(\lambda) = 0$*. As a consequence zero-parameters are isolated.*

CHAPTER

SIX

ON BOUNDEDNESS OF ZEROS OF THE INDEPENDENCE POLYNOMIAL OF TORI

6.1 Introduction

6.1.1 Main results

The independence polynomial of a finite simple graph $G = (V, E)$ is defined by

$$Z_G(\lambda) = \sum_I \lambda^{|I|},$$

where the summation runs over all *independent* subsets $I \subseteq V$. Besides its relevance in graph theory, the independence polynomial is studied extensively in the statistical physics literature, where it appears as the partition function of the hard-core model, and in theoretical computer science, where one is primarily interested in the (non-)existence of efficient algorithms for the computation or approximation of Z_G.

From the physical viewpoint it is particularly interesting to consider sequences of graphs G_n that converge to a regular lattice. We will consider the integer lattice, and focus on sequences of d-dimensional tori converging to $\mathbb{Z}^d$ for $d \geq 2$, i.e. tori whose minimal cycle lengths tend to infinity. Write $\mathbb{Z}_n$ for $\mathbb{Z}/n\mathbb{Z}$. A d-dimensional torus with side lengths $\ell_1, \ldots, \ell_d$ is the Cartesian product $\mathbb{Z}_{\ell_1} \times \cdots \times \mathbb{Z}_{\ell_d}$. For technical reasons explained below we only consider tori for which all side lengths are even and call those tori *even*. The main result of this paper is the following:

Main Theorem. *Let $\mathcal{F}$ be a family of even d-dimensional tori. If $\mathcal{F}$ is balanced, then the zeros of the independence polynomials $\{Z_{\mathcal{T}} : \mathcal{T} \in \mathcal{F}\}$ are uniformly bounded. If $\mathcal{F}$ is highly unbalanced, then the zeros are not uniformly bounded.*

Here we say that a family of d-dimensional tori $\mathcal{F}$ is balanced if there exists a $C > 0$ such that for all $\mathcal{T} \in \mathcal{F}$ we have that $\ell_d \leq \exp(C \cdot \ell_1)$, where $\ell_1 \leq \cdots \leq \ell_d$ denote the side lengths of $\mathcal{T}$. On the other hand we say that the family is *highly unbalanced* if there is no uniform constant $C > 0$ such that $\ell_d \leq \exp(C \cdot (\ell_1 \cdots \ell_{d-1})^3)$ for all $\mathcal{T} \in \mathcal{F}$. Intuitively, for highly unbalanced tori the longest side length dominates to the extend that they behave more like a one-dimensional object than a d-dimensional object. In the statistical physics literature various other properties of models on unbalanced volumes are studied, see for example [BI92] and [Bea09].

We remark that a family that is not balanced is not necessarily highly unbalanced, hence the addition of the adjective highly. It is not clear to the authors that either estimate is sharp, and it would be interesting if one or both of the results could be sharpened in order to obtain a conclusive statement for all families of even tori.

6.1.2 Motivation from statistical physics

Understanding the location and distribution of zeros of the independence polynomial plays a prominent role in statistical physics. For a sequence of graphs $G_n = (V_n, E_n)$ and for $\lambda \geq 0$ the free energy per site is defined by

$$\rho(\lambda) := \lim_{n\to\infty} \frac{\log Z_{G_n}(\lambda)}{|V_n|},$$

whenever this limit exists. It was shown by Yang and Lee [YL52] that the free energy per site exists for induced subgraphs G_n of $\mathbb{Z}^d$ that converge in the sense of van Hove, i.e. sequences of graphs for which

$$\frac{|\partial V_n|}{|V_n|} \to 0.$$

It turns out that the limit also exists and agrees for many other sequences of graphs, including cylinders, i.e. products of paths and cycles, and tori, i.e. products of cycles, as long as the length of the shortest path or cycle diverges. This motivates the notion of sequences of tori converging to $\mathbb{Z}^d$. However, we emphasize that the convergence above occurs specifically for real parameters $\lambda \geq 0$.

Independence polynomials have positive coefficients and their zeros therefore never lie on the positive real axis. The location of the complex zeros is however closely related to the behavior of the normalized limit $\rho(\lambda)$. Let G_n be again a sequence of graphs converging to $\mathbb{Z}^d$ in the sense discussed above. Yang and Lee [YL52] showed that if there exists a zero-free neighborhood of the parameter $\lambda_0 \geq 0$, then ρ is analytic near λ_0. In the other direction, knowledge of the distribution of the zeros can be used to characterize the regularity near phase transitions: parameters λ_0 where the free energy is not analytic.

As remarked above, the limit behavior on the positive real axis of the normalized logarithm of the independence polynomials is to a large extent independent from the sequence of graphs. The motivating question for this work is to what extent this remains true for the distribution and location of the complex zeros of the independence polynomial. In particular we focus on the question whether the zero sets are uniformly bounded or not.

It was shown by Helmuth, Perkins and the last author [HPR19] that for sequences of *padded* induced subgraphs of $\mathbb{Z}^d$ the zeros are uniformly bounded. We recall that an induced subgraph of $\mathbb{Z}^d$ is said to be padded if all of its boundary points share the same parity.

Our main result shows that the boundedness of zeros for tori requires additional assumptions on the relative dimensions of the tori.

6.1.3 Inspiration from holomorphic dynamics

When studying the independence polynomial on sequences of graphs that are in some sense recursively defined, one can often express the independence polynomials in terms of iterates of a rational function or map. A clear example is provided by the sequence of Cayley trees of down-degree d. For this sequence of graphs the zero sets can be described using iterates of the rational function

$$f_{\lambda,d}(z) = \frac{\lambda}{(1+z)^d}.$$

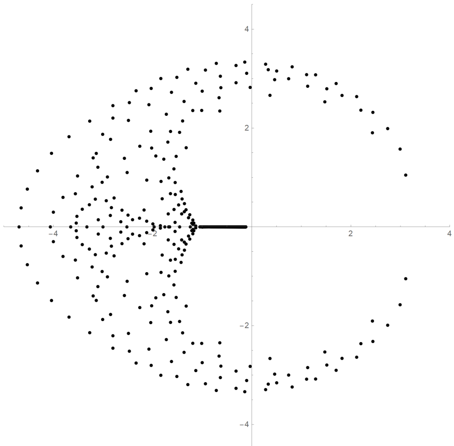
This iterative description is exploited in current work of Rivera-Letelier and Sombra [RL19] to characterize the order of the unique phase transition.

Let us describe the relationship between the independence polynomial and iterates of $f_{\lambda,d}$ in some more detail. To be more precise, for a rooted graph (G, v) we say the *occupation ratio* $R_{G,v}$ is given by the rational function

$$R_{G,v}(\lambda) = \frac{Z^{\text{in}}_{G,v}(\lambda)}{Z^{\text{out}}_{G,v}(\lambda)},$$

where $Z^{\text{in}}_{G,v}$ and $Z^{\text{out}}_{G,v}$ respectively sum only over independent subsets that do or do not contain the marked vertex v. The occupation ratio of a Cayley tree T_n of down-degree d and depth n and top vertex p is given by

$$R_{T_n,p}(\lambda) = f^{\circ n}_{\lambda,d}(\lambda) = f^{\circ(n+1)}_{\lambda,d}(0).$$



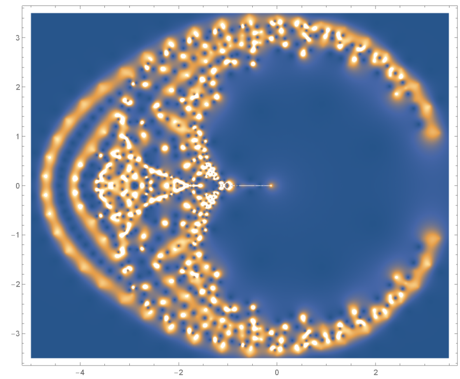


Figure 6.1: The figure on the left depicts the zeros of the independence polynomial the subgraph $B_{22}(0)$ in $\mathbb{Z}^2$. The figure on the right depicts spherical derivative of the occupation ratio of $(B_{22}(0), 0)$.

Since $Z_G = Z^{\text{in}}_{G,v} + Z^{\text{out}}_{G,v}$, the parameters for which $Z_{G,v} = 0$ are essentially equal to the parameters for which $R_{G,v} = -1$; see Chapter 5 for a more detailed discussion. As a consequence the *accumulation set* of the zeros equals the non-normality locus of the family of rational functions $R_{T_n,p}(\lambda)$, i.e. the parameters λ for which the maps $\lambda \mapsto f^{\circ n}_{\lambda,d}(0)$ does not form a normal family. Thus, zeros accumulate at parameters where the spherical derivative of $\lambda \mapsto f^{\circ n}_{\lambda,d}(0)$ is unbounded.

The relationship between the zeros of partition functions on the one hand and non-normality of a related family of rational functions or maps on the other holds in much greater generality; see for example [PR20, BHR23]. The extent to which the one-to-one correspondence also holds for specific sequences of graphs that are not recursively defined is yet to be determined.

Figure 1 contains two illustrations focusing instead on the graph $B_{22}(0)$, the induced subgraph of $\mathbb{Z}^2$ that contains all vertices of distance at most 22 to the origin. The figure on the left depicts the zeros of the independence polynomial, while the figure on the right depicts the spherical derivative of the occupation ratio, hinting at the non-normality locus of the sequence $\{B_n(0)\}_{n\in\mathbb{N}}$. We are grateful for Raymond van Venetië for writing the code used to compute the relevant independence polynomial. The two illustrations suggest a clear relationship between the zero sets and the non-normality locus. Moreover, both illustrations suggest boundedness of the zero sets; known by [HPR19] since $B_{22}(0)$ is a padded subgraph of $\mathbb{Z}^2$. Figure 6.2 depicts the zeros, on the left, and the spherical derivative of the occupation ratio, on the right, for the 18×18 torus. The resemblance

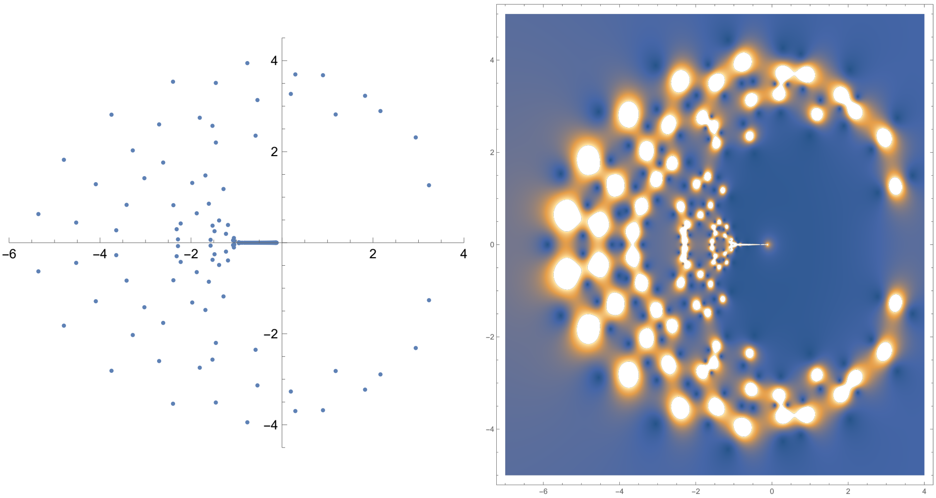


Figure 6.2: The figure on the left depicts the zeros of the independence polynomial of the 2-dimensional torus of size 18×18. The figure on the right depicts spherical derivative of the occupation ratio of this torus.

with Figure 6.1 is striking. A desire to better understand the structures that seem to appear in Figures 6.1 and 6.2 is an important motivation for the current project.

6.1.4 Proof techniques

The proof of the main theorem relies upon two different techniques: the zero-freeness for balanced tori relies heavily on Pirogov-Sinai theory, while the existence of unbounded zeros for highly unbalanced tori uses the transfer-matrix expression of the independence polynomial on tori.

Pirogov-Sinai theory

Intuitively, for $\lambda \in \mathbb{C}$ with large norm, the value of independence polynomial in λ is determined by the large independent sets. Pirogov-Sinai theory builds on this intuition [PS75]. The main idea is to study the independence polynomial as deviations from the maximal independent sets. For even tori, there are two distinct largest independent sets, one containing the even vertices of the torus and the other containing the odd vertices. The vertices where an independent set locally differs from one of these maximal independent sets will be part of so-called *contours*. The use of contours goes back to Peierls [Pei36] and was further developed by Minlos and Sinai in [MS67] and [MS68], both originally for the Ising model. Using ideas from Pirogov-Sinai theory the independence polynomial can be expressed as a partition function of a polymer model, similar to as was done in [HPR19] for padded regions in $\mathbb{Z}^d$, where the polymers will be certain sets of contours. One of the challenges in this rewriting is posed by the geometry of the torus. We deal with this by defining a suitable compatibility relation we call *torus-compatibility* and we exploit the symmetry of the torus. In our analysis, we apply Zahradník's truncated-based approach to Pirogov-Sinai theory [Zah84], and take inspiration from its usage by Borgs and Imbrie in [BI89]. The idea of this approach is to first restrict the polymer partition function to well-behaved contours, so-called *stable contours*. Then one applies the theory to the truncated partition function and with the estimates that follow one shows in fact all contours are stable, obtaining bounds for the original polymer partition function.

Transfer-matrices

The transfer-matrix method, introduced by Kramers and Wannier in [KW41a, KW41b], can be applied to rewrite the partition function of a one-dimensional lattice. It is heavily used in the literature to obtain both rigorous results and numerical approximations regarding the accumulation of zeros on physical parameters for other models; see for example [Ons44, Shr00, SS97, CS09, CS15].

In our setting we fix even integers $n_1, \dots, n_{d-1}$ and consider the sequence of tori $\mathcal{T}_n = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_{d-1}} \times \mathbb{Z}_n$. The transfer-matrix method allows us to write the independence polynomials of these tori as

$$Z_{\mathcal{T}_n}(\lambda) = \mathrm{Tr}\left(M_\lambda^n\right).$$

Here M_λ is a matrix whose entries are indexed by independent sets of the fixed torus $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_{d-1}}$ and contain monomials in λ; see Section 6.4.1 details. If we denote the (generalized) eigenvalues of M_λ by $e_1(\lambda), \dots, e_N(\lambda)$, the above equation translates to

$$Z_T(\lambda) = e_1(\lambda)^n + \cdots + e_N(\lambda)^n.$$

For $|\lambda|$ large we will show that there are two simple eigenvalues, which we denote by q^+ and q^-, of approximately the same norm that dominate the remaining eigenvalues. Normality arguments then give a relatively quick proof that zeros of $\{Z_{\mathcal{T}_n}\}_{n\geq 1}$ accumulate at ∞. This can be seen as a special case of a theorem by Sokal [Sok04]; see also [BKW78].

The normality argument does not give any bounds on how large n has to be with respect to $(n_1, \dots, n_{d-1})$ to obtain zeros of a certain magnitude. We will more thoroughly investigate the eigenvalues of M_λ, and in particular $q^\pm$, to explicitly describe such bounds. These bounds imply the unboundedness of zeros for highly unbalanced tori.

6.1.5 Implications for efficient approximation algorithms

The distribution of zeros of the independence polynomial is not only closely related to the analyticity of the limiting free energy, but also to the existence of efficient algorithms for the approximation of the independence polynomial. Indeed, let $\mathcal{G}$ be a class of bounded degree graphs. Then if $Z_G(\lambda) \neq 0$ for all $G \in \mathcal{G}$ and λ in some open set U containing 0, then by Barvinok's interpolation method [Bar16] and follow up work of Patel and the last author [PR17] there exists an algorithm that for each $\lambda \in U$ and $\varepsilon > 0$ computes on input of an n-vertex graph G from $\mathcal{G}$ a number ξ such that

$$|\xi - Z_G(\lambda)| \leq \varepsilon |Z_G(\lambda)| \tag{6.1}$$

in time polynomial in n/ε. Such an algorithm is called a *Fully Polynomial Time Approximation Scheme* or *FPTAS* for short.

Recently, Helmuth, Perkins and the last author [HPR19] were able to extend this algorithmic approach to zero-free regions that do not contain the point 0, but rather the point ∞, for certain subgraphs of the integer lattice. See also [JKP20] for extensions of this to other families of bounded degree graphs. The algorithmic results from [HPR19] also apply to the torus with all side lengths equal and of even length n, but the resulting algorithm is technically not an FPTAS, since it restricts the choice of ε to be at least e^{-cn} for some constant $c > 0$. The results of the present paper allow to remedy this and moreover extend it to non-positive evaluations and the collection of all balanced tori (at the cost of decreasing the domain).

The following result is almost a direct corollary of our main result combined with the algorithmic approach from [HPR19]; we will provide details for its proof in Section 6.5.

Proposition 6.1.1. *Let $d \in \mathbb{Z}_{\geq 2}$ and let $\mathbf{T}_d$ be a family of balanced even d-dimensional tori. Then there exists a $\Lambda > 0$ such that for each $\lambda \in \mathbb{C}$ with $|\lambda| > \Lambda$ there exists an FPTAS for approximating $Z_{\mathcal{T}}(\lambda)$ for $\mathcal{T} \in \mathbf{T}_d$.*

The interpolation method crucially depends on there being an open set not containing any zeros of the independence polynomial for graphs in the given family. There is essentially no way to circumvent this, at least for the family of all graphs of a given maximum degree $d \geq 3$, $\mathcal{G}_d$. Indeed, it was shown in Chapter 5 that the closure of the set of $\lambda \in \mathbb{C}$ for which approximating the evaluation of independence polynomial at λ (in the sense of (6.1)) is computationally hard (technically #P-hard) contains the closure of the set $\lambda \in \mathbb{C}$ for which there exists a graph $G \in \mathcal{G}_d$ such that $Z_G(\lambda) = 0$. It would be interesting to see to what extent such a result hold for more restricted families of bounded degree graphs. We suspect that, by slightly enlarging the family of highly unbalanced tori, using the techniques of Chapter 5, it can be shown that approximating the evaluation of the independence polynomial at large λ for graphs in this family (in the sense of (6.1)) is as hard as computing the evaluation exactly.

6.1.6 Questions for future work

When it comes to describing the complex zeros of the independence polynomials for graphs that converge to the integer lattice, the results in this paper barely touch the surface and raise a number of interesting questions. A first issue, already addressed above, is to close the gap between balanced and highly-unbalanced tori.

Several steps of the proof for boundedness of zeros of balanced tori rely in an essential way on the assumption that the tori are balanced. On the other hand, the highly-unbalanced assumption on the family of tori that guarantees the existence of unbounded zeros seems far from sharp, evidenced for example by the fact that the demonstrated zeros of the tori escape very rapidly in terms of the sizes of the tori. It therefore seems reasonable to expect that the balanced assumption is necessarily, while the highly-unbalanced assumption is not.

Question 6.1.2. Let $\mathcal{F}$ be a family of even d-dimensional tori for which the zeros of the independence polynomials are uniformly bounded. Is $\mathcal{F}$ necessarily balanced?

As discussed above, there are many other natural families of graphs that converge to the integer lattice, in the sense that the free energy per site converges to the same limit. Knowing that for families of induced subgraphs of $\mathbb{Z}^d$ with padded boundaries the zeros are automatically uniformly bounded, while for tori an additional assumption is required, it would be interesting to have a more general criterion that guarantees boundedness of the zero sets.

Question 6.1.3. Let $\mathcal{F}$ be a family of graphs for which the free energy per site exists and agrees with the free energy per site of d-dimensional balanced even tori. Under which conditions are the zeros of the independence polynomials uniformly bounded? Of particular interest are graphs with boundaries that are not necessarily padded, such as rectangles and cylinders.

The non-normality loci depicted in Figures 6.1 and 6.2 show a strong similarity to the non-normality loci that occur for Cayley trees of different degrees; see for example the discussion in Section 5.7 in Chapter 5. For Cayley trees the complement of the activity locus consists of infinitely many components, and apart from the two zero-free components containing 0 and ∞, each components contains exactly one zero-parameter. Almost nothing in this direction is known for graphs converging to integer lattices, and a first question in this direction is the following:

Question 6.1.4. Consider the zero-free components of the family of balanced even tori of fixed dimension. Are the zero-free components containing the points 0 and ∞ distinct?

As we remarked in Chapter 5 the zero-locus and the non-normality locus coincide for the family of all bounded degree graphs. It is not clear whether this is true for families of balanced even tori. Figure 6.2 suggests that there may be strong similarity between these two sets. While we have been able to confirm one of the suggestions in this figure, namely the boundedness of the zeros, the relation between non-normality and zeros is still completely open.

Question 6.1.5. What is the relation between the zeros of the independence polynomial and the non-normality locus of the occupation ratios for the family of balanced even tori?

6.1.7 Organization of the paper

Section 6.2 provides a self-contained background in Pirogov-Sinai theory used to prove boundedness of zeros for balanced tori in Section 6.3. We prove the unboundedness of zeros for highly unbalanced tori in Section 6.4. In Section 6.5 we finish by proving implications to efficient approximation algorithms for balanced tori.

Acknowledgment The authors are grateful to Ferenc Bencs for inspiring discussions related to the topic of this paper.

6.2 Pirogov-Sinai theory

This section provides a self-contained background in Pirogov-Sinai theory. We closely follow the framework of [HPR19], but apply it to the independence polynomial of tori, which requires several adjustments. While much of this background section is classical, proofs in the literature are often omitted or stated in a different context. For this reason this section contains several results and proofs that are not original but may be difficult to find in the literature.

In Subsection 6.2.1 we discuss contains the required background on polymer partition functions. in what follows we rewrite the independence polynomial of the torus as a suitable polymer partition function.

6.2.1 Polymer models and the Kotecký-Preiss theorem

A polymer model consists of a finite set of polymers S, an anti-reflexive and symmetric compatibility relation $\sim$ on S and a weight function $w : S \to \mathbb{C}$. We define Ω to be the set of collections of pairwise compatible polymers. The *polymer partition function* is defined as

$$Z_{\mathrm{pol}} := \sum_{\Gamma\in\Omega}\prod_{\gamma\in\Gamma} w(\gamma).$$

We note $\emptyset \in \Omega$, hence if $w(\gamma) = 0$ for all $\gamma \in S$ we see $Z_{\mathrm{pol}} = 1$.

Remark 8. Whenever f is a holomorphic function with $f(0) > 0$, we write $\log f(z)$ for a branch with $\log f(0) \in \mathbb{R}$. We will use this convention throughout the paper.

The *cluster expansion*, see for example [KP86] and Section 5.3 in [FV17], states that the polymer partition function can be expressed in terms of the following formal power series in the weights:

$$\log Z_{\mathrm{pol}} = \sum_{k\geq 1}\frac{1}{k!}\sum_{(\gamma_1,\ldots,\gamma_k)}\psi(\gamma_1,\ldots,\gamma_k)\prod_{i=1}^{k} w(\gamma_i), \tag{6.2}$$

where the sum runs over ordered k-tuples of polymers and ψ is the *Ursell function* defined as follows. Let H be the *incompatibility graph* of the polymers $\gamma_1,\ldots,\gamma_k$, i.e. the graph with vertex set $[k]$ and an edge between i and j if γ_i is incompatible with γ_j. Then

$$\psi(\gamma_1,\ldots,\gamma_k) := \sum_{\substack{E\subseteq E(H)\\ \text{spanning, connected}}} (-1)^{|E|}.$$

A multiset $\{\gamma_1,\ldots,\gamma_k\}$ of polymers is a *cluster* if its incompatibility graph is connected. For a cluster $X := \{\gamma_1,\ldots,\gamma_k\}$ of polymers we define

$$\Phi(X) = \prod_{\gamma\in S}\frac{1}{n_X(\gamma)!}\psi(\gamma_1,\ldots,\gamma_k)\prod_{i=1}^{k} w(\gamma_i),$$

with $n_X(\gamma)$ the number of times the polymer γ appears in X. Then one sees the cluster expansion (6.2) can be equivalently written as

$$\log Z_{\mathrm{pol}} = \sum_{X \text{cluster}} \Phi(X). \tag{6.3}$$

The Kotecký-Preiss theorem provides a sufficient condition on the weights that guarantees convergence of the cluster expansion, see Theorem 1 in [KP86]:

Theorem 6.2.1. *Suppose there are functions $a : S \to [0,\infty)$, $b : S \to [0,\infty)$ such that for every polymer $\gamma \in S$ we have*

$$\sum_{\gamma' \not\sim \gamma} |w(\gamma')| e^{a(\gamma')+b(\gamma')} \leq a(\gamma),$$

then $Z_{pol} \neq 0$, the cluster expansion of the polymer partition function is convergent and for any polymer $\gamma \in S$ we have

$$\sum_{X \not\sim \gamma} |\Phi(X)| e^{b(X)} \leq a(\gamma),$$

where for a cluster X we define $b(X) = \sum_{\gamma \in X} b(\gamma)$ and we write $X \not\sim \gamma$ if and only if there is a $\gamma' \in X$ with $\gamma' \not\sim \gamma$.

6.2.2 Contour representation of the independence polynomial on tori

In this section we express the independence polynomial of a torus as a sum of two polymer partition functions, using contours as polymers. This is based on ideas and definitions from Pirogov-Sinai theory [PS75, Zah84], as applied to the independence polynomial in [HPR19]. In [HPR19] the contour models are defined for padded induced subgraphs in $\mathbb{Z}^d$; we will modify the ideas and definitions such that they apply to tori instead.

Preliminaries on the topology of tori

Definition 6.2.2. We denote a d-dimensional torus by $\mathcal{T} := \mathbb{Z}_{\ell_1} \times \cdots \times \mathbb{Z}_{\ell_d}$ with $\ell_1 \leq \ell_2 \leq \cdots \leq \ell_d$ and write $|\mathcal{T}| := \prod_{i=1}^{d} \ell_i$. Let $C > 0$. A torus $\mathcal{T}$ is said to be *C-balanced* if $|\mathcal{T}| \leq e^{C\ell_1}$. We denote the set of all even d-dimensional C-balanced tori by $\mathbf{T}_d(C)$.

Note that a family of d-dimensional tori is balanced as defined in the introduction if and only if there exists a uniform $C > 0$ such that every torus in the family is C-balanced. In particular the d-dimensional torus with equal side lengths, denoted by $\mathbb{Z}_n^d$, is in $\mathbf{T}_d(1)$ for any d and any even $n \geq e^d$.

We label the vertices of $\mathcal{T}$ as $(v_1, \ldots v_d)$ with $v_i \in \{-\ell_i/2, \ldots, 0, \ldots, \ell_i/2 - 1\}$ for each $i \in [d]$. Denote the d-dimensional zero vector by $\vec{0}$. Throughout this and later sections we let $\mathcal{T}$ be an even d-dimensional torus, for a fixed $d \geq 2$. When $\mathcal{T}$ is assumed to be balanced or unbalanced we will state so explicitly.

Definition 6.2.3. We denote the ∞-neighborhood of $v \in \mathcal{T}$ by

$$N_\infty[v] := \{u \in \mathcal{T} : \|v - u\|_\infty \leq 1\}.$$

Note that each neighborhood $N_\infty[v]$ consists of 3^d distinct vertices. We say an induced subgraph $\Lambda \subset \mathcal{T}$ is ∞-connected if for each $u, v \in \Lambda$ there is a sequence $(v_0, \ldots, v_n)$ of vertices in Λ such that $v_0 = v$, $v_n = u$ and for each i we have $v_{i+1} \in N_\infty[v_i]$. Such a sequence is called an ∞-path from v to u in Λ.

We denote the set of connected components of a graph G by $\mathcal{C}(G)$.

Definition 6.2.4. For subsets $A, B \subseteq V(\mathcal{T})$ we define their *distance* and the *box-diameter* as

$$\operatorname{dist}(A, B) := \min_{a \in A,\ b \in B} \operatorname{dist}(a, b) \text{ and } \operatorname{diam}_\square(A) := \max_{i=1,\ldots,d} |A_i|,$$

where dist denotes the graph distance on $\mathcal{T}$ and A_i denotes the ith marginal of A. We define $\operatorname{diam}_\square(\emptyset) = 0$. Define the closure of A as

$$\operatorname{cl}(A) = A_1 \times \cdots \times A_d.$$

When we apply these parameters to induced subgraphs of $\mathcal{T}$ it should be read as applying it to their vertex sets.

Let ℓ_1 denote the length of the shortest side of $\mathcal{T}$. If $\operatorname{diam}_\square(A) < \ell_1$ it is not hard to see that $\mathcal{T} \setminus \operatorname{cl}(A)$ is contained in a unique connected component of $\mathcal{T} \setminus A$, which we will denote by $\operatorname{ext}(A)$. We let $\operatorname{int}(A) = \mathcal{T} \setminus (A \cup \operatorname{ext}(A))$.

The following lemma will be used implicitly several times.

Lemma 6.2.5. *Let A_1, A_2 be induced subgraphs of a torus $\mathcal{T}$ with shortest side ℓ_1 such that $\operatorname{diam}_\square(A_i) < \ell_1$, $\operatorname{dist}(A_1, A_2) \geq 2$ and both $A_1 \subseteq \operatorname{ext}(A_2)$ and $A_2 \subseteq \operatorname{ext}(A_1)$. Then $\operatorname{int}(A_1) \cap \operatorname{int}(A_2) = \emptyset$.*

Proof. Suppose for the sake of contradiction that A_1, A_2 is a counterexample for which $|\operatorname{int}(A_1)|$ is minimized. Let v be a vertex of $\operatorname{int}(A_1)$ that is connected to a vertex in A_1, say u. Because $\operatorname{dist}(A_1, A_2) \geq 2$ it follows that $u \notin A_2$. Therefore u and v lie in the same connected component of $\mathcal{T} \setminus A_2$. Because u lies in $\operatorname{ext}(A_2)$ it follows that $v \in \operatorname{ext}(A_2)$. Note also that v is not connected to an element of A_2 because $A_2 \subseteq \operatorname{ext}(A_1)$ and $v \in \operatorname{int}(A_1)$. Because $\operatorname{diam}_\square(A_1 \cup \operatorname{int}(A_1)) = \operatorname{diam}_\square(A_1)$ it follows that

- $\operatorname{diam}_\square(A_1 \cup \{v\}) < \ell$ and $\operatorname{diam}_\square(A_2) < \ell_1$;
- $\operatorname{dist}(A_1 \cup \{v\}, A_2) \geq 2$;
- both $A_1 \cup \{v\} \subseteq \operatorname{ext}(A_2)$ and $A_2 \subseteq \operatorname{ext}(A_1 \cup \{v\})$;
- $\operatorname{int}(A_1 \cup \{v\}) \cap \operatorname{int}(A_2) = \operatorname{int}(A_1) \cap \operatorname{int}(A_2)$, which is non-empty by assumption.

This is a contradiction because $\mathrm{int}(A_1 \cup \{v\})$ is strictly smaller than $\mathrm{int}(A_1)$. $\square$

Let $\Lambda \subseteq \mathcal{T}$ be an induced subgraph. We denote the boundary of Λ by $\partial\Lambda \subseteq \Lambda$, i.e. the subgraph of Λ induced by the vertices of Λ with at least one neighbor in $\mathcal{T} \setminus \Lambda$. We define $\partial^c\Lambda := \partial(\mathcal{T} \setminus \Lambda)$. Denote by $\Lambda^\circ = \Lambda \setminus \partial\Lambda$ the interior of Λ. We write $|\Lambda|$ instead of $|V(\Lambda)|$ and we write $v \in \Lambda$ instead of $v \in V(\Lambda)$.

Remark 9. Let $\mathcal{T}$ be a d-dimensional even torus with minimal side length ℓ_1. For any induced subgraph Λ in $\mathcal{T}$ with $\mathrm{diam}_\square(\Lambda) < \ell_1$, the induced subgraph $\mathrm{ext}(\Lambda) \cap \partial^c\Lambda$ is ∞-connected by Proposition B.82 in [FV17].

Lemma 6.2.6. *Let $\mathcal{T}$ be a d-dimensional even torus with minimal side length ℓ_1. Let $\Lambda_1, \dots, \Lambda_n$ and A be induced subgraphs of $\mathcal{T}$ satisfying*

1. *for each i we have $\mathrm{diam}_\square(\Lambda_i) < \ell_1$ and $\Lambda_i^\circ \subseteq A$;*
2. *for $i \neq j$ we have $\mathrm{dist}(\Lambda_i, \Lambda_j) \geq 2$;*
3. *A is ∞-connected,*

then $\cap_{i=1}^n \mathrm{ext}(\Lambda_i) \cap A$ is ∞-connected.

Proof. Take $u, v \in \cap_{i=1}^n \mathrm{ext}(\Lambda_i) \cap A$. Because A is ∞-connected, there is an ∞-path from u to v through A. Denote this path by $(a_0, \dots, a_k)$ for some $k \geq 0$, where $a_0 = u$ and $a_k = v$. If the path has empty intersection with the sets Λ_i, we are done. Let l denote the minimal index such that $a_l \in \Lambda_i$ for some i. As $u, v \in \mathrm{ext}(\Lambda_i)$, there is a minimal index m with $l < m < k$ such that $a_m \notin \Lambda_i$. As $\Lambda_i^\circ \subseteq A$, we see $a_{l-1}, a_m \in \mathrm{ext}(\Lambda_i) \cap \partial^c\Lambda_i$. We now claim there is a ∞-path from a_{l-1} to a_m which does not intersect Λ_i.

To prove this claim, first note for each i we have $\mathrm{diam}_\square(\Lambda_i) < \ell_1$ and thus the induced subgraph $\mathrm{ext}(\Lambda_i) \cap \partial^c\Lambda_i$ is ∞-connected by Remark 9. Since $\Lambda_i^\circ \subseteq A$, we see $\mathrm{ext}(\Lambda_i) \cap \partial^c\Lambda_i \subseteq A$. As for any $j \neq i$ we have $\mathrm{dist}(\Lambda_i, \Lambda_j) \geq 2$ we see $\mathrm{ext}(\Lambda_i) \cap \partial^c\Lambda_i$ does not intersect Λ_j. Hence there is a ∞-path from a_{l-1} to a_m using only vertices from $\mathrm{ext}(\Lambda_i) \cap \partial^c\Lambda_i$, and none of the vertices of this path intersect Λ_j for $j \neq i$. This proves the claim from which the lemma follows. $\square$

Contour representation of independent sets

Definition 6.2.7. Let $\Lambda \subseteq \mathcal{T}$ be an induced subgraph. A map $\sigma : V(\Lambda) \to \{0, 1\}$ is called a *feasible configuration on* Λ if $I_\sigma = \{v \in V(\Lambda) : \sigma(v) = 1\}$ is an independent set of Λ.

Given an independent set I we denote the associated feasible configuration by σ_I.

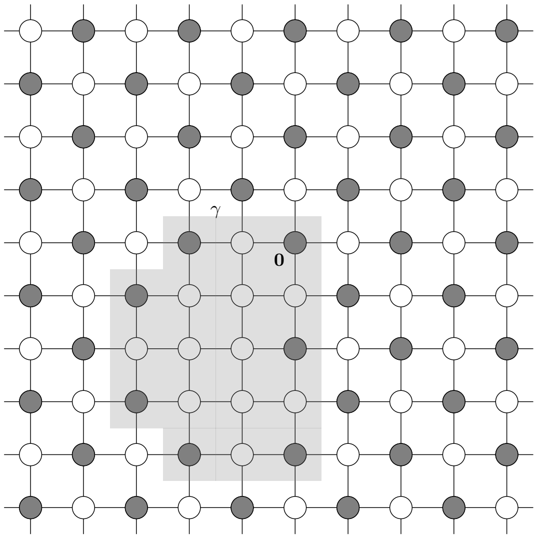


Figure 6.3: A contour γ in a 10 by 10 torus. Vertices v such that $\sigma(v) = 1$ are in dark gray and vertices v such that $\sigma(v) = 0$ are in white. The shaded gray region denotes the support of γ. The label of $\mathbb{Z}_{10}^2 \setminus \overline{\gamma}$ is even.

Definition 6.2.8. We call a vertex of $\mathcal{T}$ either *even* or *odd* if the sum of its coordinates is even or odd respectively. For an induced subgraph $\Lambda \subset \mathcal{T}$ we denote by Λ_{even} the subgraph induced by the even vertices of Λ and Λ_{odd} the subgraph induced by the odd vertices of Λ. The feasible configurations corresponding to the two maximal independent subsets of $\mathcal{T}$, consisting of either all even or all odd vertices, are denoted by σ_{even} and σ_{odd}. We refer to $\{\text{even}, \text{odd}\}$ as the set of *ground states*. Given a ground state φ, the complementary ground state will be denoted by $\overline{\varphi}$.

Definition 6.2.9. Let $\Lambda \subseteq \mathcal{T}$ be an induced subgraph. Given any feasible configuration $\sigma : V(\Lambda) \to \{0, 1\}$ we say a vertex $v \in V(\Lambda)$ is *correct* if there exists a ground state $\varphi \in \{\text{even}, \text{odd}\}$ such that for all $u \in N_\infty[v] \cap \Lambda$ we have $\sigma(u) = \sigma_\varphi(u)$, otherwise v is defined to be *incorrect*. We write $\Gamma(\Lambda, \sigma)$ for the subgraph of Λ induced by the set of incorrect vertices in Λ with respect to σ.

Definition 6.2.10. Let γ be a tuple $(\overline{\gamma}, \sigma_\gamma)$ with *support* $\overline{\gamma}$ a nonempty induced subgraph of $\mathcal{T}$ and a feasible configuration $\sigma_\gamma : V(\overline{\gamma}) \to \{0, 1\}$ for which there exists a labeling function $\text{lab}_\gamma : \mathcal{C}(\mathcal{T} \setminus \overline{\gamma}) \to \{\text{even}, \text{odd}\}$ such that the map $\hat{\sigma}_\gamma$:

$V(\mathcal{T}) \to \{0,1\}$ given by

$$\hat{\sigma}_\gamma(v) = \begin{cases} \sigma_\gamma(v) & \text{if } v \in V(\overline{\gamma}) \\ \sigma_{\mathrm{lab}_\gamma(A)}(v) & \text{if } v \in V(A) \text{ with } A \in \mathcal{C}(\mathcal{T} \setminus \overline{\gamma}) \end{cases}$$

is a feasible configuration on $\mathcal{T}$ and $\overline{\gamma} = \overline{\Gamma(\mathcal{T}, \hat{\sigma}_\gamma)}$. Let ℓ_1 denote the minimal side length of $\mathcal{T}$. We say that γ is a *small contour* if $\overline{\gamma}$ is connected and satisfies $\mathrm{diam}_\square(\overline{\gamma}) < \ell_1$. We say that γ is a *large contour* if each connected component of $\overline{\gamma}$ satisfies $\mathrm{diam}_\square(\overline{\gamma}) \geq \ell_1$. A *contour* is either a small or a large contour. Two contours γ_1, γ_2 in $\mathcal{T}$ have *compatible support* if

$$\mathrm{dist}(\overline{\gamma_1}, \overline{\gamma_2}) \geq 2.$$

See Figure 6.3 for an illustration of a contour γ in the torus $\mathbb{Z}_{10}^2$.

Remark 10. A contour $\gamma = (\overline{\gamma}, \sigma_\gamma)$ uniquely determines the labeling function, lab_γ, and the associated feasible configuration, $\hat{\sigma}_\gamma$.

Definition 6.2.11. We denote the exterior of a small contour γ by $\mathrm{ext}(\gamma)$ instead of $\mathrm{ext}(\overline{\gamma})$. The label of $\mathrm{ext}(\gamma)$ is called the *type* of γ. For a set Γ of small contours we define the exterior $\mathrm{ext}(\Gamma) = \cap_{\gamma\in\Gamma} \mathrm{ext}(\gamma)$, with the convention that $\mathrm{ext}(\emptyset) = \mathcal{T}$. For a large contour we do not define the exterior, but we artificially define the type of a large contour to be even.

For any contour γ and any ground state $\xi \in \{\text{even}, \text{odd}\}$ we define the *ξ-interior of* γ as the union over all non-exterior connected components of $\mathcal{T} \setminus \overline{\gamma}$ with label ξ, we denote this induced subgraph of $\mathcal{T}$ by $\mathrm{int}_\xi(\gamma)$. Denote the *interior* of a contour γ by $\mathrm{int}(\gamma) = \mathrm{int}_{\mathrm{even}}(\gamma) \cup \mathrm{int}_{\mathrm{odd}}(\gamma)$.

We note that the interior of any small contour γ cannot contain a connected component of a large contour because its box-diameter is strictly less than ℓ_1 (where ℓ_1 denotes the minimum side length of the underlying torus).

Definition 6.2.12. Let Γ be a set of contours with pairwise compatible supports containing at most one large contour. We say Γ is a *matching set of contours* if there is a labeling function

$$\mathrm{lab}_\Gamma : \mathcal{C}\big(\mathcal{T} \setminus \bigcup_{\gamma\in\Gamma} \overline{\gamma}\big) \to \{\text{even}, \text{odd}\}$$

such that for each $A \in \mathcal{C}\big(\mathcal{T} \setminus \bigcup_{\gamma\in\Gamma} \overline{\gamma}\big)$ and $\gamma \in \Gamma$ with $\mathrm{dist}(A, \overline{\gamma}) = 1$ we have that $\hat{\sigma}_\gamma$ is equal to $\sigma_{\mathrm{lab}(A)}$ when restricted to A.

For any contour γ the set $\Gamma = \{\gamma\}$ is a matching set of contours.

Definition 6.2.13. For non-empty Γ the labeling function lab_Γ is unique. If Γ is empty there are two possible labeling functions, namely the one that assigns either even or odd to $\mathcal{T}$. We view these as distinct matching sets of contours and denote them by $\emptyset_{\text{even}}$ and $\emptyset_{\text{odd}}$. Formally we thus define

$$\Omega_{\text{match}}(\mathcal{T}) = \{\Gamma : \Gamma \text{ a non-empty matching set of contours}\} \cup \{\emptyset_{\text{even}}\} \cup \{\emptyset_{\text{odd}}\}$$

as the set of all matching sets of contours.

See Figure 6.4 for an illustration of a matching set of contours in an 18 by 18 torus.

Definition 6.2.14. For a contour $\gamma = (\overline{\gamma}, \sigma_\gamma)$ in $\mathcal{T}$ we define the *surface energy* as

$$||\gamma|| := \frac{1}{4d} \sum_{\substack{v \in V(\overline{\gamma}) \\ \sigma_\gamma(v)=0}} \Big(2d - \sum_{u \in N(v)} \hat{\sigma}_\gamma(u)\Big).$$

For a matching set of contours Γ we define $||\Gamma|| = \sum_{\gamma\in\Gamma} ||\gamma||$.

In Theorem 6.2.15 we show the surface energy is always integer.

Theorem 6.2.15. *There is a bijection between the set of all sets of matching contours $\Omega_{match}(\mathcal{T})$ and the set of feasible configurations on an even torus $\mathcal{T}$ such that for any $\Gamma \in \Omega_{match}(\mathcal{T})$ and its corresponding feasible configuration $\tau : V(\mathcal{T}) \to \{0, 1\}$ we have*

$$||\Gamma|| = \frac{|\mathcal{T}|}{2} - |I_\tau|. \tag{6.4}$$

Proof. For $\Gamma \in \Omega_{\text{match}}(\mathcal{T})$ we define the feasible configuration τ_Γ as

$$\tau_\Gamma(v) = \begin{cases} \sigma_\gamma(v) & \text{if } v \in \overline{\gamma} \text{ for some } \gamma \in \Gamma \\ \sigma_{\text{lab}_\Gamma(A)}(v) & \text{if } v \in A \text{ for some } A \in \mathcal{C}\big(\mathcal{T} \setminus \bigcup_{\gamma\in\Gamma} \overline{\gamma}\big). \end{cases}$$

We recall here that $\Gamma \in \Omega_{\text{match}}(\mathcal{T})$ contains two copies of the empty set with either an even or an odd label. These correspond to σ_{even} and σ_{odd} respectively. It follows directly from the definition of $\Omega_{\text{match}}(\mathcal{T})$ that τ_Γ is indeed feasible.

We now define a map from the set of feasible configurations to $\Omega_{\text{match}}(\mathcal{T})$. Let $\tau : V(\mathcal{T}) \to \{0, 1\}$ be a feasible configuration. The induced subgraph $\overline{\Gamma(\mathcal{T}, \tau)}$ consists of a union of say $s \geq 0$ connected components with box-diameter strictly less than ℓ_1 and $m \geq 0$ connected components with box-diameter $\geq \ell_1$. If $s = m = 0$ then τ is equal to either σ_{even} or σ_{odd} which we map to $\emptyset_{\text{even}}$ or $\emptyset_{\text{odd}}$ respectively. If $m \geq 1$ then we denote the union of all connected components with box-diameter $\geq \ell_1$ by $\overline{\gamma_{\text{large}}}$. Denote the remaining connected components of

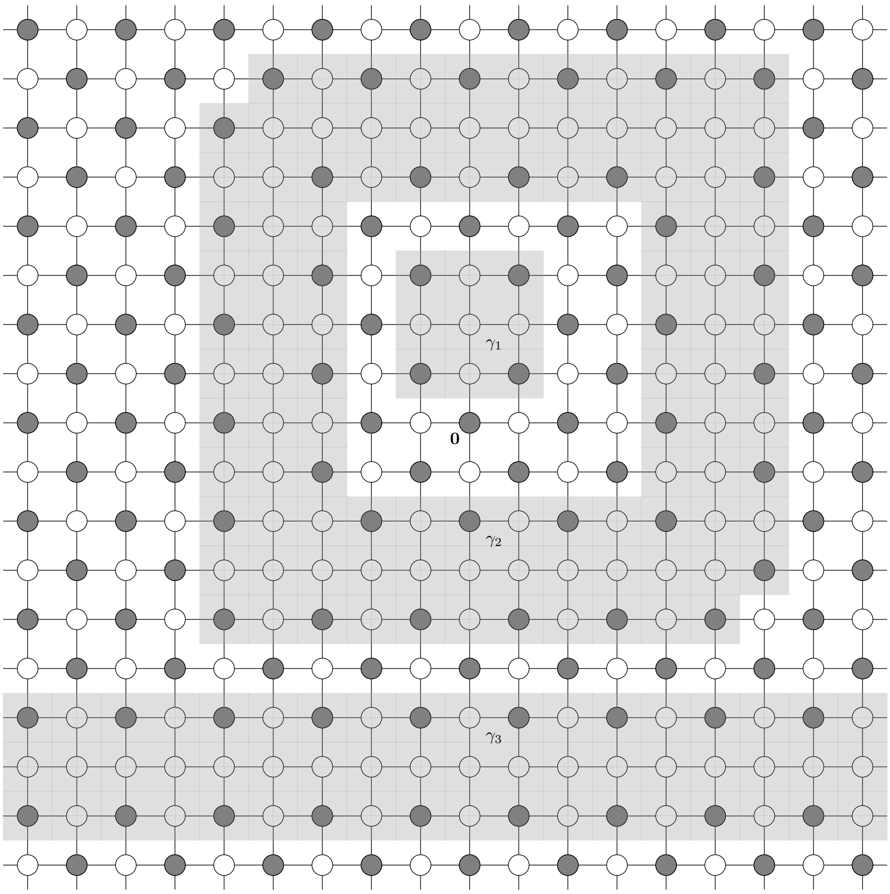


Figure 6.4: A matching set of contours in an 18 by 18 torus. The contour γ_1 is small of type even, γ_2 is small of type odd and γ_3 is large. The contours γ_2 and γ_1 lie in the odd-interior of γ_3, the contour γ_1 lies in the even-interior of γ_2.

$\overline{\Gamma(\mathcal{T},\tau)}$ by $\overline{\gamma_i}$ for $i \in \{1,\dots,s\}$. By restricting τ, we define a feasible configuration σ_γ on each support $\overline{\gamma} \in \{\overline{\gamma_{\text{large}}}, \overline{\gamma_1}, \dots, \overline{\gamma_s}\}$. We have to show that for each such $\overline{\gamma}$ we can define a labeling function lab_γ on the connected components of $\mathcal{T}\setminus\overline{\gamma}$ that makes γ into a well-defined contour.

To do this it is sufficient to show that, given $A \in \mathcal{C}(\mathcal{T}\setminus\overline{\gamma})$, there exists $\varphi \in \{\text{even}, \text{odd}\}$ such that τ restricted to the vertices of $A \cap \partial^c(\overline{\gamma})$ is equal to σ_φ. Indeed, if this is the case then we can define $\text{lab}_\gamma(A) = \varphi$. It is not hard to see that the corresponding configuration $\hat{\sigma}_\gamma$ as defined in Definition 6.2.10 is then feasible and satisfies $\overline{\gamma} = \overline{\Gamma(\mathcal{T}, \hat{\sigma}_\gamma)}$. We distinguish between two cases.

In the first case $\overline{\gamma}$ is not $\overline{\gamma_{\text{large}}}$ and $A = \text{ext}(\overline{\gamma})$. It then follows from Remark 9 that $A \cap \partial^c(\overline{\gamma})$ is a ∞-connected set of correct vertices with respect to τ. It follows that there is a unique φ such that $\tau = \sigma_\phi$ when restricted to $A \cap \partial^c(\overline{\gamma})$.

In the second case A has empty intersection with $\overline{\gamma_{\text{large}}}$ and thus any $\overline{\gamma}'$ contained in A has box-diameter strictly less than ℓ_1. Let Γ' be the collection of these $\overline{\gamma}'$. Any $\overline{\gamma}' \in \Gamma'$ must be contained in A° because otherwise $\overline{\gamma} \cup \overline{\gamma}'$ would be a single component of $\overline{\Gamma(\mathcal{T},\tau)}$. It now follows from Lemma 6.2.6 that $A' := \cap_{\gamma' \in \Gamma'} \text{ext}(\gamma') \cap A$ is ∞-connected. Because A' consists of correct vertices with respect to τ and $\partial^c(\overline{\gamma}) \cap A \subseteq A'$ it follows that there is a φ such that $\tau = \sigma_\varphi$ when restricted to $\partial^c(\overline{\gamma}) \cap A$.

We have shown that $\Gamma_\tau := \{\gamma_{\text{large}}, \gamma_1, \dots, \gamma_s\}$ is a set of contours with pairwise compatible supports. The labeling function lab_Γ that assigns to any component with vertex v the label inherited from τ shows that indeed $\Gamma_\tau \in \Omega_{\text{match}}(\mathcal{T})$. By definition the maps $\tau \mapsto \Gamma_\tau$ and $\Gamma \mapsto \tau_\Gamma$ are each others inverse.

We now prove the equality in equation (6.4). Let Γ be a set of matching contours and τ its corresponding feasible configuration. We count the number of edges in $\mathcal{T}$ in two ways. The total number of edges in $\mathcal{T}$ is $2d \cdot \frac{|\mathcal{T}|}{2}$, as there are $\frac{|\mathcal{T}|}{2}$ even vertices in $\mathcal{T}$ and each even vertex is incident to $2d$ distinct edges. The number of edges in $\mathcal{T}$ also equals the number of edges between I_τ and $\mathcal{T}\setminus I_\tau$ plus the number of edges within $\mathcal{T}\setminus I_\tau$. The number of edges between I_τ and $\mathcal{T}\setminus I_\tau$ is equal to $2d \cdot |I_\tau|$, as each vertex of I_τ has degree $2d$ and I_τ is independent. For a vertex $v \in \mathcal{T}\setminus I_\tau$ the number of neighbors of v in $\mathcal{T}\setminus I_\tau$ is $2d - \sum_{u \in N(v)} \tau(u)$, the degree of v minus the number of neighbors of v in I_τ. The number of edges within $\mathcal{T}\setminus I_\tau$ is the sum of the number of neighbors of v in $\mathcal{T}\setminus I_\tau$ over all $v \in \mathcal{T}\setminus I_\tau$ divided by 2, hence

$$\begin{aligned} |E(\mathcal{T}\setminus I_\tau)| &= \frac{1}{2} \sum_{\substack{v \in V(\mathcal{T}) \\ \tau(v)=0}} \Big(2d - \sum_{u \in N(v)} \tau(u)\Big) \\ &= \frac{1}{2} \sum_{\gamma \in \Gamma} \sum_{\substack{v \in V(\overline{\gamma}) \\ \tau(v)=0}} \Big(2d - \sum_{u \in N(v)} \tau(u)\Big) = 2d \cdot ||\Gamma||, \end{aligned}$$

where the second equality follows as each vertex outside of $\cup_{\gamma\in\Gamma}V(\overline{\gamma})$ is correct with respect to τ. From this we see

$$2d \cdot ||\Gamma|| + 2d \cdot |I_\tau| = 2d \cdot \frac{|\mathcal{T}|}{2},$$

hence $||\Gamma|| = \frac{|\mathcal{T}|}{2} - |I_\tau|$. □

Definition 6.2.16. We define the *matching contour partition function on $\mathcal{T}$* as

$$Z_{\text{match}}(\mathcal{T}; z) := \sum_{\Gamma\in\Omega_{\text{match}}(\mathcal{T})} \prod_{\gamma\in\Gamma} z^{||\gamma||}.$$

Let $Z_{\text{ind}}(G;\lambda)$ deote the independence polynomial of a graph G evaluated at λ.

Corollary 6.2.17. *We have*

$$Z_{ind}(\mathcal{T};\lambda) := \lambda^{\frac{|\mathcal{T}|}{2}} \cdot Z_{match}(\mathcal{T}; \frac{1}{\lambda}).$$

Proof. This follows from Theorem 6.2.15 and the definition of the matching contour partition function. □

As $Z_{\text{match}}(\mathcal{T}; 0) = 2 \neq 0$, the first part of the main theorem is equivalent to finding a zero free region around $z = 0$ of $Z_{\text{match}}(\mathcal{T}; z)$ for all C-balanced tori.

Contours as polymers

We next collect some definitions allowing us to split up Z_{match} up into two parts which we can then interpret as polymer partition functions.

Definition 6.2.18. We partition $\Omega_{\text{match}}(\mathcal{T})$ into three subsets. We let $\Omega^{\text{large}}_{\text{match}}(\mathcal{T})$ consists of those $\Gamma \in \Omega_{\text{match}}(\mathcal{T})$ that contain a large contour. If Γ consists of small contours we define the *type* of Γ as the label assigned to $\text{ext}(\Gamma)$ by lab_Γ. We denote those Γ with type $\varphi \in \{\text{even}, \text{odd}\}$ by $\Omega^{\varphi}_{\text{match}}(\mathcal{T})$. Note that $\emptyset_{\text{even}} \in \Omega^{\text{even}}_{\text{match}}(\mathcal{T})$, $\emptyset_{\text{odd}} \in \Omega^{\text{odd}}_{\text{match}}(\mathcal{T})$ and

$$\Omega_{\text{match}}(\mathcal{T}) = \Omega^{\text{even}}_{\text{match}}(\mathcal{T}) \cup \Omega^{\text{odd}}_{\text{match}}(\mathcal{T}) \cup \Omega^{\text{large}}_{\text{match}}(\mathcal{T}).$$

For $\varphi \in \{\text{even}, \text{odd}, \text{large}\}$ we define

$$Z^{\varphi}_{\text{match}}(\mathcal{T}; z) = \sum_{\Gamma\in\Omega^{\varphi}_{\text{match}}(\mathcal{T})} \prod_{\gamma\in\Gamma} z^{||\gamma||}.$$

Definition 6.2.19. For any $\varphi \in \{\text{even}, \text{odd}\}$ and $\Gamma \in \Omega^{\varphi}_{\text{match}}(\mathcal{T})$ we define

$$\Gamma_{\text{ext}} = \{\gamma \in \Gamma : \overline{\gamma} \subseteq \text{ext}(\gamma') \text{ for all } \gamma' \in \Gamma \text{ not equal to } \gamma\}. \tag{6.5}$$

We furthermore define

$$\Omega^{\varphi}_{\text{ext}}(\mathcal{T}) = \{\Gamma \in \Omega^{\varphi}_{\text{match}}(\mathcal{T}) : \Gamma = \Gamma_{\text{ext}}\}.$$

We will further rewrite $Z_{\text{match}}(\mathcal{T}; z)$ in order to apply the framework outlined in Section 6.2.1. The first step in rewriting is a standard technique from Pirogov-Sinai theory, analogous to what was done for finite induced subgraphs $\Lambda \subseteq \mathbb{Z}^d$ with padded boundary conditions in [HPR19].

We define a class of well-behaved induced subgraphs of tori.

Definition 6.2.20. Let $\Lambda \subseteq \mathcal{T}$ be an induced subgraph. If for any small contour γ with $\overline{\gamma} \subseteq \Lambda^\circ$ we have $\text{int}(\gamma) \subseteq \Lambda^\circ$ we say Λ is *closed under taking interiors of small contours*, or more succinctly *closed*.

Note $\mathcal{T}$ is closed, and for any contour γ the induced subgraphs $\text{int}_{\text{odd}}(\gamma)$, $\text{int}_{\text{even}}(\gamma)$ and $\text{int}(\gamma)$ are also closed.

Definition 6.2.21. Let $\Lambda \subseteq \mathcal{T}$ be an induced closed subgraph and let $\varphi \in \{\text{even,odd}\}$ be a ground state. We define

$$\Omega^{\varphi}_{\text{match}}(\Lambda) := \{\Gamma \in \Omega^{\varphi}_{\text{match}}(\mathcal{T}) : \text{ for all } \gamma \in \Gamma \text{ we have } \overline{\gamma} \subseteq \Lambda^\circ\}.$$

and

$$\Omega^{\varphi}_{\text{ext}}(\Lambda) := \{\Gamma \in \Omega^{\varphi}_{\text{ext}}(\mathcal{T}) : \text{ for all } \gamma \in \Gamma \text{ we have } \overline{\gamma} \subseteq \Lambda^\circ\}$$

We also define the *matching contour partition function* as

$$Z^{\varphi}_{\text{match}}(\Lambda; z) = \sum_{\Gamma \in \Omega^{\varphi}_{\text{match}}(\Lambda)} \prod_{\gamma \in \Gamma} z^{||\gamma||}.$$

Note if $\Lambda^\circ = \emptyset$ then $Z^{\varphi}_{\text{match}}(\Lambda; z) = 1$ as in that case $\Omega^{\varphi}_{\text{match}}(\Lambda) = \{\emptyset_\varphi\}$.

Lemma 6.2.22. *For induced closed subgraphs* $\Lambda_1, \Lambda_2 \subset \mathcal{T}$ *with* $\text{dist}(\Lambda_1, \Lambda_2) \geq 2$ *and any* $\varphi \in \{$*even, odd*$\}$ *we have*

$$Z^{\varphi}_{match}(\Lambda_1 \cup \Lambda_2; z) = Z^{\varphi}_{match}(\Lambda_1; z) \cdot Z^{\varphi}_{match}(\Lambda_1; z).$$

Proof. Note that the induced subgraph $\Lambda_1 \cup \Lambda_2$ is closed. The equality follows from the bijection between $\Omega^{\varphi}_{\text{match}}(\Lambda_1) \times \Omega^{\varphi}_{\text{match}}(\Lambda_2)$ and $\Omega^{\varphi}_{\text{match}}(\Lambda_1 \cup \Lambda_2)$ given by $(\Gamma_1, \Gamma_2) \mapsto \Gamma_1 \cup \Gamma_2$. $\square$

Lemma 6.2.23. *For any induced closed subgraph $\Lambda \subseteq \mathcal{T}$, any ground state $\varphi \in \{even, odd\}$ and any $z \in \mathbb{C}$ we have*

$$Z^{\varphi}_{match}(\Lambda; z) = \sum_{\Gamma \in \Omega^{\varphi}_{ext}(\Lambda)} \prod_{\gamma \in \Gamma} z^{||\gamma||} Z^{even}_{match}(\mathrm{int}_{even}(\gamma); z) Z^{odd}_{match}(\mathrm{int}_{odd}(\gamma); z).$$

Proof. Given an induced closed subgraph $\Lambda \subseteq \mathcal{T}$ and a set $\Gamma \in \Omega^{\varphi}_{\mathrm{ext}}(\Lambda)$ we have

$$\sum_{\substack{\Gamma' \in \Omega^{\varphi}_{\mathrm{match}}(\Lambda) \\ \Gamma'_{\mathrm{ext}} = \Gamma}} \prod_{\gamma \in \Gamma' \setminus \Gamma} z^{||\gamma||} = Z^{\mathrm{even}}_{\mathrm{match}}(\cup_{\gamma \in \Gamma} \mathrm{int}_{\mathrm{even}}(\gamma); z) Z^{\mathrm{odd}}_{\mathrm{match}}(\cup_{\gamma \in \Gamma} \mathrm{int}_{\mathrm{odd}}(\gamma); z),$$

because any $\Gamma' \in \Omega^{\varphi}_{\mathrm{match}}(\Lambda)$ with $\Gamma'_{\mathrm{ext}} = \Gamma$ gives an associated set of matching contours in

$$\Omega^{\mathrm{even}}_{\mathrm{match}}(\cup_{\gamma \in \Gamma} \mathrm{int}_{\mathrm{even}}(\gamma)) \times \Omega^{\mathrm{odd}}_{\mathrm{match}}(\cup_{\gamma \in \Gamma} \mathrm{int}_{\mathrm{odd}}(\gamma)),$$

as any non external contour $\gamma' \in \Gamma'$ lies in the interior of a unique contour $\gamma \in \Gamma$. By Lemma 6.2.22 we see for any induced closed subgraph $\Lambda \subseteq \mathcal{T}$ and any $\Gamma \in \Omega^{\varphi}_{\mathrm{ext}}(\Lambda)$ that

$$\prod_{\gamma \in \Gamma} Z^{\mathrm{even}}_{\mathrm{match}}(\mathrm{int}_{\mathrm{even}}(\gamma); z) Z^{\mathrm{odd}}_{\mathrm{match}}(\mathrm{int}_{\mathrm{odd}}(\gamma); z) =$$
$$Z^{\mathrm{even}}_{\mathrm{match}}(\cup_{\gamma \in \Gamma} \mathrm{int}_{\mathrm{even}}(\gamma); z) Z^{\mathrm{odd}}_{\mathrm{match}}(\cup_{\gamma \in \Gamma} \mathrm{int}_{\mathrm{odd}}(\gamma); z).$$

Combined, these two facts yield

$$Z^{\varphi}_{\mathrm{match}}(\Lambda; z) = \sum_{\Gamma \in \Omega^{\varphi}_{\mathrm{ext}}(\Lambda)} \sum_{\substack{\Gamma' \in \Omega^{\varphi}_{\mathrm{match}}(\Lambda) \\ \Gamma'_{\mathrm{ext}} = \Gamma}} \prod_{\gamma \in \Gamma'} z^{||\gamma||} =$$
$$\sum_{\Gamma \in \Omega^{\varphi}_{\mathrm{ext}}} Z^{\mathrm{even}}_{\mathrm{match}}(\cup_{\gamma \in \Gamma} \mathrm{int}_{\mathrm{even}}(\gamma); z) Z^{\mathrm{odd}}_{\mathrm{match}}(\cup_{\gamma \in \Gamma} \mathrm{int}_{\mathrm{odd}}(\gamma); z) \prod_{\gamma \in \Gamma} z^{||\gamma||} =$$
$$\sum_{\Gamma \in \Omega^{\varphi}_{\mathrm{ext}}(\Lambda)} \prod_{\gamma \in \Gamma} z^{||\gamma||} Z^{\mathrm{even}}_{\mathrm{match}}(\mathrm{int}_{\mathrm{even}}(\gamma); z) Z^{\mathrm{odd}}_{\mathrm{match}}(\mathrm{int}_{\mathrm{odd}}(\gamma); z).$$

□

Definition 6.2.24. We define for a contour γ in $\mathcal{T}$ of type φ the weight to be the following rational function in z

$$w(\gamma; z) := z^{||\gamma||} \frac{Z^{\overline{\varphi}}_{\mathrm{match}}(\mathrm{int}_{\overline{\varphi}}(\gamma); z)}{Z^{\varphi}_{\mathrm{match}}(\mathrm{int}_{\overline{\varphi}}(\gamma); z)}.$$

Recall that the type of a large contour is defined to be even. Note that for any induced closed subgraph $\Lambda \subset \mathcal{T}$ and any contour γ in Λ the contour γ is also a contour in $\mathcal{T}$ and hence $w(\gamma; z)$ is defined. We also note the denominator of $w(\gamma; z)$ has constant term 1 for any contour γ.

The definition of these weights is a standard trick in Pirogov-Sinai theory used to rewrite the independence polynomial as a polymer partition function; see for example [HPR19]. To also do this for tori, we define a suitable compatibility relation, which is a modification of the compatibility relation used in [HPR19] to accommodate for the large contours.

Definition 6.2.25. We define two contours γ_1, γ_2 in $\mathcal{T}$ to be *torus-compatible* if they have compatible supports and if either (1) γ_1 and γ_2 are both small and of the same type or if (2) one contour is large and the other is small and of type even. Denote by $\Upsilon^{\varphi}_{\mathrm{small}}(\mathcal{T})$ the collection of sets containing small pairwise torus-compatible contours in $\mathcal{T}$ of type φ and by $\Upsilon^{\mathrm{even}}(\mathcal{T})$ the collection of sets of torus-compatible contours in $\mathcal{T}$ of type even in which we allow both large and small contours.

Note that torus-compatibility is an anti-reflexive and symmetric relation on the set of contours.

Definition 6.2.26. Let $\Lambda \subseteq \mathcal{T}$ be an induced closed subgraph and let $\varphi \in \{\text{even,odd}\}$ be a ground state. We define

$$\Upsilon^{\varphi}_{\mathrm{small}}(\Lambda) := \{\Gamma \in \Upsilon^{\varphi}_{\mathrm{small}}(\mathcal{T}) : \text{ for all } \gamma \in \Gamma \text{ we have } \overline{\gamma} \subseteq \Lambda^{\circ}\}.$$

For any $\Gamma \in \Upsilon^{\varphi}_{\mathrm{small}}(\Lambda)$ we can define Γ_{ext} exactly how is done in (6.5) in Definition 6.2.19. It is not difficult to see that then $\Gamma_{\mathrm{ext}} \in \Omega^{\varphi}_{\mathrm{ext}}(\Lambda)$. This observation, together with Lemma 6.2.23 and the choice of weights in Definition 6.2.24, allows us to rewrite the matching contour partition function, which is a sum over matching sets of contours, as a sum over sets that only require pairwise compatibility.

Lemma 6.2.27. *Let $\Lambda \subseteq \mathcal{T}$ be an induced closed subgraph and let $\varphi \in \{even, odd\}$ be a ground state. We have for any $z \in \mathbb{C}$*

$$Z^{\varphi}_{match}(\Lambda; z) = \sum_{\Gamma \in \Upsilon^{\varphi}_{small}(\Lambda)} \prod_{\gamma \in \Gamma} w(\gamma; z).$$

Furthermore, we have

$$Z^{even}_{match}(\mathcal{T}; z) + Z^{large}_{match}(\mathcal{T}; z) = \sum_{\Gamma \in \Upsilon^{even}(\mathcal{T})} \prod_{\gamma \in \Gamma} w(\gamma; z).$$

Proof. We prove the first claim by induction on $|\Lambda|$. The base case is trivial. Suppose the claim holds for $|\Lambda'| \leq k$. Next suppose that $|\Lambda| = k+1$. By Lemma 6.2.23 we have

$$Z^{\varphi}_{\text{match}}(\Lambda; z) = \sum_{\Gamma \in \Omega^{\varphi}_{\text{ext}}(\Lambda)} \prod_{\gamma \in \Gamma} z^{||\gamma||} Z^{\varphi}_{\text{match}}(\text{int}_{\varphi}(\gamma); z) Z^{\overline{\varphi}}_{\text{match}}(\text{int}_{\overline{\varphi}}(\gamma); z),$$

which by definition of the weights is equal to

$$\sum_{\Gamma \in \Omega^{\varphi}_{\text{ext}}(\Lambda)} \prod_{\gamma \in \Gamma} w(\gamma; z) Z^{\varphi}_{\text{match}}(\text{int}(\gamma); z) =$$

$$\sum_{\Gamma \in \Omega^{\varphi}_{\text{ext}}(\Lambda)} \prod_{\gamma \in \Gamma} w(\gamma; z) \sum_{\Gamma' \in \Upsilon^{\varphi}_{\text{small}}(\text{int}(\gamma))} \prod_{\gamma' \in \Gamma'} w(\gamma'; z) = \sum_{\Gamma \in \Upsilon^{\varphi}_{\text{small}}(\Lambda)} \prod_{\gamma \in \Gamma} w(\gamma; z),$$

where the first equality uses the induction hypothesis on the induced closed subgraph $\text{int}(\gamma)$ and the last equality follows by definition of torus-compatibility. This proves the first part.

Note that $Z^{\text{large}}_{\text{match}}$ is a sum over matching set of contours that contain a large contour. Therefore we can instead write $Z^{\text{large}}_{\text{match}}$ as a sum over all large contours. Reasoning as above we obtain

$$Z^{\text{large}}_{\text{match}}(\mathcal{T}; z) = \sum_{\gamma} z^{||\gamma||} \cdot Z^{\text{even}}_{\text{match}}(\text{int}_{\text{even}}(\gamma); z) \cdot Z^{\text{odd}}_{\text{match}}(\text{int}_{\text{odd}}(\gamma_{\text{large}}); z) =$$

$$\sum_{\gamma} w(\gamma; z) \cdot Z^{\text{even}}_{\text{match}}(\text{int}(\gamma); z) = \sum_{\gamma} w(\gamma; z) \sum_{\Gamma \in \mathcal{T}^{\text{even}}_{\text{small}}(\text{int}(\gamma_{\text{large}}))} \prod_{\tau \in \Gamma} w(\tau; z),$$

where each sum is over large contours γ and where the last inequality follows as $\text{int}(\gamma)$ is an induced closed subgraph of $\mathcal{T}$. By the definition of torus-compatibility of contours we obtain

$$Z^{\text{even}}_{\text{match}}(\mathcal{T}; z) + Z^{\text{large}}_{\text{match}}(\mathcal{T}; z) = \sum_{\Gamma \in \Upsilon^{\text{even}}(\mathcal{T})} \prod_{\gamma \in \Gamma} w(\gamma; z),$$

as desired. □

Remark 11. An automorphism $t : \mathcal{T} \to \mathcal{T}$ acts on contours by pushing forward their support and pulling back their configurations and associated labels and type. We note that labels are preserved when $t(\vec{0})$ is even, and switched when $t(\vec{0})$ is odd. The surface energy is always preserved.

Lemma 6.2.28. *For all $z \in \mathbb{C}$ we have $Z^{even}_{match}(\mathcal{T}; z) = Z^{odd}_{match}(\mathcal{T}; z)$.*

Proof. Let $t : \mathcal{T} \to \mathcal{T}$ be the translation by $(0, \ldots, 0, 1)$. By Remark 11 any even contour γ corresponds to an odd contour $t(\gamma)$ with the same weight. □

Denote the set of small contours of type φ with support contained in an induced closed subgraph $\Lambda \subseteq \mathcal{T}$ by $S^{\varphi}_{\text{small}}(\Lambda)$. Denote the set of all small and large contours of even type with support contained in $\mathcal{T}$ by $S^{\text{even}}(\mathcal{T})$. Using Lemma 6.2.27 and the definition of torus-compatibility it follows that for a type $\varphi \in \{\text{even, odd}\}$ and any induced closed subgraph $\Lambda \subseteq \mathcal{T}$ the function $Z^{\varphi}_{\text{match}}(\Lambda; z)$ equals a polymer partition function as defined in Section 6.2.1 with set of polymers $S^{\varphi}_{\text{small}}(\Lambda)$ and torus-compatibility as compatibility relation on $S^{\varphi}_{\text{small}}(\Lambda)$. Similarly, we see that $Z^{\text{even}}_{\text{match}}(\mathcal{T}; z)+Z^{\text{large}}_{\text{match}}(\mathcal{T}; z)$ equals a polymer partition function with set of polymers $S^{\text{even}}(\mathcal{T})$. We observe that by Lemma 6.2.28

$$\begin{aligned} Z_{\text{match}}(\mathcal{T}; z) &= Z^{\text{odd}}_{\text{match}}(\mathcal{T}; z) + Z^{\text{even}}_{\text{match}}(\mathcal{T}; z) + Z^{\text{large}}_{\text{match}}(\mathcal{T}; z) \\ &= 2Z^{\text{even}}_{\text{match}}(\mathcal{T}; z) + Z^{\text{large}}_{\text{match}}(\mathcal{T}; z), \end{aligned}$$

giving us the promised way of writing $Z_{\text{match}}(\mathcal{T}; z)$ as the sum of two polymer partition functions. Note that we cannot view $Z_{\text{match}}(\mathcal{T}; z)$ as a single polymer partition function since it contains the occurrence of two 'distinct' empty sets of matching contours.

6.2.3 Applying the Kotecký-Preiss theorem

In order to apply the Kotecký-Preiss theorem to $Z^{\text{odd}}_{\text{match}}(\mathcal{T}; z)$ and $Z^{\text{even}}_{\text{match}}(\mathcal{T}; z) + Z^{\text{large}}_{\text{match}}(\mathcal{T}; z)$, we use Zahradník's truncated-based approach to Pirogov-Sinai theory [Zah84], which is also used in [BI89]. The idea is to first restrict to contours for which the weights respect a proper bound which helps us to check the condition of the Kotecký-Preiss theorem. This process 'truncates' the partition function. We then prove, using bounds we obtain from the Kotecký-Preiss theorem on the truncated partition function, that in fact all contours satisfy this bound. To define the bound om the weights of the contours, we need the following lemmas and definition.

Lemma 6.2.29. *Let S_m denote the set of small contours γ in $\mathcal{T}$ with support of size m containing $\vec{0}$. Then there is a constant C_d depending only on d such that $|S_m| \leq C_d^m$.*

Proof. The size of S_m is bounded by the number of connected subsets in $\mathbb{Z}^d$ of size m containing $\vec{0}$ times 2^m, as a contour is uniquely determined by its support and its configuration. In [BBR10] connected subsets of size m containing $\vec{0}$ are called *strongly-embedded lattice site animals* and in [Mad99] just *site animals.* The number of strongly embedded lattice site animals of size m grows as λ_d^m for a constant λ_d depending on d, see [Mad99] and [BBR10], which implies that there exists a constant C_d such that $|S_m| \leq C_d^m$. □

We also need a lower bound on the surface energy of contours in terms of the number of vertices in the support.

Lemma 6.2.30 (Peierls condition)**.** *Let γ be a contour in $\mathcal{T}$. For $\rho = \rho(d) := \frac{1}{2d\cdot 3^d}$ the surface energy of a contour satisfies*

$$\rho|\overline{\gamma}| \leq ||\gamma|| \leq |\overline{\gamma}|.$$

Proof. The inequality $||\gamma|| \leq |\overline{\gamma}|$ is trivial. For the other inequality, note for each incorrect vertex $v \in \overline{\gamma}$ there is at least one vertex $u \in N_\infty[v]$ and a neighbor $w \in N_\infty[v]$ of u such that $\sigma_\gamma(u) = \sigma_\gamma(w) = 0$. Hence we see that at least two of the 3^d vertices in $N_\infty[v]$ have a contribution of at least $\frac{1}{4d}$ each to $||\gamma||$. We double count this contribution at most 3^d times, as $|N_\infty[v]| = 3^d$. This yields $||\gamma|| \geq \rho|\overline{\gamma}|$ for $\rho = \rho(d) = 2 \cdot \frac{1}{4d} \cdot \frac{1}{3^d} = \frac{1}{2d\cdot 3^d}$. □

Definition 6.2.31. Define for any $d \in \mathbb{Z}_{\geq 2}$ and any $x > 0$ the real number

$$\delta_1(d, x) := e^{-(\log(2C_d)+4d+5\cdot e^{-x3^d}+x)/\rho(d)},$$

where C_d is the constant from Lemma 6.2.29 and $\rho(d) = \frac{1}{2d\cdot 3^d}$ is the constant from Lemma 6.2.30.

We can now define stability of contours.

Definition 6.2.32. Let $C > 0$ and $\mathcal{T} \in \mathbf{T}_d(C)$. We define a small contour γ in $\mathcal{T}$ to be C-*stable* if for all $|z| < \delta_1(d, C)$

$$|w(\gamma; z)| \leq |z|^{||\gamma||} e^{5e^{-C3^d}\cdot|\overline{\gamma}|}.$$

We define a large contour in $\mathcal{T}$ to be C-*stable* if for all $|z| < \delta_1(d, C)$

$$|w(\gamma; z)| \leq |z|^{||\gamma||} e^{5e^{-C3^d}\cdot|\overline{\gamma}|} \cdot e^4.$$

For an induced closed subgraph $\Lambda \subseteq \mathcal{T}$ denote by $\mathcal{C}^\varphi_\Lambda(\mathcal{T}, C)$ the set of clusters X consisting of contours γ in $\mathcal{T}$ that are small and of type φ, C-stable and satisfy $\overline{\gamma} \subseteq \Lambda^\circ$. Recall that the condition of being a small contour depends on the shortest side length ℓ_1 of $\mathcal{T}$. When $\Lambda = \mathcal{T}$ we write $\mathcal{C}^\varphi(\mathcal{T}, C)$ instead of $\mathcal{C}^\varphi_{\mathcal{T}}(\mathcal{T}, C)$.

Definition 6.2.33. Let $\mathcal{T} \in \mathbf{T}_d(C)$. For any induced closed subgraph $\Lambda \subseteq \mathcal{T}$ and ground state $\varphi \in \{\text{even, odd}\}$ we define

$$Z^\varphi_{\text{trunc}}(\Lambda; z) := \sum_{\Gamma \in \mathcal{C}^\varphi_\Lambda(\mathcal{T}, C)} \prod_{\gamma\in\Gamma} w(\gamma; z).$$

We also define
$$Z^{\text{large}}_{\text{trunc}}(\mathcal{T};z) := \sum_{\substack{\Gamma\in\Upsilon^{\text{large}}(\mathcal{T})\\ \text{all } \gamma\in\Gamma\ C\text{-stable}}} \prod_{\gamma\in\Gamma} w(\gamma;z).$$

Note each of these partition functions is a polymer partition function.

Analogous to Lemma 6.2.28 we also see
$$Z^{\text{even}}_{\text{trunc}}(\mathcal{T};z) = Z^{\text{odd}}_{\text{trunc}}(\mathcal{T};z).$$

Convergence of $\log Z^{\varphi}_{\textbf{trunc}}$

We apply the Kotecký-Preiss theorem to $Z^{\varphi}_{\text{trunc}}(\Lambda;z)$ for induced closed subgraphs $\Lambda \subseteq \mathcal{T}$ and $\varphi \in \{\text{even, odd}\}$. The set of polymers is the set of small C-stable contours of type φ in $\Lambda \subseteq \mathcal{T}$, the weights of a polymer γ is defined as $w(\gamma;z)$ and the compatibility relation is torus-compatibility. The cluster expansion takes the form
$$\log Z^{\varphi}_{\text{trunc}}(\Lambda;z) = \sum_{X\in\mathcal{C}^{\varphi}_{\Lambda}(\mathcal{T},C)} \Phi(X;z), \tag{6.6}$$

where $\Phi(X;z) = \prod_{\gamma\in\Gamma}\frac{1}{n_X(\gamma)!}\psi(\gamma_1,\dots,\gamma_n)\prod_{i=1}^{n} w(\gamma_i;z)$ is defined as in Section 6.2.1. We define the *support* of a cluster $X = \{\gamma_1,\dots,\gamma_k\}$ to be $\overline{X} = \cup_{i=1}^{k}\overline{\gamma_i}$ and we denote by $|\overline{X}|$ the size of the vertex set of $\overline{X}$. Because X is a cluster the incompatibility graph induced by $\gamma_1,\dots,\gamma_k$ is connected, which by definition of torus-compatibility implies that $\overline{X}$ is connected, because the γ_i are small contours and thus themselves connected.

Theorem 6.2.34. *Let $C > 0$, $d \in \mathbb{Z}_{\geq 2}$ and $\varphi \in \{$even, odd$\}$. Let $\mathcal{T} \in \mathbf{T}_d(C)$ and let $\Lambda \subseteq \mathcal{T}$ be any induced closed subgraph. For all $z \in \mathbb{C}$ with $|z| < \delta_1(d,C)$ the cluster expansion for $\log Z^{\varphi}_{trunc}(\Lambda;z)$ is convergent, where $\delta_1(d,C)$ defined in Definition 6.2.31. Furthermore for any $v \in \Lambda$ and any $|z| < \delta_1(d,C)$ we have*
$$\sum_{\substack{X\in\mathcal{C}^{\varphi}_{\Lambda}(\mathcal{T},C)\\ v\in\overline{X}}} |\Phi(X;z)| e^{\sum_{\gamma\in X} C|\overline{\gamma}|} \leq 2.$$

Proof. Fix $v \in \Lambda$. Define the artificial contour v_γ with support v, weight 0, and which is torus incompatible with each small contour γ for which $v \in V(\overline{\gamma})$. Add v_γ to the set of small C-stable contours of type φ in Λ. With the artificial contour added, $Z^{\varphi}_{\text{trunc}}(\Lambda;z)$ is still equal to the sum over torus-compatible collections of small contours of type φ, as the weight of v_γ is zero. For $|z| < \delta_1(d,C)$ we verify the condition of Theorem 6.2.1 with $a(\gamma) = 4d|\overline{\gamma}|$ and $b(\gamma) = C|\overline{\gamma}|$. For any contour γ

$$\begin{aligned}\sum_{\gamma' \not\sim \gamma} |w(\gamma'; z)| e^{a(\gamma')+b(\gamma')} &\le \sum_{\gamma' \not\sim \gamma} |z|^{||\gamma'||} e^{5 \cdot e^{-C3^d} \cdot |\gamma'|} e^{a(\gamma')+b(\gamma')} \\ &\le \sum_{\gamma' \not\sim \gamma} |z|^{\rho |\overline{\gamma'}|} e^{(4d+5 \cdot e^{-C3^d}+C)|\overline{\gamma'}|},\end{aligned}$$

where the sums run over non-artificial contours γ'. In the final inequality we used $||\gamma'|| \ge \rho|\overline{\gamma'}|$. Since $|z| < \delta_1(d, C)$ we have

$$\sum_{\gamma' \not\sim \gamma} |z|^{\rho |\overline{\gamma'}|} e^{(4d+5 \cdot e^{-C3^d}+C)|\overline{\gamma'}|} < \sum_{\gamma' \not\sim \gamma} e^{-\log(2C_d)|\overline{\gamma'}|}.$$

There are at most $(|\overline{\gamma}| + |\partial^c \overline{\gamma}|) C_d^m$ small contours $\gamma' \not\sim \gamma$ with $|\overline{\gamma'}| = m$, where C_d is the constant from Lemma 6.2.29. This can be seen by upper bounding the number of small contours that is torus incompatible with a single vertex and applying this bound for each vertex of $\overline{\gamma} \cup \partial^c \overline{\gamma}$. Note that $|\partial^c \overline{\gamma}| < (2d-1)|\overline{\gamma}|$. Hence

$$\begin{aligned}\sum_{\gamma' \not\sim \gamma} e^{-\log(2C_d)|\overline{\gamma'}|} &< (|\overline{\gamma}| + |\partial^c \overline{\gamma}|) \cdot \sum_{m \ge 0} (C_d)^m e^{-\log(2C_d) m} \\ &\le 2(|\overline{\gamma}| + |\partial^c \overline{\gamma}|) \le 4d|\overline{\gamma}| = a(\gamma).\end{aligned}$$

This shows the condition of Theorem 6.2.1 holds, which implies the cluster expansion is convergent for $|z| < \delta_1(d, C)$. By Theorem 6.2.1 and the definition of v_γ we have for any $v \in \Lambda$ and any $|z| < \delta_1(d, C)$ we obtain

$$\sum_{\substack{X \in \mathcal{C}^\varphi_\Lambda(\mathcal{T}, C) \\ v \in \overline{X}}} |\Phi(X; z)| e^{\sum_{\gamma \in X} C|\overline{\gamma}|} = \sum_{\substack{X \in \mathcal{C}^\varphi_\Lambda(\mathcal{T}, C) \\ X \not\sim v_\gamma}} |\Phi(X; z)| e^{b(X)} \le a(v_\gamma) = 2.$$

□

All contours are stable

To prove that all contours are stable we need some estimates on certain subseries of the cluster expansion.

Lemma 6.2.35. *Let $C > 0$ and let $\varphi \in \{even, odd\}$. Then for any $z \in \mathbb{C}$ with $|z| < \delta_1(d, C)$ the limit*

$$\lim_{n \to \infty} \sum_{\substack{X \in \mathcal{C}^\varphi(\mathbb{Z}^d_n, C) \\ \vec{0} \in \overline{X},\ |\overline{X}| < n}} \frac{\Phi(X; z)}{|\overline{X}|}$$

exists and is an analytic function of z.

Proof. First note for any $C > 0$, there is an $N = N(d) > 0$ such that for all $n \geq N$ we have $\mathbb{Z}_n^d \in \mathbf{T}_d(C)$, as for large enough n we have $e^{C\cdot n} \geq n^d$. For each $n \geq N$ and each $z \in \mathbb{C}$ with $|z| < \delta_1(d, C)$ define the series

$$S_n(z) := \sum_{\substack{X \in \mathcal{C}^\varphi(\mathbb{Z}_n^d, C) \\ \vec{0} \in \overline{X},\ |\overline{X}| < n}} \frac{\Phi(X; z)}{|\overline{X}|}.$$

By Theorem 6.2.34 we see $|S_n(z)| \leq 2$ for all $n \geq N$ and all $|z| < \delta_1(d, C)$. Thus the family of maps $\{S_n\}_{n \geq N}$ is normal on $B_{\delta_1(d,C)}$, where B_r denotes the open disk centered at 0 with radius r. For $n_2 > n_1$ any cluster X of small contours in $\mathbb{Z}_{n_1}^d$ with $|\overline{X}| < n_1$ and $\vec{0} \in \overline{X}$ can be unambiguously viewed as a cluster in $\mathbb{Z}_{n_2}^d$ with $|\overline{X}| < n_1$ and $\vec{0} \in \overline{X}$. From this and the fact that for any contour γ we have $||\gamma|| \geq \rho|\overline{\gamma}|$, where $\rho = \rho(d)$ denotes the constant from Lemma 6.2.30, we see for $n_2 > n_1$ that the first ρn_1 coefficients of the power series expansions of $S_{n_1}(z)$ and $S_{n_2}(z)$ are the same. Hence the coefficients of $S_n(z)$ are stabilizing, which implies that every convergent subsequence of S_n converges to the same limit. Normality implies that the entire sequence converges to this limit. $\square$

Definition 6.2.36. We denote the the limit function in the lemma above by $f_{\varphi,C}(z)$.

In fact, for the definition of $f_{\varphi,C}(z)$ one can take any sequence of tori $\mathcal{T} \in \mathbf{T}_d(C)$ with increasing minimal side length ℓ_1, as is implied by the following lemma.

Lemma 6.2.37. *Let $C > 0$ and let $\mathcal{T} \in \mathbf{T}_d(C)$. Denote the smallest side length of $\mathcal{T}$ by ℓ_1 and let $\varphi \in \{$even, odd$\}$. For any $|z| < \delta_1(d, C)$ we have*

$$\sum_{\substack{X \in \mathcal{C}^\varphi(\mathcal{T}, C) \\ \vec{0} \in \overline{X},\ |\overline{X}| < \ell_1}} \frac{\Phi(X; z)}{|\overline{X}|} = \sum_{\substack{X \in \mathcal{C}^\varphi(\mathbb{Z}_{\ell_1}^d, C) \\ \vec{0} \in \overline{X},\ |\overline{X}| < \ell_1}} \frac{\Phi(X; z)}{|\overline{X}|}.$$

Proof. As $\mathcal{T} \in \mathbf{T}_d(C)$ with minimal side length ℓ_1, we have $\mathbb{Z}_{\ell_1}^d \in \mathbf{T}_d(C)$. Hence Theorem 6.2.34 implies that both series are convergent for $|z| < \delta_1(d, C)$. Any cluster X in either $\mathcal{C}^\varphi(\mathbb{Z}_{\ell_1}^d, C)$ or $\mathcal{C}^\varphi(\mathcal{T}, C)$ with $\vec{0} \in \overline{X}$ and $|\overline{X}| < \ell_1$ can unambiguously be viewed as being supported on $\{-(\ell_1 - 1), \ldots, \ell_1 - 1\}^d$ because $\overline{X}$ is connected. This yields a weight preserving bijection between the two sets, which implies the equality holds for all $|z| < \delta_1(d, C)$. $\square$

The following estimate is well-known in the statistical physics literature. It is for example used in the proof of Lemma 5.3 of [BI89], though no formal proof is given there. The proof we provide here is based on Section 5.7.1 in [FV17], adapted to our setting.

Theorem 6.2.38. *Let $C > 0$ and let $\mathcal{T} \in \mathbf{T}_d(C)$. Denote the smallest side length of $\mathcal{T}$ by ℓ_1 let $\varphi \in \{even, odd\}$. Let $\Lambda \subseteq \mathcal{T}$ be an induced closed subgraph. For any $|z| < \delta_1(d, C)$ we have*

$$|\log Z^{\varphi}_{trunc}(\Lambda; z) - |\Lambda^{\circ}_{even}| f_{\varphi,C}(z) - |\Lambda^{\circ}_{odd}| f_{\overline{\varphi},C}(z)| \leq |\partial\Lambda| \cdot 2 \cdot e^{-C3^d} + |\Lambda^{\circ}| \frac{4}{\ell_1 e^{C\ell_1}},$$

where $f_{\varphi}(z)$ and $f_{\overline{\varphi}}(z)$ are the functions defined in Definition 6.2.36.

Proof. For $|z| < \delta_1(d, C)$ we have the following equalities of convergent power series

$$\log Z^{\varphi}_{\text{trunc}}(\Lambda; z) = \sum_{X \in \mathcal{C}^{\varphi}_{\Lambda}(\mathcal{T},C)} \Phi(X; z) = \sum_{v \in \Lambda^{\circ}} \sum_{\substack{X \in \mathcal{C}^{\varphi}_{\Lambda}(\mathcal{T},C) \\ v \in \overline{X}}} \frac{\Phi(X; z)}{|\overline{X}|} =$$

$$\sum_{v \in \Lambda^{\circ}} \Big(\sum_{\substack{X \in \mathcal{C}^{\varphi}(\mathcal{T},C) \\ v \in \overline{X}}} \frac{\Phi(X; z)}{|\overline{X}|} - \sum_{\substack{X \in \mathcal{C}^{\varphi}(\mathcal{T},C) \\ v \in \overline{X} \not\subset \Lambda^{\circ}}} \frac{\Phi(X; z)}{|\overline{X}|} \Big) =$$

$$|\Lambda^{\circ}_{\text{even}}| \sum_{\substack{X \in \mathcal{C}^{\varphi}(\mathcal{T},C) \\ \vec{0} \in \overline{X}}} \frac{\Phi(X; z)}{|\overline{X}|} + |\Lambda^{\circ}_{\text{odd}}| \sum_{\substack{X \in \mathcal{C}^{\overline{\varphi}}(\mathcal{T},C) \\ \vec{0} \in \overline{X}}} \frac{\Phi(X; z)}{|\overline{X}|} - \sum_{v \in \Lambda^{\circ}} \sum_{\substack{X \in \mathcal{C}^{\varphi}(\mathcal{T},C) \\ v \in \overline{X} \not\subset \Lambda^{\circ}}} \frac{\Phi(X; z)}{|\overline{X}|},$$

where in the final equality we use that a cluster of contours containing $v \in \Lambda^{\circ}$ can be translated to a cluster of contours containing $\vec{0}$; see Remark 11.

We prove the following bounds:

$$|\sum_{v \in \Lambda^{\circ}} \sum_{\substack{X \in \mathcal{C}^{\varphi}(\mathcal{T},C) \\ v \in \overline{X} \not\subset \Lambda^{\circ}}} \frac{\Phi(X; z)}{|\overline{X}|}| \leq |\partial\Lambda| \cdot 2 \cdot e^{-C3^d}, \tag{6.7}$$

and for $\xi \in \{\varphi, \overline{\varphi}\}$,

$$\left| \sum_{\substack{X \in \mathcal{C}^{\xi}(\mathcal{T},C) \\ \vec{0} \in \overline{X}}} \frac{\Phi(X; z)}{|\overline{X}|} - f_{\xi,C}(z) \right| \leq \frac{4}{\ell_1 e^{C\ell_1}}. \tag{6.8}$$

Since $|\Lambda^{\circ}| = |\Lambda^{\circ}_{\text{even}}| + |\Lambda^{\circ}_{\text{odd}}|$ these bounds together complete the proof.

To prove (6.7) we bound

$$\left|\sum_{v\in\Lambda^\circ}\sum_{\substack{X\in\mathcal{C}^\varphi(\mathcal{T},C)\\ v\in\overline{X}\not\subset\Lambda^\circ}}\frac{\Phi(X;z)}{|\overline{X}|}\right| = \left|\sum_{\substack{X\in\mathcal{C}^\varphi(\mathcal{T},C)\\ \overline{X}\not\subset\Lambda^\circ}}\sum_{v\in\Lambda^\circ}\frac{\Phi(X;z)\mathbf{1}_{\overline{X}}(v)}{|\overline{X}|}\right| =$$

$$\left|\sum_{\substack{X\in\mathcal{C}^\varphi(\mathcal{T},C)\\ \overline{X}\not\subset\Lambda^\circ,\ \overline{X}\cap\Lambda^\circ\neq\emptyset}}\frac{\Phi(X;z)|\overline{X}\cap\Lambda^\circ|}{|\overline{X}|}\right| \leq \sum_{w\in\partial\Lambda}\sum_{\substack{X\in\mathcal{C}^\varphi(\mathcal{T},C)\\ w\in\overline{X}\not\subset\Lambda^\circ}}\left|\frac{\Phi(X;z)|\overline{X}\cap\Lambda^\circ|}{|\overline{X}|}\right| \leq$$

$$\sum_{w\in\partial\Lambda}\sum_{\substack{X\in\mathcal{C}^\varphi(\mathcal{T},C)\\ w\in\overline{X}\not\subset\Lambda^\circ}}|\Phi(X;z)| \leq |\partial\Lambda|\max_{w\in\partial\Lambda}\sum_{\substack{X\in\mathcal{C}^\varphi(\mathcal{T},C)\\ w\in\overline{X}\not\subset\Lambda^\circ}}|\Phi(X;z)| \leq |\partial\Lambda|\cdot 2\cdot e^{-C3^d},$$

where the last inequality follows from Theorem 6.2.34 using that any cluster X with $w\in\overline{X}$ satisfies $\sum_{\gamma\in X}|\overline{\gamma}|\geq 3^d$.

Next we show (6.8). We split the clusters in $\mathcal{T}$ based on size and use the triangle inequality

$$\left|\sum_{\substack{X\in\mathcal{C}^\xi(\mathcal{T},C)\\ \vec{0}\in\overline{X}}}\frac{\Phi(X;z)}{|\overline{X}|}-f_{\xi,C}(z)\right| \leq \left|\sum_{\substack{X\in\mathcal{C}^\xi(\mathcal{T},C)\\ \vec{0}\in\overline{X},\ |\overline{X}|<\ell_1}}\frac{\Phi(X;z)}{|\overline{X}|}-f_{\xi,C}(z)\right|$$

$$+\left|\sum_{\substack{X\in\mathcal{C}^\xi(\mathcal{T},C)\\ \vec{0}\in\overline{X},\ |\overline{X}|\geq\ell_1}}\frac{\Phi(X;z)}{|\overline{X}|}\right| \leq \left|\sum_{\substack{X\in\mathcal{C}^\xi(\mathcal{T},C)\\ \vec{0}\in\overline{X},\ |\overline{X}|<\ell_1}}\frac{\Phi(X;z)}{|\overline{X}|}-f_{\xi,C}(z)\right| + \frac{2}{\ell_1\cdot e^{C\ell_1}} =$$

$$\left|\sum_{\substack{X\in\mathcal{C}^\xi(\mathbb{Z}^d_{\ell_1},C)\\ \vec{0}\in\overline{X},\ |\overline{X}|<\ell_1}}\frac{\Phi(X;z)}{|\overline{X}|}-f_{\xi,C}(z)\right| + \frac{2}{\ell_1\cdot e^{C\ell_1}},$$

where the last inequality follows from Theorem 6.2.34 and the last equality follows from Lemma 6.2.37.

For any $\varepsilon>0$ there is an ℓ^* large enough such that for any $|z|\leq\delta_1(d,C)$ we have

$$\left|\sum_{\substack{X\in\mathcal{C}^\xi(\mathbb{Z}^d_{\ell^*},C)\\ \vec{0}\in\overline{X},\ |\overline{X}|<\ell^*}}\frac{\Phi(X;z)}{|\overline{X}|}-f_{\xi,C}(z)\right|\leq\varepsilon,$$

by Lemma 6.2.35. By increasing ℓ^* if necessary we may assume $\ell^*>2\ell_1$. We have

$$\left| \sum_{\substack{X \in \mathcal{C}^{\xi}(\mathbb{Z}^d_{\ell_1}, C) \\ \vec{0} \in \overline{X},\ |\overline{X}| < \ell_1}} \frac{\Phi(X;z)}{|\overline{X}|} - f_{\xi,C}(z) \right| \leq \left| \sum_{\substack{X \in \mathcal{C}^{\xi}(\mathbb{Z}^d_{\ell_1}, C) \\ \vec{0} \in \overline{X},\ |\overline{X}| < \ell_1}} \frac{\Phi(X;z)}{|\overline{X}|} - \sum_{\substack{X \in \mathcal{C}^{\xi}(\mathbb{Z}^d_{\ell^*}, C) \\ \vec{0} \in \overline{X},\ |\overline{X}| < \ell^*}} \frac{\Phi(X;z)}{|\overline{X}|} \right|$$

$$+ \varepsilon = \left| \sum_{\substack{X \in \mathcal{C}^{\xi}(\mathbb{Z}^d_{\ell^*}, C) \\ \vec{0} \in \overline{X},\ \ell_1 \leq |\overline{X}| < \ell^*}} \frac{\Phi(X;z)}{|\overline{X}|} \right| + \varepsilon \leq \frac{2}{\ell_1 e^{C\ell_1}} + \varepsilon,$$

where we used Theorem 6.2.34 in the last inequality. As this holds for any $\varepsilon > 0$, we see

$$\left| \sum_{\substack{X \in \mathcal{C}^{\xi}(\mathcal{T}, C) \\ \vec{0} \in \overline{X},\ |\overline{X}| < \ell_1}} \frac{\Phi(X;z)}{|\overline{X}|} - f_{\xi,C}(z) \right| \leq \frac{2}{\ell_1 e^{C\ell_1}}.$$

This finishes the proof of (6.8). □

With this bound, we can now finally show that all contours are C-stable.

Theorem 6.2.39. *Let $C > 0$ and let $\mathcal{T} \in \mathbf{T}_d(C)$. Let $\Lambda \subseteq \mathcal{T}$ be an induced closed subgraph of $\mathcal{T}$ and let $\varphi \in \{even, odd\}$ be a ground state. For all $|z| < \delta_1(d, C)$, we have*

$$Z^{\varphi}_{match}(\Lambda; z) = Z^{\varphi}_{trunc}(\Lambda; z)$$

and

$$Z^{large}_{match}(\mathcal{T}; z) = Z^{large}_{trunc}(\mathcal{T}; z).$$

Proof. We first prove by induction on $|\Lambda|$ that all small contours of type φ are C-stable. The base case follows as the weight of an empty contour is 1. Suppose the claim holds for all Λ with $|\Lambda| \leq k$ for some $k \geq 0$. Let Λ be such that $|\Lambda| = k + 1$ and take any small contour γ of type φ in Λ. We aim to bound $|w(\gamma; z)|$ by $|z|^{||\gamma||} e^{5 \cdot e^{-C3^d} |\overline{\gamma}|}$, which shows γ is C-stable. By the induction hypothesis we see

$$\left| \frac{w(\gamma; z)}{z^{||\gamma||}} \right| = \left| \frac{Z^{\overline{\varphi}}_{\text{match}}(\text{int}_{\overline{\varphi}}(\gamma); z)}{Z^{\varphi}_{\text{match}}(\text{int}_{\overline{\varphi}}(\gamma); z)} \right| = \left| \frac{Z^{\overline{\varphi}}_{\text{trunc}}(\text{int}_{\overline{\varphi}}(\gamma); z)}{Z^{\varphi}_{\text{trunc}}(\text{int}_{\overline{\varphi}}(\gamma); z)} \right|.$$

Write $V = \text{int}_{\overline{\varphi}}(\gamma)$. For any $\varepsilon > 0$ there exists ℓ large enough such that $|V^{\circ}| \frac{4}{\ell e^{C\ell}} < \varepsilon$ and such that V is isomorphic to an induced closed subgraph of $\mathbb{Z}^d_{\ell}$. Fix such an ℓ.

Define

$$h_{\varphi,C}(V; z) = |V^{\circ}_{\text{even}}| f_{\varphi,C}(z) + |V^{\circ}_{\text{odd}}| f_{\overline{\varphi},C}(z),$$

where $f_{\varphi,C}(z)$ and $f_{\overline{\varphi},C}(z)$ denote the functions defined in Definition 6.2.36. We also write $g_C(z) = f_{\varphi,C}(z) - f_{\overline{\varphi},C}(z)$. By Theorem 6.2.34 and Lemma 6.2.35 we see for any $z \in \mathbb{C}$ with $|z| \leq \delta_1(d, C)$ we have

$$|g_C(z)| \leq |f_{\varphi,C}(z)| + |f_{\overline{\varphi},C}(z)| \leq \frac{4}{3^d} \cdot e^{-C3^d} \leq e^{-C3^d},$$

using the fact that any cluster X with $\vec{0} \in \overline{X}$ satisfies $|\overline{X}| \geq 3^d$.

Theorem 6.2.38 applied to V as a induced closed subgraph of $\mathbb{Z}_\ell^d$ now gives

$$\begin{aligned}
&\left|\frac{Z^{\overline{\varphi}}_{\text{trunc}}(V;z)}{Z^{\varphi}_{\text{trunc}}(V;z)}\right| = \left|\frac{e^{\log Z^{\overline{\varphi}}_{\text{trunc}}(V;z) - h_{\overline{\varphi},C}(V;z)}}{e^{\log Z^{\varphi}_{\text{trunc}}(V;z) - h_{\varphi,C}(V;z)}}\right| \cdot \left|\frac{e^{h_{\overline{\varphi},C}(V;z)}}{e^{h_{\varphi,C}(V;z)}}\right| \leq \\
&\frac{e^{|\log Z^{\overline{\varphi}}_{\text{trunc}}(V;z) - h_{\overline{\varphi},C}(V;z)|}}{e^{-|\log Z^{\varphi}_{\text{trunc}}(V;z) - h_{\varphi,C}(V;z)|}} \cdot \left|\frac{e^{h_{\overline{\varphi},C}(V;z)}}{e^{h_{\varphi,C}(V;z)}}\right| < \frac{e^{2e^{-C3^d}|\partial V| + |V^\circ|\frac{4}{\ell e^{C\ell}}}}{e^{-2e^{-C3^d}|\partial V| - |V^\circ|\frac{4}{\ell e^{C\ell}}}} \left|\frac{e^{h_{\overline{\varphi},C}(V;z)}}{e^{h_{\varphi,C}(V;z)}}\right| \leq \\
&e^{4e^{-C3^d}\cdot|\partial V| + |V^\circ \frac{8}{\ell e^{C\ell}}} \cdot \left|\frac{e^{h_{\overline{\varphi},C}(V;z)}}{e^{h_{\varphi,C}(V;z)}}\right|.
\end{aligned} \tag{6.9}$$

Using that $|V^\circ|\frac{4}{\ell e^{C\ell}} < \varepsilon$ and the definitions of $h_{\varphi,C}(V;z)$ and $g_C(z)$ we can further bound this by

$$e^{4\cdot e^{-C3^d}|\partial V|} \cdot e^{2\varepsilon} \cdot \left|e^{(|V^\circ_{\text{even}}| - |V^\circ_{\text{odd}}|)g_C(z)}\right| \leq e^{4\cdot e^{-C3^d}|\partial V|} \cdot e^{2\varepsilon} \cdot e^{e^{-C3^d}||V^\circ_{\text{even}}| - |V^\circ_{\text{odd}}||}. \tag{6.10}$$

We next claim that for any induced closed subgraph $V \subseteq \mathcal{T}$ it holds that

$$||V^\circ_{\text{even}}| - |V^\circ_{\text{odd}}|| < |\partial V|. \tag{6.11}$$

Indeed, define $e(A)$ for $A \subseteq V$ to be the set of edges of $\mathcal{T}$ with at least one endpoint in A. We have $|e(V)| = |e(V^\circ_{\text{even}})| + |e((\partial V)_{\text{even}})| = |e(V^\circ_{\text{odd}})| + |e((\partial V)_{\text{odd}})|$. As each vertex in V° has degree $2d$ and each vertex in ∂V has degree strictly less than $2d$ we see that

$$\begin{aligned}
||V^\circ_{\text{even}}| - |V^\circ_{\text{odd}}|| &= \frac{1}{2d}\,||e(V^\circ_{\text{even}})| - |e(V^\circ_{\text{odd}})|| \\
&= \frac{1}{2d}\,||e((\partial V)_{\text{odd}})| - |e((\partial V)_{\text{even}})|| < |\partial V|,
\end{aligned}$$

proving (6.11). Substituting (6.11) into (6.10) we get for any $\varepsilon > 0$,

$$\left|\frac{Z^{\overline{\varphi}}_{\text{trunc}}(V;z)}{Z^{\varphi}_{\text{trunc}}(V;z)}\right| < e^{2\varepsilon} \cdot e^{5\cdot e^{-C3^d}\cdot|\partial V|} \leq e^{2\varepsilon} \cdot e^{5\cdot e^{-C3^d}\cdot|\overline{\gamma}|}.$$

As $\varepsilon \to 0$ we get

$$\left| \frac{Z^{\overline{\varphi}}_{\text{trunc}}(V;z)}{Z^{\varphi}_{\text{trunc}}(V;z)} \right| \le e^{5 \cdot e^{-C3^d} \cdot |\overline{\gamma}|}$$

and hence we see that the small contour γ is C-stable.

Now let γ be a large contour. As we already proved that for any induced closed subgraph $\Lambda \subseteq \mathcal{T}$ all small contours are C-stable, we obtain

$$\left| \frac{w(\gamma;z)}{z^{||\gamma||}} \right| = \left| \frac{Z^{\text{odd}}_{\text{match}}(\text{int}_{\text{odd}}(\gamma);z)}{Z^{\text{even}}_{\text{match}}(\text{int}_{\text{odd}}(\gamma);z)} \right| = \left| \frac{Z^{\text{odd}}_{\text{trunc}}(\text{int}_{\text{odd}}(\gamma);z)}{Z^{\text{even}}_{\text{trunc}}(\text{int}_{\text{odd}}(\gamma);z)} \right|.$$

Again write $V = \text{int}_{\text{odd}}(\gamma)$, and write

$$h_{\varphi,C}(V;z) = |V^\circ_{\text{even}}| f_{\varphi,C}(z) + |V^\circ_{\text{odd}}| f_{\overline{\varphi},C}(z),$$

and $g_C(z) = f_{\varphi,C}(z) - f_{\overline{\varphi},C}(z)$. As above we have $|g_C(z)| \le e^{-C3^d}$ for any $z \in \mathbb{C}$ with $|z| \le \delta_1(d,C)$.

By Theorem 6.2.38, now applied to V as an induced closed subgraph of $\mathcal{T}$, and thus replacing ℓ by ℓ_1 in (6.9) we obtain,

$$\begin{aligned}
\left| \frac{Z^{\overline{\varphi}}_{\text{trunc}}(V;z)}{Z^{\varphi}_{\text{trunc}}(V;z)} \right| &= e^{4 \cdot e^{-C3^d}|\partial V| + |V^\circ| \frac{8}{\ell_1 e^{C\ell_1}}} \cdot \left| e^{(|V^\circ_{\text{even}}| - |V^\circ_{\text{odd}}|) g_C(z)} \right| \\
&\le e^{4 \cdot e^{-C3^d}|\partial V| + |V^\circ| \frac{8}{\ell_1 e^{C\ell_1}}} \cdot e^{e^{-C3^d}|\partial V|} \\
&\le e^{5 \cdot e^{-C3^d} \cdot |\partial V|} \cdot e^{|V^\circ| \frac{8}{\ell_1 e^{C\ell_1}}} \le e^{5 \cdot e^{-C3^d} \cdot |\overline{\gamma}|} \cdot e^4,
\end{aligned}$$

using (6.11) and the bound on $g_C(z)$ for the second to last inequality and $e^{C\ell_1} \ge |\mathcal{T}| \ge |V^\circ|$ and $\ell_1 \ge 2$ for the final inequality. Therefore each large contour γ is C-stable. $\square$

6.3 Bounded zeros for balanced tori

In this section we prove the zeros of families of balanced tori are bounded, building on the framework and results of the previous section.

Recall by Lemma 6.2.28 and Theorem 6.2.39 that we have

$$Z_{\text{match}}(\mathcal{T};z) = 2Z^{\text{even}}_{\text{trunc}}(\mathcal{T};z) + Z^{\text{large}}_{\text{trunc}}(\mathcal{T};z).$$

Our first aim this section is to bound $|Z^{\text{large}}_{\text{match}}(\mathcal{T};z))|$ away from $2|Z^{\text{even}}_{\text{trunc}}(\mathcal{T};z)|$, these bounds are the final ingredient we need to prove the zeros of families of balanced tori are bounded.

To obtain bounds on $|Z^{\mathrm{large}}_{\mathrm{match}}(\mathcal{T};z))|$ we apply the Kotecký-Preiss theorem to $\log(Z^{\mathrm{even}}_{\mathrm{match}}(\mathcal{T};z)+Z^{\mathrm{large}}_{\mathrm{match}}(\mathcal{T};z))$, which in turn means we need to bound the number of relevant contours.

Lemma 6.3.1. *Let $\mathcal{T}$ be an even d-dimensional torus. Let L_m denote the set of contours γ in $\mathcal{T}$ containing $\vec{0}$ with support of size m that are either large or small and of even type. Then we have*

$$|L_m| \le (4C_d|\mathcal{T}|^{1/\ell_1})^m,$$

where C_d is the constant from Lemma 6.2.29.

Proof. Let k denote the number of connected components of a large contour γ_{large} with $|\overline{\gamma_{\mathrm{large}}}| = m$. Each connected component of γ_{large} has size at least ℓ_1, hence $k \le \lfloor m/\ell_1 \rfloor$. Denote by m_i the size of the i-th connected component for $i \in \{1,\dots,k\}$. For each connected component of the large contour that does not contain $\vec{0}$ choose a vertex v_i of $\mathcal{T}$ in the component for $i \in \{1,\dots,k-1\}$, this can be done in $|\mathcal{T}|^{k-1}$ many ways.

Denote by P_l the set of connected large contours of size l incompatible with a specified vertex v. The number of connected sets in $\mathcal{T}$ of size l containing v is bounded by the number of connected sets of size l containing $\vec{0}$ in $\mathbb{Z}^d$. As there are at most 2^l possible feasible configurations on a set of size l, we obtain with the same argument as in Lemma 6.2.29 that $|P_l| \le C_d^l$. We apply this bound to each connected component and see the total number of large contours γ in $\mathcal{T}$ with support of size m containing $\vec{0}$ is bounded by

$$\sum_{\substack{m_1,\dots,m_k \\ \sum m_i = m \text{ and } m_i \ge \ell_1}} \prod_{i=1}^{k} C_d^{m_i}(|\mathcal{T}|)^{k-1} \le \Big(\sum_{\substack{m_1,\dots,m_k \\ \sum m_i = m}} 1\Big) C_d^m |\mathcal{T}|^{k-1} \le 4^m C_d^m |\mathcal{T}|^{k-1}. \tag{6.12}$$

Accounting also for the small even contours of size m, we get

$$|L_m| \le 4^m C_d^m (|\mathcal{T}|)^k \le (4C_d|\mathcal{T}|^{1/\ell_1})^m.$$

□

We also need a tighter bound on the absolute value of $|z|$.

Definition 6.3.2. We define for any $x > 0$ the number

$$\delta_2(d,x) = e^{-(\log(8e^x C_d)+4de^4+5\cdot e^{-x3^d})/\rho(d)},$$

where C_d is the constant from Lemma 6.2.29 and $\rho(d) = \frac{1}{2d\cdot 3^d}$ is the constant from Lemma 6.2.30.

Note that $\delta_1(d,x) > \delta_2(d,x)$ for all $d \in \mathbb{Z}_{\geq 2}$ and $x > 0$. We apply the framework of Section 6.2.1 to $\log(Z^{\text{even}}_{\text{match}}(\mathcal{T};z) + Z^{\text{large}}_{\text{match}}(\mathcal{T};z))$. In this case our polymers are contours of type even, i.e. both large and small. The weights of a contour γ equals $w(\gamma;z)$ and the compatibility relation is torus-compatibility. Denote by $\mathcal{C}^{\text{even}}_{\text{large}}(\mathcal{T})$ the set of clusters of even and large contours. The cluster expansion takes the form

$$\log(Z^{\text{even}}_{\text{match}}(\mathcal{T};z) + Z^{\text{large}}_{\text{match}}(\mathcal{T};z)) = \sum_{X \in \mathcal{C}^{\text{even}}_{\text{large}}(\mathcal{T})} \Phi(X;z), \tag{6.13}$$

where $\Phi(X;z) = \prod_{\gamma \in \Gamma} \frac{1}{n_X(\gamma)!}\psi(\gamma_1,\ldots,\gamma_n)\prod_{i=1}^n w(\gamma_i;z)$ is defined as in Section 6.2.1.

Theorem 6.3.3. *Let $C > 0$ and let $\mathcal{T} \in \mathbf{T}_d(C)$. For any $|z| < \delta_2(d,C)$ the cluster expansion for $\log(Z^{even}_{match}(\mathcal{T};z) + Z^{large}_{match}(\mathcal{T};z))$ is convergent, where $\delta_2(d,C)$ is defined in Definition 6.3.2. Furthermore for any $v \in V(\mathcal{T})$*

$$\sum_{\substack{X \in \mathcal{C}^{even}_{large}(\mathcal{T}) \\ v \in \overline{X}}} |\Phi(X;z)| \leq 4de^4.$$

Proof. Fix $v \in V(\mathcal{T})$. Define the artificial contour v_γ with support v, weight 0 and which is torus incompatible with each contour γ such that $v \in V(\overline{\gamma})$. Add v_γ to the set of contours. With the artificial contour added, $Z^{\text{even}}_{\text{match}}(\mathcal{T};z) + Z^{\text{large}}_{\text{match}}(\mathcal{T};z)$ is still equal to the sum over torus-compatible collections of large and even contours, as the weight of v_γ is zero. Throughout this proof $\sim$ denotes the relation of torus-compatibility. We verify the condition of Theorem 6.2.1 with $a(\gamma) = 4de^4|\overline{\gamma}|$ and $b(\gamma) = 0$.

Theorem 6.2.39 applies as $\delta_2(d,C) < \delta_1(d,C)$. Hence for any contour γ

$$\begin{aligned}\sum_{\gamma' \nsim \gamma} |w(\gamma';z)| e^{a(\gamma)+b(\gamma)} &\leq \sum_{\gamma' \nsim \gamma} |z|^{\rho|\overline{\gamma'}|} e^{(4de^4 + 5\cdot e^{-C3^d})|\overline{\gamma'}|} \cdot e^4 \\ &\leq \sum_{\gamma' \nsim \gamma} |z|^{\rho|\overline{\gamma'}|} e^{(4de^4 + 5\cdot e^{-C3^d})|\overline{\gamma'}|} \cdot e^4,\end{aligned}$$

where without loss of generality we may assume the first sum is over non-artificial contours γ', as $w(v_\gamma;z) = 0$. As $|z| < \delta_2(d,C) = e^{-(\log(8e^C C_d) + 4de^4 + 5\cdot e^{-C3^d})/\rho}$, we have

$$\sum_{\gamma' \nsim \gamma} |z|^{\rho|\overline{\gamma'}|} e^{(4de^4 + 5\cdot e^{-C3^d})|\overline{\gamma'}|} e^4 < e^4 \sum_{\gamma' \nsim \gamma} (8e^C C_d)^{-|\overline{\gamma'}|}.$$

There are at most $(|\overline{\gamma}| + |\partial^c \overline{\gamma}|)(4|\mathcal{T}|^{1/\ell_1} C_d)^m$ contours $\gamma' \nsim \gamma$ with $|\overline{\gamma'}| = m$, where C_d is the constant from Lemma 6.2.29, this can be seen by upper bounding how many ways a contour can be torus incompatible with a single vertex using Lemma 6.3.1 and applying this bound for each vertex of $\overline{\gamma}$. We also note that as $\mathcal{T}$ is C-balanced, we have $|\mathcal{T}|^{1/\ell_1} \leq e^C$. Hence

$$\sum_{\gamma' \not\sim \gamma} (8e^C C_d)^{-|\overline{\gamma'}|} < e^4(|\overline{\gamma}| + |\partial^c \overline{\gamma}|) \cdot \sum_{m \geq 0} (4|\mathcal{T}|^{1/\ell_1} C_d)^m (8e^C C_d)^{-m}$$
$$\leq 2de^4 |\overline{\gamma}| \cdot \sum_{m \geq 0} \left(\tfrac{1}{2}\right)^m = a(\gamma),$$

where we used $|\overline{\gamma}| + |\partial^c \overline{\gamma}| \leq 2d|\overline{\gamma}|$.

This shows the condition of Theorem 6.2.1 holds, which implies the cluster expansion is convergent for $|z| < \delta_2(d, C)$. By Theorem 6.2.1 and the definition of v_γ we have for any $v \in \mathcal{T}$ we have

$$\sum_{\substack{X \in \mathcal{C}^{\text{even}}_{\text{large}}(\mathcal{T}) \\ v \in \overline{X}}} |\Phi(X; z)| e^{\sum_{\gamma \in X} b(\gamma)} = \sum_{\substack{X \in \mathcal{C}^{\text{even}}_{\text{large}}(\mathcal{T}) \\ X \not\sim v_\gamma}} |\Phi(X; z)| \leq a(v_\gamma) = 4de^4,$$

where we can assume the clusters X do not contain v_γ, as for any cluster X containing v_γ we have $\Phi(X; z) = 0$ as $w(v_\gamma; z) = 0$. □

Lemma 6.3.4. *Let $C > 0$. The family*

$$\left\{ \frac{\log(Z^{even}_{match}(\mathcal{T}; z) + Z^{large}_{match}(\mathcal{T}; z))}{|\mathcal{T}|} \right\}_{\mathcal{T} \in \mathbf{T}_d(C)} \tag{6.14}$$

is normal on $|z| < \delta_2(d, C)$.

Proof. For any $\mathcal{T} \in \mathbf{T}_d(C)$ and any z such that $|z| < \delta_2(d, C)$ we have

$$\left| \frac{\log(Z^{\text{even}}_{\text{match}}(\mathcal{T}; z) + Z^{\text{large}}_{\text{match}}(\mathcal{T}; z))}{|\mathcal{T}|} \right| = \left| \frac{1}{|\mathcal{T}|} \sum_{v \in V(\mathcal{T})} \sum_{\substack{X \in \mathcal{C}^{\text{even}}_{\text{large}}(\mathcal{T}) \\ v \in \overline{X}}} \frac{\Phi(X; z)}{|\overline{X}|} \right|$$
$$\leq \max_{v \in V(\mathcal{T})} \sum_{\substack{X \in \mathcal{C}^{\text{even}}_{\text{large}}(\mathcal{T}) \\ v \in \overline{X}}} \frac{|\Phi(X; z)|}{|\overline{X}|} \leq 4de^4,$$

where the last inequality follows from Theorem 6.2.39, Theorem 6.3.3. Therefore the family defined in (6.14) is normal by Montel's theorem.

□

To bound $|Z^{\text{large}}(\mathcal{T}; z))|$, we show the influence of adding large contours to the even contours is negligible, for small enough z as the sizes of the tori tend to infinity.

Lemma 6.3.5. *For any $C > 0$ and any $|z| < \delta_1(d, C)$ the function*

$$f_C(z) = \lim_{\substack{|\mathcal{T}|\to\infty \\ \mathcal{T}\in\mathbf{T}_d(C)}} \frac{\log(Z^{even}_{match}(\mathcal{T}; z))}{|\mathcal{T}|}$$

is well-defined and $f_C(z) = \frac{1}{2} f_{even,C}(z) + \frac{1}{2} f_{odd,C}(z)$. For any $|z| < \delta_2(d, C)$ the function

$$g_C(z) = \lim_{\substack{|\mathcal{T}|\to\infty \\ \mathcal{T}\in\mathbf{T}_d(C)}} \frac{\log(Z^{even}_{match}(\mathcal{T}; z) + Z^{large}_{match}(\mathcal{T}; z))}{|\mathcal{T}|}$$

is well-defined and $g_C(z) = f_C(z)$.

Proof. Take any torus $\mathcal{T} \in \mathbf{T}_d(C)$ and let ℓ_1 denote the minimal side length of $\mathcal{T}$. From Theorem 6.2.39 and Theorem 6.2.38 we obtain for all $|z| < \delta_1(d, C)$

$$\left| \frac{\log Z^{\text{even}}_{\text{match}}(\mathcal{T}; z)}{|\mathcal{T}|} - \frac{1}{2} f_{\text{even},C}(z) - \frac{1}{2} f_{\text{odd},C}(z) \right| < \frac{2}{|\mathcal{T}|},$$

where in the last equality we used $\ell_1 e^{C\ell_1} \geq 2|\mathcal{T}|$, as $\ell_1 \geq 2$ and $e^{C\ell_1} \geq |\mathcal{T}|$. This implies $f_C(z)$ exists and $f_C(z) = \frac{1}{2} f_{\text{even},C}(z) + \frac{1}{2} f_{\text{odd},C}(z)$.

The first $\rho\ell_1$ terms of the power series $\log Z^{\text{even}}_{\text{match}}(\mathcal{T}; z)$ and $\log(Z^{\text{even}}_{\text{match}}(\mathcal{T}; z) + Z^{\text{large}}_{\text{match}}(\mathcal{T}; z))$ are equal, where ρ is the constant from Lemma 6.2.30 as each large contour contributes at least $z^{\rho\ell_1}$ in each cluster X containing the large contour. Therefore, as $|\mathcal{T}| \to \infty$, and hence $\ell_1 \to \infty$, the first $\rho\ell_1$ coefficients of

$$\frac{\log(Z^{\text{even}}_{\text{match}}(\mathcal{T}; z) + Z^{\text{large}}_{\text{match}}(\mathcal{T}; z))}{|\mathcal{T}|}$$

converge to the first $\rho\ell_1$ coefficients of $f_C(z)$. Lemma 6.3.4 now implies that $g_C(z)$ is well-defined on $B_{\delta_2(d,C)}(0)$ and moreover satisfies $g_C(z) = f_C(z)$. □

Remark 12. The function $f_C(z)$ is the free energy per site for the polymer model with polymers the small even contours in $\mathcal{T}$. It is related to the free energy per site for the independence polynomial defined in the introduction, which we denote by $\rho(\lambda)$. For $\lambda \in \mathbb{R}_{\geq 0}$ and $|\lambda| > 1/\delta_1(d, C)$ both functions are well-defined and satisfy $\rho(\lambda) = \frac{\lambda}{2} + f_C(\frac{1}{\lambda})$.

The following lemma provides sufficient conditions to bound $|Z^{\text{large}}_{\text{match}}(\mathcal{T}; z)|$ away from $2|Z^{\text{even}}_{\text{match}}(\mathcal{T}; z)|$, which is the final ingredient to prove zeros are bounded for C balanced tori.

Lemma 6.3.6. *Suppose there exists $\delta > 0$ and for each $n \in \mathbb{N}$ there are holomorphic functions $f_n, g_n : B_\delta(0) \to \mathbb{C}$ and functions $a, b : \mathbb{N} \to \mathbb{N}$ such that for all $z \in B_\delta(0)$ we have*

1. *the functions* $f_n(z) + z^{a(n)} g_n(z)$ *and* $f_n(z)$ *are nonzero for each* n,

2. $\lim_{n\to\infty} \frac{1}{b(n)} \cdot (\log(f_n(z) + z^{a(n)} g_n(z)) - \log f_n(z)) = 0$,

3. *there is a constant* $\kappa > 0$ *such that for all* n *we have* $a(n) > \kappa \log b(n)$,

then there is a constant $N > 0$ *such that for* $|z| < \frac{\delta}{e^{1/\kappa}}$ *and* $n \geq N$ *we have*

$$|z^{a(n)} g_n(z)| < |f_n(z)|.$$

Furthermore, the inequality in (3) is necessary.

Proof. We have

$$\log(f_n(z) + z^{a(n)} g_n(z)) - \log f_n(z) = \log\left(1 + \frac{z^{a(n)} g_n(z)}{f_n(z)}\right) = z^{a(n)} h_n(z)$$

for some convergent power series $h_n(z)$, using item (1). By item (2) we see $\lim_{n\to\infty} \frac{z^{a(n)} h_n(z)}{b(n)} = 0$ for $|z| < \delta$. Hence for any $\varepsilon > 0$ and large enough n we have $\left|\frac{z^{a(n)} h_n(z)}{b(n)}\right| < \varepsilon$. By the maximum principle we see $\left|\frac{h_n(z)}{b(n)}\right| < \frac{\varepsilon}{\delta^{a(n)}}$. Now take $|z| < \frac{\delta}{e^{1/\kappa}}$ where $\kappa > 0$ is the constant from item (3), then $|z|^{a(n)} \left|\frac{h_n(z)}{b(n)}\right| < \frac{\varepsilon}{e^{a(n)/\kappa}}$ therefore $|z^{a(n)}||h_n(z)| < \frac{b(n)}{e^{a(n)/\kappa}} \varepsilon < \varepsilon$, by item (3) of the assumptions. Therefore

$$\left|\log\left(1 + \frac{z^{a(n)} g_n(z)}{f_n(z)}\right)\right| = |z^{a(n)}||h_n(z)| < \varepsilon,$$

for large n and $|z| < \frac{\delta}{e^{1/\kappa}}$. From this we conclude $|z^{a(n)} g_n(z)| < \varepsilon |f_n(z)| < |f_n(z)|$, which finishes the first part of the proof.

To prove the estimate in (3) is sharp, let $\delta = 1$ and $a(n) = n$, choose any holomorphic map $h : \mathbb{D} \to \mathbb{C}$ and define $f_n(z) = e^{b(n)h(z)}$ and $z^n g_n(z) := f_n(z)(e^{z^n b(n)} - 1)$ so that items (1) and (2) hold. Now suppose for any $\kappa > 0$ there is an $n \geq 1$ such that $\kappa \log b(n) > n$, i.e. $b(n) > (e^{1/\kappa})^n$. We have

$$\frac{z^n g_n(z)}{f_n(z)} = e^{z^n b(n)} - 1,$$

which does not converge to 0 on any disc $B_r(0)$. In fact, by the assumption on $b(n)$, we see that for any $r > 0$ there exist infinitely many $n \geq 1$ such that $b(n) > (1/r)^n$. It follows for $z = r$ that $z^n b(n) > 1$, from which we see $e^{r^n b(n)} - 1 > e - 1 > 1$. Hence we do not have $|z^{a(n)} g_n(z)| < |f_n(z)|$ for all z small enough and n large enough. $\square$

Theorem 6.3.7. *Let $C > 0$. There exists $\delta = \delta(d, C) > 0$ such that for all $z \in \mathbb{C}$ with $|z| < \delta$ and for all tori $\mathcal{T} \in \mathbf{T}_d(C)$ we have*

$$Z_{match}(\mathcal{T}; z) \neq 0.$$

Proof. We claim there exists an $N > 0$ such that for any $z \in \mathbb{C}$ with $|z| < \frac{\delta_2(d,C)}{e^C}$ and any torus $\mathcal{T} \in \mathbf{T}_d(C)$ with $|\mathcal{T}| \geq N$ we have $Z_{\text{match}}(\mathcal{T}; z) \neq 0$. Since there are finitely many tori $\mathcal{T} \in \mathbf{T}_d(C)$ with $|\mathcal{T}| < N$, by choosing M larger if necessary the theorem follows.

To prove the claim, note $Z_{\text{match}}(\mathcal{T}; z) = 2Z^{\text{even}}_{\text{match}}(\mathcal{T}; z) + Z^{\text{large}}_{\text{match}}(\mathcal{T}; z)$, by Lemma 6.2.27 and Lemma 6.2.28. Given $d \in \mathbb{N}_{\geq 2}$ and $C > 0$ the set $\mathbf{T}_d(C)$ is countable and we can choose a bijection $h : \mathbb{N} \to \mathbf{T}_d(C)$ such that $n > m$ implies $|h(n)| \geq |h(m)|$.

We define maps $\tilde{a}, \tilde{b} : \mathbf{T}_d(C) \to \mathbb{N}$ as follows. For $\mathcal{T} \in \mathbf{T}_d(C)$ with shortest side length ℓ_1 we define $\tilde{a}(\mathcal{T}) = \lfloor \rho \ell_1 \rfloor$, where ρ is the constant from Lemma 6.2.30. Furthermore we define $\tilde{b}(\mathcal{T}) = |\mathcal{T}|$. Define maps $a, b : \mathbb{N} \to \mathbb{N}$ as $a = \tilde{a} \circ h$ and $b = \tilde{b} \circ h$.

For $n \in \mathbb{N}$ the function $Z^{\text{large}}_{\text{match}}(h(n); z)/(z^{a(n)})$ is a polynomial in z which we denote by $g_n(z)$. Write $f_n(z) = Z^{\text{even}}_{\text{match}}(h(n); z)$, thus $Z(h(n); z) = 2f_n(z) + z^{a(n)} g_n(z)$. We check the conditions of Lemma 6.3.6 for functions $f_n, g_n, a(n)$ and $b(n)$ as above with $\delta = \delta_2(d, C)$. Assumption (1) of Lemma 6.3.6 holds by Theorems 6.2.39, 6.2.34 and 6.3.3. Assumption (2) of Lemma 6.3.6 holds by choice of the bijection h and Lemma 6.3.5. Assumption (3) of Lemma 6.3.6 also holds with $\kappa = 1/C$ by definition of $\mathbf{T}_d(C)$ and the functions a and b. It follows from Lemma 6.3.6 there is a constant $M > 0$ such that for $|z| < \frac{\delta}{e^C}$ and $n \geq M$ we have $|z^{a(n)} g_n(z)| < |f_n(z)| < 2|f_n(z)|$. Hence $|Z(h(n); z)| = |2f_n(z) + z^{a(n)} g_n(z)| \geq |2|f_n(z)| - |z^{a(n)} g_n(z)|| > 0$. As for $n > m$ we have $|h(n)| \geq |h(m)|$, it follows for $N = |h(M)|$, any $z \in \mathbb{C}$ with $|z| < \frac{\delta_2(d,C)}{e^C}$ and any torus $\mathcal{T} \in \mathbf{T}_d(C)$ with $|\mathcal{T}| \geq N$ we have $Z_{\text{match}}(\mathcal{T}; z) \neq 0$. This proves the claim, completing the proof of the theorem. $\square$

Remark 13. Let $C > 0$. For $|z| < \frac{\delta_2(d,C)}{e^C}$ the limit exists

$$\lim_{\substack{|\mathcal{T}| \to \infty \\ \mathcal{T} \in \mathbf{T}_d(C)}} \frac{\log Z_{\text{match}}(\mathcal{T}; z)}{|\mathcal{T}|}$$

and converges to the function $f_C(z)$ defined in Lemma 6.3.5. For any two constants $C_1 > C_2 > 0$ and any $z \in \mathbb{C}$ with $|z| < \min(\frac{\delta_2(d,C_1)}{e^{C_1}}, \frac{\delta_2(d,C_2)}{e^{C_2}})$ we have $f_{C_1}(z) = f_{C_2}(z)$, hence the function f does not depend on C. This justifies referring to f as the limit free energy of balanced tori around infinity.

From Theorem 6.3.7, we immediately obtain the first part of the main theorem.

Theorem (First part of Main Theorem)**.** *Let $\mathcal{F}$ be a family of even d-dimensional tori. If $\mathcal{F}$ is balanced, then the zeros of the independence polynomials $\{Z_{\mathcal{T}} : \mathcal{T} \in \mathcal{F}\}$ are uniformly bounded.*

Proof. The family $\mathcal{F}$ is balanced if and only if there is a $C > 0$ such that $\mathcal{F} \subset \mathbf{T}_d(C)$. By Corollary 6.2.17 and Theorem 6.3.7, we see there exists a uniform $\Lambda(d, C) = 1/\delta(d, C)$ such that for any $\lambda \in \mathbb{C}$ with $|\lambda| > \Lambda(d, C)$ and any $\mathcal{T} \in \mathbf{T}_d(C)$ we have $Z_{\mathrm{ind}}(\mathcal{T}; \lambda) \neq 0$. □

6.4 Unbounded zeros of highly unbalanced tori

In this section we will prove that the independence polynomials of highly unbalanced tori have unbounded zeros. First we will consider tori for which all dimensions except one are constant. The fact that zeros are unbounded when the last dimension diverges will immediately imply that for sufficiently unbalanced sequences of tori the zeros are unbounded. A more careful analysis then provides explicit bounds on the required relative dimensions of the tori. The proofs in this section rely on an analysis of the corresponding transfer-matrices.

For positive integers n we will let C_n denote the cycle graph on n vertices. We let $G_1 \square G_2$ denote the cartesian product of two graphs G_1, G_2, i.e. the graph with vertex set $V(G_1) \times V(G_2)$ and $(v_1, u_1) \sim (v_2, u_2)$ iff either $v_1 = v_2$ and $u_1 \sim u_2$ in G_2 or $u_1 = u_2$ and $v_1 \sim v_2$ in G_1. What was previously denoted by $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_d}$ shall in this section be denoted by $C_{n_1} \square \cdots \square C_{n_d}$.

There will be no other partition function than the independence polynomial which, for a graph G and parameter λ, we denote by $Z(G; \lambda)$.

6.4.1 Transfer-matrix method

Fix G to be a finite graph and let $\mathcal{I}$ denote the set of its independent sets. Two independent sets $S, T \in \mathcal{I}$ are said to be *compatible* if $S \cap T = \emptyset$ and we write $S \sim T$. We let A denote the adjacency matrix of the compatibility graph, i.e. the rows and columns of A are indexed by elements of $\mathcal{I}$ and $A_{S,T} = 1$ if $S \sim T$ and $A_{S,T} = 0$ otherwise. Furthermore, for a variable λ, we let D_λ denote the diagonal matrix with $(D_\lambda)_{S,S} = \lambda^{|S|}$.

Theorem 6.4.1 (Transfer-matrix method)**.** *For any $n \in \mathbb{Z}_{\geq 1}$*

$$Z(C_n \square G; \lambda) = \mathrm{Tr}\left[(D_\lambda A)^n\right].$$

Proof. Let $\mathcal{P} \subseteq \mathcal{I}^n$ denote those tuples $(S_1, \ldots, S_n)$ for which $S_i \sim S_{i+1}$ for all $i = 1, \ldots, n$, reducing the index modulo n. The independent sets of $C_n \square G$

correspond one to one with the elements of $\mathcal{P}$. We therefore find that

$$\begin{aligned}\mathrm{Tr}\left[(D_\lambda A)^n\right] &= \sum_{(S_1,\ldots,S_n)\in\mathcal{I}^n}\prod_{i=1}^{n}(D_\lambda A)_{S_i S_{i+1}} \\ &= \sum_{(S_1,\ldots,S_n)\in\mathcal{P}} \lambda^{\sum_{i=1}^n |S_i|} = Z(C_n \square G;\lambda).\end{aligned}$$

□

Throughout this section we will frequently use that for any complex valued square matrix M and integer $n \geq 1$

$$\mathrm{Tr}(M^n) = \sum_{s \text{ eigenvalue of } M} s^n.$$

This observation reveals the strength of the transfer-matrix method. It shows that $Z(C_n \square G;\lambda)$ can be written as a simple expression in n and a fixed set of values. This motivates the study of the eigenvalues of the transfer-matrix.

Lemma 6.4.2. *Let $\lambda \in \mathbb{R}_{\geq 0}$. The eigenvalues of $D_\lambda A$ are real and there is a simple positive eigenvalue r such that $r > |s|$ for all other eigenvalues s.*

Proof. We first consider $\lambda = 0$. The only non-zero entry of the diagonal matrix D_0 is $(D_0)_{\emptyset,\emptyset}$. Therefore the matrix $D_0 A$ has rank at most 1 and thus the eigenvalue 0 appears with multiplicity at least $|\mathcal{I}| - 1$. Observe that $D_0 A e_\emptyset = e_\emptyset$ and thus 1 is an eigenvalue, which must necessarily be simple.

Now assume $\lambda > 0$. The matrix $D_\lambda A$ is conjugate to the real symmetric matrix $D_{\lambda^{1/2}} A D_{\lambda^{1/2}}$ and thus all its eigenvalues are real. The entries of $D_\lambda A$ are all non-negative and its support matrix is A. The matrix A is the adjacency matrix of a connected graph because $S \sim \emptyset$ for every $S \in \mathcal{I}$. The diagonal entry $(D_\lambda A)_{\emptyset,\emptyset} = 1$ is non-zero. These facts allow us to conclude that $D_\lambda A$ is an aperiodic irreducible matrix. The Perron–Frobenius theorem states that we can conclude that the eigenvalue of maximal norm of $D_\lambda A$ is simple and positive real. □

Corollary 6.4.3. *Let $\lambda_0 \in \mathbb{R}_{\geq 0}$. The zeros of the polynomials $\{Z(C_n \square G;\lambda)\}_{n\geq 1}$ do not accumulate on λ_0.*

Proof. According to Lemma 6.4.2 the matrix $D_{\lambda_0} A$ has a unique eigenvalue of maximal norm, which we denote by $r(\lambda_0)$. Because $r(\lambda_0)$ is simple there exists a neighborhood $U \subseteq \mathbb{C}$ of λ_0 such that $r : U \to \mathbb{C}$ is the analytic continuation of this eigenvalue, i.e. r holomorphic and $r(\lambda)$ is an eigenvalue of $D_\lambda A$ for all $\lambda \in U$.

Because the set of eigenvalues of $D_\lambda A$ moves continuously with λ there is a radius $R > 0$ and a constant $\zeta < 1$ such that $\zeta \cdot |r(\lambda)| > |s|$ for all other eigenvalues s of $D_\lambda A$ for all λ with $|\lambda - \lambda_0| \leq R$. For these λ we have

$$\left| \frac{Z(C_n \square G; \lambda)}{r(\lambda)^n} - 1 \right| = \left| \frac{Z(C_n \square G; \lambda) - r(\lambda)^n}{r(\lambda)^n} \right| = \sum_{s \neq r(\lambda)} \left(\frac{s}{r(\lambda)} \right)^n < \zeta^n \cdot (|\mathcal{I}| - 1),$$

where the sum runs over eigenvalues of $D_\lambda A$ not equal to $r(\lambda)$. For n sufficiently large the quantity on the right-hand side is strictly less than 1, which implies that $Z(C_n \square G; \lambda)$ cannot be zero. The disk of radius R around λ_0 can therefore only contain finitely many zeros. $\square$

We can deduce that the sequence $\{C_n \square G\}_{n \geq 1}$ undergoes no phase-transition. Indeed, the free energy per site converges:

$$\lim_{n \to \infty} \frac{\log(Z(C_n \square G; \lambda))}{n|V(G)|} = \frac{\log(r(\lambda))}{|V(G)|},$$

where $r(\lambda)$ is the largest eigenvalue of $D_\lambda A$. This is an analytic function of λ on $[0, \infty)$.

6.4.2 Constant width tori

We now move from general graphs to tori. Let $\mathcal{T}$ be a fixed even torus (we allow $\mathcal{T}$ to be an even cycle). We again let $\mathcal{I}$ denote the collection of independent sets of $\mathcal{T}$. We will show that the zeros of the tori $C_n \square \mathcal{T}$ are unbounded or, in other words, accumulate at ∞.

Define $\alpha = \frac{1}{2}|V(\mathcal{T})|$. There are two maximum independent sets, namely

$$S_{\text{even}} = \{v \in \mathcal{T} : v \text{ is even}\} \quad \text{and} \quad S_{\text{odd}} = \{v \in \mathcal{T} : v \text{ is odd}\}.$$

For any $S \in \mathcal{I}$ define

$$\|S\| = \alpha - |S|.$$

Although related, this definition should not be confused with the surface energy of a contour $\|\gamma\|$. We observe that $\|S_{\text{even}}\| = \|S_{\text{odd}}\| = 0$ and $\|S\| > 0$ for all other $S \in \mathcal{I}$.

We write $z = 1/\lambda$. Define the diagonal matrix $\hat{D}_z$ by $(\hat{D}_z)_{S,S} = z^{\|S\|}$ and recall that A denotes the compatibility matrix of the independent sets. We observe that

$$z^\alpha D_{1/z} = \hat{D}_z \quad \text{and} \quad \operatorname{Tr}\left[(\hat{D}_z A)^n\right] = z^{n\alpha} \cdot Z(C_n \square \mathcal{T}; 1/z).$$

From now on we let $M_z = \hat{D}_z A$. For any $S \in \mathcal{I}$ we let e_S denote the $|\mathcal{I}|$-dimensional unit vector belonging to index S. We turn our attention to the eigenvalues of M_z in a neighbourhood of $z = 0$.

Lemma 6.4.4. *There is a neighbourhood U of 0 and holomorphic functions q^+ and q^- defined on U such that*

- *$q^+(z)$ and $q^-(z)$ are eigenvalues of M_z for all $z \in U$ and*
- *$q^+(0) = 1$ and $q^-(0) = -1$ are the only non-zero eigenvalues of M_0.*

Proof. We can write

$$M_0 = e_{S_{\text{even}}} \sum_{\substack{S \in \mathcal{I} \\ S \sim S_{\text{even}}}} e_S^T + e_{S_{\text{odd}}} \sum_{\substack{S \in \mathcal{I} \\ S \sim S_{\text{odd}}}} e_S^T.$$

We see that M_0 has rank two, $M(e_{S_{\text{even}}} + e_{S_{\text{odd}}}) = e_{S_{\text{even}}} + e_{S_{\text{odd}}}$ and $M(e_{S_{\text{even}}} - e_{S_{\text{odd}}}) = -(e_{S_{\text{even}}} - e_{S_{\text{odd}}})$. Therefore $q^+(0) = 1$ and $q^-(0) = -1$ are the only two non-zero eigenvalues of M_0 and they are both simple. By the implicit function theorem these can be analytically extended to eigenvalues of M_z on a neighborhood of $z = 0$. $\square$

We will keep referring to q^+ and q^- as they are defined in Lemma 6.4.4. We can now give a reasonably short proof that the zeros of $C_n \square \mathcal{T}$ accumulate at ∞ using Montel's theorem as a black box.

Lemma 6.4.5. *Let $R > 0$. There are only finitely many n such that all zeros of $Z(C_n \square \mathcal{T}; \lambda)$ are less than R in norm.*

Proof. Let U be a connected neighborhood of $z = 0$ such that there is a $\zeta < 1$ for which $|s| < \zeta \cdot \min\{|q^+(z)|, |q^-(z)|\}$ for all other eigenvalues s of M_z for every $z \in U$. We can assume that q^+ and q^- are defined on U and that U is contained in a ball of radius $1/R$. Let N_0 be such that $\zeta^{N_0}(|\mathcal{I}| - 2) \leq 1/2$. Let $I \subseteq \mathbb{Z}_{\geq N_0}$ be the set of indices such that for $n \in I$ the polynomial $z^{n\alpha} \cdot Z(C_n \square \mathcal{T}; 1/z)$ has no zeros in $U \setminus \{0\}$. We will show that the family of functions

$$\mathcal{F} = \left\{ \frac{z^{n\alpha} \cdot Z(C_n \square \mathcal{T}; 1/z)}{q^+(z)^n} \right\}_{n \in I}$$

is a normal family on $U \setminus \{0\}$. We will do this by applying the strong version of Montel's theorem, i.e. we show that $\mathcal{F}$ avoids three values in the Riemann-sphere.

Because $z^{n\alpha} \cdot Z(C_n \square \mathcal{T}; 1/z)$ is a polynomial $f(z) \neq \infty$ for every $f \in \mathcal{F}$ and $z \in U$. By definition of I we see that $f(z) \neq 0$ for every $f \in \mathcal{F}$ and nonzero $z \in U$. We also claim that $\mathcal{F}$ avoids 1. To prove it we assume that there is a

$z \in U$ and an index $n \in I$ that show otherwise. Then

$$0 = \left| \frac{z^{n\alpha} \cdot Z(C_n \square \mathcal{T}; 1/z) - q^+(z)^n}{q^-(z)^n} \right| = \left| \frac{\mathrm{Tr}\left[(\hat{D}_z A)^n\right] - q^+(z)^n}{q^-(z)^n} \right|$$

$$= \left| 1 + \sum_{s \neq q^{\pm}(z)} \left(\frac{s}{q^-(z)} \right)^n \right| \geq 1 - \zeta^n(|\mathcal{I}| - 2) \geq 1/2,$$

where the sum runs over the eigenvalues of M_z not equal to $q^{\pm}(z)$. This is a contradiction and we can thus conclude that $\mathcal{F}$ is a normal family. We will now show that this implies that $\mathcal{F}$ is finite.

Define $\beta(z) = q^+(z)/q^-(z)$. We observe that $\beta(0) = -1$ and, by Lemma 6.4.2, $|\beta(z)| > 1$ for $z > 0$. The map β is holomorphic and non-constant and thus an open map. Let $U^+ = \{z \in U : |\beta(z)| > 1\}$ and $U^- = \{z \in U : |\beta(z)| < 1\}$. These are both open non-empty subsets of $U \setminus \{0\}$. For $z \in U^+$ we have that

$$\lim_{n\to\infty} \frac{z^{n\alpha} \cdot Z(C_n \square \mathcal{T}; 1/z)}{q^+(z)^n} = \lim_{n\to\infty} \left[\beta(z)^n + 1 + \sum_{s \neq q^{\pm}(z)} \left(\frac{s}{q^+(z)} \right)^n \right] = \infty,$$

while for $z \in U^-$ this limit is equal to 1. If $\mathcal{F}$ were to have a sequence of elements whose indices converge to ∞, it should have a subsequence that converges to a holomorphic function that is constant ∞ on U^+ and constant 1 on U^-. Because $U \setminus \{0\}$ is connected, such a function does not exist.

This shows that the index-set I is finite. It follows that there is an N_1 such that for all $n \geq N_1$ the polynomial $z^{n\alpha} \cdot Z(C_n \square \mathcal{T}; 1/z)$ has a zero $z_0 \neq 0$ in U. Therefore $\lambda_0 = 1/z_0$ is a zero of $Z(C_n \square \mathcal{T}; \lambda)$ with $|\lambda_0| > R$. $\square$

Remark 14. The proof of Corollary 6.4.3 works just as well to show that zeros of $Z(C_n \square G; \lambda)$ cannot accumulate on any λ_0 for which $D_{\lambda_0} A$ has a unique largest (in norm) eigenvalue. Similarly, the proof of Lemma 6.4.5 works to show that zeros accumulate on any parameter λ_0 for which $D_{\lambda_0} A$ has two or more simple eigenvalues $\{r_1(\lambda_0), \ldots, r_k(\lambda_0)\}$ of the same norm that are larger than all the eigenvalues if no pair of such eigenvalues persistently has the same norm. That is, if there is no distinct pair i, j and neighborhood U of λ_0 for which the analytic continuations r_i, r_j satisfy $|r_i(\lambda)| = |r_j(\lambda)|$ for all $\lambda \in U$.

This shows that, in the case that there are no eigenvalues that persistently have the same norm, the accumulation points of the zeros of $Z(C_n \square G; \lambda)$ are exactly those parameters λ_0 for which $D_{\lambda_0} A$ has two or more maximal eigenvalues of the same norm; a special case of [Sok04, Theorem 1.5]. It then follows that the set of accumulation points is a union of real algebraic curves; see Figure 6.5 for two examples.

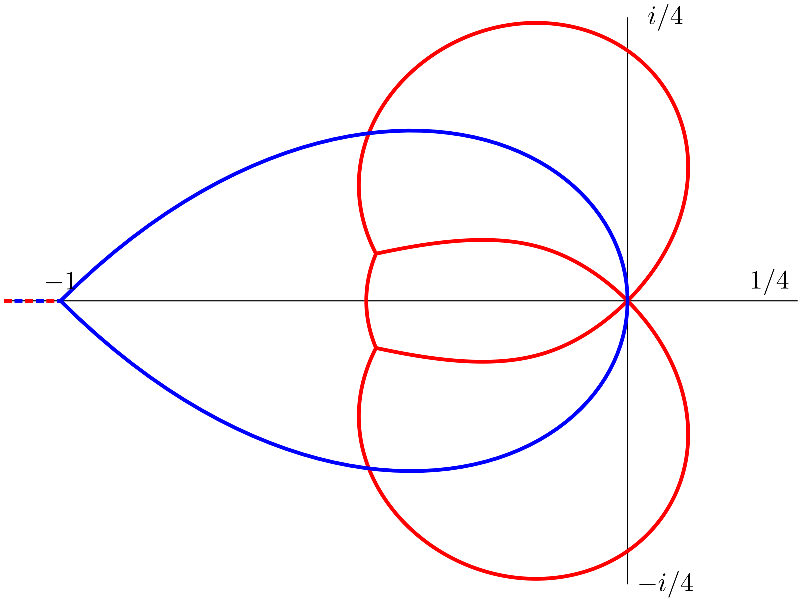


Figure 6.5: Parameters $z = 1/\lambda$ for which the transfer matrix has two maximal eigenvalues (non-persistently) of the same norm for C_2 in blue and C_4 in red. These curves are accumulation points of the zeros of the polynomials $\{Z(C_n \square C_2; \lambda)\}_{n\geq 1}$ and $\{Z(C_n \square C_4; \lambda)\}_{n\geq 1}$ respectively. The other accumulation points in λ coordinates are given by the real intervals with approximate bounds $[-1, -0.172]$ and $[-1, -0.126]$ respectively.

Let $\mathcal{T}_m = \mathbb{Z}_m^{d-1}$ and let $a_m \geq 2m$ be an even integer such that $Z(C_{a_m} \square \mathcal{T}_{2m}; \lambda)$ has a zero with norm at least m. Such an a_m exists by Lemma 6.4.5. Now $\{C_{a_m} \square \mathcal{T}_{2m}\}_{m \geq 1}$ is a sequence of tori whose sidelengths all converge to ∞ and whose zeros are unbounded. The first part of the main theorem, proved in the previous section, shows that for every $C > 0$ there are only finitely many m such that $a_m \leq e^{Cm}$, i.e. $\log(a_m) = \omega(m)$. In the next section we will show that $\log(a_m)$ can be chosen to not grow faster than $m^{3(d-1)}$.

6.4.3 Explicit bounds

The remainder of this section is dedicated to proving a more quantitative version of Lemma 6.4.5. Let $\mathcal{T}$ be an even torus, $\alpha = |V(\mathcal{T})|/2$ and $N = |\mathcal{I}(\mathcal{T})|$. We shall prove the following.

Theorem 6.4.6. *Let $R > (6N^2)^{\alpha+2}$ and $n \geq 80 \cdot R^\alpha$ then $Z(C_n \square \mathcal{T}; \lambda)$ has at least $\frac{1}{16} n R^{-\alpha}$ distinct zeros with magnitude at least R.*

Once we have proved the above, we quickly obtain a proof of the second part of the main theorem:

Theorem (Second part of Main Theorem)**.** *Let $\mathcal{F}$ be a highly unbalanced family of even tori. The zeros of the independence polynomials $\{Z(\mathcal{T}; \lambda) : \mathcal{T} \in \mathcal{F}\}$ are not uniformly bounded.*

Proof. For every $\mathcal{T} \in \mathcal{F}$ write $\ell(\mathcal{T})$ for the longest side length of $\mathcal{T}$. Furthermore, let $\mathcal{R}(\mathcal{T})$ be the torus for which $\mathcal{T} \cong C_{\ell(\mathcal{T})} \square \mathcal{R}(\mathcal{T})$. Now define

$$\mathcal{F}' = \{\mathcal{T} \in \mathcal{F} : \ell(\mathcal{T}) \geq 80 \cdot 6^{3|\mathcal{R}(\mathcal{T})|^2} \cdot 2^{6|\mathcal{R}(\mathcal{T})|^3}\}.$$

Because $\mathcal{F}$ is highly unbalanced $\mathcal{F}'$ contains infinitely many elements. We distinguish between the case where $\{\mathcal{R}(\mathcal{T}) : \mathcal{T} \in \mathcal{F}\}$ is finite or infinite.

In the former case there is a fixed torus $\mathcal{T}$ such that $\mathcal{F}$ contains infinitely many elements of the form $C_n \square \mathcal{T}$. Their zeros are unbounded according to Lemma 6.4.5.

In the latter case let $\mathcal{T}_n \in \mathcal{F}'$ be a sequence for which $|\mathcal{R}(\mathcal{T}_n)|$ tends to infinity. Let $R_{\mathcal{T}} = 6^{3|\mathcal{R}(\mathcal{T})|} \cdot 2^{6|\mathcal{R}(\mathcal{T})|}$. Because $|\mathcal{I}(\mathcal{R}(\mathcal{T}))| < 2^{|\mathcal{R}(\mathcal{T})|}$ we can apply Theorem 6.4.6 to see that $Z(\mathcal{T}; \lambda) = Z(C_{\ell(\mathcal{T})} \square \mathcal{R}(\mathcal{T}); \lambda)$ has at least one zero with magnitude at least $R_{\mathcal{T}}$ for any $\mathcal{T} \in \mathcal{F}'$. The theorem now follows from the fact that $R_{\mathcal{T}_n}$ tends to infinity. □

The remainder of this section focuses on proving Theorem 6.4.6.

The eigenvalues q^+ and q^-

We again let $\mathcal{T}$ be a fixed torus whose sidelengths are all even. We recall that we defined the rescaled transfer-matrix M_z with eigenvalues q^+, q^- holomorphic in a neighborhood of $z = 0$. We also recall the two independent sets S_{even} and S_{odd} of size α. In this section we will investigate the series expansion of $q^\pm$. For example when $\mathcal{T} = C_8$ we have

$$\begin{aligned} q^+(z) &= 1 + 4z + 6z^2 + 8z^3 + 44z^4 + \mathcal{O}(z^5) \quad \text{and} \\ q^-(z) &= -1 - 4z - 6z^2 - 8z^3 + 26z^4 + \mathcal{O}(z^5). \end{aligned}$$

We will show that the coefficient of z^m of q^+ is minus that of q^- for $m = 0, \ldots, \alpha - 1$, while the coefficients of z^α differ in magnitude. This is done so that in the end we can get a handle on the map $\beta(z) = q^+(z)/q^-(z)$ and the branches of its inverse.

For any $k \in \{0, \ldots, \alpha\}$ we define Q_k as the projection of a vector on the subspace spanned by $\{e_S\}_{\|S\|=k}$, i.e.

$$Q_k = \sum_{\substack{S \in \mathcal{I} \\ \|S\|=k}} e_S e_S^T.$$

Observe that $Q_0 + Q_1 + \cdots + Q_\alpha = I_{|\mathcal{I}|}$.

We define $v_0^+ = e_{S_{\text{even}}} + e_{S_{\text{odd}}}$ and $v_0^- = e_{S_{\text{even}}} - e_{S_{\text{odd}}}$. We also define $q_0^+ = 1$ and $q_0^- = -1$. For $n \geq 1$ recursively define the sequences of vectors $v_n^\pm$ and of integers $q_n^\pm$ by

$$v_n^\pm = q_0^\pm \Big(\sum_{k=1}^{\min(n,\alpha)} Q_k A v_{n-k}^\pm - \sum_{i=1}^{n-1} q_i^\pm v_{n-i}^\pm \Big) \quad \text{and} \quad q_n^\pm = e_{S_{\text{even}}}^T A v_n^\pm. \tag{6.15}$$

Observe that $q_n^\pm = e_{S_{\text{even}}}^T A v_n^\pm$ also holds for $n = 0$. We furthermore define the (formal) power series

$$v^\pm(z) = \sum_{n=0}^{\infty} v_n^\pm z^n \quad \text{and} \quad q^\pm(z) = \sum_{n=0}^{\infty} q_n^\pm z^n. \tag{6.16}$$

We will show that $(q^\pm, v^\pm)$ form two eigenvalue-eigenvector pairs corresponding to $q^\pm$ as defined in Lemma 6.4.4. This is will technically be an equality of formal power series until we prove that $q^\pm$ and and the entries of $v^\pm$ are analytic around 0, which we will subsequently do. We first identify a certain symmetry in the entries of v_n.

Let $\sigma \in \text{Aut}(\mathcal{T})$. For any $S \in \mathcal{I}$ we define

$$S^\sigma = \{\sigma(v) : v \in S\}.$$

The map $\varepsilon : \mathrm{Aut}(\mathcal{T}) \to \{\pm 1\}$ given by $\varepsilon(\sigma) = 1$ if $S_{\mathrm{even}}^{\sigma} = S_{\mathrm{even}}$ and $\varepsilon(\sigma) = -1$ if $S_{\mathrm{even}}^{\sigma} = S_{\mathrm{odd}}$ is a group homomorphism. An autormorphism $\sigma \in \mathrm{Aut}(\mathcal{T})$ is called even or odd according to whether $\varepsilon(\sigma) = 1$ or $\varepsilon(\sigma) = -1$ respectively. We define the permutation matrix P_σ by $P_\sigma e_S = e_{S^\sigma}$ and we observe that $P_\sigma Q_k = Q_k P_\sigma$ and $P_\sigma A = A P_\sigma$.

Lemma 6.4.7. *Let $n \in \mathbb{Z}_{\geq 0}$ and $\sigma \in \mathrm{Aut}(\mathcal{T})$. If σ is even then $P_\sigma v_n^{\pm} = v_n$, while if σ is odd then $P_\sigma v_n^{\pm} = \pm v_n^{\pm}$.*

Proof. For $n = 0$ the statement follows directly from the definitions. For $n \geq 1$ we have

$$P_\sigma v_n^{\pm} = q_0^{\pm}\Big(\sum_{k=1}^{\min(n,\alpha)} Q_k A P_\sigma v_{n-k}^{\pm} - \sum_{i=1}^{n-1} q_i^{\pm} P_\sigma v_{n-i}^{\pm} \Big).$$

The statement follows inductively. $\square$

We now prove that $(q^{\pm}, v^{\pm})$ indeed form two eigenvalue-eigenvector pairs.

Lemma 6.4.8. *As power series in z we have $M_z v^{\pm}(z) = q^{\pm}(z) v^{\pm}(z)$.*

Proof. We first claim that for any $n \in \mathbb{Z}_{\geq 0}$ we have $Q_0 A v_n^{\pm} = q_n^{\pm} v_0^{\pm}$. Let $\sigma \in \mathrm{Aut}(\mathcal{T})$ be an odd permutation. Then

$$\begin{aligned} Q_0 A v_n^{\pm} &= e_{S_{\mathrm{even}}} e_{S_{\mathrm{even}}}^T A v_n^{\pm} + e_{S_{\mathrm{odd}}} e_{S_{\mathrm{odd}}}^T A v_n^{\pm} \\ &= e_{S_{\mathrm{even}}} e_{S_{\mathrm{even}}}^T A v_n^{\pm} + e_{S_{\mathrm{odd}}} e_{S_{\mathrm{odd}}}^T P_\sigma A P_{\sigma^{-1}} v_n^{\pm} \\ &= (e_{S_{\mathrm{even}}} \pm e_{S_{\mathrm{odd}}}) e_{S_{\mathrm{even}}}^T A v_n^{\pm} \\ &= q_n^{\pm} v_0^{\pm}, \end{aligned}$$

where we have used Lemma 6.4.7 to equate $P_{\sigma^{-1}} v_n^{\pm}$ with $\pm v_n$.

We now prove the statement in the lemma. Observe that

$$M_z v^{\pm}(z) = \Big(\sum_{k=0}^{\alpha} Q_k A z^k \Big)\Big(\sum_{n=0}^{\infty} v_n^{\pm} z^n \Big) = \sum_{n=0}^{\infty} \Big[\sum_{k=0}^{\min(n,\alpha)} Q_k A v_{n-k}^{\pm} \Big] z^n.$$

Moreover,

$$q^{\pm}(z) v^{\pm}(z) = \Big(\sum_{n=0}^{\infty} q_n^{\pm} z^n \Big)\Big(\sum_{n=0}^{\infty} v_n^{\pm} z^n \Big) = \sum_{n=0}^{\infty} \Big[\sum_{i=0}^{n} q_i^{\pm} v_{n-i}^{\pm} \Big] z^n.$$

It is thus sufficient to prove that for all n

$$\sum_{i=0}^{n} q_i^{\pm} v_{n-i}^{\pm} = \sum_{k=0}^{\min(n,\alpha)} Q_k A v_{n-k}^{\pm}.$$

For $n = 0$ the statement reads $q_0^{\pm} v_0^{\pm} = Q_0 A v_0^{\pm}$, which is equivalent to the claim above for $n = 0$. For $n \geq 1$ we reason inductively as follows.

$$\begin{aligned}
\sum_{i=0}^{n} q_i^{\pm} v_{n-i}^{\pm} &= \sum_{i=1}^{n-1} q_i^{\pm} v_{n-i}^{\pm} + q_0^{\pm} v_n^{\pm} + q_n^{\pm} v_0^{\pm} \\
&= \sum_{i=1}^{n-1} q_i^{\pm} v_{n-i}^{\pm} + \left(q_0^{\pm}\right)^2 \Big(\sum_{k=1}^{\min(n,\alpha)} Q_k A v_{n-k}^{\pm} - \sum_{i=1}^{n-1} q_i^{\pm} v_{n-i}^{\pm} \Big) + Q_0 A v_n^{\pm} \\
&= \sum_{k=0}^{\min(n,\alpha)} Q_k A v_{n-k}^{\pm}.
\end{aligned}$$

This concludes the proof of the lemma. □

Now we will prove that both $q^{\pm}$ and the entries of $v^{\pm}$ are indeed analytic around $z = 0$. In what follows we let $N = |\mathcal{I}|$ so that the vectors $v_n^{\pm}$ are N-dimensional. We first prove an elementary lemma on a certain sequence of integers that will serve as an upper bound for the entries of v_n and q_n.

Lemma 6.4.9. *Define the sequence* $\{x_n\}_{n\geq 0}$ *by* $x_0 = 1$ *and for* $n \geq 1$

$$x_n = N \cdot \Big(x_{n-1} + \sum_{i=1}^{n-1} x_i x_{n-i} \Big).$$

Then $x_n \leq (6N^2)^n$.

Proof. Let $y_n(N) = x_n / N^{2n}$. We observe that $y_0(N) = 1$ and

$$y_n(N) = \frac{1}{N} y_{n-1}(N) + N \sum_{i=1}^{n-1} y_i(N) y_{n-i}(N).$$

It follows that $y_1(N) = 1/N$ and inductively $y_n(N)$ is a polynomial in $1/N$ with positive coefficients and constant term equal to zero. We can conclude that $y_n(N) \leq y_n(1)$ and thus it remains to show that $y_n(1) \leq 6^n$ for all $n \geq 0$.

We denote $y_n(1)$ by y_n and prove that

$$y_n \leq \frac{6^n}{(n+1)^2}, \tag{6.17}$$

which of course implies the desired inequality. Computer computations show that (6.17) is satisfied for $n = 1, \ldots, 199$. Suppose that (6.17) is satisfied for all values $0, \ldots, n-1$ for some $n \geq 200$. We observe

$$y_n = y_{n-1} + \sum_{i=1}^{n-1} y_i y_{n-i} \leq y_{n-1} + 2 \sum_{i=1}^{99} y_i y_{n-i} + 2 \sum_{i=100}^{\lfloor n/2 \rfloor} y_i y_{n-i}.$$

Using the induction hypothesis we find that

$$\frac{(n+1)^2}{6^n}\Big(y_{n-1}+2\sum_{i=1}^{99} y_i y_{n-i}\Big) \le (n+1)^2\Big(\frac{1}{6n^2}+2\sum_{i=1}^{99}\frac{y_i}{6^i(n+1-i)^2}\Big).$$

The right-hand side is an explicit decreasing rational function in n and thus upper bounded by the value obtained from plugging in $n = 200$, yielding an upper bound of 0.87. We also find

$$\begin{aligned}\frac{(n+1)^2}{6^n}\Big(2\sum_{i=100}^{\lfloor n/2\rfloor} y_i y_{n-i}\Big) &\le 2\sum_{i=100}^{\lfloor n/2\rfloor}\left(\frac{n+1}{(i+1)(n+1-i)}\right)^2\\ &\le 8\sum_{i=100}^{\infty}\frac{1}{(i+1)^2} \le 0.08.\end{aligned}$$

Putting these two estimates together we conclude that $y_n \le (0.87+0.08)\frac{6^n}{(n+1)^2} \le \frac{6^n}{(n+1)^2}$. □

Lemma 6.4.10. *We have $|q_n^{\pm}| \le N\cdot(6N^2)^n$ and $|(v_n^{\pm})_S| \le (6N^2)^n$ for all $n \ge 0$ and $S \in \mathcal{I}$.*

Proof. For a vector v let $|v|$ denote the vector whose entries are the magnitudes of the entries of v. For two vectors v_1, v_2 we write $v_1 \le v_2$ if the inequality holds entrywise. We let $\mathbb{1}$ denote the N-dimensional vector whose entries are all equal to 1. We inductively prove that $|v_n^{\pm}| \le x_n \cdot \mathbb{1}$ and $|q_n^{\pm}| \le N\cdot x_n$, where x_n is defined as in Lemma 6.4.9. This is sufficient by the bound proved in that lemma.

For $n = 0$ this follows by definition. For larger n we use the recursion in equation (6.15) to obtain

$$\begin{aligned}|v_n^{\pm}| &\le \sum_{k=1}^{\min(n,\alpha)} |Q_k A v_{n-k}^{\pm}| + \sum_{i=1}^{n-1}|q_i^{\pm} v_{n-i}^{\pm}|\\ &\le x_{n-1}\sum_{k=1}^{\min(n,\alpha)} Q_k A\,\mathbb{1} + \Big(\sum_{i=1}^{n-1} N x_i x_{n-i}\Big)\mathbb{1}\\ &\le N\cdot\Big(x_{n-1}+\sum_{i=1}^{n-1} x_i x_{n-i}\Big)\cdot\mathbb{1} = x_n\cdot\mathbb{1}.\end{aligned}$$

We also obtain $|q_n^{\pm}| = |e_{S_{\text{even}}}^T A v_n^{\pm}| \le e_{S_{\text{even}}}^T A\,\mathbb{1}\cdot x_n \le N\cdot x_n$. □

Corollary 6.4.11. *The functions $q^{\pm}$ and the entries of $v^{\pm}$ define holomorphic functions in a disk of radius $1/(6N^2)$. On that disk they form two eigenvalue-eigenvector pairs.*

The sum $q^+ + q^-$

Define $u_n = \frac{1}{2}(v_n^+ + v_n^-)$, $a_n = \frac{1}{2}(q_n^+ + q_n^-)$ and $b_n = \frac{1}{2}(q_n^+ - q_n^-)$. Our goal is to show that $a_n = 0$ for $n = 0, \ldots, \alpha - 1$ and $a_\alpha > 0$. We will start by deriving a useful recurrence for the u_n.

Lemma 6.4.12. *Let $\sigma \in \mathrm{Aut}(\mathcal{T})$ be an odd permutation. Then for all $n \geq 1$*

$$u_n = \sum_{k=1}^{\min(n,\alpha)} Q_k A u_{n-k}^\sigma - \sum_{i=1}^{n-1} \left(a_i u_{n-i}^\sigma + b_i u_{n-i}\right),$$

moreover,

$$a_n = e_{S_{even}}^T A u_n \quad \text{and} \quad b_n = e_{S_{odd}}^T A u_n.$$

Proof. It follows from Lemma 6.4.7 that $u_n^\sigma = \frac{1}{2}(v_n^+ - v_n^-)$ and thus $v_n^\pm = u_n \pm u_n^\sigma$. We similarly have $q_n^\pm = a_n \pm b_n$. We now use the recursion for $v_n^\pm$ defined in (6.15) to get a recursion for u_n:

$$\frac{1}{2}\Big[\Big(\sum_{k=1}^{\min(n,\alpha)} Q_k A v_{n-k}^+ - \sum_{i=1}^{n-1} q_i^+ v_{n-i}^+\Big) - \Big(\sum_{k=1}^{\min(n,\alpha)} Q_k A v_{n-k}^- - \sum_{i=1}^{n-1} q_i^- v_{n-i}^-\Big)\Big] =$$
$$\sum_{k=1}^{\min(n,\alpha)} Q_k A u_{n-k}^\sigma - \sum_{i=1}^{n-1} \frac{1}{2}\left(q_i^+ v_{n-i}^+ - q_i^- v_{n-i}^-\right).$$

The claimed recursive formula for u_n now follows from the following equality:

$$\begin{aligned}\frac{1}{2}\left(q_i^+ v_{n-i}^+ - q_i^- v_{n-i}^-\right) &= \frac{1}{2}\left[(a_i + b_i)(u_{n-i} + u_{n-i}^\sigma) - (a_i - b_i)(u_{n-i} - u_{n-i}^\sigma)\right] \\ &= a_i u_{n-i}^\sigma + b_i u_{n-i}.\end{aligned}$$

We use the part of equation (6.15) that defines $q_n^\pm$ to observe that

$$\begin{aligned}a_n &= \frac{1}{2}\left(e_{S_{\text{even}}}^T A v_n^+ + e_{S_{\text{even}}}^T A v_n^-\right) = e_{S_{\text{even}}}^T A u_n \text{ and} \\ b_n &= \frac{1}{2}\left(e_{S_{\text{even}}}^T A v_n^+ - e_{S_{\text{even}}}^T A v_n^-\right) = e_{S_{\text{even}}}^T A u_n^\sigma = e_{S_{\text{even}}}^T P_\sigma A P_{\sigma^{-1}} u_n^\sigma = e_{S_{\text{odd}}}^T A u_n.\end{aligned}$$

□

The goal is to write the elements of u_n as weighted paths of independent sets of $\mathcal{T}$; see Lemma 6.4.14. To make this formal we introduce some notation from formal language theory.

For any set F let F^* denote the set of finite words of elements of F (including the empty word denoted by $\emptyset_F$). For $f \in F$ and $w \in F^*$ we use $f \in w$ to indicate that f is a letter in the word w. For concatenation of two words $w_1, w_2 \in F^*$ we write $w_1 \cdot w_2$.

Let $\mathcal{I}_{\geq 1} = \{S \in \mathcal{I} : \|S\| \geq 1\}$. We define

$$\mathcal{P} = \mathcal{I}_{\geq 1} \times \mathbb{Z}_{\geq 1}^*.$$

For $r \in \mathbb{Z}_{\geq 1}^*$ we let $\|r\|$ denote the sum of its entries with $\|\emptyset_{\mathbb{Z}_{\geq 1}}\| = 0$. For $p \in \mathcal{P}$ of the form (S, r) we define the length and weight of p respectively as

$$\ell(p) = \|S\| + \|r\| \quad \text{and} \quad W(p) = \prod_{n \in r} (-b_n).$$

For an element $w \in \mathcal{P}^*$ we define

$$\ell(w) = \sum_{p \in w} \ell(p) \quad \text{and} \quad W(w) = \prod_{p \in w} W(p).$$

An empty sum or product we treat as 0 or 1 respectively.

Fix an odd $\sigma \in \mathrm{Aut}(\mathcal{T})$ with the property that $\sigma^2 = \mathrm{id}$, for example the automorphism $(n_1, n_2, \ldots, n_d) \mapsto (1 - n_1, n_2, \ldots, n_d)$. Define the subset $\mathcal{Q} \subseteq \mathcal{P}^*$ by

$$\begin{aligned}\mathcal{Q} = \{(S_1, r_1) \cdots (S_m, r_m) \in \mathcal{P}^* : &S_{\text{odd}} \sim S_1 \text{ and } S_i^\sigma \sim S_{i+1} \\ &\text{for all } i = 1, \ldots, m-1\} \cup \{\emptyset_{\mathcal{P}}\}.\end{aligned}$$

For any $S \in \mathcal{I}_{\geq 1}$ we let $\mathcal{Q}[S]$ denote the elements in $\mathcal{Q}$ that end in (S, r) for some r. We let $\mathcal{Q}[S_{\text{even}}] = \{\emptyset_{\mathcal{P}}\}$.

Lemma 6.4.13. *Let $S \in \mathcal{I}$ such that $S \sim S_{even}$. For any $w \in \mathcal{Q}[S]$ we have $\ell(w) \geq \alpha$, moreover if $\ell(w) = \alpha$ then $W(w) = 1$.*

Proof. Because S_{even} is not compatible with itself w is not the empty word and thus we can write $w = (S_1, r_1) \cdots (S_m, r_m)$. Let v be a vertex of $\mathcal{T}$. If $v \in S_i$ for some i then it follows from the requirement that $S_i^\sigma \sim S_{i+1}$ that $\sigma(v) \notin S_{i+1}$. Applying this fact to $\sigma(v)$ and using that σ^2 is the identity we see that $\sigma(v) \in S_i$ implies that $v \notin S_{i+1}$. The possible transitions for $(\mathbb{1}(v \in S_i), \mathbb{1}(\sigma(v) \in S_i))$ are thus given in the following diagram.

$$\begin{tikzcd}
 & (1,1) \arrow[d, leftrightarrow] & \\
(1,0) \arrow[loop left] \arrow[r, leftrightarrow] & (0,0) \arrow[loop below] \arrow[r, leftrightarrow] & (0,1) \arrow[loop right]
\end{tikzcd}$$

Assume that v is an even vertex. Because $S_1 \sim S_{\text{odd}}$ we see that $(\mathbb{1}(v \in S_1), \mathbb{1}(\sigma(v) \in S_1))$ is of the form $(*, 0)$. Similarly, because $S_m \sim S_{\text{even}}$, we see that $(\mathbb{1}(v \in S_m), \mathbb{1}(\sigma(v) \in S_m))$ is of the form $(0, *)$. It thus follows that $(\mathbb{1}(v \in S_i), \mathbb{1}(\sigma(v) \in S_i))$ takes on the value $(0,0)$ at least once more often than it takes on the value $(1,1)$. From this we can conclude that

$$\sum_{i=1}^{m} [1 - \mathbb{1}(v \in S_i) - \mathbb{1}(\sigma(v) \in S_i)] \geq 1. \tag{6.18}$$

We now find that

$$\begin{aligned} \ell(w) &= \sum_{i=1}^{m} \|S_i\| + \|r_i\| = \sum_{i=1}^{m} \left[\sum_{\substack{v \in \mathcal{T} \\ v \text{ even}}} 1 - \mathbb{1}(v \in S_i) - \mathbb{1}(\sigma(v) \in S_i) \right] + \sum_{i=1}^{m} \|r_i\| \\ &\geq \alpha + \sum_{i=1}^{m} \|r_i\| \geq \alpha, \end{aligned}$$

where we interchanged the two summations and used (6.18). We see that indeed $\ell(w) \geq \alpha$. Moreover, if $\ell(w) = \alpha$ then the final two inequalities must be equalities and thus $r_i = \emptyset_{\mathbb{Z}_{\geq 1}}$ for all i, which implies that $W(w) = 1$. □

For any $n \geq 0$ and $S \in \mathcal{I}$ define

$$\mathcal{Q}_n[S] = \{p \in \mathcal{Q}[S] : \ell(p) = n\}.$$

Lemma 6.4.14. *Let $0 \leq n \leq \alpha$ and $S \in \mathcal{I}$. Then*

$$e_S^T u_n = \sum_{w \in \mathcal{Q}_n[S]} W(w). \tag{6.19}$$

Moreover, if $n \neq \alpha$ then $a_n = 0$, while $a_\alpha \geq 1$.

Proof. By definition $a_0 = 0$ and $u_0 = e_{S_{\text{even}}}$. Moreover, $\mathcal{Q}_0[S]$ is non-empty only if $S = S_{\text{even}}$ in which case it consists of the empty word. Therefore we see that for $n = 0$ both sides of equation (6.19) are equal to 1 if $S = S_{\text{even}}$ and equal to 0 otherwise.

We will now prove the statement inductively, i.e. we let $1 \leq n \leq \alpha$ and we assume that for all values $k < n$ both (6.19) holds and $a_k = 0$.

First suppose that either $\|S\| = 0$ or $\|S\| > n$. Then it follows that $\mathcal{Q}_n[S]$ is empty and thus the right-hand side of (6.19) is equal to 0. Because $e_S^T Q_k = 0$ for $k \neq \|S\|$ we inductively obtain by Lemma 6.4.12 that in this case indeed the left-hand side is equal to

$$e_S^T u_n = -\sum_{i=1}^{n-1} b_i e_S^T u_{n-i} = 0.$$

Now suppose $1 \leq \|S\| \leq n$. We inductively find that the left-hand side of (6.19) is equal to

$$\begin{aligned}
e_S^T u_n &= e_S^T A u^\sigma_{n-\|S\|} - \sum_{i=1}^{n-1} b_i e_S^T u_{n-i} \\
&= \sum_{\substack{X \in \mathcal{I} \\ X^\sigma \sim S}} e_X^T u_{n-\|S\|} + \sum_{i=1}^{n-1} (-b_i) e_S^T u_{n-i} \\
&= \sum_{\substack{X \in \mathcal{I} \\ X^\sigma \sim S}} \sum_{w \in \mathcal{Q}_{n-\|S\|}[X]} W(w) + \sum_{i=1}^{n-1} \sum_{w \in \mathcal{Q}_{n-i}[S]} (-b_i) W(w).
\end{aligned}$$

For any $T \in \mathcal{I}$, $k \in \mathbb{Z}_{\geq 1}$ and $i \in \mathbb{Z}_{\geq 1}$ let $\mathcal{Q}_i[T, k]$ be those elements of $\mathcal{Q}_i[T]$ ending in (T, r) with r ending in k. Moreover, let $\mathcal{Q}_i[T, 0]$ denote those elements ending in $(T, \emptyset_{\mathbb{Z}_{\geq 1}})$. For $w \in \mathcal{Q}_i[T]$ we can write $w = w' \cdot (T, r)$ for some r. We let $w \oplus k$ denote the element $w' \cdot (T, r \cdot k) \in \mathcal{Q}_{i+k}[T]$. We have

$$\sum_{w \in \mathcal{Q}_n[S,0]} W(w) = \sum_{\substack{X \in \mathcal{I} \\ X^\sigma \sim S}} \sum_{w' \in \mathcal{Q}_{n-\|S\|}[X]} W(w' \cdot (S, \emptyset)) = \sum_{\substack{X \in \mathcal{I} \\ X^\sigma \sim S}} \sum_{w' \in \mathcal{Q}_{n-\|S\|}[X]} W(w').$$

While, if $i \in \{1, \dots, n-1\}$ we have

$$\sum_{w \in \mathcal{Q}_n[S,i]} W(w) = \sum_{w' \in \mathcal{Q}_{n-i}[S]} W(w \oplus i) = (-b_i) \cdot \sum_{w' \in \mathcal{Q}_{n-i}[S]} W(w).$$

We thus have

$$\begin{aligned}
\sum_{w \in \mathcal{Q}_n[S]} W(w) &= \sum_{i=0}^{n-1} \sum_{w \in \mathcal{Q}_n[S,i]} W(w) \\
&= \sum_{\substack{X \in \mathcal{I} \\ X^\sigma \sim S}} \sum_{w \in \mathcal{Q}_{n-\|S\|}[X]} W(w) + \sum_{i=1}^{n-1} \sum_{w \in \mathcal{Q}_{n-i}[S]} (-b_i) W(w),
\end{aligned}$$

which proves equality (6.19).

We now have to show that $a_n = 0$ if $n < \alpha$ and $a_\alpha \geq 1$. It follows from Lemma 6.4.12 that

$$a_n = e_{S_{\text{even}}}^T A u_n = \sum_{\substack{S \in \mathcal{I} \\ S \sim S_{\text{even}}}} e_S^T u_n = \sum_{\substack{S \in \mathcal{I} \\ S \sim S_{\text{even}}}} \sum_{w \in \mathcal{Q}_n[S]} W(w).$$

In Lemma 6.4.13 it is shown that if $S \sim S_{\text{even}}$ and $w \in Q_n[S]$, then $n \geq \alpha$. This shows that $a_n = 0$ for $n < \alpha$. Moreover, if $n = \alpha$ the lemma states that $W(w) = 1$. This shows that $a_n \geq 0$. Because $\emptyset$ is compatible with both S_{even} and S_{odd} we see that $(\emptyset, \emptyset_{\mathcal{Z}_{\geq 1}}) \in Q_\alpha[\emptyset]$ and thus we can conclude that $a_\alpha \geq 1$. □

The other eigenvalues

In this section we study the other eigenvalues of the transfer matrix M_z, i.e. those not equal to $q^{\pm}(z)$. Recall from Section 6.4.2 that $M_z = \hat{D}_z A$, where A is the compatibility matrix of the independent sets of $\mathcal{T}$ and $\hat{D}_z$ is a diagonal matrix with $(\hat{D}_z)_{S,S} = z^{\|S\|}$. In this section it will be more convenient to look at the symmetric transfer-matrix $\hat{M}_z = D_{z^{1/2}} A D_{z^{1/2}}$, where (for now) we make an arbitrary choice of $z^{1/2}$ for each z. The symmetric transfer-matrix $\hat{M}_z$ is conjugate to M_z and thus has the same eigenvalues.

Recall that the matrix $\hat{M}_z$ is N-dimensional. For this section we order the indices of the N-dimensional vectors, indexed by elements of $\mathcal{I}$, in such a way that S_{even} and S_{odd} correspond to the final two coordinates. The 2×2 submatrix of $\hat{M}_z$ induced by the final two coordinates therefore has 0s on the diagonal and 1s on the off diagonal. Every other non-zero entry of $\hat{M}_z$ is a strictly positive power of $z^{1/2}$.

For $\varepsilon > 0$ we define the forward and backward cones $C^+(\varepsilon)$ and $C^-(\varepsilon)$ by

$$C^+(\varepsilon) = \{(v_1, \dots, v_N) \in \mathbb{C}^N \ : \ \|(v_1, \dots, v_{N-2})\|_1 \leq \varepsilon \cdot \|(v_{N-1}, v_N)\|_1\}$$

and

$$C^-(\varepsilon) = \{(v_1, \dots, v_N) \in \mathbb{C}^N \ : \ \varepsilon \cdot \|(v_1, \dots, v_{N-2})\|_1 \geq \|(v_{N-1}, v_N)\|_1\}$$

For $\varepsilon < 1$ these two cones intersect only in the origin.

Lemma 6.4.15. *The symmetric transfer-matrix $\hat{M}_z$ maps $\mathbb{C}^N \backslash C^-(\varepsilon)$ into $C^+(\varepsilon)$ whenever*

$$|z| < \frac{\varepsilon^4}{N^2(1+\varepsilon)^2}.$$

Proof. Let $v = (v_1, \dots, v_N) \in \mathbb{C}^N \setminus C^-(\varepsilon)$ and write $\hat{M}_z v = w = (w_1, \dots, w_N)$. It follows that

$$\begin{aligned} \|(w_1, \dots, w_{N-2})\|_1 \leq &(N-2) \cdot \max_{j \leq N-2} |w_j| \leq (N-2)|z|^{\frac{1}{2}} \cdot \sum_{i=1}^{N} |v_i| \\ =&(N-2)|z|^{\frac{1}{2}} \|v\|_1 \qquad \leq (N-2)|z|^{\frac{1}{2}} \frac{\varepsilon+1}{\varepsilon} \|(v_{N-1}, v_N)\|_1. \end{aligned}$$

On the other hand

$$\begin{aligned}\|(w_{N-1},w_N)\|_1 &\geq |v_{N-1}|+|v_N| - 2\sum_{i=1}^{N-2}|z|^{\frac{1}{2}}|v_i|\\ &= \|(v_{N-1},v_N)\|_1 - 2|z|^{\frac{1}{2}}\|(v_1,\ldots,v_{N-2})\|_1\\ &\geq \left(1-\frac{2|z|^{\frac{1}{2}}}{\varepsilon}\right)\|(v_{N-1},v_N)\|_1.\end{aligned}$$

The inclusion $M_z(\mathbb{C}^N \setminus C^-(\varepsilon)) \subset C^+(\varepsilon)$ is therefore satisfied whenever

$$\varepsilon\Big(1-\frac{2|z|^{\frac{1}{2}}}{\varepsilon}\Big) \geq (N-2)|z|^{\frac{1}{2}}\frac{\varepsilon+1}{\varepsilon},$$

which is satisfied whenever

$$|z| \leq \frac{\varepsilon^4}{N^2(1+\varepsilon)^2}.$$

$\square$

From now on we fix $\varepsilon = \frac{1}{3}$ so that the forward and backward cones $C^+(\frac{1}{3})$ and $C^-(\frac{1}{3})$ are forward respectively backward invariant whenever $|z| < \frac{1}{144N^2}$.

Corollary 6.4.16. *For $|z| < \frac{1}{144N^2}$ the two eigenvectors $\hat{v}^+(z)$ and $\hat{v}^-(z)$ of $\hat{M}_z$ corresponding to the maximal eigenvalues $q^+(z)$ and $q^-(z)$ are contained in $C^+(\frac{1}{3})$, while all other (generalized) eigenvectors are contained in $C^-(\frac{1}{3})$.*

Proof. The statement clearly holds for $|z|$ sufficiently small. For any fixed z the entries of the matrix $\hat{M}_{\sqrt{x}z^{1/2}}$ are continuous functions of x for $x \in [0,1]$. The statement therefore follows for any $|z| < \frac{1}{144N^2}$ from the previous lemma, using the continuity of the set of eigenvectors of $\hat{M}_{\sqrt{x}z^{1/2}}$. $\square$

Lemma 6.4.17. *For $|z| < \frac{1}{144N^2}$ the absolute values of the two eigenvalues $q^+(z)$ and $q^-(z)$ are at least twice as large as the absolute value of any other eigenvalue.*

Proof. Let us first write v for one of the eigenvectors $\hat{v}^+(z)$ or $\hat{v}^-(z)$ of $\hat{M}_z$, and write $w = \hat{M}_z v$. Using that $v \in C^+(1/3)$ we obtain

$$\begin{aligned}\|(w_{N-1},w_N)\|_1 &\geq \|(v_{N-1},v_N)\|_1 - 2|z|^{\frac{1}{2}}\|(v_1,\ldots,v_{N-2})\|_1\\ &= \|(v_{N-1},v_N)\|_1 \cdot \left(1-\tfrac{1}{18N}\right),\end{aligned}$$

which implies that $|q^+(z)|$ and $|q^-(z)|$ are bounded from below by 17/18.

Now let $w = \hat{M}_z v$ for an eigenvector $v \in C^-(1/3)$. Then

$$\begin{aligned}
\|(w_1, \ldots, w_{N-2})\|_1 &\leq (N-2) \cdot \max_{j \leq N-2} \|w_j\| \\
&\leq (N-2) \left(|z|^{\frac{1}{2}} \|(v_{N-1}, v_N)\|_1 + |z| \|(v_1, \ldots, v_{N-2})\|_1 \right) \\
&\leq (N-2) \left(\tfrac{1}{36N} + \tfrac{1}{144N^2} \right) \|(v_1, \ldots, v_{N-2})\|_1 \\
&\leq \tfrac{1}{36} \|(v_1, \ldots, v_{N-2})\|_1.
\end{aligned}$$

It follows that the corresponding eigenvalue is bounded above by 1/36, which proves the statement for any $N \geq 1$. □

Proof of the main theorem

In this section we will again prove that zeros of $Z(C_n \square \mathcal{T}; \lambda)$ accumulate at ∞, as is done in Lemma 6.4.5. Similar to the proof of that lemma, we use that $\frac{z^{n\alpha} \cdot Z(C_n \square \mathcal{T}; 1/z)}{q^+(z)^n} = 1 + q^-(z)/q^+(z) + \mathcal{O}(z)$. This culminates in a proof of Theorem 6.4.6, which, as we showed in the beginning of this section, leads to a proof of the second part of the main theorem. We define $\beta(z) = q^-(z)/q^+(z)$.

Lemma 6.4.18. *Suppose $z \in \mathbb{C}$ satisfies $|z| < \frac{1}{(6N^2)^{\alpha+2}}$ then $|\beta(z) + 1| \geq \frac{1}{2}|z|^\alpha$.*

Proof. We can assume that $z \neq 0$. We have

$$\begin{aligned}
|q^+(z)| = |1 + \sum_{n=1}^{\infty} q_n^+ z^n| &\leq 1 + \sum_{n=1}^{\infty} |q_n^+||z|^n \leq 1 + \sum_{n=1}^{\infty} N \cdot (6N^2)^n |z|^n \\
&\leq 1 + N \sum_{n=1}^{\infty} \left(\frac{1}{(6N^2)^{\alpha+1}} \right)^n = 1 + \frac{N}{(6N^2)^{\alpha+1} - 1} < \frac{3}{2},
\end{aligned}$$

where we used Lemma 6.4.10 for the bound on $|q_n^+|$. We now also have

$$|\beta(z) + 1| = \left| \frac{q^+(z) + q^-(z)}{q^+(z)} \right| \geq \frac{2}{3} \left| q^+(z) + q^-(z) \right| = \frac{2}{3} |z|^\alpha \cdot \left| \frac{q^+(z) + q^-(z)}{z^\alpha} \right|.$$

We now use Lemma 6.4.14, which says that $q_n^+ + q_n^- = 0$ for $n < \alpha$, while $q_\alpha^+ + q_\alpha^- \geq 1$. We get

$$\begin{aligned}
\left| \frac{q^+(z) + q^-(z)}{z^\alpha} \right| &\geq 1 - \sum_{n=1}^{\infty} (|q_{\alpha+n}^+| + |q_{\alpha+n}^-|)|z|^n \\
&\geq 1 - 2N \sum_{n=1}^{\infty} (6N^2)^{n+\alpha} |z|^n = 1 - \frac{2N(6N^2)^\alpha}{(6N^2)^{\alpha+1} - 1} > \frac{3}{4}.
\end{aligned}$$

We therefore find that $|\beta(z) + 1| > \frac{1}{2}|z|^\alpha$. □

The following is a purely geometric lemma that will be useful in the subsequent proof.

Lemma 6.4.19. *Let $0 < \rho < 1$. The disk of radius ρ around -1 contains the sector*

$$S_\rho = \{z \in \mathbb{C} : 1 - \frac{1}{2}\rho \leq |z| \leq 1 + \frac{1}{2}\rho \text{ and } \pi - \frac{1}{5}\pi\rho \leq \arg(z) \leq \pi + \frac{1}{5}\pi\rho\}.$$

Moreover, for an integer n with $n \geq 40/\rho$ the sector S_ρ contains at least $\frac{1}{8}n\rho + 2$ distinct nth roots of unity, i.e. $\zeta \in \mathbb{C}$ such that $\zeta^n = 1$.

Proof. Take $z \in S_\rho$. We can write $-z = r(\cos(\theta) + i\sin(\theta))$ for real values r, θ with $|1 - r| \leq \frac{1}{2}\rho$ and $|\theta| \leq \frac{1}{5}\pi\rho$. We thus find

$$|1 - z|^2 = 1 - 2r\cos(\theta) + r^2 \leq (1 - r)^2 + r\theta^2 \leq \frac{1}{4}\rho^2 + \frac{3}{2}(\frac{1}{5}\pi)^2\rho^2 < \rho^2,$$

where we used that $\cos(\theta) \geq 1 - \theta^2/2$. We conclude that the distance from -1 to z is indeed less than ρ.

Now let $n \in \mathbb{Z}_{\geq 1}$. For even n the distinct roots of unity inside S_ρ are given by $-\exp(2\pi ik/n)$ for integer k satisfying $|k| \leq \frac{1}{10}\rho n$. There are $2\lfloor\frac{1}{10}\rho n\rfloor + 1$ such k. For odd n the distinct roots of unity inside S_ρ are given by $-\exp(\pi i(2k + 1)/n)$ for integer k satisfying $|2k + 1| \leq \frac{1}{5}n\rho$ there are $\lfloor\frac{1}{10}n\rho - \frac{1}{2}\rfloor + \lfloor\frac{1}{10}n\rho + \frac{1}{2}\rfloor + 1$ such k. In both cases there are at least

$$\frac{1}{5}n\rho - 1 = \frac{1}{8}n\rho + \frac{3}{40}n\rho - 1 \geq \frac{1}{8}n\rho + 2$$

roots of unity inside S_ρ. □

We can now prove Theorem 6.4.6, which we restate here for convenience.

Theorem 6.4.6. *Let $R > (6N^2)^{\alpha+2}$ and $n \geq 80 \cdot R^\alpha$ then $Z(C_n \square \mathcal{T}; \lambda)$ has at least $\frac{1}{16}nR^{-\alpha}$ distinct zeros with magnitude at least R.*

Proof. Let $B_{1/R}$ denote the disk of radius $1/R$. By Lemma 6.4.18 the image $\beta(B_{1/R})$ contains a disk of radius $\frac{1}{2}R^{-\alpha}$ around -1. By Lemma 6.4.19 this disk contains a sector $S_{\frac{1}{2}R^{-\alpha}}$ as defined in that lemma.

Let $k = \lceil\frac{1}{16}nR^{-\alpha}\rceil$. It follows from from Lemma 6.4.19 that there are at least $k + 2$ angles $\theta_1, \ldots, \theta_{k+2}$, ordered increasingly, such that $e^{in\theta_m} = 1$ and $e^{i\theta_m}$ is contained in $S_{\frac{1}{2}R^{-\alpha}}$ for all m. For $m = 1, \ldots, k + 1$ define

$$T_m = \{z \in \mathbb{C} : 1 - \frac{1}{4}R^{-\alpha} \leq |z| \leq 1 + \frac{1}{4}R^{-\alpha} \text{ and } \theta_m \leq \arg(z) \leq \theta_{m+1}\}.$$

Observe that $T_m \subseteq \beta(B_{1/R})$ for all m.

We claim that for any $w \in \partial T_m$ we have $|1+w^n| > \left(\frac{1}{2}\right)^n N$. Because $n \gg N$ clearly $\left(\frac{1}{2}\right)^n N < \frac{1}{2}$, so it will be sufficient to prove that $|1+w^n| > \frac{1}{2}$. On the radial arcs of ∂T_m we have $w^n = |w|^n$ so $|1+w^n| = 1+|w|^n > \frac{1}{2}$. If w lies in the inner circular arc of ∂T_m we have

$$|1+w^n| \geq 1-|w|^n = 1-\left(1-\frac{1}{4}R^{-\alpha}\right)^n \geq 1-\exp\left[-\frac{n}{4}R^{-\alpha}\right] \geq 1-e^{-20} > \frac{1}{2}.$$

If w lies on the outer circular arc of ∂T_m we have

$$|1+w^n| \geq |w|^n - 1 = \left(1+\frac{1}{4}R^{-\alpha}\right)^n - 1 \geq 1+\frac{n}{4}R^{-\alpha} - 1 \geq 20 > \frac{1}{2}.$$

This proves the claim.

We now recall that

$$\frac{z^{n\alpha}\cdot Z(C_n\Box\mathcal{T};1/z)}{q^+(z)^n} = 1+\beta(z)^n + \sum_{s\neq q^{\pm}(z)}\left(\frac{s}{q^+(z)}\right)^n,$$

where the sum runs over the eigenvalues of M_z not equal to $q^{\pm}(z)$. Let $Q(z)$ denote this latter sum. By Lemma 6.4.17 we have that $|Q(z)| \leq \left(\frac{1}{2}\right)^n N$ for all $z \in B_{1/R}$. Note that T_m contains an element w_0 such that $w_0^n = -1$. Consider a connected component C_m of $\beta^{-1}(T_m)$ inside $B_{1/R}$. By the maximum modulus principle C_m is simply connected and ∂C_m is mapped to ∂T_m by β. Moreover, C_m contains an element z_0 in its interior with $\beta(z_0) = w_0$. For $z \in \partial C_m$ we thus have

$$|1+\beta(z)^n| > \left(\tfrac{1}{2}\right)^n N \geq |Q(z)|,$$

while $1+\beta(z_0)^n = 0$. It follows from Rouché's theorem that $1+\beta(z)^n + Q(z)$ contains a zero inside the interior of C_m. Therefore $z^{n\alpha}\cdot Z(C_n\Box\mathcal{T};1/z)$ has $k+1$ distinct zeros inside $B_{1/R}$. As long as such a zero z is itself nonzero then $\lambda = 1/z$ is a zero of $Z(C_n\Box\mathcal{T};\lambda)$ with norm at least R. We conclude that $Z(C_n\Box\mathcal{T};\lambda)$ has at least $k = \lceil\frac{1}{16}nR^{-\alpha}\rceil$ such zeros. $\square$

This theorem leads to a proof of the second part of the main theorem as is shown in the beginning of this section.

6.5 An FPTAS for balanced tori

In this section we give a proof of Proposition 6.1.1. We will require the Newton identities that we recall here for convenience of the reader. Let $p(x) = \sum_{j=0}^n a_j x^j$

be a polynomial with positive constant term and let

$$\log p(x) = \log(a_0) + \sum_{j\geq 1} -p_j \frac{x^j}{j}$$

be the series expansion of the logarithm of p around 0. Then the Newton identities yield (cf. [PR17, Proposition 2.2])

$$ka_k = -\sum_{i=0}^{k-1} a_i p_{k-i} \tag{6.20}$$

for each $k \geq 1$, where $a_i = 0$ for $i > n$.

Proposition 6.1.1 immediately follows from the following more detailed result.

Proposition 6.5.1. *Let $d \in \mathbb{Z}_{\geq 2}$ and let $C > 0$. Let $\delta(d, C)$ be the constant from Theorem 6.3.7. For each λ such that $|\lambda| > 1/\delta(d, C)$ there exists an FPTAS for approximating $Z_{\mathcal{T}}(\lambda)$ for $\mathcal{T} \in \mathbf{T}(d, C)$.*

Proof. Let us write $p_1(z) := Z^{\text{even}}_{\text{match}}(\mathcal{T}; z) + Z^{\text{large}}_{\text{match}}(\mathcal{T}; z)$, $p_2(z) := Z^{\text{odd}}_{\text{match}}(\mathcal{T}; z)$ and $p(z) = p_1(z) + p_2(z)$. Taking $z = 1/\lambda$, it suffices to approximate $p(z)$ by Corollary 6.2.17.

Since p has no zeros in the disk of radius $\delta(d, C)$ it suffices by Barvinok's interpolation method ([Bar16, Section 2.2]) to compute an ε-approximation to $\log p(z)$. This can be done by computing the first $O(\log(n/\varepsilon))$ coefficients of the Taylor series of $\log p(z)$. By (6.20) we can compute the first m coefficients of the Taylor series of $\log p(z)$ from the first m coefficients of the polynomial $p(z)$ in $O(m^2)$ time. These coefficients in turn can be obtained from the first m coefficients of $p_1(z)$ and $p_2(z)$, which in turn, using (6.20) again, can be computed from the first m coefficients of the Taylor series of $\log p_1(z)$ and $\log p_2(z)$ in $O(m^2)$ time. To obtain an FPTAS it thus suffices to compute the first $O(\log(n/\varepsilon))$ of the Taylor series of $\log p_1(z)$ and $\log p_2(z)$ in time polynomial in n/ε.

By the cluster expansion we have power series expressions for $\log p_1(z)$ given in (6.13) and for $\log p_2(z)$ given in (6.6) using Theorem 6.2.39. From these we can extract the coefficients of the respective Taylor series. Indeed, we can restrict the sum (6.13) to clusters $X = \{\gamma_1, \ldots, \gamma_k\}$ such that $\sum_{i=1}^{k} \|\gamma_i\| \leq m$ and compute the coefficients of z^j for $j \leq m$ of this restricted series. The idea is to do this iteratively, since the weights appearing in the sum, $w(\gamma; z)$ are ratios of partition functions of smaller domains for which we can assume that we have already computed the first m coefficients of its Taylor expansion around 0.

To make this precise we need to combine some ingredients from [HPR19]. We wish to apply Theorem 2.2 from [HPR19][1]. For this we should view both p_1 and

[1]In the published version there is an error in the proof of that result, but this is corrected in a later arXiv version `arXiv:1806.11548v3`

p_2 as polymer partition functions of a collection of bounded degree graphs. For us this collection will be the collection of all induced closed subgraphs of tori contained in $\mathbf{T}_d(C)$ and denoted by $\mathfrak{G}$. (Here we maintain the information of the torus containing the closed induced subgraph.) For p_2 this is clear but for p_1 this is a bit more subtle since in [HPR19] supports of polymers are connected subgraphs of graphs in $\mathfrak{G}$. We would like to view our contours as polymers, but large contours may have disconnected support. With this change, there are some potentials issues with the proof of Theorem 2.2. We first indicate how to circumvent these issues and then verify the assumptions of that theorem.

One potential issue is in the use of [HPR19, Lemma 2.4]. We sidestep this in a similar way as in the proof of Lemma 6.3.1.

Let $G \in \mathfrak{G}$. We know that G is an induced closed subgraph of some torus $\mathcal{T}$ in $\mathbf{T}_d(C)$. Let ℓ_1 be the shortest side length of $\mathcal{T}$. Then the number of vertices of G, denoted by n, is at most $\exp(C\ell_1)$. We need to list all subgraphs H of G such that either H is connected or that each component of H has size at least ℓ_1 (since any component of a large contour has at least ℓ_1 vertices) in time $\exp(O(m))$. For connected graphs H this follows directly from [HPR19, Lemma 2.4]. We now address the listing of subgraphs H that are not necessarily connected. The number of components of such H is at most m/ℓ_1. By [HPR19, Lemma 2.4] it takes time $n\exp(O(m_i))$ to list all connected subgraphs H_i of size m_i and therefore it takes time $n^t\exp(\sum_{i=1}^t O(m_i))$ to list all subgraphs H with t components of sizes $m_1, \ldots, m_t$ respectively. Let us denote $k := \lceil m/\ell_1 \rceil$. Putting this together this gives a running time bound of

$$\begin{aligned}
&\sum_{\substack{m_1,\ldots,m_k \\ \sum m_i = m \text{ and } m_i \geq \ell_1}} \prod_{i=1}^{k} n\exp(O(m_i)) \leq \binom{m+k}{m} n^k \exp(O(m)) \\
&= n^k \exp(O(m)) \leq \exp(kC\ell_1)\exp(O(m)) = \exp(O(m)),
\end{aligned}$$

for listing these graphs, as desired.

Another potential issue is in the use of cluster graphs in the proof of [HPR19, Theorem 2.2]. In [HPR19] cluster graphs are assumed to be connected, but for us they may be disconnected (in case one of the contours in the cluster is large). In that case we have a lower bound of ℓ_1 on the size of each component. So as above we can construct the list of all cluster graphs of size $O(m)$ in time $\exp(O(m))$. With these modifications the proof of Theorem 2.2 given in [HPR19] still applies.

We next verify all the assumptions of (the modification of) Theorem 2.2 in [HPR19].

The first assumption in the theorem is clearly satisfied, since $\|\gamma\| \leq |\overline{\gamma}|$ for any contour γ.

Our weight functions satisfy Assumption 1 in [HPR19] by Lemma 6.2.30. It follows from the proof of [HPR19, Lemma 3.3] that the first m coefficients of the weights $w(\gamma; z)$ can be computed in time $\exp(m + \log|\gamma|)$. Here we need to take into account that large contours may consist of more than one component, and they should come first in the ordering of contours that is created in the proof of that lemma.

In our setting the third requirement translates that for a subgraph H of some $G \in \mathfrak{G}$ we need to be able to list all polymers whose support is equal to H in time $\exp(O(|V(H)|)$. Let $\mathcal{T}$ be the torus containing G. Let ℓ_1 denote its smallest side length. In case H is not connected we know that we are dealing with a potentially large contour, while if H is connected we have to compute its box-diameter to check whether or not the contour is large or small. This can be done in time polynomial in $|V(H)|$. If H is a candidate large contour it must have size at least ℓ_1 and since the number of vertices of G is at most $\exp(O(\ell_1) = \exp(O(|V(H)|)$, it follows that we can determine all components of $\mathcal{T} \setminus V(H)$ in time $\exp(O(|V(H)|)$. If H is a candidate small contour, we can determine all components of $\mathcal{T} \setminus V(H)$ of size bounded by $|V(H)|^d$ in time polynomial in $|V(H)|$, by breadth first search. The remaining component must then be the exterior of the candidate contour. We then go over all possible ways of assigning $0, 1$ to the vertices of $V(H)$ and types to the components, i.e. select even or odd and check whether this yields a valid configuration. For this we need to check that vertices of H are incorrect as per Definition 6.2.9. Since the number of components is at most $O(|V(H)|)$ this takes time $\exp(O(|V(H)|)$.

The fourth assumption requires zero-freeness, which follows from convergence of the cluster expansion given in Theorem 6.2.34 in combination with Theorem 6.2.39 for p_2 and in Theorem 6.3.3 for p_1.

This finishes the proof. $\square$

SUMMARY

In this dissertation I study various questions related to models from statistical physics. Another motivation for the questions I study comes from computer science. Techniques from (complex) dynamical systems are used throughout this dissertation, as the questions I study have natural related dynamical systems associated to them. In this way my work lies at the intersection of statistical physics, computer science and dynamical systems.

In Part I of this dissertation I study the antiferromagnetic Potts model, which originates in statistical physics. In particular I study the transition from multiple Gibbs measures to a unique Gibbs measure for the antiferromagnetic Potts model on the infinite regular tree. This is called a uniqueness phase transition. The uniqueness phase transition for the antiferromagnetic Potts model on the infinite regular tree is much less well understood then the ferromagnetic counterpart, see Theorem 5 in [GŠVY16].

Folklore conjecture. *Let $q, \Delta \in \mathbb{Z}_{\geq 2}$. Define $w_c := \max(1 - \frac{q}{\Delta}, 0)$. The q-state antiferromagnetic Potts model on the infinite regular tree $\mathbb{T}_\Delta$ with edge interaction w has a unique Gibbs measure if and only if*

$$\begin{cases} w > 0 & \text{for } \Delta = q, \\ w \geq w_c & \text{otherwise.} \end{cases}$$

This conjecture was confirmed for $q = 2$ and all Δ by Srivastava, Sinclair and Thurley [SST14] and for $q = 3$ and $\Delta \geq 3$ by Galanis, Goldberg and Yang [GGY18]. For random regular graphs of large enough degree, uniqueness of the Gibbs measure implies the existence an efficient randomized algorithm to approximately sample from the Gibbs measure by Theorem 2.7 in [BGG$^+$20].

In Chapter 3 I confirm the folklore conjecture for $q = 4$. The proof uses a geometric condition, which comes from analysing an associated dynamical system. This also provides a new proof of the folklore conjecture for $q = 3$.

In Chapter 4 I confirm the folklore conjecture for any integer $q \geq 5$, provided Δ is large enough. I employ a similar proof strategy as in Chapter 3, using the fact that the dynamical system is more well behaved in the limit $\Delta \to \infty$.

Part II of this dissertation concerns zeros of the independence polynomial. The independence polynomial originates in statistical physics as the partition function of the hard-core model. The location of the complex zeros of the independence polynomial is related to phase transitions in terms of the analycity of the free energy [YL52] and plays an important role in the design of efficient algorithms to approximately compute evaluations of the independence polynomial.

In Chapter 5 I directly relate the location of the complex zeros of the independence polynomial to computational hardness of approximating evaluations of the independence polynomial. This is done by moreover relating the set of zeros of the independence polynomial to chaotic behaviour of a naturally associated family of rational functions; the occupation ratios.

In Chapter 6 of this dissertation I study boundedness of zeros of the independence polynomial of tori for sequences of tori converging to the integer lattice $\mathbb{Z}^d$. It is shown that zeros are bounded for sequences of balanced tori, but unbounded for sequences of highly unbalanced tori. Here balanced means that the size of the torus is at most exponential in the shortest side length, while highly unbalanced means that the longest side length of the torus is super exponential in the product over the other side lengths cubed. For technical reasons I only consider tori for which all side lengths are even and call those tori even.

From the boundedness of zeros of balanced even tori it follows there exist efficient algorithms for approximating the independence polynomial on balanced even tori. This provides a slight improvement on the approximation algorithm in [HPR19].

SAMENVATTING

In dit proefschrift bestudeer ik verscheidene vragen gerelateerd aan modellen uit de statische fysica. Er is ook motivatie voor de vragen die ik bestudeer vanuit de theoretische informatica. Technieken uit de (complexe) dynamische systemen worden in dit proefschrift gebruikt, gezien de vragen die ik bestudeer op natuurlijke wijze een bijbehorend dynamisch systeem hebben. Op deze manier bevindt mijn werk zich in de doorsnede van de statistiche fysica, de theoretische informatica en dynamische systemen.

In Deel I van dit proefschrift bestudeer ik het antiferromagnetische Potts-model, dat zijn oorsprong kent in de statistische fysica. Meer specifiek bestudeer ik de overgang van meerdere Gibbs-maten naar een unieke Gibbs-maat voor het antiferromagnetische Potts-model op de oneindige reguliere boom. Dit wordt een uniciteitsfaseovergang genoemd. Deze uniciteitsfaseovergang voor het antiferromagnetische Potts-model op de oneindige reguliere boom is veel minder goed begrepen dan de ferromagnetische tegenhanger, zie Stelling 5 in [GŠVY16].

Folklore vermoeden. *Laat* $q, \Delta \in \mathbb{Z}_{\geq 2}$. *Definieer* $w_c := \max(1 - \frac{q}{\Delta}, 0)$. *Het antiferromagnetische Potts model met* q *toestanden op de oneindig reguliere boom* $\mathbb{T}_\Delta$ *met kantinteractieparameter* w *heeft een unieke Gibbsmaat dan en slechts dan als*

$$\begin{cases} w > 0 & \textit{voor } \Delta = q, \\ w \geq w_c & \textit{anders.} \end{cases}$$

Dit vermoeden is bewezen voor $q = 2$ en alle Δ door Srivastava, Sinclair en Thurley [SST14] en voor $q = 3$ en $\Delta \geq 3$ door Galanis, Goldberg en Yang [GGY18]. Voor random reguliere grafen van voldoende grote graad impliceert uniciteit van de Gibbs-maat het bestaan van efficiënte gerandomiseerde algoritmen om bij benadering een steekproef te nemen uit de Gibbs-maat, zie Stelling 2.7 in [BGG+20]. In Hoodstuk 3 bevestig ik het folklore vermoeden voor $q = 4$. Het bewijs gebruikt een meetkundige conditie die voortkomt uit een analyse van het bijbehorende dynamische systeem. Dit geeft ook een nieuw bewijs van het folklore vermoeden voor $q = 3$.

In Hoofdstuk 4 bevestig ik het folklore vermoeden voor elk geheel getal $q \geq 5$, onder de voorwaarde dat Δ voldoende groot is. Ik gebruik een vergelijkbare bewijsstrategie als in Hoofdstuk 3, gebruikmakend van het feit dat het dynamische systeem zich beter gedraagd in het limiet $\Delta \to \infty$.

Deel II van dit proefschrift betreft nulpunten van de onafhankelijkheidspolynoom. De onafhankelijkheidspolynoom vindt zijn oorsprong in de statische fysica als de partitiefunctie van het harde-kern-model. De ligging van de complexe nulpunten van de onafhankelijkheidspolynoom is gerelateerd aan faseovergangen in termen van de analyticiteit van de vrije energie [YL52] en speelt een belangrijke rol in het ontwerp van efficiënte algoritmen om bij benadering evaluaties van de onafhankelijkheidspolynoom uit te rekenen.

In Hoofdstuk 5 relateer ik de ligging van de complexe nulpunten van de onafhankelijkheidspolynoom aan de computationele moeilijkheid van het bij benadering evaluaties van de onafhankelijkheidspolynoom uitrekenen. Dit wordt gedaan door middel van het relateren van de verzameling van de complexe nulpunten van de onafhankelijkheidspolynoom aan het chaotische gedrag van een natuurlijke bijbehorende familie van rationale functies; de bezettingsratios.

In Hoofdstuk 6 van dit proefschrift bestudeer ik de begrensdheid van de nulpunten van de onafhankelijkheidspolynoom van tori voor rijen van tori die naar het geheeltallige rooster $\mathbb{Z}^d$ convergeren. Bewezen wordt dat nulpunten begrensd blijven voor rijen van gebalanceerde tori, maar onbegrensd zijn voor rijen van zeer ongebalanceerde tori. Hier betekent gebalanceerd dat de grootte van de torus hoogstens exponentieel is in de kortste zijdelengte, terwijl zeer ongebalanceerd betekent dat de langste zijdelengte super exponentieel is in het product over de andere zijdelengten tot de derdemacht. Wegens technische redenen bestudeer ik alleen tori waarvan alle zijdelengten even zijn en noem dergelijke tori even.

Uit de begrensdheid van de nulpunten van de onafhankelijkheidspolynoom van gebalanceerde tori volgt het bestaan van efficiënte algoritmen om bij benadering evaluaties van de onafhankelijkheidspolynoom van gebalanceerde tori uit te rekenen. Dit vormt een kleine verbetering op het benaderingsalgoritme in [HPR19].

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