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Quasi-Measurable Spaces

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We introduce the categories of quasi-measurable spaces, which are slight generalizations of the category of quasi-Borel spaces, where we now allow for general sample spaces and less restrictive random variables, quasi-measurable spaces and maps. We show that each category of quasi-measurable spaces is bi-complete and cartesian closed. We also introduce several different strong probability monads. Together these constructions provide convenient categories for higher probability theory that also support semantics of higher-order probabilistic programming languages in the same way as the category of quasi-Borel spaces does.

An important special case is the category of quasi-universal spaces, where the sample space is the set of the real numbers $\mathbb{R}$ together with the σ-algebra of all universally measurable subsets. The induced σ-algebras on those quasi-universal spaces then have explicit descriptions in terms of intersections of Lebesgue-complete σ-algebras. A central role is then played by countably separated quasi-universal spaces and universal quasi-universal spaces, which replace the role of standard Borel measurable spaces. We then go on and prove in this setting, besides others, a Fubini theorem, a disintegration theorem for Markov kernels, a Kolmogorov extension theorem and a conditional de Finetti theorem. We also translate our findings into properties of the corresponding Markov category of Markov kernels between universal quasi-universal spaces.

Furthermore, we formalize probabilistic graphical models like causal Bayesian networks in terms of quasi-universal spaces and prove a global Markov property for them. For this we need to translate the notion of transitional conditional independence into this setting and study its (asymmetric) separoid

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rules. Together with the existence of exponential objects in this category we are now able to reason about conditional independence relations between variables and causal mechanisms on equal footing. Finally, we also highlight how one can use exponential objects and random functions for counterfactual reasoning.

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Keywords: Quasi-Borel spaces, standard Borel spaces, universal measurable spaces, quasi-measurable spaces, quasi-universal spaces, convenient category, measure theory, higher probability theory, categorical probability theory, strong probability monad, Fubini theorem, disintegration of Markov kernels, Kolmogorov extension theorem, conditional de Finetti theorem, Markov category, causal Bayesian networks, transitional conditional independence, asymmetric separoid axioms, global Markov property, independent causal mechanisms, counterfactual reasoning.

Contents

1 Introduction

1.1 Motivation

Even though mathematicians have been on the quest for finding the right framework for persuing topology for more than 50 years and been rewarded with the discovery of the *convenient category* of compactly generated weakly Hausdorff spaces, quasi-topologies and the category of condensed sets, see [Spa63, Ste67, McC69, EH99, EH02, Str09, SC19, SC20], the same endavour for persuing *higher probability theory* just started rather recently. However, by mirroring the found principles for those categories of topological spaces, a major milestone was quickly achieved by the construction of the category of *quasi-Borel spaces* **QBS**, see [HKSY17, ŚKV+17, SSSW21].

The main motivation for the latter was to enable probabilistic programming to incorporate higher-order functions, see [SWY+16, HKSY17]. Even though the probability monad of Giry, see [Law62, Gir82], is strong and provides a semantic foundation for first-order probabilistic programming language, see [SWY+16], it does not support higher-order functions, because the category of measurable spaces **Meas** is not cartesian closed, see [Aum61]. The category of quasi-Borel spaces was then constructed to allow for such higher-order functions and was shown to be cartesian closed, see [HKSY17], which thus solved those problems.

Since the lack of exponential objects inside the category of measurable spaces **Meas** also provides obstacles and complications in other application domains like probabilistic graphical models and causal reasoning, see below, the authors of the paper at hand were motivated to further investigate those topics in slightly more generality under the umbrella term of the categories of *quasi-measurable spaces*.

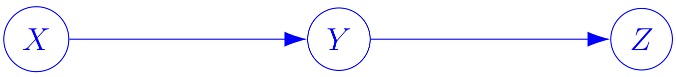


Figure 1: A Markov chain with output variables X, Y, Z. The graph allows us to read off the conditional independence: $Z \perp\!\!\!\perp X \,|\, Y$.

To motivate further, consider a Markov chain with three variables X, Y, Z, see Figure 1. Their joint distribution is then given by the factorization:

$$P(X, Y, Z) = P(Z|Y) \otimes P(Y|X) \otimes P(X). \tag{I}$$

This factorization then shows the conditional independence:

$$Z \perp\!\!\!\perp X \,|\, Y.$$

One could have as well read this off the graph in Figure 1 via the d-separation statement $Z \perp X \,|\, Y$ and the global Markov property, see e.g. [LDLL90, Lau96, Ric03, KF09, Pea09, FM17, FM18, FM20, For21]. $Z \perp\!\!\!\perp X \,|\, Y$ means that when the value of Y is provided then X has no additional information about the state of Z. From the factorization in Equation I we can even read off that the value of Z is only determined through the Markov kernel

$P(Z|Y)$ when Y is provided. So the distribution $P(X)$ of X and the Markov kernel $P(Y|X)$ of Y both have no influence on Z when conditioned on Y. So in some sense Z is independent not only of X, but also of $P(X)$ and $P(Y|X)$ when conditioned on Y. If we use $Q(X)$, $Q(Y|X)$ as variables for the values $P(X)$, $P(Y|X)$, resp., then we would like to be able, first, to formalize a conditional independence statement of form:

$$Z \perp\!\!\!\perp (X, Q(X), Q(Y|X)) \,|\, Y,$$

and, second, to read this off a graph like in Figure 2 via some global Markov property. This comes with several challenges.

First, the variables $Q(X)$ and $Q(Y|X)$ are not random variables in the usual sense as there is no distribution defined over them. The problem of treating random variables and such deterministic non-random variables on equal footing was generally solved in [For21] in the category of measurable space **Meas** by introducing the notion of *transitional conditional independence*.

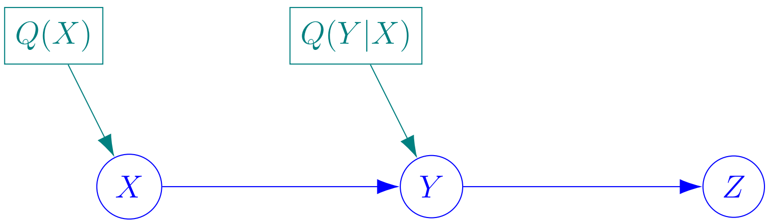


Figure 2: The partially generic causal Bayesian network with output variables X in $\mathcal{X}$ and Y in $\mathcal{Y}$ and Z in $\mathcal{Z}$ and input variables $Q(X)$ in $\mathcal{P}(\mathcal{X})$ and $Q(Y|X)$ in $\mathcal{P}(\mathcal{Y})^{\mathcal{X}}$. The graph allows us to read off the transitional conditional independence: $Z \perp\!\!\!\perp (X, Q(X), Q(Y|X)) \,|\, Y$.

Second, for a global Markov property to hold we need to incorporate variables $Q(Y|X)$ into a (causal) Bayesian network. For this, note that X and Y take values in a measurable spaces $(\mathcal{X}, \mathcal{B}_{\mathcal{X}})$ and $(\mathcal{Y}, \mathcal{B}_{\mathcal{Y}})$, resp., while $Q(Y|X)$ takes values in the space $\mathcal{L}$ of Markov kernels from $\mathcal{X}$ to $\mathcal{Y}$:

$$\mathcal{L} := \mathbf{Meas}\left((\mathcal{X}, \mathcal{B}_{\mathcal{X}}), (\mathcal{G}(\mathcal{Y}, \mathcal{B}_{\mathcal{Y}}), \mathcal{B}_{\mathcal{G}(\mathcal{Y},\mathcal{B}_{\mathcal{Y}})})\right),$$

where $\mathcal{G}(\mathcal{Y}, \mathcal{B}_{\mathcal{Y}})$ is the space of all probability measures on $(\mathcal{Y}, \mathcal{B}_{\mathcal{Y}})$ endowed with the smallest σ-algebra $\mathcal{B}_{\mathcal{G}(\mathcal{Y},\mathcal{B}_{\mathcal{Y}})}$ that makes all evaluation maps $\mu \mapsto \mu(B)$, $B \in \mathcal{B}_{\mathcal{Y}}$, measurable. So, to incorporate $Q(Y|X)$ into a Baysian network, we would need to construct a (measurable) Markov kernel from the product space $\mathcal{X} \times \mathcal{L}$ to $\mathcal{Y}$. More precisely, we would need to construct a σ-algebra $\mathcal{B}_{\mathcal{L}}$ on $\mathcal{L}$ such that the following map is measurable:

$$\text{ev}: \; (\mathcal{X} \times \mathcal{L}, \mathcal{B}_{\mathcal{X}} \otimes \mathcal{B}_{\mathcal{L}}) \to \left(\mathcal{G}(\mathcal{Y}, \mathcal{B}_{\mathcal{Y}}), \mathcal{B}_{\mathcal{G}(\mathcal{Y},\mathcal{B}_{\mathcal{Y}})}\right), \qquad (x, Q(Y|X)) \mapsto Q(Y|X = x).$$

Unfortunately and related to the fact that the category of measurable spaces **Meas** is not cartesian closed, in general, there does not exist any σ-algebra $\mathcal{B}_{\mathcal{L}}$ on $\mathcal{L}$ that makes the above map ev measurable, where $\mathcal{B}_{\mathcal{X}} \otimes \mathcal{B}_{\mathcal{L}}$ is the product σ-algebra, which is generated by the cylinder sets $A \times D$, with $A \in \mathcal{B}_\mathcal{X}$ and $D \in \mathcal{B}_\mathcal{L}$; not even in the case where the measurable spaces $\mathcal{X}$ and $\mathcal{Y}$ are (uncountable) standard Borel measurable spaces, see [Aum61].

To remedy this shortcoming we engage in studying quasi-Borel spaces and, slightly more general, *quasi-measurable spaces*. The main idea behind quasi-measurable spaces can best be described when contrasted with the classical setup of measure-theoretic probability theory:

In the *classical setup of measure-theoretic probability theory* one usually starts with the assumption of the presence of a (typically not further specified) measurable space $(\Omega, \mathcal{B}_\Omega)$, the *sample space*, and a probability measure P on it. Then one considers the *data spaces* $\mathcal{X}$, typically $\mathbb{R}$ or $\mathbb{R}^D$, where the real world measurements happen. One then specifies a σ-algebra $\mathcal{B}_\mathcal{X}$ of subsets of $\mathcal{X}$ as the collection of all admissible *events* on $\mathcal{X}$. When $\mathcal{B}_\mathcal{X}$ is given then the admissible *random variables* on $\mathcal{X}$ that are used to describe the data points or random measurements on $\mathcal{X}$ are all measurable maps X from $(\Omega, \mathcal{B}_\Omega)$ to $(\mathcal{X}, \mathcal{B}_\mathcal{X})$. The set of all admissible random variables is then:

$$\mathcal{X}^\Omega := \mathcal{F}(\mathcal{B}_\mathcal{X}) := \mathbf{Meas}\left((\Omega, \mathcal{B}_\Omega), (\mathcal{X}, \mathcal{B}_\mathcal{X})\right).$$

Note that the set of random variables is dependent on the choice of the σ-algebra. So if we change the σ-algebra $\mathcal{B}_\mathcal{X}$ we implicitely change the set of admissible random variables $\mathcal{X}^\Omega$ with it.

In the *theory of quasi-measurable spaces* we swap the roles of the set of admissible events $\mathcal{B}_\mathcal{X}$ and set of admissible random variables $\mathcal{X}^\Omega$. We again start with a sample space $(\Omega, \mathcal{B}_\Omega)$, but then directly specify the admissible set of random variables $\mathcal{X}^\Omega$ on the data space $\mathcal{X}$. This then *induces* a σ-algebra of admissible events on $\mathcal{X}$ via:

$$\mathcal{B}_\mathcal{X} := \mathcal{B}(\mathcal{X}^\Omega) := \left\{ A \subseteq \mathcal{X} \,\middle|\, \forall X \in \mathcal{X}^\Omega.\, X^{-1}(A) \in \mathcal{B}_\Omega \right\}.$$

Note that if we now change the set of admissible random variables $\mathcal{X}^\Omega$ we would implicitely change the σ-algebra $\mathcal{B}_\mathcal{X}$. Also note that the σ-algebra $\mathcal{B}_\mathcal{X}$ is now dependent on the choice of $(\Omega, \mathcal{B}_\Omega)$ and the set of random variables $\mathcal{X}^\Omega$.

The tuple $(\mathcal{X}, \mathcal{X}^\Omega)$ will then be called a *quasi-measurable space w.r.t.* $(\Omega, \mathcal{B}_\Omega)$. The properties of the corresponding category of quasi-measurable spaces **QMS** will then be dependent on the choice of the sample space $(\Omega, \mathcal{B}_\Omega)$, which thus needs to be specified and studied more closely, in contrast to the classical setup.

If we are now coming back to our example of Markov chains we will need to construct/formalize probabilistic graphical models in this setting, here *causal Bayesian networks*. Furthermore, we need to study *probability monads* and *Markov kernels* between quasi-measurable spaces. Also, to translate the mentioned notion of *transitional conditional independence* from the category of measurable spaces **Meas** to the category of quasi-measurable spaces **QMS** we need to follow the steps in [For21] and prove a *disintegration theorem* for Markov kernels in **QMS**.

For such disintegration results one usually needs close control over the σ-algebras of the involved spaces, see [For21] Cor. C.8. This is the point where the quasi-Borel spaces from [HKSY17] become rather inconvenient to work with. The reason is that the induced

σ-algebras come, as described above, as push-forward σ-algebras of the sample space along the quasi-measurable functions. This roughly means that the induced σ-algebras will be pushed onto the images of such maps and then extended by null-sets outside, without necessarily including null-sets on that image. So the induced σ-algebras will, in general and in some sense, be partially complete and partially incomplete. To overcome this inconsistency and harmonize the situation it seems natural to use sample spaces that are complete with respect to all relevant probability measures. Since, furthermore, for the disintegration theorem one requires *perfect* probability measures and *universally countably generated* σ-algebras, see [Fad85, Pac78, Fre15, For21], one then almost necessarily[1] ends up requiring a sample space that is an uncountable Polish space endowed with its σ-algebra of universally measurable subsets, which then satisfies all those constraints and requirements. The corresponding category of quasi-measurable spaces with such a sample space will be called the category of *quasi-universal spaces* **QUS**. These are the reasons we strongly engage in developing the theory for quasi-universal spaces in this paper.

1.2 Contributions

Closely following the construction of quasi-Borel spaces **QBS**, see [HKSY17], we introduce the categories of *quasi-measurable spaces* **QMS**. For this we mainly make two slight changes:

First, we set up the theory of quasi-measurable spaces to allow for different samples spaces Ω, 2.2, other than the real numbers $\mathbb{R}$ with its Borel σ-algebra.

Second, for the Definition 2.5 of quasi-measurable spaces we drop the 3rd property in the definition of quasi-Borel spaces, see [HKSY17] Def. 7. This simplifies and generalizes the theory of quasi-measurable spaces **QMS** at the same time. For our purposes, we (almost) loose none of the good properties of **QBS**, see below:

The categories of quasi-measurable spaces **QMS** are shown to have all (small) limits and colimits, see Theorem 2.38 and 2.45. Arbitrary (small) products and coproducts distribute, see Theorem 2.42. The categories of quasi-measurable spaces **QMS** are *cartesian closed*, see Theorem 2.32, thus allowing for higher-order functions and simply typed λ-calculus. We also maintain the adjunction between the category of measurable spaces **Meas** and each of the categories of quasi-measurable spaces **QMS**, see Theorem 2.16.

By not insisting on the mentioned 3rd property that quasi-Borel spaces have, [HKSY17] Def. 7, the right-adjoint $\mathcal{F}$ of that adjunction does not perserve countable coproducts of measurable spaces anymore. However, we will study quasi-measurable spaces with that property on its own in Section 4 and name them *patchable* quasi-measurable spaces. We then show in Theorem 4.10 that the corresponding category of patchable quasi-measurable spaces **PQMS** is a reflexive subcategory of **QMS**, and in fact, an exponential ideal. This allows us to just reflect back to **PQMS**, if needed, e.g. for the mentioned countable coproducts. We then also study under which constructions patchability is

[1] See Lemma 5.18.

perserved, like products and quotients, etc.

In Section 3 we introduce and study several *probability monads* $\mathcal{Q}$, $\mathcal{K}$, $\mathcal{P}$, $\mathcal{R}$, $\mathcal{S}$. Since the Giry construction $\mathcal{G}$, see [Law62, Gir82], does not interact well with the (new) structures, e.g. products, of quasi-measurable spaces we do not expect it to induce a strong monad on **QMS** and quickly drop it from further analysis. Instead, we restrict to probability measures that interact well with such products and get the noval *strong probability monad* $\mathcal{Q}$ out, see Theorem 3.16. The monads of push-forward probability measures $\mathcal{K}$, $\mathcal{P}$, $\mathcal{R}$, $\mathcal{S}$ are all slight variations of $\mathcal{P}$ on **QBS** from [HKSY17], one more general ($\mathcal{K}$), one more restrictive ($\mathcal{S}$), and one complementary ($\mathcal{R}$) to $\mathcal{P}$. Under the assumption that the sample space Ω satisfies $\Omega \times \Omega \cong \Omega$ in **QMS** all those probability monads become strong, see Theorem 3.25, and thus allow for semantics for a probabilistic programming language in the monadic style with higher-order functions similar to how the category of quasi-Borel spaces does, see [Mog91, HKSY17, ŚKV⁺17, SSSW21].

The assumption $\Omega \times \Omega \cong \Omega$ is satisfied, for instance, if Ω is a (countable) infinite product of another space Ω_0 like $\Omega \cong \Omega_0^{\mathbb{N}}$ or if Ω is an (uncountable) Polish space, like the Hilbert cube $[0,1]^{\mathbb{N}}$ or just $\mathbb{R}$, either endowed with its Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$ or its universal completion $(\mathcal{B}_{\mathbb{R}})_{\mathcal{G}}$, i.e. its σ-algebra of all universally measurable subsets.

For any uncountable Polish space with the σ-algebra of all universally measurable subsets, e.g. $(\Omega, \mathcal{B}_\Omega) = (\mathbb{R}, (\mathcal{B}_{\mathbb{R}})_{\mathcal{G}})$, we call the corresponding category the *category of quasi-universal spaces* **QUS**. This comes with further convenient properties. The induced σ-algebras $\mathcal{B}_{\mathcal{X}}$ on quasi-universal spaces $(\mathcal{X}, \mathcal{X}^\Omega)$ now have a description as the intersection of Lebesgue-complete σ-algebras, see Lemma 5.5. This becomes most pronounced for *countably separated* quasi-universal spaces, see Theorem 5.16. Note that giving such a clear description of the induced σ-algebras was not possible for quasi-Borel spaces, see [HKSY17]. We also show that the strong probability monads of push-forward probability measures $\mathcal{K}$, $\mathcal{P}$, $\mathcal{R}$, $\mathcal{S}$ all agree for quasi-universal spaces, see Theorem 5.29. Furthermore, we are able to prove several important theorems for quasi-universal spaces. This ranges from a *Fubini Theorem* 5.33, over a *Disintegration Theorem* 5.35 for Markov kernels and *Kolmogorov Extension Theorems* 5.40 and 5.41, to a *Conditional De Finetti Theorem* 5.47.

Quasi-universal spaces that have similar good properties like standard Borel spaces have are called *universal quasi-universal spaces* and its category is abbreviated as **UQUS**. Universal quasi-universal spaces are exactly the retracts of the sample space Ω in **QUS**, see Lemma 5.22. We then also express the found properties of **UQUS** in terms of *categorical probability theory*, see Theorem 5.46:

The category of universal quasi-universal spaces $(\mathbf{UQUS}, \times, \mathbf{1})$ is a symmetric cartesian (but not closed) monoidal category with countable products and countable coproducts. The triple $(\mathcal{P}, \delta, \mathbb{M})$ is a strong affine symmetric monoidal/commutative monad on $(\mathbf{UQUS}, \times, \mathbf{1})$. Its Kleisli category $\mathrm{Kl}(\mathcal{P})$ on **UQUS** is an a.s.-compatibly representable Markov category with conditionals and Kolmogorov powers, see [Kle65, CJ19, FR20, Fri20, FGPR20, FGP21].

In Section 6 we study conditional independence relations and causal models in the category of quasi-universal spaces **QUS**. There we translate the notion of *transitional conditional independence* in [For21] from **Meas** to **QUS**, see Definition 6.5, and show all the corresponding *(asymmetric) separoid rules* in Theorem 6.7. These are then used to prove the *global Markov property* for *causal Bayesian networks*, see Theorem 6.19, which relates the underlying graphical structure to the transitional conditional independencies of its variables via a d-separation criterion.

We then introduce the notion of *partially generic causal Bayesian networks*, see Defintion 6.21. Finally, with this and together with the strong probability monad $\mathcal{P}$, the exponential objects $\mathcal{P}(\mathcal{X})^{\mathcal{Z}}$ in **QUS**, and the global Markov property we now arrive at our goal and can formulate conditional independence relations between variables and causal mechanisms on equal footing and reason with them graphically as wanted and explained in the beginning with the example of Markov chains, see Figure 2.

2 The Category of Quasi-Measurable Spaces

In this section we will construct the *category of quasi-measurable spaces* **QMS** w.r.t. a fixed sample space Ω. These construction closely follow the constructions of quasi-Borel spaces from [HKSY17], also see [ŚKV+17, SSSW21], but where we now allow for different sample spaces Ω other than the set of real numbers $\mathbb{R}$. We will have products, coproducts, limits, colimits and function spaces/exponential objects, etc., in **QMS**.

2.1 Quasi-Measurable Spaces

In this subsection we will go through the basic definitions of the sample space Ω, quasi-measurable spaces and quasi-measurable maps. This will then constitute a category, the category of *quasi-measurable spaces* **QMS** w.r.t. Ω.

Notation 2.1. *For two sets $\mathcal{X}$ and $\mathcal{Y}$ we denote by:*

$$[\mathcal{X} \to \mathcal{Y}] := \mathbf{Sets}(\mathcal{X}, \mathcal{Y}) := \{f : \mathcal{X} \to \mathcal{Y}\}$$

the set of all (set-theoretic) maps from $\mathcal{X}$ to $\mathcal{Y}$.

2.1.1 The Sample Space

Here we will introduce the basic requirements for the *sample space* Ω for the category of quasi-measurable spaces **QMS**.

Notation 2.2 (The sample space)**.** *In the following let $(\Omega, \mathcal{B}_\Omega, \Omega^\Omega)$ be a triple consisting of:*[2]

1. *a measurable space $(\Omega, \mathcal{B}_\Omega)$, i.e. $\mathcal{B}_\Omega$ is a σ-algebra of subsets of Ω,*
2. *a set of functions $\Omega^\Omega \subseteq \mathbf{Meas}\left((\Omega, \mathcal{B}_\Omega), (\Omega, \mathcal{B}_\Omega)\right)$ that:*

[2] We will later further assume that $\Omega \cong \Omega \times \Omega$. This will be needed to get well-defined and strong probability monads.

3. *is closed under composition:*

$$\circ : \Omega^\Omega \times \Omega^\Omega \to \Omega^\Omega, \qquad (\varphi_1, \varphi_2) \mapsto \varphi_1 \circ \varphi_2,$$

4. *contains the constant maps:*

$$\Omega^{\mathbf{1}} := [\mathbf{1} \to \Omega] \circ [\Omega \to \mathbf{1}] \subseteq \Omega^\Omega,$$

5. *and the identity map* $\mathrm{id}_\Omega \in \Omega^\Omega$.

Remark 2.3. 1. *If* $(\Omega, \mathcal{B}_\Omega)$ *is a given measurable space then a valid choice for* Ω^Ω *in* 2.2 *is:*

$$\Omega^\Omega := \mathcal{F}(\mathcal{B}_\Omega) := \mathbf{Meas}\left((\Omega, \mathcal{B}_\Omega), (\Omega, \mathcal{B}_\Omega)\right).$$

2. *If* Ω *is a set and* Ω^Ω *a set of functions from* Ω *to* Ω *that is closed under composition and that contains the constant maps* $\Omega^{\mathbf{1}}$ *and* $\mathrm{id}_\Omega \in \Omega^\Omega$ *then a valid choice for* $\mathcal{B}_\Omega$ *in* 2.2 *is:*

$$\mathcal{B}_\Omega := \sigma\left(\left\{\varphi_2^{-1}\left(\varphi_1(\Omega)\right) \,\middle|\, \varphi_1, \varphi_2 \in \Omega^\Omega\right\}\right),$$

or even its universal completion. We clearly then have:

$$\Omega^\Omega \subseteq \mathbf{Meas}\left((\Omega, \mathcal{B}_\Omega), (\Omega, \mathcal{B}_\Omega)\right).$$

Example 2.4. *Let* $\Omega = \mathbb{R}$ *and* $\Omega^\Omega = \mathbf{Top}(\mathbb{R}, \mathbb{R})$ *the set of continuous maps from* $\mathbb{R}$ *to* $\mathbb{R}$*, which is closed under composition and contains the identity and all constant maps. Then we have that:*

$$\mathcal{B}_\Omega := \sigma\left(\left\{\varphi_2^{-1}\left(\varphi_1(\Omega)\right) \,\middle|\, \varphi_1, \varphi_2 \in \Omega^\Omega\right\}\right),$$

is the Borel σ*-algebra of* $\mathbb{R}$*, and:*

$$\mathcal{B}_\Omega := \left(\left\{\varphi_2^{-1}\left(\varphi_1(\Omega)\right) \,\middle|\, \varphi_1, \varphi_2 \in \Omega^\Omega\right\}\right)_{\mathcal{G}},$$

the σ*-algebra of all universally measurable subsets of* $\mathbb{R}$*. In any case, we have:*

$$\Omega^\Omega \subseteq \mathbf{Meas}\left((\Omega, \mathcal{B}_\Omega), (\Omega, \mathcal{B}_\Omega)\right).$$

Proof. Let $\varphi_1, \varphi_2 \in \mathbf{Top}(\mathbb{R}, \mathbb{R})$. Then $\varphi_1(\mathbb{R})$ is the countable union of closed compact subsets of $\mathbb{R}$. So $\varphi_2^{-1}(\varphi_1(\mathbb{R})) \subseteq \mathbb{R}$ is a countable union of closed subsets of $\mathbb{R}$ thus a Borel subset of $\mathbb{R}$. Since we can generate all intervals $(a, b) = \varphi_1(\mathbb{R})$ for appropriate φ_1 and with $\varphi_2 = \mathrm{id}_\mathbb{R}$ we generate the whole Borel σ-algebra of $\mathbb{R}$. ☐

2.1.2 Quasi-Measurable Spaces

Here we will introduce the most important definition of this paper: *quasi-measurable spaces*. The definition resembles the definition of quasi-Borel spaces from [HKSY17] Def. 7. There is one main difference though: we do not require our quasi-measurable spaces to satisfy the 3rd property of [HKSY17] Def. 7. That property will separately be studied in Section 4 in Definition 4 under the name of *patchable* quasi-measurable spaces. The reason we can drop that property for the most part of this paper is that almost all constructions we encounter in this paper work without that property. The only point where it matters is clarified in Lemma 4.12.

Definition 2.5 (Quasi-measurable spaces)**.** *A* quasi-measurable space $(\mathcal{X}, \mathcal{X}^\Omega)$ *w.r.t.* (Ω, Ω^Ω) *is a set* $\mathcal{X}$ *together with a set of maps:*

$$\mathcal{X}^\Omega \subseteq [\Omega \to \mathcal{X}]$$

such that:

1. *for all* $\alpha \in \mathcal{X}^\Omega$ *and* $\varphi \in \Omega^\Omega$ *we also have:* $\alpha \circ \varphi \in \mathcal{X}^\Omega$*, in short:*

$$\mathcal{X}^\Omega \circ \Omega^\Omega \subseteq \mathcal{X}^\Omega.$$

2. *all constant maps* $\Omega \to \mathcal{X}$*, mapping all elements to one point, are in* $\mathcal{X}^\Omega$*, in short:*

$$\mathcal{X}^{\mathbf{1}} := [\mathbf{1} \to \mathcal{X}] \circ [\Omega \to \mathbf{1}] \subseteq \mathcal{X}^\Omega.$$

By abuse of notations, we will refer to $\mathcal{X}$ *as the quasi-measurable space, while actually meaning* $(\mathcal{X}, \mathcal{X}^\Omega)$ *w.r.t.* (Ω, Ω^Ω)*.*

Example 2.6. *If* $\mathcal{X}$ *is a set and* $\mathcal{X}^\Omega := [\Omega \to \mathcal{X}]$ *then* $(\mathcal{X}, \mathcal{X}^\Omega)$ *is a quasi-measurable space.*

Example 2.7 (Measurable spaces as quasi-measurable spaces)**.**

1. *If* $(\mathcal{X}, \mathcal{B}_\mathcal{X})$ *is a measurable space and:*

   $$\mathcal{X}^\Omega := \mathcal{F}(\mathcal{B}_\mathcal{X}) := \mathbf{Meas}\left((\Omega, \mathcal{B}_\Omega), (\mathcal{X}, \mathcal{B}_\mathcal{X})\right) = \{\alpha : (\Omega, \mathcal{B}_\Omega) \to (\mathcal{X}, \mathcal{B}_\mathcal{X}) \text{ measurable}\},$$

   then $(\mathcal{X}, \mathcal{X}^\Omega)$ *is a quasi-measurable space w.r.t.* Ω*.*

   Indeed, the composition of measurable maps is measurable and constant maps are measurable.

2. *Note that* (Ω, Ω^Ω) *and* $(\Omega, \mathcal{F}(\mathcal{B}_\Omega))$ *might, in general, be two different quasi-measurable spaces (w.r.t.* (Ω, Ω^Ω)*).*

3. *If* $(\mathcal{X}, \mathcal{T}_\mathcal{X})$ *is a topological space (e.g.* $\mathbb{R}^D$*, etc.) we usually endow it with the set of maps induced by its Borel* σ*-algebra. More explicitely this is:*

   $$\mathcal{X}^\Omega := \mathcal{F}(\sigma(\mathcal{T}_\mathcal{X})) = \left\{\alpha : \Omega \to \mathcal{X} \,\middle|\, \forall A \in \mathcal{T}_\mathcal{X}.\, \alpha^{-1}(A) \in \mathcal{B}_\Omega\right\}.$$

4. *If* $\mathcal{X}$ *is a countable set (e.g.* $\mathbf{1} := \{0\}$*,* $\mathbf{2} := \{0, 1\}$*,* $\mathbb{N}$*, etc.) we usually endow it with the discrete topology, power-set* σ*-algebra, and use the above quasi-measurable space structure, which reduces to the* discrete *quasi-measurable space structure, defined by:*

   $$\mathcal{X}^\Omega := \left\{\alpha : \Omega \to \mathcal{X} \,\middle|\, \forall x \in \mathcal{X}.\, \alpha^{-1}(x) \in \mathcal{B}_\Omega\right\}.$$

2.1.3 Quasi-Measurable Maps

The second ingredient for the category of quasi-measurable spaces **QMS** are its morphisms: the *quasi-measurable maps*. This again closely follows [HKSY17].

Definition 2.8 (Quasi-measurable maps)**.** *Let* $(\mathcal{Y}, \mathcal{Y}^\Omega)$ *and* $(\mathcal{Z}, \mathcal{Z}^\Omega)$ *be quasi-measurable spaces w.r.t.* Ω*. A map:*

$$h : \mathcal{Y} \to \mathcal{Z},$$

is called $\mathcal{Y}^\Omega$-$\mathcal{Z}^\Omega$-quasi-measurable, *or just* quasi-measurable, *for short, if:*

$$\forall \alpha \in \mathcal{Y}^\Omega. \quad h \circ \alpha \in \mathcal{Z}^\Omega,$$

or in short:

$$h \circ \mathcal{Y}^\Omega \subseteq \mathcal{Z}^\Omega.$$

The set of all quasi-measurable maps (under (Ω, Ω^Ω)*) will be denoted by:*

$$\mathbf{QMS}\left((\mathcal{Y}, \mathcal{Y}^\Omega), (\mathcal{Z}, \mathcal{Z}^\Omega)\right),$$

or just:

$$\mathbf{QMS}\left(\mathcal{Y}, \mathcal{Z}\right),$$

where we keep the dependence on the choice of $(\Omega, \Omega^\Omega, \mathcal{B}_\Omega)$ *implicit.*

Remark 2.9. *Note that the composition of quasi-measurable maps is quasi-measurable.*

Remark 2.10. *Let* $(\mathcal{X}, \mathcal{X}^\Omega)$ *be a quasi-measurable space. Then we have the identification:*

$$\mathbf{QMS}((\Omega, \Omega^\Omega), (\mathcal{X}, \mathcal{X}^\Omega)) = \mathcal{X}^\Omega.$$

Proof. For $f \in \mathcal{X}^\Omega \subseteq [\Omega \to \mathcal{X}]$ we by the point 1. of Definition 2.5 have:

$$f \circ \Omega^\Omega \subseteq \mathcal{X}^\Omega.$$

This directly implies the inclusion:

$$\mathcal{X}^\Omega \subseteq \mathbf{QMS}((\Omega, \Omega^\Omega), (\mathcal{X}, \mathcal{X}^\Omega)).$$

If, on the other hand, $g \in \mathbf{QMS}((\Omega, \Omega^\Omega), (\mathcal{X}, \mathcal{X}^\Omega))$ then we have:

$$g \circ \Omega^\Omega \subseteq \mathcal{X}^\Omega.$$

Since $\mathrm{id}_\Omega \in \Omega^\Omega$ we get that:

$$g = g \circ \mathrm{id}_\Omega \in \mathcal{X}^\Omega.$$

We thus also get:

$$\mathbf{QMS}((\Omega, \Omega^\Omega), (\mathcal{X}, \mathcal{X}^\Omega)) \subseteq \mathcal{X}^\Omega.$$

This shows the equality. ☐

Notation 2.11. *1. The category of all quasi-measurable spaces together with quasi-measurable maps w.r.t. $(\Omega, \Omega^\Omega, \mathcal{B}_\Omega)$ will be denoted by:*

$$\mathbf{QMS}.$$

2. The category of all patchable *(see Definition 4.2) quasi-measurable spaces w.r.t. the special choice $(\Omega, \mathcal{B}_\Omega) = (\mathbb{R}, \mathcal{B}_\mathbb{R})$ of the real numbers together with its Borel σ-algebra will be called the* category of quasi-Borel spaces*:*

$$\mathbf{QBS}.$$

3. The category of all quasi-measurable spaces w.r.t. the special choice $(\Omega, \mathcal{B}_\Omega) = (\mathbb{R}, (\mathcal{B}_\mathbb{R})_\mathcal{G})$ of the real numbers together with the σ-algebra of all universally measurable subsets will be called the category of quasi-universal (measurable) spaces*:*

$$\mathbf{QUS}.$$

2.2 Adjunction: Measurable Spaces - Quasi-Measurable Spaces

In this subsection we study how the category of measurable spaces **Meas** and the category of quasi-measurable spaces **QMS** are related to each other. The result can be expressed in an *adjunction* of two functors between those categories. The right-adjoint $\mathcal{F}$ maps a σ-algebra $\mathcal{B}_\mathcal{X}$ to all $\mathcal{B}_\Omega$-$\mathcal{B}_\mathcal{X}$-measurable maps $\mathcal{F}(\mathcal{B}_\mathcal{X})$ and the left-adjoint $\mathcal{B}$ maps a set of random variables/quasi-measurable functions $\mathcal{X}^\Omega$ to the σ-algebra $\mathcal{B}(\mathcal{X}^\Omega)$ they generate. The notion of adjoint functors, limits and colimits we are going to use were defined in [Kan58] Ch. II, also see e.g. [Joh02, ML98].

In the following let $(\Omega, \Omega^\Omega, \mathcal{B}_\Omega)$ be a fixed sample space (see Notation 2.2).

Notation 2.12 (σ-algebra induced by functions)**.** *For any set $\mathcal{X}$ and any subset of maps $\mathcal{X}^\Omega \subseteq [\Omega \to \mathcal{X}]$ we define the following σ-algebra on $\mathcal{X}$:*

$$\mathcal{B}(\mathcal{X}^\Omega) := \left\{ A \subseteq \mathcal{X} \,\middle|\, \forall \alpha \in \mathcal{X}^\Omega.\, \alpha^{-1}(A) \in \mathcal{B}_\Omega \right\},$$

which is the biggest σ-algebra $\mathcal{B}_\mathcal{X}$ on $\mathcal{X}$ such that all $\alpha \in \mathcal{X}^\Omega$ are still $\mathcal{B}_\Omega$-$\mathcal{B}_\mathcal{X}$-measurabel.

Notation 2.13 (Functions induced by σ-algebras)**.** *Let $\mathcal{X}$ be any set and $\mathcal{B}_\mathcal{X}$ any σ-algebra of subsets of $\mathcal{X}$. Then we put:*

$$\mathcal{F}(\mathcal{B}_\mathcal{X}) := \mathbf{Meas}\left((\Omega, \mathcal{B}_\Omega), (\mathcal{X}, \mathcal{B}_\mathcal{X})\right) \quad \subseteq \quad [\Omega \to \mathcal{X}].$$

which is the biggest set of $\mathcal{B}_\Omega$-$\mathcal{B}_\mathcal{X}$-measurable functions $\mathcal{X}^\Omega \subseteq [\Omega \to \mathcal{X}]$ such that $(\mathcal{X}, \mathcal{X}^\Omega)$ is still a quasi-measurable space.

Lemma 2.14. *Let $\mathcal{X}$ be a set $\mathcal{B}_\mathcal{X}$ be a σ-algebra of subsets of $\mathcal{X}$ and $\mathcal{X}^\Omega \subseteq \mathbf{Meas}((\Omega, \mathcal{B}_\Omega), (\mathcal{X}, \mathcal{B}_\mathcal{X}))$ any subset of measurable maps. Then we have the inclusions:*

$$\mathcal{B}_\mathcal{X} \subseteq \mathcal{B}(\mathcal{F}(\mathcal{B}_\mathcal{X})) \subseteq \mathcal{B}(\mathcal{X}^\Omega),$$

and:

$$\mathcal{X}^\Omega \subseteq \mathcal{F}(\mathcal{B}(\mathcal{X}^\Omega)) \subseteq \mathcal{F}(\mathcal{B}_\mathcal{X}).$$

In particular, we get:

$$\mathcal{B}(\mathcal{F}(\mathcal{B}(\mathcal{X}^\Omega))) = \mathcal{B}(\mathcal{X}^\Omega),$$

and:

$$\mathcal{F}(\mathcal{B}(\mathcal{F}(\mathcal{B}_\mathcal{X}))) = \mathcal{F}(\mathcal{B}_\mathcal{X}).$$

Proof. Since by assumption we have: $\mathcal{X}^\Omega \subseteq \mathcal{F}(\mathcal{B}_\mathcal{X})$ we get the reverse inclusion when applying $\mathcal{B}$ (smaller set of functions means less constraints on the measurable subsets):

$$\mathcal{B}(\mathcal{F}(\mathcal{B}_\mathcal{X})) \subseteq \mathcal{B}(\mathcal{X}^\Omega).$$

Since by assumption every $\alpha \in \mathcal{X}^\Omega$ is $\mathcal{B}_\Omega$-$\mathcal{B}_\mathcal{X}$-measurable we have that for every $A \in \mathcal{B}_\mathcal{X}$ that $\alpha^{-1}(A) \in \mathcal{B}_\Omega$. Since this holds for every $\alpha \in \mathcal{X}^\Omega$ we see that $A \in \mathcal{B}(\mathcal{X}^\Omega)$. This shows the inclusion:

$$\mathcal{B}_\mathcal{X} \subseteq \mathcal{B}(\mathcal{X}^\Omega).$$

Since this would also hold for the special choice $\mathcal{X}^\Omega = \mathcal{F}(\mathcal{B}_\mathcal{X})$ we get:

$$\mathcal{B}_\mathcal{X} \subseteq \mathcal{B}(\mathcal{F}(\mathcal{B}_\mathcal{X})).$$

This completes the first chain of inclusions.
By applying $\mathcal{F}$ we get the reverse inclusion (smaller σ-algebras mean less constraints on the measurability of functions):

$$\mathcal{F}(\mathcal{B}(\mathcal{X}^\Omega)) \subseteq \mathcal{F}(\mathcal{B}_\mathcal{X}).$$

Since $\mathcal{B}(\mathcal{X}^\Omega)$ is the biggest σ-algebra such that all functions from $\mathcal{X}^\Omega$ are measurable we also have the inclusion:

$$\mathcal{X}^\Omega \subseteq \mathbf{Meas}((\Omega, \mathcal{B}_\Omega), (\mathcal{X}, \mathcal{B}(\mathcal{X}^\Omega))) = \mathcal{F}(\mathcal{B}(\mathcal{X}^\Omega)).$$

This completes the second chain of inclusions.
If we use the corner case: $\mathcal{B}_\mathcal{X} := \mathcal{B}(\mathcal{X}^\Omega)$, then the first chain of inclusions become equalities:

$$\mathcal{B}(\mathcal{X}^\Omega) \subseteq \mathcal{B}(\mathcal{F}(\mathcal{B}(\mathcal{X}^\Omega))) \subseteq \mathcal{B}(\mathcal{X}^\Omega).$$

If we use the corner case: $\mathcal{X}^\Omega := \mathcal{F}(\mathcal{B}_\mathcal{X})$, then the second chain of inclusions become equalities:

$$\mathcal{F}(\mathcal{B}_\mathcal{X}) \subseteq \mathcal{F}(\mathcal{B}(\mathcal{F}(\mathcal{B}_\mathcal{X}))) \subseteq \mathcal{F}(\mathcal{B}_\mathcal{X}).$$

□

Remark 2.15. *Consider the sample space* $(\Omega, \Omega^\Omega, \mathcal{B}_\Omega)$. *Then we have:*

$$\mathcal{B}(\Omega^\Omega) = \mathcal{B}_\Omega.$$

This means that there is no ambiguity in notation for $\mathcal{B}_\Omega$. *Note that, in general, we might have:* $\mathcal{F}(\mathcal{B}_\Omega) \supsetneq \Omega^\Omega$.

Proof. By Lemma 2.14 we have:

$$\mathcal{B}_\Omega \subseteq \mathcal{B}(\Omega^\Omega).$$

On the other hand, since $\mathrm{id}_\Omega \in \Omega^\Omega$, we get for $B \in \mathcal{B}(\Omega^\Omega)$ that $B = \mathrm{id}_\Omega^{-1}(B) \in \mathcal{B}_\Omega$. This shows the other inclusion:

$$\mathcal{B}(\Omega^\Omega) \subseteq \mathcal{B}_\Omega.$$

□

Theorem 2.16 (Adjunction). *The functor:*

$$\mathcal{B}: \ \mathbf{QMS} \to \mathbf{Meas}, \quad (\mathcal{X}, \mathcal{X}^\Omega) \mapsto (\mathcal{X}, \mathcal{B}(\mathcal{X}^\Omega)), \quad \mathcal{B}(f) = f,$$

with:

$$\mathcal{B}(\mathcal{X}^\Omega) := \left\{ A \subseteq \mathcal{X} \,\middle|\, \forall \alpha \in \mathcal{X}^\Omega .\, \alpha^{-1}(A) \in \mathcal{B}_\Omega \right\},$$

is left-adjoint to the functor:

$$\mathcal{F}: \ \mathbf{Meas} \to \mathbf{QMS}, \quad (\mathcal{Y}, \mathcal{B}_\mathcal{Y}) \mapsto (\mathcal{Y}, \mathcal{F}(\mathcal{B}_\mathcal{Y})), \quad \mathcal{F}(g) = g,$$

where:

$$\mathcal{F}(\mathcal{B}_\mathcal{Y}) := \mathbf{Meas}\left((\Omega, \mathcal{B}_\Omega), (\mathcal{Y}, \mathcal{B}_\mathcal{Y})\right).$$

That means for quasi-measurable $(\mathcal{X}, \mathcal{X}^\Omega)$ and measurable $(\mathcal{Y}, \mathcal{B}_\mathcal{Y})$ we have canonical identifications:

$$\mathbf{Meas}\left((\mathcal{X}, \mathcal{B}(\mathcal{X}^\Omega)), (\mathcal{Y}, \mathcal{B}_\mathcal{Y})\right) = \mathbf{QMS}\left((\mathcal{X}, \mathcal{X}^\Omega), (\mathcal{Y}, \mathcal{F}(\mathcal{B}_\mathcal{Y}))\right).$$

Furthermore, we have:

$$\mathcal{B} \circ \mathcal{F} \circ \mathcal{B} = \mathcal{B}, \qquad \mathcal{F} \circ \mathcal{B} \circ \mathcal{F} = \mathcal{F},$$

and: $\mathcal{B}_\mathcal{Y} \subseteq \mathcal{B}\mathcal{F}(\mathcal{B}_\mathcal{Y})$ and $\mathcal{X}^\Omega \subseteq \mathcal{F}\mathcal{B}(\mathcal{X}^\Omega)$.

Proof. Most of this was already shown in Lemma 2.14.
For the adjunction let $g \in \mathbf{Meas}\left((\mathcal{X}, \mathcal{B}(\mathcal{X}^\Omega)), (\mathcal{Y}, \mathcal{B}_\mathcal{Y})\right)$ and $\alpha \in \mathcal{X}^\Omega \subseteq \mathbf{Meas}\left((\Omega, \mathcal{B}_\Omega), (\mathcal{X}, \mathcal{B}(\mathcal{X}^\Omega))\right)$. Then $g \circ \alpha \in \mathbf{Meas}\left((\Omega, \mathcal{B}_\Omega), (\mathcal{Y}, \mathcal{B}_\mathcal{Y}))\right) = \mathcal{F}(\mathcal{B}_\mathcal{Y})$. Since this holds for all $\alpha \in \mathcal{X}^\Omega$ we get that: $g \in \mathbf{QMS}\left((\mathcal{X}, \mathcal{X}^\Omega), (\mathcal{Y}, \mathcal{F}(\mathcal{B}_\mathcal{Y}))\right)$. This implies the inclusion:

$$\mathbf{Meas}\left((\mathcal{X}, \mathcal{B}(\mathcal{X}^\Omega)), (\mathcal{Y}, \mathcal{B}_\mathcal{Y})\right) \subseteq \mathbf{QMS}\left((\mathcal{X}, \mathcal{X}^\Omega), (\mathcal{Y}, \mathcal{F}(\mathcal{B}_\mathcal{Y}))\right).$$

Now let $f \in \mathbf{QMS}\left((\mathcal{X}, \mathcal{X}^\Omega), (\mathcal{Y}, \mathcal{F}(\mathcal{B}_\mathcal{Y}))\right)$ then $f \in \mathbf{Meas}\left((\mathcal{X}, \mathcal{B}(\mathcal{X}^\Omega)), (\mathcal{Y}, \mathcal{B}\mathcal{F}(\mathcal{B}_\mathcal{Y})\right) \subseteq \mathbf{Meas}\left((\mathcal{X}, \mathcal{B}(\mathcal{X}^\Omega)), (\mathcal{Y}, \mathcal{B}_\mathcal{Y})\right)$. This shows the inclusion:

$$\mathbf{Meas}\left((\mathcal{X}, \mathcal{B}(\mathcal{X}^\Omega)), (\mathcal{Y}, \mathcal{B}_\mathcal{Y})\right) \supseteq \mathbf{QMS}\left((\mathcal{X}, \mathcal{X}^\Omega), (\mathcal{Y}, \mathcal{F}(\mathcal{B}_\mathcal{Y}))\right).$$

□

Definition 2.17 (Sturdy (quasi-)measurable spaces). *We define the full subcategories:*

1. **SMeas** $:= \mathcal{B}(\mathbf{QMS})$ *of* sturdy measurable spaces *inside* **Meas**.

2. **SQMS** $:= \mathcal{F}(\mathbf{Meas})$ *of* sturdy quasi-measurable spaces *inside* **QMS**.

Remark 2.18. 1. *A measurable space* $(\mathcal{X}, \mathcal{B}_\mathcal{X})$ *is sturdy iff* $\mathcal{B}_\mathcal{X} = \mathcal{B}\mathcal{F}(\mathcal{B}_\mathcal{X})$.

2. *A quasi-measurable space* $(\mathcal{X}, \mathcal{X}^\Omega)$ *is sturdy iff* $\mathcal{X}^\Omega = \mathcal{F}\mathcal{B}(\mathcal{X}^\Omega)$.

Proof. This directly follows from:

$$\mathcal{B} \circ \mathcal{F} \circ \mathcal{B} = \mathcal{B}, \qquad \mathcal{F} \circ \mathcal{B} \circ \mathcal{F} = \mathcal{F}.$$

□

Corollary 2.19. *The adjunction:*

$$\mathbf{QMS} \underset{\mathcal{F}}{\overset{\mathcal{B}}{\rightleftarrows}} \mathbf{Meas} \qquad (\bot)$$

restricts to an equivalence on the full subcategories given by the essential images of $\mathcal{B}$ *and* $\mathcal{F}$*, resp.:*

$$\mathbf{SQMS} \underset{\mathcal{F}}{\overset{\mathcal{B}}{\rightleftarrows}} \mathbf{SMeas}. \qquad (\bot)$$

Proof. Every adjunction $L \dashv R$ restricts to an equivalence on the full subcategories given by those objects where the unit $\eta : \mathrm{id} \to RL$, the counit $\varepsilon : LR \to \mathrm{id}$, resp., is an isomorphism. In our case this follows from:

$$\mathcal{B} \circ \mathcal{F} \circ \mathcal{B} = \mathcal{B}, \qquad \mathcal{F} \circ \mathcal{B} \circ \mathcal{F} = \mathcal{F}.$$

□

Remark 2.20. *The main point of Corollary 2.19 is that sturdy measurable spaces can be identified with sturdy quasi-measurable spaces and their relations are the same either considered as sturdy measurable spaces or as sturdy quasi-measurable spaces.*

Example 2.21. 1. *If* $\mathcal{F}(\mathcal{B}_\Omega) = \Omega^\Omega$ *and* $(\mathcal{X}, \mathcal{B}_\mathcal{X}) \cong (\Omega, \mathcal{B}_\Omega)$ *(in* **Meas***) then* $(\mathcal{X}, \mathcal{B}_\mathcal{X}) \in$ **SMeas**.

2. *The latter, for instance, is the case if* $(\Omega, \mathcal{B}_\Omega) = (\mathbb{R}, \mathcal{B}_\mathbb{R})$ *with the Borel* σ*-algebra and* $(\mathcal{X}, \mathcal{B}_\mathcal{X})$ *is any (uncountable) standard measurable space, e.g. if* $\mathcal{X} = \mathbb{R}^\mathbb{N} \cong \mathbb{R}$ *or* $\mathcal{X} = \mathbb{R}^D \cong \mathbb{R}$ *is the product space with the Borel* σ*-algebra.*

3. *If* $(\mathcal{X}, \mathcal{B}_\mathcal{X})$ *is discrete in the sense that* $\mathcal{B}_\mathcal{X} = \{A \subseteq \mathcal{X}\}$ *is the power-set* σ*-algebra then* $(\mathcal{X}, \mathcal{B}_\mathcal{X}) \in$ **SMeas** *(e.g. any countable discrete space). Indeed,* $\mathcal{B}\mathcal{F}(\mathcal{B}_\mathcal{X}) \supseteq \mathcal{B}_\mathcal{X}$ *must then be an equality.*

4. *If* $(\Omega, \mathcal{B}_\Omega) = (\mathbb{R}, (\mathcal{B}_\mathbb{R})_\mathcal{G})$ *is endowed with the* σ*-algebra of universally measurable subsets and* $\mathcal{X} = \mathbb{R}^\mathbb{N} \cong \mathbb{R}$ *or* $\mathcal{X} = \mathbb{R}^D \cong \mathbb{R}$ *is the product space with the universal completion of the Borel* σ*-algebra then* $(\mathcal{X}, \mathcal{B}_\mathcal{X}) \in \mathbf{SMeas}$*.*

Theorem 2.22. *The full subcategory* $\mathbf{SQMS} = \mathcal{F}(\mathbf{Meas})$ *of sturdy quasi-measurable spaces inside the category of all quasi-measurable spaces* $\mathbf{QMS}$ *is reflexiv. The reflector is given by:*

$$\mathcal{FB} : \mathbf{QMS} \to \mathbf{SQMS}, \qquad (\mathcal{X}, \mathcal{X}^\Omega) \to (\mathcal{X}, \mathcal{FB}(\mathcal{X}^\Omega)), \qquad \mathcal{FB}(f) = f,$$

and preserves all colimits and coproducts (indexed over sets).

Proof. Let $(\mathcal{X}, \mathcal{X}^\Omega)$ be a quasi-measurable space and $(\mathcal{Y}, \mathcal{Y}^\Omega)$ a sturdy quasi-measurable space. Then we need to show that that $\mathcal{FB}$ is left-adjoint to the forgetful functor $\mathbf{SQMS} \hookrightarrow \mathbf{QMS}$, i.e. that we have a natural identification:

$$\mathbf{QMS}\left((\mathcal{X}, \mathcal{FB}(\mathcal{X}^\Omega)), (\mathcal{Y}, \mathcal{Y}^\Omega)\right) = \mathbf{QMS}\left((\mathcal{X}, \mathcal{X}^\Omega), (\mathcal{Y}, \mathcal{Y}^\Omega)\right).$$

Let $f \in \mathbf{QMS}\left((\mathcal{X}, \mathcal{FB}(\mathcal{X}^\Omega)), (\mathcal{Y}, \mathcal{Y}^\Omega)\right)$ then we have:

$$f \circ \mathcal{X}^\Omega \subseteq f \circ \mathcal{FB}(\mathcal{X}^\Omega) \subseteq \mathcal{Y}^\Omega.$$

This shows: $f \in \mathbf{QMS}\left((\mathcal{X}, \mathcal{X}^\Omega), (\mathcal{Y}, \mathcal{Y}^\Omega)\right)$.
For the reverse inclusion consider $g \in \mathbf{QMS}\left((\mathcal{X}, \mathcal{X}^\Omega), (\mathcal{Y}, \mathcal{Y}^\Omega)\right)$. Then g is $\mathcal{B}(\mathcal{X}^\Omega)$-$\mathcal{B}(\mathcal{Y}^\Omega)$-measurable. Every $\alpha \in \mathcal{FB}(\mathcal{X}^\Omega)$ is, by definition, $\mathcal{B}_\Omega$-$\mathcal{B}(\mathcal{X}^\Omega)$-measurable. So the composition $g \circ \alpha$ is $\mathcal{B}_\Omega$-$\mathcal{B}(\mathcal{Y}^\Omega)$-measurable. So we get:

$$g \circ \alpha \in \mathcal{FB}(\mathcal{Y}^\Omega) = \mathcal{Y}^\Omega.$$

This shows that $g \in \mathbf{QMS}\left((\mathcal{X}, \mathcal{FB}(\mathcal{X}^\Omega)), (\mathcal{Y}, \mathcal{Y}^\Omega)\right)$.
This shows that $\mathbf{SQMS}$ is a reflexiv subcategory of $\mathbf{QMS}$.
As a left-adjoint $\mathcal{FB}$ always preserves arbitrary colimits like coproducts (indexed over sets). ☐

Remark 2.23. *Similarly, for a measurable space* $(\mathcal{Y}, \mathcal{B}_\mathcal{Y})$ *and a sturdy measurable space* $(\mathcal{X}, \mathcal{B}_\mathcal{X})$ *we have the natural identification:*

$$\mathbf{Meas}\left((\mathcal{X}, \mathcal{B}_\mathcal{X}), (\mathcal{Y}, \mathcal{B}_\mathcal{Y})\right) = \mathbf{Meas}\left((\mathcal{X}, \mathcal{B}_\mathcal{X}), (\mathcal{Y}, \mathcal{BF}(\mathcal{B}_\mathcal{Y}))\right).$$

Proof. Let $g \in \mathbf{Meas}\left((\mathcal{X}, \mathcal{B}_\mathcal{X}), (\mathcal{Y}, \mathcal{B}_\mathcal{Y})\right)$ and $B \in \mathcal{BF}(\mathcal{B}_\mathcal{Y})$ and $\alpha \in \mathcal{F}(\mathcal{B}_\mathcal{X})$. Then $g \circ \alpha \in \mathcal{F}(\mathcal{B}_\mathcal{Y})$. Since $B \in \mathcal{BF}(\mathcal{B}_\mathcal{Y})$ we get:

$$\alpha^{-1}(g^{-1}(B)) \in \mathcal{B}_\Omega.$$

Since this holds for all $\alpha \in \mathcal{F}(\mathcal{B}_\mathcal{X})$ we get:

$$g^{-1}(B) \in \mathcal{BF}(\mathcal{B}_\mathcal{X}) = \mathcal{B}_\mathcal{X}.$$

This shows that $g \in \mathbf{Meas}\left((\mathcal{X}, \mathcal{B}_\mathcal{X}), (\mathcal{Y}, \mathcal{BF}(\mathcal{B}_\mathcal{Y}))\right)$.
The other inclusion is clear. ☐

2.3 Products of Quasi-Measurable Spaces

In this subsection we define the categorical *product* of quasi-measurable spaces. The construction again closely follows [HKSY17].

Definition 2.24 (Product of quasi-measurable spaces)**.** *Let $(\mathcal{X}_i, \mathcal{X}_i^\Omega)$, $i \in I$, be a family of quasi-measurable spaces indexed by a set I. Then we endow the set-theoretic product:*

$$\prod_{i \in I} \mathcal{X}_i = \{x = (x_i)_{i \in I} \,|\, \forall i \in I.\, x_i \in \mathcal{X}_i\},$$

with the following set of maps:

$$\left(\prod_{i \in I} \mathcal{X}_i\right)^\Omega := \left\{ \alpha : \Omega \to \prod_{i \in I} \mathcal{X}_i \,\middle|\, \forall i \in I.\, \alpha_i := \mathrm{pr}_i \circ \alpha \in \mathcal{X}_i^\Omega \right\},$$

where pr_i is the canonical projection map onto $\mathcal{X}_i$. In short, we can also write this as:

$$\left(\prod_{i \in I} \mathcal{X}_i\right)^\Omega = \prod_{i \in I} \mathcal{X}_i^\Omega,$$

by identifying α with $(\alpha_i)_{i \in I}$.
We thus define the product of quasi-measurable spaces w.r.t. Ω *as:*

$$\prod_{i \in I} (\mathcal{X}_i, \mathcal{X}_i^\Omega) := \left(\prod_{i \in I} \mathcal{X}_i, \prod_{i \in I} \mathcal{X}_i^\Omega \right)$$

Lemma 2.25. *The product of quasi-measurable spaces w.r.t. Ω:*

$$\prod_{i \in I} (\mathcal{X}_i, \mathcal{X}_i^\Omega) := \left(\prod_{i \in I} \mathcal{X}_i, \prod_{i \in I} \mathcal{X}_i^\Omega \right)$$

is a quasi-measurable space w.r.t. Ω.

Proof. We check the points from Definition 2.5:

1. For $\alpha = (\alpha_i)_{i \in I} \in \prod_{i \in I} \mathcal{X}_i^\Omega$ and $\varphi \in \Omega^\Omega$ we by assumption have:

$$\forall i \in I.\, \alpha_i \circ \varphi \in \mathcal{X}_i^\Omega.$$

   It follows that:

$$\alpha \circ \varphi = (\alpha_i \circ \varphi)_{i \in I} \in \prod_{i \in I} \mathcal{X}_i^\Omega.$$

2. If $\alpha = (\alpha_i)_{i \in I}$ is a constant map, then every α_i is a constant map. By assumption such constant $\alpha_i \in \mathcal{X}_i^\Omega$. It then follows that also $\alpha \in \prod_{i \in I} \mathcal{X}_i^\Omega$.

This shows that $\prod_{i\in I}(\mathcal{X}_i, \mathcal{X}_i^\Omega)$ is a quasi-measurable space. □

Lemma 2.26. *The product of quasi-measurable spaces:*

$$\prod_{i\in I}(\mathcal{X}_i, \mathcal{X}_i^\Omega) := \left(\prod_{i\in I}\mathcal{X}_i, \prod_{i\in I}\mathcal{X}_i^\Omega\right)$$

defines a categorical product in **QMS***, i.e. for every quasi-measurable space* $(\mathcal{Z}, \mathcal{Z}^\Omega)$ *the natural map:*

$$\mathbf{QMS}\left(\mathcal{Z}, \prod_{i\in I}\mathcal{X}_i\right) \xrightarrow{\sim} \prod_{i\in I}\mathbf{QMS}\left(\mathcal{Z}, \mathcal{X}_i\right), \qquad g \mapsto (\mathrm{pr}_i \circ g)_{i\in I}$$

is a bijection.

Proof. First note that the canonical projection map:

$$\mathrm{pr}_k : \prod_{i\in I}(\mathcal{X}_i, \mathcal{X}_i^\Omega) \to (\mathcal{X}_k, \mathcal{X}_k^\Omega),$$

is a quasi-measurable map for every $k \in I$.
Next let $(\mathcal{Z}, \mathcal{Z}^\Omega)$ be another quasi-measurable space and:

$$g_k : (\mathcal{Z}, \mathcal{Z}^\Omega) \to (\mathcal{X}_k, \mathcal{X}_k^\Omega)$$

a quasi-measurable map for each $k \in I$. Then $g := (g_i)_{i\in I}$ is the only (set theoretic) map that makes the following diagram commute for all $k \in I$:

$$\begin{array}{ccc} (\mathcal{Z}, \mathcal{Z}^\Omega) & \xrightarrow{\exists! g} & \prod_{i\in I}(\mathcal{X}_i, \mathcal{X}_i^\Omega) \\ & \searrow^{g_k} & \downarrow \mathrm{pr}_k \\ & & (\mathcal{X}_k, \mathcal{X}_k^\Omega). \end{array}$$

It is left to check that g is a quasi-measurable map. If $\gamma \in \mathcal{Z}^\Omega$ by assumption we know that $g_k \circ \gamma \in \mathcal{X}_k^\Omega$ for all $k \in I$. It thus follows that:

$$g \circ \gamma = (g_i \circ \gamma)_{i\in I} \in \prod_{i\in I}\mathcal{X}_i^\Omega.$$

This thus shows the claim. □

Lemma 2.27 ($\mathcal{F}$ on products of measurable spaces)**.** *Let* $(\mathcal{X}_i, \mathcal{B}_i)$ *measurable spaces,* $i \in I$*. Then we have:*

$$\mathcal{F}\left(\bigotimes_{i\in I}\mathcal{B}_i\right) = \prod_{i\in I}\mathcal{F}(\mathcal{B}_i).$$

This means that the functor $\mathcal{F}$: **Meas** → **QMS** *preserves arbitrary products.*

Proof. Right adjoints always preserve arbitrary limits, like products.
A more concrete proof goes as follows. By the univsersal property of the product (σ-algebra), the adjunction from Theorem 2.16 and that $\mathcal{B}_\Omega = \mathcal{B}(\Omega^\Omega)$ (see Remark 2.15) we get:

$$\begin{aligned}
\mathcal{F}\left(\bigotimes_{i\in I}\mathcal{B}_i\right) &= \mathbf{Meas}\left((\Omega,\mathcal{B}_\Omega),\left(\prod_{i\in I}\mathcal{X}_i,\bigotimes_{i\in I}\mathcal{B}_i\right)\right)\\
&= \prod_{i\in I}\mathbf{Meas}\left((\Omega,\mathcal{B}_\Omega),(\mathcal{X}_i,\mathcal{B}_i)\right)\\
&= \prod_{i\in I}\mathbf{Meas}\left((\Omega,\mathcal{B}(\Omega^\Omega)),(\mathcal{X}_i,\mathcal{B}_i)\right)\\
&= \prod_{i\in I}\mathbf{QMS}\left((\Omega,\Omega^\Omega),(\mathcal{X}_i,\mathcal{F}(\mathcal{B}_i))\right)\\
&= \prod_{i\in I}\mathcal{F}(\mathcal{B}_i).
\end{aligned}$$

□

2.4 Function Spaces of Quasi-Measurable Spaces

In this subsection we will introduce the *function space* of quasi-measurable spaces. It will serve as an *exponential object/internal hom* in the category of quasi-measurable spaces **QMS**, making **QMS** cartesian closed. The construction again closely follows [HKSY17].

Definition 2.28 (Function space of quasi-measurable spaces)**.** *Let $(\mathcal{X},\mathcal{X}^\Omega)$ and $(\mathcal{Y},\mathcal{Y}^\Omega)$ be two quasi-measurable spaces w.r.t. Ω. We then define their* function space *as:*

$$\mathcal{Y}^\mathcal{X} := \mathbf{QMS}((\mathcal{X},\mathcal{X}^\Omega),(\mathcal{Y},\mathcal{Y}^\Omega)).$$

We further endow it with the following set of functions:

$$\left(\mathcal{Y}^\mathcal{X}\right)^\Omega := \left\{\beta:\Omega\to\mathcal{Y}^\mathcal{X}\,\middle|\,\forall\varphi\in\Omega^\Omega\,\forall\alpha\in\mathcal{X}^\Omega.\ \beta(\varphi)(\alpha)\in\mathcal{Y}^\Omega\right\}.$$

Lemma 2.29. *Let $(\mathcal{X},\mathcal{X}^\Omega)$ and $(\mathcal{Y},\mathcal{Y}^\Omega)$ be two quasi-measurable spaces. Then the function space $\left(\mathcal{Y}^\mathcal{X},\left(\mathcal{Y}^\mathcal{X}\right)^\Omega\right)$ is a quasi-measurable space.*

Proof. We check the points from Definition 2.5:

1. If $\beta\in\left(\mathcal{Y}^\mathcal{X}\right)^\Omega$ and $\varphi',\varphi\in\Omega^\Omega$ and $\alpha\in\mathcal{X}^\Omega$ then we get:

$$(\beta\circ\varphi')(\varphi)(\alpha) = \beta(\varphi'\circ\varphi)(\alpha)\in\mathcal{Y}^\Omega,$$

since $\varphi'\circ\varphi\in\Omega^\Omega$. By definition we then get:

$$\beta\circ\varphi'\in\left(\mathcal{Y}^\mathcal{X}\right)^\Omega.$$

2. If $\beta : \Omega \to \mathcal{Y}^{\mathcal{X}}$ is constant then there is a $\varphi \in \mathcal{Y}^{\mathcal{X}}$ such that for all ω we have $\beta(\omega) = \varphi$. Since:
$$\varphi \in \mathcal{Y}^{\mathcal{X}} = \mathbf{QMS}(\mathcal{X}, \mathcal{Y}),$$
we have for all $\alpha \in \mathcal{X}^{\Omega}$ that $\varphi(\alpha) \in \mathcal{Y}^{\Omega}$. So for $\varphi \in \Omega^{\Omega}$, $\alpha \in \mathcal{X}^{\Omega}$ we have:
$$\beta(\varphi)(\alpha) = \varphi(\alpha) \in \mathcal{Y}^{\Omega}.$$
By definition of $\left(\mathcal{Y}^{\mathcal{X}}\right)^{\Omega}$ we get that: $\beta \in \left(\mathcal{Y}^{\mathcal{X}}\right)^{\Omega}$.

This shows the claims. □

Lemma 2.30. *Let $(\mathcal{X}, \mathcal{X}^{\Omega})$ and $(\mathcal{Y}, \mathcal{Y}^{\Omega})$ be two quasi-measurable spaces. Then $\left(\mathcal{Y}^{\mathcal{X}}, \left(\mathcal{Y}^{\mathcal{X}}\right)^{\Omega}\right)$ has the universal property of a function space in the category* **QMS** *of all quasi-measurable spaces w.r.t. Ω, i.e. for any other quasi-measurable space $(\mathcal{Z}, \mathcal{Z}^{\Omega})$ the canonical curry map:*
$$\mathbf{QMS}(\mathcal{Z} \times \mathcal{X}, \mathcal{Y}) \xrightarrow{\sim} \mathbf{QMS}(\mathcal{Z}, \mathcal{Y}^{\mathcal{X}}), \qquad f \mapsto (z \mapsto (x \mapsto f(z, x))),$$
is a natural bijection.

Proof. For the universal property now consider a fixed quasi-measurable map $g : \mathcal{Z} \to \mathcal{Y}^{\mathcal{X}}$. Then the map $\tilde{g} : \mathcal{Z} \times \mathcal{X} \to \mathcal{Y}$ given by:
$$\tilde{g}(z, x) := g(z)(x),$$
is quasi-measurable. Indeed, for all $\varphi \in \mathcal{Z}^{\Omega}$ and $\alpha \in \mathcal{X}^{\Omega}$ we have that:
$$\tilde{g}(\varphi, \alpha) = g(\varphi(\mathrm{id}_{\Omega}))(\alpha) \in \mathcal{Y}^{\Omega}.$$
The inverse construction is given for quasi-measurable $f : \mathcal{Z} \times \mathcal{X} \to \mathcal{Y}$ by $\hat{f} : \mathcal{Z} \to \mathcal{Y}^{\mathcal{X}}$ via:
$$\hat{f}(z)(x) := f(z, x).$$
To show the quasi-measurability of $\hat{f}$ let $\gamma \in \mathcal{Z}^{\Omega}$ and $\varphi \in \Omega^{\Omega}$ and $\alpha \in \mathcal{X}^{\Omega}$. We need to show that:
$$\hat{f}(\gamma(\varphi))(\alpha) = f(\gamma(\varphi), \alpha) \in \mathcal{Y}^{\Omega},$$
which holds by assumption on f and since $\gamma \circ \varphi \in \mathcal{Z}^{\Omega}$.
It is then easy to see that $g \mapsto \tilde{g}$ and $f \mapsto \hat{f}$ are inverse to each other.
Finally, we mention that this bijection is natural in all three arguments, which can just be checked via composition. □

Remark 2.31. *Let $(\mathcal{X}, \mathcal{X}^{\Omega})$ and $(\mathcal{Y}, \mathcal{Y}^{\Omega})$ be two quasi-measurable spaces w.r.t. Ω.*

1. *Note that by Remark 2.10 we can identify:*
   a) *$\mathcal{Y}^{\Omega}$ from Definition 2.5,*
   b) **QMS**$((\Omega, \Omega^{\Omega}), (\mathcal{Y}, \mathcal{Y}^{\Omega}))$ *from Definition 2.8,*
   c) *$\mathcal{Y}^{\Omega} = \mathcal{Y}^{\mathcal{X}}$ for $(\mathcal{X}, \mathcal{X}^{\Omega}) = (\Omega, \Omega^{\Omega})$ in Definition 2.28.*

2. *By Lemma 2.30 we can identify:* $(\mathcal{Y}^{\mathcal{X}})^{\Omega} = \mathcal{Y}^{\Omega \times \mathcal{X}}$.

3. *The* evaluation map *is quasi-measurable:*

$$\mathrm{ev} : \mathcal{Y}^{\mathcal{X}} \times \mathcal{X} \to \mathcal{Y}, \qquad (\varphi, x) \mapsto \varphi(x).$$

This follows from the identification of ev *and* id *in:*

$$\mathrm{ev} \in \mathbf{QMS}(\mathcal{Y}^{\mathcal{X}} \times \mathcal{X}, \mathcal{Y}) \cong \mathbf{QMS}(\mathcal{Y}^{\mathcal{X}}, \mathcal{Y}^{\mathcal{X}}) \ni \mathrm{id}$$

Theorem 2.32. *The category* **QMS** *together with (finite) products* $\times$ *and the one-point quasi-measurable space* $\mathbf{1} := \{0\}$*, which is a terminal object, is a cartesian closed (symmetric braided strict monoidal) category.*

Proof. The product $\times$ constitutes a *functor* from the product category to **QMS**:

$$\times : \mathbf{QMS} \times \mathbf{QMS} \to \mathbf{QMS}, \qquad (f : \mathcal{X} \to \mathcal{Y}, g : \mathcal{W} \to \mathcal{Z}) \mapsto (f \times g : \mathcal{X} \times \mathcal{W} \to \mathcal{Y} \times \mathcal{Z}).$$

The *unit object* is $\mathbf{1} \in \mathbf{QMS}$.
We have an *associator* natural isomorphism, given for all triples as:

$$A_{\mathcal{X},\mathcal{Y},\mathcal{Z}} = \mathrm{id} : (\mathcal{X} \times \mathcal{Y}) \times \mathcal{Z} \cong \mathcal{X} \times (\mathcal{Y} \times \mathcal{Z}),$$

which in this case is always the identity (definition of *strict*).
We have the *left and right unitor* natural isomorphisms, on objects given as:

$$L_{\mathcal{X}} = \mathrm{pr}_{\mathcal{X}} : \mathbf{1} \times \mathcal{X} \cong \mathcal{X}, \qquad R_{\mathcal{X}} = \mathrm{pr}_{\mathcal{X}} : \mathcal{X} \times \mathbf{1} \cong \mathcal{X},$$

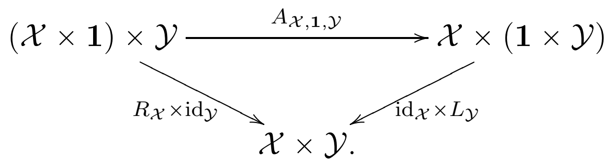
which here are the canonical projections on the first and second component, resp.
We have the *braiding* natural isomorphism, on objects given as:

$$B_{\mathcal{X},\mathcal{Y}} : \mathcal{X} \times \mathcal{Y} \cong \mathcal{Y} \times \mathcal{X},$$

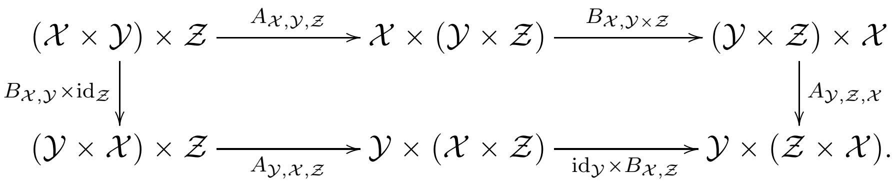
which here are given by swapping first and second entry.
Furthermore, the associator satisfies the *pentagon identity* (associativity law), which states the commutativity of the following diagram:

$$\begin{array}{ccccc}
 & & (\mathcal{W} \times \mathcal{X}) \times (\mathcal{Y} \times \mathcal{Z}) & & \\
 & \overset{A_{\mathcal{W}\times\mathcal{X},\mathcal{Y},\mathcal{Z}}}{\nearrow} & & \overset{A_{\mathcal{W},\mathcal{X},\mathcal{Y}\times\mathcal{Z}}}{\searrow} & \\
((\mathcal{W} \times \mathcal{X}) \times \mathcal{Y}) \times \mathcal{Z} & & & & \mathcal{W} \times (\mathcal{X} \times (\mathcal{Y} \times \mathcal{Z})) \\
\downarrow {\scriptstyle A_{\mathcal{W},\mathcal{X},\mathcal{Y}} \times \mathrm{id}_{\mathcal{Z}}} & & & & \uparrow {\scriptstyle \mathrm{id}_{\mathcal{W}} \times A_{\mathcal{X},\mathcal{Y},\mathcal{Z}}} \\
(\mathcal{W} \times (\mathcal{X} \times \mathcal{Y})) \times \mathcal{Z} & & \xrightarrow[A_{\mathcal{W},\mathcal{X}\times\mathcal{Y},\mathcal{Z}}]{} & & \mathcal{W} \times ((\mathcal{X} \times \mathcal{Y}) \times \mathcal{Z}).
\end{array}$$

Indeed, since the associator is given by the identity there is nothing much to show.
The left and right unitors together with the associator satisfy the *triangle identity*, which states the commutativity of the following diagram:



The braiding and the associator satisfy the (first) *hexagon identity*, which states the commutativity of the following diagram:



Finally, the braiding satisfies the *symmetry* relation:

$$B_{\mathcal{Y},\mathcal{X}} \circ B_{\mathcal{X},\mathcal{Y}} = \mathrm{id}_{\mathcal{X}\times\mathcal{Y}},$$

which in this case needs no explanation.
It follows that $(\mathbf{QMS}, \times, \mathbf{1})$ is a symmetric (braided strict) monoidal category. Since the monoidal structure is given by the category-theoretic product it is a (symmetric) *cartesian (monoidal) category*. Since by Lemma 2.30 $\mathbf{QMS}$ is *cartesian closed*, i.e. it is closed w.r.t. its cartesian monoidal structure, i.e. there is always an exponential object $\mathcal{Z}^{\mathcal{Y}} = \mathbf{QMS}(\mathcal{Y}, \mathcal{Z})$ and a natural adjunction isomorphism:

$$\mathbf{QMS}(\mathcal{X} \times \mathcal{Y}, \mathcal{Z}) \cong \mathbf{QMS}(\mathcal{X}, \mathcal{Z}^{\mathcal{Y}}),$$

that is natural in all three arguments. ◻

Lemma 2.33. *Let* $(\mathcal{X}, \mathcal{X}^\Omega)$, $(\mathcal{Y}, \mathcal{Y}^\Omega)$, $(\mathcal{Y}, \mathcal{Y}^\Omega)$ *be quasi-measurable spaces. Consider a quasi-measurable map:*

$$f : \mathcal{X} \to \mathcal{Y}.$$

Then the induced map:

$$f^{\mathcal{Z}} : \mathcal{X}^{\mathcal{Z}} \to \mathcal{Y}^{\mathcal{Z}}, \quad \alpha \mapsto f \circ \alpha,$$

is a well-defined quasi-measurable map.
Furthermore, the well-defined map:

$$\mathbf{QMS}(\mathcal{X}, \mathcal{Y}) \to \mathbf{QMS}(\mathcal{X}^{\mathcal{Z}}, \mathcal{Y}^{\mathcal{Z}}), \qquad f \mapsto f^{\mathcal{Z}},$$

is quasi-measurable.

Proof. $f^{\mathcal{Z}}$ is well-defined, i.e. $f \circ \alpha \in \mathcal{Y}^{\mathcal{Z}}$ for $\alpha \in \mathcal{X}^{\mathcal{Z}}$. Indeed, the compositum of two quasi-measurable maps is quasi-measurable:

$$\mathcal{Z} \xrightarrow{\alpha} \mathcal{X} \xrightarrow{f} \mathcal{Y}.$$

Furthermore, $f^{\mathcal{Z}}$ is quasi-measurable. If $\alpha \in (\mathcal{X}^{\mathcal{Z}})^{\Omega} = \mathcal{X}^{\Omega \times \mathcal{Z}}$ then $f \circ \alpha \in \mathcal{Y}^{\Omega \times \mathcal{Z}} = (\mathcal{Y}^{\mathcal{Z}})^{\Omega}$ by the same argument as above. This shows that the map:

$$(\_)^{\mathcal{Z}} : \ \mathbf{QMS}(\mathcal{X}, \mathcal{Y}) \to \mathbf{QMS}(\mathcal{X}^{\mathcal{Z}}, \mathcal{Y}^{\mathcal{Z}}), \qquad f \mapsto f^{\mathcal{Z}},$$

is well-defined. To show that it is quasi-measurable let $h \in \mathbf{QMS}(\Omega \times \mathcal{X}, \mathcal{Y})$. We need to show that: $h^{\mathcal{Z}} \in \mathbf{QMS}(\Omega \times \mathcal{X}^{\mathcal{Z}}, \mathcal{Y}^{\mathcal{Z}})$, which reduces to show that the (well-defined) map:

$$\Omega \times \mathcal{X}^{\mathcal{Z}} \times \mathcal{Z} \to \mathcal{Y}, \qquad (\omega, \alpha, z) \mapsto h(\omega, \alpha(z)),$$

is quasi-measurable. For this let $\varphi \in \Omega^{\Omega}$, $\alpha \in \mathcal{X}^{\Omega \times \mathcal{Z}}$ and $\gamma \in \mathcal{Z}^{\Omega}$. It is then left to show that the map:

$$\Omega \to \mathcal{Y}, \qquad \omega \mapsto h(\varphi(\omega), \alpha(\omega, \gamma(\omega)),$$

is quasi-measurable, which is true as the composition of the quasi-measurable maps:

$$\Omega \xrightarrow{\mathrm{id}_\Omega \times \mathrm{id}_\Omega \times \gamma} \Omega \times (\Omega \times \mathcal{Z}) \xrightarrow{\varphi \times \alpha} \Omega \times \mathcal{X} \xrightarrow{h} \mathcal{Y}.$$

This shows the claim. □

Remark 2.34 (Change of sample space)**.** *To be explicit let* **QMS** *be the category of quasi-measurable spaces w.r.t. the sample space* $(\Omega, \mathcal{B}_\Omega, \Omega^\Omega)$*. Now consider a quasi-measurable space* $(\mathcal{W}, \mathcal{W}^\Omega)$ *in* **QMS***. Its σ-algebra is then* $\mathcal{B}_{\mathcal{W}} := \mathcal{B}(\mathcal{W}^\Omega)$*. With the usual notations we have:*

$$\mathcal{W}^{\mathcal{W}} = \mathbf{QMS}\left((\mathcal{W}, \mathcal{W}^\Omega), (\mathcal{W}, \mathcal{W}^\Omega)\right) \subseteq \mathbf{Meas}\left((\mathcal{W}, \mathcal{B}_{\mathcal{W}}), (\mathcal{W}, \mathcal{B}_{\mathcal{W}})\right).$$

By the conditions 2.2 then $(\mathcal{W}, \mathcal{B}_{\mathcal{W}}, \mathcal{W}^{\mathcal{W}})$ *qualifies as a sample space for its own category of quasi-measurable space* $\widetilde{\mathbf{QMS}}$ *(w.r.t.* $(\mathcal{W}, \mathcal{B}_{\mathcal{W}}, \mathcal{W}^{\mathcal{W}})$*). We then have a functor, see Lemma 2.33:*

$$\mathbf{QMS} \to \widetilde{\mathbf{QMS}}, \qquad (\mathcal{X}, \mathcal{X}^\Omega) \mapsto (\mathcal{X}, \mathcal{X}^{\mathcal{W}}), \qquad g \mapsto g,$$

where we use:

$$\mathcal{X}^{\mathcal{W}} = \mathbf{QMS}\left((\mathcal{W}, \mathcal{W}^\Omega), (\mathcal{X}, \mathcal{X}^\Omega)\right).$$

Note that:

$$\mathcal{X}^{\mathcal{W}} \circ \mathcal{W}^{\mathcal{W}} \subseteq \mathcal{X}^{\mathcal{W}}, \qquad \mathcal{X}^{\mathbf{1}} \subseteq \mathcal{X}^{\mathcal{W}}.$$

So we are able to functorially shift from the sample space $(\Omega, \mathcal{B}_\Omega, \Omega^\Omega)$ *to the sample space* $(\mathcal{W}, \mathcal{B}_{\mathcal{W}}, \mathcal{W}^{\mathcal{W}})$*.*

Lemma 2.35. *Let $(\mathcal{X}, \mathcal{X}^\Omega)$ and $(\mathcal{Z}, \mathcal{Z}^\Omega)$ be quasi-measurable spaces. Then the map:*

$$i: \mathcal{X}^{\mathcal{Z}} \to \prod_{z \in \mathcal{Z}} \mathcal{X}, \qquad g \mapsto (g(z))_{z \in \mathcal{Z}},$$

is a well-defined, injective quasi-measurable map.
If, furthermore, $\mathcal{X}^\Omega = \mathcal{F}(\mathcal{B}_\mathcal{X})$ and $(\mathcal{Z}, \mathcal{Z}^\Omega)$ is countable with the discrete quasi-measurable space structure then i is an isomorphism of quasi-measurable spaces.

Proof. It is clear that i is a well-defined map and injective. Now let $\alpha \in \left(\mathcal{X}^{\mathcal{Z}}\right)^\Omega \cong \left(\mathcal{X}^\Omega\right)^{\mathcal{Z}}$. Under this identification we see that for every $z \in \mathcal{Z}$ we have that $\alpha(z) \in \mathcal{X}^\Omega$ and thus:

$$(\alpha(z))_{z \in \mathcal{Z}} \in \prod_{z \in \mathcal{Z}} \mathcal{X}^\Omega.$$

This shows that i is quasi-measurable and thus the claim.
Now assume that $(\mathcal{Z}, \mathcal{Z}^\Omega)$ is countable and discrete, w.l.o.g. $(\mathcal{Z}, \mathcal{Z}^\Omega) = (\mathbb{N}, \mathbb{N}^\Omega)$ and that $\mathcal{X}^\Omega = \mathbf{Meas}((\Omega, \mathcal{B}_\Omega), (\mathcal{X}, \mathcal{B}_\mathcal{X}))$ holds. Then consider the map:

$$j: \prod_{n \in \mathbb{N}} \mathcal{X} \to \mathcal{X}^\mathbb{N}, \qquad (x_n)_{n \in \mathbb{N}} \mapsto x = (n \mapsto x_n).$$

This is well-defined. Indeed, since $\mathbb{N}$ is a discrete measurable space the map:

$$x: \mathbb{N} \to \mathcal{X}, \qquad n \mapsto x_n,$$

is measurable for every choice of $(x_n)_{n \in N}$. Since also each $\beta \in \mathbb{N}^\Omega$ is measurable we have that: $x \circ \beta \in \mathbf{Meas}((\Omega, \mathcal{B}_\Omega), (\mathcal{X}, \mathcal{B}_\mathcal{X})) = \mathcal{X}^\Omega$. This shows that j is well-defined.
Now consider the induced map:

$$j_*: \prod_{n \in \mathbb{N}} \mathcal{X}^\Omega \to (\mathcal{X}^\mathbb{N})^\Omega, \qquad (\alpha_n)_{n \in \mathbb{N}} \mapsto \alpha = ((\omega, n) \mapsto \alpha_n(\omega)).$$

We need to show that α is quasi-measurable:

$$\alpha: \Omega \times \mathbb{N} \to \mathcal{X}.$$

For this let $\varphi \in \Omega^\Omega$ and $\beta \in \mathbb{N}^\Omega$ and $A \in \mathcal{B}_\mathcal{X}$. Then we get:

$$\begin{aligned} \alpha(\varphi, \beta)^{-1}(A) &= \left\{\omega \in \Omega \,\middle|\, \alpha_{\beta(\omega)}(\varphi(\omega)) \in A\right\} \\ &= \bigcup_{n \in \mathbb{N}} \varphi^{-1}(\alpha_n^{-1}(A)) \cap \beta^{-1}(n) \\ &\in \mathcal{B}_\Omega, \end{aligned}$$

because φ, β and each α_n is measurable. It follows that:

$$\alpha(\varphi, \beta) \in \mathbf{Meas}((\Omega, \mathcal{B}_\Omega), (\mathcal{X}, \mathcal{B}_\mathcal{X})) = \mathcal{X}^\Omega.$$

So j is a well-defined quasi-measurable map. It is easily seen that j and i are inverse to each other. This shows the claim. ∎

2.5 Limits of Quasi-Measurable Spaces

In this subsection we will quickly show that we have all (small) *limits* inside the category of quasi-measurable spaces **QMS**. Since we already have products we mainly only need to look into *equalizers*.

Definition 2.36 (Equalizer). *Let* $(\mathcal{X}, \mathcal{X}^\Omega)$ *and* $(\mathcal{Y}, \mathcal{Y}^\Omega)$ *two quasi-measurable spaces and*

$$f_1, f_2 : (\mathcal{X}, \mathcal{X}^\Omega) \to (\mathcal{Y}, \mathcal{Y}^\Omega),$$

two quasi-measurable maps. Then we define the equalizer *of* f_1 *and* f_2 *as:*

$$\mathrm{Eq}(f_1, f_2) := \{x \in \mathcal{X} \mid f_1(x) = f_2(x)\},$$

which we endow with the quasi-measurable functions:

$$\mathrm{Eq}(f_1, f_2)^\Omega := \left\{\alpha : \Omega \to \mathrm{Eq}(f_1, f_2) \,\middle|\, \iota \circ \alpha \in \mathcal{X}^\Omega\right\} = \iota^* \mathcal{X}^\Omega,$$

where $\iota : \mathrm{Eq}(f_1, f_2) \hookrightarrow \mathcal{X}$ *is the inclusion map.*

Lemma 2.37. $(\mathrm{Eq}(f_1, f_2), \mathrm{Eq}(f_1, f_2)^\Omega)$ *is quasi-measurable space and satisfies the universal property of an equalizer in the category of quasi-measurable spaces* **QMS**.

Proof. That $(\mathrm{Eq}(f_1, f_2), \mathrm{Eq}(f_1, f_2)^\Omega)$ is quasi-measurable space can be checked directly or via Lemma 2.56. Now consider a quasi-measurable map:

$$g : (\mathcal{Z}, \mathcal{Z}^\Omega) \to (\mathcal{X}, \mathcal{X}^\Omega),$$

such that $f_1 \circ g = f_2 \circ g$. Then for all $z \in \mathcal{Z}$ we get that:

$$f_1 \circ g(z) = f_2 \circ g(z),$$

which implies: $g(z) \in \mathrm{Eq}(f_1, f_2)$. By the same argument we get:

$$g \circ \mathcal{Z}^\Omega \subseteq \iota^* \mathcal{X}^\Omega = \mathrm{Eq}(f_1, f_2)^\Omega.$$

So g uniquely factorizes through the inclusion:

$$\iota : (\mathrm{Eq}(f_1, f_2), \mathrm{Eq}(f_1, f_2)^\Omega) \hookrightarrow (\mathcal{X}, \mathcal{X}^\Omega).$$

This shows that $(\mathrm{Eq}(f_1, f_2), \mathrm{Eq}(f_1, f_2)^\Omega)$ hat the universal property of an equalizer in **QMS**. □

Theorem 2.38. *The category* **QMS** *is complete, i.e. it contains all (small)*[3] *limits.*

Proof. **QMS** contains all (small) products by Lemma 2.26 and equalizers by Lemma 2.37, thus all (small) limits by the existence theorem for limits, see [ML98] Thm. V.2.1. □

[3] *Small* means indexed over sets.

Example 2.39. 1. *The terminal object* **1** *is a limit in* **QMS**.

2. *Equalizers (with possibly more than two maps) are limits in* **QMS**.
3. *Products are limits in* **QMS**.
4. *Pull-backs aka fibre products, e.g.* $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$*, are limits in* **QMS**.
5. *Inverse/projective limits (over directed sets) are limits in* **QMS**.

2.6 Coproduct of Quasi-Measurable Spaces

In this subsection we define the categorical *coproduct* of quasi-measurable spaces. Note that the coproduct we define here is different from the coproduct defined in [HKSY17]. The reason is that we did not insist on the 3rd property in Def. 7 from [HKSY17], making our coproduct in some sense more "rigid". The difference will be highlighted and resolved later in Theorem 4.10 and Lemma 4.12.

Definition 2.40 (Coproduct of quasi-measurable spaces)**.** *Let* $(\mathcal{X}_i, \mathcal{X}_i^\Omega)$*,* $i \in I$*, be a family of quasi-measurable spaces indexed by a set* I*. Then we endow the set-theoretic coproduct:*

$$\coprod_{i \in I} \mathcal{X}_i = \{(i, x) \mid i \in I, x \in \mathcal{X}_i\},$$

with the set of maps:

$$\left(\coprod_{i \in I} \mathcal{X}_i\right)^\Omega := \left\{\mathrm{incl}_i \circ \alpha_i \,\middle|\, \alpha_i \in \mathcal{X}_i^\Omega,\ i \in I\right\} = \coprod_{i \in I} \mathcal{X}_i^\Omega \qquad \subseteq \qquad \left[\Omega \to \coprod_{i \in I} \mathcal{X}_i\right],$$

where the canonical inclusion maps for $k \in I$ *are given by:*

$$\mathrm{incl}_k : \mathcal{X}_k \to \coprod_{i \in I} \mathcal{X}_i, \qquad x \mapsto (k, x).$$

This then defines the coproduct of quasi-measurable spaces*:*

$$\coprod_{i \in I} (\mathcal{X}_i, \mathcal{X}_i^\Omega) := \left(\coprod_{i \in I} \mathcal{X}_i, \left(\coprod_{i \in I} \mathcal{X}_i\right)^\Omega\right).$$

Lemma 2.41. *The coproduct of quasi-measurable spaces:*

$$\coprod_{i \in I} (\mathcal{X}_i, \mathcal{X}_i^\Omega) = \left(\coprod_{i \in I} \mathcal{X}_i, \coprod_{i \in I} \mathcal{X}_i^\Omega\right)$$

is a quasi-measurable space. Furthermore, it has the universal property of a coproduct in the category **QMS** *of quasi-measurable spaces, i.e. for every quasi-measurable space* $(\mathcal{Z}, \mathcal{Z}^\Omega)$ *the natural map:*

$$\mathbf{QMS}\left(\coprod_{i \in I} \mathcal{X}_i, \mathcal{Z}\right) \xrightarrow{\sim} \prod_{i \in I} \mathbf{QMS}\left(\mathcal{X}_i, \mathcal{Z}\right), \qquad g \mapsto (g \circ \mathrm{incl}_i)_{i \in I}$$

is a bijection.

Proof. That $\coprod_{i\in I}(\mathcal{X}_i, \mathcal{X}_i^\Omega)$ is a quasi-measurable space is clear as all quasi-measurable maps are closed under $\circ\Omega^\Omega$ and all constant maps are contained.
It is also clear the canonical inclusion maps;

$$\mathrm{incl}_k : \mathcal{X}_k \to \coprod_{i\in I} \mathcal{X}_i, \qquad x \mapsto (k, x),$$

are quasi-measurable because by definition of $\left(\coprod_{i\in I} \mathcal{X}_i\right)^\Omega$.
For the universal property let $(\mathcal{Z}, \mathcal{Z}^\Omega)$ be another quasi-measurable space and $g_i : (\mathcal{X}_i, \mathcal{X}_i^\Omega) \to (\mathcal{Z}, \mathcal{Z}^\Omega)$ quasi-measurable maps for $i \in I$. Then the map:

$$g := \coprod_{i\in I} g_i : \coprod_{i\in I} \mathcal{X}_i \to \mathcal{Z}, \qquad (k, x) \mapsto g_k(x),$$

is the only (set-theoretic) map that makes the following diagrams commute for all $k \in I$:

$$\begin{array}{ccc} (\mathcal{X}_k, \mathcal{X}_k^\Omega) & & \\ \downarrow{\scriptstyle \mathrm{incl}_k} & \searrow^{g_k} & \\ \coprod_{i\in I}(\mathcal{X}_i, \mathcal{X}_i^\Omega) & \xrightarrow[\exists! g]{} & (\mathcal{Z}, \mathcal{Z}^\Omega). \end{array}$$

We are left to check that g is a quasi-measurable map. We already have by assumption that:

$$g \circ \mathrm{incl}_k \circ \alpha_k = g_k \circ \alpha_k \in \mathcal{Z}^\Omega,$$

for every $k \in I$ and $\alpha_k \in \mathcal{X}_k^\Omega$. So we get:

$$g \circ \left(\coprod_{i\in I} \mathcal{X}_i\right)^\Omega = \left\{ g \circ \mathrm{incl}_i \circ \alpha_i \,\middle|\, \alpha_i \in \mathcal{X}_i^\Omega,\ i \in I \right\} \subseteq \mathcal{Z}^\Omega.$$

This shows the claim. □

Theorem 2.42 (Distributive law for quasi-measurable spaces)**.** *Let I be an (index) set and for $i \in I$ let J_i be another (index) set. For every $i \in I$ and $j_i \in J_i$ let $(\mathcal{X}_{i,j_i}, \mathcal{X}_{i,j_i}^\Omega)$ be a quasi-measurable space. Then we have a canonical well-defined isomorphism of quasi-measurable spaces:*

$$\coprod_{j\in\prod_{i\in I} J_i} \left(\prod_{i\in I} \mathcal{X}_{i,j_i}\right) \cong \prod_{i\in I}\left(\coprod_{j_i\in J_i} \mathcal{X}_{i,j_i}\right), \qquad ((j_i)_{i\in I}, (x_i)_{i\in I}) \mapsto (j_i, x_i)_{i\in I}\,.$$

Proof. This is clearly a bijection. Similarly note that this map induces the bijection:

$$\begin{aligned} \left(\coprod_{j\in\prod_{i\in I} J_i} \left(\prod_{i\in I} \mathcal{X}_{i,j_i}\right)\right)^\Omega &= \coprod_{j\in\prod_{i\in I} J_i} \left(\prod_{i\in I} \mathcal{X}_{i,j_i}\right)^\Omega &&= \coprod_{j\in\prod_{i\in I} J_i} \left(\prod_{i\in I} \mathcal{X}_{i,j_i}^\Omega\right) \\ &\cong \prod_{i\in I}\left(\coprod_{j_i\in J_i} \mathcal{X}_{i,j_i}^\Omega\right) &&= \prod_{i\in I}\left(\coprod_{j_i\in J_i} \mathcal{X}_{i,j_i}\right)^\Omega \\ &= \left(\prod_{i\in I}\left(\coprod_{j_i\in J_i} \mathcal{X}_{i,j_i}\right)\right)^\Omega. \end{aligned}$$

This shows the claim. ☐

2.7 Colimits of Quasi-Measurable Spaces

In this subsection we will quickly show that we have all (small) *colimits* inside the category of quasi-measurable spaces **QMS**. Since we already have coproducts we mainly only need to look into *coequalizers*.

Definition 2.43 (Coequalizer). *Let* $(\mathcal{X}, \mathcal{X}^\Omega)$ *and* $(\mathcal{Y}, \mathcal{Y}^\Omega)$ *two quasi-measurable spaces and*

$$f_1, f_2 : (\mathcal{X}, \mathcal{X}^\Omega) \to (\mathcal{Y}, \mathcal{Y}^\Omega),$$

two quasi-measurable maps. Then we define the coequalizer *of* f_1 *and* f_2 *as the quotient:*

$$\mathrm{CoEq}(f_1, f_2) := \mathcal{Y}/\sim,$$

where the equivalence relation is the smallest equivalence relation such that:

$$f_1(x) \sim f_2(x)$$

for $x \in \mathcal{X}$*. The quotient map:*

$$t : \mathcal{Y} \twoheadrightarrow \mathrm{CoEq}(f_1, f_2),$$

endows $\mathrm{CoEq}(f_1, f_2)$ *with the following quasi-measurable functions:*

$$\mathrm{CoEq}(f_1, f_2)^\Omega := t \circ \mathcal{Y}^\Omega.$$

Lemma 2.44. *The coequalizer* $(\mathrm{CoEq}(f_1, f_2), \mathrm{CoEq}(f_1, f_2)^\Omega)$ *is quasi-measurable space and has the universal property of an equalizer in the category of quasi-measurable spaces* **QMS**.

Proof. That $(\mathrm{CoEq}(f_1, f_2), \mathrm{CoEq}(f_1, f_2)^\Omega)$ is quasi-measurable space can easily be checked. Consider a quasi-measurable map:

$$g : (\mathcal{Y}, \mathcal{Y}^\Omega) \to (\mathcal{Z}, \mathcal{Z}^\Omega),$$

such that $g \circ f_1 = g \circ f_2$. Then for every $x \in \mathcal{X}$ we get:

$$g \circ f_1(x) = g \circ f_2(x).$$

So there exists a unique (set-theoretic) map:

$$h : \mathrm{CoEq}(f_1, f_2) \to \mathcal{Z}, \qquad h \circ t = g.$$

We then get:

$$h \circ \mathrm{CoEq}(f_1, f_2)^\Omega = h \circ t \circ \mathcal{Y}^\Omega = g \circ \mathcal{Y}^\Omega \subseteq \mathcal{Z}^\Omega.$$

This shows that h is quasi-measurable and thus that $(\mathrm{CoEq}(f_1, f_2), \mathrm{CoEq}(f_1, f_2)^\Omega)$ has the universal property of an equalizer in **QMS**. ☐

Theorem 2.45. *The category* **QMS** *is cocomplete, i.e. it contains all (small)*[3] *colimits.*

Proof. **QMS** contains all (small) coproducts by Lemma 2.41 and coequalizers by Lemma 2.44, thus all (small) colimits by the existence theorem of colimits, see [ML98] Thm. V.2.1 (dual version). ☐

Example 2.46.
1. *Coproducts are colimits in* **QMS**.
2. *Coequalizers (over possible more than two maps) are colimits in* **QMS**.
3. *Pushouts are colimits in* **QMS**.
4. *Direct limits (over directed sets) are colimits in* **QMS**.

2.8 The Sigma-Algebra as a Quasi-Measurable Space

In this subsection we show that the *induced σ-algebra* $\mathcal{B}_\mathcal{X}$ of a quasi-measurable space $(\mathcal{X}, \mathcal{X}^\Omega)$ can be turned into a quasi-measurable space $(\mathcal{B}_\mathcal{X}, (\mathcal{B}_\mathcal{X})^\Omega)$ in its own right. This allows us to compare σ-algebras and quasi-measurable spaces on the same footing inside **QMS**. Note that this was not possible inside the category of measurable spaces **Meas** (without difficulties or topological considerations).

Definition 2.47 (The σ-algebra of a quasi-measurable space)**.** *Let $(\mathcal{X}, \mathcal{X}^\Omega)$ be a quasi-measurable space. Then $\mathcal{X}$ is endowed with the following σ-algebra:*

$$\mathcal{B}_\mathcal{X} := \mathcal{B}(\mathcal{X}^\Omega) := (\mathcal{X}^\Omega)_*\mathcal{B}_\Omega := \left\{ A \subseteq \mathcal{X} \,\middle|\, \forall \alpha \in \mathcal{X}^\Omega.\, \alpha^{-1}(A) \in \mathcal{B}_\Omega \right\}.$$

Similarly, we get the same construction on the product $\Omega \times \mathcal{X}$:

$$\mathcal{B}_{\Omega\times\mathcal{X}} := \mathcal{B}(\Omega^\Omega \times \mathcal{X}^\Omega) = \left\{ D \subseteq \Omega \times \mathcal{X} \,\middle|\, \forall \varphi \in \Omega^\Omega\, \forall \alpha \in \mathcal{X}^\Omega.\, (\varphi, \alpha)^{-1}(D) \in \mathcal{B}_\Omega \right\}.$$

Using these notations we define the quasi-measurable space structure on $\mathcal{B}_\mathcal{X}$ as follows:

$$(\mathcal{B}_\mathcal{X})^\Omega := \{\Psi : \Omega \to \mathcal{B}_\mathcal{X} \,|\, \exists D \in \mathcal{B}_{\Omega\times\mathcal{X}}.\, \forall \omega \in \Omega.\, \Psi(\omega) = D_\omega\},$$

where we define the section D_ω as:

$$D_\omega := \{x \in \mathcal{X} \,|\, (\omega, x) \in D\} \in \mathcal{B}_\mathcal{X}.$$

Lemma 2.48. *Let $(\mathcal{X}, \mathcal{X}^\Omega)$ be a quasi-measurable space. Then also its σ-algebra $\mathcal{B}_\mathcal{X} = \mathcal{B}(\mathcal{X}^\Omega)$ forms a well-defined quasi-measurable space $(\mathcal{B}_\mathcal{X}, (\mathcal{B}_\mathcal{X})^\Omega)$.*

Proof. We first note that for $D \in \mathcal{B}_{\Omega\times\mathcal{X}}$ each section D_ω lies in $\mathcal{B}_\mathcal{X}$. For this consider $\alpha \in \mathcal{X}^\Omega$ and the constant map $\varphi = \omega$. Since $\varphi \in \Omega^\Omega$ and by definition of $\mathcal{B}_{\Omega\times\mathcal{X}}$ we then have that:

$$\alpha^{-1}(D_\omega) = (\varphi, \alpha)^{-1}(D) \in \mathcal{B}_\Omega.$$

Since this holds for all $\alpha \in \mathcal{X}^\Omega$ we get that $D_\omega \in \mathcal{B}_\mathcal{X}$. This shows the claim that the map $\omega \mapsto D_\omega$ is a well-defined map $\Omega \to \mathcal{B}_\mathcal{X}$ for $D \in \mathcal{B}_{\Omega\times\mathcal{X}}$.
We now check the points from Definition 2.5:

1. Let $\varphi \in \Omega^\Omega$ and $\Psi \in (\mathcal{B}_\mathcal{X})^\Omega$ be given via the sections of $D \in \mathcal{B}_{\Omega\times\mathcal{X}}$. Consider:
$$\tilde{D} := (\varphi, \mathrm{id}_\mathcal{X})^{-1}(D) = \{(\omega, x) \in \Omega \times \mathcal{X} \,|\, (\varphi(\omega), x) \in D\} \subseteq \Omega \times \mathcal{X}.$$
It is then $\tilde{D} \in \mathcal{B}_{\Omega\times\mathcal{X}}$. Indeed, for every $\tilde{\varphi} \in \Omega^\Omega$ and $\tilde{\alpha} \in \mathcal{X}^\Omega$ we have:
$$(\tilde{\varphi}, \tilde{\alpha})^{-1}(\tilde{D}) = (\varphi \circ \tilde{\varphi}, \tilde{\alpha})^{-1}(D) \in \mathcal{B}_\Omega.$$
Since this holds for all $\tilde{\varphi} \in \Omega^\Omega$ and $\tilde{\alpha} \in \mathcal{X}^\Omega$ we have that: $\tilde{D} \in \mathcal{B}_{\Omega\times\mathcal{X}}$.
With these considerations we then get:
$$\Psi \circ \varphi(\omega) = D_{\varphi(\omega)} = \{x \in \mathcal{X} \,|\, (\varphi(\omega), x) \in D\} = \tilde{D}_\omega.$$
So $\Psi \circ \varphi$ is given via the sections of $\tilde{D} \in \mathcal{B}_{\Omega\times\mathcal{X}}$. So we can conclude that $\Psi \circ \varphi \in (\mathcal{B}_\mathcal{X})^\Omega$.

2. Let $\Psi : \Omega \to \mathcal{B}_\mathcal{X}$ be the constant map $\omega \mapsto A \in \mathcal{B}_\mathcal{X}$. Then $D := \Omega \times A \in \mathcal{B}_{\Omega\times\mathcal{X}}$. Indeed, for $\varphi \in \Omega^\Omega$ and $\alpha \in \mathcal{X}^\Omega$ we get:
$$(\varphi, \alpha)^{-1}(D) = \{\omega \in \Omega \,|\, (\varphi(\omega), \alpha(\omega)) \in \Omega \times A\} = \alpha^{-1}(A) \in \mathcal{B}_\Omega.$$
Since this holds for all $\varphi \in \Omega^\Omega$ and $\alpha \in \mathcal{X}^\Omega$ we get $D \in \mathcal{B}_{\Omega\times\mathcal{X}}$.
With this we get for all $\omega \in \Omega$:
$$D_\omega = \{x \in \mathcal{X} \,|\, (\omega, x) \in \Omega \times A\} = A = \Psi(\omega).$$
So $\Psi \in (\mathcal{B}_\mathcal{X})^\Omega$.

This shows that $(\mathcal{B}_\mathcal{X}, (\mathcal{B}_\mathcal{X})^\Omega)$ is a well-defined quasi-measurable space. □

Lemma 2.49. *Let $(\mathcal{X}, \mathcal{X}^\Omega)$ and $(\mathcal{Y}, \mathcal{Y}^\Omega)$ be a quasi-measurable space. Then the following map:*
$$\mathcal{B}_\mathcal{Y} \times \mathcal{Y}^\mathcal{X} \to \mathcal{B}_\mathcal{X}, \qquad (B, f) \mapsto f^{-1}(B).$$
is quasi-measurable.

Proof. The map is clearly well-defined (image in $\mathcal{B}_\mathcal{X}$). The same holds for the induced map:
$$\mathcal{B}_{\Omega\times\mathcal{Y}} \times \mathcal{Y}^{\Omega\times\mathcal{X}} \to \mathcal{B}_{\Omega\times\mathcal{X}}, \qquad (B, f) \mapsto (\mathrm{id}_\Omega \times f)^{-1}(B),$$
where:
$$\mathrm{id}_\Omega \times f : \; \Omega \times \mathcal{X} \to \Omega \times \mathcal{Y}, \qquad (\omega, x) \mapsto (\omega, f(\omega, x)).$$
This shows the claim. □

2.9 The Indicator Functions of a Quasi-Measurable Space

In this subsection we look at a special case of function spaces: the *function space of indicator functions* $\mathbf{2}^{\mathcal{X}}$. We will see that indicator functions precisely indicate the measurable sets inside quasi-measurable spaces. Even more, we will show that the quasi-measurable space of the induced σ-algebra is isomorphic to the function space of indicator functions.

Notation 2.50. *We consider the set:*

$$\mathbf{2} := \{0, 1\},$$

and endow it with the quasi-measurable functions:

$$\mathbf{2}^{\Omega} := \mathcal{F}(\mathcal{B}_{\mathbf{2}}) = \mathbf{Meas}((\Omega, \mathcal{B}_{\Omega}), (\mathbf{2}, \mathcal{B}_{\mathbf{2}})) = \left\{ \psi : \Omega \to \mathbf{2} \,\middle|\, \psi^{-1}(1) \in \mathcal{B}_{\Omega} \right\}.$$

In other words, we endow it with all measurable functions $\psi : \Omega \to \mathbf{2}$, where we consider $\mathbf{2}$ to be discrete.
We will in the following always consider $\mathbf{2}$ as this quasi-measurable space $(\mathbf{2}, \mathbf{2}^{\Omega})$.
If now $(\mathcal{X}, \mathcal{X}^{\Omega})$ is another quasi-measurable space then the function space $(\mathbf{2}^{\mathcal{X}}, (\mathbf{2}^{\mathcal{X}})^{\Omega})$ is also a well-defined quasi-measurable space.

Lemma 2.51. *Let $(\mathcal{X}, \mathcal{X}^{\Omega})$ be a quasi-measurable space. Then the indicator function (inclusion check) induces an isomorphism of quasi-measurable spaces:*

$$\mathbb{1} : (\mathcal{B}_{\mathcal{X}}, (\mathcal{B}_{\mathcal{X}})^{\Omega}) \to (\mathbf{2}^{\mathcal{X}}, (\mathbf{2}^{\mathcal{X}})^{\Omega}) \qquad A \mapsto \mathbb{1}_A.$$

Proof. We first show that $\mathbb{1}$ is a well-defined map. For this let $A \in \mathcal{B}_{\mathcal{X}} = \mathcal{B}(\mathcal{X}^{\Omega})$. Clearly, $\mathbb{1}_A \in [\mathcal{X} \to \mathbf{2}]$. To check that $\mathbb{1}_A \in \mathbf{2}^{\mathcal{X}}$ let $\alpha \in \mathcal{X}^{\Omega}$ and note that:

$$\mathbb{1}_A \circ \alpha = \mathbb{1}_{\alpha^{-1}(A)} : \Omega \to \mathbf{2}.$$

Since $\alpha^{-1}(A) \in \mathcal{B}_{\Omega}$ we have that $\mathbb{1}_{\alpha^{-1}(A)}$ is measurable and thus $\mathbb{1}_A \circ \alpha \in \mathbf{2}^{\Omega}$. Since this holds for all $\alpha \in \mathcal{X}^{\Omega}$ we get that $\mathbb{1}_A \in \mathbf{QMS}(\mathcal{X}, \mathbf{2}) = \mathbf{2}^{\mathcal{X}}$. So $\mathbb{1}$ is well-defined.
As a next step we show that $\mathbb{1}$ is a quasi-measurable map. For this let $\psi \in (\mathcal{B}_{\mathcal{X}})^{\Omega}$ be given by the sections of $D \in \mathcal{B}_{\Omega \times \mathcal{X}}$. Then note that:

$$(\mathbb{1} \circ \psi(\omega))(x) = \mathbb{1}_{\psi(\omega)}(x) = \mathbb{1}_{D_\omega}(x) = \mathbb{1}_D(\omega, x).$$

Via (un-)currying we can thus identify $\mathbb{1} \circ \psi$ and $\mathbb{1}_D$. Since $D \in \mathcal{B}_{\Omega \times \mathcal{X}}$ the same arguments as above show that: $\mathbb{1}_D \in \mathbf{2}^{\Omega \times \mathcal{X}}$. Via (un-)currying this shows that $\mathbb{1} \circ \psi \in (\mathbf{2}^{\mathcal{X}})^{\Omega}$. Since this holds for all $\psi \in (\mathcal{B}_{\mathcal{X}})^{\Omega}$ we have that $\mathbb{1}$ is a quasi-measurable map.
We now claim that the inverse to $\mathbb{1}$ is the map:

$$\rho : \mathbf{2}^{\mathcal{X}} \to \mathcal{B}_{\mathcal{X}}, \quad \chi \mapsto \chi^{-1}(1).$$

We first show that ρ is a well-defined map. For this let $\chi : \mathcal{X} \to \mathbf{2}$ be an element of $\mathbf{2}^{\mathcal{X}}$. Then for all functions $\alpha \in \mathcal{X}^{\Omega}$ we have that $\chi \circ \alpha \in \mathbf{2}^{\Omega}$. This means that $\alpha^{-1}(\chi^{-1}(1)) \in \mathcal{B}_{\Omega}$ for all $\alpha \in \mathcal{X}^{\Omega}$, which implies:

$$\chi^{-1}(1) \in \mathcal{B}_{\mathcal{X}}.$$

This shows that ρ is well-defined.
Next we show that ρ is quasi-measurable. For this we again identify $(\mathbf{2}^{\mathcal{X}})^\Omega$ with $\mathbf{2}^{\Omega\times\mathcal{X}}$ and $(\mathcal{B}_\mathcal{X})^\Omega$ with $\mathcal{B}_{\Omega\times\mathcal{X}}$. Then let $\xi \in \mathbf{2}^{\Omega\times\mathcal{X}}$. The same arguments as above show that $\xi^{-1}(1) \in \mathcal{B}_{\Omega\times\mathcal{X}}$, which then represents the corresponding function: $\omega \mapsto \xi^{-1}(1)_\omega$ in $(\mathcal{B}_\mathcal{X})^\Omega$. Translating back, this means we have a well-defined map:

$$\rho_* : (\mathbf{2}^{\mathcal{X}})^\Omega \to (\mathcal{B}_\mathcal{X})^\Omega, \quad \xi \mapsto (\omega \mapsto \xi^{-1}(1)_\omega = \{x \in \mathcal{X} \,|\, \xi(\omega)(x) = 1\}),$$

implying that: $\rho \circ (\mathbf{2}^{\mathcal{X}})^\Omega \subseteq (\mathcal{B}_\mathcal{X})^\Omega$. This shows that ρ is a well-defined quasi-measurable map.
Finally, we show that $\mathbb{1}$ and ρ are inverse to each other.

$$\rho \circ \mathbb{1} : \mathcal{B}_\mathcal{X} \to \mathcal{B}_\mathcal{X}, \quad A \mapsto (\mathbb{1}_A)^{-1}(1) = A,$$

and:

$$\mathbb{1} \circ \rho : \mathbf{2}^{\mathcal{X}} \to \mathbf{2}^{\mathcal{X}}, \quad \chi \mapsto \mathbb{1}_{\chi^{-1}(1)} = \chi.$$

Since these relations also hold for $\mathcal{B}_{\Omega\times\mathcal{X}}$ and $\mathbf{2}^{\Omega\times\mathcal{X}}$ similarly, we get that $\rho \circ \mathbb{1} = \mathrm{id}_{(\mathcal{B}_\mathcal{X},(\mathcal{B}_\mathcal{X})^\Omega)}$ and $\mathbb{1} \circ \rho = \mathrm{id}_{(\mathbf{2}^{\mathcal{X}},(\mathbf{2}^{\mathcal{X}})^\Omega)}$. This shows the claim. □

2.10 Image and Pre-Image Quasi-Measurable Spaces

In this subsection we will shortly look at how one can transfer the structure of a quasi-measurable space to another set through maps in and out of that space.

Definition 2.52. *Let $(\mathcal{Z}, \mathcal{Z}^\Omega)$ be a quasi-measurable space and $\mathcal{Y}$ a set and*

$$g : \mathcal{Z} \to \mathcal{Y}$$

a map. Then we define the image or push-forward quasi-measurable space *structure $(\mathcal{Y}, g_*\mathcal{Z}^\Omega)$ via:*

$$g_*\mathcal{Z}^\Omega := \left(g \circ \mathcal{Z}^\Omega\right) \cup \left(\mathcal{Y}^{\mathbf{1}}\right).$$

Note that $g_*\mathcal{Z}^\Omega$ is the smallest set of maps $\mathcal{Y}^\Omega \subseteq [\Omega \to \mathcal{Y}]$ such that g becomes a $\mathcal{Z}^\Omega$-$\mathcal{Y}^\Omega$-quasi-measurable map.

Remark 2.53. *Note that $g \circ \mathcal{Z}^\Omega$ might not satisfy all the points from Definition 2.5. For instance, if g is not surjective, then there are constant maps $\beta : \Omega \to \mathbf{1} \to \mathcal{Y}$ that do not factorize through g.*

Lemma 2.54. *Let $(\mathcal{Z}, \mathcal{Z}^\Omega)$ be a quasi-measurable space and $\mathcal{Y}$ a set and*

$$g : \mathcal{Z} \to \mathcal{Y}$$

a surjective *map. Then $(\mathcal{Y}, \mathcal{Y}^\Omega)$ with $\mathcal{Y}^\Omega := g \circ \mathcal{Z}^\Omega$ is quasi-measurable space.*

Proof. We check the the points from Definition 2.5:

1. Let $\beta = g \circ \gamma \in g \circ \mathcal{Z}^\Omega$ and $\varphi \in \Omega^\Omega$. Then $\gamma \circ \varphi \in \mathcal{Z}^\Omega$ and thus $g \circ (\gamma \circ \varphi) \in g \circ \mathcal{Z}^\Omega$.
2. Let $y \in \mathcal{Y}$. Since g is surjective there exists a $z \in \mathcal{Z}$ such that $g(z) = y$. Then the constant map $\gamma = z$ exists in $\mathcal{Z}^\Omega$. Then $g \circ \gamma$ is the constant map y and is in $g \circ \mathcal{Z}^\Omega$.

□

Definition 2.55. *Let $(\mathcal{Y}, \mathcal{Y}^\Omega)$ be a quasi-measurable space and $\mathcal{X}$ a set and*

$$f : \mathcal{X} \to \mathcal{Y}$$

a map. Then we define the pre-image or pull-back quasi-measurable space *structure $(\mathcal{X}, f^*\mathcal{Y}^\Omega)$ via:*

$$f^*\mathcal{Y}^\Omega := \left\{\alpha : \Omega \to \mathcal{X} \,\middle|\, f \circ \alpha \in \mathcal{Y}^\Omega\right\}.$$

Lemma 2.56. *Let $(\mathcal{Y}, \mathcal{Y}^\Omega)$ be a quasi-measurable space and $\mathcal{X}$ a set and*

$$f : \mathcal{X} \to \mathcal{Y}$$

a map. Then the pull-back $(\mathcal{X}, f^*\mathcal{Y}^\Omega)$ is a quasi-measurable space w.r.t. Ω.
Note that $f^*\mathcal{Y}^\Omega$ is the biggest set of maps $\mathcal{X}^\Omega \subseteq [\Omega \to \mathcal{X}]$ such that f becomes a $\mathcal{X}^\Omega$-$\mathcal{Y}^\Omega$-quasi-measurable map.

Proof. We check the the points from Definition 2.5:

1. Let $\alpha \in f^*\mathcal{Y}^\Omega$ and $\varphi \in \Omega^\Omega$. Then $f \circ \alpha \in \mathcal{Y}^\Omega$ and by assumption $f \circ \alpha \circ \varphi \in \mathcal{Y}^\Omega$ as well. By definition of $f^*\mathcal{Y}^\Omega$ we have that $\alpha \circ \varphi \in f^*\mathcal{Y}^\Omega$.
2. If $\alpha : \Omega \to \mathcal{X}$ is constant, then $f \circ \alpha : \Omega \to \mathcal{Y}$ is constant as well. So we have $f \circ \alpha \in \mathcal{Y}^\Omega$ by assumption and thus $\alpha \in f^*\mathcal{Y}^\Omega$.

□

Lemma 2.57. *Let $(\mathcal{X}, \mathcal{X}^\Omega)$ and $(\mathcal{Y}, \mathcal{Y}^\Omega)$ be two quasi-measurable spaces and $\mathcal{Z} \subseteq \mathcal{Y}$ a subset with inclusion map $\iota : \mathcal{Z} \to \mathcal{Y}$. Further, let:*

$$f : (\mathcal{X}, \mathcal{X}^\Omega) \to (\mathcal{Y}, \mathcal{Y}^\Omega)$$

be a quasi-measurable map with $f(\mathcal{X}) \subseteq \mathcal{Z}$. Then f induces a quasi-measurable map:

$$f : (\mathcal{X}, \mathcal{X}^\Omega) \to (\mathcal{Z}, \iota^*\mathcal{Y}^\Omega),$$

with:

$$\iota^*\mathcal{Y}^\Omega = \left\{\beta \in \mathcal{Y}^\Omega \,\middle|\, \beta(\Omega) \subseteq \mathcal{Z}\right\} = [\Omega \to \mathcal{Z}] \cap \mathcal{Y}^\Omega.$$

Proof. If $f(\mathcal{X}) \subseteq \mathcal{Z}$ then for any $\alpha \in \mathcal{X}^\Omega$ we have that:

$$f \circ \alpha(\Omega) \subseteq f(\mathcal{X}) \subseteq \mathcal{Z}.$$

So $f \circ \alpha \in [\Omega \to \mathcal{Z}] \cap \mathcal{Y}^\Omega = \iota^*\mathcal{Y}^\Omega$. This shows the claim. □

Lemma 2.58. *Let* $t: (\mathcal{X}, \mathcal{X}^\Omega) \to (\mathcal{Y}, \mathcal{Y}^\Omega)$ *be quasi-measurable with:*

$$\mathcal{Y}^\Omega = t_* \mathcal{X}^\Omega := (t \circ \mathcal{X}^\Omega) \cup \mathcal{Y}^{\mathbf{1}}.$$

Then we have:

$$t_* \mathcal{B}(\mathcal{X}^\Omega) = \mathcal{B}(t \circ \mathcal{X}^\Omega) = \mathcal{B}(t_* \mathcal{X}^\Omega) = \mathcal{B}(\mathcal{Y}^\Omega).$$

Proof.

$$\begin{aligned} t_* \mathcal{B}(\mathcal{X}^\Omega) &:= \left\{ B \subseteq \mathcal{Y} \,\middle|\, t^{-1}(B) \in \mathcal{B}(\mathcal{X}^\Omega) \right\} \\ &= \left\{ B \subseteq \mathcal{Y} \,\middle|\, \forall \alpha \in \mathcal{X}^\Omega .\, \alpha^{-1}(t^{-1}(B)) \in \mathcal{B}_\Omega \right\} \\ &= \left\{ B \subseteq \mathcal{Y} \,\middle|\, \forall \beta \in t \circ \mathcal{X}^\Omega .\, \beta^{-1}(B) \in \mathcal{B}_\Omega \right\} = \mathcal{B}(t \circ \mathcal{X}^\Omega) \\ &= \left\{ B \subseteq \mathcal{Y} \,\middle|\, \forall \beta \in (t \circ \mathcal{X}^\Omega) \cup \mathcal{Y}^{\mathbf{1}} .\, \beta^{-1}(B) \in \mathcal{B}_\Omega \right\} \\ &= \mathcal{B}(t_* \mathcal{X}^\Omega). \end{aligned}$$

□

2.11 Quotient Spaces and Embeddings between Quasi-Measurable Spaces

In this subsection we quickly look at monomorphisms, epimorphisms, embeddings and quotients of quasi-measurable spaces.

Lemma 2.59 (Injective maps are monomorphisms, monomorphisms are injective)**.** *A quasi-measurable map* $i: (\mathcal{X}, \mathcal{X}^\Omega) \to (\mathcal{Y}, \mathcal{Y}^\Omega)$ *is a* monomorphism *in* **QMS** *if and only if* i *is injective (as a map).*

Proof. Assume that i is a monomorphism and $i(x_1) = i(x_2)$ for $x_i \in \mathcal{X}$, $i = 1, 2$. Then consider the constant maps: $g_i := x_i$, $i = 1, 2$. Clearly: $g_i \in \mathbf{QMS}(\mathcal{Z}, \mathcal{X})$. Then we have $i \circ g_1 = i \circ g_2$, which by the assumption that i is a monomorphisms implies: $g_1 = g_2$, and thus $x_1 = x_2$.
Let i now be injective and $g_1, g_2 \in \mathbf{QMS}(\mathcal{Z}, \mathcal{X})$ such that $i \circ g_1 = i \circ g_2$. For every $z \in \mathcal{Z}$ we then get: $i(g_1(z)) = i(g_2(z))$. Since i is injective we get that: $g_1(z) = g_2(z)$ for every $z \in \mathcal{Z}$. So $g_1 = g_2$ as maps and thus as quasi-measurable maps. □

Lemma 2.60 (Surjective maps are epimorphisms)**.** *A surjective quasi-measurable map* $t: (\mathcal{X}, \mathcal{X}^\Omega) \to (\mathcal{Y}, \mathcal{Y}^\Omega)$ *is an* epimorphism *in* **QMS***.*

Proof. Assume that t is surjective and let $h_i: (\mathcal{Y}, \mathcal{Y}^\Omega) \to (\mathcal{Z}, \mathcal{Z}^\Omega)$, $i = 1, 2$, be quasi-measurable maps with $h_1 \circ t = h_2 \circ t$. Since t is surjective, for every $y \in \mathcal{Y}$ there is an $x \in \mathcal{X}$ with $t(x) = y$. Then we get:

$$h_1(y) = h_1 \circ t(x) = h_2 \circ t(x) = h_1(y).$$

Since this holds for every $y \in \mathcal{Y}$ we get $h_1 = h_2$ as maps and thus as quasi-measurable maps. □

Lemma 2.61 (Some epimorphisms are surjective). *Let* $t: (\mathcal{X}, \mathcal{X}^\Omega) \to (\mathcal{Y}, \mathcal{Y}^\Omega)$ *be an* epimorphism *in* **QMS** *with* $t(\mathcal{X}) \in \mathcal{B}_\mathcal{Y}$. *Then* t *is surjective.*

Proof. Since $t(\mathcal{X}) \in \mathcal{B}_\mathcal{Y}$ we have that the indicator function $\mathbb{1}_{t(\mathcal{X})} \in \mathbf{2}^\mathcal{Y}$. We also have that the constant map $\mathbb{1}_\mathcal{Y} \in \mathbf{2}^\mathcal{Y}$, which maps everything to $1 \in \mathbf{2}$. Then for all $x \in \mathcal{X}$ we get:

$$\mathbb{1}_{t(\mathcal{X})} \circ t(x) = 1 = \mathbb{1}_\mathcal{Y} \circ t(x).$$

This implies the equality of quasi-measurable maps:

$$\mathbb{1}_{t(\mathcal{X})} \circ t = \mathbb{1}_\mathcal{Y} \circ t.$$

Since t is an epimorphism we get the equality:

$$\mathbb{1}_{t(\mathcal{X})} = \mathbb{1}_\mathcal{Y},$$

which when evaluated at $y \in \mathcal{Y} \backslash t(\mathcal{X})$ would give the contradiction $0 = 1$. Thus $\mathcal{Y} \backslash t(\mathcal{X}) = \varnothing$ and t must be surjective. ☐

Definition 2.62 (Quotient of a quasi-measurable space). *A quasi-measurable space* $(\mathcal{Y}, \mathcal{Y}^\Omega)$ *is called a* quotient *of the quasi-measurable space* $(\mathcal{X}, \mathcal{X}^\Omega)$ *if there exists a surjective quasi-measurable map, called the* quotient map*:*

$$t: (\mathcal{X}, \mathcal{X}^\Omega) \twoheadrightarrow (\mathcal{Y}, \mathcal{Y}^\Omega),$$

such that:

$$t_* \mathcal{X}^\Omega := t \circ \mathcal{X}^\Omega := \left\{ t \circ \alpha : \Omega \to \mathcal{Y} \,\middle|\, \alpha \in \mathcal{X}^\Omega \right\} = \mathcal{Y}^\Omega.$$

Definition 2.63 (Embedded quasi-measurable subspace). *A measurable space* $(\mathcal{X}, \mathcal{X}^\Omega)$ *is called an* embedded quasi-measurable subspace *of the quasi-measurable space* $(\mathcal{Y}, \mathcal{Y}^\Omega)$ *if there exists an injective quasi-measurable map, called the* embedding*:*

$$i: (\mathcal{X}, \mathcal{X}^\Omega) \hookrightarrow (\mathcal{Y}, \mathcal{Y}^\Omega),$$

such that:

$$\mathcal{X}^\Omega = i^* \mathcal{Y}^\Omega := \left\{ \alpha : \Omega \to \mathcal{X} \,\middle|\, i \circ \alpha \in \mathcal{Y}^\Omega \right\} = \left\{ \beta \in \mathcal{Y}^\Omega \,\middle|\, \beta(\Omega) \subseteq \mathcal{X} \right\}.$$

3 Probability Monads on the Category of Quasi-Measurable Spaces

In this section we will introduce several different *spaces of probability measures* for a quasi-measurable space: $\mathcal{G}(\mathcal{X})$, $\mathcal{Q}(\mathcal{X})$, $\mathcal{K}(\mathcal{X})$, $\mathcal{P}(\mathcal{X})$, $\mathcal{R}(\mathcal{X})$, $\mathcal{S}(\mathcal{X})$. These notions will in general all be different. To study their functorial properties we need to review the notion of a *monad.*

3.1 Monads

In this subsection we remind the reader of the definition of a (strong) monad. For literature about monads see [Koc70, Koc72, Str72, ML98, Mog91]. As a first example we will recall the strong probability monad of Giry, see [Law62, Gir82].

Definition 3.1 (Monad, see [Koc70, Koc72, ML98, Mog91])**.** *Let $\mathcal{C}$ be a category. A* monad *on $\mathcal{C}$ is a triple $(\mathcal{Q}, \delta, \mathbb{M})$ consisting of:*

1. *a functor $\mathcal{Q}: \mathcal{C} \to \mathcal{C}$,*
2. *a natural transformation $\delta: \mathrm{id}_{\mathcal{C}} \to \mathcal{Q}$,*
3. *a natural transformation $\mathbb{M}: \mathcal{Q}^2 := \mathcal{Q} \circ \mathcal{Q} \to \mathcal{Q}$,*

such that:

a. *$\mathbb{M} \circ \mathcal{Q}\mathbb{M} = \mathbb{M} \circ \mathbb{M}\mathcal{Q}$ as natural transformations $\mathcal{Q}^3 \to \mathcal{Q}$,*

b. *$\mathbb{M} \circ \mathcal{Q}\delta = \mathbb{M} \circ \delta\mathcal{Q} = \mathrm{id}_{\mathcal{Q}}$ as natural transformations $\mathcal{Q} \to \mathcal{Q}$.*

A monad $(\mathcal{Q}, \delta, \mathbb{M})$ on a category $\mathcal{C}$ is called affine*, see [Koc71, Lin79, Jac16], if $\mathcal{C}$ has a terminal object $\mathbf{1}$ and the unique morphism:*

$$! : \mathcal{Q}(\mathbf{1}) \to \mathbf{1}$$

is an isomorphism.

Definition 3.2 (Strong monad, see [Koc72, Mog91] Def. 3.2)**.** *A* strong monad *over a monoidal category $(\mathcal{C}, \times, \mathbf{1})$ is a monad $(\mathcal{Q}, \delta, \mathbb{M})$ on $\mathcal{C}$ together with a natural transformation, called* strength*:*

$$\tau_{\mathcal{X},\mathcal{Y}}: \mathcal{X} \times \mathcal{Q}(\mathcal{Y}) \to \mathcal{Q}(\mathcal{X} \times \mathcal{Y}),$$

such that:

1. *Left unitor and strength satisfy the commutative diagram:*

$$\begin{array}{ccc} \mathbf{1}\times\mathcal{Q}(\mathcal{X}) & \xrightarrow{\tau_{\mathbf{1},\mathcal{X}}} & \mathcal{Q}(\mathbf{1}\times\mathcal{X}) \\ & {\scriptstyle L_{\mathcal{Q}(\mathcal{X})}}\searrow \quad \swarrow{\scriptstyle \mathcal{Q}(L_{\mathcal{X}})} & \\ & \mathcal{Q}(\mathcal{X}). & \end{array}$$

2. *Associator and strength satisfy the commutative diagram:*

$$\begin{array}{ccccc} (\mathcal{X}\times\mathcal{Y})\times\mathcal{Q}(\mathcal{Z}) & & \xrightarrow{\tau_{\mathcal{X}\times\mathcal{Y},\mathcal{Z}}} & & \mathcal{Q}\left((\mathcal{X}\times\mathcal{Y})\times\mathcal{Z}\right) \\ \downarrow{\scriptstyle A_{\mathcal{X},\mathcal{Y},\mathcal{Q}(\mathcal{Z})}} & & & & \downarrow{\scriptstyle \mathcal{Q}(A_{\mathcal{X},\mathcal{Y},\mathcal{Z}})} \\ \mathcal{X}\times(\mathcal{Y}\times\mathcal{Q}(\mathcal{Z})) & \xrightarrow[\mathrm{id}_{\mathcal{X}}\times\tau_{\mathcal{Y},\mathcal{Z}}]{} & \mathcal{X}\times\mathcal{Q}(\mathcal{Y}\times\mathcal{Z}) & \xrightarrow[\tau_{\mathcal{X},\mathcal{Y}\times\mathcal{Z}}]{} & \mathcal{Q}(\mathcal{X}\times(\mathcal{Y}\times\mathcal{Z})). \end{array}$$

3. *Monad unit δ and strength satisfy the commutative diagram:*

$$\begin{array}{ccc} & \mathcal{X} \times \mathcal{Y} & \\ {\scriptstyle \mathrm{id}_{\mathcal{X}} \times \delta_{\mathcal{Y}}} \swarrow & & \searrow {\scriptstyle \delta_{\mathcal{X} \times \mathcal{Y}}} \\ \mathcal{X} \times \mathcal{Q}(\mathcal{Y}) & \xrightarrow[\tau_{\mathcal{X},\mathcal{Y}}]{} & \mathcal{Q}(\mathcal{X} \times \mathcal{Y}). \end{array}$$

4. *Monad action $\mathbb{M}$ commutes with strength, expressed via the commutative diagram:*

$$\begin{array}{ccccc} \mathcal{X} \times \mathcal{Q}(\mathcal{Q}(\mathcal{Y})) & \xrightarrow{\tau_{\mathcal{X},\mathcal{Q}(\mathcal{Y})}} & \mathcal{Q}(\mathcal{X} \times \mathcal{Q}(\mathcal{Y})) & \xrightarrow{\mathcal{Q}(\tau_{\mathcal{X},\mathcal{Y}})} & \mathcal{Q}(\mathcal{Q}(\mathcal{X} \times \mathcal{Y})) \\ {\scriptstyle \mathrm{id}_{\mathcal{X}} \times \mathbb{M}_{\mathcal{Y}}} \downarrow & & & & \downarrow {\scriptstyle \mathbb{M}_{\mathcal{X} \times \mathcal{Y}}} \\ \mathcal{X} \times \mathcal{Q}(\mathcal{Y}) & & \xrightarrow[\tau_{\mathcal{X},\mathcal{Y}}]{} & & \mathcal{Q}(\mathcal{X} \times \mathcal{Y}). \end{array}$$

Example 3.3 (The strong probability monad of Giry, [Law62,Gir82,Sat18])**.** *Let* $(\mathbf{Meas}, \times, \mathbf{1})$ *be the category of all measurable spaces and measurable maps, where the cartesian monoidal structure $\times$ is given by the usual product-σ-algebra:*

$$(\mathcal{X}, \mathcal{B}_{\mathcal{X}}) \times (\mathcal{Y}, \mathcal{B}_{\mathcal{Y}}) = (\mathcal{X} \times \mathcal{Y}, \mathcal{B}_{\mathcal{X}} \otimes \mathcal{B}_{\mathcal{Y}}),$$

with:

$$\mathcal{B}_{\mathcal{X}} \otimes \mathcal{B}_{\mathcal{Y}} = \sigma\left(\{A \times B \,|\, A \in \mathcal{B}_{\mathcal{X}}, B \in \mathcal{B}_{\mathcal{Y}}\}\right).$$

It is known that **Meas** *is bi-complete, i.e. it has all small limits and colimits, but it is not cartesian closed, see [Aum61]. The* Giry monad $(\mathcal{G}, \delta, \mathbb{M})$ *is given by:*

$$\mathcal{G}(\mathcal{X}, \mathcal{B}_{\mathcal{X}}) := \{\mu : \ \mathcal{B}_{\mathcal{X}} \to [0,1] \text{ probability measure}\},$$

endowed with the smallest σ-algebra that makes evaluation maps measurable:

$$\mathcal{B}_{\mathcal{G}(\mathcal{X},\mathcal{B}_{\mathcal{X}})} := \sigma\left(\left\{\mathrm{ev}_A^{-1}((t,1]) \,\middle|\, A \in \mathcal{B}_{\mathcal{X}}, t \in \mathbb{R}\right\}\right).,$$

where the evaluation maps are given by:

$$\mathrm{ev}_A : \ \mathcal{G}(\mathcal{X}, \mathcal{B}_{\mathcal{X}}) \to [0,1], \qquad \mathrm{ev}_A(\mu) := \mu(A).$$

The monad unit δ is given by mapping points to their Dirac delta point measures:

$$\delta : \ (\mathcal{X}, \mathcal{B}_{\mathcal{X}}) \to \left(\mathcal{G}(\mathcal{X}, \mathcal{B}_{\mathcal{X}}), \mathcal{B}_{\mathcal{G}(\mathcal{X},\mathcal{B}_{\mathcal{X}})}\right), \qquad x \mapsto \delta_x = (A \mapsto \mathbb{1}_A(x)).$$

The monad action $\mathbb{M}$ is given (while suppressing the σ-algebras for readability) via:

$$\mathbb{M} : \ \mathcal{G}(\mathcal{G}(\mathcal{X})) \to \mathcal{G}(\mathcal{X}), \qquad \mathbb{M}(\pi)(A) := \int_{\mathcal{G}(\mathcal{X})} \mathrm{ev}_A(\mu)\, \pi(d\mu).$$

The Giry monad is affine and strong, see also [Gir82, Pan09, Jac16], and its strength is given by:

$$\tau_{\mathcal{X},\mathcal{Y}} : \ \mathcal{X} \times \mathcal{G}(\mathcal{Y}) \to \mathcal{G}(\mathcal{X} \times \mathcal{Y}), \qquad (x, \mu) \mapsto \delta_x \otimes \mu,$$

The Giry monad supports first order semantics of contiuous probabilistic programming languages, but not higher-order ones, as **Meas** *is not cartesian closed, see [Aum61].*

3.2 Strong Probability Monad $\mathcal{Q}$

In this subsection we will highlight a novel and strong probability monad $\mathcal{Q}$. In contrast to the Giry construction $\mathcal{G}$ the probability monad $\mathcal{Q}$ is well-behaved w.r.t. the categorical constructions inside the category of quasi-measurable spaces **QMS**, e.g. w.r.t. products and σ-algebras.

Definition 3.4 (The spaces of probability measures for quasi-measurable spaces)**.** *Let $(\mathcal{X}, \mathcal{X}^\Omega)$ be a quasi-measurable space and $\mathcal{B}_\mathcal{X} := \mathcal{B}(\mathcal{X}^\Omega)$ its induced σ-algebra, which we will also consider as the quasi-measurable space $(\mathcal{B}_\mathcal{X}, (\mathcal{B}_\mathcal{X})^\Omega)$. Then we define:*

$$\begin{aligned}
\mathcal{G}(\mathcal{X}, \mathcal{X}^\Omega) &:= \mathcal{G}(\mathcal{X}, \mathcal{B}_\mathcal{X}) = \{\mu : \mathcal{B}_\mathcal{X} \to [0,1] \text{ probability measure}\}, \\
\mathcal{G}(\mathcal{X}, \mathcal{X}^\Omega)^\Omega &:= \mathcal{F}\left(\mathcal{B}_{\mathcal{G}(\mathcal{X},\mathcal{B}_\mathcal{X})}\right) \\
&= \left\{\kappa : \Omega \to \mathcal{G}(\mathcal{X}, \mathcal{X}^\Omega) \,\middle|\, \forall A \in \mathcal{B}_\mathcal{X}.\, (\omega \mapsto \kappa(\omega)(A)) \in [0,1]^\Omega\right\}, \\
\mathcal{Q}(\mathcal{X}, \mathcal{X}^\Omega) &:= \mathcal{G}(\mathcal{X}, \mathcal{X}^\Omega) \cap [0,1]^{\mathcal{B}_\mathcal{X}} \\
&= \left\{\mu : \mathcal{B}_\mathcal{X} \to [0,1] \text{ probability measure} \,\middle|\, \forall D \in \mathcal{B}_{\Omega\times\mathcal{X}}.\, (\omega \mapsto \mu(D_\omega)) \in [0,1]^\Omega\right\}, \\
\mathcal{Q}(\mathcal{X}, \mathcal{X}^\Omega)^\Omega &:= [\Omega \to \mathcal{Q}(\mathcal{X}, \mathcal{X}^\Omega)] \cap ([0,1]^{\mathcal{B}_\mathcal{X}})^\Omega \\
&= \left\{\kappa : \Omega \to \mathcal{Q}(\mathcal{X}, \mathcal{X}^\Omega) \,\middle|\, \forall \varphi \in \Omega^\Omega.\, \forall D \in \mathcal{B}_{\Omega\times\mathcal{X}}.\, (\omega \mapsto \kappa(\varphi(\omega))(D_\omega)) \in [0,1]^\Omega\right\}.
\end{aligned}$$

Remark 3.5. *Let $(\mathcal{X}, \mathcal{X}^\Omega)$ be a quasi-measurable space.*

1. *$(\mathcal{G}(\mathcal{X}), \mathcal{G}(\mathcal{X})^\Omega)$ is a quasi-measurable space as it is induced from the underlying measurable space endowed with the smallest σ-algebra that makes all evaluation maps measurable, see Example 2.7.*

2. *$(\mathcal{Q}(\mathcal{X}), \mathcal{Q}(\mathcal{X})^\Omega)$ is a quasi-measurable space with the subspace structure of $[0,1]^{\mathcal{B}_\mathcal{X}}$, also see Lemma 2.57.*

3. *It is then clear that the evaluation map:*

$$\text{ev} :\ \mathcal{Q}(\mathcal{X}) \times \mathcal{B}_\mathcal{X} \xrightarrow{\text{incl}\times\text{id}} [0,1]^{\mathcal{B}_\mathcal{X}} \times \mathcal{B}_\mathcal{X} \xrightarrow{\text{ev}} [0,1], \quad (\mu, A) \mapsto \mu(A),$$

is quasi-measurable.

4. *Note that this might not be true for $\mathcal{G}(\mathcal{X})$ in place of $\mathcal{Q}(\mathcal{X})$. This is the reason that we will focus on $\mathcal{Q}(\mathcal{X})$ in the framework of quasi-measurable spaces.*

5. *The inclusion map $(\mathcal{Q}(\mathcal{X}), \mathcal{Q}(\mathcal{X})^\Omega) \to (\mathcal{G}(\mathcal{X}), \mathcal{G}(\mathcal{X})^\Omega)$ is quasi-measurable.*

Indeed, for $\kappa \in \mathcal{Q}(\mathcal{X})^\Omega$ and $A \in \mathcal{B}_\mathcal{X}$ and $\omega \in \Omega$ we have that $\Omega \times A \in \mathcal{B}_{\Omega\times\mathcal{X}}$, that $(\Omega \times A)_\omega = A$ and the map:

$$\omega \mapsto \kappa(\omega)(A) = \kappa(\text{id}_\Omega(\omega))((\Omega \times A)_\omega)$$

is measurable.

6. *If for some reason we have that $\mathcal{B}_{\Omega\times\mathcal{X}} = \mathcal{B}_\Omega \otimes \mathcal{B}_\mathcal{X}$ then we have the equality:*
$$(\mathcal{Q}(\mathcal{X}), \mathcal{Q}(\mathcal{X})^\Omega) = (\mathcal{G}(\mathcal{X}), \mathcal{G}(\mathcal{X})^\Omega).$$
Indeed, for $A \in \mathcal{B}_\mathcal{X}$ and $B \in \mathcal{B}_\Omega$ the map:
$$\omega \mapsto \kappa(\varphi(\omega))((B \times A)_\omega) = \kappa(\varphi(\omega))(A) \cdot \mathbb{1}_B(\omega),$$
is measurable if $\omega \mapsto \kappa(\omega)(A)$ is.

7. *The mentioned condition holds, for instance, for $\Omega = \mathcal{X} = \mathbb{R}$ with the Borel-structure.*

8. *Similarly, if $\mathcal{B}_\Omega = (\mathcal{B}_\Omega)_\mathcal{G}$ and $\mathcal{B}_{\Omega\times\mathcal{X}} = (\mathcal{B}_\Omega \otimes \mathcal{B}_\mathcal{X})_\mathcal{G}$ then we also have equality:*
$$(\mathcal{Q}(\mathcal{X}), \mathcal{Q}(\mathcal{X})^\Omega) = (\mathcal{G}(\mathcal{X}), \mathcal{G}(\mathcal{X})^\Omega).$$
This holds for instance for $\Omega = \mathcal{X} = \mathbb{R}$ with the universally measurable structure as we will see later in Lemma 5.28.

Lemma 3.6. *Let $(\mathcal{X}, \mathcal{X}^\Omega)$ be a quasi-measurable space. Then the integration map:*
$$[0,\infty]^\mathcal{X} \times \mathcal{Q}(\mathcal{X}) \to [0,\infty], \quad (f,\mu) \mapsto \int f\, d\mu,$$
is quasi-measurable.

Proof. Let $g \in [0,\infty]^{\Omega\times\mathcal{X}}$ and $\nu \in \mathcal{Q}(\mathcal{X})^\Omega$. Then we need to show that the map:
$$\omega \mapsto \int g(\omega,x)\,\nu(\omega)(dx)$$
is $\mathcal{B}_\Omega$-$\mathcal{B}_{[0,\infty]}$-measurable, where $\mathcal{B}_{[0,\infty]}$ is the Borel σ-algebra of $[0,\infty]$. Since g is quasi-measurable it is $\mathcal{B}_{\Omega\times\mathcal{X}}$-$\mathcal{B}([0,\infty]^\Omega)$-measurable. Since the Borel σ-algebra $\mathcal{B}_{[0,\infty]} \subseteq \mathcal{B}([0,\infty]^\Omega)$ we see that g is $\mathcal{B}_{\Omega\times\mathcal{X}}$-$\mathcal{B}_{[0,\infty]}$-measurable. By Theorem 1.96 in [Kle20] there are (at most) countably many measurable sets $E_n \in \mathcal{B}_{\Omega\times\mathcal{X}}$ and $a_n \in [0,\infty]$ for $n \in \mathbb{N}$ such that:
$$g(\omega,x) = \sum_{n\in\mathbb{N}} a_n \cdot \mathbb{1}_{E_n}(\omega,x).$$
By definition of $\mathcal{Q}(\mathcal{X})^\Omega$ we already know that:
$$\omega \mapsto \int \mathbb{1}_{E_n}(\omega,x)\,\kappa(\omega)(dx) = \kappa(\omega)(E_{n,\omega})$$
is $\mathcal{B}_\Omega$-$\mathcal{B}_{[0,\infty]}$-measurable for every $n \in \mathbb{N}$. So we immediately get that the map:
$$\omega \mapsto \int g(\omega,x)\,\kappa(\omega)(dx) = \sum_{n\in\mathbb{N}} a_n \cdot \int \mathbb{1}_{E_n}(\omega,x)\,\kappa(\omega)(dx),$$
is also $\mathcal{B}_\Omega$-$\mathcal{B}_{[0,\infty]}$-measurable. This shows the claim. ☐

Lemma 3.7. *Let $(\mathcal{X}, \mathcal{X}^\Omega)$, $(\mathcal{Y}, \mathcal{Y}^\Omega)$, $(\mathcal{Z}, \mathcal{Z}^\Omega)$ be quasi-measurable spaces. Then the map:*

$$[0,\infty]^{\mathcal{X}\times\mathcal{Y}\times\mathcal{Z}} \times \mathcal{Q}(\mathcal{X})^{\mathcal{Y}\times\mathcal{Z}} \times \mathcal{Q}(\mathcal{Y})^{\mathcal{Z}} \times \mathcal{Z} \to [0,\infty], \qquad (g,\mu,\nu,z) \mapsto \int\int g(x,y,z)\,\mu(y,z)(dx)\,\nu(z)(dy),$$

is a well-defined quasi-measurable map.

Proof. By Lemma 3.6 the map:

$$[0,\infty]^{\mathcal{X}} \times \mathcal{Q}(\mathcal{X}) \xrightarrow{\int} [0,1], \qquad (f,\mu) \mapsto \int f d\mu$$

is quasi-measurable. By Lemma 2.33 then also the exponentiated map;

$$[0,\infty]^{\mathcal{X}\times\mathcal{Y}} \times \mathcal{Q}(\mathcal{X})^{\mathcal{Y}} \xrightarrow{\int} [0,\infty]^{\mathcal{Y}},$$

is quasi-measurable, where we used the universal property of the product $(\mathcal{W}\times\mathcal{U})^{\mathcal{Y}} = \mathcal{W}^{\mathcal{Y}} \times \mathcal{U}^{\mathcal{Y}}$ on the left. Again, by Lemma 3.6 also the integration map:

$$[0,\infty]^{\mathcal{Y}} \times \mathcal{Q}(\mathcal{Y}) \to [0,\infty], \quad (h,\nu) \mapsto \int h\,d\nu,$$

is quasi-measurable. So the composition of the two constructions above:

$$\left([0,\infty]^{\mathcal{X}\times\mathcal{Y}} \times \mathcal{Q}(\mathcal{X})^{\mathcal{Y}}\right) \times \mathcal{Q}(\mathcal{Y}) \xrightarrow{\int\times\mathrm{id}} [0,\infty]^{\mathcal{Y}} \times \mathcal{P}(\mathcal{Y}) \xrightarrow{\int} [0,\infty],$$

is quasi-measurable as well. Exponentiating with $\mathcal{Z}$, again by Lemma 2.33, then gives the quasi-measurable map:

$$[0,\infty]^{\mathcal{X}\times\mathcal{Y}\times\mathcal{Z}} \times \mathcal{Q}(\mathcal{X})^{\mathcal{Y}\times\mathcal{Z}} \times \mathcal{Q}(\mathcal{Y})^{\mathcal{Z}} \to [0,\infty]^{\mathcal{Z}},$$

$$(g,\mu,\nu) \mapsto \left(z \mapsto \int\int g(x,y,z)\,\mu(y,z)(dx)\,\nu(z)(dy)\right),$$

which induces the quasi-measurable adjoint evaluation map:

$$[0,\infty]^{\mathcal{X}\times\mathcal{Y}\times\mathcal{Z}} \times \mathcal{Q}(\mathcal{X})^{\mathcal{Y}\times\mathcal{Z}} \times \mathcal{Q}(\mathcal{Y})^{\mathcal{Z}} \times \mathcal{Z} \to [0,\infty],$$

$$(g,\mu,\nu,z) \mapsto \int\int g(x,y,z)\,\mu(y,z)(dx)\,\nu(z)(dy).$$

This shows the claim. ☐

Lemma 3.8. *Let $(\mathcal{X}, \mathcal{X}^\Omega)$, $(\mathcal{Y}, \mathcal{Y}^\Omega)$, $(\mathcal{Z}, \mathcal{Z}^\Omega)$ be quasi-measurable spaces. Then the map:*

$$\mathcal{Q}(\mathcal{X})^{\mathcal{Y}\times\mathcal{Z}} \times \mathcal{Q}(\mathcal{Y})^{\mathcal{Z}} \times \mathcal{B}_{\mathcal{X}\times\mathcal{Y}\times\mathcal{Z}} \times \mathcal{Z} \to [0,1], \qquad (\mu,\nu,D,z) \mapsto \int \mu(y,z)(D_{y,z})\,\nu(z)(dy),$$

is a well-defined quasi-measurable map.

Proof. This directly follows from Lemma 3.7 together with the quasi-measurable map:

$$\mathbb{1}: \mathcal{B}_{\mathcal{X}\times\mathcal{Y}\times\mathcal{Z}} \to [0,1]^{\mathcal{X}\times\mathcal{Y}\times\mathcal{Z}}, \qquad D \mapsto \mathbb{1}_D.$$

□

Proposition 3.9. *Let* $(\mathcal{X}, \mathcal{X}^\Omega)$, $(\mathcal{Y}, \mathcal{Y}^\Omega)$, $(\mathcal{Z}, \mathcal{Z}^\Omega)$ *be quasi-measurable spaces. Then the map:*

$$\otimes: \mathcal{Q}(\mathcal{X})^{\mathcal{Y}\times\mathcal{Z}} \times \mathcal{Q}(\mathcal{Y})^{\mathcal{Z}} \to \mathcal{Q}(\mathcal{X}\times\mathcal{Y})^{\mathcal{Z}}, \quad (\mu\otimes\nu)(z)(D) := \int \mu(y,z)(D_y)\,\nu(z)(dy),$$

is a well-defined quasi-measurable map.

Proof. By Lemma 3.8 we get a quasi-measurable map:

$$\otimes: \mathcal{Q}(\mathcal{X})^{\mathcal{Y}\times\mathcal{Z}} \times \mathcal{Q}(\mathcal{Y})^{\mathcal{Z}} \to \left([0,1]^{\mathcal{B}_{\mathcal{X}\times\mathcal{Y}}}\right)^{\mathcal{Z}}.$$

By Lemma 2.57 we only need to check that the image evaluated at each z always is a probability measure on $\mathcal{B}_{\mathcal{X}\times\mathcal{Y}}$, which is obvious. □

Lemma 3.10. *Let* $(\mathcal{X}, \mathcal{X}^\Omega)$, $(\mathcal{Y}, \mathcal{Y}^\Omega)$ *be quasi-measurable spaces. Then the push-forward map is quasi-measurable:*

$$\mathrm{pf}: \mathcal{Y}^{\mathcal{X}} \times \mathcal{Q}(\mathcal{X}) \to \mathcal{Q}(\mathcal{Y}), \qquad (f,\mu) \mapsto f_*\mu.$$

Proof. The following map is quasi-measurable:

$$\mathcal{Q}(\mathcal{X}) \times \mathcal{Y}^{\mathcal{X}} \times \mathcal{B}_{\mathcal{Y}} \to [0,1], \qquad (\mu, f, B) \mapsto \mu(f^{-1}(B)).$$

Indeed, it is the composition of the following quasi-measurable map from Lemma 2.49:

$$\mathcal{Y}^{\mathcal{X}} \times \mathcal{B}_{\mathcal{Y}} \to \mathcal{B}_{\mathcal{X}}, \qquad (f, B) \mapsto f^{-1}(B),$$

and the quasi-measurable evaluation map:

$$\mathcal{Q}(\mathcal{X}) \times \mathcal{B}_{\mathcal{X}} \to [0,1], \qquad (\mu, A) \mapsto \mu(A).$$

By adjuction we get the quasi-measurable map:

$$\mathcal{Q}(\mathcal{X}) \times \mathcal{Y}^{\mathcal{X}} \to [0,1]^{\mathcal{B}_{\mathcal{Y}}}, \qquad (\mu, f) \mapsto f_*\mu = (B \mapsto \mu(f^{-1}(B))).$$

By Lemma 2.57 is left to show that $f_*\mu$ is a probability measure, which is clear. So the claims are shown. □

Remark 3.11. *1. By Theorem 2.32 the triple* $(\mathbf{QMS}, \times, \mathbf{1})$ *is a cartesian closed (symmetric strict) monoidal category, where* $\mathbf{1}$ *is the one-point space, which is also a terminal object of* $\mathbf{QMS}$.

2. $\mathcal{Q}:$ **QMS** $\to$ **QMS** *is a functor by Lemma 3.10:*

$$(f: \mathcal{X} \to \mathcal{Y}) \mapsto (\mathcal{Q}(f): \mathcal{Q}(\mathcal{X}) \to \mathcal{Q}(\mathcal{Y}),\ \mu \mapsto f_*\mu)\,.$$

Lemma 3.12. *Let $(\mathcal{X}, \mathcal{X}^\Omega)$ be a quasi-measurable space. Then the map:*

$$\delta: \mathcal{X} \to \mathcal{Q}(\mathcal{X}), \qquad x \mapsto \delta_x = (A \mapsto \mathbb{1}_A(x)),$$

is a well-defined quasi-measurable map.
Furthermore, if we have a quasi-measurable map:

$$f: (\mathcal{X}, \mathcal{X}^\Omega) \to (\mathcal{Y}, \mathcal{Y}^\Omega),$$

then we get a commutive diagram of quasi-measurable maps:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} \\ \mathcal{Q}(\mathcal{X}) & \xrightarrow{f_*} & \mathcal{Q}(\mathcal{Y}). \end{array}$$

In other words, $\delta: \mathrm{id}_{\mathbf{QMS}} \to \mathcal{Q}$ is a natural transformation of endo-functors of **QMS***.*

Proof. To show that δ is well-defined we need to show that $\delta_x \in \mathcal{Q}(\mathcal{X})$ for $x \in \mathcal{X}$. For this let $D \in \mathcal{B}_{\Omega \times \mathcal{X}}$. Then the map:

$$\omega \mapsto \delta_x(D_\omega) = \mathbb{1}_D(\omega, x) = \mathbb{1}_{D_x}(\omega),$$

is measurable since $D_x \in \mathcal{B}_\Omega$. Thus we have $\delta_x \in \mathcal{Q}(\mathcal{X})$.
To show that δ is quasi-measurable let $\alpha \in \mathcal{X}^\Omega$ and $\varphi \in \Omega^\Omega$ and again $D \in \mathcal{B}_{\Omega\times\mathcal{X}}$. Then we see that the map:

$$\omega \mapsto \delta_{\alpha\circ\varphi(\omega)}(D_\omega) = \mathbb{1}_D(\omega, \alpha(\varphi(\omega))) = \mathbb{1}_{(\mathrm{id}_\Omega, \alpha\circ\varphi)^{-1}(D)}(\omega),$$

is measurable as $\alpha \circ \varphi \in \mathcal{X}^\Omega$. It follows that $\delta \circ \alpha \in \mathcal{Q}(\mathcal{X})^\Omega$ and thus $\delta: \mathcal{X} \to \mathcal{Q}(\mathcal{X})$ quasi-measurable.
To check the commutativity of the diagram observe that for $B \in \mathcal{B}_\mathcal{Y}$:

$$f_*\delta_x(B) = \delta_x(f^{-1}(B)) = \mathbb{1}_{f^{-1}(B)}(x) = \mathbb{1}_B(f(x)) = \delta_{f(x)}(B).$$

□

Lemma 3.13. *Let $(\mathcal{X}, \mathcal{X}^\Omega)$ be a quasi-measurable space. Then the map:*

$$\mathbb{M}_\mathcal{X}: \mathcal{Q}(\mathcal{Q}(\mathcal{X})) \to \mathcal{Q}(\mathcal{X}), \qquad \pi \mapsto \mathbb{M}_\mathcal{X}(\pi) = \left(A \mapsto \int \mathrm{ev}(\mu, A)\, \pi(d\mu)\right),$$

is a well-defined quasi-measurable map.
Furthermore, if we have a quasi-measurable map:

$$f: (\mathcal{X}, \mathcal{X}^\Omega) \to (\mathcal{Y}, \mathcal{Y}^\Omega),$$

then we get a commutive diagram of quasi-measurable maps:

$$\begin{array}{ccc} \mathcal{Q}(\mathcal{Q}(\mathcal{X})) & \xrightarrow{\mathcal{Q}(\mathcal{Q}(f))} & \mathcal{Q}(\mathcal{Q}(\mathcal{Y})) \\ \downarrow{\scriptstyle \mathbb{M}_{\mathcal{X}}} & & \downarrow{\scriptstyle \mathbb{M}_{\mathcal{Y}}} \\ \mathcal{Q}(\mathcal{X}) & \xrightarrow[\mathcal{Q}(f)]{} & \mathcal{Q}(\mathcal{Y}). \end{array}$$

In other words, $\mathbb{M} : \mathcal{Q}^2 \to \mathcal{Q}$ *is a natural transformation of endo-functors of* **QMS**.

Proof. It is easy to see that $\mathbb{M}_{\mathcal{X}}(\pi)$ is a probability measure on $\mathcal{B}_{\mathcal{X}}$. To check that $\mathbb{M}_{\mathcal{X}}$ is quasi-measurable let e be the map:

$$e : \mathcal{B}_{\mathcal{X}} \to [0,1]^{\mathcal{Q}(\mathcal{X})}, \qquad e(A)(\mu) := \mathrm{ev}(\mu, A),$$

which is quasi-measurable as the adjoint to the quasi-measurable evaluation map (see Remark 3.5):

$$\mathrm{ev} : \mathcal{Q}(\mathcal{X}) \times \mathcal{B}_{\mathcal{X}} \to [0,1].$$

Since by Lemma 3.6 also the integration pairing is quasi-measurable the following composition of quasi-measurable maps is also quasi-measurable:

$$\int \circ (\mathrm{id} \times e) : \mathcal{Q}(\mathcal{Q}(\mathcal{X})) \times \mathcal{B}_{\mathcal{X}} \to [0,1], \qquad (\pi, A) \mapsto \int \mathrm{ev}(\mu, A)\, \pi(d\mu).$$

Taking the adjoint we get the quasi-measurable map:

$$\mathbb{M}_{\mathcal{X}} : \mathcal{Q}(\mathcal{Q}(\mathcal{X})) \to [0,1]^{\mathcal{B}_{\mathcal{X}}},$$

whose image lies inside $[0,1]^{\mathcal{B}_{\mathcal{X}}} \cap \mathcal{G}(\mathcal{X}) = \mathcal{Q}(\mathcal{X})$.
To check the commutative diagram let $B \in \mathcal{B}_{\mathcal{Y}}$. Then we have with $f_* := \mathcal{Q}(f)$ and $f_{**} := \mathcal{Q}^2(f) := \mathcal{Q}(\mathcal{Q}(f))$:

$$\begin{aligned} \mathbb{M}_{\mathcal{Y}}(f_{**}\pi)(B) &= \int \mathrm{ev}(\nu, B)\, (f_{**}\pi)(d\nu) \\ &= \int \mathrm{ev}(f_*\mu, B)\, \pi(d\mu) \\ &= \int \mathrm{ev}(\mu, f^{-1}(B))\, \pi(d\mu) \\ &= \mathbb{M}_{\mathcal{X}}(\pi)(f^{-1}(B)) \\ &= f_*\mathbb{M}_{\mathcal{X}}(\pi)(B). \end{aligned}$$

This shows the claim. □

Proposition 3.14. *The triple* $(\mathcal{Q}, \delta, \mathbb{M})$ *defines a monad on* **QMS**.

Proof. By Lemma 3.10 we know that $\mathcal{Q} : \mathbf{QMS} \to \mathbf{QMS}$ is a functor. By Lemma 3.12 we know that $\delta : \mathrm{id}_{\mathbf{QMS}} \to \mathcal{Q}$ is a natural transformation. By Lemma 3.13 we know that $\mathbb{M} : \mathcal{Q}^2 \to \mathcal{Q}$ is a natural transformation. So we are left to check the following coherence conditions:

1. $\mathbb{M} \circ \mathcal{Q}\mathbb{M} = \mathbb{M} \circ \mathbb{M}\mathcal{Q}$ as natural transformations $\mathcal{Q}^3 \to \mathcal{Q}$,

2. $\mathbb{M} \circ \mathcal{Q}\delta = \mathbb{M} \circ \delta\mathcal{Q} = \mathrm{id}_{\mathcal{Q}}$ as natural transformations $\mathcal{Q} \to \mathcal{Q}$.

For simple notations note that:

$$\int \mathbb{1}_A(x)\, \mathbb{M}_{\mathcal{X}}(\pi)(dx) = \mathbb{M}_{\mathcal{X}}(\pi)(A) = \int\int \mathbb{1}_A(x)\, \mu(dx)\, \pi(d\mu),$$

or in short: $\mathbb{M}_{\mathcal{X}}(\pi)(dx) = \int \mu(dx)\, \pi(d\mu)$.
For the first let $\tau \in \mathcal{Q}(\mathcal{Q}(\mathcal{Q}(\mathcal{X})))$. Then we have for $A \in \mathcal{B}_{\mathcal{X}}$:

$$\begin{aligned}
(\mathbb{M}_{\mathcal{X}} \circ \mathcal{Q}(\mathbb{M}_{\mathcal{X}})(\tau))(A) &= \int\int \mathbb{1}_A(x)\, \nu(dx)\, ((\mathbb{M}_{\mathcal{X}})_*\tau)(d\nu) \\
&= \int\int \mathbb{1}_A(x)\, \mathbb{M}_{\mathcal{X}}(\pi)(dx)\, \tau(d\pi) \\
&= \int\int\int \mathbb{1}_A(x)\, \mu(dx)\, \pi(d\mu)\, \tau(d\pi) \\
&= \int\int \mathbb{1}_A(x)\, \mu(dx)\, (\mathbb{M}_{\mathcal{Q}(\mathcal{X})}(\tau))(d\mu) \\
&= (\mathbb{M}_{\mathcal{X}} \circ \mathbb{M}_{\mathcal{Q}(\mathcal{X})}(\tau))(A).
\end{aligned}$$

It is easily seen that $\mathbb{M} \circ \mathcal{Q}\mathbb{M}$ is a natural transformation: $\mathcal{Q}^3 \to \mathcal{Q}$. Indeed, for quasi-measurable $f : \mathcal{X} \to \mathcal{Y}$ we can use the above and the functoriality rules from Lemmata 3.12 and 3.13:

$$\begin{aligned}
\mathbb{M}_{\mathcal{Y}} \circ \mathcal{Q}(\mathbb{M}_{\mathcal{Y}}) \circ \mathcal{Q}^3(f) &= \mathbb{M}_{\mathcal{Y}} \circ \mathcal{Q}(\mathbb{M}_{\mathcal{Y}} \circ \mathcal{Q}^2(f)) \\
&= \mathbb{M}_{\mathcal{Y}} \circ \mathcal{Q}(\mathcal{Q}(f) \circ \mathbb{M}_{\mathcal{X}}) \\
&= \mathbb{M}_{\mathcal{Y}} \circ \mathcal{Q}^2(f) \circ \mathcal{Q}(\mathbb{M}_{\mathcal{X}}) \\
&= \mathcal{Q}(f) \circ \mathbb{M}_{\mathcal{X}} \circ \mathcal{Q}(\mathbb{M}_{\mathcal{X}}).
\end{aligned}$$

To show the coherence conditions 2) let $\nu \in \mathcal{Q}(\mathcal{X})$. Then we get for $A \in \mathcal{B}_{\mathcal{X}}$:

$$\begin{aligned}
\mathbb{M}_{\mathcal{X}}(\delta_\nu)(A) &= \int \mathrm{ev}(\mu, A)\, \delta_\nu(d\mu) \\
&= \mathrm{ev}(\nu, A) \\
&= \nu(A), \\
\mathbb{M}_{\mathcal{X}}(\mathcal{Q}(\delta)(\nu))(A) &= \int \mathrm{ev}(\mu, A)\, \mathcal{Q}(\delta)(\nu)(d\mu) \\
&= \int \mathrm{ev}(\mu, A)\, (\delta_*\nu)(d\mu) \\
&= \int \mathrm{ev}(\delta_x, A)\, \nu(dx) \\
&= \int \mathbb{1}_A(x)\, \nu(dx) \\
&= \nu(A).
\end{aligned}$$

Note that in the first set of equations δ is the map:

$$\delta:\ \mathcal{Q}(\mathcal{X}) \to \mathcal{Q}(\mathcal{Q}(\mathcal{X})), \quad \nu \mapsto \delta_\nu.$$

In the second set of equations we use the usual:

$$\delta:\ \mathcal{X} \to \mathcal{Q}(\mathcal{X}), \quad x \mapsto \delta_x,$$

where then:

$$\mathcal{Q}(\delta) = \delta_*:\ \mathcal{Q}(\mathcal{X}) \to \mathcal{Q}(\mathcal{Q}(\mathcal{X})), \quad \nu \mapsto \delta_*\nu,$$

is its push-forward, where we then used the usual substitution rule. □

Lemma 3.15. *Let $(\mathcal{X}, \mathcal{X}^\Omega)$ and $(\mathcal{Y}, \mathcal{Y}^\Omega)$ be two quasi-measurable spaces. Then the following map:*

$$\tau_{\mathcal{X},\mathcal{Y}}:\ \mathcal{X} \times \mathcal{Q}(\mathcal{Y}) \to \mathcal{Q}(\mathcal{X} \times \mathcal{Y}), \qquad (x, \mu) \mapsto \delta_x \otimes \mu = (D \mapsto \mu(D_x)),$$

is a well-defined quasi-measurable map.
Furthermore, if $f:\ \mathcal{X} \to \mathcal{X}'$ and $g:\ \mathcal{Y} \to \mathcal{Y}'$ are quasi-measurable we get a commutative diagram of quasi-measurable spaces:

$$\begin{array}{ccc} \mathcal{X} \times \mathcal{Q}(\mathcal{Y}) & \xrightarrow{\tau_{\mathcal{X},\mathcal{Y}}} & \mathcal{Q}(\mathcal{X} \times \mathcal{Y}) \\ {\scriptstyle f \times \mathcal{Q}(g)}\downarrow & & \downarrow{\scriptstyle \mathcal{Q}(f \times g)} \\ \mathcal{X}' \times \mathcal{Q}(\mathcal{Y}') & \xrightarrow[\tau_{\mathcal{X}',\mathcal{Y}'}]{} & \mathcal{Q}(\mathcal{X}' \times \mathcal{Y}'). \end{array}$$

Proof. It is easily seen that $\tau(x, \mu)$ is a probability measure, as it is the push-forward of μ along the quasi-measurable slice map:

$$\iota_x:\ \mathcal{Y} \to \mathcal{X} \times \mathcal{Y}, \quad y \mapsto (x, y).$$

τ is quasi-measurable as the adjoint to the compositions of the following quasi-measurable maps:

$$\mathcal{X} \times \mathcal{B}_{\mathcal{X}\times\mathcal{Y}} \times \mathcal{Q}(\mathcal{Y}) \to (\mathcal{X} \times \mathcal{Y})^{\mathcal{Y}} \times \mathcal{B}_{\mathcal{X}\times\mathcal{Y}} \times \mathcal{Q}(\mathcal{Y}) \to \mathcal{B}_{\mathcal{Y}} \times \mathcal{Q}(\mathcal{Y}) \xrightarrow{\text{ev}} [0,1],$$

which is composed of the quasi-measurable map:

$$\mathcal{X} \to (\mathcal{X} \times \mathcal{Y})^{\mathcal{Y}}, \qquad x \mapsto \iota_x,$$

together with the quasi-measurable map (see Lemma 2.49):

$$(\mathcal{X} \times \mathcal{Y})^{\mathcal{Y}} \times \mathcal{B}_{\mathcal{X}\times\mathcal{Y}} \to \mathcal{B}_{\mathcal{Y}}, \qquad (f, D) \mapsto f^{-1}(D),$$

and the quasi-measurable evaluation map (see Remark 3.5). It follows that

$$\tau:\ \mathcal{X} \times \mathcal{Q}(\mathcal{Y}) \to [0,1]^{\mathcal{B}_{\mathcal{X}\times\mathcal{Y}}}, \quad (x, \mu) \mapsto (D \mapsto \mu(D_x)),$$

is quasi-measurable and with image in $\mathcal{Q}(\mathcal{X} \times \mathcal{Y})$. So it is a well-defined quasi-measurable map until there (see Lemma 2.57).
To check that the diagram is commutative let $x \in \mathcal{X}$ and $\mu \in \mathcal{Q}(\mathcal{Y})$ and $D' \in \mathcal{B}_{\mathcal{X}' \times \mathcal{Y}'}$. Then we get:

$$\begin{aligned}
\tau_{\mathcal{X}',\mathcal{Y}'}(f(x), \mathcal{Q}(g)(\mu))(D') &= \mathcal{Q}(g)(\mu)(D'_{f(x)}) \\
&= \mu(g^{-1}(D'_{f(x)})) \\
&= \mu((f \times g)^{-1}(D')_x)) \\
&= \tau_{\mathcal{X},\mathcal{Y}}(x, \mu))((f \times g)^{-1}(D')) \\
&= \mathcal{Q}(f \times g)(\tau_{\mathcal{X},\mathcal{Y}}(x, \mu))(D').
\end{aligned}$$

□

This shows the claim.

Theorem 3.16. *The triple* $(\mathcal{Q}, \delta, \mathbb{M})$ *defines a* strong monad *on the (cartesian closed) monoidal category* $(\mathbf{QMS}, \times, \mathbf{1})$.

Proof. By Theorem 2.32 is a (cartesian closed) monoidal category $(\mathbf{QMS}, \times, \mathbf{1})$. By Proposition 3.14 the triple $(\mathcal{Q}, \delta, \mathbb{M})$ defines a monad structure on $\mathbf{QMS}$. We then define the *strength* of the monad via τ from Lemma 3.15, where it was also shown that τ is a natural transformation $(\_) \times \mathcal{Q}(\_) \to \mathcal{Q}(\_ \times \_)$. We are left to check its coherence conditions.
Left unitor and strength satisfy the commutative diagram:

$$\begin{array}{ccc}
\mathbf{1} \times \mathcal{Q}(\mathcal{X}) & \xrightarrow{\tau_{\mathbf{1},\mathcal{X}}} & \mathcal{Q}(\mathbf{1} \times \mathcal{X}) \\
& {\scriptstyle L_{\mathcal{Q}(\mathcal{X})}} \searrow \quad \swarrow {\scriptstyle \mathcal{Q}(L_{\mathcal{X}})} & \\
& \mathcal{Q}(\mathcal{X}). &
\end{array}$$

Associator and strength satisfy the commutative diagram:

$$\begin{array}{ccccc}
(\mathcal{X} \times \mathcal{Y}) \times \mathcal{Q}(\mathcal{Z}) & & \xrightarrow{\tau_{\mathcal{X} \times \mathcal{Y},\mathcal{Z}}} & & \mathcal{Q}\left((\mathcal{X} \times \mathcal{Y}) \times \mathcal{Z}\right) \\
\Big\downarrow {\scriptstyle A_{\mathcal{X},\mathcal{Y},\mathcal{Q}(\mathcal{Z})}} & & & & \Big\downarrow {\scriptstyle \mathcal{Q}(A_{\mathcal{X},\mathcal{Y},\mathcal{Z}})} \\
\mathcal{X} \times (\mathcal{Y} \times \mathcal{Q}(\mathcal{Z})) & \xrightarrow[\mathrm{id}_{\mathcal{X}} \times \tau_{\mathcal{Y},\mathcal{Z}}]{} & \mathcal{X} \times \mathcal{Q}(\mathcal{Y} \times \mathcal{Z}) & \xrightarrow[\tau_{\mathcal{X},\mathcal{Y} \times \mathcal{Z}}]{} & \mathcal{Q}(\mathcal{X} \times (\mathcal{Y} \times \mathcal{Z})).
\end{array}$$

Monad unit δ and strength satisfy the commutative diagram:

$$\begin{array}{ccc}
& \mathcal{X} \times \mathcal{Y} & \\
& {\scriptstyle \mathrm{id}_{\mathcal{X}} \times \delta_{\mathcal{Y}}} \swarrow \quad \searrow {\scriptstyle \delta_{\mathcal{X} \times \mathcal{Y}}} & \\
\mathcal{X} \times \mathcal{Q}(\mathcal{Y}) & \xrightarrow[\tau_{\mathcal{X},\mathcal{Y}}]{} & \mathcal{Q}(\mathcal{X} \times \mathcal{Y}).
\end{array}$$

Monad action $\mathbb{M}$ commutes with strength, expressed via the commutative diagram:

$$\begin{array}{ccccc} \mathcal{X} \times \mathcal{Q}(\mathcal{Q}(\mathcal{Y})) & \xrightarrow{\tau_{\mathcal{X},\mathcal{Q}(\mathcal{Y})}} & \mathcal{Q}(\mathcal{X} \times \mathcal{Q}(\mathcal{Y})) & \xrightarrow{\mathcal{Q}(\tau_{\mathcal{X},\mathcal{Y}})} & \mathcal{Q}(\mathcal{Q}(\mathcal{X} \times \mathcal{Y})) \\ \downarrow{\scriptstyle \mathrm{id}_{\mathcal{X}} \times \mathbb{M}_{\mathcal{Y}}} & & & & \downarrow{\scriptstyle \mathbb{M}_{\mathcal{X}\times\mathcal{Y}}} \\ \mathcal{X} \times \mathcal{Q}(\mathcal{Y}) & & \xrightarrow[\tau_{\mathcal{X},\mathcal{Y}}]{} & & \mathcal{Q}(\mathcal{X} \times \mathcal{Y}). \end{array}$$

To check this let $x \in \mathcal{X}$ and $\pi \in \mathcal{Q}(\mathcal{Q}(\mathcal{Y}))$ and $D \in \mathcal{B}_{\mathcal{X}\times\mathcal{Y}}$. Then we get:

$$\begin{aligned} &\mathbb{M}_{\mathcal{X}\times\mathcal{Y}} \left(\mathcal{Q}(\tau_{\mathcal{X},\mathcal{Y}})(\tau_{\mathcal{X},\mathcal{Q}(\mathcal{Y})}(x,\pi))\right)(D) \\ &= \int \mathrm{ev}(\rho, D)\, \mathcal{Q}(\tau_{\mathcal{X},\mathcal{Y}})(\tau_{\mathcal{X},\mathcal{Q}(\mathcal{Y})}(x,\pi))(d\rho) \\ &= \int \mathrm{ev}(\tau_{\mathcal{X},\mathcal{Y}}(t,\nu), D)\, (\tau_{\mathcal{X},\mathcal{Q}(\mathcal{Y})}(x,\pi))(d(t,\nu)) \\ &= \int \mathrm{ev}(\tau_{\mathcal{X},\mathcal{Y}}(x,\nu), D)\, \pi(d\nu) \\ &= \int \mathrm{ev}(\nu, D_x)\, \pi(d\nu) \\ &= \mathbb{M}_{\mathcal{Y}}(\pi)(D_x) \\ &= \tau_{\mathcal{X},\mathcal{Y}}(x, \mathbb{M}_{\mathcal{Y}}(\pi))(D). \end{aligned}$$

This shows the claim. □

3.3 Strong Probability Monads $\mathcal{K}$, $\mathcal{P}$, $\mathcal{R}$, $\mathcal{S}$

In this subsection we will introduce different versions of the *probability monad of push-forward probability measures* on quasi-measurable spaces: $\mathcal{K}$, $\mathcal{P}$, $\mathcal{R}$, $\mathcal{S}$. $\mathcal{P}$ will resemble the probability monad $\mathcal{P}$ from [HKSY17]. $\mathcal{K}$ will be a bit more general than $\mathcal{P}$, while $\mathcal{S}$ will be a bit more restrictive than $\mathcal{P}$. $\mathcal{R}$ will in some sense be complementary to $\mathcal{P}$. We then study under which conditions these probability monads agree and also when they become strong. The main requirement will be that the sample space Ω satisfies $\Omega \times \Omega \cong \Omega$ in **QMS**.

Definition 3.17 (The space of push-forward probability measures)**.** *Let $(\mathcal{X}, \mathcal{X}^\Omega)$ be a quasi-measurable space and $\mathcal{B}_\mathcal{X} := \mathcal{B}(\mathcal{X}^\Omega)$ the induced σ-algebra. Then we define the quasi-measurable space of push-forward probability measures on $(\mathcal{X}, \mathcal{X}^\Omega)$:*

$$\mathcal{K}(\mathcal{X}) := \mathrm{pf}\left(\mathcal{X}^\Omega \times \mathcal{Q}(\Omega)\right) \subseteq \mathcal{Q}(\mathcal{X}),$$

with the quotient quasi-measurable space structure, where we used the quasi-measurable push-forward map, see Lemma 3.10:

$$\mathrm{pf}: \ \mathcal{X}^\Omega \times \mathcal{Q}(\Omega) \to \mathcal{Q}(\mathcal{X}).$$

More concretely, we have:

$$\begin{aligned}\mathcal{K}(\mathcal{X}) &= \left\{\alpha_*\mu : \mathcal{B}_\mathcal{X} \to [0,1] \,\middle|\, \alpha \in \mathcal{X}^\Omega, \mu \in \mathcal{Q}(\Omega)\right\},\\ \mathcal{K}(\mathcal{X})^\Omega &= \mathrm{pf} \circ \left(\left(\mathcal{X}^\Omega\right)^\Omega \times \mathcal{Q}(\Omega)^\Omega\right)\\ &= \left\{\alpha_*\kappa : \Omega \to \mathcal{K}(\mathcal{X}) \,\middle|\, \alpha \in \left(\mathcal{X}^\Omega\right)^\Omega, \kappa \in \mathcal{Q}(\Omega)^\Omega\right\}.\end{aligned}$$

The $\alpha_*\kappa$ for $\alpha \in \left(\mathcal{X}^\Omega\right)^\Omega$ and $\kappa \in \mathcal{Q}(\Omega)^\Omega$ in $\mathcal{K}(\mathcal{X})^\Omega$ is given by:

$$(\alpha_*\kappa)(\omega)(A) = \alpha(\omega)_*\kappa(\omega)(A) = \kappa(\omega)\left(\alpha(\omega)^{-1}(A)\right) = \kappa(\omega)\left(\{\omega' \in \Omega \,|\, \alpha(\omega)(\omega') \in A\}\right).$$

We then also define the quasi-measurable spaces of probability measures:

$$\begin{aligned}\mathcal{P}(\mathcal{X}) &:= \mathcal{K}(\mathcal{X}),\\ \mathcal{P}(\mathcal{X})^\Omega &:= \mathrm{pf} \circ \left(\left(\mathcal{X}^\Omega\right)^{\mathbf{1}} \times \mathcal{Q}(\Omega)^\Omega\right) \subseteq \mathcal{K}(\mathcal{X})^\Omega,\\ \mathcal{R}(\mathcal{X}) &:= \mathcal{K}(\mathcal{X}),\\ \mathcal{R}(\mathcal{X})^\Omega &:= \mathrm{pf} \circ \left(\left(\mathcal{X}^\Omega\right)^\Omega \times \mathcal{Q}(\Omega)^{\mathbf{1}}\right) \subseteq \mathcal{K}(\mathcal{X})^\Omega,\\ \mathcal{S}(\mathcal{X}) &:= \mathcal{K}(\mathcal{X}),\\ \mathcal{S}(\mathcal{X})^\Omega &:= \mathrm{pf} \circ \left(\left(\mathcal{X}^\Omega\right)^{\mathbf{1}} \times \left(\Omega^\Omega\right)^\Omega \times \mathcal{Q}(\Omega)^{\mathbf{1}}\right)\\ &= \left\{\alpha_*\phi_*\nu : \Omega_1 \to \mathcal{S}(\mathcal{X}) \,\middle|\, \alpha \in \mathcal{X}^{\Omega_3}, \phi \in \left(\Omega_3^{\Omega_2}\right)^{\Omega_1}, \nu \in \mathcal{Q}(\Omega_2)\right\}\\ &\subseteq \mathcal{P}(\mathcal{X})^\Omega \cap \mathcal{R}(\mathcal{X})^\Omega,\end{aligned}$$

with indices for clarity: $\Omega_i := \Omega$.

Lemma 3.18. *Let $(\mathcal{X}, \mathcal{X}^\Omega)$ be a quasi-measurable space. Then the spaces of push-forward probability measures $(\mathcal{K}(\mathcal{X}), \mathcal{K}(\mathcal{X})^\Omega)$ and $(\mathcal{P}(\mathcal{X}), \mathcal{P}(\mathcal{X})^\Omega)$ and $(\mathcal{R}(\mathcal{X}), \mathcal{R}(\mathcal{X})^\Omega)$ and $(\mathcal{S}(\mathcal{X}), \mathcal{S}(\mathcal{X})^\Omega)$ are all quasi-measurable spaces. Furthermore, all the inclusion maps are quasi-measurable:*

$$\mathcal{S}(\mathcal{X}) \subseteq \mathcal{P}(\mathcal{X}) \subseteq \mathcal{K}(\mathcal{X}) \subseteq \mathcal{Q}(\mathcal{X}) \subseteq \mathcal{G}(\mathcal{X}), \quad \mathcal{S}(\mathcal{X}) \subseteq \mathcal{R}(\mathcal{X}) \subseteq \mathcal{K}(\mathcal{X}).$$

Proof. This immediately follows from Lemma 3.10 and Lemma 2.54. ∎

Lemma 3.19. *Let $(\mathcal{X}, \mathcal{X}^\Omega)$ be a quasi-measurable space.*

1. *Assume that there exists an an isomorphism of quasi-measurable spaces:*

$$\theta : \Omega \cong \Omega \times \Omega.$$

Then we have the equalities and inclusions of sets:

$$\mathcal{S}(\mathcal{X})^\Omega = \mathcal{R}(\mathcal{X})^\Omega \subseteq \mathcal{P}(\mathcal{X})^\Omega = \mathcal{K}(\mathcal{X})^\Omega.$$

2. *Assume that the following push-forward map is surjective:*

$$\mathrm{pf}: \left(\Omega^\Omega\right)^\Omega \times \mathcal{Q}(\Omega) \twoheadrightarrow \mathcal{Q}(\Omega)^\Omega, \qquad (\gamma,\nu) \mapsto \gamma_*\nu = (\omega \mapsto \gamma(\omega)_*\nu)\,.$$

Then we have the equalities and inclusions of sets:

$$\mathcal{S}(\mathcal{X})^\Omega = \mathcal{P}(\mathcal{X})^\Omega \subseteq \mathcal{R}(\mathcal{X})^\Omega = \mathcal{K}(\mathcal{X})^\Omega.$$

Proof. 1.) Let $\alpha_*\kappa \in \mathcal{K}(\mathcal{X})^{\Omega_1}$ with $\alpha \in \left(\mathcal{X}^{\Omega_2}\right)^{\Omega_1}$ and $\kappa \in \mathcal{Q}(\Omega_2)^{\Omega_1}$, where we use indices for clarity: $\Omega_i := \Omega$. In the following we will identify all maps with their (un)curried form. Define $\phi_1 \in \left(\Omega_1^{\Omega_1}\right)^{\Omega_1}$ and $\phi_2 \in \left(\Omega_2^{\Omega_2}\right)^{\Omega_1}$ on elements via:

$$\begin{aligned}\phi_1(\omega_1)(\omega_1') &:= \omega_1,\\ \phi_2(\omega_1)(\omega_2) &:= \omega_2.\end{aligned}$$

Let $\nu \in \mathcal{Q}(\Omega)$ be any probability measure. Note that $\phi_2(\omega_1) = \mathrm{id}_{\Omega_2}$ and:

$$\phi_1(\omega_1)_*\nu = \delta_{\omega_1}, \qquad (\phi_1(\omega_1) \times \phi_2(\omega_1))_*(\nu \otimes \kappa)(\omega_1) = \delta_{\omega_1} \otimes \kappa(\omega_1).$$

With this we then get:

$$\begin{aligned}\alpha(\omega_1)_*\kappa(\omega_1) &= \alpha_*\left(\delta_{\omega_1} \otimes \kappa(\omega_1)\right)\\ &= \alpha_*\phi(\omega_1)_*\left(\nu \otimes \kappa(\omega_1)\right)\\ &= \alpha_*\theta_*\theta_*^{-1}\phi(\omega_1)_*\theta_*\theta_*^{-1}\left(\nu \otimes \kappa(\omega_1)\right)\\ &= (\alpha \circ \theta)_*\left(\theta^{-1} \circ \phi(\omega_1) \circ \theta\right)_*\left(\theta_*^{-1}\left(\nu \otimes \kappa(\omega_1)\right)\right),\end{aligned}$$

with $(\alpha \circ \theta) \in \mathcal{X}^{\Omega_3}$, $(\theta^{-1} \circ \phi \circ \theta) \in \left(\Omega_3^{\Omega_3}\right)^{\Omega_1}$ and $\theta_*^{-1}(\nu \otimes \kappa) \in \mathcal{Q}(\Omega_3)^{\Omega_1}$. This shows that: $\alpha_*\kappa \in \mathcal{P}(\mathcal{X})^\Omega$. So we showed the inclusion (and thus equality):

$$\mathcal{K}(\mathcal{X})^\Omega \subseteq \mathcal{P}(\mathcal{X})^\Omega.$$

The same arguments hold for $\kappa \in \mathcal{Q}(\Omega)$. Then $\theta_*^{-1}(\nu \otimes \kappa) \in \mathcal{Q}(\Omega_3)$. This then shows the inclusion (and thus equality):

$$\mathcal{R}(\mathcal{X})^\Omega \subseteq \mathcal{S}(\mathcal{X})^\Omega.$$

2.) Again let $\alpha_*\kappa \in \mathcal{K}(\mathcal{X})^{\Omega_1}$ with $\alpha \in \left(\mathcal{X}^{\Omega_2}\right)^{\Omega_1}$ and $\kappa \in \mathcal{Q}(\Omega_2)^{\Omega_1}$. By assumption there exist $\phi \in \left(\Omega_2^{\Omega_3}\right)^{\Omega_1}$ and $\nu \in \mathcal{Q}(\Omega_3)$ such that for all $\omega_1 \in \Omega_1$:

$$\kappa(\omega_1) = \phi(\omega_1)_*\nu.$$

Define $\alpha \circ \phi \in \left(\mathcal{X}^{\Omega_3}\right)^{\Omega_1}$ on elements via:

$$(\alpha \circ \phi)(\omega_1)(\omega_3) := \alpha(\omega_1)(\phi(\omega_1)(\omega_3)).$$

This gives us:

$$\alpha(\omega_1)_*\kappa(\omega_1) = \alpha(\omega_1)_*\phi(\omega_1)_*\nu = (\alpha \circ \phi)(\omega_1)_*\nu.$$

This shows that $\alpha_*\kappa = (\alpha \circ \phi)_*\nu \in \mathcal{R}(\mathcal{X})^{\Omega_1}$. With this we get the inclusions (and thus equality):

$$\mathcal{K}(\mathcal{X})^\Omega \subseteq \mathcal{R}(\mathcal{X})^\Omega.$$

The same proof for $\alpha \in \mathcal{X}^{\Omega_2}$ shows:

$$\alpha_*\kappa(\omega_1) = \alpha_*\phi(\omega_1)_*\nu,$$

and thus the inclusion (and equality):

$$\mathcal{P}(\mathcal{X})^\Omega \subseteq \mathcal{S}(\mathcal{X})^\Omega.$$

This shows the claims. □

Lemma 3.20. *Let $(\mathcal{X}, \mathcal{X}^\Omega)$ be a quasi-measurable space and $f : \mathcal{X} \to \mathcal{Y}$ a quasi-measurable map. Then we get a commutative diagrams of quasi-measurable maps:*

$$\begin{array}{ccccccccc}
\mathcal{S}(\mathcal{X}) & \xrightarrow{\mathrm{incl}_\mathcal{X}} & \mathcal{P}(\mathcal{X}) & \xrightarrow{\mathrm{incl}_\mathcal{X}} & \mathcal{K}(\mathcal{X}) & \xrightarrow{\mathrm{incl}_\mathcal{X}} & \mathcal{Q}(\mathcal{X}) & \xrightarrow{\mathrm{incl}_\mathcal{X}} & \mathcal{G}(\mathcal{X}) \\
\downarrow{\scriptstyle \mathcal{S}(f)} & & \downarrow{\scriptstyle \mathcal{P}(f)} & & \downarrow{\scriptstyle \mathcal{K}(f)} & & \downarrow{\scriptstyle \mathcal{Q}(f)} & & \downarrow{\scriptstyle \mathcal{G}(f)} \\
\mathcal{S}(\mathcal{Y}) & \xrightarrow[\mathrm{incl}_\mathcal{Y}]{} & \mathcal{P}(\mathcal{Y}) & \xrightarrow[\mathrm{incl}_\mathcal{Y}]{} & \mathcal{K}(\mathcal{Y}) & \xrightarrow[\mathrm{incl}_\mathcal{Y}]{} & \mathcal{Q}(\mathcal{Y}) & \xrightarrow[\mathrm{incl}_\mathcal{Y}]{} & \mathcal{G}(\mathcal{Y}).
\end{array}$$

The same holds true if we replace $\mathcal{P}$ with $\mathcal{R}$.

Proof. That the horizontal lines are inclusions and quasi-measurable is clear. Since the push-forward is at each vertical map given by:

$$(f_*\mu)(B) := \mu(f^{-1}(B)),$$

the commutativity is clear as soon as all vertical maps are shown to be well-defined (with the corresponding codomain) and quasi-measurable.
For $\mathcal{Q}(f)$ both claims were shown in Lemma 3.10.
$\mathcal{G}(f)$ is clearly well-defined since $f_*\mu$ is a probability measure on $\mathcal{B}_\mathcal{Y}$. To see that $\mathcal{G}(f)$ is quasi-measurable let $\kappa \in \mathcal{G}(\mathcal{X})^\Omega$ and $B \in \mathcal{B}_\mathcal{Y}$. Then $f^{-1}(B) \in \mathcal{B}_\mathcal{X}$ and the map:

$$\Omega \to [0,1], \qquad \omega \mapsto (f_*\kappa)(\omega)(B) = \kappa(\omega)(f^{-1}(B)),$$

is $\mathcal{B}_\Omega$-$\mathcal{B}_{[0,1]}$-measurable by assumption on κ. So $f_*\kappa \in \mathcal{G}(\mathcal{Y})^\Omega$.
To show that $\mathcal{K}(f), \mathcal{P}(f), \mathcal{R}(f)$ are well-defined consider $\alpha_*\mu$ with $\alpha \in \mathcal{X}^\Omega$ and $\mu \in \mathcal{Q}(\Omega)$. Since f is quasi-measurable we have $f \circ \alpha \in \mathcal{Y}^\Omega$ and thus:

$$f_*(\alpha_*\mu) = (f \circ \alpha)_*\mu \in \mathcal{K}(\mathcal{Y}) = \mathcal{P}(\mathcal{Y}) = \mathcal{R}(\mathcal{Y}).$$

To show that $\mathcal{K}(f)$ is quasi-measurable let now $\alpha \in \left(\mathcal{X}^{\Omega_2}\right)^{\Omega_1}$ and $\kappa \in \mathcal{Q}(\Omega_2)^{\Omega_1}$. Again, since f is quasi-measurable we get that the composition:

$$f \circ \alpha : \ \Omega_2 \times \Omega_1 \xrightarrow{\alpha} \mathcal{X} \xrightarrow{f} \mathcal{Y},$$

is quasi-measurable and thus $f \circ \alpha \in \left(\mathcal{Y}^{\Omega_2}\right)^{\Omega_1}$. So we get:

$$f_*(\alpha_*\kappa)(\omega_1) = f_*\alpha(\omega_1)_*\kappa(\omega_1) = (f \circ \alpha)(\omega_1)_*\kappa(\omega_1).$$

So it follows that:

$$f_*(\alpha_*\kappa) = (f \circ \alpha)_*\kappa \in \mathcal{K}(\mathcal{Y})^{\Omega_1}.$$

Note that if in the above we have that $\alpha \in \left(\mathcal{X}^{\Omega_2}\right)^{\mathbf{1}}$ then also $f \circ \alpha \in \left(\mathcal{Y}^{\Omega_2}\right)^{\mathbf{1}}$, showing that $\mathcal{P}(f)$ is quasi-measurable.
Similarly, if $\kappa \in \mathcal{Q}(\Omega_2)^{\mathbf{1}}$ then it stays that way for the push-forward, so clearly $\mathcal{R}(f)$ is quasi-measurable.
For $\mathcal{S}(f)$ note that we can use the previous arguments to get:

$$f \circ \mathcal{S}(\mathcal{X})^\Omega = f \circ \left(\mathcal{P}(\mathcal{X})^\Omega \cap \mathcal{R}(\mathcal{X})^\Omega\right) \subseteq \mathcal{P}(\mathcal{Y})^\Omega \cap \mathcal{R}(\mathcal{Y})^\Omega = \mathcal{S}(\mathcal{Y})^\Omega.$$

□

Lemma 3.21. *Let $(\mathcal{X}, \mathcal{X}^\Omega)$ be a quasi-measurable space. Then the map:*

$$\delta: \mathcal{X} \to \mathcal{S}(\mathcal{X}), \qquad x \mapsto \delta_x = (A \mapsto \mathbb{1}_A(x)),$$

is a well-defined quasi-measurable map.
Furthermore, if we have a quasi-measurable map:

$$f: (\mathcal{X}, \mathcal{X}^\Omega) \to (\mathcal{Y}, \mathcal{Y}^\Omega),$$

then we get a commutive diagram of quasi-measurable maps:

$$\begin{array}{ccccccccccc}
\mathcal{X} & \xrightarrow{\delta} & \mathcal{S}(\mathcal{X}) & \xrightarrow{i_\mathcal{X}} & \mathcal{P}(\mathcal{X}) & \xrightarrow{i_\mathcal{X}} & \mathcal{K}(\mathcal{X}) & \xrightarrow{i_\mathcal{X}} & \mathcal{Q}(\mathcal{X}) & \xrightarrow{i_\mathcal{X}} & \mathcal{G}(\mathcal{X}) \\
\downarrow f & & \downarrow \mathcal{S}(f) & & \downarrow \mathcal{P}(f) & & \downarrow \mathcal{K}(f) & & \downarrow \mathcal{Q}(f) & & \downarrow \mathcal{G}(f) \\
\mathcal{Y} & \xrightarrow{\delta} & \mathcal{S}(\mathcal{Y}) & \xrightarrow[i_\mathcal{Y}]{} & \mathcal{P}(\mathcal{Y}) & \xrightarrow[i_\mathcal{Y}]{} & \mathcal{K}(\mathcal{Y}) & \xrightarrow[i_\mathcal{Y}]{} & \mathcal{Q}(\mathcal{Y}) & \xrightarrow[i_\mathcal{Y}]{} & \mathcal{G}(\mathcal{Y}).
\end{array}$$

In other words, we have natural transformations of endo-functors of **QMS***:*

$$\mathrm{id}_{\mathbf{QMS}} \xrightarrow{\delta} \mathcal{S} \xrightarrow{\text{incl}} \mathcal{P} \xrightarrow{\text{incl}} \mathcal{K} \xrightarrow{\text{incl}} \mathcal{Q} \xrightarrow{\text{incl}} \mathcal{G}.$$

All the above statements also hold true if we replace $\mathcal{P}$ by $\mathcal{R}$.

Proof. To show that δ is well-defined we need to show that $\delta_{\mathcal{X},x} \in \mathcal{S}(\mathcal{X}) = \mathcal{K}(\mathcal{X})$ for $x \in \mathcal{X}$. For any $\nu \in \mathcal{Q}(\Omega)$ and the constant map $x^{\mathbf{1}} \in \mathcal{X}^{\mathbf{1}}$ with value $x \in \mathcal{X}$ we get:

$$\delta_{\mathcal{X},x} = x^{\mathbf{1}}_*\nu \in \mathcal{S}(\mathcal{X}) = \mathcal{K}(\mathcal{X}).$$

To show that $\delta_\mathcal{X}$ is quasi-measurable let $\alpha \in \mathcal{X}^\Omega$ and define: $\phi(\omega_1)(\omega_2) := \omega_1$. Then we get:

$$\begin{aligned}
(\alpha_*\phi_*\nu)(\omega_1) &= \alpha_*\phi(\omega_1)_*\nu \\
&= \alpha_*\delta_{\Omega,\omega_1} \\
&= \delta_{\mathcal{X},\alpha(\omega_1)} \\
&= (\delta_\mathcal{X} \circ \alpha)(\omega_1).
\end{aligned}$$

So we get: $\delta_{\mathcal{X}} \circ \alpha = \alpha_* \phi_* \nu \in \mathcal{S}(\mathcal{X})^\Omega$.
To check the commutativity of the diagrams, by Lemma 3.20 we only need to consider the first one with $\mathcal{S}$. Observe that:

$$f_* \delta_{\mathcal{X},x} = f_* x^{\mathbf{1}}_* \nu = f(x)^{\mathbf{1}}_* \nu = \delta_{\mathcal{Y},f(x)} \in \mathcal{S}(\mathcal{Y}) = \mathcal{K}(\mathcal{Y}),$$

where with $x^{\mathbf{1}}$ and $f(x)^{\mathbf{1}}$ we mean the corresponding constant maps with those values. This shows the claims. □

Lemma 3.22. *Let $(\mathcal{X}, \mathcal{X}^\Omega)$ be a quasi-measurable space. Assume that there exists an an isomorphism of quasi-measurable spaces:*

$$\theta: \ \Omega \cong \Omega \times \Omega.$$

Then the map:

$$\mathbb{M}_{\mathcal{X}}: \ \mathcal{K}(\mathcal{K}(\mathcal{X})) \to \mathcal{K}(\mathcal{X}), \qquad \pi \mapsto \mathbb{M}_{\mathcal{X}}(\pi) = \left(A \mapsto \int \mathrm{ev}(\mu, A)\, \pi(d\mu) \right),$$

is a well-defined quasi-measurable map.
Furthermore, if we have a quasi-measurable map:

$$f: \ (\mathcal{X}, \mathcal{X}^\Omega) \to (\mathcal{Y}, \mathcal{Y}^\Omega),$$

then we get a commutive diagram of quasi-measurable maps:

$$\begin{array}{ccc} \mathcal{K}(\mathcal{K}(\mathcal{X})) & \xrightarrow{\mathcal{K}(\mathcal{K}(f))} & \mathcal{K}(\mathcal{K}(\mathcal{Y})) \\ \downarrow{\scriptstyle \mathbb{M}_{\mathcal{X}}} & & \downarrow{\scriptstyle \mathbb{M}_{\mathcal{Y}}} \\ \mathcal{K}(\mathcal{X}) & \xrightarrow[\mathcal{K}(f)]{} & \mathcal{K}(\mathcal{Y}). \end{array}$$

In other words, $\mathbb{M}: \mathcal{K}^2 \to \mathcal{K}$ is a natural transformation of endo-functors of **QMS**.
Finally, all the analogous statements hold as well when $\mathcal{K}$ is replaced by $\mathcal{R}$ or $\mathcal{P}$ or $\mathcal{S}$.

Proof. The proof that $\mathbb{M}_{\mathcal{X}}$ maps to $\mathcal{K}(\mathcal{X})$ follows the same lines as the proof that it is quasi-measurable. So we directly assume that $\pi \in \mathcal{K}(\mathcal{K}(\mathcal{X}))^{\Omega_1}$. Then there are $\rho \in \mathcal{Q}(\Omega_2)^{\Omega_1}$ and $\psi \in \left(\mathcal{K}(\mathcal{X})^{\Omega_2}\right)^{\Omega_1}$ such that:

$$\pi(\omega_1) = (\psi_* \rho)(\omega_1) = \psi(\omega_1)_* \rho(\omega_1) \in \mathcal{K}(\mathcal{K}(\mathcal{X})).$$

Since by assumption we have $\Omega_1 \times \Omega_2 \cong \Omega$ we get that:

$$\psi \in \left(\mathcal{K}(\mathcal{X})^{\Omega_2}\right)^{\Omega_1} \cong \mathcal{K}(\mathcal{X})^\Omega.$$

So there are $\alpha \in \left(\left(\mathcal{X}^{\Omega_3}\right)^{\Omega_2}\right)^{\Omega_1}$ and $\kappa \in \mathcal{Q}(\Omega_3)^{\Omega_1 \times \Omega_2}$ such that:

$$\psi(\omega_1)(\omega_2) = \alpha(\omega_1)(\omega_2)_* \kappa(\omega_1, \omega_2) \in \mathcal{K}(\mathcal{X}).$$

Then we get with $A \in \mathcal{B}_{\mathcal{X}}$ and $\omega_1 \in \Omega_1$:

$$\begin{aligned}
\mathbb{M}_{\mathcal{X}}(\pi)(\omega_1)(A) &= \int \mathrm{ev}(\mu, A)\, \pi(\omega_1)(d\mu) \\
&= \int \mathrm{ev}(\mu, A)\, (\psi_* \rho)(\omega_1)(d\mu) \\
&= \int \mathrm{ev}(\mu, A)\, \psi(\omega_1)_* \rho(\omega_1)(d\mu) \\
&= \int \mathrm{ev}(\psi(\omega_1)(\omega_2), A)\, \rho(\omega_1)(d\omega_2) \\
&= \int \mathrm{ev}(\alpha(\omega_1)(\omega_2)_* \kappa(\omega_1, \omega_2), A)\, \rho(\omega_1)(d\omega_2) \\
&= \int \kappa(\omega_1, \omega_2)(\alpha(\omega_1)(\omega_2)^{-1}(A))\, \rho(\omega_1)(d\omega_2) \\
&= \int \kappa(\omega_1, \omega_2)(\alpha^{-1}(A)_{\omega_1, \omega_2})\, \rho(\omega_1)(d\omega_2) \\
&= (\kappa \otimes \rho)(\omega_1)(\alpha^{-1}(A)_{\omega_1}) \\
&= (\kappa \otimes \rho)(\omega_1)(\alpha(\omega_1)^{-1}(A)) \\
&= \alpha(\omega_1)_* (\kappa \otimes \rho)(\omega_1)(A),
\end{aligned}$$

where now $\kappa \otimes \rho \in \mathcal{Q}(\Omega_3 \times \Omega_2)^{\Omega_1} \cong \mathcal{Q}(\Omega)^{\Omega_1}$ and $\alpha \in \left(\left(\mathcal{X}^{\Omega_3}\right)^{\Omega_2}\right)^{\Omega_1} \cong \left(\mathcal{X}^{\Omega}\right)^{\Omega_1}$. It follows that:

$$\mathbb{M}_{\mathcal{X}}(\pi) = \alpha_* (\kappa \otimes \rho) \in \mathcal{K}(\mathcal{X})^{\Omega_1}.$$

This shows that $\mathbb{M}_{\mathcal{X}}$ is well-defined and quasi-measurable.
To show the commutative diagram we apply the above calculation to:

$$f_{**}\pi = (f_* \psi)_* \rho = (f_* \alpha_* \kappa)_* \rho = ((f \circ \alpha)_* \kappa)_* \rho.$$

So replacing α with $f \circ \alpha$ in the calculation from above we get:

$$\mathbb{M}_{\mathcal{Y}}(f_{**}\pi) = (f \circ \alpha)_* (\kappa \otimes \rho) = f_* \alpha_* (\kappa \otimes \rho) = f_* \mathbb{M}_{\mathcal{X}}(\pi).$$

This shows the claim for $\mathcal{K}$.
For the corresponding statements for $\mathcal{P}$ note that ψ will not be dependent on ω_1 and thus α will not be dependent on ω_2 and ω_1 and κ will not be dependent on ω_2. In this case we do not even need the isomorphism $\Omega \times \Omega \cong \Omega$ and we get:

$$\mathbb{M}_{\mathcal{X}}(\pi) = \alpha_* (\kappa \circ \rho) \in \mathcal{P}(\mathcal{X})^{\Omega_1}.$$

For the $\mathcal{R}$ case note that ρ will not be dependent on ω_1 and κ will not be dependent on ω_2 and ω_3, which shows that $\kappa \otimes \rho$ will not be dependent on ω_1, which in this case shows that:

$$\mathbb{M}_{\mathcal{X}}(\pi) = \alpha_* (\kappa \otimes \rho) \in \mathcal{R}(\mathcal{X})^{\Omega_1}.$$

The statement for $\mathcal{S}$ is the same as for $\mathcal{R}$, see Lemma 3.19. □

Proposition 3.23. *The triple $(\mathcal{P}, \delta, \mathbb{M})$ defines a monad on* **QMS**. *If, furthermore, we have an isomorphism of quasi-measurable spaces:*

$$\theta : \ \Omega \cong \Omega \times \Omega,$$

then also the triples $(\mathcal{K}, \delta, \mathbb{M})$ and $(\mathcal{R}, \delta, \mathbb{M})$ and $(\mathcal{S}, \delta, \mathbb{M})$ each define a monad on **QMS**.

Proof. By Lemma 3.20 we know that $\mathcal{P} : \mathbf{QMS} \to \mathbf{QMS}$ is a functor. By Lemma 3.21 we know that $\delta : \mathrm{id}_{\mathbf{QMS}} \to \mathcal{P}$ is a natural transformation. By Lemma 3.22 we know that $\mathbb{M} : \mathcal{P}^2 \to \mathcal{P}$ is a natural transformation. Note that for $\mathcal{P}$ the isomorphism $\Omega \times \Omega \cong \Omega$ was not needed. So we are left to check the following coherence conditions:

1. $\mathbb{M} \circ \mathcal{P}\mathbb{M} = \mathbb{M} \circ \mathbb{M}\mathcal{P}$ as natural transformations $\mathcal{P}^3 \to \mathcal{P}$,
2. $\mathbb{M} \circ \mathcal{P}\delta = \mathbb{M} \circ \delta\mathcal{P} = \mathrm{id}_{\mathcal{P}}$ as natural transformations $\mathcal{P} \to \mathcal{P}$.

These calculations follow exactly the same as in Proposition 3.14.
All the same arguments also hold for $(\mathcal{K}, \delta, \mathbb{M})$ and $(\mathcal{R}, \delta, \mathbb{M})$ and $(\mathcal{S}, \delta, \mathbb{M})$ under the assumed isomorphism. ☐

Theorem 3.24. *Let $(\mathcal{X}, \mathcal{X}^\Omega)$, $(\mathcal{Y}, \mathcal{Y}^\Omega)$, $(\mathcal{Z}, \mathcal{Z}^\Omega)$ be quasi-measurable spaces. Assume that there exists an an isomorphism of quasi-measurable spaces:*

$$\theta : \ \Omega \cong \Omega \times \Omega.$$

Then the maps:

$$\otimes : \ \mathcal{K}(\mathcal{X})^{\mathcal{Y}\times\mathcal{Z}} \times \mathcal{K}(\mathcal{Y})^{\mathcal{Z}} \to \mathcal{K}(\mathcal{X}\times\mathcal{Y})^{\mathcal{Z}}, \quad (\mu\otimes\nu)(z)(D) := \int \mu(y,z)(D_y)\,\nu(z)(dy),$$

$$\otimes : \ \mathcal{P}(\mathcal{X})^{\mathcal{Y}\times\mathcal{Z}} \times \mathcal{P}(\mathcal{Y})^{\mathcal{Z}} \to \mathcal{P}(\mathcal{X}\times\mathcal{Y})^{\mathcal{Z}}, \quad (\mu\otimes\nu)(z)(D) := \int \mu(y,z)(D_y)\,\nu(z)(dy),$$

$$\otimes : \ \mathcal{R}(\mathcal{X})^{\mathcal{Y}\times\mathcal{Z}} \times \mathcal{R}(\mathcal{Y})^{\mathcal{Z}} \to \mathcal{R}(\mathcal{X}\times\mathcal{Y})^{\mathcal{Z}}, \quad (\mu\otimes\nu)(z)(D) := \int \mu(y,z)(D_y)\,\nu(z)(dy),$$

$$\otimes : \ \mathcal{S}(\mathcal{X})^{\mathcal{Y}\times\mathcal{Z}} \times \mathcal{S}(\mathcal{Y})^{\mathcal{Z}} \to \mathcal{S}(\mathcal{X}\times\mathcal{Y})^{\mathcal{Z}}, \quad (\mu\otimes\nu)(z)(D) := \int \mu(y,z)(D_y)\,\nu(z)(dy),$$

are well-defined quasi-measurable maps.

Proof. To keep better track of the interplay between different versions of Ω in this proof we will index them as $\Omega_k := \Omega$ for $k = 1, 2, \ldots$.
To show that $\otimes$ is well-defined let $\mu \in \mathcal{K}(\mathcal{X})^{\mathcal{Y}\times\mathcal{Z}}$ and $\nu \in \mathcal{K}(\mathcal{Y})^{\mathcal{Z}}$. We then need to show that for every $z \in \mathcal{Z}$:

$$(\mu\otimes\nu)(z) \in \mathcal{K}(\mathcal{X}\times\mathcal{Y}),$$

and that for every $\gamma \in \mathcal{Z}^{\Omega_1}$ we have:

$$(\mu\otimes\nu)\circ\gamma \in \mathcal{K}(\mathcal{X}\times\mathcal{Y})^{\Omega_1}.$$

The latter means that we need to show that for every $\gamma \in \mathcal{Z}^{\Omega_1}$ there exist $\pi \in \mathcal{Q}(\Omega_5)^{\Omega_1}$ and $\psi \in \left((\mathcal{X} \times \mathcal{Y})^{\Omega_5}\right)^{\Omega_1}$ such that:

$$(\mu \otimes \nu) \circ \gamma = \psi_* \pi \in \mathcal{K}(\mathcal{X} \times \mathcal{Y})^{\Omega_1}.$$

Since the first statement can be considered a special case of the second one (with constant $\gamma = z \in \mathcal{Z}^{\mathbf{1}} \subseteq \mathcal{Z}^{\Omega_1}$ and constant $\pi \in \mathcal{Q}(\Omega_2)^{\mathbf{1}} \subseteq \mathcal{Q}(\Omega_2)^{\Omega_1}$) we directly focus on arbitrary $\gamma \in \mathcal{Z}^{\Omega_1}$.
By composing $\nu \in \mathcal{K}(\mathcal{Y})^{\mathcal{Z}}$ with $\gamma \in \mathcal{Z}^{\Omega_1}$ then have that $\nu \circ \gamma \in \mathcal{K}(\mathcal{Y})^{\Omega_1}$. So by definition of $\mathcal{K}(\mathcal{Y})^{\Omega_1}$ there exist $\rho \in \mathcal{Q}(\Omega_2)^{\Omega_1}$ and $\beta \in \left(\mathcal{Y}^{\Omega_2}\right)^{\Omega_1}$ such that for every $\omega_1 \in \Omega_1$:

$$\nu \circ \gamma(\omega_1) = \beta(\omega_1)_* \rho(\omega_1).$$

Then note that $\beta \times \gamma$ is given by:

$$\beta \times \gamma : \ \Omega_2 \times \Omega_1 \to \mathcal{Y} \times \mathcal{Z}, \qquad (\omega_2, \omega_1) \mapsto (\beta(\omega_1)(\omega_2), \gamma(\omega_1)).$$

Together with the isomorphism $\theta : \ \Omega_4 \cong \Omega_2 \times \Omega_1$ we get the quasi-measurable map:

$$(\beta \times \gamma) \circ \theta : \ \Omega_4 \xrightarrow{\theta} \Omega_2 \times \Omega_1 \xrightarrow{\beta \times \gamma} \mathcal{Y} \times \mathcal{Z}, \qquad \omega_4 \mapsto (\beta(\theta_2(\omega_4))(\theta_1(\omega_4)), \gamma(\theta_2(\omega_4))).$$

Similarly to before by composing the latter map with $\mu \in \mathcal{K}(\mathcal{X})^{\mathcal{Y} \times \mathcal{Z}}$ we have that:

$$\mu \circ (\beta \times \gamma) \circ \theta \in \mathcal{K}(\mathcal{X})^{\Omega_4}.$$

So by definition of $\mathcal{K}(\mathcal{X})^{\Omega_4}$ there exist $\varepsilon \in \left(\mathcal{X}^{\Omega_3}\right)^{\Omega_4}$ and $\lambda \in \mathcal{Q}(\Omega_3)^{\Omega_4}$ such that for all $\omega_4 \in \Omega_4$:

$$\mu \circ (\beta \times \gamma) \circ \theta(\omega_4) = \varepsilon(\omega_4)_* \lambda(\omega_4).$$

For every $(\omega_2, \omega_1) \in \Omega_2 \times \Omega_1$ we take $\omega_4 := \theta^{-1}(\omega_2, \omega_1)$ in the above equation and get:

$$\mu \circ (\beta \times \gamma)(\omega_2, \omega_1) = \varepsilon(\theta^{-1}(\omega_2, \omega_1))_* \lambda(\theta^{-1}(\omega_2, \omega_1)).$$

We then define $\kappa \in \mathcal{Q}(\Omega_3)^{\Omega_2 \times \Omega_1}$ via the composition of quasi-measurable maps:

$$\kappa := \lambda \circ \theta^{-1} : \ \Omega_2 \times \Omega_1 \to \Omega_4 \to \mathcal{Q}(\Omega_3),$$

and $\alpha \in \left(\mathcal{X}^{\Omega_3}\right)^{\Omega_2 \times \Omega_1}$ via the composition of quasi-measurable maps:

$$\alpha := \varepsilon \circ \theta^{-1} : \ \Omega_2 \times \Omega_1 \to \Omega_4 \to \mathcal{X}^{\Omega_3}.$$

We then get for all $(\omega_2, \omega_1) \in \Omega_2 \times \Omega_1$:

$$\mu \circ (\beta \times \gamma)(\omega_2, \omega_1) = \alpha(\omega_2, \omega_1)_* \kappa(\omega_2, \omega_1).$$

With this so defined $\kappa \in \mathcal{Q}(\Omega_3)^{\Omega_2 \times \Omega_1}$ and $\rho \in \mathcal{Q}(\Omega_2)^{\Omega_1}$ we get that $\kappa \otimes \rho \in \mathcal{Q}(\Omega_3 \times \Omega_2)^{\Omega_1}$. We then again use the isomorphism $\theta : \ \Omega_5 \cong \Omega_3 \times \Omega_2$ and define:

$$\pi := \theta_*^{-1}(\kappa \otimes \rho) \in \mathcal{Q}(\Omega_5)^{\Omega_1}.$$

Furthermore, we define the quasi-measurable map $(\alpha \times \beta) \in \left((\mathcal{X} \times \mathcal{Y})^{\Omega_3 \times \Omega_2}\right)^{\Omega_1}$ via:

$$(\alpha \times \beta)(\omega_1)(\omega_3, \omega_2) := (\alpha(\omega_2, \omega_1)(\omega_3), \beta(\omega_1)(\omega_2)) \in \mathcal{X} \times \mathcal{Y},$$

and the quasi-measurable map $\psi \in \left((\mathcal{X} \times \mathcal{Y})^{\Omega_5}\right)^{\Omega_1}$ via:

$$\psi(\omega_1)(\omega_5) := (\alpha \times \beta)(\omega_1) \circ \theta(\omega_5) = (\alpha(\theta_2(\omega_5), \omega_1)(\theta_3(\omega_5)), \beta(\omega_1)(\theta_2(\omega_5))) \in \mathcal{X} \times \mathcal{Y}.$$

With all these settings and $D \in \mathcal{B}_{\mathcal{X} \times \mathcal{Y}}$ and $\omega_1 \in \Omega_1$ we get the calculations:

$$\begin{aligned}
(\mu \otimes \nu)(\gamma(\omega_1))(D) &= \int \mu(y, \gamma(\omega_1))(D_y)\, \nu(\gamma(\omega_1))(dy) \\
&= \int \mu(y, \gamma(\omega_1))(D_y)\, (\beta(\omega_1)_* \rho(\omega_1))(dy) \\
&= \int \mu(\beta(\omega_1)(\omega_2), \gamma(\omega_1))(D_{\beta(\omega_1)(\omega_2)})\, \rho(\omega_1)(d\omega_2) \\
&= \int \mu\left((\beta \times \gamma)(\omega_2, \omega_1)\right)(D_{\beta(\omega_1)(\omega_2)})\, \rho(\omega_1)(d\omega_2) \\
&= \int \alpha(\omega_2, \omega_1)_* \kappa(\omega_2, \omega_1)(D_{\beta(\omega_1)(\omega_2)})\, \rho(\omega_1)(d\omega_2) \\
&= \int \kappa(\omega_2, \omega_1)(\alpha(\omega_2, \omega_1)^{-1}(D_{\beta(\omega_1)(\omega_2)}))\, \rho(\omega_1)(d\omega_2) \\
&= \int \kappa(\omega_2, \omega_1)((\alpha \times \beta)(\omega_1)^{-1}(D)_{\omega_2})\, \rho(\omega_1)(d\omega_2) \\
&= (\kappa \otimes \rho)(\omega_1)((\alpha \times \beta)(\omega_1)^{-1}(D)) \\
&= (\alpha \times \beta)(\omega_1)_* (\kappa \otimes \rho)(\omega_1)(D) \\
&= (\alpha \times \beta)(\omega_1)_* \circ \theta_* \circ \theta_*^{-1} (\kappa \otimes \rho)(\omega_1)(D) \\
&= \psi(\omega_1)_* \pi(\omega_1)(D).
\end{aligned}$$

So we get that:

$$(\mu \otimes \nu) \circ \gamma = \psi_* \pi \in \mathcal{K}(\mathcal{X} \times \mathcal{Y})^{\Omega_1}.$$

This shows that $\otimes$ is a well-defined map.
To show that $\otimes$ is quasi-measurable we need to use the exaxt same arguments as above but where we replace $\mathcal{Z}$ with $\Omega \times \mathcal{Z}$ and $\gamma \in \mathcal{Z}^\Omega$ with $\varphi \times \gamma \in \Omega^\Omega \times \mathcal{Z}^\Omega$ everywhere. This then shows the claim for $\mathcal{K}(\mathcal{X} \times \mathcal{Y})^{\mathcal{Z}}$.
To show the claim for $\mathcal{P}(\mathcal{X} \times \mathcal{Y})^{\mathcal{Z}}$ the same proof applies by noticing that if β is not dependent on ω_1 and ε not dependent on ω_4 then α does not depend on (ω_2, ω_1) and thus $\alpha \times \beta$ and ψ will not depend on ω_1. It then follows:

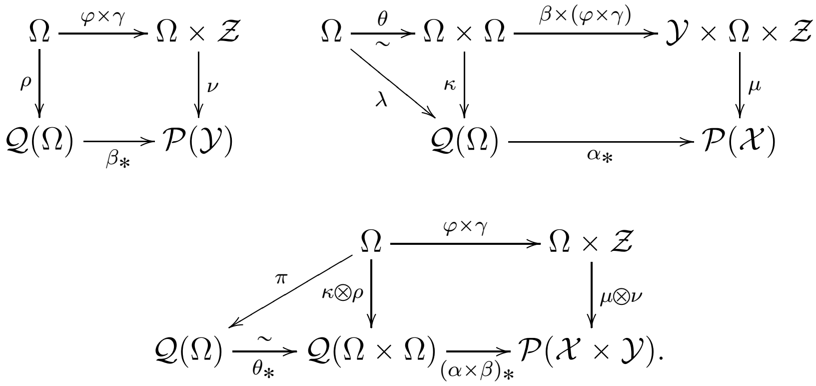
$$(\mu \otimes \nu) \circ \gamma = \psi_* \pi \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})^{\Omega_1}.$$

Similarly the claim for $\mathcal{R}(\mathcal{X} \times \mathcal{Y})^{\mathcal{Z}}$ follows by the same arguments by noticing that if ρ is not dependent on ω_1 and λ not dependent on ω_4 then κ is not dependent on (ω_2, ω_1) and thus $\kappa \otimes \rho$ and π will not depend on ω_1. It then follows that:

$$(\mu \otimes \nu) \circ \gamma = \psi_* \pi \in \mathcal{R}(\mathcal{X} \times \mathcal{Y})^{\Omega_1}.$$

Finally, the claim for $\mathcal{S}$ follows from the the case of $\mathcal{R}$, see Lemma 3.19.

The case for $\mathcal{P}(\mathcal{X}\times\mathcal{Y})^{\mathcal{Z}}$ can be visualized via the commutative diagrams:



∎

Theorem 3.25. *Assume that there exists an an isomorphism of quasi-measurable spaces:*

$$\theta:\ \Omega\cong\Omega\times\Omega.$$

Then the triples $(\mathcal{K},\delta,\mathbb{M})$, $(\mathcal{P},\delta,\mathbb{M})$, $(\mathcal{R},\delta,\mathbb{M})$, $(\mathcal{S},\delta,\mathbb{M})$ *each define a* strong monad *on the (cartesian closed) monoidal category* $(\mathbf{QMS},\times,\mathbf{1})$.

Proof. We define the strength of the monad as follows:

$$\tau_{\mathcal{X},\mathcal{Y}}:\ \mathcal{X}\times\mathcal{P}(\mathcal{Y})\to\mathcal{P}(\mathcal{X}\times\mathcal{Y}),\qquad (x,\mu)\mapsto\delta_x\otimes\mu,$$

which is a well-defined quasi-measurable map by Theorem 3.24. The rest follows the same steps as in Theorem 3.16. For this note that all involved maps are already shown to be well-defined and quasi-measurable. One then only needs to check those coherence equations on elements. This follows exactly the same as in Theorem 3.16.
Also note that the *bind/extension operation*:

$$(\_)_\circ:\ \mathcal{P}(\mathcal{Y})^{\mathcal{X}}\to\mathcal{P}(\mathcal{Y})^{\mathcal{P}(\mathcal{X})},\qquad \kappa\mapsto\kappa_\circ,\qquad \kappa_\circ(\mu)(B):=(\kappa\circ\mu)(B):=\int\kappa(x)(B)\,\mu(dx),$$

is a well-defined quasi-measurable map (as the adjunction of the push-forward of $\otimes$, see Theorem 3.24), which is functorial in $\mathcal{X}$ and $\mathcal{Y}$.
The same arguments hold for the other monads. ∎

Lemma 3.26. *Let* $(\mathcal{X},\mathcal{Y}^{\Omega})$, $(\mathcal{X},\mathcal{Y}^{\Omega})$ *be quasi-measurable spaces. Then the push-forward map:*

$$\mathrm{pf}:\ \mathcal{Y}^{\mathcal{X}}\times\mathcal{K}(\mathcal{X})\to\mathcal{K}(\mathcal{Y}),\qquad (f,\mu)\mapsto f_*\mu,$$

is a well-defined and quasi-measurable map. The same holds true for $\mathcal{R}$ *in place of* $\mathcal{K}$.

Proof. We already know from Lemma 3.20 that the map is well-defined. Consider the following commutative diagram of quasi-measurable maps, see 3.10:

$$\begin{CD} \mathcal{Y}^{\mathcal{X}} \times \mathcal{X}^{\Omega} \times \mathcal{Q}(\Omega) @>{(\_\circ\_)\times \mathrm{id}}>> \mathcal{Y}^{\Omega} \times \mathcal{Q}(\Omega) \\ @V{\mathrm{id}\times \mathrm{pf}}VV @VV{\mathrm{pf}}V \\ \mathcal{Y}^{\mathcal{X}} \times \mathcal{K}(\mathcal{X}) @>>{\mathrm{pf}}> \mathcal{Q}(\mathcal{Y}). \end{CD}$$

This then induces the following commutative diagram of quasi-measurable maps:

$$\begin{CD} \left(\mathcal{Y}^{\mathcal{X}}\right)^{\Omega_1} \times \left(\mathcal{X}^{\Omega_2}\right)^{\Omega_1} \times \mathcal{Q}(\Omega_2)^{\Omega_1} @>{(\_\circ\_)\times \mathrm{id}}>> \left(\mathcal{Y}^{\Omega_2}\right)^{\Omega_1} \times \mathcal{Q}(\Omega_2)^{\Omega_1} \\ @V{\mathrm{id}\times \mathrm{pf}}VV @VV{\mathrm{pf}}V \\ \left(\mathcal{Y}^{\mathcal{X}}\right)^{\Omega_1} \times \mathcal{K}(\mathcal{X})^{\Omega_1} @>>{\mathrm{pf}}> \mathcal{Q}(\mathcal{Y})^{\Omega_1}. \end{CD}$$

The vertical left map is surjective by definition of $\mathcal{K}(\mathcal{X})^{\Omega_1}$. This shows that:

$$\mathrm{pf}\left(\left(\mathcal{Y}^{\mathcal{X}}\right)^{\Omega_1} \times \mathcal{K}(\mathcal{X})^{\Omega_1}\right) \subseteq \mathrm{pf}\left(\left(\mathcal{Y}^{\Omega_2}\right)^{\Omega_1} \times \mathcal{Q}(\Omega_2)^{\Omega_1}\right) =: \mathcal{K}(\mathcal{Y})^{\Omega_1}.$$

This shows that the bottom push-forward map lands in $\mathcal{K}(\mathcal{Y})^{\Omega_1}$, showing our claim. The same arguments hold for $\mathcal{R}$ instead of $\mathcal{K}$ via the diagram:

$$\begin{CD} \left(\mathcal{Y}^{\mathcal{X}}\right)^{\Omega_1} \times \left(\mathcal{X}^{\Omega_2}\right)^{\Omega_1} \times \mathcal{Q}(\Omega_2)^{\mathbf{1}} @>{(\_\circ\_)\times \mathrm{id}}>> \left(\mathcal{Y}^{\Omega_2}\right)^{\Omega_1} \times \mathcal{Q}(\Omega_2)^{\mathbf{1}} \\ @V{\mathrm{id}\times \mathrm{pf}}VV @VV{\mathrm{pf}}V \\ \left(\mathcal{Y}^{\mathcal{X}}\right)^{\Omega_1} \times \mathcal{R}(\mathcal{X})^{\Omega_1} @>>{\mathrm{pf}}> \mathcal{Q}(\mathcal{Y})^{\Omega_1}. \end{CD}$$

This shows the claims. □

Remark 3.27. *Note that the proof of Lemma 3.26 would not work for $\mathcal{P}$ (or $\mathcal{S}$) instead of $\mathcal{K}$ because in the following diagram the top composition map would not be well-defined:*

$$\begin{CD} \left(\mathcal{Y}^{\mathcal{X}}\right)^{\Omega_1} \times \left(\mathcal{X}^{\Omega_2}\right)^{\mathbf{1}} \times \mathcal{Q}(\Omega_2)^{\Omega_1} @>{(\_\circ\_)\times \mathrm{id}}>> \left(\mathcal{Y}^{\Omega_2}\right)^{\mathbf{1}} \times \mathcal{Q}(\Omega_2)^{\Omega_1} \\ @V{\mathrm{id}\times \mathrm{pf}}VV @VV{\mathrm{pf}}V \\ \left(\mathcal{Y}^{\mathcal{X}}\right)^{\Omega_1} \times \mathcal{P}(\mathcal{X})^{\Omega_1} @>>{\mathrm{pf}}> \mathcal{Q}(\mathcal{Y})^{\Omega_1}, \end{CD}$$

since for $f \in \left(\mathcal{Y}^{\mathcal{X}}\right)^{\Omega_1}$ and $\alpha \in \left(\mathcal{X}^{\Omega_2}\right)^{\mathbf{1}}$ one would have that:

$$(f \circ \alpha)(\omega_1)(\omega_2) = f(\omega_1)(\alpha(\omega_2))$$

depended on $\omega_1 \in \Omega_1$, which one needs to avoid for the image to land in:

$$\mathrm{pf}\left(\left(\mathcal{Y}^{\Omega_2}\right)^{\mathbf{1}} \times \mathcal{Q}(\Omega_2)^{\Omega_1}\right) =: \mathcal{P}(\mathcal{Y})^{\Omega_1}.$$

4 The Category of Patchable Quasi-Measurable Spaces

In this section we will study the 3rd property in the definition of quasi-Borel spaces from Def. 7 in [HKSY17] transferred to quasi-measuable spaces. We will call quasi-measurable spaces that satsify that property *patchable*. Note that until now and througout the whole Sections 2 and 3 the requirement of patchability was never needed. The main reason one would anyways impose patchability onto quasi-measurable spaces is stated in Lemma 4.12, namely to preserve countable coproducts when going from measurable spaces to quasi-measurable spaces via the functor $\mathcal{F}$. We will see in Theorem 4.10 that the category of patchable quasi-measurable spaces **PQMS** is a reflexive subcategory of the category of quasi-measurable spaces **QMS**. Furthermore, it will turn out that **PQMS** is an exponential ideal of **QMS** and thus cartesian closed on its own.

Remark 4.1. *For the purpose of this section we will in the following assume that the sample space* $(\Omega, \Omega^\Omega, \mathcal{B}_\Omega)$ *satisfies:*

$$\Omega^\Omega = \mathcal{F}(\mathcal{B}_\Omega) = \mathbf{Meas}\left((\Omega, \mathcal{B}_\Omega), (\Omega, \mathcal{B}_\Omega)\right).$$

4.1 Patchable Quasi-Measurable Spaces

Here we shortly define the notion of *patchable* quasi-measurable spaces. It resembles the 3rd property of quasi-Borel spaces from Def. 7 in [HKSY17].

Definition 4.2 (Patchable quasi-measurable spaces)**.** *We call a quasi-measurable space* $(\mathcal{X}, \mathcal{X}^\Omega)$ patchable *if for every countable disjoint decomposition:* $\Omega = \dot{\bigcup}_{i \in I} C_i$ *with* $C_i \in \mathcal{B}_\Omega$*,* $i \in I \subseteq \mathbb{N}$*, and for* $\alpha_i \in \mathcal{X}^\Omega$*,* $i \in I$*, we have that:*

$$\alpha := \dot{\bigcup_{i \in I}} \alpha_i|_{C_i} \in \mathcal{X}^\Omega,$$

where, more precisely, α *is defined as:*

$$\alpha(\omega) := \alpha_i(\omega) \quad \text{for} \quad \omega \in C_i.$$

Example 4.3.

1. *If* $\mathcal{X}$ *is a set and* $\mathcal{X}^\Omega := [\Omega \to \mathcal{X}]$ *then* $(\mathcal{X}, \mathcal{X}^\Omega)$ *is a patchable quasi-measurable space.*

2. *If* $(\mathcal{X}, \mathcal{X}^\Omega)$ *is a quasi-measurable space with* $\mathcal{X}^\Omega = \mathcal{F}(\mathcal{B}_\mathcal{X})$ *for any* σ*-algebra* $\mathcal{B}_\mathcal{X}$ *then* $(\mathcal{X}, \mathcal{X}^\Omega)$ *is patchable.*

   Indeed, if $\alpha := \dot{\bigcup}_{i \in I} \alpha_i|_{C_i} \in \mathcal{X}^\Omega$ *with* $\alpha_i \in \mathcal{X}^\Omega$ *and* $D \in \mathcal{B}_\mathcal{X}$ *and* $C_i \in \mathcal{B}_\Omega$*,* $i \in I \subseteq \mathbb{N}$*, then:*

   $$\alpha^{-1}(D) = \bigcup_{i \in I} \left(C_i \cap \alpha_i^{-1}(D)\right) \in \mathcal{B}_\Omega.$$

   So α *is measurable and thus* $\alpha \in \mathcal{F}(\mathcal{B}_\mathcal{X}) = \mathcal{X}^\Omega$*.*

4.2 Heredity of Patchability

In this subsection we study under which conditions the property of being patchable is inherited by categorical constructions, like products or quotients, etc.

Lemma 4.4. *Let $(\mathcal{Y}, \mathcal{Y}^\Omega)$ be a patchable quasi-measurable space.*

1. *Let $g: \mathcal{X} \to \mathcal{Y}$ be any map. Then $(\mathcal{X}, g^*\mathcal{Y}^\Omega)$ is also patchable quasi-measurable space.*
2. *Let $h: \mathcal{Y} \twoheadrightarrow \mathcal{Z}$ be any surjective map. Then $(\mathcal{Z}, h \circ \mathcal{Y}^\Omega)$ is also a patchable quasi-measurable space.*

Proof. 1.) We clearly have $\mathcal{X}^{\mathbf{1}} \subseteq g^*\mathcal{Y}^\Omega$. If $\varphi \in \Omega^\Omega$ then:

$$\alpha \in g^*\mathcal{Y}^\Omega \quad \Longrightarrow \quad g \circ \alpha \in \mathcal{Y}^\Omega \quad \Longrightarrow \quad g \circ \alpha \circ \varphi \in \mathcal{Y}^\Omega \quad \Longrightarrow \quad \alpha \circ \varphi \in g^*\mathcal{Y}^\Omega.$$

Let $\alpha_i \in g^*\mathcal{Y}^\Omega$ then $g \circ \alpha_i \in \mathcal{Y}^\Omega$, which implies for every countable measurable disjoint union decomposition $\Omega = \dot{\bigcup}_{i \in I} C_i$ that:

$$g \circ \dot{\bigcup_{i \in I}} \alpha_i|_{C_i} = \dot{\bigcup_{i \in I}} (g \circ \alpha_i)|_{C_i} \in \mathcal{Y}^\Omega.$$

Thus: $\dot{\bigcup}_{i \in I} \alpha_i|_{C_i} \in g^*\mathcal{Y}^\Omega$. This shows the first claim.
2.) Since h is surjective we have $\mathcal{Z}^{\mathbf{1}} \subseteq h \circ \mathcal{Y}^\Omega$. We also have:

$$(h \circ \mathcal{Y}^\Omega) \circ \Omega^\Omega = h \circ (\mathcal{Y}^\Omega \circ \Omega^\Omega) \subseteq h \circ \mathcal{Y}^\Omega.$$

Similarly to the first point we have that:

$$\dot{\bigcup_{i \in I}} (h \circ \beta_i)|_{C_i} = h \circ \dot{\bigcup_{i \in I}} \beta_i|_{C_i} \in h \circ \mathcal{Y}^\Omega,$$

because if $\beta_i \in \mathcal{Y}^\Omega$, $i \in I$, then also $\dot{\bigcup}_{i \in I} \beta_i|_{C_i} \in \mathcal{Y}^\Omega$. This shows the second claim. ☐

Lemma 4.5 (Products of patchable quasi-measurable spaces are patchable)**.** *If $(\mathcal{X}_i, \mathcal{X}_i^\Omega)$ are patchable quasi-measuruable spaces for $i \in I$ and any index set I then also their product of quasi-measurable spaces is patchable:*

$$\prod_{i \in I} (\mathcal{X}_i, \mathcal{X}_i^\Omega).$$

Proof. If we consider a measurable disjoint decomposition $\Omega = \dot{\bigcup}_{j \in J} C_j$ with countable $J \subseteq \mathbb{N}$ and:

$$\alpha^j = (\alpha_i^j)_{i \in I} \in \prod_{i \in I} \mathcal{X}_i^\Omega,$$

for $j \in J$, then:

$$\alpha^j|_{C_j} = (\alpha_i^j|_{C_j})_{i \in I},$$

and thus:

$$\dot{\bigcup_{j\in J}} \alpha^j|_{C_j} = \left(\dot{\bigcup_{j\in J}} \alpha_i^j|_{C_j}\right)_{i\in I}.$$

Since by assumption $\dot{\bigcup}_{j\in J} \alpha_i^j|_{C_j} \in \mathcal{X}_i^\Omega$, for every $i \in I$ we get:

$$\dot{\bigcup_{j\in J}} \alpha^j|_{C_j} = \left(\dot{\bigcup_{j\in J}} \alpha_i^j|_{C_j}\right)_{i\in I} \in \prod_{i\in I} \mathcal{X}_i^\Omega.$$

□

Remark 4.6. *1. Also (small) limits and equalizers of patchable quasi-measurable spaces (taken in* **QMS***) are patchable.*

2. Note that colimits like coproducts of patchable quasi-measurable spaces might not be patchable.

Lemma 4.7 (Exponential ideal property)**.** *Let $(\mathcal{X}, \mathcal{X}^\Omega)$ and $(\mathcal{Y}, \mathcal{Y}^\Omega)$ be two quasi-measurable spaces, where $(\mathcal{Y}, \mathcal{Y}^\Omega)$ is patchable. Then also $\left(\mathcal{Y}^\mathcal{X}, \left(\mathcal{Y}^\mathcal{X}\right)^\Omega\right)$ is patchable.*

Proof. Consider a measurable disjoint union decomposition $\Omega = \dot{\bigcup}_{j\in J} C_j$ with countable $J \subseteq \mathbb{N}$ and $\beta_j \in \left(\mathcal{Y}^\mathcal{X}\right)^\Omega$ for $j \in J$. We then consider:

$$\beta := \dot{\bigcup_{j\in J}} \beta_j|_{C_j} \in [\Omega \to \mathcal{Y}^\mathcal{X}].$$

Let $\varphi \in \Omega^\Omega$ and $\alpha \in \mathcal{Y}^\mathcal{X}$. We then have for $\omega \in \varphi^{-1}(C_j)$:

$$\beta(\varphi(\omega))(\alpha(\omega)) = \beta_j(\varphi(\omega))(\alpha(\omega)).$$

Note that we get the measurable disjoint union decomposition $\Omega = \dot{\bigcup}_{j\in J} \varphi^{-1}(C_j)$. Furthermore, we know that $\beta_j(\varphi)(\alpha) \in \mathcal{Y}^\Omega$ for all $j \in J$ by assumption. Because $\mathcal{Y}$ is patchable we then get that:

$$\beta(\varphi)(\alpha) = \dot{\bigcup_{j\in J}} \beta_j(\varphi)(\alpha)|_{\varphi^{-1}(C_j)} \in \mathcal{Y}^\Omega.$$

Since this holds for all φ and α the above shows that:

$$\dot{\bigcup_{j\in J}} \beta_j|_{C_j} = \beta \in \left(\mathcal{Y}^\mathcal{X}\right)^\Omega.$$

□

Example 4.8. *Let $(\mathcal{X}, \mathcal{X}^\Omega)$ be a (not necessarily patchable) quasi-measurable space.*

1. *The space of indicator functions* $(\mathbf{2}^{\mathcal{X}}, (\mathbf{2}^{\mathcal{X}})^{\Omega})$ *is always a patchable quasi-measurable space.*

2. *The corresponding* σ*-algebra* $(\mathcal{B}_{\mathcal{X}}, (\mathcal{B}_{\mathcal{X}})^{\Omega})$ *is always a patchable quasi-measurable space.*

Proof. $(\mathbf{2}, \mathbf{2}^{\Omega}) = (\mathbf{2}, \mathcal{F}(\mathcal{B}_{\mathbf{2}}))$ is patchable by Example 4.3 and thus $(\mathbf{2}^{\mathcal{X}}, (\mathbf{2}^{\mathcal{X}})^{\Omega})$ as well by Lemma 4.7.
Since $(\mathcal{B}_{\mathcal{X}}, (\mathcal{B}_{\mathcal{X}})^{\Omega}) \cong (\mathbf{2}^{\mathcal{X}}, (\mathbf{2}^{\mathcal{X}})^{\Omega})$ by Lemma 2.48 also $(\mathcal{B}_{\mathcal{X}}, (\mathcal{B}_{\mathcal{X}})^{\Omega})$ is patchable. □

4.3 Reflector to Patchable Quasi-Measurable Spaces

In this subsection we construct the *reflector* $\mathcal{L}$ from the category of quasi-measurable spaces **QMS** to the category of *patchable* quasi-measurable spaces **PQMS**.

Lemma 4.9 (Reflector to patchable quasi-measurable spaces)**.** *Let* $\mathcal{X}$ *be a set and* $\mathcal{H} \subseteq [\Omega \to \mathcal{X}]$ *any subset of functions,* $\mathcal{X}^{\mathbf{1}} := [\mathbf{1} \to \mathcal{X}] \circ [\Omega \to \mathbf{1}]$ *the set of constant maps to* $\mathcal{X}$*. Then define:*

$$\mathcal{L}(\mathcal{H}) := \bigcap_{\substack{\mathcal{G} \subseteq [\Omega \to \mathcal{X}] \\ \mathcal{X}^{\mathbf{1}} \subseteq \mathcal{G} \\ \mathcal{G} \circ \Omega^{\Omega} \subseteq \mathcal{G} \\ \mathcal{G}\, \text{Def. } 4.2}} \mathcal{G} \qquad \subseteq [\Omega \to \mathcal{X}],$$

the intersection of all set of functions $\mathcal{G} \subseteq [\Omega \to \mathcal{X}]$ *that:*

1. *contains* $\mathcal{H}$*,*

2. *contains the constant maps* $\mathcal{X}^{\mathbf{1}}$*,*

3. *is closed under right composition with elements* $\varphi \in \Omega^{\Omega}$*,*

4. *contains every* $\alpha = \dot{\bigcup}_{i \in I} \alpha_i|_{C_i}$ *if every* $\alpha_i \in \mathcal{G}$ *and* $\Omega = \dot{\bigcup}_{i \in I} C_i$ *is a countable measurable disjoint union decomposition,* $i \in I \subseteq \mathbb{N}$*.*

Then $\mathcal{L}(\mathcal{H})$ *satisfies all these points as well and* $(\mathcal{X}, \mathcal{L}(\mathcal{H}))$ *is a well-define* patchable *quasi-measurable space.*

Proof. First, the intersection is non-empty as $\mathcal{G} = [\Omega \to \mathcal{X}]$ satisfies all points.
We clearly have $\mathcal{X}^{\mathbf{1}} \cup \mathcal{H} \subseteq \mathcal{L}(\mathcal{H})$ if $\mathcal{X}^{\mathbf{1}} \cup \mathcal{H} \subseteq \mathcal{G}$ for all $\mathcal{G}$ of consideration.
If $\alpha \in \mathcal{L}(\mathcal{H})$ and $\varphi \in \Omega^{\Omega}$ then $\alpha \circ \varphi \in \mathcal{G}$ for all those $\mathcal{G}$ and thus $\alpha \circ \varphi \in \mathcal{L}(\mathcal{H})$.
If $\alpha_i \in \mathcal{L}(\mathcal{H})$ for $i \in I \subseteq \mathbb{N}$ then $\alpha_i \in \mathcal{G}$ for $i \in I$. So we get $\alpha := \dot{\bigcup}_{i \in I} \alpha_i|_{C_i} \in \mathcal{G}$ for all $\mathcal{G}$ of consideration, which implies $\alpha = \dot{\bigcup}_{i \in I} \alpha_i|_{C_i} \in \mathcal{L}(\mathcal{H})$. □

Theorem 4.10. *The full subcategory* **PQMS** *of all patchable quasi-measurable spaces inside the cartesian closed category* **QMS** *of all quasi-measurable spaces is a reflexive subcategory, an exponential ideal and cartesian closed in itself. In particular, the reflector* $\mathcal{L}$: **QMS** $\to$ **PQMS** *preserves finite products and arbitrary (small) colimits and coproducts.*

Furthermore, **PQMS** *has all (small) limits and products: the* **QMS***-products/limits of diagrams in* **PQMS** *are already in* **PQMS***.* **PQMS** *has all (small) colimits and coproducts, which are given by applying the reflector* $\mathcal{L}$ *to the corresponding* **QMS***-colimits/coproducts:*

$$\coprod_{i\in I}^{\mathbf{PQMS}} (\mathcal{X}_i, \mathcal{X}_i^\Omega) = \mathcal{L}\left(\coprod_{i\in I}^{\mathbf{QMS}} (\mathcal{X}_i, \mathcal{X}_i^\Omega)\right) = \left(\coprod_{i\in I} \mathcal{X}_i, \mathcal{L}\left(\coprod_{i\in I} \mathcal{X}_i^\Omega\right)\right)$$

Proof. The reflector is given via Lemma 4.9:

$$\mathcal{L}:\ \mathbf{QMS} \to \mathbf{PQMS}, \qquad (\mathcal{X}, \mathcal{X}^\Omega) \mapsto (\mathcal{X}, \mathcal{L}(\mathcal{X}^\Omega)), \qquad \mathcal{L}(g) = g.$$

We have to show that $\mathcal{L}$ is left-adjoint to the forgetful functor $\mathbf{PQMS} \hookrightarrow \mathbf{QMS}$, i.e. that for every quasi-measurable $(\mathcal{X}, \mathcal{X}^\Omega)$ and every patchable quasi-measurable space $(\mathcal{Y}, \mathcal{Y}^\Omega)$ we have the natural identification:

$$\mathbf{QMS}\left((\mathcal{X}, \mathcal{L}(\mathcal{X}^\Omega)), (\mathcal{Y}, \mathcal{Y}^\Omega)\right) = \mathbf{QMS}\left((\mathcal{X}, \mathcal{X}^\Omega), (\mathcal{Y}, \mathcal{Y}^\Omega)\right).$$

Indeed, if $f \in \mathbf{QMS}\left((\mathcal{X}, \mathcal{L}(\mathcal{X}^\Omega)), (\mathcal{Y}, \mathcal{Y}^\Omega)\right)$ then:

$$f \circ \mathcal{X}^\Omega \subseteq f \circ \mathcal{L}(\mathcal{X}^\Omega) \subseteq \mathcal{Y}^\Omega,$$

which implies: $f \in \mathbf{QMS}\left((\mathcal{X}, \mathcal{X}^\Omega), (\mathcal{Y}, \mathcal{Y}^\Omega)\right)$.
For the reverse inclusion let $g \in \mathbf{QMS}\left((\mathcal{X}, \mathcal{X}^\Omega), (\mathcal{Y}, \mathcal{Y}^\Omega)\right)$. Then we get:

$$g \circ \mathcal{X}^\Omega \subseteq \mathcal{Y}^\Omega,$$

which is equivalent to:

$$\mathcal{X}^\Omega \subseteq g^*\mathcal{Y}^\Omega := \left\{\alpha:\ \Omega \to \mathcal{X} \,\middle|\, g \circ \alpha \in \mathcal{Y}^\Omega\right\}.$$

Since $g^*\mathcal{Y}^\Omega$ satisfies all 4 points from Lemma 4.9 by Lemma 4.4 we get:

$$\mathcal{L}(\mathcal{X}^\Omega) \subseteq g^*\mathcal{Y}^\Omega,$$

which is equivalent to:

$$g \circ \mathcal{L}(\mathcal{X}^\Omega) \subseteq \mathcal{Y}^\Omega.$$

This implies: $g \in \mathbf{QMS}\left((\mathcal{X}, \mathcal{L}(\mathcal{X}^\Omega)), (\mathcal{Y}, \mathcal{Y}^\Omega)\right)$.
To show that **PQMS** is an exponential ideal of **QMS** we need to show that for every $\mathcal{X} \in \mathbf{QMS}$ and $\mathcal{Y} \in \mathbf{PQMS}$ also $\mathcal{Y}^\mathcal{X} \in \mathbf{PQMS}$. This was already shown in Lemma 4.7.
That $\mathcal{L}$ preserves finite products follows from the last statement and [Joh02] Prp. A4.3.1.

$$\mathcal{L}\left(\prod_{n=1}^{N} (\mathcal{X}_n, \mathcal{X}_n^\Omega)\right) = \prod_{n=1}^{N} (\mathcal{X}_n, \mathcal{L}(\mathcal{X}_n^\Omega)).$$

Since $\mathcal{L}$ is a left-adjoint it automatically preserves all colimits like coproducts.
Since **QMS** has all small colimits by Theorem 2.45 also **PQMS** has all small colimits by applying $\mathcal{L}$ afterwards.
Since **PQMS** inherits all products, see Lemma 4.5, and equalizers, see Lemma 2.37 and check that equalizers of patchable quasi-measurable spaces are patchable, it then also inherits all limits from **QMS**. ◻

Remark 4.11. *1. For a quasi-measurable space $(\mathcal{X}, \mathcal{X}^\Omega)$ we always have:*

$$\mathcal{X}^\Omega \subseteq \mathcal{L}(\mathcal{X}^\Omega) \subseteq \mathcal{F}\mathcal{B}(\mathcal{X}^\Omega) \subseteq [\Omega \to \mathcal{X}].$$

2. **SQMS** $\subseteq$ **PQMS** $\subseteq$ **QMS** *are reflexive full subcategories.*

3. *For quasi-universal spaces we even have the full subcategories:*

$$\mathbf{UQUS} \subseteq \mathbf{SQUS} \subseteq \mathbf{PQUS} \subseteq \mathbf{QUS},$$

where **UQUS** *is the full subcategory of all universal quasi-universal spaces, see Definition 5.22 and Corollary 5.27 later on, which might not be reflexive inside* **QUS**.

Lemma 4.12 (Countable coproducts)**.** *Let I be a countable set and $(\mathcal{X}_i, \mathcal{B}_i)$ measurable spaces for $i \in I$. Then we have:*

$$\mathcal{L}\left(\coprod_{i\in I}(\mathcal{X}_i, \mathcal{F}(\mathcal{B}_i))\right) = \mathcal{F}\left(\coprod_{i\in I}(\mathcal{X}_i, \mathcal{B}_i)\right),$$

where on the lhs the reflector $\mathcal{L}$ is applied to the **QMS**-*coproduct, which together is the* **PQMS**-*coproduct, of the individual spaces after applying $\mathcal{F}$, and on the rhs we have $\mathcal{F}$ applied to the coproduct in* **Meas**.
In other words, $\mathcal{F}$, when seen as a functor from **Meas** *to* **PQMS**, *preserves countable coproducts.*

Proof. Remember that the coproduct σ-algebra is:

$$\mathcal{B}_{\coprod_{i\in I}(\mathcal{X}_i,\mathcal{B}_i)} := \left\{ D \subseteq \coprod_{i\in I} \mathcal{X}_i \;\middle|\; \forall i \in I.\, \mathrm{incl}_i^{-1}(D) \in \mathcal{B}_i \right\}.$$

Since the inclusion are measurable:

$$\mathrm{incl}_k : (\mathcal{X}_k, \mathcal{B}_k) \to \coprod_{i\in I}(\mathcal{X}_i, \mathcal{B}_i),$$

we get by applying $\mathcal{F}$ the quasi-measurable maps:

$$\mathrm{incl}_k : (\mathcal{X}_k, \mathcal{F}(\mathcal{B}_k)) \to \mathcal{F}\left(\coprod_{i\in I}(\mathcal{X}_i, \mathcal{B}_i)\right).$$

By the universal property of the coproduct we then have a quasi-measurable (identity) map:

$$\mathrm{id} : \coprod_{i\in I}(\mathcal{X}_i, \mathcal{F}(\mathcal{B}_i)) \to \mathcal{F}\left(\coprod_{i\in I}(\mathcal{X}_i, \mathcal{B}_i)\right).$$

Since the rhs is a patchable quasi-measurable space we then also get the well-defined quasi-measurable map:

$$\mathrm{id}: \mathcal{L}\left(\coprod_{i\in I}(\mathcal{X}_i, \mathcal{F}(\mathcal{B}_i))\right) \to \mathcal{F}\left(\coprod_{i\in I}(\mathcal{X}_i, \mathcal{B}_i)\right).$$

This shows the inclusion:

$$\mathcal{L}\left(\coprod_{i\in I} \mathcal{F}(\mathcal{B}_i)\right) \subseteq \mathcal{F}\left(\mathcal{B}_{\coprod_{i\in I}(\mathcal{X}_i,\mathcal{B}_i)}\right).$$

To show the reverse inclusion consider:

$$\alpha \in \mathcal{F}\left(\mathcal{B}_{\coprod_{i\in I}(\mathcal{X}_i,\mathcal{B}_i)}\right) = \mathbf{Meas}\left((\Omega, \mathcal{B}_\Omega), \left(\coprod_{i\in I} \mathcal{X}_i, \mathcal{B}_{\coprod_{i\in I}\mathcal{X}_i}\right)\right).$$

Since $\mathcal{X}_k \in \mathcal{B}_{\coprod_{i\in I}(\mathcal{X}_i,\mathcal{B}_i)}$ we get that:

$$C_k := \alpha^{-1}(\mathcal{X}_k) \in \mathcal{B}_\Omega.$$

We can then define:

$$\alpha_k : \Omega \to \mathcal{X}_k, \qquad \alpha_k|_{C_k} := \alpha|_{C_k}, \qquad \alpha_k|_{\Omega\setminus C_k} := x_k \in \mathcal{X}_k.$$

Since α is measurable and $C_k \in \mathcal{B}_\Omega$ then also α_k is measurable. So $\alpha_k \in \mathcal{F}(\mathcal{B}_k)$ for $k \in I$. By definition of the coproduct in **QMS** we get for every $k \in I$ that:

$$\mathrm{incl}_k \circ \alpha_k \in \coprod_{i\in I} \mathcal{F}(\mathcal{B}_i).$$

Furthermore, we get by definition of $\mathcal{L}$ that:

$$\alpha = \dot{\bigcup_{i\in I}}(\mathrm{incl}_i \circ \alpha_i|_{C_i}) \in \mathcal{L}\left(\coprod_{i\in I} \mathcal{F}(\mathcal{B}_i)\right).$$

This shows that:

$$\mathcal{F}\left(\mathcal{B}_{\coprod_{i\in I}(\mathcal{X}_i,\mathcal{B}_i)}\right) \subseteq \mathcal{L}\left(\coprod_{i\in I} \mathcal{F}(\mathcal{B}_i)\right),$$

and thus the claim. □

Remark 4.13. *The strongest argument to work inside the category of patchable quasi-measurable spaces* **PQMS** *rather than inside the category of all quasi-measurable spaces* **QMS** *is in fact Lemma 4.12, which asserts that when going from* **Meas** *to* **PQMS** *via the functor $\mathcal{F}$ one preserves countable coproducts (as well as all products):*

$$\coprod_{n\in\mathbb{N}}^{\mathbf{PQMS}} (\mathcal{X}_n, \mathcal{F}(\mathcal{B}_n)) = \mathcal{L}\left(\coprod_{n\in\mathbb{N}}^{\mathbf{QMS}} (\mathcal{X}_n, \mathcal{F}(\mathcal{B}_n))\right) = \mathcal{F}\left(\coprod_{n\in\mathbb{N}}^{\mathbf{Meas}} (\mathcal{X}_n, \mathcal{B}_n)\right).$$

Furthermore, one does not loose anything over **QMS** *as one also has all (small) limits and colimits in* **PQMS***, which is also cartesian closed. On the downside one needs to require and check the condition from Definition 4.2 through all constructions and examples, although that condition does not seem to play any role besides Lemma 4.12, which also states how to recover from the disalignment by applying the reflector $\mathcal{L}$.*

4.4 Patchable Probability Spaces

In this subsection we study under which conditions our probability monads become patchable.

Lemma 4.14 (Spaces of probability measures $\mathcal{G}$, $\mathcal{Q}$, $\mathcal{K}$)**.** *Let $(\mathcal{X}, \mathcal{X}^\Omega)$ be a (not necessarily patchable) quasi-measurable space. Then the following points hold:*

1. *$(\mathcal{G}(\mathcal{X}), \mathcal{G}(\mathcal{X})^\Omega)$ is patchable.*
2. *$(\mathcal{Q}(\mathcal{X}), \mathcal{Q}(\mathcal{X})^\Omega)$ is patchable.*

If $(\mathcal{X}, \mathcal{X}^\Omega)$ is patchable then also the following holds:

3. *$(\mathcal{K}(\mathcal{X}), \mathcal{K}(\mathcal{X})^\Omega)$ is patchable.*

Proof. Since $\mathcal{G}(\mathcal{X})^\Omega = \mathcal{F}(\mathcal{B}_{\mathcal{G}(\mathcal{X})})$, where:

$$\mathcal{B}_{\mathcal{G}(\mathcal{X})} := \sigma\left(\left\{\mathrm{ev}_A^{-1}((t,1]) \,\middle|\, A \in \mathcal{B}_\mathcal{X}, t \in \mathbb{R}\right\}\right),$$

it is clear by Example 4.3 that $(\mathcal{G}(\mathcal{X}), \mathcal{G}(\mathcal{X})^\Omega)$ is patchable.
For $\mathcal{Q}(\mathcal{X})$ first note that $([0,1], [0,1]^\Omega) = ([0,1], \mathcal{F}(\mathcal{B}_{[0,1]}))$ is patchable by Example 4.3, and thus by Lemma 4.7 also $[0,1]^{\mathcal{B}_\mathcal{X}}$. Since $\mathcal{Q}(\mathcal{X})$ carries the subspace quasi-measurable space structure of $[0,1]^{\mathcal{B}_\mathcal{X}}$ by Lemma 4.4 also $(\mathcal{Q}(\mathcal{X}), \mathcal{Q}(\mathcal{X})^\Omega)$ is patchable.
Now assume that $(\mathcal{X}, \mathcal{X}^\Omega)$ is patchable.
$\mathcal{K}(\mathcal{X})$ carries the quotient quasi-measurable space structure of $\mathcal{X}^\Omega \times \mathcal{Q}(\Omega)$ along the push-forward map pf. Since $\mathcal{X}$ is patchable so is $\mathcal{X}^\Omega$ by Lemma 4.7. By the point above $\mathcal{Q}(\Omega)$ is patchable as well. By Lemma 4.5 then also their product $\mathcal{X}^\Omega \times \mathcal{Q}(\Omega)$ is patchable. As a quotient of the latter by Lemma 4.4 also the space $(\mathcal{K}(\mathcal{X}), \mathcal{K}(\mathcal{X})^\Omega)$ becomes patchable.

□

Lemma 4.15 (The space of probability measures $\mathcal{P}$)**.** *Let $(\mathcal{X}, \mathcal{X}^\Omega)$ be a patchable quasi-measurable space and assume that there exists an isomorphism of measurable spaces:*

$$\coprod_{n \in \mathbb{N}} (\Omega, \mathcal{B}_\Omega) \cong (\Omega, \mathcal{B}_\Omega).$$

Further assume that $\Omega^\Omega = \mathcal{F}(\mathcal{B}_\Omega)$. Then the space of push-forward probability measures $(\mathcal{P}(\mathcal{X}), \mathcal{P}(\mathcal{X})^\Omega)$ is a patchable quasi-measurable space.

Proof. Consider a measurable disjoint union decomposition $\Omega = \dot{\bigcup}_{j \in J} C_j$ with countable $J \subseteq \mathbb{N}$ and $\nu_j = \alpha_{j,*}\kappa_j \in \mathcal{P}(\mathcal{X})^\Omega$ with $\alpha_j \in \mathcal{X}^\Omega$ and $\kappa_j \in \mathcal{Q}(\Omega)^\Omega$. By the made assumption we also have a countable measurable disjoint union decomposition:

$$\Omega = \dot{\bigcup_{n \in \mathbb{N}}} D_n,$$

with measurable isomorphisms: $\Omega \cong D_n$ for $n \in \mathbb{N}$. Consider those measurable isomorphisms composed with the measurable inclusion map:

$$d_n : \ \Omega \cong D_n \hookrightarrow \Omega.$$

Then we get that $d_n \in \mathcal{F}(\mathcal{B}_\Omega) = \Omega^\Omega$.
Define the map:

$$\kappa := \dot{\bigcup_{j \in J}} d_{j,*}\kappa_j|_{C_j} : \ \Omega = \dot{\bigcup_{j \in J}} C_j \to \mathcal{Q}\left(\dot{\bigcup_{n \in \mathbb{N}}} D_n\right) = \mathcal{Q}(\Omega).$$

Note that each $d_{j,*}\kappa_j$ is supported on D_j.
To check that $\kappa \in \mathcal{Q}(\Omega)^\Omega$ let $\varphi \in \Omega^\Omega$ and $E \in \mathcal{B}_{\Omega \times \Omega}$. Then the map:

$$\varphi^{-1}(C_i) \ni \omega \mapsto \kappa(\varphi(\omega))(E_\omega) = d_{i,*}\kappa_i(\varphi(\omega))(E_\omega) = \kappa_i(\varphi(\omega))((\mathrm{id}_\Omega, d_i)^{-1}(E)_\omega),$$

is seen to be measurable by the assumption that $\kappa_i \in \mathcal{Q}(\Omega)^\Omega$, thus $\kappa \in \mathcal{Q}(\Omega)^\Omega$.
We now pick a fixed point $\omega_0 \in \Omega$ and $x_0 \in \mathcal{X}$ and define for $i \in J$:

$$d_i^- : \Omega = \dot{\bigcup_{n \in \mathbb{N}}} D_n \to \Omega, \qquad d_i^-|_{D_i} := d_i^{-1}, \qquad d_i^-|_{D_j} := \omega_0, \ j \neq i.$$

For $n \in \mathbb{N} \backslash J$ we define the constant maps: $d_n^- := \omega_0$ and $\alpha_n := x_0$. Then all d_n^- are measurable and thus $d_n^- \in \mathcal{F}(\mathcal{B}_\Omega) = \Omega^\Omega$. Since all $\alpha_n \in \mathcal{X}^\Omega$ we get that $\alpha_n \circ d_n^- \in \mathcal{X}^\Omega$ for all $n \in \mathbb{N}$. With these settings we can then put:

$$\alpha := \dot{\bigcup_{n \in \mathbb{N}}} (\alpha_n \circ d_n^-)|_{D_n} : \ \Omega = \dot{\bigcup_{n \in \mathbb{N}}} D_n \to \mathcal{X}.$$

It is then $\alpha \in \mathcal{X}^\Omega$ since $\mathcal{X}^\Omega$ is patchable. Furthermore, for $\omega \in C_i$, $i \in J$, we get:

$$\alpha_*\kappa(\omega) = \alpha_{i,*}\, d_{i,*}^-\, d_{i,*}\, \kappa_i(\omega) = \alpha_{i,*}\kappa_i(\omega) = \nu_i(\omega).$$

So we get that:

$$\dot{\bigcup_{j \in J}} \nu_j|_{C_j} = \alpha_*\kappa \in \mathcal{P}(\mathcal{X})^\Omega,$$

with $\alpha \in \mathcal{X}^\Omega$ and $\kappa \in \mathcal{Q}(\Omega)^\Omega$, which shows the claim. □

Remark 4.16. *It has not yet been investigated under which conditions* $(\mathcal{R}(\mathcal{X}), \mathcal{R}(\mathcal{X})^\Omega)$ *or* $(\mathcal{S}(\mathcal{X}), \mathcal{S}(\mathcal{X})^\Omega)$ *become patchable. We will see in Theorem 5.29 however that for the category of quasi-universal spaces* **QUS** *we have:*

$$\mathcal{S} = \mathcal{P} = \mathcal{R} = \mathcal{K},$$

which then will be covered by the results above.

5 The Category of Quasi-Universal Spaces

In this section we specialize to the category of quasi-measurable spaces with the special choice of a sample space. If the sample space Ω is an uncountable Polish space, like $\mathbb{R}$ or the Hilbert cube $[0,1]^{\mathbb{N}}$, endowed with the σ-algebra of all universally measurable subsets then we call the corresponding quasi-measurable spaces *quasi-universal spaces* in analogy to quasi-Borel spaces from [HKSY17] and the corresponding category the *category of quasi-universal spaces* **QUS**. We will say more when going through the subsections.

5.1 Completions of Sigma-Algebras

Since the σ-algebras of quasi-universal spaces will turn out to be intersection of (Lebesgue) complete σ-algebras we here recap the notion of (Lebesgue) completions of σ-algebras and show how measurable maps and Markov kernels behave under such completions.

Definition 5.1 (Completions)**.** *Let $\mathcal{X}$ be a set and $\mathcal{E}$ a set of subsets of $\mathcal{X}$, e.g. $\mathcal{E}$ a σ-algebra of subsets of $\mathcal{X}$.*

1. *We abbreviate the set of all probability measures of $(\mathcal{X}, \sigma(\mathcal{E}))$ as:*

$$\mathcal{G}(\mathcal{X}, \mathcal{E}) := \mathcal{G}(\mathcal{X}, \sigma(\mathcal{E})) := \{\mu : \ \sigma(\mathcal{E}) \to [0,1] \quad \textit{probability measure}\} .$$

2. *For any non-empty subset $\mathcal{P} \subseteq \mathcal{G}(\mathcal{X}, \mathcal{B}_{\mathcal{X}})$ with $\mathcal{E} \subseteq \mathcal{B}_{\mathcal{X}}$ we define the* $\mathcal{P}$-completion *of $\mathcal{E}$ as:*

$$(\mathcal{E})_{\mathcal{P}} := \bigcap_{\mu \in \mathcal{P}} (\sigma(\mathcal{E}))_{\mu|_{\sigma(\mathcal{E})}} ,$$

   i.e. the intersection of all Lebesgue completions of the σ-algebra $\sigma(\mathcal{E})$ generated by $\mathcal{E}$ w.r.t. the restrictions of every probability measure μ in $\mathcal{P}$ to $\sigma(\mathcal{E})$.

3. *For $\mathcal{P} = \varnothing$ we put:*

$$(\mathcal{E})_{\varnothing} := \{A \subseteq \mathcal{X}\} ,$$

   the power-set of $\mathcal{X}$.

4. *The* universal completion *of $\mathcal{E}$ is defined as:*

$$(\mathcal{E})_{\mathcal{G}} := (\mathcal{E})_{\mathcal{G}(\mathcal{X},\mathcal{E})} := \bigcap_{\mu \in \mathcal{G}(\mathcal{X}, \sigma(\mathcal{E}))} (\sigma(\mathcal{E}))_{\mu} ,$$

   i.e. $(\mathcal{E})_{\mathcal{G}} = (\mathcal{E})_{\mathcal{P}}$ with the special choice $\mathcal{P} := \mathcal{G}(\mathcal{X}, \mathcal{E})$.

5. *We say that $\mathcal{E}$ or $(\mathcal{X}, \mathcal{E})$, resp., is* $\mathcal{P}$-complete *if $\mathcal{E} = (\mathcal{E})_{\mathcal{P}}$.*

6. *We say that $\mathcal{E}$ or $(\mathcal{X}, \mathcal{E})$, resp., is* universally complete *if $\mathcal{E} = (\mathcal{E})_{\mathcal{G}}$.*

Lemma 5.2. *Let* $f: (\mathcal{X}, \mathcal{B}_\mathcal{X}) \to (\mathcal{Y}, \mathcal{B}_\mathcal{Y})$ *be a measurable map and*

$$f_*: \mathcal{G}(\mathcal{X}, \mathcal{B}_\mathcal{X}) \to \mathcal{G}(\mathcal{Y}, \mathcal{B}_\mathcal{Y}), \qquad \mu \mapsto f_*\mu,$$

the (measurable) induced map. Let $\mathcal{Q} \subseteq \mathcal{G}(\mathcal{Y}, \mathcal{B}_\mathcal{Y})$ *and* $\mathcal{P} \subseteq (f_*)^{-1}(\mathcal{Q})$ *subsets. Then* f *is also* $(\mathcal{B}_\mathcal{X})_\mathcal{P}$*-*$(\mathcal{B}_\mathcal{Y})_\mathcal{Q}$*-measurable. In particular, this holds for:*

$$f: (\mathcal{X}, (\mathcal{B}_\mathcal{X})_{(f_*)^{-1}(\mathcal{Q})}) \to (\mathcal{Y}, (\mathcal{B}_\mathcal{Y})_\mathcal{Q}), \qquad f: (\mathcal{X}, (\mathcal{B}_\mathcal{X})_\mathcal{P}) \to (\mathcal{Y}, (\mathcal{B}_\mathcal{Y})_{f_*(\mathcal{P})}).$$

Proof. First remember that for every $\mu \in \mathcal{G}(\mathcal{X}, \mathcal{B}_\mathcal{X})$ we have that f is $(\mathcal{B}_\mathcal{X})_\mu$-$(\mathcal{B}_\mathcal{Y})_{f_*\mu}$-measurable. Then for $\mu \in \mathcal{P}$ we have by assumption that $f_*\mu \in \mathcal{Q}$. So for $B \in (\mathcal{B}_\mathcal{Y})_\mathcal{Q} \subseteq (\mathcal{B}_\mathcal{Y})_{f_*\mu}$ we get that $f^{-1}(B) \in (\mathcal{B}_\mathcal{X})_\mu$ for all $\mu \in \mathcal{P}$, thus $f^{-1}(B) \in (\mathcal{B}_\mathcal{X})_\mathcal{P}$. ☐

Proposition 5.3 (See [For21] Thm. B.42 and [Res77] Thm. 4)**.** *Let* $(\mathcal{X}, \mathcal{B}_\mathcal{X})$ *be a measurable space and* $\mathcal{G}(\mathcal{X}, \mathcal{B}_\mathcal{X})$ *the space of all probability measures on* $(\mathcal{X}, \mathcal{B}_\mathcal{X})$ *endowed with the smallest* σ*-algebra* $\mathcal{B}_{\mathcal{G}(\mathcal{X},\mathcal{B}_\mathcal{X})}$ *such that all evaluation maps:*

$$\mathrm{ev}_A: \mathcal{G}(\mathcal{X}, \mathcal{B}_\mathcal{X}) \to [0,1], \qquad \mu \mapsto \mu(A),$$

are measurable for all $A \in \mathcal{B}_\mathcal{X}$*. Let* $\mathcal{P} \subseteq \mathcal{G}(\mathcal{X}, \mathcal{B}_\mathcal{X})$ *be a fixed non-empty subset of probability measures endowed with a* σ*-algebra* $\mathcal{B}_\mathcal{P}$ *that contains the subspace* σ*-algebra* $\mathcal{B}_{\mathcal{G}(\mathcal{X},\mathcal{B}_\mathcal{X})|\mathcal{P}}$ *(e.g.* $\mathcal{B}_\mathcal{P} = \mathcal{B}_{\mathcal{G}(\mathcal{X},\mathcal{B}_\mathcal{X})|\mathcal{P}}$*).*[4] *Consider the map:*

$$\mathbb{M}_\iota: \mathcal{G}(\mathcal{P}, \mathcal{B}_\mathcal{P}) \xrightarrow{\iota_*} \mathcal{G}\left(\mathcal{G}(\mathcal{X}, \mathcal{B}_\mathcal{X}), \mathcal{B}_{\mathcal{G}(\mathcal{X},\mathcal{B}_\mathcal{X})}\right) \xrightarrow{\mathbb{M}} \mathcal{G}(\mathcal{X}, \mathcal{B}_\mathcal{X}), \quad \mathbb{M}_\iota(\pi)(A) := \int_\mathcal{P} \mathrm{ev}_A(\mu)\, \pi(d\mu),$$

and let $\mathcal{M} \subseteq \mathbb{M}_\iota^{-1}(\mathcal{P}) \subseteq \mathcal{G}(\mathcal{P}, \mathcal{B}_\mathcal{P})$ *be a subset*[5]*.*
Then for every $B \in (\mathcal{B}_\mathcal{X})_\mathcal{P}$ *the evaluation map:*

$$\mathrm{ev}_B: \mathcal{P} \to [0,1], \qquad \mu \mapsto \mu(B),$$

is π*-measurable for every* $\pi \in \mathcal{M}$*. In short:*

$$\mathrm{ev}_B: (\mathcal{P}, (\mathcal{B}_\mathcal{P})_\mathcal{M}) \to \left([0,1], \mathcal{B}_{[0,1]}\right), \qquad \mu \mapsto \mu(B),$$

is measurable for every $B \in (\mathcal{B}_\mathcal{X})_\mathcal{P}$*. Or, even shorter, the inclusion map*

$$(\mathcal{P}, (\mathcal{B}_\mathcal{P})_\mathcal{M}) \hookrightarrow \mathcal{G}(\mathcal{X}, (\mathcal{B}_\mathcal{X})_\mathcal{P})$$

is well-defined and measurable.
In particular, the map:

$$\mathrm{ev}_B: \left(\mathcal{G}(\mathcal{X}, \mathcal{B}_\mathcal{X}), (\mathcal{B}_{\mathcal{G}(\mathcal{X},\mathcal{B}_\mathcal{X})})_\mathcal{G}\right) \to \left([0,1], \mathcal{B}_{[0,1]}\right), \qquad \mu \mapsto \mu(B),$$

is (universally) measurable for all $B \in (\mathcal{B}_\mathcal{X})_\mathcal{G}$*.*

[4] So the inclusion map $\iota: (\mathcal{P}, \mathcal{B}_\mathcal{P}) \overset{\iota}{\hookrightarrow} \left(\mathcal{G}(\mathcal{X}, \mathcal{B}_\mathcal{X}), \mathcal{B}_{\mathcal{G}(\mathcal{X},\mathcal{B}_\mathcal{X})}\right)$ is measurable.
[5] We can allow for $\mathcal{M} = \varnothing$ if we define $(\mathcal{B}_\mathcal{P})_\varnothing := \{D \subseteq \mathcal{P}\}$ to be the power-set of $\mathcal{P}$ in the following.

Proof. Let $\alpha \in \mathbb{R}$ and $\pi \in \mathcal{M}$. We need to show that $\mathrm{ev}_B^{-1}((\alpha, 1]) \in (\mathcal{B}_\mathcal{P})_\pi$. Now let $\nu := \mathbb{M}_\iota(\pi) \in \mathcal{P}$, which for $A \in \mathcal{B}_\mathcal{X}$ is given by:

$$\nu(A) = \mathbb{M}_\iota(\pi)(A) := \int_\mathcal{P} \mathrm{ev}_A(\mu)\, \pi(d\mu).$$

Note that the map $\mathcal{P} \xhookrightarrow{\iota} \mathcal{G}(\mathcal{X}, \mathcal{B}_\mathcal{X}) \xrightarrow{\mathrm{ev}_A} [0,1]$ is measurable for $A \in \mathcal{B}_\mathcal{X}$ and the above integral is well-defined.
Since $B \in (\mathcal{B}_\mathcal{X})_\mathcal{P} \subseteq (\mathcal{B}_\mathcal{X})_\nu$ there exist $A_1, A_2 \in \mathcal{B}_\mathcal{X}$ such that:

$$A_1 \subseteq B \subseteq A_2, \qquad \nu(A_2 \backslash A_1) = 0.$$

The last condition shows that:

$$0 = \nu(A_2 \backslash A_1) = \int_\mathcal{P} \mathrm{ev}_{A_2 \backslash A_1}(\mu)\, \pi(d\mu).$$

Since $\mathrm{ev}_{A_2 \backslash A_1}(\mu) \geqslant 0$ and $\mathrm{ev}_{A_2 \backslash A_1} : \mathcal{P} \to [0,1]$ is measurable we have that:

$$C := \mathrm{ev}_{A_2 \backslash A_1}^{-1}((0,1]) = \left\{\mu \in \mathcal{P} \,\middle|\, \mathrm{ev}_{A_2 \backslash A_1}(\mu) > 0\right\} \in \mathcal{B}_\mathcal{P}, \qquad \text{and} \qquad \pi(C) = 0.$$

We now have the inclusions:

$$\mathrm{ev}_{A_1}^{-1}((\alpha, 1]) \subseteq \mathrm{ev}_B^{-1}((\alpha, 1]) \subseteq \mathrm{ev}_{A_2}^{-1}((\alpha, 1]),$$

with $\mathrm{ev}_{A_i}^{-1}((\alpha, 1]) \in \mathcal{B}_\mathcal{P}$, $i = 1, 2$, and:

$$\mathrm{ev}_{A_2}^{-1}((\alpha, 1]) \backslash \mathrm{ev}_{A_1}^{-1}((\alpha, 1]) \subseteq \mathrm{ev}_{A_2 \backslash A_1}^{-1}((0, 1]) = C.$$

Since $\pi(C) = 0$ we also have that:

$$\pi\left(\mathrm{ev}_{A_2}^{-1}((\alpha, 1]) \backslash \mathrm{ev}_{A_1}^{-1}((\alpha, 1])\right) = 0.$$

This shows that $\mathrm{ev}_B^{-1}((\alpha, 1]) \in (\mathcal{B}_\mathcal{P})_\pi$. Since this holds for all $\pi \in \mathcal{M}$ we get that:

$$\mathrm{ev}_B^{-1}((\alpha, 1]) \in (\mathcal{B}_\mathcal{P})_\mathcal{M}.$$

This shows that the map:

$$\mathrm{ev}_B : \ (\mathcal{P}, (\mathcal{B}_\mathcal{P})_\mathcal{M}) \to \left([0,1], \mathcal{B}_{[0,1]}\right),$$

is measurable for all $B \in (\mathcal{B}_\mathcal{X})_\mathcal{P}$ with $\mathcal{M} \subseteq \mathbb{M}_\iota^{-1}(\mathcal{P})$. ☐

Theorem 5.4 (Markov kernels under completions)**.** *Let $\kappa : \ (\mathcal{Z}, \mathcal{B}_\mathcal{Z}) \to \mathcal{G}(\mathcal{X}, \mathcal{B}_\mathcal{X})$ be a (measurable) Markov kernel and $\mathcal{P} \subseteq \mathcal{G}(\mathcal{X}, \mathcal{B}_\mathcal{X})$ a subset of probability measures such that:*

$$\kappa(\mathcal{Z}) := \{\kappa(z) \,|\, z \in \mathcal{Z}\} \subseteq \mathcal{P}.$$

Then the following induced Markov kernel is well-defined and measurable:

$$\kappa: \ (\mathcal{Z}, (\mathcal{B}_\mathcal{Z})_{(\kappa_\circ)^{-1}(\mathcal{P})}) \to \mathcal{G}(\mathcal{X}, (\mathcal{B}_\mathcal{X})_\mathcal{P}),$$

where we endow the spaces of probability measures with the smallest σ-algebras such that all evaluation maps ev_A are measurable where A is running through the indicated σ-algebras, and where $\kappa_\circ$ is the following induced (measurable) "bind" map:

$$\kappa_\circ: \ \mathcal{G}(\mathcal{Z}, \mathcal{B}_\mathcal{Z}) \xrightarrow{\kappa_*} \mathcal{G}\left(\mathcal{G}(\mathcal{X}, \mathcal{B}_\mathcal{X}), \mathcal{B}_{\mathcal{G}(\mathcal{X},\mathcal{B}_\mathcal{X})}\right) \xrightarrow{\mathbb{M}} \mathcal{G}(\mathcal{X}, \mathcal{B}_\mathcal{X}), \quad \kappa_\circ(\nu)(A) = \int \kappa(z)(A)\, \nu(dz).$$

In particular, we have the corner cases for $\mathcal{P} = \kappa(\mathcal{Z})$ and for $\mathcal{P} = \mathcal{G}(\mathcal{X}, \mathcal{B}_\mathcal{X})$:

$$\kappa: \ (\mathcal{Z}, (\mathcal{B}_\mathcal{Z})_{(\kappa_\circ)^{-1}(\kappa(\mathcal{Z}))}) \to \mathcal{G}(\mathcal{X}, (\mathcal{B}_\mathcal{X})_{\kappa(\mathcal{Z})}), \qquad \kappa: \ (\mathcal{Z}, (\mathcal{B}_\mathcal{Z})_\mathcal{G}) \to \mathcal{G}(\mathcal{X}, (\mathcal{B}_\mathcal{X})_\mathcal{G}).$$

Proof. We use Lemma 5.2 and Proposition 5.3 with the choices:

$$\mathcal{B}_\mathcal{P} := \mathcal{B}_{\mathcal{G}(\mathcal{X},\mathcal{B}_\mathcal{X})|\mathcal{P}}, \qquad \mathcal{M} := \mathbb{M}_\iota^{-1}(\mathcal{P}) \subseteq \mathcal{G}(\mathcal{P}, \mathcal{B}_\mathcal{P}).$$

Since $\kappa(\mathcal{Z}) \subseteq \mathcal{P}$ the map κ factorizes into the following measurable maps:

$$\kappa: \ (\mathcal{Z}, \mathcal{B}_\mathcal{Z}) \xrightarrow{\kappa} (\mathcal{P}, \mathcal{B}_\mathcal{P}) \xhookrightarrow{\iota} \left(\mathcal{G}(\mathcal{X}, \mathcal{B}_\mathcal{X}), \mathcal{B}_{\mathcal{G}(\mathcal{X},\mathcal{B}_\mathcal{X})}\right).$$

This induces the following measurable factorization of $\kappa_\circ$:

$$\kappa_\circ: \ \mathcal{G}(\mathcal{Z}, \mathcal{B}_\mathcal{Z}) \xrightarrow{\kappa_*} \mathcal{G}(\mathcal{P}, \mathcal{B}_\mathcal{P}) \xrightarrow{\iota_*} \mathcal{G}\left(\mathcal{G}(\mathcal{X}, \mathcal{B}_\mathcal{X}), \mathcal{B}_{\mathcal{G}(\mathcal{X},\mathcal{B}_\mathcal{X})}\right) \xrightarrow{\mathbb{M}} \mathcal{G}(\mathcal{X}, \mathcal{B}_\mathcal{X}),$$

showing that: $\mathcal{Q} := (\kappa_\circ)^{-1}(\mathcal{P}) = (\kappa_*)^{-1}(\mathcal{M})$.
Then we get a commutative diagram of measurable maps:

$$\begin{array}{ccccc}
(\mathcal{Z}, (\mathcal{B}_\mathcal{Z})_\mathcal{Q}) & \xrightarrow{\kappa} & (\mathcal{P}, (\mathcal{B}_\mathcal{P})_\mathcal{M}) & \xhookrightarrow{\text{incl}} & \mathcal{G}(\mathcal{X}, (\mathcal{B}_\mathcal{X})_\mathcal{P}) \\
{\scriptstyle \mathrm{id}_\mathcal{Z}} \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{id}_\mathcal{P}} & & \big\downarrow {\scriptstyle \mathrm{res}} \\
(\mathcal{Z}, \mathcal{B}_\mathcal{Z}) & \xrightarrow[\kappa]{} & (\mathcal{P}, \mathcal{B}_\mathcal{P}) & \xhookrightarrow[\iota]{} & \mathcal{G}(\mathcal{X}, \mathcal{B}_\mathcal{X}).
\end{array}$$

Indeed, the upper left map is well-defined and measurable by Lemma 5.2 using $\kappa_*(\mathcal{Q}) \subseteq \mathcal{M}$ and and the upper right map by Proposition 5.3 using $\mathcal{M} \subseteq \mathbb{M}_\iota^{-1}(\mathcal{P})$. Since the composition of the upper maps is our induced Markov kernel, the claim is shown. ☐

5.2 Quasi-Universal Spaces

5.2.1 The Sample Space for Quasi-Universal Spaces

The category of *quasi-universal spaces* **QUS** is per definition the category of quasi-measurable spaces **QMS** with the following special choice of a sample space:

$$(\Omega, \mathcal{B}_\Omega, \Omega^\Omega) = \left(\mathbb{R}^\mathbb{N}, \left(\bigotimes_{n\in\mathbb{N}} \mathcal{B}_\mathbb{R}\right)_\mathcal{G}, \mathcal{F}(\mathcal{B}_\Omega)\right),$$

where $\mathcal{B}_\Omega$ is the σ-algebra of all universally measurable subsets of Ω and:

$$\Omega^\Omega = \mathcal{F}(\mathcal{B}_\Omega) = \mathbf{Meas}\left((\Omega, \mathcal{B}_\Omega), (\Omega, \mathcal{B}_\Omega)\right).$$

Since there exists a (Borel) isomorphism $\mathbb{R}^\mathbb{N} \cong \mathbb{R}$ we can equally well use:

$$(\Omega, \mathcal{B}_\Omega, \Omega^\Omega) = (\mathbb{R}, (\mathcal{B}_\mathbb{R})_\mathcal{G}, \mathcal{F}(\mathcal{B}_\Omega)),$$

or any other uncountable Polish space, e.g. the Hilbert cube, endowed with the σ-algebra of all universally measurable subsets. The reason we chose $\mathbb{R}^\mathbb{N}$ is that it is rather elementary to see that $\Omega \times \Omega \cong \Omega$ (in **Meas**) without the use of complicated Borel isomorphisms. It is also easy to model countably many random experiments with this sample space, which could capture all stochasticity in reality.

The isomorphism $\Omega \times \Omega \cong \Omega$ shows that the probability monads $(\mathcal{Q}, \delta, \mathbb{M})$, $(\mathcal{K}, \delta, \mathbb{M})$, $(\mathcal{P}, \delta, \mathbb{M})$, $(\mathcal{R}, \delta, \mathbb{M})$, $(\mathcal{S}, \delta, \mathbb{M})$ are all strong on the cartesian closed category $(\mathbf{QUS}, \times, \mathbf{1})$ by Theorems 3.16 and 3.25. In Theorem 5.29 we will ses that we can actually identify on $(\mathbf{QUS}, \times, \mathbf{1})$:

$$(\mathcal{S}, \delta, \mathbb{M}) = (\mathcal{P}, \delta, \mathbb{M}) = (\mathcal{R}, \delta, \mathbb{M}) = (\mathcal{K}, \delta, \mathbb{M}).$$

So only the strong probability monads $(\mathcal{P}, \delta, \mathbb{M})$ and $(\mathcal{Q}, \delta, \mathbb{M})$ remain of interest on $(\mathbf{QUS}, \times, \mathbf{1})$.

The only difference to the sample space $(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mathcal{F}(\mathcal{B}_\mathbb{R}))$ of the category of *quasi-Borel spaces* **QBS** from [HKSY17] is then the use of the universal completion $(\mathcal{B}_\mathbb{R})_\mathcal{G}$ of the Borel σ-algebra $\mathcal{B}_\mathbb{R}$. The advantage of using universally complete sample spaces is that the σ-algebras on the spaces $(\mathcal{X}, \mathcal{X}^\Omega)$, which come as the push-forward of $\mathcal{B}_\Omega$, can more explicitely be described as intersections of complete σ-algebras, see Lemma 5.5:

$$\mathcal{B}(\mathcal{X}^\Omega) = \mathcal{X}^\Omega_* \mathcal{B}_\Omega = \bigcap_{\substack{\alpha \in \mathcal{X}^\Omega \\ \mu \in \mathcal{Q}(\Omega)}} \alpha_*(\mathcal{B}_\mathbb{R})_\mu.$$

This relation will be even more accentuated when one looks at countably separated quasi-universal spaces, see Theorem 5.16. Note that the push-forward $\alpha_*\mathcal{B}_\mathbb{R}$ would push the σ-algebra $\mathcal{B}_\mathbb{R}$ onto the image $\alpha(\Omega)$ and then extend everything by null-sets outside the image. So it is natural to require that the σ-algebra also contains all null-sets inside the image $\alpha(\Omega)$. So working with complete σ-algebras $(\mathcal{B}_\mathbb{R})_\mu$ and their push-forward σ-algebras $\alpha_*(\mathcal{B}_\mathbb{R})_\mu$, which are then also complete, harmonizes things.

Another difference between **QBS** and **QUS** is that for quasi-universal spaces we do not require the spaces to be *patchable*, see Definition 4.2, in contrast to quasi-Borel spaces, which by definition in [HKSY17] are always *patchable*. The only reason, for our purposes here, that one would require patchable spaces is to preserve countable coproducts during the transition from measurable spaces to quasi-universal spaces, see Lemma 4.12. For all other purposes it seems more general and simple to proceed without that property.

5.2.2 The Sigma-Algebras of Quasi-Universal Spaces

In this subsection we will give a description of the induced σ-algebras of quasi-universal spaces in terms of intersections of complete σ-algebras.

Lemma 5.5 (The σ-algebras of quasi-universal spaces)**.** *Let $(\mathcal{X}, \mathcal{X}^\Omega)$ be a quasi-universal space. Then $\mathcal{B}(\mathcal{X}^\Omega)$ is universally complete and complete w.r.t. the set of push-forward probability measures:*

$$\mathcal{P}(\mathcal{X}, \mathcal{X}^\Omega) := \left\{\alpha_*\mu : \ \mathcal{B}(\mathcal{X}^\Omega) \to [0,1] \,\middle|\, \alpha \in \mathcal{X}^\Omega, \mu \in \mathcal{G}(\Omega, \mathcal{B}_\Omega)^{6}\right\},$$

i.e. we have:

$$\mathcal{B}(\mathcal{X}^\Omega) = \mathcal{B}(\mathcal{X}^\Omega)_{\mathcal{G}} = \mathcal{B}(\mathcal{X}^\Omega)_{\mathcal{P}(\mathcal{X},\mathcal{X}^\Omega)} = \bigcap_{\substack{\mu \in \mathcal{G}(\Omega,\mathcal{B}_\Omega) \\ \alpha \in \mathcal{X}^\Omega}} \alpha_*(\mathcal{B}_\Omega)_\mu.$$

Note that each $\alpha_*(\mathcal{B}_\Omega)_\mu$ is a complete σ-algebra on $\mathcal{X}$ w.r.t. the push-forward probability measure $\alpha_*\mu$.

Proof. Since the set of all probability measures contains all push-forward probability measures:

$$\mathcal{G}(\mathcal{X}, \mathcal{B}(\mathcal{X}^\Omega)) =: \mathcal{G}(\mathcal{X}, \mathcal{X}^\Omega) \supseteq \mathcal{P}(\mathcal{X}, \mathcal{X}^\Omega),$$

we have the chain of inclusions:

$$\begin{aligned}
\mathcal{B}(\mathcal{X}^\Omega)_{\mathcal{P}(\mathcal{X},\mathcal{X}^\Omega)} \supseteq \mathcal{B}(\mathcal{X}^\Omega)_{\mathcal{G}} \supseteq \mathcal{B}(\mathcal{X}^\Omega)& \\
&= \bigcap_{\alpha \in \mathcal{X}^\Omega} \alpha_*\mathcal{B}_\Omega \\
&= \bigcap_{\alpha \in \mathcal{X}^\Omega} \alpha_*(\mathcal{B}_\Omega)_{\mathcal{G}} \\
&= \bigcap_{\alpha \in \mathcal{X}^\Omega} \alpha_* \bigcap_{\mu \in \mathcal{G}(\Omega,\mathcal{B}_\Omega)} (\mathcal{B}_\Omega)_\mu \\
&= \bigcap_{\alpha \in \mathcal{X}^\Omega} \bigcap_{\mu \in \mathcal{G}(\Omega,\mathcal{B}_\Omega)} \alpha_*(\mathcal{B}_\Omega)_\mu \\
&\supseteq \bigcap_{\alpha \in \mathcal{X}^\Omega} \bigcap_{\mu \in \mathcal{G}(\Omega,\mathcal{B}_\Omega)} (\mathcal{B}(\mathcal{X}^\Omega))_{\alpha_*\mu} \\
&= \bigcap_{\nu \in \mathcal{P}(\mathcal{X},\mathcal{X}^\Omega)} (\mathcal{B}(\mathcal{X}^\Omega))_\nu \\
&= \mathcal{B}(\mathcal{X}^\Omega)_{\mathcal{P}(\mathcal{X},\mathcal{X}^\Omega)}.
\end{aligned}$$

This shows equality and thus the claim. □

Remark 5.6. *If we take $\Omega = \mathbb{R}$ and $\mathcal{B}_\mathbb{R} \subseteq \mathcal{B}_\Omega$ the Borel σ-algebra, we get:*

$$\mathcal{B}(\mathcal{X}^\Omega) = \bigcap_{\alpha \in \mathcal{X}^\Omega} \bigcap_{\mu \in \mathcal{G}(\mathbb{R},\mathcal{B}_\mathbb{R})} \alpha_*(\mathcal{B}_\mathbb{R})_\mu.$$

[6] We will see in Lemma 5.28 that $\mathcal{G}(\Omega, \mathcal{B}_\Omega) = \mathcal{Q}(\Omega, \Omega^\Omega)$ as sets and quasi-universal spaces and the ambiguity in the notation for $\mathcal{P}(\mathcal{X}, \mathcal{X}^\Omega)$ will disappear.

5.3 Countably Separated Quasi-Universal Spaces

In this subsection we study *countably separated* quasi-universal spaces. The reason is that those spaces have a very convenient descriptions for their σ-algebra. It turns out that such countably separated quasi-unviersal spaces $(\mathcal{X}, \mathcal{X}^\Omega)$ will satsify:

$$\mathcal{B}(\mathcal{X}^\Omega) = (\mathcal{E})_{\mathcal{P}(\mathcal{X},\mathcal{X}^\Omega)},$$

for every countable subset $\mathcal{E} \subseteq \mathcal{B}(\mathcal{X}^\Omega)$ that separates the points of $\mathcal{X}$. Surprisingly, that means that those σ-algebras are then automatically countably *generated* up to some form of completion. Many things will simplify for those spaces.

5.3.1 Countably Separated Spaces

Here will go through the definition of *countably separated* spaces and variants thereof.

Definition 5.7. *Let $\mathcal{X}$ be a set and $\mathcal{E}$ a set of subsets of $\mathcal{X}$.*

1. *$(\mathcal{X}, \mathcal{E})$, or just $\mathcal{E}$, is called* separated *if $\mathcal{E}$ separates the points of $\mathcal{X}$. This means that for every $x_1, x_2 \in \mathcal{X}$ with $x_1 \neq x_2$ there exists an $A \in \mathcal{E}$ such that $x_1 \in A$ and $x_2 \notin A$, or, $x_1 \notin A$ and $x_2 \in A$.*

2. *$(\mathcal{X}, \mathcal{E})$, or just $\mathcal{E}$, is called a* countably separated *if there exists a countable $\mathcal{E}' \subseteq \mathcal{E}$ that separates the points of $\mathcal{X}$.*

3. *$(\mathcal{X}, \mathcal{E})$, or just $\mathcal{E}$, is called a* universally countably separated *if $(\mathcal{X}, (\mathcal{E})_\mathcal{G})$ is countably separated.*

4. *A quasi-measurable space $(\mathcal{X}, \mathcal{X}^\Omega)$ is called* (countably) separated *if $(\mathcal{X}, \mathcal{B}(\mathcal{X}^\Omega))$ is in the above definition.*

5. *A quasi-measurable space $(\mathcal{X}, \mathcal{X}^\Omega)$ is called* quasi-separated *if:*

$$\Delta_\mathcal{X} := \{(x,x) \in \mathcal{X} \times \mathcal{X} \,|\, x \in \mathcal{X}\} \in \mathcal{B}_{\mathcal{X}\times\mathcal{X}} := \mathcal{B}(\mathcal{X}^\Omega \times \mathcal{X}^\Omega).$$

Remark 5.8.

1. *Every countably separated (quasi-)measurable space is separated and universally countably separated.*

2. *Every countably separated quasi-measurable space is also quasi-separated.*

3. *The countable product of countably separated (quasi-)measurable spaces is countably separated.*

Proof. The first claim is clear.
For the second claim note that by [Bog07] Thm. 6.5.7 for countably separated spaces we have:

$$\Delta_\mathcal{X} \in \mathcal{B}_\mathcal{X} \otimes \mathcal{B}_\mathcal{X} \subseteq \mathcal{B}_{\mathcal{X}\times\mathcal{X}}.$$

For the third claim take $\mathcal{E} := \{\mathrm{pr}_n^{-1}(A_n) \,|\, n \in \mathbb{N}, A_n \in \mathcal{E}_n\}$ on the product if $\mathcal{E}_n$ is countable and separates the points of $\mathcal{X}_n$, $n \in \mathbb{N}$. ☐

Definition 5.9 (Separated quotient of a (quasi-)measurable space)**.** *Let* $(\mathcal{X}, \mathcal{B}_\mathcal{X})$, $(\mathcal{X}, \mathcal{X}^\Omega)$, *resp., be a (quasi-)measurable space. Then the quotient:*

$$t : \mathcal{X} \twoheadrightarrow \tilde{\mathcal{X}} := \mathcal{X}/\sim,$$

together with the quotient σ-algebra $\mathcal{B}_{\tilde{\mathcal{X}}} := t_*\mathcal{B}_\mathcal{X}$*, the quotient functions* $\tilde{\mathcal{X}}^\Omega := t \circ \mathcal{X}^\Omega$*, resp., will be called the* separated quotient *of* $(\mathcal{X}, \mathcal{B}_\mathcal{X})$, $(\mathcal{X}, \mathcal{X}^\Omega)$*, resp. Here the equivalence relation is given by:*

$$x_1 \sim x_2 \quad :\Longleftrightarrow \quad \forall A \in \mathcal{B}_\mathcal{X}. \quad \{x_1, x_2\} \subseteq A \ \vee \ \{x_1, x_2\} \subseteq A^\mathsf{c}.$$

Also note for the quasi-measurable space $(\mathcal{X}, \mathcal{X}^\Omega)$ *that we take* $\mathcal{B}_\mathcal{X} := \mathcal{B}(\mathcal{X}^\Omega)$ *and that:*

$$t_*\mathcal{B}_\mathcal{X} = t_*\mathcal{B}(\mathcal{X}^\Omega) = \mathcal{B}(t \circ \mathcal{X}^\Omega) = \mathcal{B}(\tilde{\mathcal{X}}^\Omega) = \mathcal{B}_{\tilde{\mathcal{X}}}.$$

Remark 5.10. *Let* $(\mathcal{X}, \mathcal{B}_\mathcal{X})$, $(\mathcal{X}, \mathcal{X}^\Omega)$*, resp., be a (quasi-)measurable space. Then the quotient map of the separated quotient:*

$$t : \mathcal{X} \twoheadrightarrow \tilde{\mathcal{X}} := \mathcal{X}/\sim,$$

induces a bijection:

$$t^* : \mathcal{B}_{\tilde{\mathcal{X}}} \to \mathcal{B}_\mathcal{X}, \qquad \tilde{A} \mapsto t^{-1}(\tilde{A}).$$

Furthermore, every section s *of* t*, which exists by the axiom of choice:*

$$s : \tilde{\mathcal{X}} \to \mathcal{X}, \qquad t(s(\tilde{x})) = \tilde{x},$$

is measurable.

The following Lemmata might be of interest to understand the role of countably separated quasi-universal spaces better:

Lemma 5.11. *Let* $(\mathcal{Y}, \mathcal{Y}^\Omega)$ *be a quasi-universal space. Then the following statements are equivalent:*

1. $(\mathcal{Y}, \mathcal{Y}^\Omega)$ *is countably separated.*
2. *There exists an injective measurable map:*
$$\iota : (\mathcal{Y}, \mathcal{B}(\mathcal{Y}^\Omega)) \hookrightarrow (\mathbb{R}, \mathcal{B}_\mathbb{R}).$$
3. *There exists an injective measurable map:*
$$\iota : (\mathcal{Y}, \mathcal{B}(\mathcal{Y}^\Omega)) \hookrightarrow (\Omega, \mathcal{B}_\Omega).$$
4. *There exists an injective quasi-measurable map:*
$$\iota : (\mathcal{Y}, \mathcal{Y}^\Omega) \hookrightarrow (\Omega, \Omega^\Omega).$$

5. *There exists a mono-morphism* $(\mathcal{Y}, \mathcal{Y}^\Omega) \hookrightarrow (\Omega, \Omega^\Omega)$ *in* **QUS**.

Proof. 4. $\iff$ 5.: These are the same statements.
1. $\iff$ 2. $\iff$ 3.: This follows from [Bog07] Thm. 6.5.7 and the fact that $\mathcal{B}(\mathcal{Y}^\Omega)$ is universally complete by Lemma 5.5.
3. $\implies$ 4.: Apply functor $\mathcal{F}$ and use $\mathcal{Y}^\Omega \subseteq \mathcal{F}\mathcal{B}(\mathcal{Y}^\Omega)$.
4. $\implies$ 3.: Apply functor $\mathcal{B}$. □

Lemma 5.12. *Let* $(\mathcal{Y}, \mathcal{Y}^\Omega)$ *be a quasi-universal space. Then the following statements are equivalent:*

1. *There exists a countably generated* σ*-algebra* $\mathcal{E}_\mathcal{Y}$ *that separates the points of* $\mathcal{Y}$ *such that* $\mathcal{F}(\mathcal{E}_\mathcal{Y}) = \mathcal{Y}^\Omega$.

2. *There exists a universally countably generated and separated* σ*-algebra* $\mathcal{E}_\mathcal{Y}$ *on* $\mathcal{Y}$ *such that* $\mathcal{F}(\mathcal{E}_\mathcal{Y}) = \mathcal{Y}^\Omega$.

3. *There exists an injective quasi-measurable map:*

$$\iota : \ (\mathcal{Y}, \mathcal{Y}^\Omega) \hookrightarrow (\Omega, \Omega^\Omega),$$

such that: $\mathcal{Y}^\Omega = i^*\Omega^\Omega$.

4. *There exists an embedding* $(\mathcal{Y}, \mathcal{Y}^\Omega) \hookrightarrow (\Omega, \Omega^\Omega)$ *in* **QUS**.

Proof. 3. $\iff$ 4.: The first is the definition of the latter.
Now let $\mathcal{B}_\mathbb{R} \subseteq \mathcal{B}_\Omega$ be the Borel σ-algebra of $\mathbb{R}$.
2. $\iff$ 3.: Since $\mathcal{B}_\Omega$ is universally complete we have: $\mathcal{F}(\mathcal{E}_\mathcal{Y}) = \mathcal{F}((\mathcal{E}_\mathcal{Y})_\mathcal{G})$.
1. $\implies$ 3.: $\mathcal{E}_\mathcal{Y}$ induces an injective measurable map:

$$\iota : \ (\mathcal{Y}, \mathcal{E}_\mathcal{Y}) \hookrightarrow (\mathbb{R}, \mathcal{B}_\mathbb{R}),$$

such that $\mathcal{E}_\mathcal{Y} = \iota^*\mathcal{B}_\mathbb{R}$, see [Bog07] Thm. 6.5.8. Applying the functor $\mathcal{F}$ gives us an injective quasi-measurable map:

$$\iota : \ (\mathcal{Y}, \mathcal{Y}^\Omega) = (\mathcal{Y}, \mathcal{F}(\mathcal{E}_\mathcal{Y})) \hookrightarrow (\mathbb{R}, \mathcal{F}(\mathcal{B}_\mathbb{R})) = (\Omega, \Omega^\Omega).$$

So we have the inclusion $\mathcal{Y}^\Omega \subseteq \iota^*\Omega^\Omega$. For the reverse inclusion let: $\beta \in \iota^*\Omega^\Omega$. Then $\beta \in [\Omega \to \mathcal{Y}]$ with:

$$\iota \circ \beta \in \Omega^\Omega = \mathcal{F}(\mathcal{B}_\mathbb{R}) = \mathbf{Meas}\left((\Omega, \mathcal{B}_\Omega), (\mathbb{R}, \mathcal{B}_\mathbb{R})\right).$$

But this means that β is $\mathcal{B}_\Omega$-$\iota^*\mathcal{B}_\mathbb{R}$-measurable. Since $\mathcal{E}_\mathcal{Y} = \iota^*\mathcal{B}_\mathbb{R}$ we have that $\beta \in \mathcal{F}(\mathcal{E}_\mathcal{Y})$. So we get $\iota^*\Omega^\Omega \subseteq \mathcal{F}(\mathcal{E}_\mathcal{Y}) = \mathcal{Y}^\Omega$.
3. $\implies$ 1.: We assume that we have an injective quasi-measurable map:

$$\iota : \ (\mathcal{Y}, \mathcal{Y}^\Omega) \hookrightarrow (\Omega, \Omega^\Omega),$$

such that: $\mathcal{Y}^\Omega = i^*\Omega^\Omega$. Applying the functor $\mathcal{B}$ to it gives an injective measurable map:

$$\iota : \ (\mathcal{Y}, \mathcal{B}(\mathcal{Y}^\Omega)) \hookrightarrow (\Omega, \mathcal{B}_\Omega).$$

We put:

$$\mathcal{E}_\mathcal{Y} := \iota^*\mathcal{B}_\mathbb{R} \subseteq \iota^*\mathcal{B}_\Omega \subseteq \mathcal{B}(\mathcal{Y}^\Omega).$$

Then clearly $\mathcal{E}_\mathcal{Y}$ separates the points of $\mathcal{Y}$. Furthermore, we have:

$$\mathcal{Y}^\Omega \subseteq \mathcal{F}\mathcal{B}(\mathcal{Y}^\Omega) \subseteq \mathcal{F}(\mathcal{E}_\mathcal{Y}) = \mathcal{F}((\mathcal{E}_\mathcal{Y})_\mathcal{G}).$$

For the reverse inclusion let $\beta \in \mathcal{F}(\mathcal{E}_\mathcal{Y}) = \mathcal{F}((\mathcal{E}_\mathcal{Y})_\mathcal{G})$. Since ι is $\mathcal{E}_\mathcal{Y}$-$\mathcal{B}_\mathbb{R}$-measurable it is also $(\mathcal{E}_\mathcal{Y})_\mathcal{G}$-$\mathcal{B}_\Omega$-measurable. So the composition $\iota \circ \beta$ is $\mathcal{B}_\Omega$-$\mathcal{B}_\Omega$-measurable. So $\iota \circ \beta \in \Omega^\Omega$ and thus $\beta \in \iota^*\Omega^\Omega$. This shows: $\mathcal{F}(\mathcal{E}_\mathcal{Y}) \subseteq \iota^*\Omega^\Omega = \mathcal{Y}^\Omega$. □

5.3.2 The Sigma-Algebra of Countably Separated Quasi-Universal Spaces

In this subsection we will highlight the structure of the induced σ-algebra for countably separated quasi-universal spaces.

Lemma 5.13. *Let $(\mathcal{Z}, \mathcal{B}_\mathcal{Z}, \mu)$ be a perfect probability space and $(\mathcal{X}, \mathcal{B}_\mathcal{X})$ be a countably separated measurable space and $f : \mathcal{Z} \to \mathcal{X}$ a measurable map. Then we have:*

$$f_*(\mathcal{B}_\mathcal{Z})_\mu = (\mathcal{B}_\mathcal{X})_{f_*\mu},$$

where the index refers to the (Lebesgue) completion w.r.t. the corresponding probability measure.

Proof. By the measurability of f we always have the inclusion:

$$f_*(\mathcal{B}_\mathcal{Z})_\mu \supseteq (\mathcal{B}_\mathcal{X})_{f_*\mu}.$$

Since $(\mathcal{X}, \mathcal{B}_\mathcal{X})$ is countably separated by [Bog07] Thm. 6.5.7 there exists an injective $\mathcal{B}_\mathcal{X}$-$\mathcal{B}_\mathbb{R}$-measurable map:

$$j : \ \mathcal{X} \to \mathbb{R}.$$

So we have the chain of inclusions:

$$j_*f_*(\mathcal{B}_\mathcal{Z})_\mu \supseteq j_*(\mathcal{B}_\mathcal{X})_{f_*\mu} \supseteq (\mathcal{B}_\mathbb{R})_{j_*f_*\mu}.$$

Since μ is perfect we have the equality:

$$j_*f_*(\mathcal{B}_\mathcal{Z})_\mu = (\mathcal{B}_\mathbb{R})_{j_*f_*\mu},$$

which then implies the equality:

$$j_*f_*(\mathcal{B}_\mathcal{Z})_\mu = j_*(\mathcal{B}_\mathcal{X})_{f_*\mu}.$$

Since j is injective we get:

$$f_*(\mathcal{B}_\mathcal{Z})_\mu = j^*j_*f_*(\mathcal{B}_\mathcal{Z})_\mu = j^*j_*(\mathcal{B}_\mathcal{X})_{f_*\mu} = (\mathcal{B}_\mathcal{X})_{f_*\mu}.$$

This shows the claim.
We quickly check that for an injective map $j : \mathcal{X} \to \mathcal{Y}$ we always have:

$$j^* j_* \mathcal{B} = \mathcal{B}.$$

First, let $C \in j^* j_* \mathcal{B}$. Then there exists $B \in j_* \mathcal{B}$ such that $C = j^{-1}(B)$. By definition of $j_* \mathcal{B}$ we have that:

$$C = j^{-1}(B) \in \mathcal{B}.$$

This shows:

$$j^* j_* \mathcal{B} \subseteq \mathcal{B}.$$

Now let $A \subseteq \mathcal{B}$ then by the injectivity of j:

$$j^{-1}(j(A)) = A \in \mathcal{B}.$$

So by definition of $j_* \mathcal{B}$ we get that $j(A) \in j_* \mathcal{B}$ and thus:

$$A = j^{-1}(j(A)) \in j^* j_* \mathcal{B}.$$

This shows:

$$\mathcal{B} \subseteq j^* j_* \mathcal{B}.$$

and thus the claim. □

Lemma 5.14. *Let $(\Omega, \mathcal{B}_\Omega)$ be a measurable space and $\mathcal{P}$ be a set of probability measures on $(\Omega, \mathcal{B}_\Omega)$ such that:*

$$\mathcal{B}_\Omega \supseteq (\mathcal{B}_\Omega)_\mathcal{P} := \bigcap_{\mu \in \mathcal{P}} (\mathcal{B}_\Omega)_\mu .$$

Let $(\mathcal{X}, \mathcal{B}_\mathcal{X})$ be a measurable space and $\mathcal{F} \subseteq \mathbf{Meas}\left((\Omega, \mathcal{B}_\Omega), (\mathcal{X}, \mathcal{B}_\mathcal{X})\right)$ be any subset of measurable maps. Let $\mathcal{Q}$ be any set of probability measures on $(\mathcal{X}, \mathcal{B}_\mathcal{X})$ that contains the push-forward probability measures:

$$\mathcal{Q} \supseteq \mathcal{P}(\mathcal{F}) := \left\{ \alpha_* \mu : \mathcal{B}_\mathcal{X} \to [0,1] \,\middle|\, \alpha \in \mathcal{F}, \mu \in \mathcal{P} \right\}.$$

Then we have the inclusion:

$$(\mathcal{B}_\mathcal{X})_\mathcal{Q} \subseteq \mathcal{B}(\mathcal{F}).$$

Proof. We have the following chain of inclusions:

$$\begin{aligned}
\mathcal{B}(\mathcal{F}) &:= \left\{ A \subseteq \mathcal{X} \,\middle|\, \forall \alpha \in \mathcal{F}.\, \alpha^{-1}(A) \in \mathcal{B}_\Omega \right\} \\
&= \bigcap_{\alpha \in \mathcal{F}} \alpha_* \mathcal{B}_\Omega \\
&\supseteq \bigcap_{\alpha \in \mathcal{F}} \alpha_* \bigcap_{\mu \in \mathcal{P}} (\mathcal{B}_\Omega)_\mu \\
&= \bigcap_{\alpha \in \mathcal{F}} \bigcap_{\mu \in \mathcal{P}} \alpha_* (\mathcal{B}_\Omega)_\mu \\
&\supseteq \bigcap_{\alpha \in \mathcal{F}} \bigcap_{\mu \in \mathcal{P}} (\mathcal{B}_\mathcal{X})_{\alpha_* \mu} \\
&= \bigcap_{\nu \in \mathcal{P}(\mathcal{F})} (\mathcal{B}_\mathcal{X})_\nu \\
&\supseteq \bigcap_{\nu \in \mathcal{Q}} (\mathcal{B}_\mathcal{X})_\nu \\
&=: (\mathcal{B}_\mathcal{X})_\mathcal{Q}.
\end{aligned}$$

□

Proposition 5.15. *Let* $(\Omega, \mathcal{B}_\Omega)$ *be a measurable space and* $\mathcal{P}$ *be a set of* perfect *probability measures on* $(\Omega, \mathcal{B}_\Omega)$*. Let* $(\mathcal{X}, \mathcal{B}_\mathcal{X})$ *be a* countably separated *measurable space and* $\mathcal{E} \subseteq \mathcal{B}_\mathcal{X}$ *be any countable subset that separates the points of* $\mathcal{X}$*. Let* $\mathcal{F} \subseteq \mathbf{Meas}\left((\Omega, \mathcal{B}_\Omega), (\mathcal{X}, \mathcal{B}_\mathcal{X})\right)$ *be any subsets of measurable maps. We abbreviate the set of push-forward measures:*

$$\mathcal{P}(\mathcal{F}) := \{\alpha_* \mu : \mathcal{B}_\mathcal{X} \to [0,1] \,|\, \alpha \in \mathcal{F}, \mu \in \mathcal{P}\}.$$

Then we have the equalities:

$$(\mathcal{E})_{\mathcal{P}(\mathcal{F})} = (\mathcal{B}_\mathcal{X})_{\mathcal{P}(\mathcal{F})} = \mathcal{B}(\mathcal{F})_{\mathcal{P}(\mathcal{F})},$$

where the index $\mathcal{P}(\mathcal{F})$ *refers to the intersections of all (Lebesgue) completions w.r.t.* $\nu \in \mathcal{P}(\mathcal{F})$*.*
If, furthermore, we have that $\mathcal{B}_\Omega$ *is complete w.r.t.* $\mathcal{P}$*, i.e. that:*

$$\mathcal{B}_\Omega = (\mathcal{B}_\Omega)_\mathcal{P} := \bigcap_{\mu \in \mathcal{P}} (\mathcal{B}_\Omega)_\mu,$$

then we also get the equality:

$$\mathcal{B}(\mathcal{F}) = (\mathcal{E})_{\mathcal{P}(\mathcal{F})}.$$

Proof. We clearly have the chain of inclusions and equalities:

$$\begin{aligned}
\mathcal{E} \subseteq \mathcal{B}_{\mathcal{X}} \subseteq \mathcal{B}(\mathcal{F}) &:= \left\{ A \subseteq \mathcal{X} \,\middle|\, \forall \alpha \in \mathcal{F}.\, \alpha^{-1}(A) \in \mathcal{B}_\Omega \right\} \\
&= \bigcap_{\alpha \in \mathcal{F}} \alpha_* \mathcal{B}_\Omega \\
&\subseteq \bigcap_{\alpha \in \mathcal{F}} \alpha_* (\mathcal{B}_\Omega)_{\mathcal{P}} \\
&= \bigcap_{\alpha \in \mathcal{F}} \alpha_* \bigcap_{\mu \in \mathcal{P}} (\mathcal{B}_\Omega)_\mu \\
&= \bigcap_{\alpha \in \mathcal{F}} \bigcap_{\mu \in \mathcal{P}} \alpha_* (\mathcal{B}_\Omega)_\mu \\
&= \bigcap_{\alpha \in \mathcal{F}} \bigcap_{\mu \in \mathcal{P}} (\mathcal{E})_{\alpha_* \mu} \\
&= \bigcap_{\nu \in \mathcal{P}(\mathcal{F})} (\mathcal{E})_\nu \\
&= (\mathcal{E})_{\mathcal{P}(\mathcal{F})} ,
\end{aligned}$$

where the equality: $\alpha_* (\mathcal{B}_\Omega)_\mu = (\mathcal{E})_{\alpha_* \mu}$ comes from Lemma 5.13 and the fact that $\mathcal{E}$ is countably separated and that every $\alpha \in \mathcal{F}$ is a $\mathcal{B}_\Omega$-$\sigma(\mathcal{E})$-measurable map and that every $\mu \in \mathcal{P}$ is perfect. Together we get the inclusions:

$$\mathcal{E} \subseteq \mathcal{B}_{\mathcal{X}} \subseteq \mathcal{B}(\mathcal{F}) \subseteq (\mathcal{E})_{\mathcal{P}(\mathcal{F})} ,$$

where the latter is an equality if $\mathcal{B}_\Omega = (\mathcal{B}_\Omega)_{\mathcal{P}}$. Taking the completions w.r.t. $\mathcal{P}(\mathcal{F})$ we get:

$$(\mathcal{E})_{\mathcal{P}(\mathcal{F})} \subseteq (\mathcal{B}_{\mathcal{X}})_{\mathcal{P}(\mathcal{F})} \subseteq \mathcal{B}(\mathcal{F})_{\mathcal{P}(\mathcal{F})} \subseteq (\mathcal{E})_{\mathcal{P}(\mathcal{F})} ,$$

which imply the equalities:

$$(\mathcal{E})_{\mathcal{P}(\mathcal{F})} = (\mathcal{B}_{\mathcal{X}})_{\mathcal{P}(\mathcal{F})} = \mathcal{B}(\mathcal{F})_{\mathcal{P}(\mathcal{F})}.$$

This shows the claim. □

Theorem 5.16. *Let $(\mathcal{X}, \mathcal{X}^\Omega)$ be a countably separated quasi-universal space. Then we have:*

$$\mathcal{B}(\mathcal{X}^\Omega) = (\mathcal{E})_{\mathcal{P}(\mathcal{X}, \mathcal{X}^\Omega)} ,$$

for every countable subset $\mathcal{E} \subseteq \mathcal{B}(\mathcal{X}^\Omega)$ that separates the points of $\mathcal{X}$, where:

$$\mathcal{P}(\mathcal{X}, \mathcal{X}^\Omega) := \left\{ \alpha_* \nu : \ \mathcal{B}(\mathcal{X}^\Omega) \to [0,1] \,\middle|\, \alpha \in \mathcal{X}^\Omega, \nu \in \mathcal{G}(\Omega, \mathcal{B}_\Omega)^{6} \right\} .$$

Proof. This follows directly from Proposition 5.15 together with the fact that every probability measure on $(\Omega, \mathcal{B}_\Omega)$ is perfect and $(\Omega, \mathcal{B}_\Omega)$ is universally complete. □

Corollary 5.17. *Let $(\mathcal{X}, \mathcal{X}^\Omega)$ be a countably separated quasi-universal space. Then the quasi-universal space of probability measures:*

$$\mathcal{P}(\mathcal{X}, \mathcal{X}^\Omega)$$

is countably separated as well.

Proof. By Theorem 5.16 we have that $\mathcal{B}(\mathcal{X}^\Omega) = (\mathcal{E})_{\mathcal{P}(\mathcal{X},\mathcal{X}^\Omega)}$ for a countable algebra $\mathcal{E} \subseteq \mathcal{B}(\mathcal{X}^\Omega)$. Then we have that:

$$\mathcal{E}_{\mathcal{P}} := \left\{ \mathrm{ev}_A^{-1}\left((t,1]\right) \,\middle|\, A \in \mathcal{E}, t \in \mathbb{Q} \right\} \subseteq \mathcal{B}\left(\mathcal{P}(\mathcal{X}, \mathcal{X}^\Omega)^\Omega\right)$$

is countable and separates the points of $\mathcal{P}(\mathcal{X}, \mathcal{X}^\Omega)$. Indeed, two measures μ_1, μ_2 agree on $\sigma(\mathcal{E})$ if they agree on the generating π-system $\mathcal{E}$. Since $\mu_1, \mu_2 \in \mathcal{P}(\mathcal{X}, \mathcal{X}^\Omega)$ they uniquely extend to $\mathcal{B}(\mathcal{X}^\Omega) = (\mathcal{E})_{\mathcal{P}(\mathcal{X},\mathcal{X}^\Omega)}$ and thus are then equal there as well.
Note that the above set $\mathcal{E}_{\mathcal{P}}$ really lies inside the σ-algebra $\mathcal{B}\left(\mathcal{P}(\mathcal{X}, \mathcal{X}^\Omega)^\Omega\right)$ as the evaluation maps ev_A are measurable. ☐

5.4 Universal (Quasi-)Measurable Spaces

In this section we will gather properties of *universal measurable spaces* and *universal quasi-universal spaces*. The former were studied in [For21]. They will be the most well-behaved quasi-universal spaces and play a similar role in the category of quasi-universal spaces **QUS** as standard Borel measurable spaces play inside the category of measurable spaces **Meas** or quasi-Borel spaces **QBS**. They also allow us to translate (some) results from **Meas** to **QUS**. We will see that this mainly entails checking which properties of standard Borel measurable spaces are preserved under universal completion.

5.4.1 Universal Measurable Spaces

Here we quickly review *universal measurable spaces*, see [For21] Appendix B, which are measurable spaces that are isomorphic to a universally measurable subset of $\mathbb{R}$, thus generalizing standard Borel measurable spaces and analytic measurable spaces. One design choice one has to make is if one explicitely wants their σ-algebra, per definition, to be universally complete or if one implicitely builds this into the definition. Since we do not want to study more (co)reflexive subcategories of **Meas**, here given by the universal completion, and we want to be able to say: "Standard and analytic measurable spaces are universal." and "Countable products of universal measurable spaces are universal." (without changing the product-σ-algebra), etc., we allow for non-universally complete universal measurable spaces, which then get their convenient properties only after universal completion.

Definition/Lemma 5.18 (Universal measurable spaces, see [For21] Appendix B)**.** *Let $\mathcal{B}_{\mathbb{R}}$ be the Borel σ-algebra on $\mathbb{R}$ and $(\Omega, \mathcal{B}_\Omega) = (\mathbb{R}, (\mathcal{B}_{\mathbb{R}})_{\mathcal{G}})$ its universal completion. A measurable space $(\mathcal{X}, \mathcal{B}_{\mathcal{X}})$ is called* universal measurable space *if it satisfies any of the following equivalent statements:*

1. *There exist* $\mathcal{Z} \in \mathcal{B}_\Omega$ *and an isomorphism of measurable spaces:*

$$(\mathcal{X}, (\mathcal{B}_\mathcal{X})_\mathcal{G}) \cong (\mathcal{Z}, \mathcal{B}_{\Omega|\mathcal{Z}}).$$

2. $(\mathcal{X}, \mathcal{B}_\mathcal{X})$ *is universally countably generated, (universally countably) separated and perfect (i.e. every probability measure* $\mu \in \mathcal{G}(\mathcal{X}, \mathcal{B}_\mathcal{X})$ *is perfect).*

3. *There exists a countable subset* $\mathcal{E} \subseteq (\mathcal{B}_\mathcal{X})_\mathcal{G}$ *that separates the points of* $\mathcal{X}$ *such that* $(\mathcal{E})_\mathcal{G} = (\mathcal{B}_\mathcal{X})_\mathcal{G}$ *and every probability measure* $\mu \in \mathcal{G}(\mathcal{X}, \mathcal{E})$ *is perfect.*

4. $(\mathcal{X}, (\mathcal{B}_\mathcal{X})_\mathcal{G})$ *is a retract of* $(\Omega, \mathcal{B}_\Omega)$ *in* **Meas***, i.e. there are measurable maps* $i : (\mathcal{X}, (\mathcal{B}_\mathcal{X})_\mathcal{G}) \to (\Omega, \mathcal{B}_\Omega)$ *and* $r : (\Omega, \mathcal{B}_\Omega) \to (\mathcal{X}, (\mathcal{B}_\mathcal{X})_\mathcal{G})$ *such that* $r \circ i = \mathrm{id}_\mathcal{X}$.

Proof. 2. $\iff$ 3.: This is just an explicit reformulation. Note that if $(\mathcal{E})_\mathcal{G} = (\mathcal{B}_\mathcal{X})_\mathcal{G}$ then we have the equality of sets:

$$\mathcal{G}(\mathcal{X}, \mathcal{E}) = \mathcal{G}(\mathcal{X}, (\mathcal{E})_\mathcal{G}) = \mathcal{G}(\mathcal{X}, (\mathcal{B}_\mathcal{X})_\mathcal{G}) = \mathcal{G}(\mathcal{X}, \mathcal{B}_\mathcal{X}).$$

Also note that a probability measure is perfect if and only if its completion is perfect.
1. $\implies$ 4.: Let i be the composition of the measurable isomorphism and the measurable inclusion map:

$$i : (\mathcal{X}, (\mathcal{B}_\mathcal{X})_\mathcal{G}) \cong (\mathcal{Z}, \mathcal{B}_{\Omega|\mathcal{Z}}) \hookrightarrow (\Omega, \mathcal{B}_\Omega).$$

Define the map r via:

$$r : (\Omega, \mathcal{B}_\Omega) \to (\mathcal{X}, (\mathcal{B}_\mathcal{X})_\mathcal{G}), \qquad r|_\mathcal{Z} := i^{-1}, \qquad r|_{\Omega \setminus \mathcal{Z}} := x_0 \in \mathcal{X}.$$

Since $\mathcal{Z} \in \mathcal{B}_\Omega$ the map r is measurable. It is clear that $r \circ i = \mathrm{id}_\mathcal{X}$. This shows that $(\mathcal{X}, (\mathcal{B}_\mathcal{X})_\mathcal{G})$ is a retract of $(\Omega, \mathcal{B}_\Omega)$.
4. $\implies$ 3.: Put $\mathcal{E} := i^*\mathcal{B}_\mathbb{R} \subseteq (\mathcal{B}_\mathcal{X})_\mathcal{G}$. Then $\mathcal{E}$ is countably generated and separates the points of $\mathcal{X}$ since i is injective. So $(\mathcal{B}_\mathcal{X})_\mathcal{G}$ is universally countably separated. Let $\mu \in \mathcal{G}(\mathcal{X}, \mathcal{E})$. Furthermore, i is then $\mathcal{E}$-$\mathcal{B}_\mathbb{R}$-measurable and thus $(\mathcal{E})_\mathcal{G}$-$\mathcal{B}_\Omega$-measurable. Note that r is $\mathcal{B}_\Omega$-$(\mathcal{B}_\mathcal{X})_\mathcal{G}$-measurable. Together we get for $A \in (\mathcal{B}_\mathcal{X})_\mathcal{G}$ that $r^{-1}(A) \in \mathcal{B}_\Omega$ and thus:

$$A = \mathrm{id}_\mathcal{X}^{-1}(A) = i^{-1}(r^{-1}(A)) \in (\mathcal{E})_\mathcal{G}.$$

This shows the inclusions:

$$(\mathcal{B}_\mathcal{X})_\mathcal{G} \subseteq (\mathcal{E})_\mathcal{G} \subseteq (\mathcal{B}_\mathcal{X})_\mathcal{G},$$

and thus equality. This shows that $(\mathcal{X}, \mathcal{B}_\mathcal{X})$ is universally countably generated. Let $\mu \in \mathcal{G}(\mathcal{X}, (\mathcal{B}_\mathcal{X})_\mathcal{G})$. Then $i_*\mu \in \mathcal{G}(\Omega, \mathcal{B}_\Omega)$. Since all probability measures on $(\Omega, \mathcal{B}_\Omega)$ are perfect so is $i_*\mu$. Since also push-forwards of perfect probability measures are perfect also:

$$\mu = \mathrm{id}_{\mathcal{X},*}\mu = r_*(i_*\mu),$$

is perfect. This shows that all probability measures on $(\mathcal{X}, \mathcal{B}_\mathcal{X})$ are perfect.
3. $\implies$ 1.: Let $\mathcal{E} \subseteq (\mathcal{B}_\mathcal{X})_\mathcal{G}$ be a countably generated σ-algebra that separates the points

of $\mathcal{X}$ such that $(\mathcal{E})_{\mathcal{G}} = (\mathcal{B}_{\mathcal{X}})_{\mathcal{G}}$ and $(\mathcal{X}, \mathcal{E})$ is perfect. $\mathcal{E}$ then induces an injective $\mathcal{E}$-$\mathcal{B}_{\mathbb{R}}$-measurable map $i: \mathcal{X} \to \Omega$ such that $\mathcal{E} = i^*\mathcal{B}_{\mathbb{R}}$ (see [Bog07] Thm. 6.5.8.). If we put $\mathcal{Z} := i(\mathcal{X})$ then i induces an isomorphism of measurable spaces:

$$(\mathcal{X}, \mathcal{E}) \cong (\mathcal{Z}, \mathcal{B}_{\mathbb{R}|\mathcal{Z}}).$$

Since $(\mathcal{X}, \mathcal{E})$ and thus $(\mathcal{Z}, \mathcal{B}_{\mathbb{R}|\mathcal{Z}})$ is perfect we get that $\mathcal{Z} := i(\mathcal{X}) \in \mathcal{B}_{\Omega}$ (see [Bog07] Thm. 7.5.7 or [Dar71, Saz62] Lem. 3). With $\mathcal{Z} \in \mathcal{B}_{\Omega}$ we get that $(\mathcal{B}_{\mathbb{R}|\mathcal{Z}})_{\mathcal{G}} = \mathcal{B}_{\Omega|\mathcal{Z}}$ and thus an measurable isomorphism:

$$(\mathcal{X}, (\mathcal{E})_{\mathcal{G}}) \cong (\mathcal{Z}, (\mathcal{B}_{\mathbb{R}|\mathcal{Z}})_{\mathcal{G}}) = (\mathcal{Z}, \mathcal{B}_{\Omega|\mathcal{Z}}),$$

with $\mathcal{Z} \in \mathcal{B}_{\Omega}$.
This shows all claims. □

Remark 5.19 (See [For21] Appendix B)**.** 1. *The countable product of universal measurable spaces is a universal measurable space (with the product σ-algebra, i.e. in* **Meas***).*

2. *The countable coproduct (in* **Meas***) of universal measurable spaces is a universal measurable space.*

3. *Every universally measurable subset of a universal measurable space is a universal measurable space.*

4. $(\Omega, \mathcal{B}_{\Omega}) = (\mathbb{R}, (\mathcal{B}_{\mathbb{R}})_{\mathcal{G}})$ *is a universal measurable space.*

5. *Every standard (Borel) measurable space and every analytic measurable space is a universal measurable space.*

6. *Every countable discrete measurable space is a universal measurable space.*

7. *Every Radon Hausdorff space that has a countable network of universally measurable subsets is a universal measurable space together with (the universal completion of) its Borel σ-algebra.*

8. *Every Polish space is a universal measurable space together with (the universal completion of) its Borel σ-algebra.*

Notation 5.20 (The category of universally complete universal measurable spaces)**.** *We abbreviate the full subcategory of all universally complete universal measurable spaces inside* **Meas** *as* **UMeas***.*

Lemma 5.21. *Let $(\mathcal{X}, \mathcal{B}_{\mathcal{X}})$ be a universal measurable space then we get that:*

$$(\mathcal{X}, \mathcal{B}\mathcal{F}(\mathcal{B}_{\mathcal{X}})) = (\mathcal{X}, (\mathcal{B}_{\mathcal{X}})_{\mathcal{G}})$$

is the universal completion of $(\mathcal{X}, \mathcal{B}_{\mathcal{X}})$. Furthermore, we have the equality of sets:

$$\mathcal{G}(\mathcal{X}, \mathcal{B}_{\mathcal{X}}) = \mathcal{G}(\mathcal{X}, \mathcal{X}^{\Omega}) = \mathcal{P}(\mathcal{X}, \mathcal{X}^{\Omega}) := \left\{\alpha_*\nu : \mathcal{B}(\mathcal{X}^{\Omega}) \to [0,1] \,\middle|\, \alpha \in \mathcal{X}^{\Omega}, \nu \in \mathcal{G}(\Omega, \mathcal{B}_{\Omega})^{6}\right\},$$

where we put: $\mathcal{X}^{\Omega} := \mathcal{F}(\mathcal{B}_{\mathcal{X}})$.

Proof. By Definition/Lemma 5.18 there are measurable maps:

$$\mathrm{id}_{\mathcal{X}}:\ (\mathcal{X},(\mathcal{B}_{\mathcal{X}})_{\mathcal{G}}) \xrightarrow{i} (\Omega,\mathcal{B}_\Omega) \xrightarrow{r} (\mathcal{X},(\mathcal{B}_{\mathcal{X}})_{\mathcal{G}}),$$

which show the inclusions and thus equalities:

$$(\mathcal{B}_{\mathcal{X}})_{\mathcal{G}} \supseteq i^*\mathcal{B}_\Omega \supseteq i^*r^*(\mathcal{B}_{\mathcal{X}})_{\mathcal{G}} = \mathrm{id}_{\mathcal{X}}^*(\mathcal{B}_{\mathcal{X}})_{\mathcal{G}} = (\mathcal{B}_{\mathcal{X}})_{\mathcal{G}}.$$

Applying the functor $\mathcal{BF}$ to the above gives the measurable maps:

$$\mathrm{id}_{\mathcal{X}}:\ (\mathcal{X},\mathcal{BF}(\mathcal{B}_{\mathcal{X}})) \xrightarrow{i} (\Omega,\mathcal{B}_\Omega) \xrightarrow{r} (\mathcal{X},\mathcal{BF}(\mathcal{B}_{\mathcal{X}})).$$

For the latter note that since $\mathcal{B}_\Omega$ is universally complete we have $\mathcal{F}(\mathcal{B}_{\mathcal{X}}) = \mathcal{F}((\mathcal{B}_{\mathcal{X}})_{\mathcal{G}})$, and also: $\mathcal{BF}(\mathcal{B}_\Omega) = \mathcal{BFB}(\Omega^\Omega) = \mathcal{B}(\Omega^\Omega) = \mathcal{B}_\Omega$.
So we get the inclusions and thus equalities:

$$\mathcal{BF}(\mathcal{B}_{\mathcal{X}}) \supseteq i^*\mathcal{B}_\Omega \supseteq i^*r^*\mathcal{BF}(\mathcal{B}_{\mathcal{X}}) = \mathrm{id}_{\mathcal{X}}^*\mathcal{BF}(\mathcal{B}_{\mathcal{X}}) = \mathcal{BF}(\mathcal{B}_{\mathcal{X}}).$$

Together we then get:

$$(\mathcal{B}_{\mathcal{X}})_{\mathcal{G}} = i^*\mathcal{B}_\Omega = \mathcal{BF}(\mathcal{B}_{\mathcal{X}}).$$

This shows the first claim.
Now let $\mathcal{X}^\Omega := \mathcal{F}(\mathcal{B}_{\mathcal{X}})$ and $\mu \in \mathcal{G}(\mathcal{X},\mathcal{X}^\Omega)$. Then we get:

$$\mu = r_*(i_*\mu) \in \mathcal{P}(\mathcal{X},\mathcal{X}^\Omega) \subseteq \mathcal{G}(\mathcal{X},\mathcal{X}^\Omega),$$

since $r \in \mathcal{F}(\mathcal{B}_{\mathcal{X}}) = \mathcal{X}^\Omega$ and $i_*\mu \in \mathcal{G}(\Omega,\mathcal{B}_\Omega)$. This shows:

$$\mathcal{G}(\mathcal{X},\mathcal{X}^\Omega) = \mathcal{P}(\mathcal{X},\mathcal{X}^\Omega).$$

This shows the second claim. □

5.4.2 Universal Quasi-Universal Spaces

Here we now give a definition for *universal quasi-universal spaces*, which are now objects in the category of quasi-universal spaces **QUS**. We then show how they correspond to universal measurable spaces. This then allows us to translate properties of standard Borel measurable spaces to universal measurable spaces to universal quasi-universal spaces.

Definition/Lemma 5.22. *A quasi-universal space* $(\mathcal{X},\mathcal{X}^\Omega)$ *will be called* universal quasi-universal space *if it satisfies any of the following equivalent conditions:*

1. $(\mathcal{X},\mathcal{B}(\mathcal{X}^\Omega))$ *is a (universally complete) universal measurable space, see Definition/Lemma 5.18, and* $\mathcal{FB}(\mathcal{X}^\Omega) = \mathcal{X}^\Omega$.
2. *There exists a* σ*-algebra* $\mathcal{B}_{\mathcal{X}}$ *on* $\mathcal{X}$ *such that* $(\mathcal{X},\mathcal{B}_{\mathcal{X}})$ *is a universal measurable space, see Definition/Lemma 5.18, and* $\mathcal{X}^\Omega = \mathcal{F}(\mathcal{B}_{\mathcal{X}})$.

3. *$(\mathcal{X}, \mathcal{X}^\Omega)$ is a retract of (Ω, Ω^Ω) in* **QUS**, *i.e. there exist quasi-measurable maps $i: (\mathcal{X}, \mathcal{X}^\Omega) \to (\Omega, \Omega^\Omega)$ and $r: (\Omega, \Omega^\Omega) \to (\mathcal{X}, \mathcal{X}^\Omega)$ such that $r \circ i = \mathrm{id}_\mathcal{X}$.*

Proof. 3. $\Longrightarrow$ 1.: Consider the composition of quasi-measurable maps:

$$\mathrm{id}_\mathcal{X}: (\mathcal{X}, \mathcal{X}^\Omega) \xrightarrow{i} (\Omega, \Omega^\Omega) \xrightarrow{r} (\mathcal{X}, \mathcal{X}^\Omega).$$

This gives us the inclusions and thus equalities:

$$\mathcal{X}^\Omega = \mathrm{id}_\mathcal{X} \circ \mathcal{X}^\Omega = r \circ i \circ \mathcal{X}^\Omega \subseteq r \circ \Omega^\Omega \subseteq \mathcal{X}^\Omega.$$

Applying the functor $\mathcal{B}$ to the above gives the composition of measurable maps:

$$\mathrm{id}_\mathcal{X}: (\mathcal{X}, \mathcal{B}(\mathcal{X}^\Omega)) \xrightarrow{i} (\Omega, \mathcal{B}_\Omega) \xrightarrow{r} (\mathcal{X}, \mathcal{B}(\mathcal{X}^\Omega)).$$

So $(\mathcal{X}, \mathcal{B}(\mathcal{X}^\Omega))$ is a retract of $(\Omega, \mathcal{B}_\Omega)$ in **Meas** and universally complete by Lemma 5.5, showing that $(\mathcal{X}, \mathcal{B}(\mathcal{X}^\Omega))$ is a (universally complete) universal measurable space. Again, applying the functor $\mathcal{F}$ to the above gives the composition of quasi-measurable maps:

$$\mathrm{id}_\mathcal{X}: (\mathcal{X}, \mathcal{F}\mathcal{B}(\mathcal{X}^\Omega)) \xrightarrow{i} (\Omega, \Omega^\Omega) \xrightarrow{r} (\mathcal{X}, \mathcal{F}\mathcal{B}(\mathcal{X}^\Omega)).$$

This gives us the inclusions and thus equalities:

$$\mathcal{F}\mathcal{B}(\mathcal{X}^\Omega) = \mathrm{id}_\mathcal{X} \circ \mathcal{F}\mathcal{B}(\mathcal{X}^\Omega) = r \circ i \circ \mathcal{F}\mathcal{B}(\mathcal{X}^\Omega) \subseteq r \circ \Omega^\Omega \subseteq \mathcal{F}\mathcal{B}(\mathcal{X}^\Omega).$$

Together we get:

$$\mathcal{X}^\Omega = r \circ \Omega^\Omega = \mathcal{F}\mathcal{B}(\mathcal{X}^\Omega).$$

1. $\Longrightarrow$ 2.: Pick $\mathcal{B}_\mathcal{X} := \mathcal{B}(\mathcal{X}^\Omega)$.
2. $\Longrightarrow$ 1.: If $(\mathcal{X}, \mathcal{B}_\mathcal{X})$ is a universal measurable space with $\mathcal{X}^\Omega = \mathcal{F}(\mathcal{B}_\mathcal{X})$ then by Lemma 5.21 we have that $(\mathcal{X}, \mathcal{B}(\mathcal{X}^\Omega)) = (\mathcal{X}, \mathcal{B}\mathcal{F}(\mathcal{B}_\mathcal{X}))$ is a (universally complete) universal measurable space and $(\mathcal{B}_\mathcal{X})_\mathcal{G} = \mathcal{B}\mathcal{F}(\mathcal{B}_\mathcal{X})$. Applying $\mathcal{F}$ to the latter we get:

$$\mathcal{X}^\Omega = \mathcal{F}(\mathcal{B}_\mathcal{X}) = \mathcal{F}((\mathcal{B}_\mathcal{X})_\mathcal{G}) = \mathcal{F}\mathcal{B}\mathcal{F}(\mathcal{B}_\mathcal{X}) = \mathcal{F}\mathcal{B}(\mathcal{X}^\Omega).$$

1. $\Longrightarrow$ 3.: Since $(\mathcal{X}, \mathcal{B}(\mathcal{X}^\Omega))$ is a (univerally complete) universal measurable space we by Definition/Lemma 5.18 have a composition of measurable spaces:

$$\mathrm{id}_\mathcal{X}: (\mathcal{X}, \mathcal{B}(\mathcal{X}^\Omega)) \xrightarrow{i} (\Omega, \mathcal{B}_\Omega) \xrightarrow{r} (\mathcal{X}, \mathcal{B}(\mathcal{X}^\Omega)).$$

Applying the functor $\mathcal{F}$ to it and expoiting $\mathcal{X}^\Omega = \mathcal{F}\mathcal{B}(\mathcal{X}^\Omega)$ gives a composition of quasi-measurable maps:

$$\mathrm{id}_\mathcal{X}: (\mathcal{X}, \mathcal{X}^\Omega) = (\mathcal{X}, \mathcal{F}\mathcal{B}(\mathcal{X}^\Omega)) \xrightarrow{i} (\Omega, \Omega^\Omega) \xrightarrow{r} (\mathcal{X}, \mathcal{F}\mathcal{B}(\mathcal{X}^\Omega)) = (\mathcal{X}, \mathcal{X}^\Omega).$$

This shows that $(\mathcal{X}, \mathcal{X}^\Omega)$ is a retract of (Ω, Ω^Ω) in **QUS**. □

Lemma 5.23. *Let $(\mathcal{X}, \mathcal{X}^\Omega)$ be a universal quasi-universal space. Then we have the equality of sets:*

$$\mathcal{G}(\mathcal{X}, \mathcal{X}^\Omega) = \mathcal{P}(\mathcal{X}, \mathcal{X}^\Omega) := \left\{\alpha_* \nu : \mathcal{B}(\mathcal{X}^\Omega) \to [0,1] \,\middle|\, \alpha \in \mathcal{X}^\Omega, \nu \in \mathcal{G}(\Omega, \mathcal{B}_\Omega)^{6}\right\}.$$

Proof. This follows from Definition/Lemma 5.22 and Lemma 5.21. □

Lemma 5.24. *Let J be a countable set (e.g. $J = \mathbb{N}$). Then there is an isomorphism of quasi-universal spaces (in* **QUS***):*

$$(\Omega, \Omega^\Omega) \cong \prod_{j \in J} (\Omega, \Omega^\Omega).$$

Proof. Since J is countable by [Fre15] 424C there exists an isomorphism of measurable spaces (in **Meas**):

$$(\mathbb{R}, \mathcal{B}_\mathbb{R}) \cong \prod_{j \in J} (\mathbb{R}, \mathcal{B}_\mathbb{R}) = \left(\prod_{j \in J} \mathbb{R}, \bigotimes_{j \in J} \mathcal{B}_\mathbb{R}\right).$$

We now apply the functor $\mathcal{F}$ and exploit that $\mathcal{F}$ as a right-adjoint preserves products, see Lemma 2.27:

$$(\mathbb{R}, \mathcal{F}(\mathcal{B}_\mathbb{R})) \cong \left(\prod_{j \in J} \mathbb{R}, \mathcal{F}\left(\bigotimes_{j \in J} \mathcal{B}_\mathbb{R}\right)\right) = \left(\prod_{j \in J} \mathbb{R}, \prod_{j \in J} \mathcal{F}(\mathcal{B}_\mathbb{R})\right).$$

Since $(\Omega, \mathcal{B}_\Omega) = (\mathbb{R}, (\mathcal{B}_\mathbb{R})_\mathcal{G})$ is universally complete we have:

$$\mathcal{F}(\mathcal{B}_\mathbb{R}) = \mathcal{F}((\mathcal{B}_\mathbb{R})_\mathcal{G}) = \mathcal{F}(\mathcal{B}_\Omega) = \Omega^\Omega.$$

This gives us the desired isomorphism of quasi-universal spaces:

$$(\Omega, \Omega^\Omega) = (\mathbb{R}, \mathcal{F}(\mathcal{B}_\mathbb{R})) \cong \left(\prod_{j \in J} \mathbb{R}, \prod_{j \in J} \mathcal{F}(\mathcal{B}_\mathbb{R})\right) = \left(\prod_{j \in J} \Omega, \prod_{j \in J} \Omega^\Omega\right) = \prod_{j \in J} (\Omega, \Omega^\Omega).$$

This shows the claim. □

Lemma 5.25 (Countable products of universal quasi-universal spaces)**.** *Let $(\mathcal{X}_j, \mathcal{X}_j^\Omega)$ be universal quasi-universal spaces for $j \in J$ for a countable index set J. Then the product of quasi-universal spaces (in* **QUS***):*

$$\prod_{j \in J} (\mathcal{X}_j, \mathcal{X}_j^\Omega) = \left(\prod_{j \in J} \mathcal{X}_j, \prod_{j \in J} \mathcal{X}_j^\Omega\right)$$

is again a universal quasi-universal space.

Proof. By Definition/Lemma 5.22 we have compositions of quasi-measurable maps for $j \in J$:

$$\mathrm{id}_{\mathcal{X}_j} : (\mathcal{X}_j, \mathcal{X}_j^\Omega) \xrightarrow{i_j} (\Omega, \Omega^\Omega) \xrightarrow{r_j} (\mathcal{X}_j, \mathcal{X}_j^\Omega).$$

Taking the product of all those maps for $j \in J$ gives the composition of quasi-measurable maps:

$$\mathrm{id}_{\prod_{j\in J} \mathcal{X}_j} : \prod_{j\in J}(\mathcal{X}_j, \mathcal{X}_j^\Omega) \xrightarrow{(i_j)_{j\in J}} \prod_{j\in J}(\Omega, \Omega^\Omega) \xrightarrow{(r_j)_{j\in J}} \prod_{j\in J}(\mathcal{X}_j, \mathcal{X}_j^\Omega).$$

Since J is countable by Lemma 5.24 we now have an isomorphism of quasi-universal spaces:

$$\prod_{j\in J}(\Omega, \Omega^\Omega) \cong (\Omega, \Omega^\Omega),$$

showing together with the above that $\prod_{j\in J}(\mathcal{X}_j, \mathcal{X}_j^\Omega)$ is a retract of (Ω, Ω^Ω) in **QUS**. This shows the claim that $\prod_{j\in J}(\mathcal{X}_j, \mathcal{X}_j^\Omega)$ is also a universal quasi-universal space. ☐

Remark 5.26 (Countable coproducts of universal quasi-universal spaces)**.** *The countable coproduct of universal measurable spaces is a universal measurable space. The functor $\mathcal{F}$ only preserves countable coproducts if one reflects back to the category of patchable quasi-universal spaces* **PQUS***, see Lemma 4.12. So we get that if $(\mathcal{X}_i, \mathcal{X}_i^\Omega)$ is a countable family of universal quasi-universal spaces then*

$$(\mathcal{X}, \mathcal{X}^\Omega) := \coprod_{i\in I}^{\mathbf{PQUS}} (\mathcal{X}_i, \mathcal{X}_i^\Omega) = \mathcal{L}\left(\coprod_{i\in I}^{\mathbf{QUS}} (\mathcal{X}_i, \mathcal{X}_i^\Omega)\right) = \left(\coprod_{i\in I} \mathcal{X}_i, \mathcal{L}\left(\coprod_{i\in I} \mathcal{X}_i^\Omega\right)\right)$$

is also a universal quasi-universal space. In other words, the countable **PQUS***-coproduct of universal quasi-universal spaces is a universal quasi-universal space.*

Corollary 5.27. *The functors $\mathcal{F}$ and $\mathcal{B}$ from Theorem 2.16 establish an equivalence between the full subcategory of universally complete universal measurable spaces* **UMeas** *inside* **Meas** *and the full subcategory of universal quasi-universal spaces* **UQUS** *inside* **QUS***.*

Proof. This immediately follows from Theorem 2.16 and Lemmata 5.18, 5.21, and 5.22. Note that $\mathcal{BF}$ and $\mathcal{FB}$ do not change the maps. Furthermore, $\mathcal{BF} = \mathrm{id}_{\mathbf{UMeas}}$, since by Lemma 5.21:

$$\mathcal{BF}(\mathcal{B}_\mathcal{X}) = (\mathcal{B}_\mathcal{X})_\mathcal{G} = \mathcal{B}_\mathcal{X},$$

if $(\mathcal{X}, \mathcal{B}_\mathcal{X})$ is a universally complete universal measurable space. We also have that $\mathcal{FB} = \mathrm{id}_{\mathbf{UQUS}}$, since:

$$\mathcal{FB}(\mathcal{X}^\Omega) = \mathcal{X}^\Omega,$$

by Definition/Lemma 5.22 of universal quasi-universal spaces. ☐

5.5 Spaces of Probability Measures for Quasi-Universal Spaces

This subsection builds on the general Subsection 3.3 of the probability monads of push-forward probability measures. We will show here that the studied (strong) probability monads $\mathcal{K}$, $\mathcal{P}$, $\mathcal{R}$, $\mathcal{S}$ will all agree on the category of quasi-universal spaces **QUS**, see Theorem 5.29. We then go on to show in Theorem 5.30 that for *universal* quasi-universal spaces all defined probability spaces even agree with $\mathcal{G}$ and $\mathcal{Q}$, as sets and as quasi-measurable spaces, which allows us to translate properties of standard Borel measurable spaces to universal quasi-universal spaces.

Lemma 5.28. *Let $(\mathcal{X}, \mathcal{X}^\Omega)$ be a universal quasi-universal space (e.g. $(\mathcal{X}, \mathcal{X}^\Omega) = (\Omega, \Omega^\Omega)$) then we have the equality of quasi-universal spaces:*

$$\mathcal{G}(\mathcal{X}, \mathcal{X}^\Omega) = \mathcal{Q}(\mathcal{X}, \mathcal{X}^\Omega).$$

Proof. The inclusion $\mathcal{Q}(\mathcal{X}, \mathcal{X}^\Omega) \subseteq \mathcal{G}(\mathcal{X}, \mathcal{X}^\Omega)$ is quasi-measurable by Remark 3.5. Now let $\mu \in \mathcal{G}(\mathcal{X}, \mathcal{X}^\Omega)$ and $\kappa \in \mathcal{G}(\mathcal{X}, \mathcal{X}^\Omega)^\Omega$. For $D \in \mathcal{B}_{\Omega\times\mathcal{X}}$ and $\varphi \in \Omega^\Omega$ we need to show that the maps:

$$\Omega \to [0,1], \qquad \omega \mapsto \mu(D_\omega); \qquad \omega \mapsto \kappa(\varphi(\omega))(D_\omega),$$

are $\mathcal{B}_\Omega$-$\mathcal{B}_{[0,1]}$-measurable. Since the former can be viewed as a special case of the latter we only focus on the κ case. The latter map can then be factorized as:

$$h:\ \Omega \xrightarrow{\varphi\times\kappa} \Omega \times \mathcal{G}(\mathcal{X}, \mathcal{B}_\mathcal{X}) \xrightarrow{\delta\times(\mathrm{id}_\mathcal{X})_*} \mathcal{G}(\Omega \times \mathcal{X}, \mathcal{B}_\Omega \otimes \mathcal{B}_\mathcal{X}) \xrightarrow{\mathrm{ev}_D} [0,1],$$

$$\omega \mapsto (\delta_\omega \otimes \kappa(\varphi(\omega)))(D) = \kappa(\varphi(\omega))(D_\omega).$$

The left map is measurable. The middle map is measurable as the strength of the Giry monad, see [Gir82] or also [For21] Lem. B.39. The type of measurability of the evaluation map ev_D on the right depends on the kind of the subset D, see Proposition 5.3. We need to analyse this further.
Since countable products of universal measurable spaces are universal measurable spaces (see Remark 5.19 and Lemma 5.25) we get from Lemma 5.21:

$$D \in \mathcal{B}_{\Omega\times\mathcal{X}} = \mathcal{B}(\Omega^\Omega \times \mathcal{X}^\Omega) = \mathcal{B}(\mathcal{F}(\mathcal{B}_\Omega) \times \mathcal{F}(\mathcal{B}_\mathcal{X})) = \mathcal{B}\mathcal{F}(\mathcal{B}_\Omega \otimes \mathcal{B}_\mathcal{X}) = (\mathcal{B}_\Omega \otimes \mathcal{B}_\mathcal{X})_\mathcal{G}.$$

Proposition 5.3 shows that ev_D and thus h is universall measurable, i.e. $(\mathcal{B}_\Omega)_\mathcal{G}$-$\mathcal{B}_{[0,1]}$-measurable. Since $(\mathcal{B}_\Omega)_\mathcal{G} = \mathcal{B}_\Omega$ it is even $\mathcal{B}_\Omega$-$\mathcal{B}_{[0,1]}$-measurable. This shows that $\kappa \in \mathcal{Q}(\mathcal{X}, \mathcal{X}^\Omega)^\Omega$. Similarly, $\mu \in \mathcal{Q}(\mathcal{X}, \mathcal{X}^\Omega)$. This shows the wanted equality of quasi-measurable spaces:

$$\mathcal{G}(\mathcal{X}, \mathcal{X}^\Omega) = \mathcal{Q}(\mathcal{X}, \mathcal{X}^\Omega),$$

and thus the claim. □

Theorem 5.29. *Let $(\mathcal{X}, \mathcal{X}^\Omega)$ be a quasi-universal space. Then we have the equality of quasi-universal spaces:*

$$\mathcal{S}(\mathcal{X}, \mathcal{X}^\Omega) = \mathcal{P}(\mathcal{X}, \mathcal{X}^\Omega) = \mathcal{R}(\mathcal{X}, \mathcal{X}^\Omega) = \mathcal{K}(\mathcal{X}, \mathcal{X}^\Omega).$$

If, furthermore, $\nu = U[0,1]$ is the uniform distribution on $\Omega \cong [0,1]$ then we can represent:

$$\mathcal{P}(\mathcal{X}, \mathcal{X}^\Omega) = \left\{ \alpha_* \nu \,\middle|\, \alpha \in \mathcal{X}^\Omega \right\}, \qquad \mathcal{P}(\mathcal{X}, \mathcal{X}^\Omega)^\Omega = \left\{ \alpha_* \phi_* \nu \,\middle|\, \alpha \in \mathcal{X}^\Omega, \phi \in \left(\Omega^\Omega\right)^\Omega \right\}$$ [7].

In particular, the following map is a well-defined (surjective) quotient map of quasi-universal spaces:

$$\mathcal{X}^\Omega \twoheadrightarrow \mathcal{P}(\mathcal{X}, \mathcal{X}^\Omega), \qquad \alpha \mapsto \alpha_* \nu.$$

Proof. First, it is important to note that by Lemma 5.28 we have that $\nu \in \mathcal{G}(\Omega) = \mathcal{Q}(\Omega)$. Next we apply Lemma 3.19: By Lemma 5.24 we have an isomorphism of quasi-universal spaces:

$$\Omega \cong \Omega \times \Omega.$$

By Lemma 3.19 we get the inclusions and equalities:

$$\mathcal{S}(\mathcal{X})^\Omega = \mathcal{R}(\mathcal{X})^\Omega \subseteq \mathcal{P}(\mathcal{X})^\Omega = \mathcal{K}(\mathcal{X})^\Omega.$$

Next we will argue that the following push-forward map is surjective:

$$\mathrm{pf}: \; \left(\Omega^\Omega\right)^\Omega \times \{\nu\} \hookrightarrow \left(\Omega^\Omega\right)^\Omega \times \mathcal{Q}(\Omega) \to \mathcal{Q}(\Omega)^\Omega.$$

Let $\kappa \in \mathcal{Q}(\Omega)^\Omega$ and $\phi \in \left(\Omega^\Omega\right)^\Omega$ its conditional quantile function (using the fixed isomorphism $\Omega \cong [0,1]$). Then it is a well known fact that $\kappa = \phi_* \nu$, see e.g. [For21] Appendix G or [Č82, Kal21]. So the map from above is surjective.
By Lemma 3.19 we then get the inclusions and equalities:

$$\mathcal{S}(\mathcal{X})^\Omega = \mathcal{P}(\mathcal{X})^\Omega \subseteq \mathcal{R}(\mathcal{X})^\Omega = \mathcal{K}(\mathcal{X})^\Omega.$$

This already shows the equalities of quasi-universal spaces:

$$\mathcal{S}(\mathcal{X}) = \mathcal{P}(\mathcal{X}) = \mathcal{R}(\mathcal{X}) = \mathcal{K}(\mathcal{X}).$$

To show that the mentioned map is a quotient map just note that:

$$\alpha_* \kappa = \alpha_* \phi_* \nu = (\alpha \circ \phi)_* \nu,$$

for $\alpha \in \mathcal{X}^\Omega$, $\kappa \in \mathcal{Q}(\Omega)^\Omega$, $\phi \in \left(\Omega^\Omega\right)^\Omega$ and $\alpha \circ \phi \in \left(\mathcal{X}^\Omega\right)^\Omega$ given by:

$$(\alpha \circ \phi)(\omega_1)(\omega_2) = \alpha\left(\phi(\omega_1)(\omega_2)\right).$$

This shows all the claims. ☐

[7] We could here also directly absorb ϕ into α and write $\alpha \in \left(\mathcal{X}^\Omega\right)^\Omega$. We could also require $\alpha \in \mathcal{X}^\Omega$ and restrict ϕ further to be a conditional quantile function, see proof.

Theorem 5.30. *Let $(\mathcal{X}, \mathcal{X}^\Omega)$ be a universal quasi-universal space (e.g. $(\mathcal{X}, \mathcal{X}^\Omega) = (\Omega, \Omega^\Omega)$) then we have the equality of quasi-universal spaces:*

$$\mathcal{G}(\mathcal{X}, \mathcal{X}^\Omega) = \mathcal{Q}(\mathcal{X}, \mathcal{X}^\Omega) = \mathcal{P}(\mathcal{X}, \mathcal{X}^\Omega) = \mathcal{K}(\mathcal{X}, \mathcal{X}^\Omega) = \mathcal{R}(\mathcal{X}, \mathcal{X}^\Omega) = \mathcal{S}(\mathcal{X}, \mathcal{X}^\Omega).$$

Furthermore, $\mathcal{G}(\mathcal{X}, \mathcal{X}^\Omega)$ is also a universal quasi-universal space.

Proof. By Lemma 5.28 we have the equality of quasi-universal spaces:

$$\mathcal{G}(\mathcal{X}, \mathcal{X}^\Omega) = \mathcal{Q}(\mathcal{X}, \mathcal{X}^\Omega).$$

Then by Lemma 5.23 and Lemma 3.18 we have the equality and the inclusion of sets:

$$\mathcal{P}(\mathcal{X}, \mathcal{X}^\Omega) = \mathcal{G}(\mathcal{X}, \mathcal{X}^\Omega), \qquad \mathcal{P}(\mathcal{X}, \mathcal{X}^\Omega)^\Omega \subseteq \mathcal{G}(\mathcal{X}, \mathcal{X}^\Omega)^\Omega.$$

By Theorem 5.29 we already have the equalities of quasi-universal spaces:

$$\mathcal{S}(\mathcal{X}, \mathcal{X}^\Omega) = \mathcal{P}(\mathcal{X}, \mathcal{X}^\Omega) = \mathcal{R}(\mathcal{X}, \mathcal{X}^\Omega) = \mathcal{K}(\mathcal{X}, \mathcal{X}^\Omega).$$

To show the reverse inclusion $\mathcal{G}(\mathcal{X}, \mathcal{X}^\Omega)^\Omega \subseteq \mathcal{P}(\mathcal{X}, \mathcal{X}^\Omega)^\Omega$ let $\kappa \in \mathcal{G}(\mathcal{X}, \mathcal{X}^\Omega)^\Omega$. We can then consider, similar to Lemma 5.21, an inclusion $i: \mathcal{X} \hookrightarrow \Omega$ with $\mathcal{B}_\mathcal{X} = \mathcal{B}_{\Omega|\mathcal{X}}$ and:

$$\alpha: \Omega \to \mathcal{X}, \qquad \alpha|_\mathcal{X} := \mathrm{id}_\mathcal{X}, \qquad \alpha|_{\Omega\setminus\mathcal{X}} := x_0 \in \mathcal{X}.$$

Since $\mathcal{X}, \Omega\setminus\mathcal{X} \in \mathcal{B}_\Omega$ we have that $\alpha \in \mathcal{F}(\mathcal{B}_\mathcal{X}) = \mathcal{X}^\Omega$ and $\alpha \circ i = \mathrm{id}_\mathcal{X}$. Then $i_*\kappa \in \mathcal{G}(\Omega, \Omega^\Omega)^\Omega = \mathcal{Q}(\Omega, \Omega^\Omega)^\Omega$ and:

$$\kappa = \alpha_*(i_*\kappa) \in \mathcal{P}(\mathcal{X}, \mathcal{X}^\Omega)^\Omega.$$

This then shows the first claim.
For the claim that $\mathcal{G}(\mathcal{X}, \mathcal{X}^\Omega)$ is also a universal quasi-universal space first note that $\mathcal{G}(\mathcal{X}, \mathcal{X}^\Omega)^\Omega = \mathcal{F}\left(\mathcal{B}_{\mathcal{G}(\mathcal{X},\mathcal{B}_\mathcal{X})}\right)$, where the σ-algebra is the smallest σ-algebra that makes all evaluation maps measurable. It was shown in [For21] Cor. B.49 that $(\mathcal{G}(\mathcal{X}, \mathcal{B}_\mathcal{X}), \mathcal{B}_{\mathcal{G}(\mathcal{X},\mathcal{B}_\mathcal{X})})$ is a universal measurable spaces if $(\mathcal{X}, \mathcal{B}_\mathcal{X})$ is one. Then Definition/Lemma 5.22 shows that:

$$\left(\mathcal{G}(\mathcal{X}, \mathcal{X}^\Omega), \mathcal{G}(\mathcal{X}, \mathcal{X}^\Omega)^\Omega\right) = \left(\mathcal{G}(\mathcal{X}, \mathcal{B}_\mathcal{X}), \mathcal{F}\left(\mathcal{B}_{\mathcal{G}(\mathcal{X},\mathcal{B}_\mathcal{X})}\right)\right)$$

is a universal quasi-universal space. ◻

Corollary 5.31. *Let $(\mathcal{X}, \mathcal{X}^\Omega)$ and $(\mathcal{Z}, \mathcal{Z}^\Omega)$ universal quasi-universal spaces. Then we have a commutative diagram of quasi-measurable maps:*

$$\begin{array}{ccc} \left(\mathcal{X}^\mathcal{Z}\right)^\Omega & \xrightarrow{(\_)_*\nu} & \mathcal{P}(\mathcal{X}^\mathcal{Z}) \\ \text{swap} \| & & \downarrow \\ \left(\mathcal{X}^\Omega\right)^\mathcal{Z} & \xrightarrow[{[(\_)_*\nu]^\mathcal{Z}}]{} & \mathcal{P}(\mathcal{X})^\mathcal{Z}, \end{array}$$

where the horizontal maps are induced by $\psi \mapsto \psi_*\nu$, where ν is the uniform distribution on $\Omega \cong [0,1]$, and where the right vertical map is:

$$\psi_*\nu \mapsto (z \mapsto \psi_*(z)\nu).$$

Furthermore, if either $(\mathcal{X}, \mathcal{X}^\Omega)$ or $(\mathcal{Z}, \mathcal{Z}^\Omega)$ is a universal quasi-universal space then all maps are (surjective) quotient maps of quasi-universal spaces.

Proof. It is clear that all maps are well-defined and quasi-measurable, the diagram commutes and that the top horizontal map is a quotient map, see Theorem 5.29. It is left to show that the bottom horizontal map is a quotient map, i.e. that it is surjective for both $\mathcal{Z}$ and $\Omega \times \mathcal{Z}$ in the exponent.
1.) Assume that $\mathcal{X}$ is a univeral quasi-universal space. By Theorem 5.30 we then have that: $(\mathcal{P}(\mathcal{X}), \mathcal{P}(\mathcal{X})^\Omega) = (\mathcal{G}(\mathcal{X}), \mathcal{F}(\mathcal{B}_{\mathcal{G}(\mathcal{X})}))$. By Definition/Lemma 5.22 We also have quasi-measurable maps $\mathcal{X} \xrightarrow{\iota} \Omega \xrightarrow{r} \mathcal{X}$ with $r \circ \iota = \mathrm{id}_\mathcal{X}$. Now let ρ be given by the adjuction, Theorem 2.16:

$$\rho \in \mathcal{P}(\mathcal{X})^\mathcal{Z} = \mathbf{QUS}(\mathcal{Z}, \mathcal{P}(\mathcal{X})) = \mathbf{Meas}(\mathcal{Z}, \mathcal{G}(\mathcal{X})).$$

Let $\beta : \ \mathcal{Z} \times \Omega \to \Omega$ be the conditional quantile function of the Markov kernel $\iota_*\rho$, which is measurable, thus $\beta \in \left(\Omega^\Omega\right)^\mathcal{Z}$. We have that $\beta_*\nu = \iota_*\rho$, see e.g. [Č82, Kal21] or [For21] Appendix G. Then $(r \circ \beta)_*\nu = r_*\iota_*\rho = \rho$ and $(r \circ \beta) \in \left(\mathcal{X}^\Omega\right)^\mathcal{Z}$. This shows the surjectivity of $\left(\mathcal{X}^\Omega\right)^\mathcal{Z} \to \mathcal{P}(\mathcal{X})^\mathcal{Z}$. By replacing $\mathcal{Z}$ with $\Omega \times \mathcal{Z}$ we can use the same arguments to see that this is actually a quotient map.
This shows the claim if $(\mathcal{X}, \mathcal{X}^\Omega)$ is a universal quasi-universal space.
2.) Now assume that $(\mathcal{Z}, \mathcal{Z}^\Omega)$ is a universal quasi-universal space and $\rho \in \mathcal{P}(\mathcal{X})^\mathcal{Z}$. By Definition/Lemma 5.22 we get quasi-measurable maps $\mathcal{Z} \xrightarrow{\iota} \Omega \xrightarrow{r} \mathcal{Z}$ with $r \circ \iota = \mathrm{id}_\mathcal{Z}$. Then we have:

$$\rho \circ r \in \mathcal{P}(\mathcal{Z})^\Omega = \left\{\alpha_*\nu \,\middle|\, \alpha \in \left(\mathcal{X}^\Omega\right)^\Omega\right\}.$$

So $\rho \circ r = \alpha_*\nu$ and we get by composing with ι:

$$\rho = \rho \circ r \circ \iota = (\alpha \circ \iota)_*\nu,$$

with now $\alpha \circ \iota \in \left(\mathcal{X}^\Omega\right)^\mathcal{Z}$. This shows the surjectivity of $\left(\mathcal{X}^\Omega\right)^\mathcal{Z} \to \mathcal{P}(\mathcal{X})^\mathcal{Z}$. By replacing $\mathcal{Z}$ with $\Omega \times \mathcal{Z}$ we can use the same arguments to see that this is actually a quotient map. For this note that by Lemma 5.25 the product $\Omega \times \mathcal{Z}$ is again a universal quasi-universal space. This shows the claim. ☐

Remark 5.32. *To understand the difference between the objects $\mathcal{P}(\mathcal{X}^\mathcal{Z})$ and $\mathcal{P}(\mathcal{X})^\mathcal{Z}$ from Corollary 5.31 let $\mathcal{Z} = \{0, 1, 2\}$. Then an element $P(X) \in \mathcal{P}(\mathcal{X}^\mathcal{Z})$ corresponds to encoding the joint distribution:*

$$P(X_0, X_1, X_2).$$

This is in contrast to an element $P(X|Z) \in \mathcal{P}(\mathcal{X})^\mathcal{Z}$, which would correspond to encoding the tuple of marginal distributions:

$$\left(P(X_0), P(X_1), P(X_2)\right).$$

The quasi-measurable map: $\mathcal{P}(\mathcal{X}^{\mathcal{Z}}) \twoheadrightarrow \mathcal{P}(\mathcal{X})^{\mathcal{Z}}$ *then corresponds to mapping the joint distribution to the tuple of marginal distributions:*

$$P(X_0, X_1, X_2) \mapsto (P(X_0), P(X_1), P(X_2)) .$$

5.6 Fubini Theorem for Probability Monad $\mathcal{P}$ on Quasi-Universal Spaces

We show that integration w.r.t. product probability measures on products of quasi-universal spaces commute. Such results are known as *Fubini theorems.*

Theorem 5.33 (Fubini Theorem for Markov kernels)**.** *Let* $(\mathcal{X}, \mathcal{X}^\Omega)$, $(\mathcal{Y}, \mathcal{Y}^\Omega)$, $(\mathcal{U}, \mathcal{U}^\Omega)$, $(\mathcal{Z}, \mathcal{Z}^\Omega)$ *be quasi-universal spaces then the following diagram is commutative:*

$$\begin{array}{ccccc}
\mathcal{P}(\mathcal{X})^{\mathcal{U}} \times \mathcal{P}(\mathcal{Y})^{\mathcal{Z}} & & \xrightarrow{\quad\otimes\quad} & & \mathcal{P}(\mathcal{X} \times \mathcal{Y})^{\mathcal{U} \times \mathcal{Z}} \\
\Big\downarrow {\scriptstyle \mathrm{swap}_{\mathcal{P}(\mathcal{X})^{\mathcal{U}}, \mathcal{P}(\mathcal{Y})^{\mathcal{Z}}}} & & & & \Big\downarrow {\scriptstyle \mathrm{swap}_{(\mathcal{X},\mathcal{Y}),*}} \\
\mathcal{P}(\mathcal{Y})^{\mathcal{Z}} \times \mathcal{P}(\mathcal{X})^{\mathcal{U}} & \xrightarrow[\otimes]{} & \mathcal{P}(\mathcal{Y} \times \mathcal{X})^{\mathcal{Z} \times \mathcal{U}} & \xrightarrow[\mathrm{swap}^*_{\mathcal{U},\mathcal{Z}}]{} & \mathcal{P}(\mathcal{Y} \times \mathcal{X})^{\mathcal{U} \times \mathcal{Z}}.
\end{array}$$

More concretely, for $\mu \in \mathcal{P}(\mathcal{X})^{\mathcal{U}}$, $\nu \in \mathcal{P}(\mathcal{Y})^{\mathcal{Z}}$, $u \in \mathcal{U}$, $z \in \mathcal{Z}$ *and* $f \in [0, \infty]^{\mathcal{X} \times \mathcal{Y}}$ *we have:*

$$\int \int f(x, y)\, \mu(u)(dx)\, \nu(z)(dy) = \int \int f(x, y)^{8}\, \nu(z)(dy)\, \mu(u)(dx).$$

Proof. It was already shown that all maps are quasi-measurable, see Theorem 3.24 and Lemma 3.20. We only need to check the commutativity on elements and fix $u \in \mathcal{U}$ and $z \in \mathcal{Z}$. So we can w.l.o.g. ignore $\mathcal{U}$ and $\mathcal{Z}$.
Let $\mu = \alpha_* \kappa \in \mathcal{P}(\mathcal{X})$ with $\alpha \in \mathcal{X}^{\Omega_2}$, $\kappa \in \mathcal{Q}(\Omega_2)$ and $\nu = \beta_* \rho \in \mathcal{P}(\mathcal{Y})$ with $\beta \in \mathcal{Y}^{\Omega_1}$, $\rho \in \mathcal{Q}(\Omega_1)$ and $D \in \mathcal{B}_{\mathcal{Y} \times \mathcal{X}}$, where Ω_2, Ω_1 are copies of $\Omega = \mathbb{R}$, with an index for readability. Then we use the superscript $(\_)^s$ to indicate swaps:

$$D^s := \{(x, y) \in \mathcal{X} \times \mathcal{Y} \,|\, (y, x) \in D\} .$$

We need to show:

$$(\nu \otimes \mu)(D) = (\mu \otimes \nu)(D^s).$$

We first show the analogon for κ and ρ. For this note that by Lemma 5.21:

$$\mathcal{B}_{\Omega_1 \times \Omega_2} = (\mathcal{B}_{\Omega_1} \otimes \mathcal{B}_{\Omega_2})_{\mathcal{G}} = (\{B \times A \,|\, B \in \mathcal{B}_{\Omega_1}, A \in \mathcal{B}_{\Omega_2}\})_{\mathcal{G}} .$$

We trivially have the required equality on the elements of the generating π-system:

$$(\rho \otimes \kappa)(B \times A) = \rho(B) \cdot \kappa(A) = \kappa(A) \cdot \rho(B) = (\kappa \otimes \rho)(A \times B) = (\kappa \otimes \rho)((B \times A)^s).$$

[8] One would more consistently use $f^s(y, x) := f(x, y)$ here in this place.

This equality then extends to its universal completion and we get for all $E \in \mathcal{B}_{\Omega_1 \times \Omega_2}$:

$$(\rho \otimes \kappa)(E) = (\kappa \otimes \rho)(E^s).$$

For the case of μ and ν we first note the following:

$$\begin{aligned}\alpha^{-1}(D^s_{\beta(\omega_1)}) &= \{\omega_2 \in \Omega_2 \,|\, (\alpha(\omega_2), \beta(\omega_1)) \in D^s\} \\ &= \left\{\omega_2 \in \Omega_2 \,\middle|\, (\omega_2, \omega_1) \in (\alpha \times \beta)^{-1}(D^s)\right\} \\ &= (\alpha \times \beta)^{-1}(D^s)_{\omega_1}.\end{aligned}$$

Also note that:

$$\begin{aligned}(\alpha \times \beta)^{-1}(D^s) &= \{(\omega_2, \omega_1) \in \Omega_2 \times \Omega_1 \,|\, (\beta(\omega_1), \alpha(\omega_2)) \in D\} \\ &= (\beta \times \alpha)^{-1}(D)^s.\end{aligned}$$

Then we get:

$$\begin{aligned}&(\mu \otimes \nu)(D^s) \\ &= \int \mu(D^s_y)\, \nu(dy) \\ &= \int (\alpha_* \kappa)(D^s_y)\, (\beta_* \rho)(dy) \\ &= \int \kappa\left(\alpha^{-1}(D^s_{\beta(\omega_1)})\right)\, \rho(d\omega_1) \\ &= \int \kappa\left((\alpha \times \beta)^{-1}(D^s)_{\omega_1}\right)\, \rho(d\omega_1) \\ &= (\kappa \otimes \rho)\left((\alpha \times \beta)^{-1}(D^s)\right) \\ &= (\kappa \otimes \rho)\left((\beta \times \alpha)^{-1}(D)^s\right) \\ &= (\rho \otimes \kappa)\left((\beta \times \alpha)^{-1}(D)\right) \\ &= \int \rho\left((\beta \times \alpha)^{-1}(D)_{\omega_2}\right)\, \kappa(d\omega_2) \\ &= \int \rho\left(\beta^{-1}(D_{\alpha(\omega_2)})\right)\, \kappa(d\omega_2) \\ &= \int (\beta_* \rho)(D_x)\, (\alpha_* \kappa)(dx) \\ &= \int \nu(D_x)\, \mu(dx) \\ &= (\nu \otimes \mu)(D).\end{aligned}$$

This shows the claim for indicator functions $\mathbb{1}_D(y, x)$.
The claim for $f \in [0, \infty]^{\mathcal{X} \times \mathcal{Y}}$ follows from [Kle20] Thm. 1.96 giving us the representation:

$$f(x, y) = \sum_{n \in \mathbb{N}} c_n \cdot \mathbb{1}_{D^s_{(n)}}(x, y),$$

with $D_{(n)} \in \mathcal{B}_{\mathcal{Y} \times \mathcal{X}}$, $n \in \mathbb{N}$. The linearity of the integral implies the claim. □

5.7 Disintegration of Markov Kernels between Quasi-Universal Spaces

In this subsection we will show under which conditions a Markov kernel on a product space of quasi-universal spaces can be *disintegrated* into a marginal part and a conditional part such that the chain rule holds. For this we use the a bit more suggestive notation $K(X, Y|Z)$ for Markov kernels to make it easier to indicate the marginals $K(Y|Z)$, which usually would need to be written as a push-forward, and the chain rule for conditioning like $K(X, Y|Z) = K(X|Y, Z) \otimes K(Y|Z)$, which quickly became unreadable otherwise. We then discuss the (essential) uniqueness of such factorizations. We first recall the strongest disintegration theorem for measurable spaces from [For21].

Theorem 5.34 (See [For21] Cor. C.8). *Let $(\mathcal{X}, \mathcal{B}_\mathcal{X})$, $(\mathcal{Y}, \mathcal{B}_\mathcal{Y})$ and $(\mathcal{Z}, \mathcal{B}_\mathcal{Z})$ be measurable spaces, where $(\mathcal{X}, \mathcal{B}_\mathcal{X})$ is perfect and universally countably generated (e.g. a universal measurable space) and $(\mathcal{Y}, \mathcal{B}_\mathcal{Y})$ universally countably generated (e.g. countably generated). Let:*

$$K(X, Y|Z): \ (\mathcal{Z}, \mathcal{B}_\mathcal{Z}) \to \mathcal{G}(\mathcal{X} \times \mathcal{Y}, \mathcal{B}_\mathcal{X} \otimes \mathcal{B}_\mathcal{Y}),$$

be measurable. Then there exists a (universally) measurable:

$$K(X|Y, Z): \ (\mathcal{Y} \times \mathcal{Z}, (\mathcal{B}_\mathcal{Y} \otimes \mathcal{B}_\mathcal{Z})_\mathcal{G}) \to \mathcal{G}(\mathcal{X}, \mathcal{B}_\mathcal{X}),$$

such that:

$$K(X, Y|Z) = K(X|Y, Z) \otimes K(Y|Z),$$

as Markov kernels:

$$(\mathcal{Z}, \mathcal{B}_\mathcal{Z}) \to \mathcal{G}(\mathcal{X} \times \mathcal{Y}, \mathcal{B}_\mathcal{X} \otimes \mathcal{B}_\mathcal{Y}),$$

where $K(Y|Z)$ is the marginal Markov kernel.

Theorem 5.35 (Disintegration of Markov kernels between quasi-universal spaces). *Let $(\mathcal{X}, \mathcal{X}^\Omega)$, $(\mathcal{Y}, \mathcal{Y}^\Omega)$ and $(\mathcal{Z}, \mathcal{Z}^\Omega)$ be quasi-universal spaces. Assume that $(\mathcal{Y}, \mathcal{Y}^\Omega)$ is* countably separated. *In addition, assume one of the following points:*

1. *$(\mathcal{X}, \mathcal{X}^\Omega)$ is a* universal *quasi-universal space, or:*
2. *$(\mathcal{Z}, \mathcal{Z}^\Omega)$ is a* universal *quasi-universal space (e.g. $\mathcal{Z} \cong \mathbf{1}$, $\mathbb{N}$, Ω, etc.).*

Then the quasi-measurable map given by the product of Markov kernels:

$$\otimes: \ \mathcal{P}(\mathcal{X})^{\mathcal{Y} \times \mathcal{Z}} \times \mathcal{P}(\mathcal{Y})^\mathcal{Z} \to \mathcal{P}(\mathcal{X} \times \mathcal{Y})^\mathcal{Z},$$

is a (surjective) quotient map.

Proof. Let $K(X, Y|Z) \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})^\mathcal{Z}$. Then the marginal $K(Y|Z) \in \mathcal{P}(\mathcal{Y})^\mathcal{Z}$.
1.) Assume that $(\mathcal{X}, \mathcal{X}^\Omega)$ is a universal quasi-universal space and $(\mathcal{Y}, \mathcal{Y}^\Omega)$ countably separated. Then by Theorem 5.30 we have the identification of quasi-universal spaces:

$$\mathcal{P}(\mathcal{X}) = \mathcal{K}(\mathcal{X}) = \mathcal{Q}(\mathcal{X}) = \mathcal{G}(\mathcal{X}).$$

In particular, $\mathcal{P}(\mathcal{X})^\Omega = \mathcal{G}(\mathcal{X})^\Omega$ consist of all measurable maps $\Omega \to \mathcal{G}(\mathcal{X}, \mathcal{B}_\mathcal{X})$.
Also by Lemma 5.5 and Theorem 5.16 there exists a countably generated σ-algebras $\mathcal{E}_\mathcal{X} \subseteq \mathcal{B}_\mathcal{X}$ and $\mathcal{E}_\mathcal{Y} \subseteq \mathcal{B}_\mathcal{Y}$ such that:

a.) $\mathcal{E}_\mathcal{X}$ separates the points of $\mathcal{X}$ such that $(\mathcal{E}_\mathcal{X})_\mathcal{G} = \mathcal{B}_\mathcal{X}$, and:

b.) $\mathcal{E}_\mathcal{Y}$ separates the points of $\mathcal{Y}$ such that $(\mathcal{E}_\mathcal{Y})_{\mathcal{P}(\mathcal{Y},\mathcal{Y}^\Omega)} = \mathcal{B}_\mathcal{Y}$.

Then consider the composition with the measurable restriction maps:

$$K(X,Y|Z): \ \mathcal{Z} \to \mathcal{P}(\mathcal{X}\times\mathcal{Y}) \to \mathcal{G}(\mathcal{X}\times\mathcal{Y},\mathcal{B}_{\mathcal{X}\times\mathcal{Y}}) \to \mathcal{G}(\mathcal{X}\times\mathcal{Y},\mathcal{B}_\mathcal{X}\otimes\mathcal{E}_\mathcal{Y}).$$

By Theorem 5.34 and the fact that $(\mathcal{X},\mathcal{B}_\mathcal{X})$ is a universal measurable space there exists a measurable:

$$K(X|Y,Z): \ (\mathcal{Y}\times\mathcal{Z},(\mathcal{E}_\mathcal{Y}\otimes\mathcal{B}_\mathcal{Z})_\mathcal{G}) \to \mathcal{G}(\mathcal{X},\mathcal{B}_\mathcal{X}),$$

such that for all $z\in\mathcal{Z}$:

$$K(X,Y|Z=z) = K(X|Y,Z=z)\otimes K(Y|Z=z).$$

as measures on $\mathcal{B}_\mathcal{X}\otimes\mathcal{E}_\mathcal{Y}$. By Lemma 5.5 we get that:

$$(\mathcal{E}_\mathcal{Y}\otimes\mathcal{B}_\mathcal{Z})_\mathcal{G} \subseteq (\mathcal{B}_{\mathcal{Y}\times\mathcal{Z}})_\mathcal{G} = \mathcal{B}_{\mathcal{Y}\times\mathcal{Z}}.$$

Let $\beta\in\mathcal{Y}^\Omega$ and $\gamma\in\mathcal{Z}^\Omega$ then the composition:

$$(\Omega,\mathcal{B}_\Omega) \xrightarrow{\beta\times\gamma} (\mathcal{Y}\times\mathcal{Z},\mathcal{B}_{\mathcal{Y}\times\mathcal{Z}}) \xrightarrow{\mathrm{id}_{\mathcal{Y}\times\mathcal{Z}}} (\mathcal{Y}\times\mathcal{Z},(\mathcal{E}_\mathcal{Y}\otimes\mathcal{B}_\mathcal{Z})_\mathcal{G}) \xrightarrow{K(X|Y,Z)} \mathcal{G}(\mathcal{X},\mathcal{B}_\mathcal{X}) = \mathcal{P}(\mathcal{X})$$

is measurable and thus an element in $\mathcal{P}(\mathcal{X})^\Omega$ by the remarks above. Since this holds for all $\beta\in\mathcal{Y}^\Omega$ and $\gamma\in\mathcal{Z}^\Omega$ we get that:

$$K(X|Y,Z)\in\mathcal{P}(\mathcal{X})^{\mathcal{Y}\times\mathcal{Z}}.$$

So for every $z\in\mathcal{Z}$ we have that: $K(X,Y|Z=z)\in\mathcal{P}(\mathcal{X}\times\mathcal{Y})$ and $K(X|Y,Z=z)\otimes K(Y|Z=z)\in\mathcal{P}(\mathcal{X}\times\mathcal{Y})$ and both agree on $\mathcal{B}_\mathcal{X}\otimes\mathcal{E}_\mathcal{Y}$. Since $\mathcal{B}_\mathcal{X}$ and $\mathcal{E}_\mathcal{Y}$ are countably separated also the product $\mathcal{B}_\mathcal{X}\otimes\mathcal{E}_\mathcal{Y}$ is. Then again by Lemma 5.5 we get that:

$$\mathcal{B}_{\mathcal{X}\times\mathcal{Y}} = (\mathcal{B}_\mathcal{X}\otimes\mathcal{E}_\mathcal{Y})_{\mathcal{P}(\mathcal{X}\times\mathcal{Y})}.$$

This shows that the equality:

$$K(X,Y|Z=z) = K(X|Y,Z=z)\otimes K(Y|Z=z)$$

uniquely extends from $\mathcal{B}_\mathcal{X}\otimes\mathcal{E}_\mathcal{Y}$ to $(\mathcal{B}_\mathcal{X}\otimes\mathcal{E}_\mathcal{Y})_{\mathcal{P}(\mathcal{X}\times\mathcal{Y})} = \mathcal{B}_{\mathcal{X}\times\mathcal{Y}}$.[9] This shows the surjectivity of the map under the first assumptions.
Since the same arguments hold also when $\mathcal{Z}$ is replaced by $\mathcal{Z}\times\Omega$ the map is even a quotient map, i.e. surjective on the level of quasi-measurable functions.
2.) Now assume that $(\mathcal{Z},\mathcal{Z}^\Omega)$ is a universal quasi-universal space and $(\mathcal{Y},\mathcal{Y}^\Omega)$ countably

[9]Even though $\mathcal{Q}(\mathcal{X}\times\mathcal{Y})$ is defined on the same σ-algebra $\mathcal{B}_{\mathcal{X}\times\mathcal{Y}}$ its probabillity measures might not come as completions of $\mathcal{B}_\mathcal{X}\otimes\mathcal{E}_\mathcal{Y}$ unless they are from $\mathcal{P}(\mathcal{X}\times\mathcal{Y})$. This is where the arguments for $\mathcal{Q}$ would break.

separated. Then w.l.o.g. we can assume $\mathcal{Z} \in \mathcal{B}_\Omega$ and $\mathcal{B}_\mathcal{Z} = \mathcal{B}_{\Omega|\mathcal{Z}}$. Let $\iota: \mathcal{Z} \hookrightarrow \Omega$ be the (quasi-)measurable inclusion map and define the measurable projection map:

$$\gamma: \Omega \twoheadrightarrow \mathcal{Z}, \qquad \gamma_{|\mathcal{Z}} := \mathrm{id}_\mathcal{Z}, \qquad \gamma_{|\Omega\setminus\mathcal{Z}} := z_0 \in \mathcal{Z}.$$

Then $\gamma \in \mathcal{F}(\mathcal{B}_\mathcal{Z}) = \mathcal{Z}^\Omega$, since $\mathcal{Z}$ is a universal quasi-universal space, and: $\gamma \circ \iota = \mathrm{id}_\mathcal{Z}$. We then consider the composition of quasi-measurable maps:

$$K(X, Y|W): \ \Omega \xrightarrow{\gamma} \mathcal{Z} \xrightarrow{K(X,Y|Z)} \mathcal{P}(\mathcal{X} \times \mathcal{Y}),$$

which shows that $K(X, Y|W) \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})^\Omega$. By definition there exist $\alpha \in \mathcal{X}^\Omega$ and $\beta \in \mathcal{Y}^\Omega$ and $K(U|W) \in \mathcal{G}(\Omega)^\Omega$ such that:

$$K(X, Y|Z = \gamma) = K(X, Y|W) \stackrel{!}{=} K(\alpha(U), \beta(U)|W) := (\alpha \times \beta)_* K(U|W).$$

Then consider:

$$K(U, Y|Z) := K(U, \beta(U)|W = \iota(Z)) := (\mathrm{id}_\Omega \times \beta\iota)_* K(U|W = \iota(Z)) \in \mathcal{P}(\Omega \times \mathcal{Y})^\mathcal{Z}.$$

Note that we have:

$$(\alpha\iota \times \mathrm{id}_\mathcal{Y})_* K(U, Y|Z) = K(X, Y|Z).$$

Then the Y-marginal of $K(U, Y|Z)$ agrees with the Y-marginal of $K(X, Y|Z)$, because:

$$K(Y|W = \iota(z)) = K(Y|Z = \gamma(\iota(z)) = K(Y|Z = z).$$

Since Ω is a universal quasi-universal space by the first point we now have:

$$K(U, Y|Z) = K(U|Y, Z) \otimes K(Y|Z) \in \mathcal{P}(\Omega \times \mathcal{Y})^\Omega,$$

with $K(U|Y, Z) \in \mathcal{P}(\Omega)^{\mathcal{Y}\times\mathcal{Z}}$ and $K(Y|Z) \in \mathcal{P}(\mathcal{Y})^\mathcal{Z}$. We now put:

$$K(X|Y, Z) := (\alpha\iota)_* K(U|Y, Z) \in \mathcal{P}(\mathcal{X})^{\mathcal{Y}\times\mathcal{Z}}.$$

With this we get

$$K(X, Y|Z) = (\alpha\iota \times \mathrm{id}_\mathcal{Y})_* K(U, Y|Z) = (\alpha\iota)_* K(U|Y, Z) \otimes K(Y|Z) = K(X|Y, Z) \otimes K(Y|Z).$$

This shows the surjectivity for the second case.
To see that these maps are quotient maps again note that we can use the same proof by replacing $\mathcal{Z}$ with $\mathcal{Z} \times \Omega$. For this note that $\mathcal{Z} \times \Omega$ is again a universal quasi-universal space if $\mathcal{Z}$ is, see Lemma 5.25. □

Remark 5.36. *1. Theorem 5.35 would also hold if one replaces universal quasi-measurable spaces with perfect universally countably generated ones as in Theorem 5.34. For this to work one uses the separated quotient and a section, see Definition 5.9 Remark 5.10. Note that the separated quotient of a perfect universally countably generated quasi-universal space is a universal quasi-universal space.*

2. *It might even be possible to weaken the assumption in Theorem 5.35 from universal to countably separated quasi-universal spaces and/or other weaker assumptions. We have not further investigated this.*

Theorem 5.37 (Essential uniqueness of factorizations). *Let $(\mathcal{X}, \mathcal{X}^\Omega)$, $(\mathcal{Y}, \mathcal{Y}^\Omega)$ and $(\mathcal{Z}, \mathcal{Z}^\Omega)$ be quasi-universal spaces. Let $\nu \in \mathcal{P}(\mathcal{Y})^{\mathcal{Z}}$ and $\mu_1, \mu_2 \in \mathcal{P}(\mathcal{X})^{\mathcal{Y}\times\mathcal{Z}}$ such that:*

$$\mu_1 \otimes \nu = \mu_2 \otimes \nu \quad \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})^{\mathcal{Z}}.$$

For every $D \in \mathcal{B}_{\mathcal{X}\times\mathcal{Y}\times\mathcal{Z}}$ we have that:

$$N_D := \{(y,z) \in \mathcal{Y} \times \mathcal{Z} \,|\, \mu_1(y,z)(D_{y,z}) \neq \mu_2(y,z)(D_{y,z})\}$$

is a ν-null set in $\mathcal{B}_{\mathcal{Y}\times\mathcal{Z}}$, i.e. $\nu(z)(N_{D,z}) = 0$ for all $z \in \mathcal{Z}$.
If, furthermore, $(\mathcal{X}, \mathcal{X}^\Omega)$ is countably separated then $\mu_1 = \mu_2$ ν-almost-surely, i.e., more precisely, that there exists a set $N \in \mathcal{B}_{\mathcal{Y}\times\mathcal{Z}}$ such that for all $z \in \mathcal{Z}$:

$$\nu(z)(N_z) = 0,$$

where $N_z := \{y \in \mathcal{Y} \,|\, (y,z) \in N\}$, and such that for all $(y,z) \notin N$ we have:

$$\mu_1(y,z) = \mu_2(y,z).$$

Proof. First consider for $D \in \mathcal{B}_{\mathcal{X}\times\mathcal{Y}\times\mathcal{Z}}$ the set:

$$N_D^> := \{(y,z) \in \mathcal{Y} \times \mathcal{Z} \,|\, \mu_1(y,z)(D_{y,z}) > \mu_2(y,z)(D_{y,z})\}$$

Then define:

$$D^> := D \cap (\mathcal{X} \times N_D^>) \in \mathcal{B}_{\mathcal{X}\times\mathcal{Y}\times\mathcal{Z}}.$$

With this we then get:

$$\begin{aligned} 0 &= (\mu_1 \otimes \nu)(z)(D_z^>) - (\mu_2 \otimes \nu)(z)(D_z^>) \\ &= \int\int \mathbb{1}_{N_D^>}(y,z) \cdot \mathbb{1}_D(x,y,z)\ (\mu_1(y,z)(dx) - \mu_2(y,z)(dx))\ \nu(z)(dy) \\ &= \int \mathbb{1}_{N_D^>}(y,z) \cdot \underbrace{(\mu_1(y,z)(D_{y,z}) - \mu_2(y,z)(D_{y,z}))}_{>0 \text{ for } (y,z)\in N_D^>}\ \nu(z)(dy). \end{aligned}$$

It follows from this that $\nu(z)(N_{D,z}^>) = 0$ for all $z \in \mathcal{Z}$.
We get by a symmetric argument that $\nu(z)(N_{D,z}^<) = 0$ and thus $\nu(z)(N_{D,z}) = 0$ for all $z \in \mathcal{Z}$. This shows the first claim.
If now $(\mathcal{X}, \mathcal{X}^\Omega)$ is countably separated then by Lemma 5.5 there exists a countable algebra $\mathcal{E} \subseteq \mathcal{B}_{\mathcal{X}}$ that separates the points of $\mathcal{X}$ and such that:

$$\mathcal{B}_{\mathcal{X}} = (\mathcal{E})_{\mathcal{P}(\mathcal{X})}.$$

Then define the (countable) union:

$$N := \bigcup_{E \in \mathcal{E}} N_{E \times \mathcal{Y} \times \mathcal{Z}} \in \mathcal{B}_{\mathcal{Y} \times \mathcal{Z}}.$$

Similar to above we then have that $\nu(z)(N_z) = 0$ for every $z \in \mathcal{Z}$ and also that for every $E \in \mathcal{E}$ and $(y, z) \notin N$:

$$\mu_1(y, z)(E) = \mu_2(y, z)(E).$$

Since $\mathcal{E}$ is a π-system, $\mathcal{B}_{\mathcal{X}} = (\mathcal{E})_{\mathcal{P}(\mathcal{X})}$ and $\mu_1(y, z), \mu_2(y, z) \in \mathcal{P}(\mathcal{X})$ the above equality uniquely extends to all $A \in \mathcal{B}_{\mathcal{X}}$ and $(y, z) \notin N$:

$$\mu_1(y, z)(A) = \mu_2(y, z)(A).$$

This shows the second claim. □

5.8 Kolmogorov Extension Theorems for Quasi-Universal Spaces

In this subsection we study how sequences of probability distributions and Markov kernels defined on finite products of quasi-universal spaces can be extended to an infinite product if they are at least consistent with each other. Such results are usually called *Kolmogorov extension theorems.*

Theorem 5.38 (Kolmogorov's extension theorem for Markov kernels, see [Kol33, Fre15] 454D-G, also see [FR20] Ex. 3.6)**.** *In the following let $(\mathcal{X}_j, \mathcal{B}_j)$, $j \in J$, be a family of measurable spaces indexed over any set J. For a subset $K \subseteq J$ we put:*

$$\mathcal{X}_K := \prod_{k \in K} \mathcal{X}_k, \qquad \mathcal{B}_K := \bigotimes_{k \in K} \mathcal{B}_k := \sigma\left(\mathcal{E}_K\right),$$

where $\mathcal{E}_K$ is given by all finite cylinder sets:

$$\mathcal{E}_K := \left\{ \mathrm{pr}_L^{-1}(\times_{l \in L} A_l) \subseteq \mathcal{X}_K \,\middle|\, A_l \in \mathcal{B}_l, L \subseteq K, \#L < \infty \right\},$$

and with the canonical projection maps:

$$\mathrm{pr}_L : \mathcal{X}_K \to \mathcal{X}_L, \qquad (x_k)_{k \in K} \mapsto (x_l)_{l \in L}.$$

Let $(\mathcal{Z}, \mathcal{B}_{\mathcal{Z}})$ be another measurable space. For every finite subset $K \subseteq J$ consider a (measurable) Markov kernel:

$$Q_L(X_K|Z) : (\mathcal{Z}, \mathcal{B}_{\mathcal{Z}}) \to \mathcal{G}(\mathcal{X}_K, \mathcal{B}_K),$$

such that for every finite subset $L \subseteq K$ we have the consistency relation:

$$\mathrm{pr}_{L,*} Q_K(X_K|Z) = Q_L(X_L|Z),$$

and such that for every $z \in \mathcal{Z}$ and $j \in J$ the marginal probability measure $Q_j(X_j|Z = z)$ is perfect. Then there exists a unique Markov kernel:

$$Q_J(X_J|Z) : (\mathcal{Z}, \mathcal{B}_{\mathcal{Z}}) \to \mathcal{G}(\mathcal{X}_J, \mathcal{B}_J)$$

such that for all finite subsets $L \subseteq J$ we have:

$$\mathrm{pr}_{L,*} Q_J(X_J|Z) = Q_L(X_L|Z).$$

Furthermore, for every $z \in \mathcal{Z}$ the probability measure $Q_J(X_J|Z = z)$ is perfect and measurable as a map in z.

Proof. For every fixed $z \in \mathcal{Z}$ we individually apply Kolmogorov's extension theorem, see [Kol33, Fre15] 454D-G and we get a unique probability measure: $Q_J(X_J|Z = z)$ on $\mathcal{B}_J$ such that for all finite subsets $L \subseteq J$ we have:

$$\mathrm{pr}_{L,*} Q_J(X_J|Z = z) = Q_L(X_L|Z = z).$$

Furthermore, $Q_J(X_J|Z = z)$ is perfect.
We are left to show that the map $z \mapsto Q_J(X_J|Z = z)$ is measurable. For this consider the set:

$$\mathcal{D} := \left\{ D \in \mathcal{B}_J \,\middle|\, (z \mapsto Q_J(X_J \in D|Z = z)) \text{ is } \mathcal{B}_{\mathcal{Z}}\text{-}\mathcal{B}_{[0,1]}\text{-measurable} \right\}.$$

Then $\mathcal{E}_J \subseteq \mathcal{D}$. Indeed, for a finite subset $L \subseteq J$ and $A_L = \times_{l \in L} A_l$ with $A_l \in \mathcal{B}_l$ we have by the consistency relation:

$$Q_J(X_J \in \mathrm{pr}_L^{-1}(A_L)|Z) = Q_L(X_L \in A_L|Z),$$

which by assumption on $Q_L(X_L|Z)$ is measurable.
It is also easily seen that $\mathcal{D}$ is Dynkin system (aka λ-system), as it is closed under complements and disjoint unions and contains $\emptyset$. Since $\mathcal{E}_J$ is a closed under intersections (aka a π-system), we get from Dynkin's π-λ-theorem, see [Kle20] Theorem 1.19, that:

$$\mathcal{B}_J = \sigma(\mathcal{E}_J) \subseteq \mathcal{D}.$$

This shows that $Q_J(X_J|Z)$ is measurable and thus the claim. ☐

Remark 5.39. *The Markov kernel $Q_J(X_J|Z)$ from Theorem 5.38 can uniquely be extended from $\mathcal{B}_J$ to its universal completion $(\mathcal{B}_J)_{\mathcal{G}}$. By Proposition 5.3 we then get that $Q_J(X_J|Z)$ still is (universally) $(\mathcal{B}_{\mathcal{Z}})_{\mathcal{G}}$-$\mathcal{B}_{\mathcal{G}(\mathcal{X}_J,(\mathcal{B}_J)_{\mathcal{G}})}$-measurable as a map:*

$$Q_J(X_J|Z): \ (\mathcal{Z}, (\mathcal{B}_{\mathcal{Z}})_{\mathcal{G}}) \to \mathcal{G}(\mathcal{X}_J, (\mathcal{B}_J)_{\mathcal{G}}).$$

Clearly, it also stays perfect and is consistent with its finite marginals.

Theorem 5.40 (Kolmogorov's extension theorem for Markov kernels for universal quasi-universal spaces)**.** *Let $(\mathcal{X}_n, \mathcal{X}_n^{\Omega})$, $n \in \mathbb{N}$, be a sequence of universal quasi-universal spaces and $(\mathcal{Z}, \mathcal{Z}^{\Omega})$ any quasi-universal space. Assume we have $Q_n(X_{0:n}|Z) \in \mathcal{P}(\mathcal{X}_{0:n})^{\mathcal{Z}}$ for every $n \in \mathbb{N}$ such that for every $n \in \mathbb{N}$:*

$$\mathrm{pr}_{0:n,*} Q_{n+1}(X_{0:n+1}|Z) = Q_n(X_{0:n}|Z).$$

Then there exists a unique $Q(X_{\mathbb{N}}|Z) \in \mathcal{P}(\mathcal{X}_{\mathbb{N}})^{\mathcal{Z}}$ such that for every $n \in \mathbb{N}$:

$$\mathrm{pr}_{0:n,*} Q(X_{\mathbb{N}}|Z) = Q_n(X_{0:n}|Z),$$

where $(\mathcal{X}_{\mathbb{N}}, \mathcal{X}_{\mathbb{N}}^{\Omega}) := \prod_{n \in \mathbb{N}} (\mathcal{X}_n, \mathcal{X}_n^{\Omega})$.

Proof. By Lemma 5.22 we have that $\mathcal{B}_n := \mathcal{B}(\mathcal{X}_n^\Omega)$ makes $(\mathcal{X}_n, \mathcal{B}_n)$ a universal measurable spaces with $\mathcal{F}(\mathcal{B}_n) = \mathcal{X}_n^\Omega$. Put $\mathcal{B}_\mathcal{Z} := \mathcal{B}(\mathcal{Z}^\Omega)$ and note that this is universally complete. Since countable products of universal measurable spaces are universal, see Lemma 5.25, we have that:

$$(\mathcal{B}_{0:n})_\mathcal{G} = \left(\bigotimes_{k=0}^n \mathcal{B}_k\right)_\mathcal{G} = \mathcal{B}\left(\prod_{k=0}^n \mathcal{X}_k^\Omega\right) = \mathcal{B}(\mathcal{X}_{0:n}^\Omega).$$

Then we get the compatible measurable Markov kernels:

$$Q_n(X_{0:n}|Z): \ (\mathcal{Z}, \mathcal{B}_\mathcal{Z}) \to \mathcal{P}(\mathcal{X}_{0:n}) = \mathcal{G}(\mathcal{X}_{0:n}, \mathcal{B}_{0:n}),$$

where the latter identification comes from by Theorem 5.30. Then note that every probability measure on a universal measurable space $(\mathcal{X}_n, \mathcal{B}_n)$ is perfect. Then by Theorem 5.38 and Remark 5.39 we get a unique measurable Markov kernel:

$$Q(X_\mathbb{N}|Z): \ (\mathcal{Z}, \mathcal{B}_\mathcal{Z}) \to \mathcal{G}(\mathcal{X}_\mathbb{N}, (\mathcal{B}_\mathbb{N})_\mathcal{G}),$$

where $\mathcal{B}_\mathbb{N} := \bigotimes_{n\in\mathbb{N}} \mathcal{B}_n$, such that for all $n \in \mathbb{N}$:

$$\mathrm{pr}_{0:n,*} Q(X_\mathbb{N}|Z) = Q_n(X_{0:n}|Z).$$

We are left to show that $Q(X_\mathbb{N}|Z) \in \mathcal{P}(\mathcal{X}_\mathbb{N})^\mathcal{Z}$. For this let $\gamma \in \mathcal{Z}^\Omega \subseteq \mathcal{F}(\mathcal{B}_\mathcal{Z})$. Then the composition:

$$Q(X_\mathbb{N}|Z) \circ \gamma$$

is measurable. We thus get:

$$Q(X_\mathbb{N}|Z) \circ \gamma \in \mathcal{G}(\mathcal{X}_\mathbb{N})^\Omega = \mathcal{P}(\mathcal{X}_\mathbb{N})^\Omega,$$

since $\mathcal{X}_\mathbb{N}$ is a universal quasi-universal space, again by Theorem 5.30. Since this holds for all $\gamma \in \mathcal{Z}^\Omega$ we get:

$$Q(X_\mathbb{N}|Z) \in \mathbf{QUS}(\mathcal{Z}, \mathcal{P}(\mathcal{X}_\mathbb{N})) = \mathcal{P}(\mathcal{X}_\mathbb{N})^\mathcal{Z}.$$

This shows the claim. ☐

Theorem 5.41 (Kolmogorov's extension theorem for Markov kernels for countably separated quasi-universal spaces)**.** *Let $(\mathcal{X}_n, \mathcal{X}_n^\Omega)$, $n \in \mathbb{N}$, be a sequence of countably separated quasi-universal spaces and $(\mathcal{Z}, \mathcal{Z}^\Omega)$ a universal quasi-universal space. Assume we have $Q_n(X_{0:n}|Z) \in \mathcal{P}(\mathcal{X}_{0:n})^\mathcal{Z}$ for every $n \in \mathbb{N}$ such that for every $n \in \mathbb{N}$:*

$$\mathrm{pr}_{0:n,*} Q_{n+1}(X_{0:n+1}|Z) = Q_n(X_{0:n}|Z).$$

Then there exists a unique $Q(X_\mathbb{N}|Z) \in \mathcal{P}(\mathcal{X}_\mathbb{N})^\mathcal{Z}$ such that for every $n \in \mathbb{N}$:

$$\mathrm{pr}_{0:n,*} Q(X_\mathbb{N}|Z) = Q_n(X_{0:n}|Z),$$

where $(\mathcal{X}_\mathbb{N}, \mathcal{X}_\mathbb{N}^\Omega) := \prod_{n\in\mathbb{N}} (\mathcal{X}_n, \mathcal{X}_n^\Omega)$.

Proof. Since $(\mathcal{Z}, \mathcal{Z}^\Omega)$ is a universal quasi-universal space by Lemma 5.22 there exists $\gamma \in \mathcal{Z}^\Omega$ and $\iota \in \Omega^\mathcal{Z}$ such that $\gamma \circ \iota = \mathrm{id}_\mathcal{Z}$.
We then have that:

$$Q_n(X_{0:n}|W) := Q_n(X_{0:n}|Z) \circ \gamma \in \mathcal{P}(\mathcal{X}_{0:n})^\Omega,$$

which still satify the consistency relations for $n \in \mathbb{N}$.
By the disintegration Theorem 5.35 the following map is a surjective quotient map:

$$\otimes : \mathcal{P}(\mathcal{X}_{n+1})^{\mathcal{X}_{0:n} \times \Omega} \times \mathcal{P}(\mathcal{X}_{0:n})^\Omega \twoheadrightarrow \mathcal{P}(\mathcal{X}_{0:n+1})^\Omega.$$

So there exist $Q_{n+1}(X_{n+1}|X_{0:n}, W) \in \mathcal{P}(\mathcal{X}_{n+1})^{\mathcal{X}_{0:n} \times \Omega}$, such that:

$$Q_{n+1}(X_{n+1}|X_{0:n}, W) \otimes Q_n(X_{0:n}|W) = Q_{n+1}(X_{0:n+1}|W).$$

By definition of $\mathcal{P}(\mathcal{X}_{n+1})$ and the isomorphism $\Omega^n \cong \Omega$ we can inductively pick $\alpha_{n+1} \in \left(\mathcal{X}_{n+1}^{\Omega_{0:n}}\right)^\Omega$ and $\kappa_{n+1}(W_{n+1}|W_{0:n}, W) \in \mathcal{Q}(\Omega_{n+1})^{\Omega_{0:n} \times \Omega}$ such that:

$$\begin{aligned}\alpha_{n+1,*}\kappa_{n+1}(W_{n+1}|W_{0:n}, W) &= Q_{n+1}(X_{n+1}|W_{0:n}, W) \\ &:= Q_{n+1}(X_{n+1}|X_{0:n}, W) \circ (\alpha_{0:n} \times \mathrm{id}_\Omega)(W_{0:n}, W).\end{aligned}$$

By Kolmogorov's extension theorem for universal quasi-universal spaces, see Theorem 5.40, there now exists a unique $\kappa(W_\mathbb{N}|W) \in \mathcal{Q}(\Omega^\mathbb{N})^\Omega \cong \mathcal{Q}(\Omega)^\Omega$ such that for all $n \in N$:

$$\mathrm{pr}_{0:n,*}\kappa(W_\mathbb{N}|W) = \bigotimes_{j=n}^{0} \kappa_j(W_j|W_{0:j-1}, W).$$

We then put:

$$\alpha := (\alpha_n)_{n\in\mathbb{N}} \in \left(\mathcal{X}_\mathbb{N}^{\Omega^\mathbb{N}}\right)^\Omega \cong \left(\mathcal{X}^\Omega\right)^\Omega.$$

Then:

$$Q(X_\mathbb{N}|W) := \alpha_*\kappa(W_\mathbb{N}|W) \in \mathcal{P}(\mathcal{X}_\mathbb{N})^\Omega,$$

and thus:

$$Q(X_\mathbb{N}|Z) := Q(X_\mathbb{N}|W = \iota(Z)) := Q(X_\mathbb{N}|W) \circ \iota \in \mathcal{P}(\mathcal{X}_\mathbb{N})^\mathcal{Z}.$$

By construction we then have that for all $n \in \mathbb{N}$:

$$\mathrm{pr}_{0:n,*}Q(X_\mathbb{N}|Z) = Q_n(X_{0:n}|Z).$$

We are now left to show the uniqueness. Since the spaces $\mathcal{X}_n$ are countably separated there are countably generated σ-algebras $\mathcal{E}_n \subseteq \mathcal{B}_n := \mathcal{B}(\mathcal{X}_n^\Omega)$ that separate the points of $\mathcal{X}_n$. Then $\mathcal{E}_\mathbb{N} := \bigotimes_{n\in\mathbb{N}} \mathcal{E}_n$ is countably generated and separates the points of $\mathcal{X}_\mathbb{N} := \prod_{n\in\mathbb{N}} \mathcal{X}_n$. Then $Q(X_\mathbb{N}|Z)$, when restricted to $\mathcal{E}_\mathbb{N}$, is the unique extension of the $Q_n(X_{0:n}|Z)$, when restricted to $\bigotimes_{j=0}^n \mathcal{E}_j$. Since by Lemma 5.5 we have:

$$\mathcal{B}(\mathcal{X}_\mathbb{N}^\Omega) = (\mathcal{E}_\mathbb{N})_{\mathcal{P}(\mathcal{X}_\mathbb{N})},$$

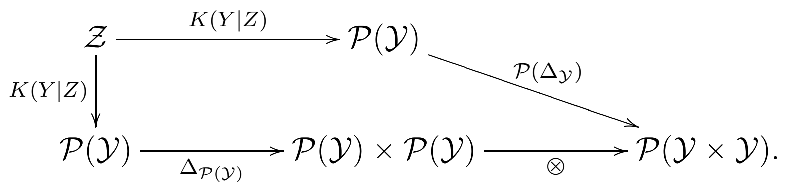
the Markov kernel $Q(X_\mathbb{N}|Z)$ uniquely extends from $\mathcal{E}_\mathbb{N}$ to $\mathcal{B}(\mathcal{X}_\mathbb{N}^\Omega)$.
This shows the claim. □

5.9 Deterministic Markov Kernels between Quasi-Universal Spaces

In this subsection we characterize Markov kernels between quasi-universal spaces that are in some sense *deterministic*. We go through a-priori different notions of deterministic Markov kernels and give a sufficient criterion when they become equivalent.

Definition/Lemma 5.42 (Deterministic Markov kernels, see [RR81, Fri20] Def. 10.1)**.** *Let* $(\mathcal{Y}, \mathcal{Y}^\Omega)$ *and* $(\mathcal{Z}, \mathcal{Z}^\Omega)$ *be a quasi-universal spaces and* $(\mathcal{Z}, \mathcal{Z}^\Omega)$ *any quasi-universal space and* $K(Y|Z) \in \mathcal{P}(\mathcal{Y})^\mathcal{Z}$ *a Markov kernel. Then each statement for* $K(Y|Z)$ *will imply the one below:*

1. $K(Y|Z)$ *is* function-deterministic*, i.e. there exists a quasi-measurable map* $g \in \mathcal{Y}^\mathcal{Z}$ *such that:*
$$K(Y|Z) = \delta_g(Y|Z) \quad \in \mathcal{P}(\mathcal{Y})^\mathcal{Z}.$$
2. $K(Y|Z)$ *is* copy-deterministic*, i.e. we have the relation:*
$$K(Y_1|Z = z) \otimes K(Y_2|Z = z) = K(Y_1, Y_2|Z = z) \quad \in \mathcal{P}(\mathcal{Y} \times \mathcal{Y})^\mathcal{Z},$$
where Y_1, Y_2 *are copies of* Y *with an index for readability, i.e. the following diagram commutes:*



3. $K(Y|Z)$ *is* 0-1-deterministic*, i.e. for all* $z \in \mathcal{Z}$ *and* $B \in \mathcal{B}_\mathcal{Y}$*:*
$$K(Y \in B|Z = z) \in \{0, 1\}.$$

Proof. 1. $\implies$ 2.: Let $D \in \mathcal{B}_{\mathcal{Y}\times\mathcal{Y}}$ and $z \in \mathcal{Z}$. Then:
$$\begin{aligned}
&(\delta_g(Y_1|Z = z) \otimes \delta_g(Y_2|Z = z))(D) \\
&= \int \delta_{g(z)}(D_{y_2})\, \delta_{g(z)}(dy_2) \\
&= \int \mathbb{1}_D(g(z), y_2)\, \delta_{g(z)}(dy_2) \\
&= \mathbb{1}_D(g(z), g(z)) \\
&= \delta_g((Y_1, Y_2) \in D|Z = z).
\end{aligned}$$

2. $\implies$ 3.: We have by assumption for every $B \in \mathcal{B}_\mathcal{Y}$:
$$K(Y \in B|Z = z) \cdot K(Y \in B|Z = z) = K(Y \in B, Y \in B|Z = z) = K(Y \in B|Z = z).$$

Solving for $K(Y \in B|Z = z)$ shows:

$$K(Y \in B|Z = z) \in \{0, 1\}.$$

This shows the claims. □

Lemma 5.43. *Let $K(Y|Z) : (\mathcal{Z}, \mathcal{B}_{\mathcal{Z}}) \to \mathcal{G}(\mathcal{Y}, \mathcal{B}_{\mathcal{Y}})$ be a measurable and 0-1-deterministic Markov kernel.*

1. *If $(\mathcal{Y}, \mathcal{B}_{\mathcal{Y}})$ is countably generated then there exists a measurable $g : (\mathcal{Z}, \mathcal{B}_{\mathcal{Z}}) \to (\mathcal{Y}, \mathcal{B}_{\mathcal{Y}})$ such that: $K(Y|Z) = \delta_g(Y|Z)$.*
2. *If $(\mathcal{Y}, \mathcal{B}_{\mathcal{Y}})$ is universally countably generated then there exists a universally measurable $g : (\mathcal{Z}, (\mathcal{B}_{\mathcal{Z}})_{\mathcal{G}}) \to (\mathcal{Y}, (\mathcal{B}_{\mathcal{Y}})_{\mathcal{G}})$ such that: $K(Y|Z) = \delta_g(Y|Z)$.*

Furthermore, if $(\mathcal{Y}, \mathcal{B}_{\mathcal{Y}})$ is separated then the map g from above is unique.

Proof. 1.) By assumption we have a countable algebra $\mathcal{E} \subseteq \mathcal{B}_{\mathcal{Y}}$ such that $\sigma(\mathcal{E}) = \mathcal{B}_{\mathcal{Y}}$. Then for every $z \in \mathcal{Z}$ we pick an arbitrary point (using the axiom of choice):

$$g(z) \in C_z := \bigcap_{\substack{B \in \mathcal{E} \\ K(Y \in B|Z=z)=1.}} B \in \mathcal{B}_{\mathcal{Y}}.$$

Note that since $\mathcal{E}$ is countable the intersection C_z lies in $\mathcal{B}_{\mathcal{Y}}$. Furthermore, $C_z \neq \emptyset$. Otherwise we would get the contradiction:

$$1 = K(Y \in C_z^{\complement}|Z = z) \leqslant \sum_{\substack{B \in \mathcal{E} \\ K(Y \in B|Z=z)=1.}} K(Y \in B^{\complement}|Z = z) = 0.$$

To check that g is measurable we only need to check this for $B \in \mathcal{E}$. We then get

$$\begin{aligned} g^{-1}(B) &= \{z \in \mathcal{Z} \,|\, g(z) \in B\} \\ &= \{z \in \mathcal{Z} \,|\, C_z \subseteq B\} \\ &= \{z \in \mathcal{Z} \,|\, K(Y \in B|Z = z) = 1\} \\ &= K(Y \in B|Z)^{-1}(1) \\ &\in \mathcal{B}_{\mathcal{Z}}, \end{aligned}$$

since the Markov kernel is a measurable map for each B. This shows the measurability of g. Similarly, for $B \in \mathcal{E}$ we have:

$$\delta_g(Y \in B|Z = z) = \mathbb{1}_B(g(z)) = K(Y \in B|Z = z).$$

Since $\mathcal{E}$ is a generating π-system the equality also holds for all $B \in \sigma(\mathcal{E}) = \mathcal{B}_{\mathcal{Y}}$. This shows the first claim.
2.) When $K(Y|Z)$ is extended to $(\mathcal{B}_{\mathcal{Y}})_{\mathcal{G}}$ then it is still $(\mathcal{B}_{\mathcal{Z}})_{\mathcal{G}}$-measurable by Proposition 5.3. For fixed $z \in \mathcal{Z}$ and $C \in (\mathcal{B}_{\mathcal{Z}})_{\mathcal{G}}$ we can write:

$$C = B \triangle N,$$

with $B \in \mathcal{B}_{\mathcal{Y}}$ and N a $K(Y|Z = z)$-null set. So $K(Y \in C|Z = z) \in \{0, 1\}$ for all $C \in (\mathcal{B}_{\mathcal{Y}})_{\mathcal{G}}$ and $z \in \mathcal{Z}$ as well.
By assumption there now exists a countably generated σ-algebra $\mathcal{E} \subseteq (\mathcal{B}_{\mathcal{Y}})_{\mathcal{G}}$ such that $(\mathcal{E})_{\mathcal{G}} = (\mathcal{B}_{\mathcal{Y}})_{\mathcal{G}}$. By the same arguments as above we get a $(\mathcal{B}_{\mathcal{Z}})_{\mathcal{G}}$-$\mathcal{E}$-measurable map g such that:

$$\delta_g(Y|Z) = K(Y|Z),$$

when restricted to $\mathcal{E}$. This equality then uniquely extends to $(\mathcal{E})_{\mathcal{G}} = (\mathcal{B}_{\mathcal{Y}})_{\mathcal{G}}$. Note that g is also $(\mathcal{B}_{\mathcal{Z}})_{\mathcal{G}}$-$(\mathcal{E})_{\mathcal{G}}$-measurable, which is the same as being $(\mathcal{B}_{\mathcal{Z}})_{\mathcal{G}}$-$(\mathcal{B}_{\mathcal{Y}})_{\mathcal{G}}$-measurable. This shows the second claim.
For the uniqueness let g_1, g_2 be two different (universally) measurable maps such that:

$$\delta_{g_1}(Y|Z) = \delta_{g_2}(Y|Z).$$

Then there exists a point $z \in \mathcal{Z}$ such that $g_1(z) \neq g_2(z)$. Since $(\mathcal{Y}, \mathcal{B}_{\mathcal{Y}})$ is separated there exists $B \in \mathcal{B}_{\mathcal{Y}}$ such that $g_1(z) \in B$ and $g_2(z) \notin B$, which implies:

$$\delta_{g_1}(Y \in B|Z = z) = 1 \neq 0 = \delta_{g_2}(Y \in B|Z = z),$$

in contradiction to the assumption. So $g_1 = g_2$. $\square$

Theorem 5.44. *Let $(\mathcal{Y}, \mathcal{Y}^\Omega)$ and $(\mathcal{Z}, \mathcal{Z}^\Omega)$ be quasi-universal spaces and $K(Y|Z) \in \mathcal{P}(\mathcal{Y})^{\mathcal{Z}}$. Assume that $(\mathcal{Y}, \mathcal{Y}^\Omega)$ is countably separated with $\mathcal{FB}(\mathcal{Y}^\Omega) = \mathcal{Y}^\Omega$ (e.g. $(\mathcal{Y}, \mathcal{Y}^\Omega)$ a universal quasi-universal space).*[10] *Then the following statements are equivalent:*

1. *$K(Y|Z)$ is* copy-deterministic*, i.e.:*

$$K(Y_1|Z) \otimes K(Y_2|Z) = K(Y_1, Y_2|Z) \quad \in \mathcal{P}(\mathcal{Y} \times \mathcal{Y})^{\mathcal{Z}},$$

   where Y_1 and Y_2 are copies of Y.

2. *$K(Y|Z)$ is* 0-1-deterministic*, i.e. for all $B \in \mathcal{B}_{\mathcal{Y}}$ and $z \in \mathcal{Z}$:*

$$K(Y \in B|Z = z) \in \{0, 1\}.$$

3. *$K(Y|Z)$ is* function-deterministic*, i.e. there exists a (unique) quasi-measurable map $g \in \mathcal{Y}^{\mathcal{Z}}$ such that:*

$$K(Y|Z) = \delta_g(Y|Z) \quad \in \mathcal{P}(\mathcal{Y})^{\mathcal{Z}}.$$

Proof. Two directions follow from Lemma 5.42. For the remaining claim let $K(Y|Z)$ be 0-1-deterministic. We need to be careful with handling the σ-algebras in the following. Let $\mathcal{B}_{\mathcal{Z}} := \mathcal{B}(\mathcal{Z}^\Omega)$ and $\mathcal{B}_{\mathcal{Y}} := \mathcal{B}(\mathcal{Y}^\Omega)$. Since $(\mathcal{Y}, \mathcal{Y}^\Omega)$ is countably separated we have a countably generated σ-algebra $\mathcal{E}_{\mathcal{Y}} \subseteq \mathcal{B}_{\mathcal{Y}}$ that separates the points of $\mathcal{Y}$ such that

[10] Note that all quasi-universal spaces that have embeddings $(\mathcal{Y}, \mathcal{Y}^\Omega) \hookrightarrow (\Omega, \Omega^\Omega)$ in **QUS** satisfy these conditions, see Lemma 5.12. Also all (univerally) countably separated measurable spaces satisfy these conditions after applying the functor $\mathcal{F}$.

$\mathcal{B}_\mathcal{Y} = (\mathcal{E}_\mathcal{Y})_{\mathcal{P}(\mathcal{Y})}$, see Lemma 5.5. Because of $\mathcal{B}_\mathcal{Y} = (\mathcal{E}_\mathcal{Y})_{\mathcal{P}(\mathcal{Y})}$ we know that the following composition of (measurable) inclusion and restriction maps is injective:

$$\text{res}: \; (\mathcal{P}(\mathcal{Y}), \mathcal{B}(\mathcal{P}(\mathcal{Y})^\Omega)) \xrightarrow{\text{incl}} (\mathcal{G}(\mathcal{Y}), \mathcal{B}(\mathcal{G}(\mathcal{Y})^\Omega)) \xrightarrow{\text{res}} (\mathcal{G}(\mathcal{Y}, \mathcal{E}_\mathcal{Y}), \mathcal{B}_{\mathcal{G}(\mathcal{Y},\mathcal{E}_\mathcal{Y})}),$$

where $\mathcal{B}_{\mathcal{G}(\mathcal{Y},\mathcal{E}_\mathcal{Y})}$ is the σ-algebra generated by the evaluation maps ev_B for $B \in \mathcal{E}_\mathcal{Y}$, which is countably generated (since $\mathcal{E}_\mathcal{Y}$ is) and separates the points of $\mathcal{G}(\mathcal{Y}, \mathcal{E}_\mathcal{Y})$. The σ-algebra $\mathcal{B}_{\mathcal{P}(\mathcal{Y})} := \text{res}^*\mathcal{B}_{\mathcal{G}(\mathcal{Y},\mathcal{E}_\mathcal{Y})} \subseteq \mathcal{B}(\mathcal{P}(\mathcal{Y})^\Omega)$ is then also countably generated and separates the points of $\mathcal{P}(\mathcal{Y})$, since res is injective. Again by Lemma 5.5 we then have that $\mathcal{B}(\mathcal{P}(\mathcal{Y})^\Omega) = \left(\mathcal{B}_{\mathcal{P}(\mathcal{Y})}\right)_{\mathcal{P}(\mathcal{P}(\mathcal{Y}))}$. Also note that: $(\mathcal{B}_\mathcal{Z})_{\mathcal{P}(\mathcal{Z})} = \mathcal{B}_\mathcal{Z}$.
We then have the composition of measurable maps:

$$(\mathcal{Z}, \mathcal{B}_\mathcal{Z}) \xrightarrow{K(Y|Z)} (\mathcal{P}(\mathcal{Y}), \mathcal{B}_{\mathcal{P}(\mathcal{Y})}) \xhookrightarrow{\text{res}} (\mathcal{G}(\mathcal{Y}, \mathcal{E}_\mathcal{Y}), \mathcal{B}_{\mathcal{G}(\mathcal{Y},\mathcal{E}_\mathcal{Y})}).$$

Since $K(Y|Z)$ is 0-1-deterministic and $\mathcal{E}_\mathcal{Y}$ countably generated by Lemma 5.43 there exists a (unique) $\mathcal{B}_\mathcal{Z}$-$\mathcal{E}_\mathcal{Y}$-measurable map g such that for $z \in \mathcal{Z}$:

$$K(Y|Z = z) = \delta_g(Y|Z = z) \quad \in \mathcal{G}(\mathcal{Y}, \mathcal{E}_\mathcal{Y}),$$

suppressing mentioning res for brevity. So we get the following commutative diagram of measurable maps:

$$\begin{array}{ccc}
(\mathcal{Z}, \mathcal{B}_\mathcal{Z}) & \xrightarrow{\quad K(Y|Z) \quad} & (\mathcal{P}(\mathcal{Y}), \mathcal{B}_{\mathcal{P}(\mathcal{Y})}) \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \text{res}} \\
(\mathcal{Y}, \mathcal{E}_\mathcal{Y}) & \xhookrightarrow[\quad \delta \quad]{} & (\mathcal{G}(\mathcal{Y}, \mathcal{E}_\mathcal{Y}), \mathcal{B}_{\mathcal{G}(\mathcal{Y},\mathcal{E}_\mathcal{Y})}).
\end{array}$$

We now want to show that g is even $\mathcal{B}_\mathcal{Z}$-$\mathcal{B}_\mathcal{Y}$-measurable. Since $\mathcal{B}_\mathcal{Y} = (\mathcal{E}_\mathcal{Y})_{\mathcal{P}(\mathcal{Y})}$ we want to complete $\mathcal{E}_\mathcal{Y}$ w.r.t. $\mathcal{P}(\mathcal{Y})$ and show that g stays measurable even after completion. By Lemma 5.2 and $(\mathcal{B}_\mathcal{Z})_{\mathcal{P}(\mathcal{Z})} = \mathcal{B}_\mathcal{Z}$ it is enough to show that $g_*(\mathcal{P}(\mathcal{Z})) \subseteq \mathcal{P}(\mathcal{Y})$. We start by noting that the equality $K(Y|Z) = \delta_g(Y|Z)$ induces the identity:

$$\delta_* \circ g_* = \text{res}_* \circ K(Y|X)_* : \; \mathcal{G}(\mathcal{Z}, \mathcal{B}_\mathcal{Z}) \to \mathcal{G}\left(\mathcal{G}(\mathcal{Y}, \mathcal{E}_\mathcal{Y}), \mathcal{B}_{\mathcal{G}(\mathcal{Y},\mathcal{E}_\mathcal{Y})}\right).$$

We can apply the monad action:

$$\mathbb{M}: \; \mathcal{G}\left(\mathcal{G}(\mathcal{Y}, \mathcal{E}_\mathcal{Y}), \mathcal{B}_{\mathcal{G}(\mathcal{Y},\mathcal{E}_\mathcal{Y})}\right) \to \mathcal{G}(\mathcal{Y}, \mathcal{E}_\mathcal{Y}),$$

and use the monad rule that $\mathbb{M} \circ \delta_* = \text{id}$ to get the equality (suppressing res_* for brevity):

$$g_* = \mathbb{M} \circ K(Y|X)_* : \; \mathcal{G}(\mathcal{Z}, \mathcal{B}_\mathcal{Z}) \to \mathcal{G}(\mathcal{Y}, \mathcal{E}_\mathcal{Y}).$$

Since $K(Y|Z): \; (\mathcal{Z}, \mathcal{Z}^\Omega) \to (\mathcal{P}(\mathcal{Y}), \mathcal{P}(\mathcal{Y})^\Omega)$ is quasi-measurable, we get that $K(Y|Z)_*(\mathcal{P}(\mathcal{Z})) \subseteq \mathcal{P}(\mathcal{P}(\mathcal{Y}))$. Since $(\mathcal{P}, \delta, \mathbb{M})$ is a monad we also get that $\mathbb{M}: \; \mathcal{P}(\mathcal{P}(\mathcal{Y})) \to \mathcal{P}(\mathcal{Y})$ is a well-defined map, showing that: $\mathbb{M}(\mathcal{P}(\mathcal{P}(\mathcal{Y}))) \subseteq \mathcal{P}(\mathcal{Y})$. Note that $\mathbb{M}$ for $\mathcal{G}$ and $\mathbb{M}$ for $\mathcal{P}$

commute with res, i.e. if $\pi \in \mathcal{P}(\mathcal{P}(\mathcal{Y}))$ and $B \in \mathcal{E}_\mathcal{Y}$ we get:

$$\begin{aligned}\mathbb{M}_\mathcal{G}(\mathrm{res}_*\pi)(B) &= \int_{\mathcal{G}(\mathcal{Y},\mathcal{E}_\mathcal{Y})} \mathrm{ev}_B(\mu)\,(\mathrm{res}_*\pi)(d\mu)\\ &= \int_{\mathcal{P}(\mathcal{Y})} \mathrm{ev}_B(\mathrm{res}(\nu))\,\pi(d\nu)\\ &= \int_{\mathcal{P}(\mathcal{Y})} \mathrm{ev}_B(\nu)\,\pi(d\nu)\\ &= \mathbb{M}_\mathcal{P}(\pi)(B).\end{aligned}$$

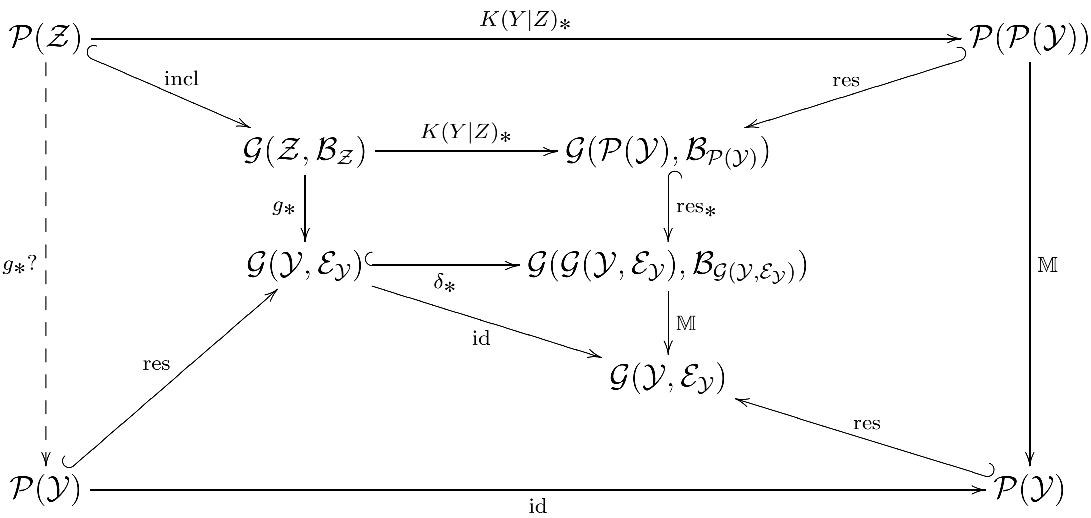
Without making further distinctions between the $\mathbb{M}$'s we then arrive at:

$$g_*\left(\mathcal{P}(\mathcal{Z})\right) = \mathbb{M} \circ K(Y|Z)_*\left(\mathcal{P}(\mathcal{Z})\right) \subseteq \mathbb{M}\left(\mathcal{P}(\mathcal{P}(\mathcal{Y}))\right) \subseteq \mathcal{P}(\mathcal{Y}).$$

Since g is $\mathcal{B}_\mathcal{Z}$-$\mathcal{E}_\mathcal{Y}$-measurable it is then, because of $g_*\left(\mathcal{P}(\mathcal{Z})\right) \subseteq \mathcal{P}(\mathcal{Y})$, by Lemma 5.2, also $(\mathcal{B}_\mathcal{Z})_{\mathcal{P}(\mathcal{Z})}$-$(\mathcal{E}_\mathcal{Y})_{\mathcal{P}(\mathcal{Y})}$-measurable. Since $(\mathcal{B}_\mathcal{Z})_{\mathcal{P}(\mathcal{Z})} = \mathcal{B}_\mathcal{Z}$ and $(\mathcal{E}_\mathcal{Y})_{\mathcal{P}(\mathcal{Y})} = \mathcal{B}_\mathcal{Y}$ we get that g is $\mathcal{B}_\mathcal{Z}$-$\mathcal{B}_\mathcal{Y}$-measurable.
Now let $\gamma \in \mathcal{Z}^\Omega \subseteq \mathcal{F}(\mathcal{B}_\mathcal{Z})$. Then we have $g \circ \gamma \in \mathcal{F}(\mathcal{B}_\mathcal{Y}) = \mathcal{Y}^\Omega$, where the last equality holds by assumption. This shows that $g \in \mathcal{Y}^\mathcal{Z}$ and thus: $\delta_g(Y|Z) \in \mathcal{P}(\mathcal{Y})^\mathcal{Z}$. Both $K(Y|Z)$ and $\delta_g(Y|Z)$ are in $\mathcal{P}(\mathcal{Y})^\mathcal{Z}$ and we already have the equality: $K(Y|Z) = \delta_g(Y|Z)$ on $\mathcal{E}_\mathcal{Y}$. Since $(\mathcal{E}_\mathcal{Y})_{\mathcal{P}(\mathcal{Y})} = \mathcal{B}_\mathcal{Y}$ the above equality uniquely extends to $\mathcal{B}_\mathcal{Y}$ and we get:

$$K(Y|Z) = \delta_g(Y|Z) \quad \in \mathcal{P}(\mathcal{Y})^\mathcal{Z}.$$

This shows that $K(Y|Z)$ is function-deterministic.



□

Remark 5.45. *One can prove an "almost-sure" version of Theorem 5.44 w.r.t. some given Markov kernel* $Q(Z|T) \in \mathcal{P}(\mathcal{Z})^\mathcal{T}$ *under the same conditions. For this one needs to gather the countably many null-sets* N_B *for each* $B \in \mathcal{E}_\mathcal{Y}$*, which is then again a null-set w.r.t.* $Q(Z|T)$*.*

5.10 The Markov Category of Markov Kernels between Universal Quasi-Universal Spaces

Here we quickly summarize our findings in terms of categorical probability theory without going into the details of these definitions. They can be found in [FR20, Fri20, FGPR20, FGP21]. As a Corollary we get a *conditional de Finetti theorem* out.

Theorem 5.46 (The Markov category of Markov kernels between universal quasi-universal spaces)**.** *The category of universal quasi-universal spaces* $(\mathbf{UQUS}, \times, \mathbf{1})$ *is a symmetric cartesian (but not closed) monoidal category with countable products and countable coproducts. The triple* $(\mathcal{P}, \delta, \mathbb{M})$ *is a strong affine symmetric monoidal/commutative monad on* $(\mathbf{UQUS}, \times, \mathbf{1})$*. Its Kleisli category* $\mathrm{Kl}(\mathcal{P})$ *on* $\mathbf{UQUS}$ *is an a.s.-compatibly representable Markov category with conditionals and Kolmogorov powers.*

Proof. We apply Prop. 3.1. from [FGPR20].
Strong monad $\mathcal{P}$: See Theorem 3.25. Note also that $\mathcal{K} = \mathcal{P} = \mathcal{R} = \mathcal{Q} = \mathcal{G} = \mathcal{S}$ for universal quasi-universal spaces by Theorem 5.30 and that $\mathcal{P}(\mathcal{X})$ stays inside $\mathbf{UQUS}$.
Symmetry/commutativity: See Fubini Theorem 5.33.
Conditionals: See Disintegration Theorem 5.34.
Kolmogorov powers: See Kolmogorov's Extension Theorem 5.40.
Deterministic morphisms/Markov kernels: See Theorem 5.44.
A.s.-compatible representability: See Remark 5.45 and essential uniqueness of factorizations Theorem 5.37.
All other aspects are easy to see or follow from general considerations. □

Corollary 5.47 (Conditional De Finetti Theorem, see [DF37, Kal06, FGP21])**.** *Let* $(\mathcal{X}, \mathcal{X}^\Omega)$ *be a universal quasi-universal space and* $(\mathcal{Z}, \mathcal{Z}^\Omega)$ *another quasi-universal space. For a Markov kernel* $Q(X_\mathbb{N}|Z) \in \mathcal{P}(\mathcal{X}^\mathbb{N})^\mathcal{Z}$ *the following are equivalent:*

1. $Q(X_\mathbb{N}|Z)$ *is* exchangable*, i.e. invariant under all finite permutations* $\sigma : \mathbb{N} \xrightarrow{\sim} \mathbb{N}$.
2. *There exist a quasi-universal space* $\mathcal{S}$ *and* $Q(X|S) \in \mathcal{P}(\mathcal{X})^\mathcal{S}$ *and* $P(S|Z) \in \mathcal{P}(\mathcal{S})^\mathcal{Z}$ *such that:*
$$Q(X_\mathbb{N}|Z) = (\otimes_{n\in\mathbb{N}} Q(X_n|S)) \circ P(S|Z).$$

If this is the case one can w.l.o.g. choose $\mathcal{S} = \mathcal{P}(\mathcal{X})$ *and*

$$Q(X|S) = \mathrm{id}_{\mathcal{P}(\mathcal{X})} : \mathcal{S} = \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X}), \qquad Q(X \in A|S = P) := P(A).$$

Proof. This directly follows from the synthetic version of de Finetti's Theorem in categorical probability theory, see [FGP21], together with Theorem 5.46. Note that for our case we do not need to require that $(\mathcal{Z}, \mathcal{Z}^\Omega)$ is a *universal* quasi-universal space as all required results work without that assumption. □

6 Causal Models

This section is dedicated to the notion of conditional independence and causal models. More clearly, we will translate the notion of *transitional conditional independence* studied in [For21] from **Meas** to **QUS**. This allows us to mix random variables with deterministic non-random variables in conditional independence statements. Furthermore, we will engage in causal graphical models and demonstrate our results on *causal Bayesian networks*. Generalizations to other probabilistic graphical models will be apparent. We also re-formulate the notion of *strong ignorability* using random functions with values in function spaces of quasi-universal spaces.

6.1 Transitional Conditional Independence

In this subsection we translate all structures related to *transitional conditional independence* defined for measurable spaces in [For21] to quasi-universal spaces. We will also investigate the *(asymmetric) separoid rules*.

Notation 6.1 (Transition probability space)**.** *In the following we fix two quasi-universal spaces* $(\mathcal{T}, \mathcal{T}^\Omega)$ *and* $(\mathcal{W}, \mathcal{W}^\Omega)$*. The generic choice is to take* $(\mathcal{W}, \mathcal{W}^\Omega) = (\Omega, \Omega^\Omega)$*, but we allow more general spaces for now. We also fix a Markov kernel* $\mathbf{K}(W|T) \in \mathcal{P}(\mathcal{W})^{\mathcal{T}}$*. We will call the tuple* $(\mathcal{W} \times \mathcal{T}, \mathbf{K}(W|T))$ *our* transition probability space*.*

Definition 6.2 (Transitional random variable)**.** *A* transitional random variable *is a Markov kernel from our transition probability space* $(\mathcal{W} \times \mathcal{T}, \mathbf{K}(W|T))$ *to any other quasi-universal space* $(\mathcal{X}, \mathcal{X}^\Omega)$*:*

$$\mathbf{X}(X|W, T) \in \mathcal{P}(\mathcal{X})^{\mathcal{W} \times \mathcal{T}}.$$

Example 6.3 (Special transitional random variables of importance)**.** *We denote the:*

1. *canonical projection onto* $\mathcal{T}$ *as:*

$$T := \mathrm{pr}_{\mathcal{T}} : \ \mathcal{W} \times \mathcal{T} \to \mathcal{T}, \qquad T(w, t) := t,$$

   and put:

$$\mathbf{T}(T|W, T) = \boldsymbol{\delta}(T|W, T) = \delta \circ T \in \mathcal{P}(\mathcal{T})^{\mathcal{W} \times \mathcal{T}}.$$

2. *constant transitional random variable as:*

$$\boldsymbol{\delta}_0 : \ \mathcal{W} \times \mathcal{T} \to \mathcal{P}(\mathbf{1}) = \mathbf{1} = \{0\}.$$

Notation 6.4. *For two transitional random variables* $\mathbf{X} \in \mathcal{P}(\mathcal{X})^{\mathcal{W} \times \mathcal{T}}$ *and* $\mathbf{Y} \in \mathcal{P}(\mathcal{Y})^{\mathcal{W} \times \mathcal{T}}$ *we define their joint Markov kernel as:*

$$\mathbf{K}(X, Y|T) := (\mathbf{X}(X|W, T) \otimes \mathbf{Y}(Y|W, T)) \circ \mathbf{K}(W|T) \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})^{\mathcal{T}}$$

We then define the relation:

$$\mathbf{X} \precsim_{\mathbf{K}} \mathbf{Y} \qquad :\Longleftrightarrow \qquad \exists \varphi \in \mathcal{X}^{\mathcal{Y}}. \quad \mathbf{K}(X, Y|T) = \boldsymbol{\delta}_\varphi(X|Y) \otimes \mathbf{K}(Y|T).$$

Definition 6.5 (Transitional conditional independence)**.** *For our transition probability space* $(\mathcal{W} \times \mathcal{T}, \mathbf{K}(W|T))$ *and three transitional random variables* $\mathbf{X} \in \mathcal{P}(\mathcal{X})^{\mathcal{W}\times\mathcal{T}}$ *and* $\mathbf{Y} \in \mathcal{P}(\mathcal{Y})^{\mathcal{W}\times\mathcal{T}}$ *and* $\mathbf{Z} \in \mathcal{P}(\mathcal{Z})^{\mathcal{W}\times\mathcal{T}}$ *we consider their joint Markov kernel:*

$$\mathbf{K}(X,Y,Z|T) := (\mathbf{X}(X|W,T) \otimes \mathbf{Y}(Y|W,T) \otimes \mathbf{Z}(Z|W,T)) \circ \mathbf{K}(W|T) \in \mathcal{P}(\mathcal{X} \times \mathcal{Y} \times \mathcal{Z})^{\mathcal{T}}.$$

We then say that $\mathbf{X}$ *is* independent of $\mathbf{Y}$ conditioned on $\mathbf{Z}$ *w.r.t.* $\mathbf{K}$*, in symbols:*

$$\mathbf{X} \underset{\mathbf{K}}{\perp\!\!\!\perp} \mathbf{Y} \,|\, \mathbf{Z},$$

if there exists a Markov kernel $\mathbf{Q}(X|Z) \in \mathcal{P}(\mathcal{X})^{\mathcal{Z}}$ *such that:*

$$\mathbf{K}(X,Y,Z|T) = \mathbf{Q}(X|Z) \otimes \mathbf{K}(Y,Z|T) \in \mathcal{P}(\mathcal{X} \times \mathcal{Y} \times \mathcal{Z})^{\mathcal{T}}.$$

Lemma 6.6. *Consider transitional random variables* $\mathbf{X} \in \mathcal{P}(\mathcal{X})^{\mathcal{W}\times\mathcal{T}}$ *and* $\mathbf{Y} \in \mathcal{P}(\mathcal{Y})^{\mathcal{W}\times\mathcal{T}}$ *and* $\mathbf{Z} \in \mathcal{P}(\mathcal{Z})^{\mathcal{W}\times\mathcal{T}}$*. Assume that:*

$$\Delta_{\mathcal{X}} := \{(x,x) \in \mathcal{X} \times \mathcal{X} \,|\, x \in \mathcal{X}\} \in \mathcal{B}_{\mathcal{X}\times\mathcal{X}}.$$ [11]

Then we have the implication:

$$\mathbf{X} \precsim_{\mathbf{K}} \mathbf{Z} \qquad \Longrightarrow \qquad \mathbf{X} \underset{\mathbf{K}}{\perp\!\!\!\perp} \mathbf{Y} \,|\, \mathbf{Z}.$$

Proof. Assume $\mathbf{X} \precsim_{\mathbf{K}} \mathbf{Z}$. Then there exists a $\varphi \in \mathcal{X}^{\mathcal{Z}}$ such that:

$$\mathbf{K}(X,Z|T) = \boldsymbol{\delta}_{\varphi}(X|Z) \otimes \mathbf{K}(Z|T).$$

Since $\Delta_{\mathcal{X}} \in \mathcal{B}_{\mathcal{X}\times\mathcal{X}}$ we know that:

$$M := \{(x,z) \in \mathcal{X} \times \mathcal{Z} \,|\, x \neq \varphi(z)\} = (\mathrm{id}_{\mathcal{X}} \times \varphi)^{-1}(\Delta_{\mathcal{X}}^{\complement}) \in \mathcal{B}_{\mathcal{X}\times\mathcal{Z}}.$$

Then M is a $\mathbf{K}(X,Z|T)$-null set. Indeed:

$$\begin{aligned}\mathbf{K}\left((X,Z) \in M|T\right) &= \int \boldsymbol{\delta}_{\varphi}(X \in M_z|Z=z)\, \mathbf{K}(Z \in dz|T) \\ &= \int \mathbb{1}_{M_z}(\varphi(z))\, \mathbf{K}(Z \in dz|T) \\ &= \int 0\, \mathbf{K}(Z \in dz|T) \\ &= 0.\end{aligned}$$

Now consider the Markov kernel $\mathbf{K}(X_1, X_2, Z, Y|T) \in \mathcal{P}\left(\mathcal{X} \times \mathcal{X} \times \mathcal{Z} \times \mathcal{Y}\right)^{\mathcal{T}}$ with $X_1 := X$ and $X_2 := \varphi(Z)$ and the set:

$$N := \{(x_1,x_2,z,y) \in \mathcal{X} \times \mathcal{X} \times \mathcal{Z} \times \mathcal{Y} \,|\, x_1 \neq x_2\} = \Delta_{\mathcal{X}}^{\complement} \times \mathcal{Z} \times \mathcal{Y} \in \mathcal{B}_{\mathcal{X}\times\mathcal{X}\times\mathcal{Z}\times\mathcal{Y}}.$$

[11] If $(\mathcal{X}, \mathcal{X}^{\Omega})$ is countably separated then we have: $\Delta_{\mathcal{X}} \in \mathcal{B}_{\mathcal{X}} \otimes \mathcal{B}_{\mathcal{X}} \subseteq \mathcal{B}_{\mathcal{X}\times\mathcal{X}}$.

Then we have:

$$\mathbf{K}\left((X_1, X_2, Z, Y) \in N|T\right) = \mathbf{K}\left((X, Z, Y) \in M \times \mathcal{Y}|T\right) = 0.$$

So for every $D \in \mathcal{B}_{\mathcal{X}\times\mathcal{X}\times\mathcal{Z}\times\mathcal{Y}}$ we get:

$$\begin{aligned}\mathbf{K}\left((X_1, X_2, Z, Y) \in D|T\right) &= \mathbf{K}\left((X_1, X_2, Z, Y) \in D \cap N^{\mathsf{c}}|T\right)\\ &= \mathbf{K}\left((X_2, X_1, Z, Y) \in D \cap N^{\mathsf{c}}|T\right)\\ &= \mathbf{K}\left((X_2, X_1, Z, Y) \in D|T\right).\end{aligned}$$

For $D = \mathcal{X} \times E$ with $E \in \mathcal{B}_{\mathcal{X}\times\mathcal{Z}\times\mathcal{Y}}$ this gives:

$$\begin{aligned}\left(\boldsymbol{\delta}_\varphi(X|Z) \otimes \mathbf{K}(Y, Z|T)\right)(E) &= \mathbf{K}\left((X_2, Z, Y) \in E|T\right)\\ &= \mathbf{K}\left((X_1, Z, Y) \in E|T\right)\\ &= \mathbf{K}\left((X, Z, Y) \in E|T\right).\end{aligned}$$

So this implies that we have:

$$\mathbf{K}(X, Y, Z|T) = \boldsymbol{\delta}_\varphi(X|Z) \otimes \mathbf{K}(Y, Z|T),$$

for every transitional random variable $\mathbf{Y}$. If we pick $\mathbf{Q}(X|Z) := \boldsymbol{\delta}_\varphi(X|Z)$ we showed:

$$\mathbf{X} \underset{\mathbf{K}}{\perp\!\!\!\perp} \mathbf{Y} \,|\, \mathbf{Z},$$

which is the claim. □

Theorem 6.7 (Separoid rules for transitional conditional independence)**.** *Consider a transition probability space $(\mathcal{W} \times \mathcal{T}, \mathbf{K}(W|T))$ and transitional random variables $\mathbf{X} \in \mathcal{P}(\mathcal{X})^{\mathcal{W}\times\mathcal{T}}$ and $\mathbf{Y} \in \mathcal{P}(\mathcal{Y})^{\mathcal{W}\times\mathcal{T}}$ and $\mathbf{Z} \in \mathcal{P}(\mathcal{Z})^{\mathcal{W}\times\mathcal{T}}$ and $\mathbf{U} \in \mathcal{P}(\mathcal{U})^{\mathcal{W}\times\mathcal{T}}$. Then the ternary relation $\perp\!\!\!\perp = \perp\!\!\!\perp_{\mathbf{K}(W|T)}$ satisfies the following rules:*

a) Extended Left Redundancy (if $\Delta_{\mathcal{X}} \in \mathcal{B}_{\mathcal{X}\times\mathcal{X}}$[11]):

$\mathbf{X} \precsim_{\mathbf{K}} \mathbf{Z} \implies \mathbf{X} \perp\!\!\!\perp \mathbf{Y} \,|\, \mathbf{Z}.$

Proof: Lemma 6.6.

b) $\mathbf{T}$-Restricted Right Redundancy (for $\mathcal{Z}$ countably separated and either $\mathcal{X}$ or $\mathcal{T}$ universal):

$\mathbf{X} \perp\!\!\!\perp \boldsymbol{\delta}_0 \,|\, \mathbf{Z} \otimes \mathbf{T}$ *always holds.*

Proof: Disintegration Theorem 5.35: $\mathbf{K}(X, Z|T) = \mathbf{Q}(X|Z, T) \otimes \mathbf{K}(Z|T)$.

c) Left Decomposition:

$\mathbf{X} \otimes \mathbf{U} \perp\!\!\!\perp \mathbf{Y} \,|\, \mathbf{Z} \implies \mathbf{U} \perp\!\!\!\perp \mathbf{Y} \,|\, \mathbf{Z}.$

Proof: *Take the marginal* $\mathbf{Q}(U|Z)$ *of* $\mathbf{Q}(X, U|Z)$.

d) *Right Decomposition:*

$$\mathbf{X} \perp\!\!\!\perp \mathbf{Y} \otimes \mathbf{U} \,|\, \mathbf{Z} \implies \mathbf{X} \perp\!\!\!\perp \mathbf{U} \,|\, \mathbf{Z}.$$

Proof: *Take* $\mathbf{Q}(X|Z)$ *again.*

e) $\mathbf{T}$*-Inverted Right Decomposition:*

$$\mathbf{X} \perp\!\!\!\perp \mathbf{Y} \,|\, \mathbf{Z} \implies \mathbf{X} \perp\!\!\!\perp \mathbf{T} \otimes \mathbf{Y} \,|\, \mathbf{Z}.$$

Proof: *Take* $\mathbf{Q}(X|Z)$ *again.*

f) *Left Weak Union (for* $\mathcal{U}$ *countably separated and either* $\mathcal{X}$ *or* $\mathcal{Z}$ *universal):*

$$\mathbf{X} \otimes \mathbf{U} \perp\!\!\!\perp \mathbf{Y} \,|\, \mathbf{Z} \implies \mathbf{X} \perp\!\!\!\perp \mathbf{Y} \,|\, \mathbf{U} \otimes \mathbf{Z}.$$

Proof: *Disintegration Theorem 5.35:* $\mathbf{Q}(X, U|Z) = \mathbf{Q}(X|U, Z) \otimes \mathbf{Q}(U|Z)$.

g) *Right Weak Union:*

$$\mathbf{X} \perp\!\!\!\perp \mathbf{Y} \otimes \mathbf{U} \,|\, \mathbf{Z} \implies \mathbf{X} \perp\!\!\!\perp \mathbf{Y} \,|\, \mathbf{U} \otimes \mathbf{Z}.$$

Proof: *Take* $\mathbf{Q}(X|U, Z) := \mathbf{Q}(X|Z)$.

h) *Left Contraction:*

$$(\mathbf{X} \perp\!\!\!\perp \mathbf{Y} \,|\, \mathbf{U} \otimes \mathbf{Z}) \wedge (\mathbf{U} \perp\!\!\!\perp \mathbf{Y} \,|\, \mathbf{Z}) \implies \mathbf{X} \otimes \mathbf{U} \perp\!\!\!\perp \mathbf{Y} \,|\, \mathbf{Z}.$$

Proof: *Take* $\mathbf{Q}_1(X|U, Z) \otimes \mathbf{Q}_2(U|Z)$.

i) *Right Contraction:*

$$(\mathbf{X} \perp\!\!\!\perp \mathbf{Y} \,|\, \mathbf{U} \otimes \mathbf{Z}) \wedge (\mathbf{X} \perp\!\!\!\perp \mathbf{U} \,|\, \mathbf{Z}) \implies \mathbf{X} \perp\!\!\!\perp \mathbf{Y} \otimes \mathbf{U} \,|\, \mathbf{Z}.$$

Proof: *Take* $\mathbf{Q}_2(X|Z)$ *and use essential uniqueness of factorizations Theorem 5.37.*

j) *Right Cross Contraction:*

$$(\mathbf{X} \perp\!\!\!\perp \mathbf{Y} \,|\, \mathbf{U} \otimes \mathbf{Z}) \wedge (\mathbf{U} \perp\!\!\!\perp \mathbf{X} \,|\, \mathbf{Z}) \implies \mathbf{X} \perp\!\!\!\perp \mathbf{Y} \otimes \mathbf{U} \,|\, \mathbf{Z}.$$

Proof: *Take* $\mathbf{Q}_1(X|U, Z) \circ \mathbf{Q}_2(U|Z)$ *and use essential uniqueness of factorizations Theorem 5.37.*

k) *Flipped Left Cross Contraction:*

$$(\mathbf{X} \perp\!\!\!\perp \mathbf{Y} \,|\, \mathbf{U} \otimes \mathbf{Z}) \wedge (\mathbf{Y} \perp\!\!\!\perp \mathbf{U} \,|\, \mathbf{Z}) \implies \mathbf{Y} \perp\!\!\!\perp \mathbf{X} \otimes \mathbf{U} \,|\, \mathbf{Z}.$$

Proof: *Take* $\mathbf{Q}_2(Y|Z)$.

Proof. The constructions follow the same lines as in [For21] Theorem 3.11 and Appendix E. Note that for Left Weak Union we need the disintegration $\mathbf{Q}(X, U|Z) = \mathbf{Q}(X|U, Z) \otimes \mathbf{Q}(U|Z)$. That is the reason we need to make the extra assumptions on the underlying spaces, see disintegration Theorem 5.35. Similarly, for $\mathbf{T}$-Restricted Right Redundancy we need the disintegration $\mathbf{K}(X, Z|T) = \mathbf{K}(X|Z, T) \otimes \mathbf{K}(Z|T)$. The corresponding conditions on the underlying spaces could be weakened if one could find weaker conditions under which disintegrations exist, see again Theorem 5.35. ☐

6.2 Conditional Directed Acyclic Graphs (CDAGs)

In this subsection we will gather all purely graph theoretical structures that are needed for causal Bayesian networks later on. The main focus is on the definition of *conditional directed acyclic graphs (CDAGs)* and on the recap of *d-separation* in such graphs.

Definition 6.8 (Conditional directed acyclic graphs (CDAGs))**.** *A* conditional directed acyclic graph (CDAG) $\mathbf{G} = (J, V, E)$ *consists of two (disjoint) sets of vertices/nodes: the set of* input nodes J*, the set of* output nodes V*; and a set of* directed edges*:*

$$E \subseteq \{w \rightarrowtail v \,|\, w \in J \cup V, v \in V\},$$

such that there are no directed non-trivial cycles*. Note that - per definition - there won't be any arrow heads pointing to input nodes $j \in J$.*

Notation 6.9. *For a CDAG we will suggestively write:* $\mathbf{G}(V|\,\mathrm{do}(J)) := (J, V, E) = \mathbf{G}$*, where the sets of edges E are implicit. We will also write: $v \in \mathbf{G}$ to mean $v \in J \cup V$, $v_1 \leftarrowtail v_2 \in \mathbf{G}$ to mean $v_2 \rightarrowtail v_1 \in E$ and $v_1 \ast\!\!-\!\!\ast v_2 \in \mathbf{G}$ to mean that either $v_1 \leftarrowtail v_2 \in \mathbf{G}$ or $v_1 \rightarrowtail v_2 \in \mathbf{G}$, etc.*

Definition 6.10 (Walks)**.** *Let $\mathbf{G} = \mathbf{G}(V|\,\mathrm{do}(J))$ be a CDAG and $v, w \in \mathbf{G}$.*

1. *A* walk *from v to w in $\mathbf{G}$ is a finite sequence of nodes and edges*

$$v = v_0 \ast\!\!-\!\!\ast v_1 \ast\!\!-\!\!\ast \cdots \ast\!\!-\!\!\ast v_{n-1} \ast\!\!-\!\!\ast v_n = w$$

   in $\mathbf{G}$ for some $n \geqslant 0$, i.e. such that for every $k = 1, \dots, n$ we have that $v_{k-1} \ast\!\!-\!\!\ast v_k \in \mathbf{G}$, and with $v_0 = v$ and $v_n = w$.

   In the definition of walk the appearance of the same nodes several times is allowed. Also the trivial walk *consisting of a single node $v_0 \in \mathbf{G}$ (without any edges) is allowed as well (if $v = w$).*

2. *A* directed walk *from v to w in $\mathbf{G}$ is of the form:*

$$v = v_0 \rightarrowtail v_1 \rightarrowtail \cdots \rightarrowtail v_{n-1} \rightarrowtail v_n = w,$$

   for some $n \geqslant 0$, where all arrow heads point in the direction of w and there are no arrow heads pointing back.

Definition 6.11. *For a CDAG* $\mathbf{G} = \mathbf{G}(V|\operatorname{do}(J))$ *and* $v \in \mathbf{G}$ *we define the sets:*

1. Parents*:* $\operatorname{Pa}^{\mathbf{G}}(v) := \{w \in \mathbf{G} \,|\, w \rightarrowtail v \in \mathbf{G}\}$,
2. Children*:* $\operatorname{Ch}^{\mathbf{G}}(v) := \{w \in \mathbf{G} \,|\, v \rightarrowtail w \in \mathbf{G}\}$,
3. Ancestors*:* $\operatorname{Anc}^{\mathbf{G}}(v) := \{w \in \mathbf{G} \,|\, \exists$ *directed walk in* $\mathbf{G}: w \rightarrowtail \cdots \rightarrowtail v\}$,
4. Descendents*:* $\operatorname{Desc}^{\mathbf{G}}(v) := \{w \in \mathbf{G} \,|\, \exists$ *directed walk in* $\mathbf{G}: v \rightarrowtail \cdots \rightarrowtail w\}$,

We extend these notions to sets $A \subseteq J \cup V$ *by taking the union, e.g.* $\operatorname{Anc}^{\mathbf{G}}(A) = \bigcup_{v \in A} \operatorname{Anc}^{\mathbf{G}}(v)$.

Remark 6.12 (Acyclicity)**.** *The requirement that a CDAG* $\mathbf{G} = \mathbf{G}(V|\operatorname{do}(J))$ *has no non-trivial cycles in the above terms means that the only directed walk from a node* $v \in \mathbf{G}$ *to itself inside* $\mathbf{G}$ *is trivial.*

Definition 6.13 (Topological order)**.** *Let* $\mathbf{G} = \mathbf{G}(V|\operatorname{do}(J))$ *be a CDAG. A* topological order *of* $\mathbf{G}$ *is a total order* $<$ *of* $J \cup V$ *such that for all* $v, w \in \mathbf{G}$*:*

$$v \in \operatorname{Pa}^{\mathbf{G}}(w) \implies v < w.$$

A topological order always exists for every CDAG.

Definition 6.14 (Extended CDAG)**.** *Let* $\mathbf{G}(V|\operatorname{do}(J))$ *be a CDAG and* $W \subseteq V$ *a subset. Then the* extended CDAG *w.r.t.* W *is the CDAG* $\mathbf{G}(V|\operatorname{do}(J \,\dot\cup\, I_W))$ *that arises from* $\mathbf{G}(V|\operatorname{do}(J))$ *by adding the new nodes* I_w *for* $w \in W$ *to* J *and the new edges* $I_w \rightarrowtail w$ *to* E.

Here we will review the definition of d-separation in a CDAG, see [Pea09], also see [Ric03,FM17,FM18,FM20] for extensions and variations of d-separation to other graphs.

Definition 6.15 (d-blocked walks)**.** *Let* $\mathbf{G} = \mathbf{G}(V|\operatorname{do}(J))$ *be a CDAG and* $C \subseteq J \cup V$ *a subset of nodes and* π *a walk in* $\mathbf{G}(V|\operatorname{do}(J))$*:* $\pi = (v_0 \ast\!\!-\!\!\ast \cdots \ast\!\!-\!\!\ast v_n)$.

1. *We say that the walk* π *is* C-d-blocked *or* d-blocked by C *if either:*
   a) $v_0 \in C$ *or* $v_n \in C$ *or:*
   b) *there are two adjacent edges in* π *of one of the following forms:*

$$\begin{array}{rcl}
\textit{left chain:} & v_{k-1} \leftarrowtail v_k \leftarrowtail v_{k+1} & \textit{with} \quad v_k \in C, \\
\textit{right chain:} & v_{k-1} \rightarrowtail v_k \rightarrowtail v_{k+1} & \textit{with} \quad v_k \in C, \\
\textit{fork:} & v_{k-1} \leftarrowtail v_k \rightarrowtail v_{k+1} & \textit{with} \quad v_k \in C, \\
\textit{collider:} & v_{k-1} \rightarrowtail v_k \leftarrowtail v_{k+1} & \textit{with} \quad v_k \notin C.
\end{array}$$

In short, π *is* C*-d-blocked if it either has a* collider *not in* C *or a* non-collider *in* C.

2. *We say that the walk* π *is* C-d-open *if it is not* C*-d-blocked.*

Definition 6.16 (d-separation)**.** *Let* $\mathbf{G} = \mathbf{G}(V|\operatorname{do}(J))$ *be a CDAG and* $A, B, C \subseteq J \cup V$ *(not necessarily disjoint) subset of nodes. We then say that:*

1. A is d-separated from B given C in $\mathbf{G}$, *in symbols:* $A \underset{\mathbf{G}(V|\,\mathrm{do}(J))}{\perp} B \,|\, C$,

   if every walk from a node in A to a node in $J \cup B$[12] *is d-blocked by C.*

2. *Otherwise we write:* $A \underset{\mathbf{G}(V|\,\mathrm{do}(J))}{\not\perp} B \,|\, C$.

3. *Special case:* $A \underset{\mathbf{G}(V|\,\mathrm{do}(J))}{\perp} B \quad :\iff \quad A \underset{\mathbf{G}(V|\,\mathrm{do}(J))}{\perp} B | \emptyset$.

6.3 Causal Bayesian Networks and the Global Markov Property

In this subsection we will introduce *causal Bayesian networks* for quasi-universal spaces. The main result is the *global Markov property* for causal Bayesian networks, which relates d-separation statements in the graph to conditional independence statements between the variables. Furthermore, we will introduce *partially generic causal Bayesian networks* where some of the causal mechanisms are not provided and are filled in with generic, most general ones. Since this fill-in will lead to a well-defined causal Bayesian network we also get a global Markov property for partially generic causal Bayesian networks, now allowing us to relate d-separation statements in the extended graph to conditional independence statements between variables and causal mechanisms. Note that this is only possible because in **QUS** we have exponential objects, transitional conditional independence still works with all its separoid rules for **QUS** and the global Markov property can be shown. It is important to see that this kind of reasoning would break inside the usual category of measurable spaces **Meas**.

Definition 6.17 (Causal Bayesian network)**.** *A* causal Bayesian network (CBN) $\mathbf{M}$ *consists of:*

1. *a (finite) conditional directed acyclic graph (CDAG):* $\mathbf{G} = \mathbf{G}(V|\,\mathrm{do}(J))$,

2. *input variables X_j, $j \in J$, and (stochastic) output variables X_v, $v \in V$,*

3. *a quasi-universal space $\mathcal{X}_v$ for every $v \in J \,\dot\cup\, V$,*

4. *a Markov kernel for every $v \in V$, suggestively written as:*

$$\mathbf{P}_v \left(X_v \middle| \mathrm{do}\left(X_{\mathrm{Pa}^{\mathbf{G}}(v)} \right) \right) \in \mathcal{P}(\mathcal{X}_v)^{\mathcal{X}_{\mathrm{Pa}^{\mathbf{G}}(v)}},$$

   where we write for $D \subseteq J \,\dot\cup\, V$:

$$\begin{aligned} \mathcal{X}_D &:= \prod_{v \in D} \mathcal{X}_v, & \mathcal{X}_\emptyset &:= \mathbf{1} = \{0\}, \\ X_D &:= (X_v)_{v \in D}, & X_\emptyset &:= 0, \\ x_D &:= (x_v)_{v \in D}, & x_\emptyset &:= 0. \end{aligned}$$

[12]Note the inclusion of J here. This is done to get similar asymmetric separoid rules to transitional conditional independence in order to have a more "nice" looking global Markov property later on.

By abuse of notation, we denote the causal Bayesian network as:

$$\mathbf{M}(V|\,\mathrm{do}(J)) = \left(\mathbf{G}(V|\,\mathrm{do}(J)), \left(\mathbf{P}_v\left(X_v\middle|\,\mathrm{do}\left(X_{\mathrm{Pa}^{\mathbf{G}}(v)}\right)\right)\right)_{v\in V}\right).$$

Notation 6.18. *Any CBN* $\mathbf{M} = \mathbf{M}(V|\,\mathrm{do}(J))$ *comes with its* joint Markov kernel*:*

$$\mathbf{P}(X_V|\,\mathrm{do}(X_J)) \in \mathcal{P}(\mathcal{X}_V)^{\mathcal{X}_J},$$

given by:

$$\mathbf{P}(X_V|\,\mathrm{do}(X_J)) := \bigotimes_{v\in V}^{>} \mathbf{P}_v\left(X_v\middle|\,\mathrm{do}\left(X_{\mathrm{Pa}^{\mathbf{G}}(v)}\right)\right),$$

where the product $\otimes^{>}$ *is taken in reverse order of a fixed topological order* $<$*, i.e. children appear only on the left of their parents in the product. Note that the joint Markov kernels does actually not depend on the actual choice of the topological order, if one swaps the corresponding arguments and spaces into the right positions accordingly, see Fubini Theorem 5.33.*

Theorem 6.19 (Global Markov property for causal Bayesian networks)**.** *Consider a causal Bayesian network* $\mathbf{M}(V|\,\mathrm{do}(J))$ *with graph* $\mathbf{G}(V|\,\mathrm{do}(J))$ *and joint Markov kernel:* $\mathbf{P}(X_V|\,\mathrm{do}(X_J))$*. Assume that* $\mathcal{X}_v$ *is a universal quasi-universal space for every* $v \in V$*. Then for all* $A, B, C \subseteq J \,\dot\cup\, V$ *(not-necessarily disjoint) we have the implication:*

$$A \underset{\mathbf{G}(V|\,\mathrm{do}(J))}{\perp} B\,|\,C \qquad \Longrightarrow \qquad X_A \underset{\mathbf{P}(X_V|\,\mathrm{do}(X_J))}{\perp\!\!\!\perp} X_B\,|\,X_C.$$

Recall that we have - per definition - an implicit dependence on J*,* X_J*, resp., in the second argument on each side.*

Proof. The proof only uses the (asymmetric) separoid rules for d-separation and transitional conditional independence from Theorem 6.7, see [For21] Theorem 6.3 and Appendix J. It is only the rule Left Weak Union that requires the spaces $\mathcal{X}_v$ for $v \in V$ to be universal quasi-universal spaces, which is needed for the disintegration Theorem 5.35 to work. Also note that the use of Extended Left Redundancy is not really necessary as one only needs the statements of the form $X_A \perp\!\!\!\perp X_B\,|\,X_A$, which can directly be verified using $\mathbf{Q}(X_A|X_A) := \boldsymbol{\delta}(X_A|X_A)$. □

Remark 6.20. *The global Markov property says that already checking the graphical criterion* $A \underset{\mathbf{G}(V|\,\mathrm{do}(J))}{\perp} B\,|\,C$ *is enough to get the existence of a Markov kernel, suggestively written as:* $\mathbf{P}\left(X_A|\cancel{X_B}, X_{C\cap V}, \mathrm{do}(X_{C\cap J}), \cancel{\mathrm{do}(X_J)}\right)$*, such that:*

$$\mathbf{P}(X_A, X_B, X_C|\,\mathrm{do}(X_J)) = \mathbf{P}\left(X_A|\cancel{X_B}, X_{C\cap V}, \mathrm{do}(X_{C\cap J}), \cancel{\mathrm{do}(X_J)}\right) \otimes \mathbf{P}(X_B, X_C|\,\mathrm{do}(X_J)).$$

Definition 6.21 (The partially generic causal Bayesian network)**.** *Let* $\mathbf{G} = \mathbf{G}(V|\,\mathrm{do}(J))$ *be a CDAG and* $\mathcal{X}_v$ *for* $v \in J \,\dot\cup\, V$ *quasi-universal spaces,* $W \subseteq V$ *a subset and* $\mathbf{P}_v \in \mathcal{P}(\mathcal{X}_v)^{\mathcal{X}_{\mathrm{Pa}^{\mathbf{G}}(v)}}$ *for* $v \in V \backslash W$ *given Markov kernels. Then the* partially generic causal Bayesian network *w.r.t. those choices is the causal Bayesian network* $\mathbf{M}(V|\,\mathrm{do}(J \,\dot\cup\, I_W))$

with the extended CDAG $\mathbf{G}(V|\,\mathrm{do}(J \dot\cup I_W))$*, see Definition 6.14, the given spaces* $\mathcal{X}_v$ *for* $v \in J \dot\cup V$ *and the following spaces for* $w \in W$*:*

$$\mathcal{X}_{I_w} := \mathcal{P}(\mathcal{X}_w)^{\mathcal{X}_{\mathrm{Pa}^{\mathbf{G}}(w)}},$$

and the generic Markov kernels *defined by:*

$$\mathbf{P}_w\left(X_w \in A_w|\,\mathrm{do}(X_{\mathrm{Pa}^G(w)} = x_{\mathrm{Pa}^G(w)}, X_{I_w} = \mathbf{Q}_w)\right) := \mathbf{Q}_w\left(X_w \in A_w|\,\mathrm{do}(X_{\mathrm{Pa}^G(w)} = x_{\mathrm{Pa}^G(w)})\right).$$

Note that: $\mathbf{P}_w \in \mathcal{P}(\mathcal{X}_w)^{\mathcal{X}_{\mathrm{Pa}^{\mathbf{G}}(w)} \times \mathcal{X}_{I_w}}$ *for* $w \in W$*.*

Remark 6.22. *1. Note that the construction of the "generic" Markov kernels* $\mathbf{P}_w$ *in the Definition 6.21 of partially generic causal Bayesian networks would not have been possible in the category of measurable spaces* **Meas** *as the spaces of measurable maps or spaces of Markov kernels do not carry well-behaved* σ*-algebras, not even for standard (Borel) measurable spaces, see [Aum61]. This was one of the core motivations for formalizing probabilistic graphical models inside the cartesian closed category of quasi-universal spaces* **QUS***.*

2. *The partially generic causal Bayesian network, see Definition 6.21, is again a causal Bayesian network, see Definition 6.17, and thus follows the global Markov property w.r.t. the extended graph* $\mathbf{G}(V|\,\mathrm{do}(J \dot\cup I_W))$*, see Theorem 6.19: If* $\mathcal{X}_v$ *is a universal quasi-universal space for every* $v \in V$ *then for all* $A, B, C \subseteq V \dot\cup I_W \dot\cup J$ *we have the implication:*

$$A \underset{\mathbf{G}(V|\,\mathrm{do}(J \dot\cup I_W))}{\perp} B\,|\,C \qquad \Longrightarrow \qquad X_A \underset{\mathbf{P}(X_V|\,\mathrm{do}(X_J, X_{I_W}))}{\perp\!\!\!\perp} X_B\,|\,X_C.$$

   This will allow us to reason about the (conditional) independence of variables X_v *for* $v \in J \dot\cup V$*, w.r.t. "mechanisms"* $\mathbf{Q}_w$ *for* $w \in W$*.*

3. *It is easy to extend the global Markov property, Theorem 6.19, to more general (causal) graphical models like causal Bayesian networks with latent confounders or even structural causal models that allow for cycles, latent confounders and selection bias. For this one mainly would need to adjust the graphical part, e.g. use the more general* σ*-separation, see [FM17, FM18, FM20, For21]. This was also discussed in [For21] Remark K.8:*

$$A \overset{\sigma}{\underset{\mathbf{G}(V|S,\mathrm{do}(J))}{\perp}} B\,|\,C \qquad \Longrightarrow \qquad X_A \underset{\mathbf{P}(X_V, X_S|\,\mathrm{do}(X_J))}{\perp\!\!\!\perp} X_B\,|\,X_S, X_C.$$

Example 6.23 (Markov chain)**.** *Consider Markov chain from Figure 1. We need to specify Markov kernels* $P(X) \in \mathcal{P}(\mathcal{X})$ *and* $P(Y|X) \in \mathcal{P}(\mathcal{Y})^{\mathcal{X}}$ *and* $P(Z|Y) \in \mathcal{P}(\mathcal{Z})^{\mathcal{Y}}$*. The joint distribution is then given by:*

$$P(X, Y, Z) = P(Z|Y) \otimes P(Y|X) \otimes P(X).$$

We then have that:

$$P(X, Y, Z) = P(Z|Y) \otimes P(X, Y),$$

which shows the transitional conditional independence:

$$Z \perp\!\!\!\perp X \,|\, Y.$$

This means that when the value of Y is provided then the value of X has no additional information about the state of Z.
Note that the global Markov property Theorem 6.19 would have allowed us to conclude the same as we also have the corresponding d-separation: $Z \perp X \,|\, Y$.
Since only the Markov kernel $P(Z|Y)$ determines the state of Z when Y is given we actually could make the stronger statement that even the distribution $P(X)$ and the Markov kernel $P(Y|X)$ have no influence on Z when Y is given. Unfortunately, this cannot directly be seen or read off from Figure 1.
However, such reasoning is possible with the use of the partially generic causal Bayesian network, see Figure 2. Since the global Markov property, see Theorem 6.19, also applies here the d-separation $Z \perp (X, Q(X), Q(Y|X)) \,|\, Y$ gives us the transitional conditional independence:

$$Z \perp\!\!\!\perp (X, Q(X), Q(Y|X)) \,|\, Y.$$

Note that we use $Q(X)$ and $Q(Y|Z)$, resp., as a variables and $P(X)$ and $P(Y|Z)$, resp., as specific values of those variables. So that transitional conditional independence now exactly states that when Y is given the value of Z is not determined by the values of X, $Q(X)$ and $Q(Y|X)$, as we expected and wanted, and without computing factorizations of type $P(X, Y, Z) = P(Z|Y) \otimes P(X, Y)$ by hand.

6.4 Counterfactual Reasoning

One of the core assumptions for counterfactual reasoning in the potential outcome framework is the assumption of *strong ignorability*, see [Rub78, RR83]:

$$X \perp\!\!\!\perp (Y_0, Y_1) \,|\, Z.$$

The usual meaning of the variables is the following. Z is usually a random vector of all *observed features* of a patient, whom we want to prescribe one of two treatments from $\mathcal{X} = \{0, 1\}$. The variable X is then the measured *treatment variable*, which can take the values from $\mathcal{X}$. The variables Y_0 and Y_1 are the potential outcomes if we forced the patient to take the corresponding treatment, 0 or 1. The assumption of strong ignorability can then be interpreted as that besides Z there are no further (unobserved) confounding variables, or that Z is *deconfounding*.

If one now wanted to generalize this scenario to more than two possible treatments and thus potential outcomes, i.e. $\#\mathcal{X} > 2$, or even $\mathcal{X} = \mathbb{R}$, one quickly runs into the problem of even stating the assumption of strong ignorability measure-theoretically rigorously. Many authors then put this into the following notation:

$$X \perp\!\!\!\perp (Y_x)_{x \in \mathcal{X}} \,|\, Z.$$

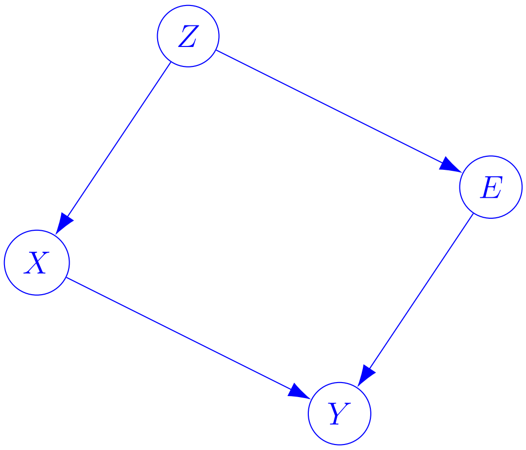


Figure 3: The causal Bayesian network encoding the strong ignorability assumption $X \perp\!\!\!\perp E \,|\, Z$. The random variable E takes values in the function space $\mathcal{Y}^{\mathcal{X}}$ and $Y = E(X)$. The potential outcomes after intervening on X with values $x \in \mathcal{X}$ are then $Y_x = E(x)$ with values in $\mathcal{Y}$.

This notation indicates that $(Y_x)_{x\in\mathcal{X}}$ would live on some product space $\prod_{x\in\mathcal{X}} \mathcal{Y}$. Those authors then use those variables in a way that one implicitely needs to assume that the map:

$$\mathrm{ev}: \ \mathcal{X} \times \prod_{x\in\mathcal{X}} \mathcal{Y} \to \mathcal{Y}, \qquad (\hat{x}, (y_x)_{x\in\mathcal{X}}) \mapsto y_{\hat{x}},$$

is measurable, even when $\mathcal{X}$ is an uncountable (standard Borel) measurable space. This runs into similar problems as discussed in the introduction, see Section 1 and [Aum61].

To overcome this issue it might be beneficial to work in the category of quasi-universal spaces **QUS** instead and replace the random variable $(Y_x)_{x\in\mathcal{X}}$, which has values in $\prod_{x\in\mathcal{X}} \mathcal{Y}$, with a random variable E that has values in the function space $\mathcal{Y}^{\mathcal{X}}$. Note that its marginal distribution is then an element $P(E) \in \mathcal{P}(\mathcal{Y}^{\mathcal{X}})$[13]. Then one could easily state the assumption of *strong ignorability* as:

$$X \perp\!\!\!\perp E \,|\, Z.$$

This would solve the measure-theoretic problems and we would get: $Y_x = E(x)$ as the potential outcomes for $x \in \mathcal{X}$ and $Y = E(X)$ in the observational scenario.

If strong ignorability holds and we, furthermore, assume that $\mathcal{Z}$ is countably separated then we can apply the disintegration theorem, see Theorem 5.35, to get the factorization of the joint distribution of all variables:

$$P(X, Y, E, Z) = P(Y|X, E) \otimes P(X|Z) \otimes P(E|Z) \otimes P(Z).$$

This factorization can also be modelled with the use of a causal Bayesian network, see Figure 3. A similar graphical model, but without the variable E or only after marginalizing out E, has been proposed before, see [Pea09].

[13] Carefully distinguish between $\mathcal{P}(\mathcal{Y}^{\mathcal{X}})$ and $\mathcal{P}(\mathcal{Y})^{\mathcal{X}}$, see Corollary 5.31 and Remark 5.32.

This example shows how one could, even in more generality, use random functions like E with values in a function space $\mathcal{Y}^{\mathcal{X}}$ to model potential outcomes and counterfactual relationships. This could also lead to new graphical representations of counterfactuals similar to single world intervention graphs, see [RR13a, RR13b, MSR19]. We leave this for future research.

7 Discussion

7.1 Results

Motivated by the example of Markov chains 6.23, where we wanted to be able to graphically check which variables X_v are (conditionally) independent from certain Markov kernels $Q(X_v | \operatorname{do}(X_{\operatorname{Pa}^{\mathbf{G}}(v)}))$, we investigated how one could allow variables in probabilistic graphical models to take values in the (huge) space of Markov kernels $\mathcal{P}(\mathcal{X}_v)^{\mathcal{X}_{\operatorname{Pa}^{\mathbf{G}}(v)}}$. For this we needed to study the cartesian closed category of *quasi-Borel spaces* from [HKSY17]. During that process we generalized that theory to allow for general samples spaces and we also simplified definitions to be less restrictive. We then called those categories the *categories of quasi-measurable spaces* **QMS**. We showed that those categories are also cartesian closed, contain all (small) limits and colimits, and products and coproducts distribute.
We then systematically studied different *probability monads* on those categories of quasi-measurable spaces. One corresponds to $\mathcal{P}$, which was introduced in [HKSY17], the others, $\mathcal{Q}$, $\mathcal{K}$, $\mathcal{R}$, $\mathcal{S}$, are either noval or slight variation of the first one. It turned out that those probability monads are either strong without further assumptions, $\mathcal{Q}$, or under the assumption that the sample space satisfies: $\Omega \times \Omega \cong \Omega$, which holds for (countably) infinite product spaces like $\Omega = \Omega_0^{\mathbb{N}}$, but also for all uncountable standard Borel measurable spaces, like $\mathbb{R}$ or the Hilbert cube $[0,1]^{\mathbb{N}}$, endowed either with their Borel σ-algebra or the σ-algebra of all universally measurable subsets.
We then went on and specialized to such a sample space $\mathbb{R}^{\mathbb{N}}$, or equivalently $\mathbb{R}$, endowed with the σ-algebra of all universally measurable subsets. We called this the *category of quasi-universal spaces* **QUS**. It turns out that the induced σ-algebras on quasi-universal spaces are themselves quasi-universal spaces and have simple descriptions in terms of intersections of Lebegue-complete σ-algebras, in contrast to the σ-algebras of quasi-Borel spaces. This becomes even more pronounced when one studies the σ-algebras of *countably separated* quasi-universal spaces. A further special role was played by *universal quasi-universal spaces*, which come with similar good properties as standard Borel measurable spaces have.
The strong probability monads $\mathcal{K}$, $\mathcal{R}$, $\mathcal{P}$, $\mathcal{S}$ all agree on the cartesian closed category of quasi-universal spaces **QUS**. They have similar properties and semantics for higher probability theory like $\mathcal{P}$ does for quasi-Borel spaces.
Furthermore, we proved for quasi-universal spaces a Fubini theorem, and, under certain conditions, a theorem about the disintegration of Markov kernels, Kolmogorov extension theorems, a conditional de Finetti theorem, etc.

We then translated many of the properties of universal quasi-universal spaces into properties of the Kleisli category of their Markov kernels w.r.t. the proposed probability monad.
Finally, we formalized transitional conditional independence, see [For21], and causal Bayesian networks inside the category of quasi-universal spaces and proved a global Markov property for them. This then allowed us to solve the problem of our motivating example, namely to include the "mechanisms" of a given causal Bayesian network as variables/nodes on the same standing as other variables. This then allows us to check and reason about (conditional) independences between variables X_v and mechanisms/Markov kernels $Q_w(X_w|X_{\mathrm{Pa}^{\mathbf{G}}(w)})$ graphically.

7.2 Different Sample Spaces

After studying the properties of the category of quasi-universal spaces **QUS** it seems to the author that taking an uncountable Polish space endowed with its σ-algebra of all universally measurable subsets as the sample space Ω for a category of quasi-measurable spaces **QMS** is an optimal choice. We again highlight the arguments below.

It seems unavoidable to ask for an isomorphisms $\Omega \times \Omega \cong \Omega$ to have well-behaved probability monads, see Theorem 3.24 and Theorem 3.25. In this case one then has the choice to either take $\mathcal{R}$ or $\mathcal{P}$ for the monads of push-forward probability measures, as the others agree with those, see Lemma 3.19.

If one is interested in Kolmogorov extension theorems, see Theorem 5.38 and [Kol33, Fre15] 454D-G, or disintegration theorems, see Theorem 5.34 and [Fad85, Pac78, Fre15, For21] Cor. C.8, one might want or need to restrict to *perfect* probability measures or even countably compact ones. Even though the product of perfect probability measures is perfect again, the arbitrary mixture of perfect measures might not be perfect anymore, see [Ram79]. To avoid this problem one option would be to use a sample space where all probability distributions (on the product) are perfect, e.g. $\mathbb{R}$. Alternatively, one could restrict to the probability monad $\mathcal{R}$ instead of $\mathcal{P}$, because when using $\mathcal{R}$ one only mixes probability measures via push-forwards of products of probability measures, see proof of Theorem 3.24, which would then be perfect again, if one could resolve the ambiguity between **QMS**-products and **Meas**-products for the sample space Ω (as done for **QUS** and **QBS**), see [Fre15] 451J.

To avoid requiring that all probability measures of $\mathcal{Q}(\Omega)$ are perfect one could restrict to the subset $\mathcal{V}(\Omega) \subseteq \mathcal{Q}(\Omega)$ of all *perfect* probability measures ν on $(\Omega, \mathcal{B}_\Omega)$ that also lie in $\mathcal{Q}(\Omega)$:

$$\mathcal{V}(\Omega) := \left\{\nu : \mathcal{B}_\Omega \to [0,1] \text{ perfect prob. measure} \,\middle|\, \forall D \in \mathcal{B}_{\Omega\times\Omega}.\ (\omega \mapsto \nu(D_\omega)) \in [0,1]^\Omega\right\}.$$

We can then define, similar to $\mathcal{R}$:

$$\mathcal{V}(\mathcal{X}) := \left\{\alpha_*\nu \,\middle|\, \alpha \in \mathcal{X}^\Omega, \nu \in \mathcal{V}(\Omega)\right\},$$
$$\mathcal{V}(\mathcal{X})^\Omega := \left\{\alpha_*\nu \,\middle|\, \alpha \in \left(\mathcal{X}^\Omega\right)^\Omega, \nu \in \mathcal{V}(\Omega)\right\}.$$

Since push-forwards of perfect measures are perfect again, see [Fre15] 451E, there is no clash in notations here.

One would similar to Lemma 3.26 also have a well-defined quasi-measurable push-forward map:

$$\mathrm{pf}:\ \mathcal{Y}^{\mathcal{X}} \times \mathcal{V}(\mathcal{X}) \to \mathcal{V}(\mathcal{Y}), \qquad (f,\mu) \mapsto f_*\mu.$$

Furthermore, under $\Omega \times \Omega \cong \Omega$ (and if **QMS**-products of perfect measures on Ω are perfect again) we would also have the quasi-measurable map, see Theorem 3.24:

$$\otimes:\ \mathcal{V}(\mathcal{X})^{\mathcal{Y}\times\mathcal{Z}} \times \mathcal{V}(\mathcal{Y})^{\mathcal{Z}} \to \mathcal{V}(\mathcal{X}\times\mathcal{Y})^{\mathcal{Z}}, \quad (\mu\otimes\nu)(z)(D) := \int \mu(y,z)(D_y)\,\nu(z)(dy).$$

Under the same condition $(\mathcal{V},\delta,\mathbb{M})$ would be a strong probability monad by the same proofs as used for $\mathcal{R}$, see Theorem 3.25.

If one, in addition, wanted to have a similar description of the induced σ-algebras as in Lemma 5.5 and Theorem 5.16 one then would need to require that $\mathcal{B}_\Omega$ is complete w.r.t. $\mathcal{V}(\Omega)$:

$$\mathcal{B}_\Omega = (\mathcal{B}_\Omega)_{\mathcal{V}(\Omega)}.$$

Finally, the same arguments would hold if we replaced the word “perfect” with “countably compact” everywhere, see [Fre15] 451J, 451K, 452R, 454A.

If one wants to make use of the disintegration theorems, see Theorem 5.34, one would need to require that $(\Omega,\mathcal{B}_\Omega)$ is universally countably generated and that all probability measures are perfect. If one also wants $\mathcal{B}_\Omega$ to separate the points of Ω one necessarily arrives at that $(\Omega,\mathcal{B}_\Omega)$ is a universal measurable space, see Lemma 5.18. If one also wants the convenient descriptions of the induced σ-algebras, see Lemma 5.5 and Theorem 5.16, one needs $(\Omega,\mathcal{B}_\Omega)$ to be universally complete, i.e. $\mathcal{B}_\Omega = (\mathcal{B}_\Omega)_{\mathcal{G}}$. So we arrive at requiring that $(\Omega,\mathcal{B}_\Omega)$ is a universally closed universal measurable space, under which certainly the universal completions of Polish spaces are the best behaved ones. The isomorphism $\Omega\times\Omega\cong\Omega$ shows that Ω must be infinite. If one wants to allow for non-discrete probability measures one even arrives at uncountable Polish spaces $(\Omega,\mathcal{B}_\Omega)$ endowed with their σ-algebras of all universally measurable subsets, and thus at the *category of quasi-universal spaces* **QUS**.

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