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Global algebraic Poincaré–Bendixson annulus for the Rayleigh equation

Alexander Grin✉ [1] and **Klaus R. Schneider**[2]

[1]Yanka Kupala State University of Grodno, Ozheshko Street 22, Grodno, 230023, Belarus
[2]Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstr. 39, Berlin, 10117, Germany

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Abstract. We consider the Rayleigh equation $\ddot{x} + \lambda(\dot{x}^2/3 - 1)\dot{x} + x = 0$ depending on the real parameter λ and construct a Poincaré–Bendixson annulus $\mathcal{A}_\lambda$ in the phase plane containing the unique limit cycle Γ_λ of the Rayleigh equation for all $\lambda > 0$. The novelty of this annulus consists in the fact that its boundaries are algebraic curves depending on λ. The polynomial defining the interior boundary represents a special Dulac–Cherkas function for the Rayleigh equation which immediately implies that the Rayleigh equation has at most one limit cycle. The outer boundary is the diffeomorphic image of the corresponding boundary for the van der Pol equation. Additionally we present some equations which are linearly topologically equivalent to the Rayleigh equation and provide also for these equations global algebraic Poincaré–Bendixson annuli.

Keywords: limit cycle, Rayleigh equation, Dulac–Cherkas function, Poincaré–Bendixson annulus, topologically equivalent Rayleigh systems.

2020 Mathematics Subject Classification: 34C05, 34C07, 34C23.

1 Introduction

The British physicist and Nobel prize winner J. W. Strutt, better known as Lord Rayleigh, published fundamental results to a broad spectrum of physical phenomena. In his monograph "Theory of Sounds" [18] he used the linear differential equation with constant coefficients

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + n^2x = 0$$

for the description of acoustic oscillations of a clarinet. The nonlinear modification of this equation

$$\frac{d^2x}{dt^2} + \lambda\left[\left(\frac{dx}{dt}\right)^2 \Big/ 3 - 1\right]\frac{dx}{dt} + x = 0, \tag{1.1}$$

✉Corresponding author. Email: grin@grsu.by

where λ is a real parameter, is known under the name Rayleigh equation [3, 17]. Its corresponding system

$$\begin{aligned}\frac{dx}{dt} &= -y,\\ \frac{dy}{dt} &= x + \lambda y - \lambda \frac{y^3}{3},\end{aligned} \tag{1.2}$$

which is invariant under the transformation $t \to -t, y \to -y, \lambda \to -\lambda$ has been studied by several authors [1, 2, 8, 9, 13, 15, 21–23].

The existence of a limit cycle (isolated closed orbit) of a planar autonomous system is established usually by the construction of an annulus $\mathcal{A}$ in the phase plane with the following properties: (i). $\mathcal{A}$ contains no equilibrium of the system under consideration. (ii). The boundary of $\mathcal{A}$ consists of two simple closed curves (in what follows called ovals) such that any trajectory of the considered system meeting the boundary of $\mathcal{A}$ will enter $\mathcal{A}$ either for increasing or for decreasing t. An annulus with the properties (i) and (ii) is called a Poincaré–Bendixson annulus since the application of the Poincaré–Bendixson theorem [5, 16] to that annulus provides the existence of at least one limit cycle in $\mathcal{A}$. The crucial problem in that approach is the construction of the ovals forming the boundary of $\mathcal{A}$. In numerous publications (see e.g. [5, 6, 14, 16, 19, 20]) these ovals consist of piecewise smooth curves constructed in a sophisticated way. In this paper we are concerned with the construction of such ovals which are differentiable curves having only a finite number of points where the trajectories of the underlying system touch the ovals. We call such ovals as crossing ovals. Recently, two papers have been published [7, 10] in which a procedure for the construction of algebraic crossing ovals for planar polynomial systems is described. For both papers it is characteristic that they need the approximation of at least one orbit by a polynomial in t. In what follows, we present an approach to construct algebraic crossing ovales for the Rayleigh system (1.2) and some of its topologically equivalent systems, which is completely different from that one presented in the cited papers [7, 10].

The structure of our paper is as follows: in Section 2 we describe a method for the construction of an algebraic crossing oval for a class of polynomial systems. For this reason we introduce the concept of Dulac–Cherkas functions including one method for their construction. Section 3 is devoted to the construction of a crossing oval by means of a diffeomorphically equivalent system. In Section 4 we derive some linearly diffeomorphically equivalent systems to the Rayleigh system (1.2) and present the corresponding Poincaré–Bendixson annuli.

2 Construction of an interior boundary for a Poincaré–Bendixson annulus of the Rayleigh system (1.2)

Our approach to construct an interior boundary for a Poincaré–Bendixson annulus for system (1.2) is based on the use of a Dulac–Cherkas function. For this reason we introduce in the next subsection the definition of a Dulac–Cherkas function and compose some of its properties.

2.1 Definition and properties of Dulac–Cherkas functions

We consider the planar differential system

$$\frac{dx}{dt} = P(x, y, \lambda), \qquad \frac{dy}{dt} = Q(x, y, \lambda) \tag{2.1}$$

under the assumption

(A) Let $\mathcal{G}$ be an open subset of $\mathbb{R}^2$, let Λ be some open interval, let $P, Q \in C^{1\ \ 0}_{(x,y)\ \lambda}(\mathcal{G} \times \Lambda, \mathbb{R})$.

We denote by X the vector field defined by (2.1). First we recall the definition of a Dulac function.

Definition 2.1. Suppose the assumption (A) to be valid. A function B belonging to the class $C^{1\ \ 0}_{(x,y)\ \lambda}(\mathcal{G} \times \Lambda, \mathbb{R})$ is called a Dulac function of system (2.1) in $\mathcal{G}$ for $\lambda \in \Lambda$ if the expression

$$\operatorname{div}(BX) \equiv \frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y} \equiv (\operatorname{grad} B, X) + B \operatorname{div} X$$

does not change sign in $\mathcal{G}$ and vanishes only on a set $\mathcal{N}_\lambda \subset \mathcal{G}$ of measure zero for $\lambda \in \Lambda$.

The class of Dulac functions has been generalized by L. A. Cherkas in 1997 (see [4]). The corresponding generalized Dulac function, which is called Dulac–Cherkas function nowadays, is defined as follows.

Definition 2.2. Suppose the assumption (A) to be valid. A function $\Psi \in C^{1\ \ 0}_{(x,y)\ \lambda}(\mathcal{G} \times \Lambda, \mathbb{R})$ is called a Dulac–Cherkas function of system (2.1) in $\mathcal{G}$ for $\lambda \in \Lambda$ if there exists a real number $\kappa \neq 0$ such that

$$\Phi := (\operatorname{grad} \Psi, X) + \kappa \Psi \ \operatorname{div}\ X > 0 \quad (< 0) \quad \text{in } \mathcal{G} \text{ for } \lambda \in \Lambda. \tag{2.2}$$

Remark 2.3. Condition (2.2) can be relaxed by assuming that Φ may vanish in $\mathcal{G}$ on a set $\mathcal{N}_\lambda$ of measure zero, and that no oval of this set is a limit cycle.

Remark 2.4. In case $\kappa = 1$, Ψ is a Dulac function.

For the sequel we introduce the subset $\mathcal{W}_\lambda$ of $\mathcal{G}$ defined by

$$\mathcal{W}_\lambda := \{(x, y) \in \mathcal{G} : \Psi(x, y, \lambda) = 0\}. \tag{2.3}$$

From the Definition 2.2 we get immediately

Lemma 2.5. *Suppose the assumption (A) to be valid. Let Ψ be a Dulac–Cherkas function of system (2.1) in $\mathcal{G}$ for $\lambda \in \Lambda$. Then any oval of $\mathcal{W}_\lambda$ having only a finite number of points where $(\operatorname{grad} \Psi, X)$ vanishes is a crossing oval for system (2.1) and can be used as a boundary for a Poincaré–Bendixson annulus.*

The following theorem is a special case of a more general result established in [11].

Theorem 2.6. *Suppose the assumption (A) to be valid. Let $\mathcal{G}$ be a simply connected region, let Ψ be a Dulac–Cherkas function of (2.1) in $\mathcal{G}$ for $\lambda \in \Lambda$ such that $\mathcal{W}_\lambda$ contains exactly one oval $\mathcal{O}_\lambda$ in $\mathcal{G}$. Then in the case $\kappa < 0$ system (2.1) has for $\lambda \in \Lambda$ at most one limit cycle in $\mathcal{G}$, and if it exists, it surrounds $\mathcal{W}_\lambda$ and is hyperbolic.*

This theorem implies

Corollary 2.7. *Under the assumptions of Theorem 2.6 the oval $\mathcal{O}_\lambda$ can be used as interior boundary for a Poincaré–Bendixson annulus of system (2.1) provided it is a crossing oval.*

The problem how to construct a Dulac–Cherkas function for the Rayleigh system (1.2) will be treated in the next subsection. We note that the presented procedure can be applied to a more general class of planar polynomial systems.

2.2 Construction of Dulac–Cherkas functions for system (1.2)

We consider system (1.2) in $\mathbb{R}^2$ for $\lambda > 0$. The corresponding vector field X reads

$$X(x, y, \lambda) := (-y, x + \lambda y - \lambda y^3/3). \tag{2.4}$$

We look for a Dulac–Cherkas function in the form

$$\Psi(x, y, \lambda) := \Psi_0(y, \lambda) + \Psi_1(y, \lambda)x + \Psi_2(y, \lambda)x^2, \tag{2.5}$$

where we assume that for all $\lambda > 0$ the function Ψ_2 is not identically zero.

Using (2.4) and (2.5) we obtain for the function Φ defined in (2.2) the representation

$$\Phi(x, y, \lambda, \kappa) = \sum_{k=0}^{3} \Phi_k(y, \lambda, \kappa)x^k, \tag{2.6}$$

where the functions Φ_k are defined by the relations

$$\Phi_0 := -\Psi_1 y + \Psi_0'\lambda(y - y^3/3) + \kappa\lambda\Psi_0(1 - y^2), \tag{2.7}$$
$$\Phi_1 := -2\Psi_2 y + \Psi_0' + \Psi_1'\lambda(y - y^3/3) + \kappa\lambda\Psi_1(1 - y^2), \tag{2.8}$$
$$\Phi_2 := \Psi_1' + \Psi_2'\lambda(y - y^3/3) + \kappa\lambda\Psi_2(1 - y^2), \tag{2.9}$$
$$\Phi_3 := \Psi_2', \tag{2.10}$$

where the symbol $'$ indicates the differentiation with respect to y. One approach to guarantee that Φ is a definite function in $\mathbb{R}^2$ for $\lambda > 0$ is to require Φ_k to be identically zero for $1 \leq k \leq 3$ and that Φ_0 is definite. Applying this approach we get from (2.10) the linear differential equation

$$\Psi_2' = 0, \tag{2.11}$$

such that it holds

$$\Psi_2(y, \lambda, \kappa) \equiv c_2 \neq 0. \tag{2.12}$$

Taking into account (2.11) and (2.12) we obtain from (2.9)

$$\Psi_1' + \kappa c_2\lambda(1 - y^2) = 0 \tag{2.13}$$

whose solution reads

$$\Psi_1(y, \lambda) = -\kappa c_2\lambda(y - y^3/3) + c_1. \tag{2.14}$$

Taking into account (2.14), (2.13) and (2.12) we get from (2.8)

$$\Psi_0' = 2c_2 y + (1 + \kappa)\kappa c_2\lambda^2(1 - y^2)(y - y^3/3) - \kappa c_1\lambda(1 - y^2). \tag{2.15}$$

Setting

$$\kappa = -1, \qquad c_1 = 0 \tag{2.16}$$

we have by (2.14)

$$\Psi_1(y, \lambda) = c_2\lambda(y - y^3/3) \tag{2.17}$$

and the differential equation (2.15) reads

$$\Psi_0' = 2c_2 y \tag{2.18}$$

whose solution has the form

$$\Psi_0(y,\lambda) = c_2 y^2 + c_0. \tag{2.19}$$

Using (2.16)–(2.19) we obtain from (2.7)

$$\Phi_0(y,\lambda,-1) = \frac{2}{3}\lambda c_2\left(y^4 + \frac{3c_0}{2c_2}y^2 - \frac{3c_0}{2c_2}\right). \tag{2.20}$$

Now we have to determine c_0 and c_2 such that $\Phi_0(y,\lambda,-1)$ is a definite function and that the corresponding Dulac–Cherkas function Ψ has the property that its zero-level set $\mathcal{W}_\lambda$ contains an oval surrounding the origin. Setting $c_0 = -\frac{8}{3}c_2$, where by (2.12) $c_0 \neq 0$ holds, we have

$$\Phi_0(y,\lambda,-1) = \frac{2}{3}\lambda c_2(y^4 - 4y^2 + 4) \tag{2.21}$$

which has for $\lambda > 0$ the same sign for all y and vanishes only at $y = \pm\sqrt{2}$. Thus it holds

Lemma 2.8. *The polynomial*

$$\Psi(x,y,\lambda) := c_2\left(x^2 + y^2 - \frac{8}{3} + \lambda x\left(y - \frac{y^3}{3}\right)\right) \tag{2.22}$$

is a Dulac–Cherkas function for system (1.2) in $\mathbb{R}^2$ for $\lambda > 0$.

2.3 Construction of an interior boundary for a Poincaré–Bendixson annulus of system (1.2)

The set $\mathcal{W}_\lambda$ of the Dulac–Cherkas function Ψ in (2.22) is defined by

$$\mathcal{W}_\lambda := \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 - \frac{8}{3} + \lambda x\left(y - \frac{y^3}{3}\right) = 0\right\}. \tag{2.23}$$

First we note that $\mathcal{W}_0$ is the circle $x^2 + y^2 = 8/3$. From (2.23) we get further that for all $\lambda > 0$ the set $\mathcal{W}_\lambda$ is symmetric with respect to the origin and that the intersection of $\mathcal{W}_\lambda$ with the straight lines $y = \pm\sqrt{3}$ is empty for any $\lambda > 0$. For the following we denote by $\mathcal{S}_{2\sqrt{3}}$ in $\mathbb{R}^2$ the strip symmetric to the x-axis and with thickness $2\sqrt{3}$. We obtain from (2.23) the result

Lemma 2.9. *The set $\mathcal{W}_\lambda$ defined in (2.23) consists in $\mathbb{R}^2$ for $\lambda > 0$ of three different branches: the oval $\mathcal{I}_\lambda$ surrounding the origin and located in the strip $\mathcal{S}_{2\sqrt{3}}$, the unbounded branch $\mathcal{W}^1_\lambda$ located in the first quadrant in the region $y > \sqrt{3}$ and the symmetric branch $\mathcal{W}^3_\lambda$ in the third quadrant in the region $y < -\sqrt{3}$.*

Figure 2.1 shows the branches of $\mathcal{W}_\lambda$ for $\lambda = 1.3$.

In order to prove that the oval $\mathcal{I}_\lambda$ is a crossing oval, we note that we have by (2.2)

$$(\operatorname{grad}\Psi, X)\big|_{\Psi=0} = \Phi\big|_{\Psi=0} = \Phi_0\big|_{\Psi=0}.$$

According to (2.21) there exist four points on $\mathcal{I}_\lambda$, where the vector field X touches the oval $\mathcal{I}_\lambda$. Therefore, $\mathcal{I}_\lambda$ is a crossing oval and we get from Corollary 2.7

Theorem 2.10. *The oval $\mathcal{I}_\lambda$ represents for $\lambda > 0$ an interior boundary for a Poincaré–Bendixson annulus of system (1.2).*

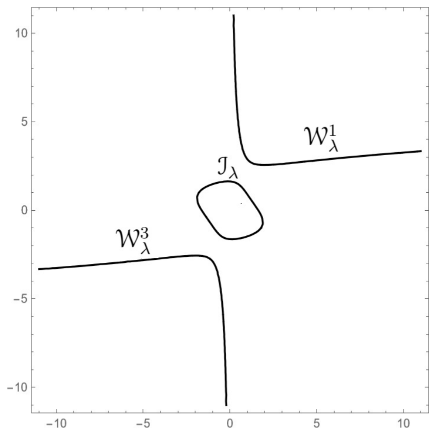


Figure 2.1: Three branches of the set $\mathcal{W}_\lambda$ including the oval $\mathcal{I}_\lambda$ for $\lambda = 1.3$.

3 Construction of an outer boundary for a Poincaré–Bendixson annulus of the Rayleigh system (1.2)

For the construction of an outer boundary of a Poincaré–Bendixson annulus for the Rayleigh system (1.2) a similar but more sophisticated procedure could be applied as it has been used for the van der Pol system in our paper [12]. In what follows we describe another approach based on the concept of diffeomorphically equivalent systems. In the following subsection we present the definition of topological equivalence of phase portraits and some important consequence.

3.1 Definition of topological equivalence and some important consequences

Our basic assumption reads as follows

($\tilde{A}$) Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be open subsets of $\mathbb{R}^2$, let Λ be some open interval, let $P_1, Q_1 \in C^{1\ \ 0}_{(x,y)\ \lambda}(\mathcal{G}_1 \times \Lambda, \mathbb{R})$, $P_2, Q_2 \in C^{1\ \ 0}_{(x,y)\ \lambda}(\mathcal{G}_2 \times \Lambda, \mathbb{R})$.

Consider the topological structure of the trajectories of the system

$$\frac{dx}{dt} = P_1(x,y,\lambda), \qquad \frac{dy}{dt} = Q_1(x,y,\lambda) \tag{3.1}$$

in $\mathcal{G}_1$ and the topological structure of the trajectories of the system

$$\frac{dx}{d\tau} = P_2(x,y,\lambda), \qquad \frac{dy}{d\tau} = Q_2(x,y,\lambda) \tag{3.2}$$

in $\mathcal{G}_2$.

Definition 3.1. Suppose assumption ($\tilde{A}$) to be valid. Let Λ_1 be a subinterval of Λ. The systems (3.1) and (3.2) are called topologically equivalent for $\lambda \in \Lambda_1$ if for $\lambda \in \Lambda_1$ there is a homeomorphism h_λ mapping $\mathcal{G}_1$ onto $\mathcal{G}_2$ and which maps the trajectories of system (3.1) onto the trajectories of system (3.2) and there is a strictly increasing homeomorphism g_λ mapping $\mathbb{R}$ onto itself such that $\tau = g_\lambda(t)$. If h_λ is a diffeomorphism then the systems are called diffeomorphically equivalent.

The following result is a consequence of the well known fact that the composition of a local diffeomorphism with a diffeomorphism is still a local diffeomorphism.

Theorem 3.2. *Suppose that the assumption* $(\tilde{A})$ *is valid and that the systems* (3.1) *and* (3.2) *are diffeomorphically equivalent for* $\lambda \in \Lambda_1$. *Let* $\mathcal{O}_\lambda$ *be a crossing oval for system* (3.1) *for* $\lambda \in \Lambda_1$. *Then the image of* $\mathcal{O}_\lambda$ *under the diffeomorphism* d_λ *is a crossing oval for system* (3.2) *for* $\lambda \in \Lambda_1$.

In order to be able to apply Theorem 3.2 for the construction of an outer boundary for a Poincaré–Bendixson annulus for the Rayleigh system (1.2) we use the following lemma.

Lemma 3.3. *The van der Pol system*

$$\begin{aligned} \frac{du}{dt} &= -v, \\ \frac{dv}{dt} &= u - \lambda(u^2-1)v \end{aligned} \tag{3.3}$$

is for $\lambda > 0$ *diffeomorphically equivalent to the Rayleigh system* (1.2).

Proof. Applying the diffeomorphism d_λ mapping $\mathbb{R}^2$ onto itself defined by

$$\begin{aligned} x &= -v + \lambda\left(\frac{u^3}{3} - u\right), \\ y &= u \end{aligned} \tag{3.4}$$

we get from (3.3)

$$\begin{aligned} \frac{dx}{dt} &= -y, \\ \frac{dy}{dt} &= x + \lambda y - \lambda\frac{y^3}{3}, \end{aligned} \tag{3.5}$$

which coincides with the Rayleigh system (1.2). □

3.2 Construction of an outer boundary for a Poincaré–Bendixson annulus of the Rayleigh system (1.2)

In the paper [12] we have proved the following result

Theorem 3.4. *For* $\lambda > 0$, *the oval*

$$\mathcal{V}_\lambda := \left\{(u,v) \in \mathbb{R}^2 : v^2 + \lambda vu\left(2 - \frac{u^2}{3}\right) + (1+\lambda^2)u^2 - \frac{7}{12}\lambda^2u^4 + \frac{\lambda^2}{18}u^6 - 8 - 3\lambda - 18\lambda^2 = 0\right\} \tag{3.6}$$

is a crossing oval forming an outer boundary of a global algebraic Poincaré–Bendixson annulus for the van der Pol system (3.3).

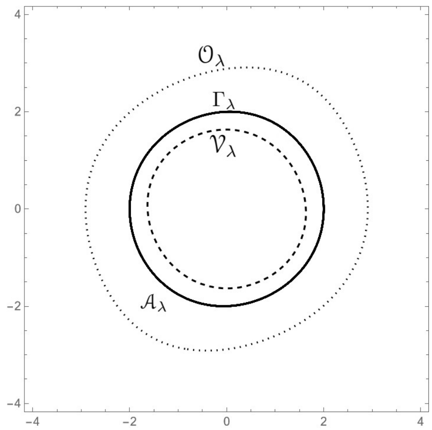
According to Lemma 3.3, the van der Pol system (3.3) is for $\lambda > 0$ diffeomorphically equivalent to the Rayleigh system (1.2), where the corresponding diffeomorphism d_λ is defined in (3.4). By Theorem 3.2, the image of the crossing oval $\mathcal{V}_\lambda$ for the van der Pol system (3.3) under the diffeomorphism d_λ is for $\lambda > 0$ a crossing oval $\mathcal{O}_\lambda$ of the Rayleigh system (1.2). From (3.4) and (3.6) we get

$$\begin{aligned}\mathcal{O}_\lambda := \Bigg\{ (x,y) : \left(-x + \lambda\left(\frac{y^3}{3} - y\right)\right)^2 + \lambda\left(-x + \lambda\left(\frac{y^3}{3} - y\right)\right)\left(2y - \frac{y^3}{3}\right) \\ + (1+\lambda^2)y^2 - \frac{7}{12}\lambda^2 y^4 + \frac{1}{18}\lambda^2 y^6 - 8 - 3\lambda - 18\lambda^2 = 0. \Bigg\}\end{aligned} \tag{3.7}$$

It can be verified that the derivative of $\mathcal{O}_\lambda$ along system (1.2) is negative on $\mathcal{O}_\lambda$ except at four points. Thus we have the result

Theorem 3.5. *The algebraic oval $\mathcal{O}_\lambda$ defined in (3.7) is for $\lambda > 0$ an algebraic crossing oval of the Rayleigh system (1.2) forming the outer boundary of a Poincaré–Bendixson annulus. Together with the algebraic oval $\mathcal{I}_\lambda$ it determines a global algebraic Poincaré–Bendixson annulus $\mathcal{A}_\lambda$ containing the unique limit cycle Γ_λ of the Rayleigh system (1.2).*

Figure 3.1 shows the Poincaré–Bendixson annulus $\mathcal{A}_\lambda$ with the limit cycle Γ_λ of system (1.2) for $\lambda = 0.1$ and $\lambda = 1.3$.



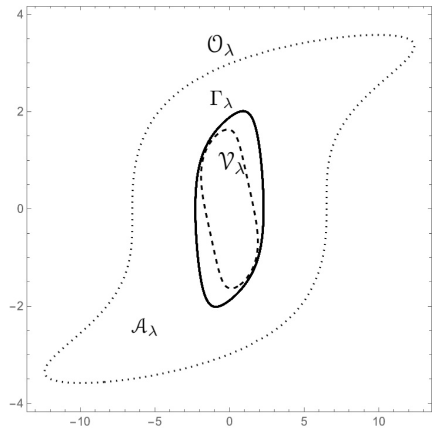


Figure 3.1: Annulus $\mathcal{A}_\lambda$ with the limit cycle Γ_λ of system (1.2) for $\lambda = 0.1$ (left) and $\lambda = 1$ (right).

4 Global algebraic Poincaré–Bendixson annuli for systems diffeomorphically equivalent to the Rayleigh system

If we apply for $\lambda > 0$ the linear diffeomorphism

$$u = \sqrt{\lambda}x, \quad v = \sqrt{\lambda}y \tag{4.1}$$

to the Rayleigh system (1.2) we obtain the system

$$\begin{aligned}\frac{du}{dt} &= -v, \\ \frac{dv}{dt} &= u + \lambda v - \frac{v^3}{3},\end{aligned} \tag{4.2}$$

which is diffeomorphically equivalent to system (1.2) for $\lambda > 0$. Thus, system (4.2) has for $\lambda > 0$ a unique limit cycle $\bar{\Gamma}_\lambda$. According to Theorem 3.2 we obtain a global algebraic Poincaré–Bendixson annulus for system (4.2) by applying the diffeomorphism (4.1) to the ovals $\mathcal{I}_\lambda$ and $\mathcal{O}_\lambda$. It holds

Theorem 4.1. *The algebraic ovals*

$$\bar{\mathcal{I}}_\lambda := \left\{(u,v) \in \mathcal{S}_{2\sqrt{3\lambda}} : u^2 + v^2 + uv\left(\lambda - \frac{v^2}{3}\right) - \frac{8}{3}\lambda = 0\right\} \tag{4.3}$$

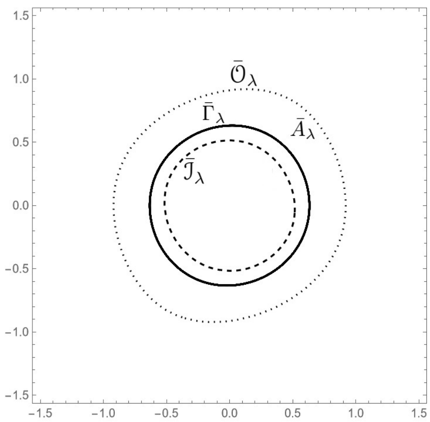
and

$$\begin{aligned}\bar{\mathcal{O}}_\lambda := &\left\{(u,v) \in \mathbb{R}^2 : \left(-u + \left(\frac{v^3}{3} - \lambda v\right)\right)^2 + \left(-u + \left(\frac{v^3}{3} - \lambda v\right)v\left(2\lambda - \frac{v^2}{3}\right)\right.\right.\\ &\left.+ (1+\lambda^2)v^2 - \frac{7}{12}\lambda v^4 + \frac{1}{18}v^6 - 8\lambda - 3\lambda^2 - 18\lambda^3 = 0\right\}\end{aligned} \tag{4.4}$$

form a global algebraic Poincaré–Bendixson annulus $\bar{A}_\lambda$ containing the unique limit cycle $\bar{\Gamma}_\lambda$ of system (4.2).

If λ tends to zero we get from (4.3) and (4.4) that both ovals shrink to the origin which reflects the property of system (4.2) that the limit cycle $\bar{\Gamma}_\lambda$ bifurcates from the origin when λ passes zero (Andronov–Hopf bifurcation). This distinguishes system (4.2) from the Rayleigh system where the limit cycle Γ_λ bifurcates from the circle $x^2 + y^2 = 2$ when λ passes zero.

Figure 4.1 shows the Poincaré–Bendixson annulus $\bar{A}_\lambda$ with the limit cycle $\bar{\Gamma}_\lambda$ of system 4.2 for $\lambda = 0.1$ and $\lambda = 1$.



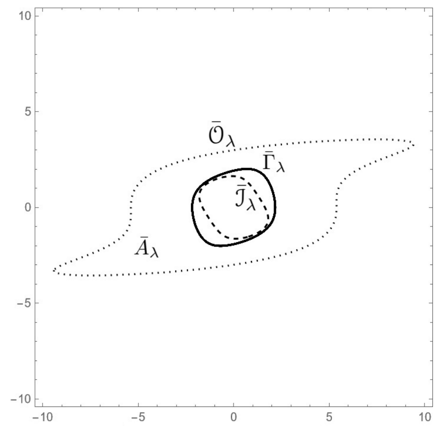


Figure 4.1: Annulus $\bar{A}_\lambda$ with the limit cycle $\bar{\Gamma}_\lambda$ of system (4.2) for $\lambda = 0.1$ (left) and $\lambda = 1$ (right).

If we apply for $\lambda > 0$ the linear diffeomorphism

$$x = \lambda u, \quad y = v, \quad t = \lambda\tau \tag{4.5}$$

to the Rayleigh system (1.2) and use the notation $\varepsilon = 1/\lambda^2$ we obtain the topologically equivalent system

$$\begin{aligned}\frac{du}{d\tau} &= -v,\\ \varepsilon\frac{dv}{d\tau} &= u + v - \frac{v^3}{3}\end{aligned} \tag{4.6}$$

which is a singularly perturbed system in case of small ε. Thus, the unique limit cycle $\hat{\Gamma}_\varepsilon$ represents a relaxation oscillation for small ε. If we apply the linear diffeomorphism (4.5) to the ovals $\mathcal{I}_\lambda$ and $\mathcal{O}_\lambda$ we obtain a global algebraic Poincaré–Bendixson annulus $\hat{\mathcal{A}}_\varepsilon$ for system (4.6).

Theorem 4.2. *The algebraic ovals*

$$\hat{\mathcal{I}}_\varepsilon := \left\{ (u,v) \in \mathcal{S}_{2\sqrt{3}} : u^2 + \varepsilon v^2 + uv\left(1 - \frac{v^2}{3}\right) - \frac{8}{3}\varepsilon = 0 \right\} \tag{4.7}$$

and

$$\begin{aligned} \hat{\mathcal{O}}_\varepsilon := \Bigg\{ (u,v) \in \mathbb{R}^2 : \\ \left(-u + v\left(\frac{v^2}{3} - 1\right)\right)(-u+v) + (1+\varepsilon)v^2 - \frac{7}{12}v^4 + \frac{1}{18}v^6 - 8\varepsilon - 3\sqrt{\varepsilon} - 18 = 0 \Bigg\} \end{aligned} \tag{4.8}$$

form a global algebraic Poincaré–Bendixson annulus $\hat{\mathcal{A}}_\varepsilon$ containing the unique limit cycle $\hat{\Gamma}_\varepsilon$ of system (4.2).

Figure 4.2 shows the Poincaré–Bendixson annulus $\hat{\mathcal{A}}_\varepsilon$ with the limit cycle $\hat{\Gamma}_\varepsilon$ of system (4.6) for $\varepsilon = 0.1$ and $\varepsilon = 2$.

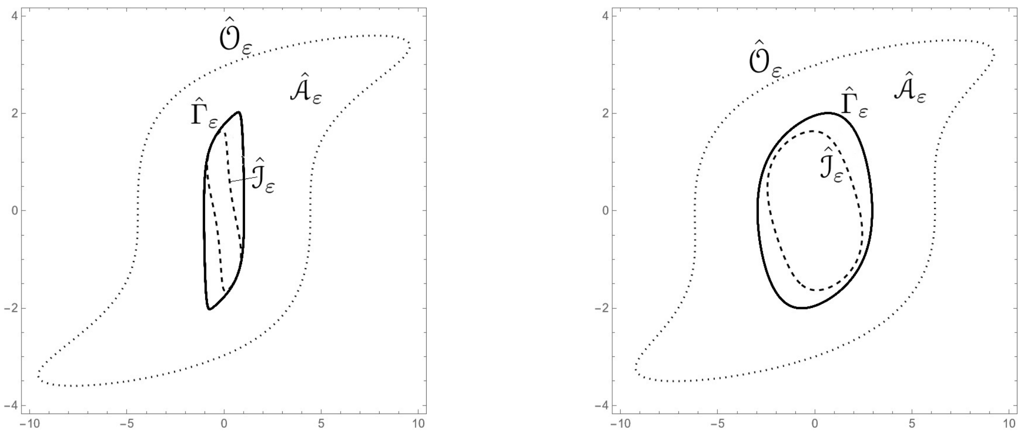


Figure 4.2: Annulus $\hat{\mathcal{A}}_\varepsilon$ with limit cycle $\hat{\Gamma}_\varepsilon$ of system (4.6) for $\varepsilon = 0.1$ (left) and $\varepsilon = 2$ (right).

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