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Periodic solution of a bioeconomic fishery model by coincidence degree theory

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Abstract. In this article we use coincidence degree theory to study the existence of a positive periodic solutions to the following bioeconomic model in fishery dynamics

$$\begin{cases} \frac{dn}{dt} = n\left(r(t)\left(1-\frac{n}{K}\right) - \frac{q(t)E}{n+D}\right), \\ \frac{dE}{dt} = E\left(\frac{A(t)q(t)}{\alpha(t)}\frac{n}{n+D} - \frac{q^2(t)}{\alpha(t)}\frac{n^2E}{(n+D)^2} - c(t)\right), \end{cases}$$

where the functions r, q, A, c and α are continuous positive T-periodic functions. This is the model of a coastal fishery represented as a single site with $n(t)$ is the fish stock biomass, and $E(t)$ is the fishing effort. Examples are given to strengthen our results.

Keywords: periodic solution, coincidence degree theory, existence of solutions.

2020 Mathematics Subject Classification: 34C25, 34A34, 34A38.

1 Introduction

In [17], Moussaoui and Auger introduced following system of three ordinary differential equation describing the fishery dynamics with price depending on supply and demand

$$\begin{cases} \frac{dn}{d\tau} = \varepsilon\left(rn\left(1-\frac{n}{K}\right) - \frac{qnE}{n+D}\right) \\ \frac{dE}{d\tau} = \varepsilon\left(-cE + p\frac{qnE}{n+D}\right) \\ \frac{dp}{d\tau} = \phi p\left(P(p) - \frac{qnE}{n+D}\right) \end{cases} \tag{1.1}$$

$n(t)$ is the fish stock biomass, $E(t)$ is the fishing effort and ($p(t)$ is the price per unit of the resource at time t). Authors assumed that the price varies at a fast time scale τ, while fish growth and investment in the fishery by boat owners occur at a slow time scale $t = \varepsilon\tau$, with $\tau \ll 1$ being a small dimensionless parameter.

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The study of existence, uniqueness and asymptotic behavior of solutions of mathematical models can be found in all applied sciences in the recent years. Many of the mathematical models occur in terms of differential equations or a system of differential equations.The increasing expansion of branches of system of differential equations has attracted many researchers to study the dynamical nature of solutions, especially, on existence and uniqueness of solutions. One of the models that attracts the attention of researchers in applied science is the bioeconomic model, similar to classical bioeconomic models of fishery dynamics [1,3].

Using regression [17], we can transform model (1.1) into the following system of two differential equations.

$$\begin{cases} \frac{dn}{dt} = n\left(r\left(1-\frac{n}{K}\right) - \frac{qE}{n+D}\right) \\ \frac{dE}{dt} = E\left(\frac{Aq}{\alpha}\frac{n}{n+D} - \frac{q^2}{\alpha}\frac{n^2E}{(n+D)^2} - c\right). \end{cases} \tag{1.2}$$

Since the variation of environment, in particular the periodic variation of the environment, play an important role in many biological and ecological system, especially, in fish stock biomass and fishing effort, it is natural to study the existence and asymptotic behavior of periodic solutions of the model (1.2). From the application point of view, only positive periodic solutions are important. Hence, it is realistic to assume the periodicity of the coefficient functions in (1.2). Thus, assuming $r, q, A, c,$ and α to be positive T-periodic functions, we have the following nonautonomous model

$$\begin{cases} \frac{dn}{dt} = n\left(r(t)\left(1-\frac{n}{K}\right) - \frac{q(t)E}{n+D}\right) \\ \frac{dE}{dt} = E\left(\frac{A(t)q(t)}{\alpha(t)}\frac{n}{n+D} - \frac{q^2(t)}{\alpha(t)}\frac{n^2E}{(n+D)^2} - c(t)\right) \end{cases} \tag{1.3}$$

where $r,\ q,\ A,\ c,$ and α are continuous positive T-periodic functions with ecological meaning as n the fish stock biomass, E the fishing effort, r fish growth rate, K carrying capacity, q catchability per fishing effort unit, D half saturation level, A carrying capacity of the market or maximum demand and α slope of the linear demand function decreasing with the price.

Setting

$$f(t,n,E) = \frac{r(t)}{K}n^2 + \frac{q(t)En}{n+D}$$

and

$$g(t,n,E) = \frac{A(t)q(t)}{\alpha(t)}\frac{nE}{n+D} - \frac{q^2(t)}{\alpha(t)}\frac{n^2E^2}{(n+D)^2},$$

we can express (1.3) into the following systems of equations

$$\begin{cases} \frac{dn}{dt} = r(t)n(t) - f(t,n(t),E(t)) \\ \frac{dE}{dt} = -c(t)E(t) + g(t,n(t),E(t)). \end{cases} \tag{1.4}$$

System of equations of the form (1.4) with general f and g have been studied by many authors [2,11,14,20–25,28] using various types of fixed point theorems to study the existence of positive T-periodic of (1.4) when f and g are positive continuous functions. Further, they were applied to many mathematical models [11,14,20–25,28] to study the existence of positive T-periodic solutions. One may refer to [19] for applications of fixed point theorems [7,9,10,12] on the existence of positive periodic solutions of mathematical models. As far as our knowledge is concerned, there exist no results on the existence and uniqueness of positive T- periodic solutions of (1.3). We have used Mawhin's coincidence degree theory to study the existence of

T-periodic solution of (1.3). Although there exist hundreds of research articles in the literature on the use of Schauder's fixed point theorem and Krasnosel'skii's fixed point theorem, the use of Mawhin's coincidence degree theory to study the existence of positive T-periodic solutions of (1.3) is relatively scarce in the literature. Previous papers based on Mawhin's coincidence degree theory for different biological models are [4–6, 8, 15, 26, 27, 29].

In order to obtain our results, we assume $r(t)$, $q(t)$, $A(t)$, $c(t)$ and $\alpha(t)$ in (1.3) are all positive T-periodic functions. Further then we assume f and g are T-periodic functions with respect to the first variable.

This work has been divided into four sections. Section 1 is Introduction. Basic theory and Mawhin's coincidence degree theory is given in Section 2. Section 3 contains the main results of this paper. Examples are given to illustrate our results. Section 4 discusses the conclusion of this article.

2 Preliminaries

Before presenting our results on the existence of periodic solution of system (1.3), We provide the essentials of the coincidence degree theory. Let Z and W be the real Banach spaces, and Let $L : \mathrm{dom}(L) \subset Z \to W$ be Fredholm operator of index zero, If $P : Z \to Z$ and $Q : W \to W$ are two continuous projectors such that $\mathrm{Im}(P) = \mathrm{Ker}(L)$, $\mathrm{Ker}(Q) = \mathrm{Im}(L)$, $Z = \mathrm{Ker}(L) \oplus \mathrm{Ker}(P)$ and $W = \mathrm{Im}(L) \oplus \mathrm{Im}(Q)$, then the inverse operator of $L|_{\mathrm{dom}(L)\cap\mathrm{Ker}(P)} : \mathrm{dom}(L) \cap \mathrm{Ker}(P) \to \mathrm{Im}(L)$ exists and is denoted by K_p (generalized inverse operator of L). If Ω is an open bounded subset of Z such that $\mathrm{dom}(L) \cap \Omega \neq 0$, the mapping $N : Z \to W$ will be called L-compact on $\overline{\Omega}$, if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N : \overline{\Omega} \to Z$ is compact. The abstract equation $Lx = Nx$ is shown to be solvable in view of [16, Theorem 2.4 on p. 84].

Theorem 2.1 ([16]). *Let L be a Fredholm operator of index zero and let N be the L-compact on $\overline{\Omega}$. Assume the following conditions are satisfied:*

1) *$Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(\mathrm{dom}(L)\backslash Ker(L)) \cap \partial\Omega] \times (0, 1)$;*
2) *$Nx \notin \mathrm{Im}(L)$ for every $x \in Ker(L) \cap \partial\Omega$;*
3) *$\deg(QN|_{\mathrm{Ker}(L)}, \mathrm{Ker}(L) \cap \Omega, 0) \neq 0$, where $Q : W \to W$ is a projector as above with $\mathrm{Im}(L) = \mathrm{Ker}(Q)$.*

Then, the equation $Lx = Nx$ has at least one solution in $\mathrm{dom}(L) \cap \overline{\Omega}$.

3 Existence of the periodic solution

For the sake of convenience and simplicity, we use the notations:

$$\overline{f} = \frac{1}{T}\int_0^T f(t)dt, \qquad f^L = \min_{t\in[0,1]} f(t), \qquad f^M = \max_{t\in[0,1]} f(t),$$

where f is a continuous t-Periodic function.

Set:

$$m_\epsilon = A^M(K + D) + \epsilon, \quad g_\epsilon = K\left(1 - \frac{q^M m_0}{Dr^L}\right) - \epsilon, \quad h_\epsilon = \frac{\alpha^L}{(q^L)^2}\left(\frac{A^L q^L g_0}{\alpha^M(K + D)} - c^M\right) - \epsilon.$$

Also, there exist positive numbers L_i $(i = 1, 2, \ldots, 4)$ such that $L_2 \leq z_1(t) \leq L_1$, $L_4 \leq z_2(t) \leq L_3$, where L_i $(i = 1, 2 \ldots, 4)$ will be calculated as in the proof of following theorem.

Theorem 3.1. *Assume the following conditions hold:*

(A1) $q^M m_0 < Dr^L$,

(A2) $c^M \alpha^M (K + D) < A^L q^L g_0$,

(A3) $\overline{A}\overline{q} - \overline{\alpha} K < \overline{\alpha}\,\overline{c} < \overline{A}\overline{q} + D\overline{r}\,\overline{q}$.

Then, system (1.3) *has at least one positive T-periodic solution*

Proof. Firstly, we make a change of variables.

Consider

$$z_1(t) = \ln n(t) \Rightarrow n(t) = e^{z_1(t)},$$
$$z_2(t) = \ln E(t) \Rightarrow E(t) = e^{z_2(t)},$$

then system (1.3) becomes

$$\begin{cases} \frac{dz_1}{dt} = r(t)\left(1 - \frac{e^{z_1(t)}}{K}\right) - \frac{q(t)e^{z_2(t)}}{e^{z_1(t)}+D}, \\ \frac{dz_2}{dt} = \frac{A(t)q(t)}{\alpha(t)} \frac{e^{z_1(t)}}{e^{z_1(t)}+D} - \frac{q^2(t)}{\alpha(t)} \frac{e^{2z_1(t)}e^{z_2(t)}}{(e^{z_1(t)}+D)^2} - c(t). \end{cases} \tag{3.1}$$

Define $Z = W = \{z = (z_1, z_2) \in (\mathbb{R}, \mathbb{R}^2) | z(t + T) = z(t)\}$, Z, W are both Banach spaces with the norm $\|\cdot\|$ as follows:

$$\|z\| = \max_{t\in[0,T]} \sum_{i=1}^{2} |z_i|, \quad z = (z_1, z_2) \in Z \text{ or } W.$$

For any $z = (z_1, z_2) \in Z$, the periodicity of system (3.1) implies

$$r(t)\left(1 - \frac{e^{z_1(t)}}{K}\right) - \frac{q(t)e^{z_2(t)}}{e^{z_1(t)} + D} = \Gamma_1(z, t),$$

$$\frac{A(t)q(t)}{\alpha(t)} \frac{e^{z_1(t)}}{e^{z_1(t)} + D} - \frac{q^2(t)}{\alpha(t)} \frac{e^{2z_1(t)}e^{z_2(t)}}{(e^{z_1(t)} + D)^2} - c(t) = \Gamma_2(z, t),$$

are T-periodic functions. In fact

$$\Gamma_1(z(t + T), t + T) = r(t)\left(1 - \frac{e^{z_1(t)}}{K}\right) - \frac{q(t)e^{z_2(t)}}{e^{z_1(t)} + D}.$$

Obviously, $\Gamma_2(z, t)$ is also periodic function by similar way.

Define operators L, P, Q as follows, respectively

$$L : \text{dom}(L) \cap Z \to W, \quad Lz = \left(\frac{dz_1}{dt}, \frac{dz_2}{dt}\right),$$

$$P\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = Q\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{T}\int_0^T z_1(t)dt \\ \frac{1}{T}\int_0^T z_2(t)dt \end{pmatrix}, \qquad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in Z = W,$$

where $\text{dom}(L) = \{z \in Z : z(t) \in C^1(\mathbb{R}, \mathbb{R}^2)\}$.

Define $N : Z \times [0,1] \to W$

$$N\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \Gamma_1(z,t) \\ \Gamma_2(z,t) \end{pmatrix}.$$

It is easy to see that

$$\mathrm{Ker}(L) = \{z \in Z \mid z = c_0,\ c_0 \in \mathbb{R}^2\},$$

and

$$\mathrm{Im}(L) = \left\{z \in W \,\middle|\, \int_0^T z(t)dt = 0\right\}$$

is closed in W. Furthermore, both P, Q are continuous projections satisfying

$$\mathrm{Im}(P) = \mathrm{Ker}(L), \qquad \mathrm{Im}(L) = \mathrm{Ker}(Q) = \mathrm{Im}(I-Q).$$

For any $z \in W$, let $z_1 = z - Qz$, we can obtain that

$$\int_0^T z_1 dp = \int_0^T z(p)dp - \int_0^T \frac{1}{T}\int_0^T z(t)dtdp = 0,$$

so $z_1 \in \mathrm{Im}(L)$. It follows that $W = \mathrm{Im}(L) + \mathrm{Im}(Q) = \mathrm{Im}(L) + \mathbb{R}^2$. Since $\mathrm{Im}(L) \cup \mathbb{R}^2 = 0$, we conclude that $W = \mathrm{Im}(L) \oplus \mathbb{R}^3$, which means $\dim\ \mathrm{Ker}(L) = \mathrm{codim}\ \mathrm{Im}(L) = \dim\ (\mathbb{R}^2) = 2$. Thus, L is a Fredholm operator of index zero, which implies that L has a unique generalized inverse operator.

Next we show that N is L-compact. Define the inverse of L as $K_P : \mathrm{Im}(L) \to\ \mathrm{Ker}(P) \cap \mathrm{dom}(L)$ and is given by

$$K_P(z) = \int_0^t z(s)ds - \frac{1}{T}\int_0^T \int_0^t z(s)dsdt.$$

Therefore, for any $z(t) \in Z$, we have

$$QN\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{T}\int_0^T \Gamma_1(z,t)dt \\ \frac{1}{T}\int_0^T \Gamma_2(z,t)dt \end{pmatrix},$$

and

$$\begin{aligned} K_P(I-Q)Nz &= \int_0^t Nz(s)ds - \frac{1}{T}\int_0^T \int_0^t Nz(s)dsdt - \frac{1}{T}\int_0^t \int_0^T QNz(s)dtds \\ &\quad + \frac{1}{T^2}\int_0^T \int_0^t \int_0^T QNz(s)dtdsdt \\ &= \int_0^t Nz(s)ds - \frac{1}{T}\int_0^T \int_0^t Nz(s)dsdt - \left(\frac{t}{T} - \frac{1}{2}\right)\int_0^T QNz(s)ds. \end{aligned}$$

Clearly, QN and $K_P(I-Q)N$ are continuous. Due to Z is Banach space, using the Arzelà–Ascoli theorem, we have that N is L-compact on $\overline{U}$ for any open bounded set $U \subset Z$. Next, in order to apply the coincidence degree theory, we need to construct an appropriate open bounded subset U. Therefore, the operator equation is defined by $Lz = \lambda Nz$, $\lambda \in (0,1)$, that is,

$$\begin{cases} \frac{dz_1}{dt} = \lambda \left[r(t) \left(1 - \frac{e^{z_1(t)}}{K} \right) - \frac{q(t)e^{z_2(t)}}{e^{z_1(t)}+D} \right], \\ \frac{dz_2}{dt} = \lambda \left[\frac{A(t)q(t)}{\alpha(t)} \frac{e^{z_1(t)}}{e^{z_1(t)}+D} - \frac{q^2(t)}{\alpha(t)} \frac{e^{2z_1(t)}e^{z_2(t)}}{(e^{z_1(t)}+D)^2} - c(t) \right]. \end{cases} \tag{3.2}$$

We assume that $z \in (z_1, z_2)^T \in Z$ is a T-periodic solution of system (3.1) for any fixed $\lambda \in (0,1)$. Now, integrating system (3.1) from 0 to T leads to

$$\begin{cases} \bar{r}T = \int_0^T \left[\frac{r(t)e^{z_1(t)}}{K} + \frac{q(t)e^{z_2}}{e^{z_1}+D} \right] dt, \\ \bar{c}T = \int_0^T \left[\frac{A(t)q(t)}{\alpha(t)} \frac{e^{z_1(t)}}{e^{z_1(t)}+D} - \frac{q^2(t)}{\alpha(t)} \frac{e^{2z_1(t)}e^{z_2(t)}}{(e^{z_1(t)}+D)^2} \right] dt. \end{cases} \tag{3.3}$$

Since $(z_1, z_2) \in Z$, there exist $\eta_i, \xi_i \in [0, T]$ such that

$$z_i(\eta_i) = \max_{t\in[0,T]} z_i(t), \qquad z_i(\xi_i) = \min_{t\in[0,T]} z_i(t), \qquad i = 1, 2.$$

Through simple analysis, we have,

$$\dot{z}_1(\eta_1) = \dot{z}_1(\xi_1) = 0, \qquad \dot{z}_2(\eta_2) = \dot{z}_2(\xi_2) = 0.$$

If we apply previous to (3.2), we obtain

$$r(\eta_1) \left(1 - \frac{e^{z_1(\eta_1)}}{K} \right) - \frac{q(\eta_1)e^{z_2(\eta_1)}}{e^{z_1(\eta_1)} + D} = 0, \tag{3.4}$$

$$- c(\eta_2) + \frac{A(\eta_2)q(\eta_2)}{\alpha(\eta_2)} \frac{e^{z_1(\eta_2)}}{e^{z_1(\eta_2)} + D} - \frac{q^2(\eta_2)}{\alpha(\eta_2)} \frac{e^{2z_1(\eta_2)}e^{z_2(\eta_2)}}{(e^{z_1(\eta_2)} + D)^2} = 0, \tag{3.5}$$

and

$$r(\xi_1) \left(1 - \frac{e^{z_1(\xi_1)}}{K} \right) - \frac{q(\xi_1)e^{z_2(\xi_1)}}{e^{z_1(\xi_1)} + D} = 0, \tag{3.6}$$

$$- c(\xi_2) + \frac{A(\xi_2)q(\xi_2)}{\alpha(\xi_2)} \frac{e^{z_1(\xi_2)}}{e^{z_1(\xi_2)} + D} - \frac{q^2(\xi_2)}{\alpha(\xi_2)} \frac{e^{2z_1(\xi_2)}e^{z_2(\xi_2)}}{(e^{z_1(\xi_2)} + D)^2} = 0. \tag{3.7}$$

From (3.4), we obtain

$$r(\eta_1) - \frac{r(\eta_1)e^{z_1(\eta_1)}}{K} > 0,$$

which implies that

$$z_1(\eta_1) < \ln(K) = L_1. \tag{3.8}$$

Considering (3.5) and (3.8), we get

$$\frac{q^2(\eta_2)}{\alpha(\eta_2)} \frac{e^{2z_1(\eta_2)}e^{z_2(\eta_2)}}{(e^{z_1(\eta_2)} + D)^2} + c(\eta_2) = \frac{A(\eta_2)q(\eta_2)}{\alpha(\eta_2)} \frac{e^{z_1(\eta_2)}}{e^{z_1(\eta_2)} + D}.$$

So, we can obtain

$$\frac{q^2(\eta_2)}{\alpha(\eta_2)} \frac{e^{2z_1(\eta_2)}e^{z_2(\eta_2)}}{(e^{z_1(\eta_2)} + D)^2} < \frac{A(\eta_2)q(\eta_2)}{\alpha(\eta_2)} \frac{e^{z_1(\eta_2)}}{e^{z_1(\eta_2)} + D},$$

or

$$\frac{qe^{z_1(\eta_2)}e^{z_2(\eta_2)}}{e^{z_1(\eta_2)}+D} < A(\eta_2),$$

or

$$e^{z_2(\eta_2)} < A^M(e^{z_1(\eta_2)}+D),$$

or

$$e^{z_2(\eta_2)} < A^M(K+D),$$

which gives

$$z_2(\eta_2) < \ln(A^M(K+D)) = \ln m_0 = L_3. \tag{3.9}$$

From (3.6) and (3.9), we can obtain

$$r(\xi_1) - \frac{r(\xi_1)e^{z_1(\xi_1)}}{K} - \frac{q(\xi_1)m_0}{D} < 0,$$

then,

$$\frac{e^{z_1(\xi_!)}}{K} > 1 - \frac{q^M m_0}{Dr^L}$$

which implies that

$$z_1(\xi_1) > \ln\left(K\left(1 - \frac{q^M m_0}{Dr^L}\right)\right) = \ln(g_0) = L_2. \tag{3.10}$$

In view of (3.7) and (3.10), we have

$$\frac{q^2(\xi_2)}{\alpha(\xi_2)}\frac{e^{2z_1(\xi_2)}e^{z_2(\xi_2)}}{(e^{z_1(\xi_2)}+D)^2} = \frac{A(\xi_2)q(\xi_2)}{\alpha(\xi_2)}\frac{e^{z_1(\xi_2)}}{e^{z_1(\xi_2)}+D} - c(\xi_2).$$

Thus,

$$\frac{q^2(\xi_2)e^{z_2(\xi_2)}}{\alpha(\xi_2)} > \frac{A^L q^L g_0}{\alpha^M(K+D)} - c^M,$$

or

$$e^{z_2(\xi_2)} > \frac{\alpha^L}{(q^2)^L}\left(\frac{A^L q^L g_0}{\alpha^M(K+D)} - c^M\right),$$

that is

$$z_2(\xi_2) > \ln\left(\frac{\alpha^L}{(q^2)^L}\left(\frac{A^L q^L g_0}{\alpha^M(K+D)} - c^M\right)\right) = \ln(h_0) = L_4. \tag{3.11}$$

Finally, from (3.8), (3.9), (3.10), (3.11), we get

$$|z_1(t)| < \max\{|L_1|, |L_2|\} = \Lambda_1,$$

$$|z_2(t)| < \max\{|L_3|, |L_4|\} = \Lambda_2.$$

where Λ_1, Λ_2 is independent of λ. Denote $\Lambda = \Lambda_1 + \Lambda_2 + \Lambda_3$ where Λ_3 is taken sufficiently large such that each solution (z_1^*, z_2^*) of following system

$$\begin{cases} \bar{r} - \frac{\bar{r}}{K}e^{z_1(t)} - \frac{\bar{q}e^{z_2(t)}}{e^{z_1(t)}+D} = 0, \\ \frac{\overline{Aq}}{\alpha(t)}\frac{e^{z_1(t)}}{e^{z_1(t)}+D} - e^{z_1(t)} + \frac{\overline{q}^2}{\alpha(t)}\frac{e^{2z_1(t)}e^{z_2(t)}}{(e^{z_1(t)}+D)^2} - \bar{c} = 0, \end{cases} \tag{3.12}$$

satisfies $|z_1^*| + |z_2^*| < \Lambda$. Now we consider $\Omega = \{(z_1, z_2)^T \in Z : \|(z_1, z_2)\| < \Lambda\}$ then it is clear that Ω satisfies the first condition of Theorem 2.1.

For the second condition of Theorem 2.1, we prove that $QN(z_1, z_2)^T \neq (0,0)^T$ for each $(z_1, z_2) \in \partial\Omega \cap \mathrm{Ker}(L)$. When $(z_1, z_2)^T \in \partial\Omega \cap \mathrm{Ker}(L) = \partial\Omega \cap \mathbb{R}^2$, $(z_1, z_2)^T$ is a constant vector in $\mathbb{R}^2$ and $|z_1| + |z_2| = \Lambda$. If the system (3.12) has a solution, then

$$QN\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \bar{r} - \frac{\bar{r}}{K}e^{z_1(t)} - \frac{\bar{q}e^{z_2(t)}}{e^{z_1(t)}+D} \\ \frac{\overline{Aq}}{\bar{\alpha}}\frac{e^{z_1(t)}}{e^{z_1(t)}+D} - e^{z_1(t)} + \frac{\bar{q}^2}{\bar{\alpha}}\frac{e^{2z_1(t)}e^{z_2(t)}}{(e^{z_1(t)}+D)^2} - \bar{c} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since, (3.12) does not have solution then, it is evident that $QN(z_1, z_2)^T \neq 0$, thus the second condition of Theorem 2.1 is satisfied. Finally, we prove that the last condition of Theorem 2.1 is satisfied, to do so, we define the following mapping $\Psi_\mu : \mathrm{dom}(L) \times [0,1] \to Z$

$$\Psi_\mu\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \bar{r} - \frac{\bar{r}}{K}e^{z_1(t)} - \frac{\bar{q}e^{z_2(t)}}{\mu e^{z_1(t)}+D} \\ \frac{\overline{Aq}}{\bar{\alpha}}\frac{e^{z_1(t)}}{e^{z_1(t)}+\mu D} - e^{z_1(t)} + \frac{\bar{q}^2}{\bar{\alpha}}\frac{e^{2z_1(t)}e^{z_2(t)}}{(e^{z_1(t)}+\mu D)^2} - \bar{c} \end{pmatrix}.$$

By using the invariance property of homotopy in topological degree theory, we get,

$$\begin{aligned} &\deg(QN(z_1, z_2)^T, \Omega \cap \mathrm{Ker}(L), (0,0)^T) \\ &\quad = \deg(\Psi(z_1, z_2, 1)^T, \Omega \cap \mathrm{Ker}(L), (0,0)^T) \\ &\quad = \deg(\Psi(z_1, z_2, \mu)^T, \Omega \cap \mathrm{Ker}(L), (0,0)^T) \\ &\quad = \deg(\Psi(z_1, z_2, 0)^T, \Omega \cap \mathrm{Ker}(L), (0,0)^T) \\ &\quad = \deg\left(\bar{r} - \frac{\bar{r}}{K}e^{z_1(t)} - \frac{\bar{q}e^{z_2(t)}}{D}, \frac{\overline{Aq}}{\bar{\alpha}} - e^{z_1(t)} + \frac{\bar{q}^2}{\bar{\alpha}}e^{z_2(t)} - \bar{c}, \Omega \cap \mathrm{Ker}(L), (0,0)^T\right) \end{aligned}$$

Furthermore, the system of algebraic equation

$$\begin{cases} \bar{r} - \frac{\bar{r}}{K}x - \frac{\bar{q}y}{D} = 0, \\ \frac{\overline{Aq}}{\bar{\alpha}} - x + \frac{\bar{q}^2}{\bar{\alpha}}y - \bar{c} = 0, \end{cases} \tag{3.13}$$

has a unique solution (x^*, y^*), where $x^* = \bar{r}(1 - \frac{\bar{\alpha}K + \bar{\alpha}\bar{c} - \overline{Aq}}{\bar{\alpha}K + D\bar{r}\bar{q}}) > 0$ and $y^* = \frac{D\bar{r}(\bar{\alpha}K + \bar{\alpha}\bar{c} - \overline{Aq})}{\bar{q}(\bar{\alpha}K + D\bar{r}\bar{q})} > 0$. Thus,

$$\begin{aligned} \deg\{QN(z_1, z_2)^T, \Omega \cap \mathrm{Ker}(L), (0,0)^T\} &= \begin{vmatrix} -\frac{\bar{r}}{K}x^* & -\frac{q}{D}y^* \\ -1 & \frac{q^2}{\alpha}y^* \end{vmatrix} \\ &= \mathrm{sgn}\left[-\bar{q}\left(\frac{\bar{q}\bar{r}x^*y^*}{\bar{\alpha}K} - \frac{y^*}{D}\right)\right] \\ &= -1 \neq 0. \end{aligned}$$

Now, all the conditions in Theorem 2.1 have been verified. This implies that system (3.1) has at least one T-periodic solution. Consequently, system (1.3) has at least one positive T-periodic solution. The theorem is proved. □

Corollary 3.1. *If $qA(K + D) < Dr$, $c\alpha(K + D) < AqK(1 - \frac{qA(K+D)}{Dr})$, and $Aq - \alpha K < \alpha c <$*

$Aq + Drq$ holds, then the system (1.2) has a positive T-periodic solution.

Example 3.1. By Corollary 3.1, the system of equations

$$\begin{cases} \frac{dn}{dt} = n\left(8\left(1-\frac{n}{1}\right) - \frac{E}{n+1/2}\right) \\ \frac{dE}{dt} = E\left(\frac{1.5\times 1}{1.5}\frac{n}{n+1/2} - \frac{1^2}{1.5}\frac{n^2E}{(n+1/2)^2} - \frac{1}{4}\right). \end{cases} \tag{3.14}$$

has a positive periodic solution.

Example 3.2. Consider the system

$$\begin{cases} \frac{dn}{dt} = n\left((21+\cos t)\left(1-\frac{n}{1.1}\right) - \frac{(1.15+\frac{1}{10}\cos t)E}{n+\frac{1}{4}}\right) \\ \frac{dE}{dt} = E\left(\frac{(1.7+\frac{1}{10}\sin t)(1.15+\frac{1}{10}cost)}{(1.5+\frac{1}{10}\cos t)}\frac{n}{n+\frac{1}{4}} - \frac{(1.15+\frac{1}{10}\cos t)^2}{(1.5+\frac{1}{10}\cos t)}\frac{n^2E}{(n+\frac{1}{4})^2} - \left(\frac{1}{4}+\frac{1}{10}\sin t\right)\right) \end{cases} \tag{3.15}$$

It is easy to obtain $\overline{q} = 1.15,\ \overline{r} = 21,\ \overline{A} = 1.7,\ \overline{\alpha} = 1.5,\ \overline{c} = 0.25,\ D = 0.25, K = 1.1,\ q^L = 1.05,\ q^M = 1.25,\ r^L = 20,\ A^L = 1.6,\ A^M = 1.8,\ \alpha^M = 1.6,\ c^M = 0.35,\ m_0 = 1.8(1.1+0.25) = 2.43,\ \ g_0 = 1.1\left(1-\frac{1.25\times 2.43}{0.25\times 20}\right) = 0.43175$. Consequently, we obtain

$$q^M m_0 = 3.0375 < Dr^L = 5,$$

$$c^M\alpha^M(K+D) = 0.7 < A^L q^L g_0 = 0.72534,$$

and

$$\overline{A}\overline{q} = 0.305 < \overline{\alpha}\overline{c} = 0.525 < \overline{A}\overline{q} + D\overline{r}\overline{q} = 7.9925.$$

It is clear that assumptions (A1), (A2), (A3) are satisfied. Hence, according to Theorem 3.1, system (3.15) has at least one positive T-periodic solution.

4 Conclusion

Using Mawhin's coincidence degree theory, we have established sufficient conditions for the existence of positive periodic solutions of the model (1.3). By formulating the model as a system of differential equations and introducing appropriate transformations, we were able to apply the coincidence degree theory and obtain our main results. The conditions (A1), (A2), and (A3) played a crucial role in establishing the existence of periodic solutions.

Set $r(t) \equiv r,\ q(t) \equiv q,\ A(t) \equiv A,\ \alpha(t) \equiv \alpha$ and $c(t) \equiv c$ be constants; then (1.3) reduces to

$$\begin{cases} \frac{dn}{dt} = n\left(r\left(1-\frac{n}{K}\right) - \frac{qE}{n+D}\right) \\ \frac{dE}{dt} = E\left(\frac{Aq}{\alpha}\frac{n}{n+D} - \frac{q^2}{\alpha}\frac{n^2E}{(n+D)^2} - c\right). \end{cases} \tag{4.1}$$

In a recent paper, Moussaoui and Auger [17], studied the equilibrium points of (4.1). They proved that if

$$Aq < \alpha c, \tag{4.2}$$

then the system (4.1) has no positive equilibrium point provided that

$$D < K, \qquad \frac{\alpha}{q} < \frac{K}{2} \tag{4.3}$$

holds. It is worth noting that our conditions (A1), (A2), and (A3) are different from the condition (4.2). By Theorem 3.1, the system (4.1) has positive T-periodic solution. We note that the condition (4.3) can be satisfied for large K. On the other hand, by Theorem 1 b) of [17], the system (4.1) has a unique positive equilibrium, which is a positive T-periodic solution of (4.1). Our Theorem 3.1 strengthens this observation.

While this research paper has successfully addressed the existence of positive periodic solutions for the bioeconomic fishery model, there are several avenues for further exploration. It would be interesting to study global attractivity and uniqueness of the solution for the system investigated in this paper. Another promising direction is to examine problem (1.3) by introducing a delay in the system, such as incorporating a time lag in fish stock biomass. Conducting further investigations in these areas have potential implications for understanding and managing fisheries dynamics.

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