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Solutions for a quasilinear elliptic problem with indefinite nonlinearity with critical growth

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Abstract. We are interested in nonhomogeneous problems with a nonlinearity that changes sign and may possess a critical growth as follows

$$\begin{cases} -\operatorname{div}\left(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\right) = \lambda|u|^{q-2}u + W(x)|u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $N \geq 2$, $1 < p \leq q < N$, $q < r \leq q^*$, $\lambda \in \mathbb{R}$ and function W is a weight function which changes sign in Ω. Using variational methods, we prove the existence of four solutions: two solutions which do not change sign and two solutions which change sign exactly once in Ω.

Keywords: subcritical and critical exponents, $p\&q$ Laplacian operator, indefinite problems.

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1 Introduction

The goal of this paper is to find nontrivial solutions for the problem

$$\begin{cases} -\operatorname{div}\left(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\right) = \lambda|u|^{q-2}u + W(x)|u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $N \geq 2$, $1 < p \leq q < N$, $q < r \leq q^*$ and $\lambda \in \mathbb{R}$, where $q^* = \frac{Nq}{N-q}$ is the critical Sobolev exponent.

We introduce the hypotheses on the function a in the sequel.

(a_1) Function $a : [0, \infty) \to \mathbb{R}$ is of class C^1 and there exist constants $k_1, k_2 \geq 0$ such that

$$k_1 t^p + t^q \leq a(t^p)t^p \leq k_2 t^p + t^q, \quad \text{for all } t > 0;$$

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(a_2) Define, for $t \geq 0$, $A(t) = \int_0^t a(s)ds$. The mapping $t \mapsto A(t^p)$ is convex on $(0, \infty)$;

(a_3) The mapping $t \mapsto \frac{a(t^p)}{t^{q-p}}$ is nonincreasing on $(0, \infty)$.

(a_4) If $1 < p \leq q \leq 2 \leq N$, then the mapping $t \mapsto a(t)$ is nondecreasing for $t > 0$. If $2 \leq p \leq q < N$, the mapping $t \mapsto a(t)t^{p-2}$ is nondecreasing for $t > 0$.

As a direct consequence of (a_3), we obtain that the function a and its derivative a' satisfy

$$a'(t)t \leq \frac{(q-p)}{p} a(t) \quad \text{for all } t > 0. \tag{1.1}$$

Now, if we define the function $h(t) = a(t)t - \frac{q}{p}A(t)$, using (1.1) we can prove that function h is nonincreasing. Then,

$$\frac{1}{q} a(t)t \leq \frac{1}{p} A(t), \quad \text{for all } t \geq 0. \tag{1.2}$$

To illustrate the degree of generality of the kind of problems studied here, and with adequate hypotheses on the functions a, which will be made clear shortly, we present some examples of problems that are also interesting from a mathematical point of view and have a wide range of applications in physics and related sciences.

Problem 1: Let $a(t) = t^{\frac{q-p}{p}}$. In this case we are studying problem as

$$\begin{cases} -\Delta_q u = \lambda |u|^{q-2}u + W(x)|u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

and it is related to the main result showed in [6]. See also the work [7].

Problem 2: Let $a(t) = 1 + t^{\frac{q-p}{p}}$. In this case we are studying problem as

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda |u|^{q-2}u + W(x)|u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_λ}$$

Problem 3: Let $a(t) = 1 + \frac{1}{(1+t)^{\frac{p-2}{p}}}$. In this case we are studying problem

$$\begin{cases} -\Delta_p u - \operatorname{div}\left(\dfrac{|\nabla u|^{p-2}\nabla u}{(1 + |\nabla u|^p)^{\frac{p-2}{p}}} \right) = \lambda |u|^{q-2}u + W(x)|u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_λ}$$

Problem 4: Let $a(t) = 1 + t^{\frac{q-p}{p}} + \frac{1}{(1+t)^{\frac{p-2}{p}}}$. In this case, we are studying problem

$$\begin{cases} -\Delta_p u - \Delta_q u - \operatorname{div}\left(\dfrac{|\nabla u|^{p-2}\nabla u}{(1 + |\nabla u|^p)^{\frac{p-2}{p}}} \right) = \lambda |u|^{q-2}u + W(x)|u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_λ}$$

Such class of problems arise from applications in physics and related sciences, such as biophysics, plasma physics and chemical reactions (for instance, see [16,17,24]).

The interest in studying nonlinear partial differential equations with $p\&q$ operator has increased because many applications arising in mathematical physics may be stated with an operator in this form. We refer the reader to the works [9–11, 15], where the authors have considered nonhomogeneous elliptic problems involving several type of function a.

Problems involving indefinite nonlinearities, that is, signal changing nonlinearities, have attracted the attention of many researchers over the past few decades, either because of their application in population dynamics describing the stationary behavior of a population in a heterogeneous environment (see [1, 19, 22, 23]) or because of their mathematical relevance. Researchers have studied this type of problem using: variational methods (see [2,3,8,12,20,21]), sub-supersolution method (see [12, 13, 20]) and Morse theory (see [2, 19]).

This paper deals with the class of problem (P_λ) that brings important characteristics, which are the nonlinearities that change signal (see the hypotheses on W below) with subcritical or critical growth and the generality of the operator that includes, for instance, $p-$Laplacian and $p\&q-$Laplacian operators. These characteristics provoke some behaviors in the geometry of the energy functional associated to problem (P_λ) which make it difficult to find nontrivial solutions. As far as we know, this is the only work that proves existence and multiplicity of ground state solutions of problem (P_λ) under our assumptions.

Let us consider a weight function $W : \Omega \to \mathbb{R}$ which changes sign in Ω. More specifically, function W satisfies

(W_1) $W \in L^\infty(\Omega)$ and the set $\Omega_+ := \{ x \in \Omega : W(x) > 0\}$ has positive measure.

It follows directly of (W_1) that

$$\lambda^* := \inf \left\{ \frac{\int_\Omega |\nabla u|^q \, dx}{\int_\Omega |u|^q \, dx} : u \in W_0^{1,q}(\Omega) \setminus \{0\} \text{ and } \int_\Omega W(x)|u|^r \, dx \geq 0 \right\} < +\infty. \tag{1.3}$$

We are going to require another important hypothesis on W. For this, let λ_1 be the first eigenvalue of the operator $(-\Delta_q)$ on Ω, with zero Dirichlet boundary condition, and let φ_1 be the first eigenfunction associated to λ_1. The weight function W satisfies only one of the following two hypotheses:

(W_2^+)

$$\int_\Omega W(x)|\varphi_1|^r \, dx > 0.$$

(W_2^-)

$$\int_\Omega W(x)|\varphi_1|^r \, dx < 0.$$

By the variational characterization of λ_1, we have

i) If the weight function W satisfies (W_1) and (W_2^+), then $\lambda^* = \lambda_1$.

ii) If the weight function W satisfies (W_1) and (W_2^-), then $\lambda^* > \lambda_1$.

We are now ready to state our first main result concerning the subcritical case.

Theorem 1.1. *Let $r < q^*$, a satisfying (a_1)–(a_4) and the weight function W satisfying (W_1), (W_2^+) or (W_2^-). Then,*

i) if $\lambda < \lambda_1$ and u is a nontrivial solution of (P_λ), then

$$\int_\Omega W(x)|u|^r dx > 0;$$

ii) if $\lambda \in (-\infty, \lambda^*)$, then problem ($P_\lambda$) has two least energy solutions which do not change sign in Ω. Moreover, if $\lambda < \lambda_1$, these two solutions are ground state solutions;

iii) if $\lambda \in (-\infty, \lambda^*)$, then problem ($P_\lambda$) has two least energy nodal solutions which change sign exactly once in Ω. Moreover, if $\lambda < \lambda_1$, these two nodal solutions are nodal ground state solutions.

Item (i) of Theorem 1.1 provides some interesting qualitative properties on nontrivial solutions of problem (P_λ). For example:

1) If $\Omega_0 := \{ x \in \Omega : W(x) = 0\} \subset \Omega$ is a domain with smooth boundary and u is a nontrivial solution of (P_λ), then $u \neq 0$ $a.e$ in $\Omega \setminus \Omega_0$;

2) If $\Omega_+ := \{ x \in \Omega : W(x) > 0\}$ and $\Omega_- := \{ x \in \Omega : W(x) < 0\}$ have positive measure, and u is a nontrivial solution of (P_λ), then u must "belong" more to Ω_+ than Ω_-, that is,

$$\int_{\Omega_+} W(x)|u|^r dx > -\int_{\Omega_-} W(x)|u|^r dx > 0;$$

3) If Ω is a symmetric set and $W \in C(\Omega)$ is an odd function, then a nontrivial solution u of (P_λ) can be neither an even nor an odd function. In fact, otherwise

$$\int_{\Omega} W(x)|u|^r dx = \int_{\Omega_+} W(x)|u|^r dx + \int_{\Omega_-} W(x)|u|^r dx = 0;$$

To illustrate this, consider $\Omega = \left\{x \in \mathbb{R}^N : \ |x| < 2\pi\right\}$ and $W : \mathbb{R}^N \to \mathbb{R}$ given by $W(x) = \cos(|x|)$.

To show the existence of solutions to the problem in the critical case, we will need to add a new hypothesis on the weight function W. The new hypothesis is as follows.

(W_3) There exists an open set $\Omega_* \subset\subset \Omega_+$ such that $|\Omega_-| > |\Omega_*|$. Moreover, there exist positive numbers $\mathcal{W}_1$ and $\mathcal{W}_2$ such that

$$\mathcal{W}_1 \geq W(x) \geq \mathcal{W}_2 > \|W^-\|_\infty, \quad \forall\, x \in \Omega_*.$$

The above hypothesis is fundamental to overcome the lack of compactness generated by the critical exponent $r = q^*$. It is important to highlight that, up to our knowledge, (W_3) is a new hypothesis in the literature, which makes it one of the relevant points of this work.

To provide an example of a function that satisfies hypothesis (W_3), just consider $\Omega = \{x \in \mathbb{R}^N : 0 \leq |x| \leq 2\pi\}$, $\Omega_* = \left\{x \in \mathbb{R}^N : \frac{\pi}{4} \leq |x| \leq \frac{3\pi}{4}\right\}$ and $W : \Omega \to \mathbb{R}$, given by

$$W(x) = \begin{cases} \sin(|x|), & |x| \leq \pi, \\ \dfrac{\sin(|x|)}{2\sqrt{2}}, & \pi \leq |x| \leq 2\pi. \end{cases}$$

Now, let $S > 0$ be the best constant of the Sobolev embedding $W_0^{1,q}(\Omega) \hookrightarrow L^{q^*}(\Omega)$. Our second main result, concerning the critical case, is the following.

Theorem 1.2. *Consider $r = q^*$ and $\lambda < \lambda_1$. Let a satisfy (a_1)–(a_4) and the weight function W satisfy (W_1), (W_3) and*

$$\int_\Omega W(x)|\varphi_1|^{q^*}\,dx < \frac{\frac{1}{N}\left(\mathcal{W}_2 - \|W^-\|_\infty\right)\left(\frac{S}{\mathcal{W}_1}\right)^{\frac{N}{q}}}{\frac{k_2}{k_1 p} + \frac{1}{N}}.$$

Then, there are two nontrivial solutions for problem (P_λ).

The paper is organized as follows: in Section 2, we will prove technical results and the first part of Theorem 1.1. In Section 3, we will demonstrate the second part of Theorem 1.1, namely, the existence of least energy solutions that do not change sign. Finally, in Section 4, we will establish the last part of Theorem 1.1, that is, the existence of least energy nodal solutions that change sign exactly once.

2 Variational framework and preliminary results

The natural space to look for weak solutions to problem (P_λ) is the Sobolev space $W_0^{1,q}(\Omega)$ with the associated norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^q dx\right)^{\frac{1}{q}}, \quad \text{for } u \in W_0^{1,q}(\Omega).$$

Since the approach is variational, consider the energy functional associated $J_\lambda : W_0^{1,q}(\Omega) \to \mathbb{R}$ given by

$$J_\lambda(u) := \frac{1}{p}\int_\Omega A\left(|\nabla u|^p\right)\,dx - \frac{\lambda}{q}\int_\Omega |u|^q\,dx - \frac{1}{r}\int_\Omega W(x)|u|^r\,dx.$$

We know that J_λ is differentiable on $W_0^{1,q}(\Omega)$ and, for all $u, v \in W_0^{1,q}(\Omega)$,

$$J'_\lambda(u)v := \int_\Omega a\left(|\nabla u|^p\right)|\nabla u|^{p-2}\nabla u \nabla v\,dx - \lambda\int_\Omega |u|^{q-2}uv\,dx - \int_\Omega W(x)|u|^{r-2}uv\,dx.$$

Thus, $u \in W_0^{1,q}(\Omega)$ is a critical point of J_λ if, and only if, u is a weak solution of problem (P_λ). Moreover, let us define the Nehari manifold

$$\mathcal{N}_\lambda := \left\{u \in W_0^{1,q}(\Omega) \ : \ J'_\lambda(u)u = 0\right\} \tag{2.1}$$

and the nodal Nehari set

$$\mathcal{N}_\lambda^\pm := \left\{u \in W_0^{1,q}(\Omega) \ : \ u^\pm \neq 0 \text{ and } J'_\lambda(u)u = 0\right\}, \tag{2.2}$$

where

$$u^+(x) := \max\{u(x),\, 0\} \quad \text{and} \quad u^-(x) := \min\{u(x),\, 0\}.$$

Notice that $u = u^+ + u^-$ and $\mathcal{N}_\lambda^\pm \subset \mathcal{N}_\lambda$.

Now we introduce some important subsets of $\mathcal{N}_\lambda$. Consider

$$\mathcal{M}_\lambda := \left\{u \in W_0^{1,q}(\Omega) : \ u \in \mathcal{N}_\lambda \text{ and } \int_\Omega W(x)|u|^r\,dx > 0\right\} \tag{2.3}$$

and

$$\mathcal{M}_\lambda^{\pm} := \left\{ u \in W_0^{1,q}(\Omega) : \ u^{\pm} \in \mathcal{N}_\lambda \text{ and } \int_\Omega W(x)|u^{\pm}|^r \, dx > 0 \right\}. \tag{2.4}$$

Since we want to use the method of minimization, we begin to study the behavior of the functional J_λ on $\mathcal{N}_\lambda$.

Proposition 2.1. *Assume that the function a satisfies (a_1)–(a_3). Then, there exist positive constants K_1, K_2 and K_3 such that the following properties hold:*

(i) $J_\lambda(u) \geq K_1\big(\frac{\lambda_1-\lambda}{\lambda_1}\big)\|u\|^q$, *for all* $u \in \mathcal{N}_\lambda$.

(ii) $\|u\| \geq K_2\big(\frac{\lambda_1-\lambda}{\lambda_1}\big)^{\frac{1}{r-q}}$, *for all* $u \in \mathcal{N}_\lambda$.

(iii) $\int_\Omega W(x)|u|^r \, dx \geq K_3\big(\frac{\lambda_1-\lambda}{\lambda_1}\big)^{\frac{r}{r-q}}$, *for all* $u \in \mathcal{N}_\lambda$.

Proof. For every $u \in \mathcal{N}_\lambda$, by (1.2), we have

$$\begin{aligned} J_\lambda(u) &= J_\lambda(u) - \frac{1}{r}J_\lambda'(u)u \\ &= \frac{1}{p}\int_\Omega A(|\nabla u|^p)dx - \frac{1}{r}\int_\Omega a\left(|\nabla u|^p\right)|\nabla u|^p dx - \lambda\left(\frac{1}{q}-\frac{1}{r}\right)\int_\Omega |u|^q dx \\ &\geq \left(\frac{1}{q}-\frac{1}{r}\right)\int_\Omega a\left(|\nabla u|^p\right)|\nabla u|^p \, dx - \lambda\left(\frac{1}{q}-\frac{1}{r}\right)\int_\Omega |u|^q dx. \end{aligned}$$

Hence, by (a_1) and the Poincaré inequality,

$$J_\lambda(u) \geq \left(\frac{r-q}{qr}\right)\left(\frac{\lambda_1-\lambda}{\lambda_1}\right)\left(\int_\Omega |\nabla u|^q dx\right). \tag{2.5}$$

Then item (i) follows.

We now prove item (ii). Taking $u \in \mathcal{N}_\lambda$, by (a_1) and the Poincaré inequality, one has

$$\begin{aligned} \int_\Omega |\nabla u|^q \, dx \leq \int_\Omega a\left(|\nabla u|^p\right)|\nabla u|^p \, dx &= \lambda\int_\Omega |u|^q \, dx + \int_\Omega W(x)|u|^r \, dx \\ &\leq \frac{\lambda}{\lambda_1}\int_\Omega |\nabla u|^q \, dx + \int_\Omega W(x)|u|^r \, dx. \end{aligned}$$

Hence,

$$\left(1-\frac{\lambda}{\lambda_1}\right)\int_\Omega |\nabla u|^q \, dx \leq \int_\Omega W(x)|u|^r \, dx. \tag{2.6}$$

Finally, using that $W \in L^\infty(\Omega)$, the Sobolev embeddings and (2.6), there exists a positive constant C_1 such that

$$\left(1-\frac{\lambda}{\lambda_1}\right)\|u\|^q \leq C_1\|u\|^r.$$

This inequality proves item (ii).

Item (iii) follows directly from inequality contained in item (ii) and by (2.6). In fact,

$$K_2\left(1-\frac{\lambda}{\lambda_1}\right) \leq \int_\Omega W(x)|u|^r \, dx. \qquad \square$$

The next result is a direct consequence of Proposition (2.1).

Corollary 2.2. *If $\lambda < \lambda_1$, then $\mathcal{N}_\lambda = \mathcal{M}_\lambda$ and $\mathcal{N}_\lambda^\pm = \mathcal{M}_\lambda^\pm$.*

Proof. By definition of $\mathcal{N}_\lambda$ and $\mathcal{M}_\lambda$, we get $\mathcal{M}_\lambda \subset \mathcal{N}_\lambda$ and $\mathcal{M}_\lambda^\pm \subset \mathcal{N}_\lambda^\pm$. The other inclusions follow from item (iii) of previous proposition. □

By the same arguments of Proposition 2.1, but using the definition of λ^* instead of Poincaré inequality, the next result follows.

Proposition 2.3. *Assume that function a satisfies (a_1)–(a_3). Then, there exist positive constants K_1, K_2 and K_3 such that the following properties hold:*

(i) $J_\lambda(u) \geq K_1\left(\frac{\lambda^*-\lambda}{\lambda^*}\right)\|u\|^q$, *for all $u \in \mathcal{M}_\lambda$.*

(ii) $\|u\| \geq K_2\left(\frac{\lambda^*-\lambda}{\lambda^*}\right)^{\frac{1}{r-q}}$, *for all $u \in \mathcal{M}_\lambda$.*

(iii) $\int_\Omega W(x)|u|^r\,dx \geq K_3\left(\frac{\lambda^*-\lambda}{\lambda^*}\right)^{\frac{r}{r-q}}$, *for all $u \in \mathcal{M}_\lambda$.*

Therefore, from Proposition 2.1 and Proposition 2.3, the following real numbers are well defined:

$$c_\lambda = \inf_{\mathcal{N}_\lambda} J_\lambda, \quad d_\lambda = \inf_{\mathcal{N}_\lambda^\pm} J_\lambda, \quad \tilde{c}_\lambda = \inf_{\mathcal{M}_\lambda} J_\lambda \quad \text{and} \quad \tilde{d}_\lambda = \inf_{\mathcal{M}_\lambda^\pm} J_\lambda. \tag{2.7}$$

Moreover, if $\lambda_1 > \lambda$, notice that Corollary 2.2 allows us called a solution of (P_λ) which is a minimizer of $\mathcal{M}_\lambda$ (or $\mathcal{M}_\lambda^\pm$) of ground state solution (or nodal ground state solution).

Lemma 2.4. *Consider $u \in W_0^{1,q}(\Omega) \setminus \{0\}$ such that $\int_\Omega W(x)|u|^r\,dx > 0$. Then, there exists a unique $t_u > 0$ satisfying*

$$J_\lambda(t_u u) := \max_{t\geq 0} J_\lambda(tu) > 0.$$

Moreover, if $J_\lambda'(u)u < 0$, then $t_u \in (0,1]$.

Proof. Let $u \in W_0^{1,q}(\Omega)\setminus\{0\}$ and $t \in (0,+\infty)$. So, by (a_1), we obtain

$$J_\lambda(tu) \leq k_2\frac{t^p}{p}\int_\Omega |\nabla u|^p\,dx + \frac{t^q}{q}\left(\int_\Omega |\nabla u|^q\,dx - \lambda\int_\Omega |u|^q\,dx\right) - \frac{t^r}{r}\int_\Omega W(x)|u|^r\,dx \tag{2.8}$$

and

$$J_\lambda(tu) \geq k_1\frac{t^p}{p}\int_\Omega |\nabla u|^p\,dx + \frac{t^q}{q}\left(\int_\Omega |\nabla u|^q\,dx - \lambda\int_\Omega |u|^q\,dx\right) - \frac{t^r}{r}\int_\Omega W(x)|u|^r\,dx. \tag{2.9}$$

Therefore,

$$\limsup_{t\to 0^+}\frac{J_\lambda(tu)}{t^p} > 0 \quad \text{and} \quad \limsup_{t\to+\infty}\frac{J_\lambda(tu)}{t^r} = -\frac{1}{r}\int_\Omega W(x)|u|^r\,dx. \tag{2.10}$$

Thus, since $\int_\Omega W(x)|u|^r\,dx > 0$, we ensure the existence of $t_u \in (0,+\infty)$ such that

$$J_\lambda(t_u u) := \max_{t\geq 0} J_\lambda(tu) > 0.$$

To guarantee that the value $t_u > 0$ is unique, let us prove that the equation $J_\lambda'(su)su = 0$ is satisfied only for $s = t_u$. Indeed, this equation is equivalent to

$$s^{r-q}\int_\Omega W(x)|u|^r\,dx + \lambda\int_\Omega |u|^q\,dx = \int_\Omega \frac{a\left(|\nabla(su)|^p\right)}{|\nabla(su)|^{q-p}}|\nabla u|^q\,dx.$$

By (a_3), the right-hand side of the equation above is a nonincreasing function on $s > 0$, while the left side, an increasing function on $s > 0$ provided $r > q$ and $\int_\Omega W(x)|u|^r\,dx > 0$. This shows the uniqueness of the value $t_u > 0$. With the same arguments, we obtain that $t_u > 1$ implies $J'_\lambda(u)u \geq 0$, and the proof of the lemma follows. □

Lemma 2.5. *If $q < r < q^*$ and $W : \Omega \to \mathbb{R}$ satisfies (W_1), then $\mathcal{M}^\pm_\lambda \neq \emptyset$ for all $\lambda \in \mathbb{R}$. Consequently, $\mathcal{M}_\lambda \neq \emptyset$ for all $\lambda \in \mathbb{R}$.*

Proof. From (W_1), we may consider two open balls B_1 and B_2 contained in Ω such that

$$B_1 \cap B_2 = \emptyset, \quad |B_1 \cap \Omega_+| > 0 \quad \text{and} \quad |B_2 \cap \Omega_+| > 0.$$

Arguing as in [6, Lemma 2.3], we have two negative solutions $u_1 \in C_0^\infty(B_1)$ and $u_2 \in C_0^\infty(B_2)$ such that

$$\int_\Omega W(x)|u_1|^r\,dx > 0 \quad \text{and} \quad \int_\Omega W(x)|u_2|^r\,dx > 0.$$

Then, by Lemma 2.4, there are $t_1,\ t_2 > 0$ such that $J'_\lambda(t_1u_1)t_1u_1 = 0$ and $J'_\lambda(t_2u_2)t_2u_2 = 0$. Using $B_1 \cap B_2 = \emptyset$, we have that

$$J'_\lambda(t_1u_1 + t_2u_2)(t_1u_1 + t_2u_2) = J'_\lambda(t_1u_1)t_1u_1 + J'_\lambda(t_2u_2)t_2u_2 = 0.$$

Hence $(t_1u_1 + t_2u_2) \in \mathcal{M}^\pm_\lambda$, which implies $\mathcal{M}^\pm_\lambda \neq \emptyset$. Since $\mathcal{M}^\pm_\lambda \subset \mathcal{M}_\lambda$, we have $\mathcal{M}_\lambda \neq \emptyset$. □

Proof of item (i) of Theorem 1.1

Proof. The proof follows directly from item (iii) of Proposition 2.1. □

3 Existence of two least energy solutions which do not change sign

In this section, we are going to show that $\tilde{c}_\lambda$ is attained by some function which is a solution of problem (P_λ). For our purposes, we write

$$J_\lambda(u) = \Phi_\lambda(u) - I(u), \quad \forall\, u \in W_0^{1,q}(\Omega),$$

where the functionals $\Phi_\lambda,\ I \in C^1(W_0^{1,q}(\Omega), \mathbb{R})$ are given by

$$\Phi_\lambda(u) := \frac{1}{p}\int_\Omega A(|\nabla u|^p)\,dx - \frac{\lambda}{q}\int_\Omega |u|^q\,dx \quad \text{and} \quad I(u) := \frac{1}{r}\int_\Omega W(x)|u|^r\,dx.$$

Let us consider the set $Y := \{u \in W_0^{1,q}(\Omega) : \int_\Omega W(x)|u|^r\,dx > 0\}$ which is an open cone of $W_0^{1,q}(\Omega)$, that is, $tu \in Y$ for every $t > 0$ and $u \in Y$.

We now present some properties of the functionals Φ_λ and I when $\lambda < \lambda^*$.

Lemma 3.1. *If $\lambda < \lambda^*$, then the following properties hold:*

(i) Φ_λ and $u \mapsto \Phi'_\lambda(u)u$ are weakly lower semicontinuous and $I'(u_n) \to I'(u)$ in $W_0^{1,q'}(\Omega)$ if $u_n \rightharpoonup u$ in $W_0^{1,q}(\Omega)$.

(ii) There exists $C_1 > 0$ such that $\Phi'_\lambda(u)u \geq C_1\|u\|^q$ for every $u \in \overline{Y}$ and $I'(u) = o(\|u\|^{q-1})$ as $u \to 0$ in $\overline{Y}$.

(iii) $I(u) = I'(u)u = 0$ *for every* $u \in \partial Y$.

(iv) $t \mapsto \frac{\Phi'_\lambda(tu)u}{t^{q-1}}$ *and* $t \mapsto \frac{I'(tu)u}{t^{q-1}}$ *are nonincreasing and increasing, respectively, in* $(0, +\infty)$ *and for every* $u \in Y$. *Moreover,*

$$\limsup_{t\to+\infty} \frac{\Phi_\lambda(tu)}{t^q} < \limsup_{t\to+\infty} \frac{I(tu)}{t^q} = +\infty.$$

(v) *If* $\lambda < \lambda_1$, *then* $I'(u)u \leq 0 < \Phi'_\lambda(u)u$ *for all* $u \in W^{1,q}(\Omega) \setminus (Y \cup \{0\})$.

Proof. To prove (i), let us consider $(u_n) \subset W_0^{1,q'}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,q'}(\Omega)$. From (a_2), it follows that

$$\int_\Omega A(|\nabla u|^p)\, dx \leq \liminf_{n\to+\infty} \int_\Omega A(|\nabla u_n|^p)\, dx. \tag{3.1}$$

$$\int_\Omega a(|\nabla u|^p)|\nabla u|^p\, dx \leq \liminf_{n\to+\infty} \int_\Omega a(|\nabla u_n|^p)|\nabla u|^p\, dx, \tag{3.2}$$

Moreover, by Sobolev embeddings, (W_1) and, up to a subsequence, we get

$$\int_\Omega |u|^q\, dx = \lim_{n\to+\infty} \int_\Omega |u_n|^q\, dx \quad \text{and} \quad \int_\Omega W(x)|u|^r\, dx = \lim_{n\to+\infty} \int_\Omega W(x)|u_n|^r\, dx. \tag{3.3}$$

Hence, by (3.1), (3.2) and (3.3), the first item is proved.

To prove (ii), arguing as Proposition 2.3, we have

$$\Phi_\lambda(u) \geq \left(\frac{\lambda^* - \lambda}{\lambda^*}\right) \|u\|^q, \quad \forall\, u \in W_0^{1,q}(\Omega). \tag{3.4}$$

On the other hand, by (W_1),

$$|I'(u)v| \leq \|W\|_\infty \left(\int_\Omega |v|^r\, dx\right)^{\frac{1}{r}} \left(\int_\Omega |u|^r\, dx\right)^{\frac{r-1}{r}},$$

and then, by Sobolev embeddings,

$$\frac{I'(u)}{\|u\|^{q-1}} \leq C\|u\|^{r-q}, \quad u \neq 0. \tag{3.5}$$

From (3.4) and (3.5) the item (ii) holds. Since $\partial Y = \{0\}$, this shows that the item (iii) holds.

Now let us prove item (iv). Since $q < r$ and $u \in Y$, we obtain

$$\frac{d}{dt}\left[\frac{I'(tu)u}{t^{q-1}}\right] = (r-q)t^{r-q-1} \int_\Omega W(x)|u|^r\, dx > 0, \quad \forall\, t \in (0, +\infty),$$

which implies that $t \mapsto \dfrac{I'(tu)u}{t^{q-1}}$ is increasing in $(0, +\infty)$ and for every $u \in Y$. Moreover,

$$\limsup_{t\to+\infty} \frac{I(tu)}{t^q} = \limsup_{t\to+\infty} \frac{t^{r-q}}{r} \int_\Omega W(x)|u|^r\, dx = +\infty \tag{3.6}$$

On the other hand, note that

$$\frac{\Phi'_\lambda(tu)u}{t^{q-1}} = \int_\Omega \frac{a(|\nabla tu|^p)}{|\nabla tu|^{q-p}}|\nabla u|^q\, dx - \lambda \int_\Omega |u|^q\, dx$$

is a nonincreasing function by (a_3). Moreover, we also have

$$\limsup_{t\to+\infty} \frac{\Phi_\lambda(tu)}{t^{q-1}} \leq \frac{1}{q}\left(\int_\Omega |\nabla u|^q\, dx - \lambda \int_\Omega |u|^q\, dx\right). \tag{3.7}$$

Then, by (3.6) and (3.7), we conclude the proof of item (iv).

To finish, if $u \in W^{1,q}(\Omega) \setminus (\Upsilon \cup \{0\})$, then $\int_\Omega W(x)|u|^r\, dx \leq 0$. Hence, by (a_1) and $\lambda < \lambda_1$,

$$I'(u)u = \int_\Omega W(x)|u|^r\, dx \leq 0 < \int_\Omega a(|\nabla u_n|^p)|\nabla u|^p\, dx - \lambda \int_\Omega |u|^q\, dx = \Phi'_\lambda(u)u$$

and the proof of the last item of the proposition is complete. □

Using the previous lemma and [14, Corollary 3.1], we obtain the next result.

Corollary 3.2. *If $\lambda < \lambda^*$, then there exists $v_\lambda \in \mathcal{M}_\lambda$ such that*

$$J_\lambda(v_\lambda) = \tilde{c}_\lambda := \inf_{u\in\mathcal{M}_\lambda} J_\lambda(u).$$

We now show that problem (P_λ) has two least energy solutions when $\lambda < \lambda^*$.

Proposition 3.3. *If $\lambda < \lambda^*$, then there exists a nontrivial function v_λ which is a least energy solution of (P_λ), and $\tilde{v}_\lambda := -v_\lambda$ is also a least energy solution of (P_λ). Moreover, if $\lambda < \lambda_1$ these solutions are ground state solutions.*

Proof. Let v_λ be the solution found in Corollary 3.2 and let us assume by contradiction that $v_\lambda^\pm \neq 0$. Since v_λ is a critical point of functional J_λ and the intersection of the support of the functions $v_\lambda^\pm$ is empty, we have that $v_\lambda^\pm \in \mathcal{N}_\lambda$. Hence,

$$c_\lambda \leq J_\lambda(v_\lambda^\pm). \tag{3.8}$$

Since Proposition 2.3 holds, then either

$$\int_\Omega W(x)|v_\lambda^+|^r\, dx > 0 \quad \text{or} \quad \int_\Omega W(x)|v_\lambda^-|^r\, dx > 0.$$

Without loss of generality, we can assume that $\int_\Omega W(x)|v_\lambda^+|^r\, dx > 0$. Then, $v_\lambda^+ \in \mathcal{M}_\lambda$ and, hence,

$$\tilde{c}_\lambda \leq J_\lambda(v_\lambda^+). \tag{3.9}$$

Therefore, by (3.8) and (3.9),

$$c_\lambda + \tilde{c}_\lambda \leq J_\lambda(v_\lambda^+) + J_\lambda(v_\lambda^-) = J_\lambda(v_\lambda) = \tilde{c}_\lambda.$$

This contradiction proves that the least energy solution does not change sign.

We may assume that v_λ is nonnegative. Then, setting $\tilde{v}_\lambda = -v_\lambda$, we have that

$$\tilde{c}_\lambda = J_\lambda(v_\lambda) = \frac{1}{p}\int_\Omega A\left(|\nabla(-v_\lambda)|^p\right)\, dx - \frac{\lambda}{q}\int_\Omega |(-v_\lambda)|^q\, dx - \frac{1}{r}\int_\Omega W(x)|(-v_\lambda)|^r\, dx = J_\lambda(\tilde{v}_\lambda).$$

Moreover, using that v_λ is a critical point of J_λ, we have for all $\varphi \in W_0^{1,q}(\Omega)$,

$$\begin{aligned}\int_\Omega a(|\nabla(-v_\lambda)|^p)|\nabla(-v_\lambda)|^{p-2}\nabla(-v_\lambda)\nabla\varphi\, dx &= \lambda \int_\Omega |(-v_\lambda)|^{q-2}(-v_\lambda)\varphi\, dx \\ &\quad + \int_\Omega W(x)|(-v_\lambda)|^{r-2}(-v_\lambda)\varphi\, dx.\end{aligned}$$

Thus, $\tilde{v}_\lambda$ is a critical point of J_λ. Therefore, problem (P_λ) has a nonpositive solution and a nonnegative solution. Furthermore, when $\lambda < \lambda_1$, by Corollary 2.2, $\mathcal{M}_\lambda = \mathcal{N}_\lambda$. Thus, these solutions are ground state solutions of (P_λ). □

3.1 Proof of item (ii) of Theorem 1.1

Proof. The proof follows directly from Corollary 3.2 and Proposition 3.3. □

4 Existence of two nodal solutions

We begin this section by showing that $\tilde{d}_\lambda$ is attained by some function which is a least energy nodal solution of problem (P_λ).

Proposition 4.1. *If $\lambda < \lambda^*$, then there exists $\tilde{w}_\lambda \in \mathcal{M}_\lambda^\pm$ such that*

$$d_\lambda := J_\lambda(\tilde{w}_\lambda).$$

Proof. Let $(w_n) \subset \mathcal{M}_\lambda^\pm$ be a minimizing sequence, that is, a sequence satisfying

$$(w_n) \subset \mathcal{M}_\lambda^\pm \quad \text{and} \quad I_\lambda(w_n) = d_\lambda + o_n(1). \tag{4.1}$$

By item (i) of Lemma 2.1, we obtain that functional J_λ is coercive on $\mathcal{M}_\lambda^\pm$, and hence (w_n) is bounded in $W_0^{1,q}(\Omega)$. Then, by Sobolev embeddings and the continuity of the maps $w \mapsto w^+$ and $w \mapsto w^-$ are continuous from $L^r(\mathbb{R}^N)$ in $L^r(\mathbb{R}^N)$ (for details, see [4, Lemma 2.3] with suitable adaptations), there exists $w_\lambda \in W_0^{1,q}(\Omega)$ such that, up to a subsequence, we have

$$\begin{cases} w_n^\pm \rightharpoonup w_\lambda^\pm & \text{in } W_0^{1,q}(\Omega), \\ w_n^\pm \to w_\lambda^\pm & a.e. \text{ in } \Omega, \\ w_n^\pm \to w_\lambda^\pm & \text{in } L^s(\Omega),\ 1 \le s < q^*. \end{cases} \tag{4.2}$$

We claim that $w_\lambda^\pm \neq 0$ and $\displaystyle\int_\Omega W(x)|w_\lambda^\pm|^r\,dx > 0$. Indeed, using that $W \in L^\infty(\Omega)$ and item (iii) of Lemma 2.1, we obtain

$$\int_\Omega |w_\lambda^\pm|^r\,dx = \lim_{n\to\infty}\int_\Omega |w_n^\pm|^r\,dx \ge \frac{K_3}{\|W\|_\infty} > 0, \tag{4.3}$$

and

$$\int_\Omega W(x)|w_\lambda^\pm|^r\,dx = \lim_{n\to\infty}\int_\Omega W(x)|w_n^\pm|^r\,dx \ge K_3 > 0, \tag{4.4}$$

that proves our claim. Therefore, by Lemma 2.4, there exists $t_\lambda^\pm \in (0,+\infty)$ such that $t_\lambda^\pm w_\lambda^\pm \in \mathcal{M}_\lambda$.

We claim that $t_\lambda^\pm \in (0,1)$. In fact, by Fatou's Lemma and (4.2), we have

$$\begin{aligned} \int_\Omega a(|w_\lambda^\pm|^p)|w_\lambda^\pm|^p\,dx &\le \liminf_{n\to+\infty}\int_\Omega a(|w_n^\pm|^p)|w_n^\pm|^p\,dx \\ &= \lim_{n\to+\infty}\left(\lambda\int_\Omega |w_n^\pm|^q\,dx + \int_\Omega W(x)|w_n^\pm|^r\,dx\right) \\ &= \lambda\int_\Omega |w_\lambda^\pm|^q\,dx + \int_\Omega W(x)|w_\lambda^\pm|^r\,dx, \end{aligned}$$

that is, $J'_\lambda(w_\lambda^\pm)w_\lambda^\pm \le 0$. Hence, by Lemma 2.4, the claim follows.

Similarly, with the same arguments of Proposition 3.2, we obtain

$$
\begin{aligned}
J_\lambda(t_\lambda^\pm w_\lambda^\pm) &= J_\lambda(t_\lambda^\pm w_\lambda^\pm) - \frac{1}{q}J'_\lambda(t_\lambda^\pm w_\lambda^\pm)t_\lambda^\pm w_\lambda^\pm \\
&= \int_\Omega \left[\frac{1}{p}A(|\nabla(t_\lambda^\pm w_\lambda^\pm)|^p) - \frac{1}{q}a(|\nabla(t_\lambda^\pm w_\lambda^\pm)|^p)|\nabla(t_\lambda^\pm w_\lambda^\pm)|^p\right] dx \\
&\quad + \left(\frac{1}{q}-\frac{1}{r}\right)\int_\Omega W(x)|t_\lambda^\pm w_\lambda^\pm|^r\, dx \\
&\le \int_\Omega \left[\frac{1}{p}A(|\nabla w_\lambda^\pm|^p) - \frac{1}{q}a(|\nabla u_\lambda^\pm|^p)|\nabla w_\lambda^\pm|^p\right] dx \qquad (4.5) \\
&\quad + \left(\frac{1}{q}-\frac{1}{r}\right)\int_\Omega W(x)|w_\lambda^\pm|^r\, dx \\
&\le \liminf_{n\to+\infty}\left[J_\lambda(w_n^\pm) - \frac{1}{q}J'_\lambda(w_n^\pm)w_n^\pm\right] = J_\lambda(w_n^\pm) + o_n(1).
\end{aligned}
$$

Then, setting $\tilde{w}_\lambda = t_\lambda^- w_\lambda^- + t_\lambda^+ w_\lambda^+$, from (4.4),

$$
\int_\Omega W(x)|\tilde{w}_\lambda|^r\, dx = \int_\Omega W(x)|t_\lambda^- w_\lambda^-|^r\, dx + \int_\Omega W(x)|t_\lambda^+ w_\lambda^+|^r\, dx \ge 2K_3 > 0,
$$

that is, $\tilde{w}_\lambda \in \mathcal{M}_\lambda^\pm$. Hence, using (4.5), we can conclude

$$
\begin{aligned}
d_\lambda = J_\lambda(\tilde{w}_\lambda) &= J_\lambda(t_\lambda^- w_\lambda^-) + J_\lambda(t_\lambda^+ w_\lambda^+) \\
&\le J_\lambda(w_n^-) + J_\lambda(w_n^+) + o_n(1) = J_\lambda(w_n) + o_n(1) = d_\lambda.
\end{aligned}
$$

Thus, the level d_λ is attained by the function $\tilde{w}_\lambda \in \mathcal{M}_\lambda^\pm$. □

Corollary 4.2. *Let $\tilde{w}_\lambda$ be a minimizer found in Propositions 4.1. Then, $\tilde{w}_\lambda$ is a critical point of J_λ and has exactly two nodal domains.*

Proof. The proof that $\tilde{w}_\lambda \in \mathcal{M}_\lambda^\pm$ is a critical point of J_λ is done using a suitable quantitative deformation lemma and Brouwer's topological degree properties. It is done, with suitable modifications, as in [5, Lemma 4.3] and [5, Theorem 1.1]. To show that the nodal solution $\tilde{w}_\lambda$ has exactly two nodal domains, or in other words it changes sign exactly once, see for instance [5, pages 1230-1232] . □

Using the same arguments as in Proposition 3.3 one can immediately prove the following result.

Corollary 4.3. *If $\lambda < \lambda^*$, then there exists a function $\tilde{w}_\lambda$ which is a nodal least energy solution of (P_λ), and $\overline{w_\lambda} := -\tilde{w}_\lambda$ is also a nodal least energy solution of (P_λ). Moreover, if $\lambda < \lambda_1$, then these solutions are ground state solutions of (P_λ).*

4.1 Proof of item (iii) of Theorem 1.1.

Proof. It follows directly from Corollaries 4.2 and 4.3.

□

5 A nontrivial solution for the indefinite critical problem

In this section we consider the following critical problem

$$\begin{cases} -\operatorname{div}\left(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\right) = \lambda|u|^{q-2}u + W(x)|u|^{q^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $N \geq 2$, $1 < p \leq q < N$ and $\lambda \in \mathbb{R}$, where $q^* = \frac{Nq}{N-q}$ is the critical Sobolev exponent. Here, we consider the associated functional $I_\lambda \in C^1(W_0^{1,q}(\Omega), \mathbb{R})$ given by

$$I_\lambda(u) := \frac{1}{p}\int_\Omega A\left(|\nabla u|^p\right)\, dx - \frac{\lambda}{q}\int_\Omega |u|^q\, dx - \frac{1}{r}\int_\Omega W(x)|u|^{q^*}\, dx.$$

Let us show that the associated functional to the indefinite critical problem has a mountain pass geometry.

Proposition 5.1. *The functional $I_\lambda : W_0^{1,q}(\Omega) \to \mathbb{R}$ satisfies the following properties:*

i) There exist positive numbers α and ρ such that

$$I_\lambda(u) \geq \rho, \quad \textit{for all } \|u\| = \rho.$$

ii) There exists a function $e \in W_0^{1,q}(\Omega)$ such that $\|e\| \geq \rho$ and

$$I_\lambda(e) < 0.$$

Proof. By (a_1) and the Poincaré inequality,

$$\begin{aligned} I_\lambda(u) &\geq \frac{k_1}{p}\int_\Omega |\nabla u|^p\, dx + \frac{1}{q}\int_\Omega |\nabla u|^q\, dx - \frac{\lambda}{q}\int_\Omega |u|^q\, dx - \int_\Omega W(x)|u|^{q^*}\, dx \\ &\geq \frac{1}{q}\left(\frac{\lambda_1 - \lambda}{\lambda_1}\right)\int_\Omega |\nabla u|^q\, dx - \int_\Omega W(x)|u|^{q^*}\, dx. \end{aligned}$$

Thus, by Sobolev embeddings and $W \in L^\infty(\Omega)$, there exists a positive constant C such that

$$I_\lambda(u) \geq \frac{1}{q}\left(\frac{\lambda_1 - \lambda}{\lambda_1}\right)\|u\|^q - C\|u\|^{q^*} = \|u\|^q\left[\frac{1}{q}\left(\frac{\lambda_1 - \lambda}{\lambda_1}\right) - C\|u\|^{q^*-q}\right].$$

Therefore, since $\lambda < \lambda_1$, we can choose $\|u\| = \rho$ small enough such that there exists $\alpha > 0$ satisfying

$$I_\lambda(u) \geq \rho, \quad \text{for all } \|u\| = \rho.$$

To prove item (ii), let us consider a nontrivial function $w \in C_0^\infty(\Omega_+) \setminus \{0\}$ and $t > 0$. Then, by (a_1),

$$\begin{aligned} I_\lambda(tw) &\leq \frac{k_2}{p}\int_\Omega |\nabla(tw)|^p\, dx + \frac{1}{q}\int_\Omega |\nabla(tw)|^q\, dx - \frac{\lambda}{q}\int_\Omega |tw|^q\, dx - \int_\Omega W(x)|tw|^{q^*}\, dx \\ &< t^{q^*}\left[t^{p-q^*}\frac{k_2}{p}\int_\Omega |\nabla w|^p\, dx + \frac{t^{q-q^*}}{q}\int_\Omega |\nabla w|^q\, dx - \int_\Omega W(x)|w|^{q^*}\, dx\right]. \end{aligned}$$

Hence, letting $t \to +\infty$,

$$\limsup_{t\to\infty} I_\lambda(tw) \leq -\infty.$$

□

Recall that, if E is a Banach space, $\Phi \in C^1(E,\mathbb{R})$ and $c \in \mathbb{R}$ we say that Φ satisfies the Palais-Smale condition at level c (shortly: Φ satisfies $(PS)_c$) if every sequence $(u_n) \in E$ such that $\Phi(u_n) \to c$ in $\mathbb{R}$ and $\Phi'(u_n) \to 0$ in E', as $n \to \infty$, admits a subsequence that converges for a critical point of Φ. This sequence is called a $(PS)_c$ sequence for Φ.

Notice that Lemma 5.1 ensures us the existence of a $(PS)_{c_\lambda}$ sequence for the functional $I_\lambda : W_0^{1,q}(\Omega) \to \mathbb{R}$, where

$$c_\lambda = \inf_{\eta\in\Gamma} \max_{t\in[0,1]} I_\lambda(\eta(t)) > 0,$$

and

$$\Gamma := \{\eta \in C([0,1],X) : \eta(0) = 0,\ I_\lambda(\eta(1)) < 0\}.$$

Lemma 5.2. *If $\lambda < \lambda_1$ and $(u_n) \subset W_0^{1,q}(\Omega)$ is a $(PS)_c$ sequence for I_λ, then (u_n) is bounded in $W_0^{1,q}(\Omega)$.*

Proof. Let $(u_n) \subset W_0^{1,q}(\Omega)$ be a $(PS)_c$ sequence for I_λ. Then, by (a_1) and Poincaré inequality for $W_0^{1,q}(\Omega)$,

$$\begin{aligned} c + o_n(1)\|u_n\| &= I_\lambda(u_n) - \frac{1}{q^*} I'_\lambda(u_n)u_n \\ &= \frac{1}{p}\int_\Omega A(|\nabla u_n|^p)\,dx - \frac{1}{q^*}\int_\Omega a(|\nabla u_n|^p)|\nabla u_n|^p\,dx - \lambda\left(\frac{1}{q} - \frac{1}{q^*}\right)\int_\Omega |u_n|^q\,dx \\ &\geq \left(\frac{1}{q} - \frac{1}{q^*}\right)\left(k_2\int_\Omega |\nabla u_n|^p\,dx + \int_\Omega |\nabla u_n|^q\,dx\right) \\ &\quad - \frac{\lambda}{\lambda_1}\left(\frac{1}{q} - \frac{1}{q^*}\right)\int_\Omega |\nabla u_n|^q\,dx. \end{aligned}$$

Hence,

$$c + o_n(1)\|u_n\| \geq \left(1 - \frac{\lambda}{\lambda_1}\right)\left(\frac{1}{q} - \frac{1}{q^*}\right)\|u_n\|^q,$$

which implies that (u_n) is bounded in $W_0^{1,q}(\Omega)$. □

Lemma 5.3. *If $\lambda < \lambda_1$, then*

i) $\int_\Omega W(x)|\varphi_1|^{q^*}\,dx > 0.$

ii) $c_\lambda < \left(\frac{k_2}{k_1 p} + \frac{1}{N}\right)\int_\Omega W(x)|\varphi_1|^{q^*}\,dx$

Proof. Using Lemma 5.1, let us consider $t_\alpha > 0$ such that $J_\lambda(t_\alpha\varphi_1) = \max_{t\geq 0} I_\lambda(t\varphi_1)$. Then, by (a_1),

$$\begin{aligned} t_\alpha^{q^*}\int_\Omega W(x)|\varphi_1|^{q^*}\,dx &= \int_\Omega a(|\nabla(t_\alpha^p\varphi_1)|^p)|\nabla(t_\alpha^p\varphi_1)|^p\,dx - \lambda t_\alpha^q\int_\Omega |\varphi_1|^q\,dx \\ &\geq k_1 t_\alpha^p\int_\Omega |\nabla\varphi_1|^p\,dx + t_\alpha^q(\lambda_1 - \lambda)\int_\Omega |\varphi_1|^q\,dx > 0. \end{aligned} \tag{5.1}$$

This shows the first item. Moreover, with the same argument as in Lemma 2.4,

$$1 \geq t_\alpha \geq \left[\frac{k_1\int_\Omega |\nabla\varphi_1|^p\,dx}{\int_\Omega W(x)|\varphi_1|^{q^*}\,dx}\right]^{\frac{1}{q^*-q}} > 0.$$

Therefore, by (a_1) and (5.1),

$$\begin{aligned} c_\lambda &\le I_\lambda(t_\alpha\varphi_1) \\ &\le k_2\frac{t_\alpha^p}{p}\int_\Omega |\nabla\varphi_1|^p\,dx + \frac{t_\alpha^q}{q}(\lambda_1-\lambda)\int_\Omega|\varphi_1|^q\,dx - \frac{t_\alpha^{q^*}}{q^*}\int_\Omega W(x)|\varphi_1|^{q^*}\,dx \\ &\le \left(\frac{k_2}{k_1 p}+\frac{1}{q}-\frac{1}{q^*}\right)t_\alpha^{q^*}\int_\Omega W(x)|\varphi_1|^{q^*}\,dx \\ &< \left(\frac{k_2}{k_1 p}+\frac{1}{N}\right)\int_\Omega W(x)|\varphi_1|^{q^*}\,dx. \end{aligned} \tag{5.2}$$

The proof of the theorem is complete. □

Proposition 5.4. *If $\lambda < \lambda_1$ and*

$$\int_\Omega W(x)|\varphi_1|^{q^*}\,dx < \frac{\frac{1}{N}(\mathcal{W}_2 - \|W^-\|_\infty)\left(\frac{S}{\mathcal{W}_1}\right)^{\frac{N}{q}}}{\frac{k_2}{k_1 p}+\frac{1}{N}}, \tag{5.3}$$

where $\mathcal{W}_1$, $\mathcal{W}_2$ are positive constants given by (W_3), then I_λ has a nontrivial critical point.

Proof. By Proposition 5.1, let $(u_n) \subset W_0^{1,q}(\Omega)$ be a $(PS)_{c_\lambda}$-sequence for functional I_λ which is bounded in $W_0^{1,q}(\Omega)$ by Lemma 5.2. Then, up to a subsequence,

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } W_0^{1,q}(\Omega), \\ u_n \to u & \text{strongly in } L^s(\Omega) \text{ for any } 1 \le s < q^*, \\ u_n(x) \to u(x) & \text{for a.e. } x \in \Omega, \end{cases} \tag{5.4}$$

for some $u \in W_0^{1,q}(\Omega)$. From the Sobolev embeddings, we can conclude that u is a critical point of I_λ.

Now we are going to show that u is nontrivial. Suppose, by contradiction, that $u = 0$ in $\Omega_* \subset\subset \Omega_+$, where Ω_* is a open set given by (W_3). Since (u_n) is bounded in $W_0^{1,q}(\Omega)$ and using the Lions's Concentration Compactness Principle [18], we may suppose that

$$|\nabla u_n|^q \rightharpoonup \mu \quad \text{and} \quad |u_n|^{q^*} \rightharpoonup \nu,$$

for some measures μ and ν. Hence, we obtain an at most countable index set Γ, sequences $(x_i) \subset \Omega_*$ and (μ_i), $(\nu_i) \subset (0,\infty)$ such that

$$\mu \ge |\nabla u|^q + \sum_{i\in\Gamma}\mu_i\delta x_i, \quad \nu = |u|^{q^*} + \sum_{i\in\Gamma}\nu_i\delta x_i \quad \text{and} \quad S\nu_i^{q/q^*} \le \mu_i, \tag{5.5}$$

for all $i \in \Gamma$, where δ_{x_i} is the Dirac mass at $x_i \in \Omega_*$ and $S > 0$ is the best constant of the Sobolev embedding $W_0^{1,q}(\Omega) \hookrightarrow L^{q^*}(\Omega)$. Thus it is sufficient to show that $\{x_i\}_{i\in\Gamma} \cap \Omega_* = \emptyset$. Then we suppose, by contradiction, that $x_i \in \Omega_*$ for some $i \in \Gamma$. Consider $R > 0$ and the function $\psi_R(x) := \psi(x_i - x)$, where $\psi \in C_0^\infty(\Omega_*, [0,1])$ is such that $\psi = 1$ on $B_R(x_i)$, $\psi = 0$ on $\Omega \backslash B_{2R}(x_i)$ and $|\nabla\psi|_\infty \le 2$. We suppose also that $R > 0$ is chosen in such way that $I'_\mu(u_n)\psi_R u_n = o_n(1)$, we obtain

$$\begin{aligned} \int_\Omega \psi_R\, a(|\nabla u_n|^p)|\nabla u_n|^p\,dx = &- \int_\Omega u_n a(|\nabla u_n|^p)|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla\psi_R\,dx \\ &+ \lambda\int_\Omega |u_n|^q\psi_R\,dx + \int_\Omega W(x)|u_n|^{q^*}\psi_R dx + o_n(1). \end{aligned}$$

One gets from the weakly convergence $u_n \rightharpoonup u = 0$ that

$$\int_\Omega u_n a(|\nabla u_n|^p)|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla \psi_R \, dx = 0 \quad \text{and} \quad \lim_{n\to\infty} \lambda \int_\Omega |u_n|^q \psi_R \, dx = 0.$$

Consequently, by (5) and (a_1), as $n \to +\infty$,

$$\int_\Omega |\nabla u_n|^q \psi_R \, dx \leq \int_\Omega a(|\nabla u_n|^p)|\nabla u_n|^p \psi_R \, dx = \int_\Omega W(x)|u_n|^{q^*} \psi_R dx + o_n(1).$$

Since ψ_R has compact support, taking $n \to \infty$ in the above expression, we have

$$\int_\Omega \psi_R d\mu \leq \int_\Omega \psi_R W(x) d\nu,$$

which implies that

$$\mu_i \leq \mathcal{W}_1 \nu_i,$$

where $\mathcal{W}_1 \geq W(x) \geq \mathcal{W}_2 > 0$ for all $x \in \Omega_* \subset\subset \Omega_+$. Since $\mu_i > 0$, then $x_i \in \Omega_*$. Therefore, from (5.5), we get

$$\left(\frac{S}{\mathcal{W}_1}\right)^{\frac{q^*}{q^*-q}} \leq \nu_i. \tag{5.6}$$

On the other hand, (u_n) is a $(PS)_{c_\lambda}$-sequence for functional I_λ then, arguing as Proposition 2.3, we have

$$\int_\Omega W(x)|u_n|^{q^*} dx + o_n(1) > 0. \tag{5.7}$$

Since sequence $u_n \rightharpoonup u = 0$ weakly in $W_0^{1,q}(\Omega)$, $\psi_R \in C_0^\infty(\Omega_*; [0,1])$ and $|\Omega_-| \geq |\Omega_*|$, we obtain

$$\begin{aligned}
c_\lambda &= I_\lambda(u_n) - \frac{1}{q} I'_\lambda(u_n) u_n \psi_R + o_n(1) \\
&= \left(\frac{1}{q} - \frac{1}{q^*}\right) \int_\Omega W(x)|u_n|^{q^*} dx + o_n(1) \\
&= \frac{1}{N} \left[\int_{\Omega_+} W(x)|u_n|^{q^*} dx + \int_{\Omega_-} W(x)|u_n|^{q^*} dx \right] + o_n(1) \\
&\geq \frac{1}{N} \left[\int_{\Omega^*} W(x)|u_n|^{q^*} dx - \|W^-\|_\infty \int_{\Omega_-} |u_n|^{q^*} dx \right] + o_n(1) \\
&\geq \frac{1}{N} \left(\mathcal{W}_2 - \|W^-\|_\infty\right) \int_{\Omega^*} |u_n|^{q^*} dx + o_n(1) \\
&\geq \frac{1}{N} \left(\mathcal{W}_2 - \|W^-\|_\infty\right) \int_{\Omega^*} \psi_R |u_n|^{q^*} dx + o_n(1).
\end{aligned}$$

Therefore, using (5.5) and (5.6), we get

$$\begin{aligned}
c_\lambda &\geq \frac{1}{N} \left(\mathcal{W}_2 - \|W^-\|_\infty\right) \sum_{i\in\Gamma} \psi_R(x_i) \nu_i \\
&= \frac{1}{N} \left(\mathcal{W}_2 - \|W^-\|_\infty\right) \nu_i \\
&\geq \frac{1}{N} \left(\mathcal{W}_2 - \|W^-\|_\infty\right) \left(\frac{S}{\mathcal{W}_1}\right)^{\frac{N}{q}}.
\end{aligned}$$

Since (5.3) holds, we obtain a contradiction by Lemma 5.3. Hence, $u \in W_0^{1,q}(\Omega)$ is a nontrivial solution. □

5.1 Proof of Theorem 1.2.

Proof. If $\lambda < \lambda_1$, by Proposition 5.1 and Lemma 5.2, we have that there exists a critical point u_λ of I_λ. Thus, if

$$\int_\Omega W(x)|\varphi_1|^{q^*}\,dx < \frac{\frac{1}{N}\left(\mathcal{W}_2 - \|W^-\|_\infty\right)\left(\frac{S}{\mathcal{W}_1}\right)^{\frac{N}{q}}}{\frac{k_2}{k_1 p} + \frac{1}{N}},$$

then, by Proposition 5.4, u_λ is nontrivial solution. Moreover, using the same arguments as in Proposition 3.3, one can immediately shows that $-u_\lambda$ is also a nontrivial solution. □

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