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On a Dirichlet boundary value problem for an Ermakov–Painlevé I equation. A Hamiltonian EPI system

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Abstract. Here, a proto-type Ermakov–Painlevé I equation is introduced and a homogeneous Dirichlet-type boundary value problem analysed. In addition, a novel Ermakov–Painlevé I system is set down which is reducible by an involutory transformation to the autonomous Ermakov–Ray–Reid system augmented by a single component Ermakov–Painlevé I equation. Hamiltonian such systems are delimited.

Keywords: Ermakov, Painlevé, Dirichlet boundary value problem, Hamiltonian system.

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1 Introduction

Ermakov in [13] in now classical work introduced a canonical nonlinear equation which has subsequently been established as the base member of two and, in general, multi-component nonlinear systems with diverse applications in both nonlinear physics and continuum mechanics [33]. Thus, in [16,17], what are now termed Ermakov–Ray–Reid systems were derived which admit a distinctive integral of motion together with concomitant nonlinear superposition principles. These two-component coupled systems arise notably in nonlinear optics as detailed in [14, 30, 31]. In [1], what constitutes a Ermakov–Ray–Reid system was derived in an application of a variational approach to the analysis of elliptic cloud evolution in a Bose-Einstein condensate.

In [24], a classical 2+1-dimensional rotating shallow water system with an underlying circular paraboloidal bottom topography was shown to admit an integrable subsystem of Ermakov–Ray–Reid type. The latter system in that context describes the time-evolution of the semi-axes of the elliptical moving shoreline on the paraboloidal basin. It is, in addition, Hamiltonian and this integral of motion allied with the admitted Ermakov invariant

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allows exact solution of the Ermakov–Ray–Reid system. The procedure adopted in [24] had its genesis in that applied in [18] to construct the general solution of an eight dimensional nonlinear dynamical system descriptive of the time-evolution of upper ocean warm-core elliptical eddies. Therein, representation of this system in terms of modulated versions of the divergence, spin, shear and normal deformation rates rendered the elliptic warm-core ring system analytically tractable. Importantly, a relevant class of exact solutions with a Ermakov connection therein termed pulsrodons was isolated which characteristically both rotate and pulsate periodically. Lyapunov stability of such pulsrodons and their duals was subsequently addressed via a Lagrangian treatment in [15]. In [34] pulsrodonic phenomena was exhibited in a 2+1-dimensional nonlinear system governing rotating homentropic magnetogasdynamics in a bounded region. In a related development [35], a 2+1-dimensional version of a non-isothermal gasdynamic system with origin in work of Dyson [12] on spinning gas clouds was investigated. It was established therein via an elliptic vortex ansatz that the system admits a Hamiltonian reduction to a particular Ermakov–Ray–Reid system when the adiabatic index $\sigma = 2$.

The preceding attest to the diverse physical applications of the two-component Ermakov–Ray–Reid systems. In the present context, such a system will be shown in an appendix to arise via reduction of a three-component hybrid Ermakov–Painlevé I system.

In [36], it was established that a symmetry reduction of a classical 2+1-dimensional N-layer hydrodynamic system leads naturally to a novel multi-component Ermakov-type system. Importantly, the latter was shown to be iteratively reducible to a system of $N - 2$ linear equations augmented by a canonical Ermakov–Ray–Reid system. Moreover sequences of such systems were shown to be linked via Darboux transformations. Novel links between multi-component Ermakov systems and classes of many-body problems were subsequently established in [19]. In [29], Ermakov-type systems in two-dimensions were constructed and multi-wave solutions of a 2+1-dimensional modulated sine-Gordon equation thereby derived. Ermakov systems of arbitrary order and dimension were constructed in [42] which inherit key characteristics of the canonical Ermakov–Ray–Reid system.

The connection between the classical Painlevé I–VI equations and symmetry reduction of solitonic systems is well-documented (see e.g. [9] and literature cited therein). Indeed, the generic properties of solitonic equations associated with admittance of linear representations [2] and Bäcklund transformations [37,41] are likewise possessed by these Painlevé equations. It is remarked that such a Bäcklund transformation admitted by Painlevé II and its iteration have application not only in soliton theory but also in the analytic treatment of important boundary value problems for the celebrated Nernst–Planck system of ion transport [7,10,26].

In [20], wave packet representations inserted into a multi-component nonlinear Schrödinger system which incorporated a de-Broglie–Bohm quantum potential term resulted in novel hybrid Ermakov–Painlevé II reductions. Therein, a pair of Ermakov–Painlevé II equations was derived as a reduction of a nonlinear elastodynamic system governing the coupled stress associated with a class of shear motions. Hybrid Ermakov–Painlevé II-IV systems have subsequently been the subject of extensive investigation in [21,22,25,38]. In particular, physical applications of Ermakov–Painlevé II equations have been shown to arise in such diverse areas as cold plasma physics [28], Korteweg capillarity theory [27] and in multi-ion Nernst-Planck systems. In the latter context, Dirichlet-type boundary value problems were analysed in [3] for a Ermakov–Painlevé II reduction of such a three-ion electrolytic system. Hybrid Ermakov–Painlevé IV systems were originally derived via symmetry reduction of a multi-component resonant derivative nonlinear Schrödinger system in [21]. In subsequent work [25], Bäcklund transformations were applied to generate classes of exact solutions of the Ermakov–Painlevé IV system via the classical Painlevé IV equation. The forms of the prototype Ermakov–Painlevé II–IV equations have been set down explicitly in [4]. Two-point boundary value problems of Dirichlet-type for the single component base Ermakov–Painlevé IV equation were analysed in [5]. In addition, therein it was established that admitted Ermakov invariants can be used in the systematic generation of a coupled Ermakov–Painlevé IV system in terms of seed solutions of the canonical Painlevé IV equation.

The nonlinear coupled systems as introduced in [16, 17] that have come to be known as Ermakov–Ray–Reid systems adopt the form

$$\ddot{x} + \omega(t)x = \frac{1}{x^2 y}\Phi(y/x),$$
$$\ddot{y} + \omega(t)y = \frac{1}{xy^2}\Psi(x/y)$$

and admit the distinctive integral of motion

$$\mathrm{I} = \frac{1}{2}(x\dot{y} - y\dot{x})^2 + \int^{y/x} \Phi(z)dz + \int^{x/y} \Psi(w)dw$$

together with concomitant nonlinear superposition principles. The latter which are characteristic of the system are not of the type generic in soliton theory which are generated via invariance under Bäcklund transformations. The classical single component Ermakov equation of [13], namely

$$\ddot{\rho} + \omega(t)\rho = \delta/\rho^3$$

admits the nonlinear superposition principle

$$\rho = \sqrt{c_1\alpha^2(t) + 2c_2\alpha(t)\beta(t) + c_3\beta^2(t)}$$

wherein $\alpha(t), \beta(t)$ are two linearly independent solutions of

$$\ddot{\sigma} + \omega(t)\sigma = 0$$

with corresponding constant Wronskian $\mathcal{W} = \alpha\dot{\beta} - \beta\dot{\alpha}$ with constants c_1 such that

$$c_1c_3 - c_2^2 = \delta/\mathcal{W}^2 \ .$$

This result and its extensions are readily derived via Lie group methods [32, 40]. The preceding nonlinear superposition principle may be applied in the systematic reduction via reciprocal transformations of Ermakov-modulated solitonic systems to their canonical unmodulated counterparts [39].

In [6], the Ermakov–Ray–Reid system was reduced to its associated autonomous form via application of a novel class of involutory transformations. It was demonstrated thereby that the system admits an underlying linear structure albeit not of the type generic to solitonic systems.

Painlevé I has been derived in [8] via the classical Lie group procedure as a symmetry reduction of the solitonic Boussinesq equation. The latter arises in diverse physical applications such as long wave propagation in shallow water hydrodynamics, nonlinear lattice theory and plasma physics. Here a proto-type Ermakov–Painlevé I equation is introduced and a homogeneous Dirichlet-type boundary value problem analysed. In addition, a novel

Ermakov–Painlevé I system is set down which is reducible via an involutory transformation to the autonomous Ermakov–Ray–Reid system augmented by a single component hybrid Ermakov–Painlevé equation. Hamiltonian such systems are delimited.

The paper is organised as follows. The next section is devoted to the search of classical solutions to a homogeneous Dirichlet problem for a Ermakov–Painlevé I equation. Furthermore, the order of the zeros at the endpoints is analysed and an upper bound for the distance between distinct solutions is obtained. The main tool is the method of upper and lower solutions, combined with a Cantor diagonal argument. Finally, a two-component Ermakov–Painlevé I system with underlying Hamiltonian structure is set down and an associated Ermakov–Ray–Reid system constructed in the Appendix.

2 A Dirichlet problem

Here, a classical solution $\rho(t)$ of the Ermakov–Painlevé I equation

$$\rho''(t) = \left[5\left(\frac{\rho'(t)}{\rho(t)}\right)^2 - t\frac{\rho(t)^4}{4}\right]\rho(t) - \frac{3}{2\rho(t)^3} \tag{2.1}$$

is sought over the interval $(0,1)$ subject to the boundary conditions

$$\rho(0) = \rho(1) = 0. \tag{2.2}$$

It is seen that the EPI equation (2.1) is invariant under $\rho \to -\rho$ and in the sequel attention is restricted to solutions $\rho(t) > 0$ of the boundary value problem determined by (2.1)–(2.2).

Theorem 2.1. *Boundary value problem* (2.1)–(2.2) *has at least one solution* $\rho \in C[0,1] \cap C^2(0,1)$ *such that* $\rho(t) > 0$ *for* $t \in (0,1)$.

To establish this result, let us recall that the transformation $w = \rho^{-4}$ yields the standard Painlevé I equation

$$w''(t) = 6w(t)^2 + t.$$

The strategy shall consist in proving the existence of a monotone sequence $0 < w_1 < w_2 \ldots$ such that

$$\begin{aligned} w_n''(t) &= 6w_n(t)^2 + t, \quad t \in (0,1) \\ w_n(0) &= w_n(1) = n \end{aligned} \tag{2.3}$$

and set ρ as the limit of the sequence $\{w_n^{-1/4}\}$. However, it is not clear *a priori* whether or not the limit function $w(t) := \lim_{n\to\infty} w_n(t)$ is continuous and satisfies $w(t) < \infty$ for all $t \in (0,1)$. In order to circumvent this impediment, we shall give a location result with the aid of the method of upper and lower solutions. The following elementary result suffices in this regard (see e.g. [11, Ch. 2]):

Lemma 2.2. *Let* $f : [0,1] \times [0,+\infty) \to \mathbb{R}$ *be continuous and let* $R, S > 0$*. Assume that the smooth functions* α, β *satisfy*

$$\alpha''(t) > f(t,\alpha(t)), \qquad \beta''(t) < f(t,\beta(t)) \qquad t \in (0,1)$$

$$\alpha(0) \le R \le \beta(0), \quad \alpha(1) \le S \le \beta(1),$$

and $0 \le \alpha(t) < \beta(t)$ *for all* $t \in (0,1)$. *Then the Dirichlet boundary value problem*

$$u''(t) = f(t, u(t)), \qquad u(0) = R, \quad u(1) = S$$

has at least one solution u *with* $\alpha(t) < u(t) < \beta(t)$ *for* $t \in (0,1)$. *If furthermore* f *is nondecreasing with respect to its second variable, then the boundary value problem has no other (positive) solutions.*

The next lemma provides an ordered couple (α_n, β_n) of positive lower and upper solutions for (2.3).

Lemma 2.3. *There exist unique* α_n, β_n *with* $0 < \alpha_n(t) < \beta_n(t) < n$ *for* $t \in (0,1)$ *such that*

$$\alpha_n''(t) = 6\alpha_n(t)^2 + 1, \qquad \beta_n''(t) = 6\beta_n(t)^2$$

$$\alpha_n(0) = \alpha_n(1) = \beta_n(0) = \beta_n(1) = n.$$

Moreover, $m_n := \min_{t\in[0,1]} \beta(t)$ *satisfies* $m_n = \beta\left(\frac{1}{2}\right)$ *and*

$$c \le m_n \le C$$

for constants $C > c > 0$ *independent of* n.

Proof. Let $u(t) := \left(t - \frac{1}{2}\right)^2$ and $v(t) \equiv n$, then, for $t \in (0,1)$,

$$u''(t) \equiv 2 > 6u(t)^2 + 1, \qquad v''(t) \equiv 0 < 6v(t)^2 < 6v(t)^2 + 1$$

and $0 \le u(t) < v(t)$. From Lemma 2.2, the existence and uniqueness of α_n between u and v follows. Next, the pair (α_n, n) is adopted as an ordered couple of a lower and an upper solution for the problem $\beta'' = 6\beta^2$ which, in turn, provides the existence and uniqueness of β_n, with $\alpha_n < \beta_n < n$.

Next, multiplication of the equality $\beta_n'' = 6\beta_n^2$ by β_n' and integration yields

$$\beta_n'(t)^2 = 4\beta_n(t)^3 + A$$

for some constant A. By virtue of convexity, it follows that β_n achieves a unique minimum value $m_n < n$ at some $t_0 \in (0,1)$. It is deduced that $A = -4m_n^3$ and

$$\beta_n'(t) = \begin{cases} -2\sqrt{\beta_n(t)^3 - m_n^3}, & t \le t_0 \\ 2\sqrt{\beta_n(t)^3 - m_n^3}, & t > t_0. \end{cases}$$

Thus, for $t \le t_0$ we obtain

$$-\int_0^t \frac{\beta_n'(s)ds}{\sqrt{\beta_n(s)^3 - m_n^3}} = 2t$$

and setting $u := \frac{\beta_n(s)}{m_n}$ it follows that

$$m_n^{-1/2} \int_{\beta_n(t)/m_n}^{n/m_n} \frac{du}{\sqrt{u^3 - 1}} = 2t.$$

Analogously, for $t > t_0$ it is seen that

$$m_n^{-1/2} \int_{\beta_n(t)/m_n}^{n/m_n} \frac{du}{\sqrt{u^3 - 1}} = 2(1 - t).$$

In particular, letting $t \to t_0$ it follows that $2t_0 = 2(1-t_0)$, that is, $t_0 = \frac{1}{2}$. Furthermore,

$$m_n^{-1/2}\int_1^{n/m_n}\frac{du}{\sqrt{u^3-1}} = 1,$$

whence

$$m_n^{1/2} = \int_1^{n/m_n}\frac{du}{\sqrt{u^3-1}} \le \int_1^{\infty}\frac{du}{\sqrt{u^3-1}} < \infty.$$

This gives the inequality $m_n \le C$ and, for $n \ge C+1$

$$m_n^{1/2} = \int_1^{n/m_n}\frac{du}{\sqrt{u^3-1}} \ge \int_1^{1+\frac{1}{C}}\frac{du}{\sqrt{u^3-1}},$$

so $m_n \ge c$ for some constant $c > 0$ independent of n. □

Remark 2.4. With regard to the preceding, the fact that the minimum m_n is achieved at $t_0 = \frac{1}{2}$ follows directly by noticing that β_n is symmetric, that is, $\beta_n(t) = \beta_n(1-t)$. Indeed, this is due to uniqueness since $\beta_n(1-t)'' = \beta_n''(1-t) = 6\beta_n(1-t)^2$ and $\beta_n(1-0) = \beta_n(1-1) = n$. A similar argument holds for α_n.

As a corollary, we obtain:

Lemma 2.5. *Boundary value problem for Painlevé I* (2.3) *has a unique positive solution w_n with $\alpha_n < w_n < \beta_n$.*

Next, we shall prove a monotonicity property.

Lemma 2.6. *The sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{w_n\}$ are strictly nondecreasing.*

Proof. The claim is here proved just for $\{w_n\}$. The other cases are analogous. Assume that $w_{n+1} - w_n$ achieves its absolute minimum at some $\hat{t}$. If $w_{n+1}(\hat{t}) < w_n(\hat{t})$, then $\hat{t} \in (0,1)$ and

$$0 \le (w_{n+1} - w_n)''(\hat{t}) = 6(w_{n+1} + w_n)(\hat{t})(w_{n+1} - w_n)(\hat{t}) < 0,$$

a contradiction. Furthermore, because $(w_{n+1} - w_n)'(\hat{t}) = 0$, it is deduced that the equality $w_{n+1}(\hat{t}) = w_n(\hat{t})$ cannot hold either, due to the uniqueness of solutions of the initial value problem for the equation $w'' = 6w^2 + t$. □

As a consequence of the preceding lemma, we may define the functions $\alpha, \beta, w : [0,1] \to [0,+\infty]$ as the respective pointwise limits of the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{w_n\}$. It is clear that $\alpha \le w \le \beta$; however, it remains to prove that $w(t)$ is finite and satisfies the Painlevé I equation for $t \in (0,1)$. With this in mind, it is noted that the monotone and bounded sequence $\{m_n\}$ converges to a value $m = \beta\left(\frac{1}{2}\right) \in (0,+\infty)$ and, for $t \in \left(0,\frac{1}{2}\right)$, the implicit formula

$$m_n^{-1/2}\int_{\beta_n(t)/m_n}^{n/m_n}\frac{du}{\sqrt{u^3-1}} = 2t.$$

implies, when passing to the limit, that

$$m^{-1/2}\int_{\beta(t)/m}^{\infty}\frac{du}{\sqrt{u^3-1}} = 2t.$$

This shows that $\beta(t)$ is finite and the same conclusion is obtained for $t \in \left(\frac{1}{2},1\right)$; thus, $w(t) < +\infty$ for all $t \in (0,1)$. Moreover, the previous identity also implies that β is smooth and satisfies the equality $\beta''(t) = 6\beta(t)^2$. Next, fix $n_0 > m$ and $a < b$ the (unique) values in $(0,1)$ such that $\alpha_{n_0}(a) = \alpha_{n_0}(b) = m$. This implies that $w_n(t) > m$ for all $n \geq n_0$ and $t \notin [a,b]$. In particular, if the absolute minimum of w_n is achieved at $t_n \in (0,1)$, then $t_n \in [a,b]$ and we may take a subsequence $t_{n_j} \to t_* \in [a,b]$. Thus, from the identity

$$w_n'(t) = \int_{t_n}^{t} (6w_n(s)^2 + s)\,ds$$

and by the monotone convergence theorem we obtain:

$$w_{n_j}'(t) \to \int_{t_*}^{t} (6w(s)^2 + s)\,ds.$$

Now writing $w_{n_j}(t) = w_{n_j}(t_*) + \int_{t_*}^{t} w_{n_j}'(s)\,ds$, it is immediately verified that w is smooth and satisfies the Painlevé I equation for all $t \in (0,1)$. Hence $\rho := w^{-1/4}$ is positive and satisfies (2.1) for $t \in (0,1)$. It remains to prove that $\rho(0^+) = \rho(1^-) = 0$. To this end, for arbitrary $M > 0$ fix $n_0 > M$ and $\delta > 0$ such that $\alpha_{n_0}(t) > M$ when $t < \delta$ or $t > 1-\delta$. Since $\{\alpha_n\}$ is increasing, it follows that $\alpha(t) > M$ when $t < \delta$ or $t > 1-\delta$; accordingly, it has been established that $\alpha(0^+) = \alpha(1^-) = +\infty$ and, consequently, ρ is extended continuously to a solution of (2.1)–(2.2).

2.1 Order of the zeros

This section is devoted to investigation of the behaviour of the classical positive solutions of (2.1)–(2.2) in the neighbourhood of the endpoints of the interval. With this in mind, set as before $w := \rho^{-4}$ satisfying the Painlevé I equation and let $t_{\min} \in (0,1)$ be the value in which the absolute minimum $w_{\min}$ of w is achieved. For $t \in (0, t_{\min})$, the inequalities $6w(t)^2 < w''(t) < 6w(t)^2 + 1$ yield

$$6w(t)^2 w'(t) > w''(t)w'(t) > 6w(t)^2 w'(t) + w'(t)$$

and, upon integration,

$$4w(t)^3 - 4w_{\min}^3 < w'(t)^2 < 4w(t)^3 + 2w(t) - [4w_{\min}^3 + 2w_{\min}].$$

Using the identity $\sqrt{A+B} - \sqrt{A} = \frac{B}{\sqrt{A+B}+\sqrt{A}}$ for $A, A+B > 0$, we may write

$$2w(t)^{3/2} - R(t) < -w'(t) < 2w(t)^{3/2} + S(t)$$

where, due to the continuity of the solution ρ, the positive functions R and S can be made arbitrarily small when t is close to 0. In fact, given $r \in (0,1)$ it suffices to fix $\delta_0 > 0$ such that $R(t), S(t) < 2rw(t)^{3/2}$ for all $t < \delta_0$. This implies, for $0 < t < \delta < \delta_0$,

$$1 - r < \left(w^{-1/2}\right)'(t) < 1 + r,$$

whence

$$(1-r)(\delta - t) < w^{-1/2}(\delta) - w^{-1/2}(t) < (1+r)(\delta - t)$$

and letting $t \to 0$ we obtain:

$$(1-r)\delta \leq w^{-1/2}(\delta) \leq (1+r)\delta$$

that is

$$\sqrt{(1-r)\delta} \leq \rho(\delta) \leq \sqrt{(1+r)\delta}.$$

Since r is arbitrary, we conclude that $\rho(\delta) \sim \sqrt{\delta}$ for small values of δ. Analogously, it is verified that $\rho(\xi) \sim \sqrt{1-\xi}$ when ξ is close to 1. The previous conclusions allow a more precise computation of the solution near the endpoints of the interval. Indeed, it is observed that the functions R and S behave respectively as

$$R(t) = R_0(t)w(t)^{-3/2}, \qquad S(t) = S_0(t)w(t)^{-1/2}$$

for some bounded positive functions R_0 and S_0, so for $0 < t < \delta$ sufficiently small it is obtained:

$$\sqrt{1 - O(\delta^4)} \leq \frac{\rho(\delta)}{\sqrt{\delta}} \leq \sqrt{1 + O(\delta^6)}.$$

In particular, this shows that $\rho(t) \sim \sqrt{t} + O(t^{5/2})$ as $t \sim 0$ and, analogously, $\rho(t) \sim \sqrt{1-t} + O((1-t)^{5/2})$ as $t \sim 1$.

2.2 The uniqueness problem

In this section, it is established that the solution given via Theorem 2.1 is maximal, that is, any other possible solution $\tilde{\rho}$ of (2.1)–(2.2) such that $\tilde{\rho} \neq \rho$ satisfies $\tilde{\rho}(t) < \rho(t)$ for all $t \in (0,1)$. Furthermore, if $\tilde{\rho}$ is the limit of a sequence of solutions of (2.1) that are strictly positive in $[0,1]$, then $\tilde{\rho} = \rho$. Accordingly, the solution obtained in the preceding sections is the only one that can be defined as the limit of approximate solutions of the non-homogeneous Dirichlet problem.

The proof of the previous assertions is deduced in a straightforward manner from the following:

Lemma 2.7. *Let $\rho_1, \rho_2 \in C^2(0,1)$ be distinct strictly positive solutions of (2.1). Then ρ_1 and ρ_2 cross each other at most in one value $t \in (0,1)$.*

Proof. Due to the uniqueness for the initial value problem, it is clear that all possible cross points are isolated. Suppose that $a < b$ are two consecutive cross points and, for example, that $\rho_1 < \rho_2$ in (a,b), then the corresponding functions $w_j := \rho_j^{-4}$ satisfy $w_1 > w_2$ and

$$(w_1 - w_2)'' = 6(w_1 + w_2)(w_1 - w_2) > 0$$

over (a,b), which contradicts the fact that $w_1 = w_2$ for $t = a, b$. □

Proposition 2.8. *Let ρ be a positive solution of (2.1)–(2.2) such that ρ is the limit of a sequence $\{\rho_n\}$ of solutions of (2.1) with $\rho_n > 0$ on $[0,1]$. If $\tilde{\rho}$ is any distinct positive solution of (2.1)–(2.2), then $\tilde{\rho}(t) < \rho(t)$ for all $t \in (0,1)$.*

Proof. Suppose that $\rho_n(t) < \tilde{\rho}(t)$ for some $t \in (0,1)$. Then, because $\rho_n(0)$ and $\rho_n(1)$ are strictly positive, it follows that ρ_n crosses $\tilde{\rho}$ in more than one point, a contradiction. This shows that $\rho_n(t) \geq \tilde{\rho}(t)$ for all t and, consequently, $\rho \geq \tilde{\rho}$. Furthermore, if $\rho(t) = \tilde{\rho}(t)$ for some t, then $\rho'(t) = \tilde{\rho}'(t)$, whence $\rho \equiv \tilde{\rho}$. □

In view of the latter result, it might be conjectured that the positive solution of (2.1)–(2.2) is, indeed, unique. However, our conclusions do not exclude the existence of "small" solutions. The next result provides a lower bound for such small solutions.

Proposition 2.9. *Let ρ be a positive solution of* (2.1)–(2.2) *and let β be defined as before. Then $\rho(t) > \beta(t)^{-1/4}$ for all $t \in (0,1)$.*

Proof. Observe, at the outset, that β is the unique positive solution of the problem $v''(t) = 6v(t)^2$ satisfying $v(0^+) = v(1^-) = +\infty$. Indeed, it is seen as before that v achieves its unique minimum at $t = \frac{1}{2}$, with

$$v\left(\tfrac{1}{2}\right)^{1/2} = \int_1^\infty \frac{du}{\sqrt{u^3-1}} = \beta\left(\tfrac{1}{2}\right)^{1/2}.$$

Since, furthermore, $v'\left(\frac{1}{2}\right) = 0 = \beta'\left(\frac{1}{2}\right)$, it is deduced that $v \equiv \beta$. Next, suppose that $\rho(t_0) \le \beta(t_0)^{-1/4}$ for some $t_0 \in (0,1)$, then $w := \rho^{-4}$ satisfies

$$w''(t) = 6w(t)^2 + t > 6w(t)^2, \qquad w(0^+) = w(1^-) = +\infty.$$

On the other hand, setting $k \ge 1$ large enough, it is verified that

$$(\beta + kw)''(t) = 6\beta(t)^2 + 6kw(t)^2 + kt < 6[\beta(t) + kw(t)]^2.$$

Thus, $(w, \beta + kw)$ is an ordered couple of a lower and an upper solution for the problem $v'' = 6v^2$ and a diagonal argument proves the existence of a solution v with $w(t) < v(t) < \beta(t) + kw(t)$ for all $t \in (0,1)$. A contradiction then arises from the fact that $v \equiv \beta$. □

As a consequence, a somewhat sharp bound for the distance between distinct solutions is readily computed. Let w be the solution of the Painlevé I equation constructed in the proof of Theorem 2.1 and define $w_{\min}$ as the minimum value of w. As previously, w is a lower solution for the problem $v'' = 6v^2$ and setting $c > 0$ it is seen that

$$(w + c)''(t) = 6w(t)^2 + t < 6[w(t) + c]^2,$$

provided that $t < 6[c^2 + 2w(t)c]$. Thus, taking

$$c := \sqrt{w_{\min}^2 + \frac{1}{6}} - w_{\min}$$

it follows that $w(t) < \beta(t) < w(t) + c$ for all $t \in (0,1)$. For instance, a rough estimation shows that, since $\beta\left(\frac{1}{2}\right)$ is approximately equal to 5.9, then the optimal value of c is smaller than 0.015. In particular, this yields the bound

$$\beta(t)^{-1/4} < \rho(t) < [\beta(t) - c]^{-1/4} \qquad t \in (0,1)$$

for all possible solutions of (2.1)–(2.2).

Appendix. A Hamiltonian hybrid Ermakov–Painlevé I system

Just as the classical Ermakov equation of [13] constitutes the base one-component reduction of the Ermakov–Ray–Reid system of [16, 17], so the nonlinear Ermakov–Painlevé I equation, which is the subject of the present paper, may be embedded in a two-component hybrid Ermakov–Painlevé I system. Ermakov–Painlevé II–IV systems and their properties have been placed in a general solitonic context in [23]. Here, by way of illustration, a two-component Ermakov–Painlevé I system with underlying Hamiltonian structure is set down and an associated Ermakov–Ray–Reid system constructed.

Here, a Ermakov–Painlevé I system is introduced according to

$$\ddot{x} + \left[-5\left(\frac{\dot{\rho}}{\rho}\right)^2 + \frac{t\rho^4}{4} + \frac{3}{2\rho^4} \right] x = \frac{1}{x^2 y}\Phi(y/x),$$
$$\ddot{y} + \left[-5\left(\frac{\dot{\rho}}{\rho}\right)^2 + \frac{t\rho^4}{4} + \frac{3}{2\rho^4} \right] y = \frac{1}{xy^2}\Psi(x/y)$$

wherein ρ is governed by the single component EPI equation

$$\ddot{\rho} + \left[-5\left(\frac{\dot{\rho}}{\rho}\right)^2 + \frac{t\rho^4}{4} + \frac{3}{2\rho^4} \right] \rho = 0\ .$$

Thus,

$$\rho\ddot{x} - \ddot{\rho}x = \frac{\rho}{x^2 y}\Phi(y/x), \quad \rho\ddot{y} - \ddot{\rho}y = \frac{\rho}{xy^2}\Psi(x/y)\ .$$

whence, on introduction of the involutory transformation

$$\left.\begin{array}{c} x^* = x/\rho\ , \quad y^* = y/\rho\ , \\ dt^* = \rho^{-2}dt \\ \rho^* = 1/\rho \end{array}\right\} \quad \mathcal{R}$$

with $\mathcal{R}^2 = \mathrm{I}$, reduction is made to the canonical autonomous Ermakov–Ray–Reid system

$$x^*_{t^*t^*} = \frac{1}{x^{*2}y^*}\Phi(y^*/x^*), \quad y^*_{t^*t^*} = \frac{1}{x^*y^{*2}}\Psi(x^*/y^*)\ .$$

If the Ermakov–Painlevé I system has the J-parametric representation

$$\ddot{x} + \left[-5\left(\frac{\dot{\rho}}{\rho}\right)^2 + \frac{t\rho^4}{4} + \frac{3}{2\rho^4} \right] x = \frac{2}{x^3}J(y/x) + \frac{y}{x^4}J'(y/x)$$
$$\ddot{y} + \left[-5\left(\frac{\dot{\rho}}{\rho}\right)^2 + \frac{t\rho^4}{4} + \frac{3}{2\rho^4} \right] y = -\frac{1}{x^3}J'(y/x)$$

augmented by the canonical single component EPI equation, then application of the involutory transformation $\mathcal{R}$ produces the parametrisation of the canonical Hamiltonian Ermakov–Ray–Reid system as set down in a nonlinear optics context in [31]

$$x^*_{t^*t^*} = \frac{2}{x^{*3}}J(y^*/x^*) + \frac{y^*}{x^{*4}}J'(y^*/x^*)\ ,$$
$$y^*_{t^*t^*} = -\frac{1}{x^{*3}}J'(y^*/x^*)\ .$$

The latter admits the Hamiltonian integral of motion

$$\mathcal{H}^* = \frac{1}{2}\left[x^{*2}_{t^*} + y^{*2}_{t^*}\right] + \frac{1}{x^{*2}}J(y^*/x^*)\ .$$

which, together with the Ermakov invariant I^* allows the systematic integration of the canonical Hamiltonian system. It is remarked that such Ermakov–Ray–Reid systems with underlying Hamiltonian structure occur in diverse physical applications.

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