FLEXIBLE PLACEMENTS OF PERIODIC GRAPHS IN THE PLANE

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Abstract. Given a periodic graph, we wish to determine via combinatorial methods whether it has perioidic embeddings in the plane that are flexible, i.e. allow motions that preserve edge-lengths and periodicity to non-congruent embeddings. By introducing NBAC-colourings for the corresponding quotient gain graphs, we identify which periodic graphs have flexible embeddings in the plane when the lattice of periodicity is fixed. We further characterise with NBAC-colourings which 1-periodic graphs have flexible embeddings in the plane with a flexible lattice of periodicity, and characterise in special cases which 2-periodic graphs have flexible embeddings in the plane with a flexible lattice of periodicity.

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1. Introduction

A *(bar-joint) framework in the plane* is a pair $(\mathcal{G}, \mathcal{P})$, where $\mathcal{G}$ is a simple graph and $\mathcal{P}$ (the *placement* of $\mathcal{G}$) is a map from $V(\mathcal{G})$ to $\mathbb{R}^2$. By considering each edge vw as a rigid bar that restricts the distance between v and w, a natural question to ask is whether or not the structure is *flexible*, i.e. does there exist a continuous path in the space of placements of $\mathcal{G}$ that preserves the edge distances but is not a rigid body motion? If the vertex set of $\mathcal{G}$ is finite and the coordinates of the vector $(\mathcal{P}(v))_{v \in V(\mathcal{G})}$ are algebraically independent over $\mathbb{Q}$, then it has been proven (first by Pollaczek-Geiringer [14] and later by Laman independently [8]) that $(\mathcal{G}, \mathcal{P})$ is *rigid* (i.e. not flexible) in the plane if and only if $\mathcal{G}$ contains a (somewhat erroneously named) *Laman graph*; a graph $\mathcal{H}$ where $|E(\mathcal{H})| = 2|V(\mathcal{H})| - 3$ and $|E(\mathcal{H}')| \leq 2|V(\mathcal{H}')| - 3$ for all subgraphs $\mathcal{H}'$ of $\mathcal{H}$. Given a graph that contains a Laman graph, there can however still exist non-generic placements that are flexible; see Figure 1.

This raises a new question; can we use combinatorial methods to determine if a graph $\mathcal{G}$ has *any* placement that defines a flexible framework $(\mathcal{G}, \mathcal{P})$? This question was answered in the positive in [5], where it was proved that a finite simple graph will have flexible placements in the plane if and only if it has a *NAC-colouring*, a surjective red-blue edge colouring where no cycle has exactly one red edge or exactly one blue edge. Detecting whether graphs have flexible placements via NAC-colourings is a very recent area of research that utilises many different areas of Algebraic Geometry, including Valuation Theory [3][4][5].

Supported by the Austrian Science Fund (FWF): P31888.

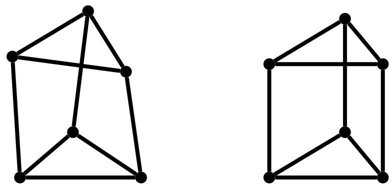

FIGURE 1. (Left): A rigid placement of $K_2 \times K_3$ in the plane. As $K_2 \times K_3$ is a Laman graph, almost all placements will give a rigid framework. (Right): A flexible placement of the same graph.

We now wish to extend the method using NAC-colourings to a frameworks with *k-periodic symmetry*, i.e. frameworks $(\mathcal{G}, \mathcal{P})$ where there exists a matrix $L \in M_{d \times k}(\mathbb{R})$ and a free group action θ of $\mathbb{Z}^k$ on $\mathcal{G}$ via graph automorphisms, such that $\mathcal{G}$ has a finite set of vertex orbits under θ and $\mathcal{P}(\theta(\gamma)v) = \mathcal{P}(v) + L.\gamma$ for all $v \in V(G)$ and $\gamma \in \mathbb{Z}^k$; we call L the *lattice* of $\mathcal{P}$, θ the *symmetry* of $\mathcal{G}$, and $\mathcal{P}$ a *k-periodic placement* of $(\mathcal{G}, \theta)$. Specifically, we wish to be able to determine if a graph $\mathcal{G}$ with symmetry θ has a k-periodic placement $\mathcal{P}$ where $(\mathcal{G}, \mathcal{P})$ can be deformed by a motion that preserves the periodic structure of $(\mathcal{G}, \mathcal{P})$, and if such a placement does exist, be able to also determine in advance whether the motion will preserve the lattice structure of $(\mathcal{G}, \mathcal{P})$.

Research into the rigidity of periodic frameworks has seen much interest in the last decade. Some of the main areas of research include combinatorial characterisations of rigid periodic graphs [2][9][16][12], periodic graphs with unique realisations [7], rigid unit modes of periodic frameworks [13][15], and rigidity under infinitesimal motions where the periodicity is relaxed somewhat [1][6][10].

Each k-periodic framework $(\mathcal{G}, \mathcal{P})$ with a given symmetry θ defines a family of *gain-equivalent* triples (G, p, L), where G is a *$\mathbb{Z}^k$-gain graph* and $p : V(G) \to \mathbb{R}^2$ (see Section 2.2 and Section 2.3 for definitions), and likewise, each such triple (G, p, L) will define a framework $(\mathcal{G}, \mathcal{P})$ with k-periodic symmetry; see [16, Section 2.2] for more details. As $\mathbb{Z}^k$-gain graphs have a finite amount of vertices but still encode all the required information needed for working with motions that preserve periodicity, we shall define a *k-periodic framework in the plane* to be a triple (G, p, L) for some $\mathbb{Z}^k$-gain graph G, and the pair (p, L) to be a *placement-lattice* of G.

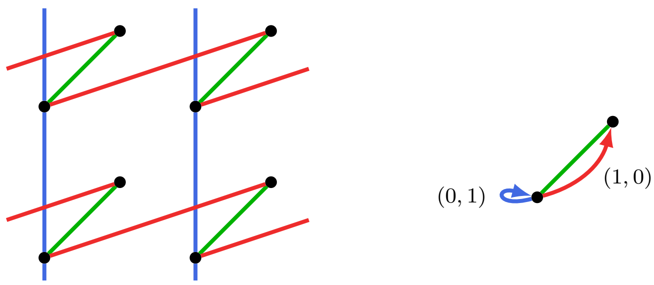

FIGURE 2. (Left): A framework $(\mathcal{G}, \mathcal{P})$ with 2-periodic symmetry. (Right): A corresponding triple (G, p, L) with $L := 2I_2$, where I_2 is the 2×2 identity matrix.

Using our new definition of k-periodic frameworks, our new question is now the following; can we use combinatorial methods to determine if a $\mathbb{Z}^k$-gain graph G has *any* placement-lattice that defines a flexible k-periodic framework (G, p, L)? We shall answer this in the positive for 1-periodic frameworks where the lattice is allowed to deform (see Theorem 5.1) and k-periodic frameworks where the lattice is fixed (see Theorem 4.1). We also obtain partial results for the more difficult case of 2-periodic frameworks where the lattice is allowed to deform (see Lemma 6.5, Theorem 7.5 and Theorem 7.8). To do this we shall introduce *NBAC-colourings* (NBAC being short for No Almost Balanced Circuit), an analogue of NAC-colourings for $\mathbb{Z}^k$-gain graphs, and characterise the types of NBAC-colourings that are generated by different motions of the framework.

The outline of the paper is as follows. In Section 2, we shall layout some background on Valuation Theory, gain graphs and periodic frameworks in both $\mathbb{R}^d$ and $\mathbb{C}^d$. In Section 3, we shall define NBAC-colourings and their various sub-types, including active NBAC-colourings, and utilise valuations to prove that flexibility will imply the existence of an NBAC-colouring. In Section 4, Section 5 and Section 6, we shall apply our methods using NBAC-colourings to fixed lattice k-periodic frameworks, flexible lattice 1-periodic frameworks and flexible lattice 2-periodic frameworks respectively, with partial results in the latter case. In Section 7, we shall prove that a full characterisation of $\mathbb{Z}^2$-gain graphs with a flexible placement-lattice is possible if we assume that our graph has at least a single loop.

2. Preliminaries

2.1. **Function fields and valuations.** We shall refer to all affine algebraic sets over $\mathbb{C}$ as algebraic sets, and we shall refer to any irreducible algebraic set as varieties. For an algebraic set V in $\mathbb{C}^n$ we define $I(V)$ to be the ideal of $\mathbb{C}^n$ that defines V. We remember that the dimension of an algebraic set is the maximal length of chains of distinct nonempty subvarieties of A. An *algebraic curve* is an affine variety of dimension 1.

Definition 2.1. Let V be a variety in the polynomial ring $\mathbb{C}[X_1, \dots, X_n]$. We define the *coordinate ring* to be the quotient $\mathbb{C}[V] := \mathbb{C}[X_1, \dots, X_n]/I(V)$ and the *field of rational functions* $\mathbb{C}(V)$ to be the field of fractions of $\mathbb{C}[V]$. Each $\hat{f}/\hat{g} \in \mathbb{C}(V)$ can, for any $f \in \hat{f}$ and $g \in \hat{g}$, be considered to be a partially defined function

$$f/g : V \to \mathbb{C}, \ x \mapsto f(x)/g(x),$$

and this function is independent of the choice of f, g. If C is an algebraic curve then $\mathbb{C}(C)$ is also referred to as a *function field*.

We remember that for a field extension K/k, an element $a \in K$ is *transcendental over* k if there is no polynomial $p \in k[X]$ with $p(a) = 0$, and *algebraic over* k otherwise.

Lemma 2.2. *Let $\mathcal{C}$ be an algebraic curve in $\mathbb{C}[X_1, \dots, X_n]$ and let $f \in \mathbb{C}[x_1, \dots, x_n]$. Then one of the following holds:*

(i) f takes an infinite amount of values on $\mathcal{C}$ and is transcendental over $\mathbb{C}$ when considered as an element of $\mathbb{C}(\mathcal{C})$.
(ii) f is constant on $\mathcal{C}$.

Proof. Suppose f takes only a finite amount of values $a_1, \dots, a_n \in \mathbb{C}$ on $\mathcal{C}$. Define the polynomial $p(X) = \prod_{i=1}^n (X - a_i) \in \mathbb{C}[X]$, then $p(f(x)) = 0$ for all $x \in \mathcal{C}$. It follows then that f is algebraic in the field extension $\mathbb{C}(\mathcal{C})/\mathbb{C}$, thus as $\mathbb{C}$ is algebraically closed, f is constant on $\mathbb{C}$.

Now suppose that f takes an infinite amount of values on $\mathcal{C}$ but is algebraic in the field extension $\mathbb{C}(\mathcal{C})/\mathbb{C}$, i.e. there exists a polynomial $q \in \mathbb{C}[X]$ so that $q(f(x)) = 0$ for all $x \in \mathcal{C}$. As q has a finite number of roots then it cannot be zero on an infinite set. This contradicts that f takes an infinite amount of values on $\mathcal{C}$, thus f is not algebraic over $\mathbb{C}$, i.e. f is transcendental over $\mathbb{C}$. $\square$

Definition 2.3. For a function field $\mathbb{C}(C)$, a function $\nu : \mathbb{C}(C) \to \mathbb{Z} \cup \{\infty\}$ is a *valuation* if

(i) $\nu(x) = \infty$ if and only if $x = 0$.
(ii) $\nu(xy) = \nu(x) + \nu(y)$.
(iii) $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$, with equality if $\nu(x) \neq \nu(y)$.
(iv) $\nu(x) = 0$ if $x \in \mathbb{C} \setminus \{0\}$.

The following is a rewording of a useful result [17, Corollary 1.1.20].

Proposition 2.4. *Let $\mathbb{C}(C)$ be a function field and suppose $f \in \mathbb{C}(\mathcal{C})$ is transcendental over $\mathbb{C}$. Then there exists a valuation ν of $\mathbb{C}(\mathcal{C})$ with $\nu(f) > 0$.*

2.2. Gain graphs.

Definition 2.5. A Γ-*gain graph* is a triple $G := (V(G), E(G), \Gamma)$, where:

(i) $V(G)$ is a finite set of *vertices*.
(ii) Γ is an additive abelian group with identity 0.
(iii) $E(G) \subset (V(G)^2 \times \Gamma)/R$ is a set of *edges*, where $(a, b, \gamma)R(c, d, \mu)$ if and only either $a = c$, $b = d$ and $\gamma = \mu$, or $a = d$, $b = c$ and $\gamma = -\mu$. We shall assume that there is no edge of the form $(v, v, 0)$. We shall, however, allow $E(G)$ to be an infinite set.

While the edges of a gain graph are not orientated, we often find it easier to assume that there is some orientation on the edges, i.e. G is directed. We may then define the *gain* of an edge (v, w, γ) to be γ. We refer the reader to Figure 3 for an example.

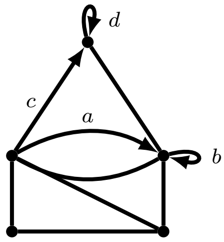

FIGURE 3. A Γ-gain graph with $a, b, c, d \in \Gamma$. We represent any edge (v, w, γ) by an arrow from v to w with a label γ, and we represent any edge $(v, w, 0)$ by an undirected edge from v to w.

Given a Γ-gain graph G, $v \in V(G)$ and $\mu \in \Gamma$, we define G^μ_u to be the Γ-gain graph with vertex set $V(G)$ and edge set

$$\{(v, w, \gamma), (u, z, \gamma + \mu) : (v, w, \gamma), (u, z, \gamma) \in E(G),\ v, w \neq u \text{ or } v = w = u,\ \text{and } z \neq u\}.$$

A *switching operation at u by μ* is the map $\phi^\mu_u : G \to G^\mu_u$ where $\phi^\mu_u(v) = v$ and

$$\phi^\mu_u(v, w, \gamma) = \begin{cases} (u, w, \gamma + \mu) & \text{if } v = u,\ w \neq u \\ (v, u, \gamma - \mu) & \text{if } v \neq u,\ w = u \\ (v, w, \gamma) & \text{if } v, w \neq u \text{ or } v = w = u. \end{cases}$$

See Figure 4 for an example of a gain switching operation at a vertex.

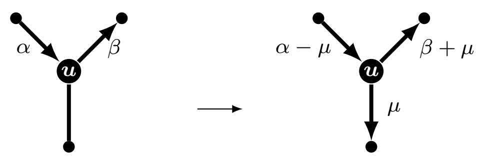

FIGURE 4. A switching operation at u by μ.

We define $\phi := \phi^{\mu_n}_{u_n} \circ \ldots \circ \phi^{\mu_1}_{u_1}$ to be a *gain equivalence*, and if G and G' are Γ-gain graphs and $G' = \phi(G)$, then we say G and G' are *gain-equivalent* (or $G \approx G'$); further, if $H \subset G$ and $H' := \phi(H)$ then we say H' is the *corresponding subgraph* of H in G'. The relation $\approx$ is an equivalence relation for gain graphs.

Remark 2.6. It is immediate that $\phi_u^{-\mu} \circ \phi_u^{\mu}(G) = G$ and any two switching operations will commute. Due to this, we shall denote

$$\prod_{i=1}^{n} \phi_{u_i}^{\mu_i} := \phi_{u_n}^{\mu_n} \circ \ldots \circ \phi_{u_1}^{\mu_1}.$$

A *walk* in G is an ordered set

$$C := (e_1, e_2, \ldots, e_{n-1},),$$

where $e_i = (v_i, v_{i+1}, \gamma_i)$ for some γ_i; we note that we orientate each edge so we have a *directed walk* from v_1 to v_n. If $v_1 = v_n$ then C is a *circuit.* For a circuit C, we define

$$\psi(C) := \gamma_1 + \gamma_2 + \ldots + \gamma_n$$

to be the *gain* of C. A circuit is *balanced* if $\psi(C) := I$, and *unbalanced* otherwise. For a connected subgraph $H \subset G$, we define the *span of* H to be the subgroup

$$\operatorname{span}(H) := \{\psi(C) : C \text{ is a circuit in } H\}.$$

If $\Gamma \cong \mathbb{Z}^k$ for some $k \in \mathbb{N}$ then we define $\operatorname{rank}(H)$ to be the rank of $\operatorname{span}(H)$. A subgraph H is *balanced* if $\operatorname{span}(H)$ is the trivial group, and *unbalanced* otherwise.

Proposition 2.7. *Let G, G' be gain-equivalent Γ-gain graphs and $H \subset G$ be a connected subgraph. If H' is the corresponding subgraph of H in G, then* $\operatorname{span}(H') = \operatorname{span}(H)$.

Proof. We note that switching operations will not change the span of a circuit. The result now follows. □

Proposition 2.8. *Let G be a Γ-gain graph and $\{H_1, \ldots, H_n\}$ a set of connected subgraphs with pairwise disjoint vertex sets. Then there exists $G' \approx G$ such that for each $i \in \{1, \ldots, n\}$, all the edges of the corresponding subgraph H_i' of H_i in G' have gain in* $\operatorname{span}(H_i)$.

Proof. Choose a spanning tree T_i for each $i \in \{1, \ldots, n\}$. We note that we may choose $G' \approx G$ such that for each corresponding subgraph T_i' of T_i in G' has only trivial gain for its edges; see [16, Section 2.4] a description of the method.

Choose any edge $e = (v, w, \gamma) \in E(H_i)$. As v, w are connected by a walk in T_i', the corresponding subgraph of T_i in G', then there exists a circuit from v to w with gain γ, thus $\gamma \in \operatorname{span}(H_i')$. By Proposition 2.7, $\operatorname{span}(H_i) = \operatorname{span}(H_i')$ as required. □

2.3. **Rigidity and flexibility for k-periodic frameworks.** Let $d \in \mathbb{N}$ and $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$. We shall define $\|\cdot\|^2 : \mathbb{K}^d \to \mathbb{K}$ to be the map with

$$\left\|(x_i)_{i=1}^d\right\|^2 := \sum_{i=1}^{d} x_i^2$$

for all $(x_i)_{i=1}^d \in \mathbb{K}^d$; for $\mathbb{K} = \mathbb{R}$ this is in fact the square of the norm, though this does not hold for $\mathbb{K} = \mathbb{C}$. It is however a quadratic form, and the isometries of $(\mathbb{K}^d, \|\cdot\|^2)$ are exactly the affine maps $x \mapsto Mx + y$, where $y \in \mathbb{K}^d$ and $M \in M_n(\mathbb{K})$ is a $d \times d$-matrix where $M^T M = I_d$.

Remark 2.9. For any matrix $M \in M_{m \times n}(\mathbb{K})$ and $x := (x_1, \ldots, x_n) \in \mathbb{K}^n$, we shall denote by $M.x$ the matrix multiplication $M[x_1 \ \ldots \ x_n]^T$.

Definition 2.10. Let $d \in \mathbb{N}$ and G be a $\mathbb{Z}^k$-gain graph for some $1 \leq k \leq d$. A *k-periodic framework in $\mathbb{K}^d$* is a triple (G, p, L) such that G is a $\mathbb{Z}^k$-gain graph, $p : V(G) \to \mathbb{K}^d$, and $L \in M_{d \times k}(\mathbb{K})$, with the assumption that if $(v, w, \gamma) \in E(G)$ then $p(v) \neq p(w) + L.\gamma$. We shall define p to be the *placement*, L to be a *lattice*, and (p, L) to be a *placement-lattice.* If L is also injective then (G, p, L) is *full*, and if $\mathbb{K} = \mathbb{R}$ then we simply refer to (G, p, L) as a k-periodic framework.

For a given $\mathbb{Z}^k$-gain graph G, we define $\mathcal{V}^d_{\mathbb{K}}(G)$ to be the space of placement-lattices of G, which we shall consider to be a subspace of $\mathbb{K}^{d|V(G)|+dk}$. We immediately note that $\mathcal{V}^d_{\mathbb{K}}(G)$ is an open non-empty subset in the Zariski topology, and if G has an edge, it is a proper subset.

Definition 2.11. Let (G,p,L) and (G,p',L') be k-periodic frameworks in $\mathbb{K}^d$. Then $(G,p,L) \sim (G,p',L')$ (or (G,p,L) and (G,p',L') are *equivalent*) if for all $(v,w,\gamma) \in E(G)$,

$$\|p(v) - p(w) - L.\gamma\|^2 = \|p'(v) - p'(w) - L'.\gamma\|^2, \tag{1}$$

and $(p,L) \sim (p',L')$ (or (p,L) and (p',L') are *congruent*) if (1) holds for all $v,w \in V(G)$ and $\gamma \in \mathbb{Z}^k$; equivalently, we may define $(p,L) \sim (p',L')$ if and only if there exists a linear isometry $M \in M_d(\mathbb{K})$ and $y \in \mathbb{K}^d$ such that $p'(v) = M.p(v) + y$ for all $v \in V(G)$ and $L' = ML$. For any $L, L' \in M_{d\times k}(\mathbb{K})$, we define to be L and L' *orthogonally equivalent* (or $L \sim L'$) if for any $\gamma, \mu \in \mathbb{Z}^k$,

$$(L.\gamma).(L.\mu) = (L'.\gamma).(L'.\mu). \tag{2}$$

We note that, by linearity, if (2) holds for all pairs of some basis of $\mathbb{Z}^k$, then it holds for all $\gamma, \mu \in \mathbb{Z}^k$. Further, if $(p,L) \sim (p',L')$ then $(G,p,L) \sim (G,p',L')$ and $L \sim L'$.

Definition 2.12. For a k-periodic framework (G,p,L) we define the algebraic subsets

$$\begin{aligned}\mathcal{V}_{\mathbb{K}}(G,p,L) &:= \left\{(p',L') \in \mathcal{V}^d_{\mathbb{K}}(G) : (G,p',L') \sim (G,p,L)\right\},\\ \mathcal{V}^f_{\mathbb{K}}(G,p,L) &:= \left\{(p',L') \in \mathcal{V}^d_{\mathbb{K}}(G) : (G,p',L') \sim (G,p,L),\ L' \sim L\right\}.\end{aligned}$$

Definition 2.13. Let (G,p,L) be a k-periodic framework in $\mathbb{K}^d$. A *flex* of (G,p,L) is a continuous path $t \mapsto (p_t, L_t)$, $t \in [0,1]$, in $\mathcal{V}_{\mathbb{K}}(G,p,L)$. If $(p_t,L_t) \in \mathcal{V}^f_{\mathbb{K}}(G,p,L)$ for all $t \in [0,1]$ then (p_t,L_t) is a *fixed lattice flex*. If $(p_t,L_t) \sim (p,L)$ for all $t \in [0,1]$ then (p_t,L_t) is *trivial*.

Remark 2.14. An equivalent definition for a trivial finite flex is as follows: (p_t,L_t) is a trivial flex of (G,p,L) if and only (p_t,L_t) is a trivial flex of (K,p,L), where K is $\mathbb{Z}^k$-gain graph with vertex set $V(G)$ and edge set

$$\left\{(v,w,\gamma) : v,w \in V(G), \gamma \in \mathbb{Z}^k, \gamma \neq 0 \text{ if } v = w\right\}.$$

Definition 2.15. Let (G,p,L) be a k-periodic framework. Then we define the following:

(i) (G,p,L) is *rigid* if all flexes of $(\tilde{G},\tilde{p})$ are trivial, and *flexible* otherwise.
(ii) (G,p,L) is *fixed lattice rigid* if all fixed lattice flexes of $(\tilde{G},\tilde{p})$ are trivial, and *fixed lattice flexible* otherwise.

Given a switching operation ϕ^μ_u of G, we may define the *framework switching operation at u by μ* to be the a map $\phi^\mu_u : \mathcal{V}^d_{\mathbb{K}}(G) \to \mathcal{V}^d_{\mathbb{K}}(\phi^\mu_u(G))$, where given the placement lattice (p',L') of $\phi^\mu_u(G)$ with

$$p'(v) := \begin{cases} p(u) + L.\mu, & \text{if } v = u \\ p(v), & \text{otherwise}\end{cases}$$

for all $v \in V^d_{\mathbb{K}}(G)$, and $L' = L$, we define $\phi^\mu_u(p,L) := (p',L')$ and $\phi^\mu_u(G,p,L) := (\phi^\mu_u(G), \phi^\mu_u(p,L))$. We define $\phi := \phi^{\mu_n}_{u_n} \circ \ldots \circ \phi^{\mu_1}_{u_1}$ to be *gain equivalence*, and if $(G',p',L) = \phi(G,p,L)$, then we say (G,p,L) and (G',p',L) are *gain-equivalent*; we shall denote this by $(G,p,L) \approx (G',p',L)$. It is immediate that $\approx$ is an equivalence relation for frameworks. We further note that $\phi^\gamma_v \circ \phi^\mu_u(G,p,L) = \phi^\mu_u \circ \phi^\gamma_v(G,p,L)$.

Remark 2.16. Let (G,p,L) and (G',p',L) be gain-equivalent k-periodic frameworks in $\mathbb{K}^d$. We note that the map ϕ^μ_u is the restriction of a linear map, thus $\mathcal{V}_{\mathbb{K}}(G,p,L)$ is isomorphic to $\mathcal{V}_{\mathbb{K}}(G',p',L')$ as algebraic sets. It follows that (G,p,L) is (fixed lattice) rigid if and only if (G',p',L') is (fixed lattice) rigid.

3. NBAC-colourings and flexibility in the plane

3.1. **NBAC-colourings.**

Definition 3.1. Let G be a Γ-gain graph with edge colouring $\delta : E(G) \to \{\text{red}, \text{blue}\}$. We define the following:

(i) $G^\delta_{\text{red}} := (V(G), \{e \in E(G) : \delta(e) = \text{red}\})$.
(ii) A *red component* is a connected component of G^δ_{red}.
(iii) A *red walk* (respectively, *red circuit*) is a walk (respectively, circuit) where every edge is red.
(iv) An *almost red circuit* is a circuit with exactly one blue edge.
(v) G^δ_{blue}, *blue components*, *blue walks*, *blue circuits*, and *almost blue circuits* are defined analogously.
(vi) We define *monochromatic components*, *monochromatic walks*, *monochromatic circuits* and *almost monochromatic circuits* to be either red or blue.

A colouring δ is a *NBAC-colouring* (*No Balanced Almost Circuits*) if it is surjective and there are no balanced almost red circuits and no balanced almost blue circuits; see Figure 5 for an example of an NBAC-colouring.

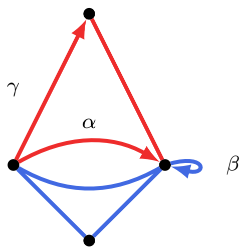

FIGURE 5. A surjective colouring δ of a Γ-gain graph. If $\alpha \neq \gamma$, $\alpha \neq 0$ and $\alpha \notin \langle\beta\rangle$ then δ is a NBAC-colouring.

If δ is a colouring of G and $G \approx H$, then by abuse of notation we shall also define δ to be a colouring for H. We note that if δ is a NBAC-colouring of G, then δ is a NBAC-colouring of H.

Definition 3.2. Let G be a $\mathbb{Z}^k$-gain graph for some $k \in \{1, 2\}$, with an NBAC-colouring δ. If G^δ_{red} (respectively, G^δ_{blue}) is unbalanced, G^δ_{blue} (respectively, G^δ_{red}) is balanced and G has no almost red (respectively, almost blue) circuits, then δ is a *fixed lattice* NBAC-colouring.

Definition 3.3. Let G be a $\mathbb{Z}$-gain graph with an NBAC-colouring δ. If both G^δ_{red} and G^δ_{blue} are balanced, then δ is a *flexible 1-lattice* NBAC-colouring.

Remark 3.4. We note that if G is a $\mathbb{Z}$-gain graph with NBAC-colouring δ, then δ cannot be both a fixed lattice NBAC-colouring and a flexible 1-lattice NBAC-colouring. We can, however, have that G has both a fixed lattice NBAC-colouring and a flexible 1-lattice NBAC-colouring; see Figure 6 for an example.

Definition 3.5. Let G be a $\mathbb{Z}^2$-gain graph with an NBAC-colouring δ. We define the following:

(i) If both G^δ_{red} and G^δ_{blue} are balanced, then δ is a *type 1 flexible 2-lattice* NBAC-colouring.
(ii) If there exists $\alpha, \beta \in \mathbb{Z}^2$ such that
- either α, β are linearly independent or exactly one of α, β is equal to $(0, 0)$,
- $\text{span}(G^\delta_{\text{red}})$ is a non-trivial subgroups of $\mathbb{Z}\alpha$ or $\alpha = (0, 0)$,
- $\text{span}(G^\delta_{\text{blue}})$ is a non-trivial subgroups of $\mathbb{Z}\beta$ or $\beta = (0, 0)$,
- there are no almost red circuits with gain in $\mathbb{Z}\alpha$, and

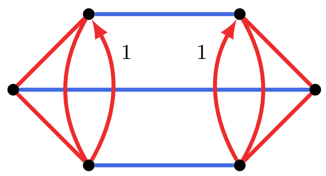

FIGURE 6. A $\mathbb{Z}$-gain graph with two different NBAC-colourings. The left is a fixed lattice NBAC-colouring, while the right is a flexible 1-lattice NBAC-colouring.

- there are no almost blue circuits with gain in $\mathbb{Z}\beta$,

then δ is a *type 2 flexible 2-lattice* NBAC-colouring.

(iii) If there exists $\alpha \in \mathbb{Z}^2 \setminus \{(0,0)\}$ such that
- $\mathrm{span}(G^\delta_{\mathrm{red}})$ and $\mathrm{span}(G^\delta_{\mathrm{red}})$ are non-trivial subgroups of $\mathbb{Z}\alpha$, and
- there are no almost monochromatic circuits with gain in $\mathbb{Z}\alpha$,

then δ is a *type 3 flexible 2-lattice* NBAC-colouring.

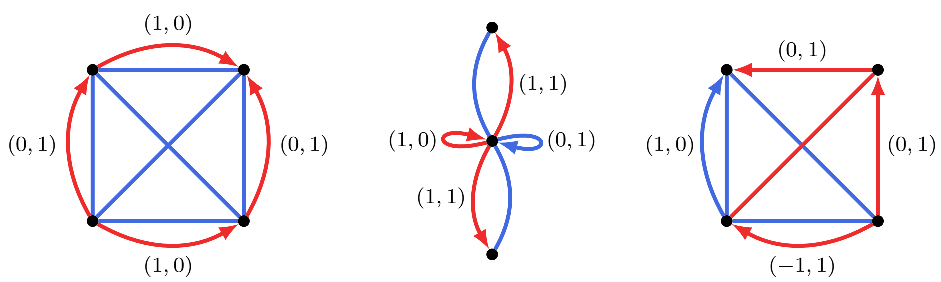

FIGURE 7. (Left): A $\mathbb{Z}^2$-gain graph with a type 1 flexible 2-lattice NBAC-colouring. (Middle): A $\mathbb{Z}^2$-gain graph with a type 2 flexible 2-lattice NBAC-colouring ($\alpha = (1,0)$, $\beta = (0,1)$). (Right): A $\mathbb{Z}^2$-gain graph with a type 3 flexible 2-lattice NBAC-colouring ($\alpha = (1,0)$).

Remark 3.6. We note that if G is a $\mathbb{Z}^2$-gain graph with NBAC-colouring δ, then
- For distinct $i, j \in \{1,2,3\}$, δ cannot be both a type i and type j flexible 2-lattice NBAC-colouring.
- δ cannot be both a fixed lattice NBAC-colouring and a type k flexible 2-lattice NBAC-colouring, unless $k = 2$; see Figure 8 for an example of an NBAC-colouring that is both.
- If $H \subset G$ that is not monochromatic and δ is a type k flexible 2-lattice NBAC-colouring for some $k \in \{1,2,3\}$, then δ restricted to H is a type k' flexible 2-lattice NBAC-colouring for some $1 \leq k' \leq k$.

3.2. **k-periodic frameworks in the plane.** Let G be a k-periodic framework for $k \in \{1,2\}$, with placement $p : V(G) \to \mathbb{R}^2$ and lattice $L \in M_{d\times k}(\mathbb{R})$; if $k = 1$ we shall define $L_1 := L.1$ and if $k = 2$ we shall define $L_1 := L.(1,0)$ and $L_2 := L.(0,1)$. By rotations and translations we may assume that for some edge $\tilde{e} := (\tilde{v}, \tilde{w}, \tilde{\gamma})$,

$$p(\tilde{v}) := (0,0), \qquad p(\tilde{w}) + L(\gamma_0) := (c, 0)$$

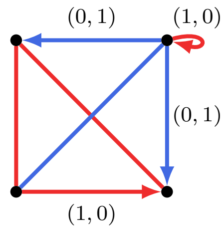

FIGURE 8. A $\mathbb{Z}^2$-gain graph with a colouring that is both a fixed lattice NBAC-colouring and a flexible 2-lattice NBAC-colouring ($\alpha = (1, 0)$, $\beta = (0, 0)$).

for some $c \in \mathbb{R}$. For each $e = (v, w, \gamma)$ with $\gamma := (\gamma_j)_{j=1}^k$, we define

$$\lambda(e) := \left\| p(v) - p(w) - \sum_{j=1}^{k} \gamma_j L_j \right\|$$

(we note that this is well-defined as (G, p, L) is a k-periodic framework in $\mathbb{R}^2$). We further define for each $1 \leq j, l \leq k$,

$$\lambda(j, l) := L_j.L_l.$$

We shall consider $\mathcal{V}^2_{\mathbb{C}}(G)$ to be the complex linear space of points

$$z = (x_v, y_v)_{v \in V(G)} \times (x_j, y_j)_{j=1}^k,$$

where $x_v, y_v, x_j, y_j \in \mathbb{C}$. We define the algebraic set $\mathcal{V}_{\tilde{e}}(G, p, L) \subset \mathcal{V}^2_{\mathbb{C}}(G)$ of all points where

$$x_{\tilde{v}} = y_{\tilde{v}} = 0, \qquad y_{\tilde{w}} + \sum_{j=1}^{k} \tilde{\gamma}_j y_j = 0.$$

and for all $e = (v, w, \gamma) \in E(G)$,

$$\left(x_v - x_w - \sum_{j=1}^{k} \gamma_j x_j \right)^2 + \left(y_v - y_w - \sum_{j=1}^{k} \gamma_j y_j \right)^2 = \lambda(e)^2. \tag{3}$$

We further define $\mathcal{V}^f_{\tilde{e}}(G, p, L)$ to be the algebraic subset of $\mathcal{V}_{\tilde{e}}(G, p, L)$ where

$$x_j x_l + y_j y_l = \lambda(j, l)^2,$$

for each $1 \leq j, l \leq k$.

Remark 3.7. We note that $\mathcal{V}_{\mathbb{C}}(G, p, L)$ is homeomorphic to

$$\mathcal{V}_{\tilde{e}}(G, p, L) \times SO(2, \mathbb{C}) \times \mathbb{C}^2,$$

as $\mathcal{V}_{\tilde{e}}(G, p, L)$ is the set of equivalent frameworks in $\mathbb{C}^2$ assuming that the edge $\tilde{e}$ is fixed. Similarly, $\mathcal{V}^f_{\mathbb{C}}(G, p, L)$ is homeomorphic to

$$\mathcal{V}^f_{\tilde{e}}(G, p, L) \times SO(2, \mathbb{C}) \times \mathbb{C}^2.$$

Given an algebraic curve $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$ and any $v, w \in V(G)$, $\gamma \in \mathbb{Z}^k$, we define the maps

$$W^{\gamma}_{v,w,}, Z^{\gamma}_{v,w} : \mathcal{C} \to \mathbb{C}$$

by the polynomials

$$W^\gamma_{v,w} := \left(x_v - x_w - \sum_{j=1}^k \gamma_j x_j\right) + i\left(y_v - y_w - \sum_{j=1}^k \gamma_j y_j\right)$$
$$Z^\gamma_{v,w} := \left(x_v - x_w - \sum_{j=1}^k \gamma_j x_j\right) - i\left(y_v - y_w - \sum_{j=1}^k \gamma_j y_j\right).$$

We further define the maps $W_j, Z_j : \mathcal{C} \to \mathbb{C}$ for $1 \leq j \leq k$ as the polynomials,

$$W_j := x_j + iy_j$$
$$Z_j := x_j - iy_j.$$

For the case of $k = 2$, we shall define for each $\gamma := (a, b) \in \mathbb{Z}^2$ the maps

$$\gamma W := aW_1 + bW_2, \qquad \gamma Z := aZ_1 + bZ_2.$$

It is immediate that $W^{-\gamma}_{w,v} = -W^\gamma_{v,w}$ and $Z^{-\gamma}_{w,v} = -Z^\gamma_{v,w}$. We note that if $e = (v, w, \gamma) \in E(G)$,

$$W^\gamma_{v,w} Z^\gamma_{v,w} = \lambda(e)^2,$$

and if $\mathcal{C} \subset \mathcal{V}^f_{\tilde{e}}(G, p, L)$ then

$$W_j.Z_j = \lambda(j, j)^2, \qquad W_j.Z_l + W_l.Z_j = 2\lambda(j, l)^2,$$

for all $1 \leq j, l \leq k$.

3.3. **Active NBAC-colourings.** Active NAC-colourings for finite simple graphs was first introduced in [4]. We shall now give an analogue of them for $\mathbb{Z}^k$-gain graphs.

Definition 3.8. Let (G, p, L) be k-periodic framework in $\mathbb{R}^2$, $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$ be an algebraic curve and δ a NBAC-colouring of G. We define δ to be an *active NBAC-colourings of* $\mathcal{C}$ if there exists a valuation ν of $\mathbb{C}(\mathcal{C})$ and $\alpha \in \mathbb{R}$ such that for each $e \in E(G)$,

$$\delta(e) = \begin{cases} \text{red if } \nu(W^\gamma_{v,w}) > \alpha, \\ \text{blue if } \nu(W^\gamma_{v,w}) \leq \alpha. \end{cases}$$

We shall define δ to be the NBAC-colouring *generated* by ν and α. For a k-periodic framework (G, p, L) in $\mathbb{R}^2$, we define δ to be an *active NBAC-colourings of* (G, p, L) if it is an active NBAC-colourings of some algebraic curve $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$ for some $\tilde{e} \in E(G)$. We define δ to be an *active NBAC-colourings of* G if it is an active NBAC-colourings of a k-periodic framework (G, p, L) in $\mathbb{R}^2$.

Remark 3.9. If δ is an active NBAC-colouring of an algebraic curve $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$ and δ' is the NBAC-colouring with $\delta'(e) \neq \delta(e)$ for all $e \in E(G)$, then δ' is also an active NBAC-colouring of $\mathcal{C}$; this can be shown in a similar way to the proof of [4, Lemma 1.13].

We shall now denote by $W^\gamma_{v,w}|_\mathcal{C}$ (respectively, $W^\gamma_{v,w}|_\mathcal{C}$) the map $W^\gamma_{v,w}$ (respectively, $Z^\gamma_{v,w}$) when considered with respect to an algebraic curve $\mathcal{C}$.

Lemma 3.10. *Let (G, p, L) be a k-periodic framework in $\mathbb{R}^2$, and $e_1, e_2 \in E(G)$, with $e_1 = (v_1, w_1, \gamma_1)$ and $e_2 = (v_2, w_2, \gamma_2)$. Then the map*

$$f_{e_1,e_2} : \mathcal{V}_{e_1}(G, p, L) \to \mathcal{V}_{e_2}(G, p, L),\ (q, M) \mapsto (R^q_{e_2}.(q(v) - q(v_2)))_{v \in V(G)} \times R^q_{e_2} M$$

is biregular, where

$$R^q_{e_2} := \frac{1}{\lambda(e_2)} \begin{bmatrix} x_{w_2} - x_{v_2} & y_{w_2} - y_{v_2} \\ -(y_{w_2} - y_{v_2}) & x_{w_2} - x_{v_2} \end{bmatrix}.$$

Further, for any algebraic curve $\mathcal{C} \subset \mathcal{V}_{e_1}(G,p,L)$ and any $v,w \in V(G)$, $\gamma \in \mathbb{Z}^k$, we have that $\mathcal{C}' := f_{e_1,e_2}(\mathcal{C})$ is an algebraic curve and

$$W^{\gamma}_{v,w}|_{\mathcal{C}'} \circ f_{e_1,e_2} = \frac{1}{\lambda(e_2)} W^{\gamma}_{v,w}|_{\mathcal{C}} Z^{\gamma_2}_{v_2,w_2}|_{\mathcal{C}}, \quad Z^{\gamma}_{v,w}|_{\mathcal{C}'} \circ f_{e_1,e_2} = \frac{1}{\lambda(e_2)} Z^{\gamma}_{v,w}|_{\mathcal{C}} W^{\gamma_2}_{v_2,w_2}|_{\mathcal{C}}. \tag{4}$$

Proof. We note that the transform $z \mapsto R^q_{e_2}.(z - q(v_2))$ will preserve distance under $\|\cdot\|^2$ in $\mathbb{C}^2$. It follows $(G, f_{e_1,e_2}(q,M))$ will be an equivalent framework to (G,q,M) where the edge e_2 has been fixed with v_2 at the origin and v_2 on the y-axis, then f_{e_1,e_2} is well-defined. Clearly f_{e_1,e_2} is regular. If we define f_{e_2,e_1} in a similar way, then f_{e_2,e_1} is the inverse of f_{e_1,e_2}, thus f_{e_1,e_2} is biregular.

Since f_{e_1,e_2} is biregular, $\mathcal{C}'$ will be an algebraic curve. Equation (4) now holds by direct computation. □

Proposition 3.11. *Let (G,p,L) be a k-periodic framework in $\mathbb{R}^2$, $e_1, e_2 \in E(G)$ with $e_1 = (v_1,w_1,\gamma_1)$ and $e_2 = (v_2,w_2,\gamma_2)$, and $\mathcal{C} \subset \mathcal{V}_{e_1}(G,p,L)$. If δ is an active NBAC-colouring of $\mathcal{C}$ then there exists an algebraic curve $\mathcal{C}' \subset \mathcal{V}_{e_2}(G,p,L)$ such that δ is an active NBAC-colouring of an algebraic curve $\mathcal{C}'$.*

Proof. Let $\mathcal{C}' := f_{e_1,e_2}(\mathcal{C})$, where f_{e_1,e_2} is the map defined in Lemma 3.10. Let ν be the valuation of $\mathcal{C}$ and $\alpha \in \mathbb{R}$ be chosen so that they generate δ. Define ν' to be the the valuation of $\mathcal{C}'$ where $\nu'(f) := \nu(f \circ f_{e_1,e_2})$ for each $f \in \mathbb{C}(\mathcal{C}')$. By Lemma 3.10,

$$\nu'\left(W^{\gamma}_{v,w}|_{\mathcal{C}'}\right) = \nu\left(W^{\gamma}_{v,w}|_{\mathcal{C}'} \circ f_{e_1,e_2}\right) = \nu\left(\frac{1}{\lambda(e_2)} W^{\gamma}_{v,w}|_{\mathcal{C}} Z^{\gamma_2}_{v_2,w_2}|_{\mathcal{C}}\right) = \nu\left(W^{\gamma}_{v,w}|_{\mathcal{C}}\right) + \nu\left(Z^{\gamma_2}_{v_2,w_2}|_{\mathcal{C}}\right).$$

If we define $\alpha' := \alpha + \nu\left(Z^{\gamma_2}_{v_2,w_2}|_{\mathcal{C}}\right)$, then ν' and α' will generate δ. □

Lemma 3.12. *Let (G,p,L) and (G',p',L) be gain equivalent frameworks with gain equivalence $\phi : \mathcal{V}^d_{\mathbb{K}}(G) \to \mathcal{V}^d_{\mathbb{K}}(G')$. If $\tilde{e} \in E(G)$ and $\tilde{e}' := \phi(\tilde{e})$, then ϕ is a biregular map with $\phi(\mathcal{V}_{\tilde{e}}(G,p,L)) = \mathcal{V}_{\tilde{e}}(G',p',L)$. Further, for any algebraic curve $\mathcal{C} \subset \mathcal{V}_{e_1}(G,p,L)$ and any $v,w \in V(G)$, $\gamma \in \mathbb{Z}^k$, we have that $\mathcal{C}' := \phi(\mathcal{C})$ is an algebraic curve and*

$$W^{\gamma}_{v,w}|_{\mathcal{C}'} \circ \phi = W^{\gamma}_{v,w}|_{\mathcal{C}}, \quad Z^{\gamma}_{v,w}|_{\mathcal{C}'} \circ \phi = Z^{\gamma}_{v,w}|_{\mathcal{C}}. \tag{5}$$

Proof. As ϕ is the restriction of an invertible linear map and it is bijective then it is a biregular map; it follows that $\phi(\mathcal{C})$ is an algebraic curve. Equation (5) now follows by direct computation. □

Proposition 3.13. *Let G and G' be gain equivalent $\mathbb{Z}^k$-gain graphs. Then δ is an active NBAC-colouring of G if and only if δ is an active NBAC-colouring of G'.*

Proof. Let δ be an active NBAC-colouring of $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G,p,L)$ generated by the valuation ν of $\mathbb{C}(\mathcal{C})$ and $\alpha \in \mathbb{R}$. Let ϕ be the gain equivalence from G to G'. We define the gain equivalent framework $(G',p',L) := \phi(G,p,L)$, the algebraic curve $\mathcal{C}' := \mathcal{C}$ (Lemma 3.12), and the valuation ν' of $\mathcal{C}'$ where $\nu'(f) := \nu(f \circ \phi)$ for each $f \in \mathbb{C}(\mathcal{C}')$. By Lemma 3.12,

$$\nu'\left(W^{\gamma}_{v,w}|_{\mathcal{C}'}\right) = \nu\left(W^{\gamma}_{v,w}|_{\mathcal{C}'} \circ \phi\right) = \nu\left(W^{\gamma}_{v,w}|_{\mathcal{C}'}\right),$$

thus ν' and α generate δ for G'. □

3.4. **Key results.** We are now ready to outline the key results that shall help us throughout the rest of the paper.

Lemma 3.14. *Let (G,p,L) be a k-periodic framework in $\mathbb{R}^2$. Then the following holds:*

(i) If (G,p,L) is flexible, there exists an algebraic curve $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G,p,L)$.

(ii) If (G,p,L) is fixed lattice flexible, there exists an algebraic curve $\mathcal{C} \subset \mathcal{V}^f_{\tilde{e}}(G,p,L)$.

Proof. (i): If (G,p) is flexible then $\mathcal{V}_{\tilde{e}}(G,p,L)$ cannot be finite. As every algebraic set that is not finite contains a variety with positive dimension and every variety with positive dimension contains an algebraic curve, the result holds.

(ii): This follows from a similar method. $\square$

Lemma 3.15. *Let $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G,p,L)$ be an algebraic curve and suppose G contains a spanning tree T that contains $\tilde{e}$ and has trivial gain for all of its edges. If $\operatorname{rank}(G) = k$ then there exists $(v,w,\gamma) \in E(G)$ such that $W^\gamma_{v,w}$ takes an infinite amount of values on $\mathcal{C}$.*

Proof. Suppose that for each $(v,w,\gamma) \in E(G)$, $W^\gamma_{v,w}$ takes a finite amount of values on $\mathcal{C}$. By Lemma 2.2, each map $W^\gamma_{v,w}$ is constant of $\mathcal{C}$; it follows that each map $Z^\gamma_{v,w}$ is also constant $\mathcal{C}$. Choose any two vertices $v, w \in V(G)$ with $v \neq w$, then there exists a unique walk $v_1, \ldots, v_n$ from v to w in T. As

$$W^0_{v,w} = \sum_{j=1}^{n-1} W^0_{v_j,v_{j+1}}, \qquad Z^0_{v,w} = \sum_{j=1}^{n-1} Z^0_{v_j,v_{j+1}},$$

both $W^0_{v,w}$ and $Z^0_{v,w}$ are constant on $\mathcal{C}$; further, as

$$(x_v - x_w) = \frac{1}{2}(W^0_{v,w} + Z^0_{v,w}), \qquad (y_v - y_w) = \frac{i}{2}(Z^0_{v,w} - W^0_{v,w}),$$

then $(x_v - x_w)$ and $(y_v - y_w)$ are also constant on $\mathcal{C}$. Since $x_{\tilde{v}}, y_{\tilde{w}}, x_{\tilde{w}}, y_{\tilde{v}}$ are constant on $\mathcal{C}$ then each x_v, y_v is constant on $\mathcal{C}$ also.

Suppose $k = 1$, then there exists an edge $e = (v,w,\gamma)$ such that $\gamma \neq 0$. By observing the maps $W^\gamma_{v,w}$ and $Z^\gamma_{v,w}$, we note that x_1 and y_1 are constant on $\mathcal{C}$ (since x_v, x_w, y_v, y_w are all constant on $\mathcal{C}$). It now follows $\mathcal{C}$ is a single point, contradicting that $\dim \mathcal{C} > 0$.

Now suppose $k = 2$. As $\operatorname{rank}(G) = k$, there exists edges (v,w,γ) and (v',w',γ') such that γ, γ' are independent. By observing the maps $W^\gamma_{v,w}, Z^\gamma_{v,w}, W^{\gamma'}_{v',w'}, Z^{\gamma'}_{v',w'}$, we note that the polynomials

$$\begin{aligned} f &:= \gamma_1 x_1 + \gamma_2 x_2, \qquad g := \gamma_1 y_1 + \gamma_2 y_2 \\ f' &:= \gamma'_1 x_1 + \gamma'_2 x_2, \qquad g' := \gamma_1 y'_1 + \gamma_2 y'_2 \end{aligned}$$

are constant on $\mathcal{C}$. As both x_1 and x_2 can be formed by linear combinations of f, f' then both are constant on $\mathcal{C}$; similarly, as both y_1 and y_2 can be formed by linear combinations of g, g' then both are also constant on $\mathcal{C}$. It now follows $\mathcal{C}$ is a single point, contradicting that $\dim \mathcal{C} > 0$. $\square$

Lemma 3.16. *Let (G,p,L) be full k-periodic framework in $\mathbb{R}^2$, $\operatorname{rank}(G) = k$ for $k \in \{1,2\}$ and $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G,p,L)$ an algebraic curve. Suppose G contains a balanced spanning tree T with $\tilde{e} \in T$, and there exists $a := (a_1,a_2,\alpha) \in E(G)$ such that $W^\alpha_{a_1,a_2}$ takes an infinite amount of values on $\mathcal{C}$. Then there exists a valuation ν of $\mathbb{C}(\mathcal{C})$ such that the colouring $\delta : E(G) \to \{red, blue\}$ given by*

$$\delta(e) := \begin{cases} \textit{red if } \nu(W^\gamma_{v,w}) > 0, \\ \textit{blue if } \nu(W^\gamma_{v,w}) \leq 0. \end{cases}$$

for each $e = (v,w,\gamma)$, is an NBAC-colouring of G; further, $\delta(\tilde{e}) = blue$ and $\delta(a) = red$.

Proof. By Lemma 2.2, $W^\alpha_{a_1,a_2}$ is transcendental over $\mathbb{C}$, thus by Proposition 2.4, there exists a valuation ν of $\mathbb{C}(\mathcal{C})$ such that $\nu(W^\alpha_{a_1,a_2}) > 0$. As $\tilde{e}$ is fixed and $\lambda(\tilde{e}) \neq 0$, $\nu(W^{\tilde{\gamma}}_{\tilde{v},\tilde{w}}) = 0$. As $W^\gamma_{v,w} Z^\gamma_{v,w}$ is constant for each $(v,w,\gamma) \in E(G)$ then $\nu(W^\gamma_{v,w}) = -\nu(Z^\gamma_{v,w})$. Let $\delta : E(G) \to \{\text{red}, \text{blue}\}$ be the corresponding colouring. It follows that a is red and $\tilde{e}$ is blue, thus δ is surjective.

Suppose there exists a balanced almost red circuit $(e_1, \ldots, e_n)$ in G with $e_j := (v_j, v_{j+1}, \gamma_j)$, $v_{n+1} = v_1$ and $\delta(e_n) = \text{blue}$, then

$$\nu(W^{\gamma_n}_{v_1,v_n}) = \nu\left(\sum_{j=1}^{n-1} W^{\gamma_j}_{v_j,v_{j+1}}\right) \geq \min\left\{\nu(W^{\gamma_j}_{v_j,v_{j+1}}) : j = 1, \ldots, n-1\right\} > 0,$$

however this contradicts that $\nu(W^{\gamma_n}_{v_1,v_n}) \leq 0$. Now suppose instead that $(e_1, \ldots, e_n)$ is a balanced almost blue circuit and $\delta(e_n) = \text{red}$, then

$$\nu(Z^{\gamma_n}_{v_1,v_n}) = \nu\left(\sum_{j=1}^{n-1} Z^{\gamma_j}_{v_j,v_{j+1}}\right) \geq \min\left\{\nu(Z^{\gamma_j}_{v_j,v_{j+1}}) : j = 1, \ldots, n-1\right\} \geq 0,$$

however this contradicts that $\nu(Z^{\gamma_n}_{v_1,v_n}) < 0$. □

Definition 3.17. For any two edges e_1, e_2 of a k-periodic framework (G, p, L) in $\mathbb{R}^2$ with $e_i := (v_i, w_i, \gamma_i)$ for each $i \in \{1, 2\}$, we define the *angle function of* e_1, e_2 to be the map

$$A_{e_1,e_2} : \mathcal{V}^2_{\mathcal{C}}(G) \to \mathbb{C}, \; (p', L') \mapsto (p'(v_1) - p'(w_1) - L'.\gamma_1).(p'(v_2) - p'(w_2) - L'.\gamma_2).$$

Remark 3.18. We note that for any $\tilde{e} \in E(G)$ and any algebraic curve $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$,

$$A_{e_1,e_2}|_{\mathcal{C}} = W^{\gamma_1}_{v_1,w_1} Z^{\gamma_2}_{v_2,w_2} + Z^{\gamma_1}_{v_1,w_1} W^{\gamma_2}_{v_2,w_2}.$$

Further, if $(p, L) \sim (p', L')$, then $A_{e_1,e_2}(p, L) = A_{e_1,e_2}(p', L')$; this is as linear isometries of $(\mathbb{C}^2, \|\cdot\|^2)$ will preserve the bilinear form associated to $\|\cdot\|^2$.

Lemma 3.19. *Let (G, p, L) be k-periodic framework in $\mathbb{R}^2$ for $k \in \{1, 2\}$, $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$ be an algebraic curve and $e_1, e_2 \in E(G)$, with $e_j := (v_j, w_j, \gamma_j)$ for each $j \in \{1, 2\}$. If $\delta(e_1) = \delta(e_2)$ for all active NBAC-colourings of $\mathcal{C}$, then $A_{e_1,e_2}|_{\mathcal{C}}$ is constant.*

Proof. As A_{e_1,e_2} is invariant for congruent placement-lattices, by Proposition 3.11, we may assume $\tilde{e} = e_1$. We note the map

$$(p', L') \mapsto p'(v_1) - p'(w_1) - L'.\gamma_1 \tag{6}$$

is constant on $\mathcal{C}$, and $W^{\gamma_1}_{v_1,w_1}$ is constant also. Suppose $A^{\mathcal{C}}_{e_1,e_2}|_{\mathcal{C}}$ is not constant, then as (6) is constant,

$$(p', L') \mapsto p'(v_2) - p'(w_2) - L'.\gamma_2$$

is not constant on $\mathcal{C}$; this implies that $W^{\gamma_2}_{v_2,w_2}$ takes an infinite amount of values over $\mathcal{C}$. By Lemma 2.2, $W^{\gamma_2}_{v_2,w_2}$ is transcendental over $\mathbb{C}$, thus by Proposition 2.4, there exists a valuation ν of $\mathbb{C}(\mathcal{C})$ such that $\nu(W^{\gamma_2}_{v_2,w_2}) > 0$. As $W^{\gamma_1}_{v_1,w_1}$ is constant and non-zero (since (G, p, L) is a framework and $e_1 \in E(G)$) then $\nu(W^{\gamma_1}_{v_1,w_1}) = 0$. Following the method of the proof of Lemma 3.16, we may define an active NBAC-colouring δ of $\mathcal{C}$ with $\delta(e_1) \neq \delta(e_2)$. □

Lemma 3.20. *Let (G, p, L) be k-periodic framework in $\mathbb{R}^2$ for $k \in \{1, 2\}$, and $\tilde{e}, e_1, e_2 \in E(G)$. If A_{e_1,e_2} is not constant on $\mathcal{V}_{\tilde{e}}(G, p, L)$ then there exists an algebraic curve $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$ such that $A_{e_1,e_2}|_{\mathcal{C}}$ is not constant.*

Proof. As A_{e_1,e_2} is not constant on $\mathcal{V}_{\tilde{e}}(G, p, L)$ then there exists a point $(p', L') \in \mathcal{V}_{\tilde{e}}(G, p, L)$ such that $A_{e_1,e_2}(p', L') \neq A_{e_1,e_2}(p, L)$. By [11, Lemma pg. 56], there exists an algebraic curve $\mathcal{C}$ that contains (p, L) and (p', L') as required. □

Proposition 3.21. *Let (G, p, L) be k-periodic framework in $\mathbb{R}^2$ for $k \in \{1, 2\}$, and $\tilde{e}, e_1, e_2 \in E(G)$. Then $\delta(e_1) = \delta(e_2)$ for all active NBAC-colourings δ of (G, p, L) if and only if A_{e_1,e_2} is constant on $\mathcal{V}_{\tilde{e}}(G, p, L)$.*

Proof. Suppose $\delta(e_1) = \delta(e_2)$ for all active NBAC-colourings δ of (G, p, L). By Lemma 3.19, $A_{e_1,e_2}|_{\mathcal{C}}$ is constant for any algebraic curve $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$. By Lemma 3.20, it follows that A_{e_1,e_2} is constant on $\mathcal{V}_{\tilde{e}}(G, p, L)$.

Suppose there exists an algebraic curve $\mathcal{C}$ and active NBAC-colouring δ of $\mathcal{C}$ generated by ν, α, such that $\delta(e_1) \neq \delta(e_2)$. Let $e_j = (v_j, w_j, \gamma_j)$ for $j \in \{1, 2\}$, then without loss of generality we may assume $\nu(W^{\gamma_1}_{v_1,w_1}) \leq \alpha < \nu(W^{\gamma_2}_{v_2,w_2})$. We now note

$$\nu(A_{e_1,e_2}|_{\mathcal{C}}) = \nu(W^{\gamma_1}_{v_1,w_1} Z^{\gamma_2}_{v_2,w_2} + Z^{\gamma_1}_{v_1,w_1} W^{\gamma_2}_{v_2,w_2}) = \nu(W^{\gamma_1}_{v_1,w_1}) - \nu(W^{\gamma_2}_{v_2,w_2}) < 0,$$

thus $A_{e_1,e_2}|_{\mathcal{C}}$ is not constant. It follows immediately that A_{e_1,e_2} is not constant on $\mathcal{V}_{\tilde{e}}(G, p, L)$ as required. □

We shall end this section by defining a graph operation we shall use later in both Lemma 5.3 and Lemma 6.5.

Definition 3.22. Let (G, p, L) be a k-periodic framework in $\mathbb{R}^2$. We define a k-periodic framework (G', p', L) in $\mathbb{R}^2$ to be a *Hennenberg 1 extension of* (G, p, L) *at* v_1, v_2 *by* γ_1, γ_2 if

$$V(G') := V(G) \cup \{v_0\}, \qquad E(G') := E(G) \cup \{(v_0, v_i, \gamma_i) : i = 1, 2\}$$

and $p'(v) = p(v)$ for all $v \in V(G)$. If $v_1 = v_2$, we define (G', p', L) to be a *Hennenberg 1 extension of* (G, p, L) *at* v_1 *by* γ_1, γ_2 (Figure 9).

FIGURE 9. (Left): A Henneberg 1 extension of (G, p, L) at v_1, v_2 by γ_1, γ_2. (Right): A Henneberg 1 extension of (G, p, L) at v_1 by γ_1, γ_2.

Lemma 3.23. *Let* (G, p, L) *be a* k*-periodic framework in* $\mathbb{R}^2$ *with flex* (p_t, L_t)*,* $t \in [0, 1]$*. Assume that* $\|L_t.\gamma\| \neq 0$ *for all* $t \in [0, 1]$*. Then there exists a Henneberg 1 extension* (G', p', L) *of* (G, p, L) *at* v_1 *by* $0, \gamma$ *with non-trivial flex* (p'_t, L_t) *such that* p'_t *restricted to* $V(G)$ *is the placement* p_t *for each* $t \in [0, 1]$*.*

We remember that the function atan2 : $\mathbb{R}^2 \setminus \{(0, 0)\} \to (-\pi, \pi]$ is given by

$$\text{atan2}(y, x) := \begin{cases} \arctan(y/x) & \text{if } x > 0 \\ \arctan(y/x) + \pi & \text{if } x < 0, y \geq 0 \\ \arctan(y/x) - \pi & \text{if } x < 0, y < 0 \\ \frac{\pi}{2} & \text{if } x = 0, y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0, y < 0. \end{cases}$$

Proof. Define $(x_t, y_t) := L_t.\gamma$. As $[0, 1]$ is compact, we may choose

$$r \geq \frac{1}{2}\sqrt{x_t^2 + y_t^2}. \tag{7}$$

We now define the continuous function $f : [0, 1] \to \mathbb{R}/2\pi\mathbb{Z}$, where

$$f(t) := \cos^{-1}\left(-\frac{1}{2r}\sqrt{x_t^2 + y_t^2}\right) + \text{atan2}(y_t, x_t).$$

As x_t and y_t cannot both be equal to 0 for any $t \in [0,1]$ (since $\|L_t.\gamma\| \neq 0$ for all $t \in [0,1]$), and we chose r such that (7) holds then $f(t)$ is well-defined. We note that

$$x_t \cos f(t) + y_t \sin f(t) = \sqrt{x_t^2 + y_t^2} \cos\left(f(t) - \operatorname{atan2}(y_t, x_t)\right) = \frac{-(x_t^2 + y_t^2)}{2r},$$

thus by some rearranging we have

$$(x_t + r\cos f(t))^2 + (y_t + r \sin f(t))^2 = r^2.$$

Define for each $t \in [0,1]$ the placement-lattice (p'_t, L_t) of G', with $p'_t(v) = p_t(v)$ for all $v \in V(G)$ and $p'_t(v_0) = p_t(v_1) + (r\cos f(t), r \sin f(t))$. It follows that for all $t \in [0,1]$,

$$\|p'_t(v_0) - p'_t(v_1)\|^2 = \|p'_t(v_0) - p'_t(v_1) - L_t.\gamma\|^2 = r^2,$$

and so we define $(G', p', L) := (G', p'_0, L_0)$ as required. □

4. Characterising flexible fixed lattice frameworks

For this section we shall prove the following result.

Theorem 4.1. *Let G be a connected $\mathbb{Z}^k$-gain graph for $k \in \{1,2\}$. Then there exists a placement-lattice (p,L) of G in $\mathbb{R}^2$ such that (G,p,L) is a fixed lattice flexible full k-periodic framework if and only if either:*

(i) G has a fixed lattice NBAC-colouring,
(ii) G is balanced.

We shall first need to prove three results; a pair of necessity lemmas that prove a fixed-lattice flexible k-periodic framework will have a G has a fixed lattice NBAC-colouring or have an unbalanced graph (see Lemma 4.3 for $k=1$ and Lemma 4.6 for $k=2$), and a construction lemma to prove that we can construct a fixed-lattice flexible framework given either a balanced graph or a graph with a flexible 1-lattice NBAC-colouring (see Lemma 5.6).

4.1. Necessary conditions for fixed lattice flexibility.

Lemma 4.2. *Let (G,p,L) be full 1-periodic framework in $\mathbb{R}^2$ where G is connected and unbalanced, and let $\mathcal{C} \subset \mathcal{V}^f_{\bar{e}}(G,p,L)$ be an algebraic curve. Then every active NBAC-colouring of $\mathcal{C}$ is a fixed lattice NBAC-colouring.*

Proof. Let δ be an active NBAC-colouring of $\mathcal{C}$ generated by the valuation ν and $\alpha \in \mathbb{R}$. As $\mathcal{C} \subset \mathcal{V}^f_{\bar{e}}(G,p,L)$, $W_1 Z_1 = \|L.1\|^2$. Since $W_1 Z_1$ is constant, then $\nu(W_1) = -\nu(Z_1)$. We shall assume $\nu(W_1) > \alpha$ as the proof for the case $\nu(W_1) \leq \alpha$ follows from a similar method.

Suppose there exists an almost red circuit $C := (e_1, \ldots, e_n)$ in G with $e_j := (v_j, v_{j+1}, \gamma_j)$, $v_{n+1} = v_1$ and $\delta(e_n) = $ blue. As δ is an NBAC-colouring, we must have that $\gamma := \psi(C) \neq 0$. It then follows

$$\nu(W^{\gamma_n}_{v_1,v_n}) = \nu\left(\sum_{j=1}^{n-1} W^{\gamma_j}_{v_j,v_{j+1}} + \gamma W_1\right) \geq \min\left\{\nu(W^{\gamma_j}_{v_j,v_{j+1}}), \nu(W_1) : j = 1, \ldots, n-1\right\} > \alpha,$$

however this contradicts that $\nu(W^{\gamma_n}_{v_1,v_n}) \leq \alpha$.

Now suppose there exists an unbalanced blue circuit $C := (e_1, \ldots, e_n)$ in G with $\gamma := \psi(C)$, $e_j := (v_j, v_{j+1}, \gamma_j)$ and $v_{n+1} = v_1$. We note

$$\nu(-\gamma Z_1) = \nu\left(\sum_{j=1}^{n} Z^{\gamma_j}_{v_j,v_{j+1}}\right) \geq \min\{\nu(Z^{\gamma_j}_{v_j,v_{j+1}}) : j = 1, \ldots, n-1\} \geq \alpha,$$

contradicting that $\nu(Z_1) < \alpha$. □

We are now ready to prove our first necessity lemma.

Lemma 4.3. *Let (G,p,L) be full 1-periodic framework in $\mathbb{R}^2$. If (G,p,L) is fixed lattice flexible then either G has an active fixed lattice NBAC-colouring, G is balanced or G is disconnected.*

Proof. Suppose G is unbalanced and connected. It follows from Proposition 2.8 that we may assume G contains a spanning tree T where every edge has trivial gain and $\tilde{e} \in T$, since by Proposition 3.13, if an equivalent graph to G has an active NBAC-colouring then so does G. By Lemma 3.14 (ii), there exists an algebraic curve $\mathcal{C} \subset \mathcal{V}^f_{\tilde{e}}(G,p,L)$. By Lemma 3.15, there exists $a := (a_1, a_2, \alpha) \in E(G)$ such that $W^{\alpha}_{a_1,a_2}$ is not constant on $\mathcal{C}$. By Lemma 3.16, there exists an active NBAC-colouring δ of $\mathcal{C}$, thus by Lemma 4.2, δ is a fixed lattice NBAC-colouring as required. □

Lemma 4.4. *Let (G,p,L) be a full 2-periodic framework in $\mathbb{R}^2$, $\mathcal{C} \subset \mathcal{V}^f_{\tilde{e}}(G,p,L)$ be an algebraic curve and suppose the function field $\mathbb{C}(\mathcal{C})$ has valuation ν. Then the following holds:*

(i) $\nu(W_1) = -\nu(Z_1)$, $\nu(W_2) = -\nu(Z_2)$ and $\nu(W_j.Z_l + W_k.Z_l) = 0$.
(ii) $\nu(W_1) = \nu(W_2)$ and $\nu(Z_1) = \nu(Z_2)$.
(iii) For all $\gamma \in \mathbb{Z}^2$, $\nu(\gamma Z) = -\nu(\gamma W)$.
(iv) For any $\gamma \in \mathbb{Z}^2$ and $\alpha \in \mathbb{R}$, if $\nu(W_1) > \alpha$, then $\nu(\gamma W) > \alpha$, and if $\nu(W_1) \leq \alpha$, then $\nu(\gamma W) \leq \alpha$.

Proof. (i): As $\mathcal{C} \subset \mathcal{V}^f_{\tilde{e}}(G,p,L)$ then

$$W_1 Z_1 = \lambda(1,1)^2, \quad W_2 Z_2 = \lambda(2,2)^2, \quad W_j.Z_l + W_k.Z_l = 2\lambda(1,2)^2,$$

thus all are non-zero and constant. Since $\nu(f) = 0$ for all non-zero and constant $f \in \mathbb{C}(\mathcal{C})$, the result follows.

(ii): For any $1 \leq j, l \leq k$,

$$\nu(W_j.Z_l + W_k.Z_l) \geq \min\{\nu(W_j) - \nu(W_l), \nu(W_l) - \nu(W_j)\}$$

with equality if $\nu(W_j) \neq \nu(W_l)$. If $\nu(W_j) \neq \nu(W_l)$, then $\nu(W_j.Z_l + W_k.Z_l) < 0$, contradicting that $\nu(W_j.Z_l + W_k.Z_l) = 0$ (as $\mathcal{C} \subset \mathcal{V}^f_{\tilde{e}}(G,p,L)$), thus $\nu(W_j) = \nu(W_l)$.

(iii): Let $\gamma := (\gamma_1, \gamma_2)$ and define

$$\begin{aligned} g &:= (\gamma_1 W_1 + \gamma_2 W_2)(\gamma_1 Z_1 + \gamma_2 Z_2)) \\ &= \gamma_1^2 W_1 Z_1 + \gamma_2^2 W_2 Z_2 + \gamma_1\gamma_2(W_1 Z_2 + W_2 Z_1) \\ &= (\gamma_1 x_1 + \gamma_2 x_2)^2 + (\gamma_1 y_1 + \gamma_2 y_2)^2. \end{aligned}$$

As W_1Z_1, W_2Z_2 and $W_1Z_2 + W_2Z_1$ are all constant (since $\mathcal{C} \subset \mathcal{V}^f_{\tilde{e}}(G,p,L)$), then g is constant. We further note that $g = 0$ if and only if the vectors (x_1, y_1) and (x_2, y_2) are linearly dependent for all points in $\mathcal{C}$. As this would contradict that (G,p,L) is full, $\nu(g) = 0$. The result now follows.

(iv): Let $\gamma := (\gamma_1, \gamma_2)$. By (i) and (ii), $\nu(W_1) = \nu(W_2)$. If $\nu(W_1) > \alpha$, then

$$\nu(\gamma_1 W_1 + \gamma_2 W_2) \geq \min\{\nu(W_1), \nu(W_2)\} > \alpha,$$

while if $\nu(W_1) \leq \alpha$, then by (iii),

$$\nu(\gamma_1 W_1 + \gamma_2 W_2) = -\nu(\gamma_1 Z_1 + \gamma_2 Z_2) \leq -\min\{\nu(Z_1), \nu(Z_2)\} = \max\{\nu(W_1), \nu(W_2)\} \leq \alpha.$$

□

Lemma 4.5. *Let (G,p,L) be full 2-periodic framework in $\mathbb{R}^2$ where G is a connected graph and* $\operatorname{rank}(G) = 2$, *and let $\mathcal{C} \subset \mathcal{V}^f_{\tilde{e}}(G,p,L)$ be an algebraic curve. Then every active NBAC-colouring of $\mathcal{C}$ is a fixed lattice NBAC-colouring.*

Proof. Let δ be an active NBAC-colouring of $\mathcal{C}$ with corresponding valuation ν and non-zero $\alpha \in \mathbb{R}$. By Lemma 4.4 (i) and Lemma 4.4 (ii), $\nu(W_1) = \nu(W_2)$, $\nu(Z_1) = -\nu(W_1)$ and $\nu(Z_2) = -\nu(W_2)$. We shall assume $\nu(W_1) > \alpha$ as the proof for the case $\nu(W_1) \leq \alpha$ follows from a similar method.

Suppose there exists an almost red circuit $(e_1, \ldots, e_n)$ in G with $e_j := (v_j, v_{j+1}, \gamma_j)$, $v_{n+1} = v_1$, $\gamma := \psi(C)$ and $\delta(e_n) = \text{blue}$, then,

$$W^{\gamma_n}_{v_1,v_n} = \sum_{j=1}^{n-1} W^{\gamma_j}_{v_j,v_{j+1}} + \gamma W$$

By Lemma 4.4 (iv),

$$\nu(W^{\gamma_n}_{v_1,v_n}) \geq \min\left\{\nu(W^{\gamma_j}_{v_j,v_{j+1}}), \gamma W : j = 1, \ldots, n-1\right\} > \alpha,$$

however this contradicts that $\nu(W^{\gamma_n}_{v_1,v_n}) \leq \alpha$.

Now suppose there exists an unbalanced blue circuit $C := (e_1, \ldots, e_n)$ in G with $e_j := (v_j, v_{j+1}, \gamma_j)$, $v_{n+1} = v_1$ and $\gamma := \psi(C)$. We note

$$\nu(-\gamma Z) = \nu\left(\sum_{j=1}^{n} Z^{\gamma_j}_{v_j,v_{j+1}}\right) \geq \min\{\nu(Z^{\gamma_j}_{v_j,v_{j+1}}) : j = 1, \ldots, n-1\} \geq \alpha.$$

By Lemma 4.4 (iii) and Lemma 4.4 (iv), $\nu(-\gamma Z) < \alpha$, a contradiction. □

We are now ready to prove our final necessity lemma.

Lemma 4.6. *Let (G, p, L) be full 2-periodic framework in $\mathbb{R}^2$. If (G, p, L) is fixed lattice flexible then either G has an active fixed lattice NBAC-colouring, G is balanced or G is disconnected.*

Proof. Suppose $\operatorname{rank} G = 1$ and G is connected. We note that any 2-periodic framework with rank 1 is fixed lattice flexible if and only if it is fixed lattice flexible when considered as a 1-periodic framework. By Lemma 4.3, G has an active fixed lattice NBAC-colouring.

Suppose $\operatorname{rank} G = 2$ and G is connected. It follows from Proposition 2.8 and Proposition 3.13 that we may assume G contains a spanning tree T where every edge has trivial gain and $\tilde{e} \in T$. By Lemma 3.14 (ii), there exists an algebraic curve $\mathcal{C} \subset \mathcal{V}^f_{\tilde{e}}(G, p, L)$. By Lemma 3.15, there exists $a := (a_1, a_2, \alpha) \in E(G)$ such that $W^{\alpha}_{a_1,a_2}$ is not constant on $\mathcal{C}$. By Lemma 3.16, there exists an active NBAC-colouring δ of $\mathcal{C}$, then by Lemma 4.5, δ is a fixed lattice NBAC-colouring as required. □

4.2. **Constructing fixed lattice flexible frameworks.**

Lemma 4.7. *Let G be a connected $\mathbb{Z}^k$-gain graph for $k \in \{1, 2\}$. If G has a fixed lattice NBAC-colouring δ then there exists a full placement-lattice (p, L) of G in $\mathbb{R}^2$ such that (G, p, L) is fixed lattice flexible.*

Proof. The proof for $k = 1$ is identical to that for $k = 2$ except we have $L := [c\ 0]^T$ for some irrational $c > 0$. Due to this, we shall only prove the case for $k = 2$.

We may assume without loss of generality that G^{δ}_{blue} is unbalanced and G^{δ}_{blue} is balanced; further, by Proposition 2.8, we may assume all edges of G^{δ}_{red} have trivial gain. Let $R_1, \ldots, R_n$ be the red connected components and $B_1, \ldots, B_m$ be the blue connected components. As δ is a NBAC-colouring, there exists a blue edge $\tilde{e} \in E(G)$; by reordering the blue components we may assume the end points of $\tilde{e}$ lie in B_1.

Choose any two irrational points $c_1, c_2 > 0$. We define the placement-lattice (p, L) of G with

$$p(v) := (x, y), \qquad L := \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$$

for $v \in V(R_x) \cap V(B_y)$.

We shall now prove (G, p, L) is a well-defined k-periodic framework. Suppose there exists a red edge $e := (v, w, \gamma) \in E(G)$ such that $p(v) = p(w) + L.\gamma$. As e is red then $\gamma = (0, 0)$, thus $p(v) = p(w)$. It follows that for some $1 \leq x \leq n$ and $1 \leq y \leq m$, we have $v, w \in V(R_x) \cap V(B_y)$, thus there exists a blue path $(e_1, \dots, e_n)$ that starts at w and ends at v. We note, however, that $(e_1, \dots, e_n, e)$ is an almost blue circuit, contradicting that δ is a fixed-lattice NBAC-colouring.

Now suppose exists a blue edge $e := (v, w, \gamma) \in E(G)$ with $\gamma = (\gamma_1, \gamma_2)$ such that $p(v) = p(w) + L.\gamma$, then $p(v) = p(w) + (\gamma_1 c_1, \gamma_2 c_2)$. As both c_1, c_2 are irrational then $\gamma_1, \gamma_2 = 0$, thus $p(v) = p(w)$. This implies that for some $1 \leq x \leq n$ and $1 \leq y \leq m$, we have $v, w \in V(R_x) \cap V(B_y)$, and there exists a red path $(e_1, \dots, e_n)$ that starts at w and ends at v. We note, however, that $(e_1, \dots, e_n, e)$ is a balanced almost red circuit (since all red edges have trivial gain), contradicting that δ is an NBAC-colouring. It now follows (G, p, L) is a full k-periodic framework.

Define the motion (p_t, L_t), $t \in [0, 1]$, where for $p(v) = (x, y)$,

$$p_t(v) := (x + y \sin t, y \cos t),$$

and $L_t = L$. Choose any $t \in [0, 1]$ and $e := (v, w, \gamma) \in E(G)$, with $\gamma = (\gamma_1, \gamma_2)$, $p(v) = (x, y)$ and $p(w) = (x', y')$. Suppose $\delta(e) = \text{red}$, then $x' = x$, and $\gamma = (0, 0)$ (as all red edges have trivial gain); it follows then that

$$\|p_t(v) - p_t(w) - L_t.\gamma\|^2 = ((y - y')\sin t)^2 + ((y - y')\cos t)^2 = (y - y')^2.$$

Now suppose $\delta(e) = \text{blue}$, then $y' = y$. We now note

$$\|p_t(v) - p_t(w) - L_t.\gamma\|^2 = (x - x' + \gamma_1 c_1)^2 + (\gamma_2 c_2)^2.$$

It follows that $(G, p_t, L_t) \sim (G, p, L)$ for all $t \in [0, 1]$, thus (p_t, L_t) is a fixed lattice flex of (G, p, L). As the edge $\tilde{e}$ is fixed then (p_t, L_t) is non-trivial, thus (G, p, L) is fixed lattice flexible as required. We refer the reader to Figure 10 for an example of the construction described. $\square$

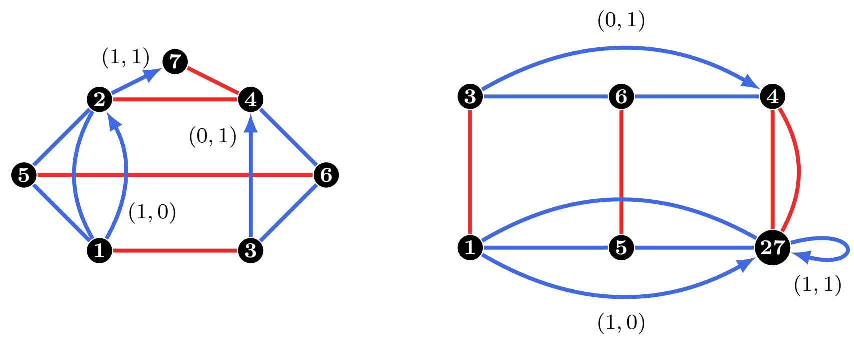

FIGURE 10. (Left): A $\mathbb{Z}^2$-gain graph G with a fixed lattice NBAC-colouring. (Right): The constructed full 2-periodic framework (G, p, L) in $\mathbb{R}^2$. We note that even though we place (2) and (7) at the same point in $\mathbb{R}^2$, $p(2) \neq p(7) + L.(1, 1)$.

Lemma 4.8. *Let G be a $\mathbb{Z}^k$-gain graph for $k \in \{1, 2\}$. If G is balanced then there exists a full placement-lattice (p, L) of G in $\mathbb{R}^2$ such that (G, p, L) is fixed lattice flexible.*

Proof. By Proposition 2.8, we may assume every edge of G has trivial gain. Choose any injective map p and any full lattice L. We may now define the fixed lattice flex (p_t, L_t) for $t \in [0, 2\pi]$, where $p_t = p$ and

$$L_t = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} L.$$

$\square$

We may now combined the results of this section to prove Theorem 4.1

Proof of Theorem 4.1. If (G, p, L) is a fixed lattice flexible full k-periodic framework, by Lemma 4.3 if $k = 1$ or Lemma 4.6 if $k = 2$, either G has a fixed lattice NBAC-colouring or G is balanced.

If G has a fixed lattice NBAC-colouring, by Lemma 4.7, there exists a fixed lattice flexible full k-periodic framework (G, p, L) in $\mathbb{R}^2$.

If G is balanced, by Lemma 4.8, there exists a fixed lattice flexible full k-periodic framework (G, p, L) in $\mathbb{R}^2$. □

5. Characterising flexible 1-periodic frameworks

In this section we shall prove the following theorem.

Theorem 5.1. *Let G be a connected $\mathbb{Z}$-gain graph. Then there exists a full placement-lattice (p, L) of G in $\mathbb{R}^2$ such that (G, p, L) is a flexible full 1-periodic framework if and only if either:*
(i) G has a fixed lattice NBAC-colouring,
(ii) G has a flexible 1-lattice NBAC-colouring,
(iii) G is balanced.

Fortunately, much of the required work has been dealt with in Section 4, since fixed lattice flexible 1-periodic frameworks are a subclass of flexible 1-periodic frameworks. We shall only need to prove two results; a necessity lemma that proves a flexible 1-periodic framework will have one of the required characteristics (see Lemma 5.2), and a construction lemma to prove that we can construct a flexible 1-periodic framework given a graph with a flexible 1-lattice NBAC-colouring (see Lemma 5.6).

5.1. Necessary conditions for 1-periodic flexibility.

Lemma 5.2. *Let (G, p, L) be a 1-periodic framework in $\mathbb{R}^2$ with edge $(v, w, \gamma) \in E(G)$ for some $\gamma \neq 0$, $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$ be an algebraic curve and ν a valuation of $\mathcal{C}$. Suppose $x_v - x_w$ and $y_v - y_w$ are constant on $\mathcal{C}$, then $W^{\gamma}_{v,w}$ is constant if and only if $\mathcal{C} \subset \mathcal{V}^f_{\tilde{e}}(G, p, L)$.*

Proof. We note that $W^{\gamma}_{v,w}$ is constant if and only if $Z^{\gamma}_{v,w}$ is also constant as $W^{\gamma}_{v,w} Z^{\gamma}_{v,w}$ is constant. As $x_v - x_w$ and $y_v - y_w$ are constant then $W^{\gamma}_{v,w}$ and $Z^{\gamma}_{v,w}$ are constant if and only if both $x_1 + iy_1$ and $x_1 - iy_1$ are constant, which in turn is equivalent to both x_1, y_1 being constant. The result now follows. □

We are now ready for our necessity lemma.

Lemma 5.3. *Let (G, p, L) be a full 1-periodic framework in $\mathbb{R}^2$. If (G, p, L) is flexible then G either has an active fixed lattice NBAC-colouring, an active flexible 1-lattice NBAC-colouring, G is balanced, or G is disconnected.*

Proof. Suppose G is connected and unbalanced, then we have two possibilities.

(G contains a pair of parallel edges): By our choice of $\tilde{e}$, we may assume $\tilde{e}$ and $\tilde{f}$ are the pair of parallel edges on $\tilde{v}, \tilde{w}$, with $\psi(\tilde{f}) = \mu \neq 0$. It follows from Proposition 2.8 and Proposition 3.13 that we may assume G contains a spanning tree T where every edge has trivial gain and $\tilde{e} \in T$. By Lemma 3.14 (ii), there exists an algebraic curve $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$.

Suppose $\mathcal{C} \subset \mathcal{V}^f_{\tilde{e}}(G, p, L)$, then by Lemma 4.3, G has an active fixed lattice NBAC-colouring δ. By Lemma 5.2, we note that we must have $\delta(\tilde{e}) = \delta(\tilde{f})$.

Now suppose $\mathcal{C} \not\subset \mathcal{V}^f_{\tilde{e}}(G, p, L)$. By Lemma 5.2, $W^{\tilde{f}}_{\tilde{v},\tilde{w}}$ is not constant on $\mathbb{C}(\mathcal{C})$. Let ν the valuation of $\mathbb{C}(\mathcal{C})$ and δ the NBAC-colouring given by Lemma 3.16 with $a := \tilde{f}$. By our choice of valuation, $\nu(W^0_{\tilde{v},\tilde{w}}) = 0$ and $\nu(W^{\mu}_{\tilde{v},\tilde{w}}) > 0$; it follows immediately that $\nu(Z^0_{\tilde{v},\tilde{w}}) = 0$ and $\nu(Z^{\mu}_{\tilde{v},\tilde{w}}) < 0$ as both $W^0_{\tilde{v},\tilde{w}} Z^0_{\tilde{v},\tilde{w}}$ and $W^{\mu}_{\tilde{v},\tilde{w}} Z^{\mu}_{\tilde{v},\tilde{w}}$ are constant. As $\mu W_1 = W^0_{\tilde{v},\tilde{w}} - W^{\mu}_{\tilde{v},\tilde{w}}$ then $\nu(W_1) = \nu(W^0_{\tilde{v},\tilde{w}}) = 0$. Similarly, as $\mu Z_1 = Z^0_{\tilde{v},\tilde{w}} - Z^{\mu}_{\tilde{v},\tilde{w}}$ then $\nu(Z_1) = \nu(Z^{\mu}_{\tilde{v},\tilde{w}}) < 0$.

Suppose G has an unbalanced monochromatic circuit $C := (e_1, \ldots, e_n)$, where for each j we have $e_j = (v_j, v_{j+1}, \gamma_j)$ and $v_{n+1} = v_1$. If C is red, then

$$\nu(W_1) = \nu(-\psi(C)W_1) = \nu\left(\sum_{j=1}^{n} W^{\gamma_j}_{v_j,v_{j+1}}\right) \geq \min\left\{\nu(W^{\gamma_j}_{v_j,v_{j+1}}) : 1 \leq j \leq n\right\} > 0,$$

contradicting that $\nu(W_1) = 0$. If C is blue, then

$$\nu(Z_1) = \nu(-\psi(C)Z_1) = \nu\left(\sum_{j=1}^{n} Z^{\gamma_j}_{v_j,v_{j+1}}\right) \geq \min\{\nu(Z^{\gamma_j}_{v_j,v_{j+1}}) : 1 \leq j \leq n\} \geq 0,$$

contradicting that $\nu(Z_1) < 0$. It now follows that δ is an active flexible 1-lattice NBAC-colouring.

(G does not contain a pair of parallel edges): By Lemma 3.23, there exists a Henneberg 1 extension (G', p', L) of (G, p, L) at v_1 by $0, 1$ such that (G', p', L) is flexible and $\mathcal{V}^f_{\tilde{e}}(G', p', L) \neq \mathcal{V}_{\tilde{e}}(G', p', L)$; we shall define these new edges by $\tilde{e}, \tilde{f}$, with $\psi(\tilde{e}) = 0$ and $\psi(\tilde{f}) = 1$. As G' contains a double edge, either G' has an active flexible 1-lattice NBAC-colouring δ' with $\delta'(\tilde{e}) = \text{blue}$ and $\delta'(\tilde{f}) = \text{red}$, or G' has an active fixed lattice lattice NBAC-colouring δ'' with $\delta''(\tilde{e}) = \delta''(\tilde{f}) = \text{blue}$.

Suppose G' has a colouring δ' as described above. Let δ be the colouring of G with $\delta(e) := \delta'(e)$ for all $e \in E(G)$. We note that δ is an active flexible 1-lattice NBAC-colouring if and only if δ' is not monochromatic on the subgraph G of G'. As G is unbalanced, δ' cannot be monochromatic on G, thus is an active flexible 1-lattice NBAC-colouring of G.

Now suppose G' has a colouring δ'' as described above. Let δ be the colouring of G with $\delta(e) := \delta''(e)$ for all $e \in E(G)$. We note that δ is an active fixed lattice NBAC-colouring if and only if δ' is not monochromatic on the subgraph G of G'. If δ' is monochromatic on G, then as $\delta'(\tilde{e}) = \delta'(\tilde{f}) = \text{blue}$ and G is unbalanced, we must have $\delta(G) = \text{blue}$, however this would contradict that $\delta'(G') = \{\text{red}, \text{blue}\}$. It now follows that δ is an active fixed lattice NBAC-colouring of G. $\square$

5.2. **Constructing flexible frameworks from flexible 1-lattice NBAC-colourings.**

Lemma 5.4. *Let G be a $\mathbb{Z}$-gain graph with a flexible 1-lattice NBAC-colouring. Then there exists $G' \approx G$ such that each blue edge has trivial gain and no red edge has trivial gain.*

Proof. As G^{δ}_{blue} is balanced, by Proposition 2.8, we may suppose all blue edges of G have trivial gain. Let $B_1, \ldots, B_n$ be the blue components of G and choose $\mu \in \mathbb{N}$ such that $\mu > |\gamma|$ for all $(v, w, \gamma) \in E(G)$. We now define

$$G' := \left(\prod_{i=1}^{n} \prod_{v \in B_i} \phi^{i\mu}_v\right)(G).$$

We first note that any blue edge of G' will have trivial gain since both of its ends will lie in the same blue component. Choose a red edge $(v, w, \gamma) \in E(G)$ and suppose $v \in B_i$ and $w \in B_j$. We note that

$$\left(\prod_{i=1}^{n} \prod_{v \in B_i} \phi^{i\mu}_v\right)(v, w, \gamma) = \phi^{i\mu}_v \circ \phi^{j\mu}_w(v, w, \gamma) = (v, w, \gamma + (i - j)\mu).$$

As $\mu > |\gamma|$ and $i - j \in \mathbb{Z}$ then $\gamma + (i - j)\mu = 0$ if and only if $\gamma = 0$ and $i = j$. If $v, w \in B_i$ and $\gamma = 0$ then there would exist a balanced almost blue circuit as v, w are connected by a blue path and all blue edges of G have trivial gain, thus $\gamma + (i - j)\mu \neq 0$ as required. $\square$

Lemma 5.5. *Let H be a balanced $\mathbb{Z}$-gain graph. Then there exists a placement q of H in $\mathbb{Z}$ such that for all $(v,w,\gamma) \in E(H)$, $q(w) - q(v) = 2\gamma$.*

Proof. We may suppose without loss of generality that H is connected. Choose a spanning tree T of H. It is immediate that we may choose a placement q of T that satisfies the condition $q(w) - q(v) = 2\gamma$ for all $(v,w,\gamma) \in E(T)$. Choose an edge $e = (a,b,\mu) \in E(H) \setminus E(T)$, then there exists a path $(e_1,\ldots,e_{n-1})$ in T with $e_i = (v_i, v_{i+1}, \gamma_i)$, $v_1 = b$ and $v_n = a$. As H is balanced, $\psi(e_1,\ldots,e_{n-1}) = -\mu$, thus by our choice of q,

$$q(b) - q(a) = \left(\sum_{i=1}^{n-1} q(v_{i+1}) - q(v_i)\right) = -2\psi(e_1,\ldots,e_{n-1}) = 2\mu.$$

□

We our now ready to prove our construction lemma.

Lemma 5.6. *Let G be a $\mathbb{Z}$-gain graph with a flexible 1-lattice NBAC-colouring δ. Then there exists a full placement-lattice (p,L) of G in $\mathbb{R}^2$ such that (G,p,L) is flexible full 1-periodic framework.*

Proof. By Lemma 5.4, we may assume all blue edges of G have trivial gain and all red edges have non-trivial gain. Let $R_1,\ldots,R_n$ be the red components of G and define E_j to be the set of edges (v,w,γ) in G^δ_{red} with $v,w \in R_j$. By Lemma 5.5, for each R_j there exists a placement q_j in $\mathbb{R}$ where $q_j(w) - q_j(v) = 2\gamma$ for all $(v,w,\gamma) \in E_j$. We now define for each $t \in [0,2\pi]$ the full placement-lattice (p_t, L_t) of G in $\mathbb{R}^2$, with

$$p_t(v) := (q_j(v), j), \qquad L_t.1 := (-2 + \cos t, \sin t)$$

for $v \in R_j$ and $t \in [0,2\pi]$. We shall denote $(p,L) := (p_0, L_0)$.

To see that (p,L) is a well-defined placement-lattice, choose any $e = (v,w,\gamma)$ and suppose that $p(v) = p(w) + L.\gamma$. It follows $v,w \in R_j$ and $q_j(v) - q_j(w) = \gamma$. Suppose $\delta(e) = \text{red}$, then $\gamma \neq 0$. By our choice of q_j we have $q_j(v) - q_j(w) = -2\gamma$, a contradiction. Now suppose $\delta(e) = \text{blue}$, then $\gamma = 0$. As $v,w \in R_j$, there exists a red path $(e_1,\ldots,e_{n-1})$ with $e_j = (v_j, v_{j+1}, \gamma_j) \in E_j$, $v_1 = w$ and $v_n = v$. Since $q_j(v) = q_j(w)$, $\sum_{j=1}^{n-1} \gamma_j = 0$, however we note that $(e_1,\ldots,e_{n-1},e)$ is a balanced almost red circuit, contradicting that δ is a NBAC-colouring.

Choose any $e = (v,w,\gamma)$. If $\delta(e) = \text{blue}$ then $\gamma = 0$. As $p_t = p$ then for each $t \in [0,2\pi]$,

$$\|p_t(v) - p_t(w) - L_t.\gamma\|^2 = \|p(v) - p(v)\|^2.$$

If $\delta(e) = \text{red}$ then $v,w \in R_j$, thus for each $t \in [0,2\pi]$,

$$\|p_t(v) - p_t(w) - L_t.\gamma\|^2 = (-(q_j(w) - q_j(v)) + 2\gamma - \gamma\cos t)^2 + (\gamma \sin t)^2 = \gamma^2.$$

It follows that (p_t, L_t) is a flex of (G,p,L) as required. We refer the reader to Figure 11 for an example of the construction. □

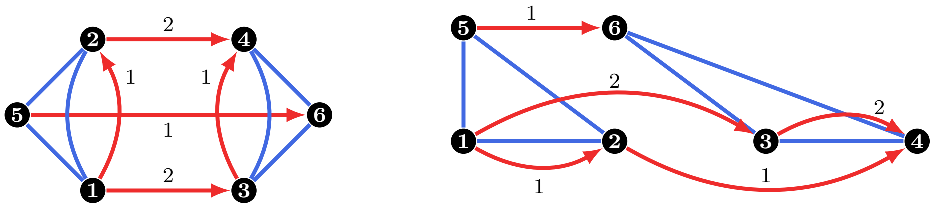

FIGURE 11. (Left): A $\mathbb{Z}$-gain graph with a flexible 1-lattice NBAC-colouring. (Right): The constructed full 1-periodic framework in $\mathbb{R}^2$.

We are now ready to prove the main theorem of this section.

Proof of Theorem 5.1. Suppose (G, p, L) is flexible. By Lemma 5.3, either G is balanced, G has a fixed lattice NBAC-colouring, or G has a flexible 1-periodic NBAC-colouring.

If G is balanced, then by Lemma 4.8, G has a flexible full placement-lattice in $\mathbb{R}^2$.

If G has a fixed lattice NBAC-colouring, then by Lemma 4.7, G has a flexible full placement-lattice in $\mathbb{R}^2$.

If G has a flexible 1-lattice NBAC-colouring, then by Lemma 5.6, G has a flexible full placement-lattice in $\mathbb{R}^2$. □

6. Characterising flexible 2-periodic frameworks

Unlike with 1-periodic frameworks, a full characterisation of $\mathbb{Z}^2$-gain graphs with flexible 2-periodic full placements in the plane via NBAC-colourings is unknown. We would conjecture the following.

Conjecture 6.1. *Let G be a connected $\mathbb{Z}^2$-gain graph. Then there exists a full placement-lattice (p, L) of G in $\mathbb{R}^2$ such that (G, p, L) is a flexible full 2-periodic framework if and only if either:*

(i) G has a type 1 flexible 2-lattice NBAC-colouring,

(ii) G has a type 2 flexible 2-lattice NBAC-colouring,

(iii) G has a type 3 flexible 2-lattice NBAC-colouring,

(iv) G has a fixed lattice NBAC-colouring,

(v) $\operatorname{rank}(G) < 2$.

We are able to obtain the required necessity lemma and most of the required construction lemmas, however a construction of a flexible full 2-periodic framework from a type 3 flexible 2-lattice NBAC-colouring is currently unknown. In this section we shall, however, outline some partial results regarding $\mathbb{Z}^2$-gain graphs, in particular, Lemma 6.5, Lemma 6.6, Lemma 6.9 and Lemma 6.12. We shall discuss some other possible conjectures at the end of the section, and later in Section 7 we shall obtain analogues of Theorem 5.1 for certain types of graphs; see Theorem 7.5 and Theorem 7.8.

6.1. Necessary conditions for 2-periodic flexibility.

For any $\gamma = (a, b) \in \mathbb{Z}^2$, we remember the notation $\gamma W := aW_1 + bW_2$ and $\gamma Z := aZ_1 + bZ_2$.

Lemma 6.2. *Let (G, p, L) be a 2-periodic framework in $\mathbb{R}^2$ with edge $(v, w, \gamma) \in E(G)$ for some $\gamma = (\gamma_1, \gamma_2) \neq (0, 0)$, $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$ be an algebraic curve and ν a valuation of $\mathcal{C}$. Suppose $x_v - x_w$ and $y_v - y_w$ are constant on $\mathcal{C}$, then $W^{\gamma}_{v,w}$ is constant if and only if*

$$(\gamma_1 x_1 + \gamma_2 x_2)^2 + (\gamma_1 y_1 + \gamma_2 y_2)^2$$

is constant.

Proof. We note that $W^{\gamma}_{v,w}$ is constant if and only if $Z^{\gamma}_{v,w}$ is also constant as $W^{\gamma}_{v,w} Z^{\gamma}_{v,w}$ is constant. As $x_v - x_w$ and $y_v - y_w$ are constant then $W^{\gamma}_{v,w}$ and $Z^{\gamma}_{v,w}$ are constant if and only if both $(\gamma_1 x_1 + \gamma_2 x_2) + i(\gamma_1 y_1 + \gamma_2 y_2)$ and $(\gamma_1 x_1 + \gamma_2 x_2) - i(\gamma_1 y_1 + \gamma_2 y_2)$ are constant, which in turn is equivalent to both $(\gamma_1 x_1 + \gamma_2 x_2)$, $(\gamma_1 y_1 + \gamma_2 y_2)$ being constant. The result now follows. □

Lemma 6.3. *Let (G, p, L) be a full 2-periodic framework in $\mathbb{R}^2$ and $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$ be an algebraic curve. Suppose the function field $\mathbb{C}(\mathcal{C})$ has valuation ν and for some $\mu \in \mathbb{Z}^2$,*

$$\nu(\mu W) = 0, \qquad \nu(\mu Z) < 0.$$

Then one of the following cases holds:

(i) For all $\gamma \in \mathbb{Z}^2 \setminus \{(0, 0)\}$,

$$\nu(\gamma W) \leq 0, \qquad \nu(\gamma Z) < 0.$$

(ii) There exists $\alpha, \beta \in \mathbb{Z}^2$ with at least one non-zero such that for all $\gamma \in \mathbb{Z}^2 \setminus (\mathbb{Z}\alpha \cup \mathbb{Z}\beta)$,

$$\nu(\gamma W) \leq 0, \qquad \nu(\gamma Z) < 0,$$

for all $\gamma \in \mathbb{Z}\alpha \setminus \{(0,0)\}$,

$$\nu(\gamma W) > 0, \qquad \nu(\gamma Z) < 0,$$

and for all $\gamma \in \mathbb{Z}\beta \setminus \{(0,0)\}$,

$$\nu(\gamma W) \leq 0, \qquad \nu(\gamma Z) \geq 0,$$

(iii) There exists $\alpha \in \mathbb{Z}^2 \setminus \{(0,0)\}$ such that for all $\gamma \in \mathbb{Z}^2 \setminus \mathbb{Z}\alpha$,

$$\nu(\gamma W) \leq 0, \qquad \nu(\gamma Z) < 0,$$

and for all $\gamma \in \mathbb{Z}\alpha \setminus \{(0,0)\}$,

$$\nu(\gamma W) > 0, \qquad \nu(\gamma Z) \geq 0.$$

Proof. Choose $\lambda \in \mathbb{Z}^2$ such that μ and λ are linearly independent.

If $\nu(\mu W) \neq \nu(\lambda W)$, then we note that for all $\gamma \in \mathbb{Z}^2 \setminus \{(0,0)\}$ with $\gamma = a\mu + b\lambda$,

$$\nu(\gamma W) = \nu((a\mu + b\lambda)W) = \min\{\nu(\mu W), \nu(\lambda W)\} \leq 0;$$

similarly, if $\nu(\mu Z) \neq \nu(\lambda Z)$, then $\nu(\gamma Z) < 0$ for all $\gamma \in \mathbb{Z}^2$.

If $\nu(\mu W) = \nu(\lambda W)$, then there can exist $\alpha \in \mathbb{Z}^2 \setminus \{(0,0)\}$ that is pairwise independent of μ, λ such that $\nu(\alpha W) > 0$. We note that α is unique up to scalar multiplication, as if there exists $\gamma \in \mathbb{Z}^2 \setminus \mathbb{Z}\alpha$ such that $\nu(\gamma W) > 0$ also, then we may choose $A, B \in \mathbb{R}$ such that $A\alpha + B\gamma = \mu$, and note that

$$\nu(\mu W) \geq \min\{\nu(\alpha W), \nu(\gamma W)\} > 0,$$

contradicting that $\nu(\mu W) = 0$. Likewise, if $\nu(\mu Z) = \nu(\lambda Z)$, then there can exist at most one $\beta \in \mathbb{Z}^2 \setminus \{0\}$ such that $\nu(\alpha W) > 0$.

We now check the cases:

- Suppose $\nu(\mu W) \neq \nu(\lambda W)$ and $\nu(\mu Z) \neq \nu(\lambda Z)$.
  - Case (i) holds if $\nu(\lambda W), \nu(\lambda Z) < 0$.
  - Case (ii) holds if $\nu(\lambda W) < 0 < \nu(\lambda Z)$ or $\nu(\lambda Z) < 0 < \nu(\lambda W)$.
  - Case (iii) holds if $\nu(\lambda W), \nu(\lambda Z) > 0$.
- Suppose $\nu(\mu W) = \nu(\lambda W)$ and $\nu(\mu Z) \neq \nu(\lambda Z)$.
  - Case (i) holds if holds if α doesn't exist and $\nu(\mu Z) < 0$.
  - Case (ii) holds otherwise.
- Suppose $\nu(\mu W) \neq \nu(\lambda W)$ and $\nu(\mu Z) = \nu(\lambda Z)$.
  - Case (i) holds if holds if $\nu(\mu W) < 0$ and β doesn't exist.
  - Case (ii) holds otherwise.
- Suppose $\nu(\mu W) = \nu(\lambda W)$ and $\nu(\mu Z) = \nu(\lambda Z)$.
  - Case (i) holds if α, β don't exist.
  - Case (ii) holds if α exists and β doesn't exist, α doesn't exist and β exists, or if α, β exist and $\alpha \neq \beta$.
  - Case (iii) holds if α, β exist and $\alpha = \beta$.

□

Lemma 6.4. *Let (G, p, L) be a full 2-periodic framework in $\mathbb{R}^2$ and $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$ is an algebraic curve. Further, suppose G contains a pair of parallel edges (v, w, γ) and (v, w, γ') such that $\gamma - \gamma' = (\lambda_1, \lambda_2)$.*

$$(\lambda_1 x_1 + \lambda_2 x_2)^2 + (\lambda_1 y_1 + \lambda_2 y_2)^2$$

is not constant on $\mathcal{C}$. Then one of the following holds:

(i) G has an active type 1 flexible 2-lattice NBAC-colouring,

(ii) G has an active type 2 flexible 2-lattice NBAC-colouring,
(iii) G has an active type 3 flexible 2-lattice NBAC-colouring.

Proof. By our choice of $\tilde{e}$, we may assume $\tilde{e}$ and $\tilde{f}$ are the pair of parallel edges on $\tilde{v}, \tilde{w}$, with $\psi(\tilde{f}) = \mu$ for some $\mu = (\mu_1, \mu_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}$. It follows from Proposition 2.8 and Proposition 3.13 that we also may assume G contains a spanning tree T where every edge has trivial gain and $\tilde{e} \in T$. Since μ is the difference in gains of $\tilde{e}, \tilde{f}$, then

$$(\mu_1 x_1 + \mu_2 x_2)^2 + (\mu_1 y_1 + \mu_2 y_2)^2$$

is not constant. By Lemma 6.2, $W^{\mu}_{\tilde{v},\tilde{w}}$ is not constant on $\mathbb{C}(\mathcal{C})$. Let ν be the valuation of $\mathbb{C}(\mathcal{C})$ and δ be the active NBAC-colouring given by Lemma 3.16 with $a := \tilde{f}$.

We note that $\nu(W^0_{\tilde{v},\tilde{w}}) = 0$ and $\nu(W^{\mu}_{\tilde{v},\tilde{w}}) > 0$. As $\mu W = W^0_{\tilde{v},\tilde{w}} - W^{\mu}_{\tilde{v},\tilde{w}}$ then

$$\nu\left(\mu W\right) = \nu\left(W^0_{\tilde{v},\tilde{w}}\right) = 0.$$

Similarly, as $\mu Z = Z^0_{\tilde{v},\tilde{w}} - Z^{\mu}_{\tilde{v},\tilde{w}}$ then

$$\nu\left(\mu Z\right) = \nu\left(Z^{\mu}_{\tilde{v},\tilde{w}}\right) < 0.$$

Let case (i), case (ii) and case (iii) refer to the three possibilities given by Lemma 6.3. We shall now proceed to prove that case (i) implies G has a type 1 flexible 2-lattice NBAC-colouring, case (ii) implies G has either a type 1 or type 2 flexible 2-lattice NBAC-colouring, and case (iii) implies G has either a type 1, type 2 or a type 3 flexible 2-lattice NBAC-colouring.

(Case (i) holds): Suppose G has an unbalanced monochromatic circuit $C := (e_1, \ldots, e_n)$, where for each j we have $e_j = (v_j, v_{j+1}, \gamma_j)$ and $v_{n+1} = v_1$, and define $\gamma := \psi(C)$. If C is red, then

$$\nu(\gamma W) = \nu\left(-\sum_{j=1}^{n} W^{\gamma_j}_{v_j,v_{j+1}}\right) \geq \min\{\nu(W^{\gamma_j}_{v_j,v_{j+1}}) : 1 \leq j \leq n\} > 0,$$

contradicting that $\nu(\gamma W) \leq 0$. If C is blue, then

$$\nu(\gamma Z) = \nu\left(-\sum_{j=1}^{n} Z^{\gamma_j}_{v_j,v_{j+1}}\right) \geq \min\{\nu(Z^{\gamma_j}_{v_j,v_{j+1}}) : 1 \leq j \leq n\} \geq 0,$$

contradicting that $\nu(\gamma Z) < 0$. It now follows that δ is a type 1 flexible 2-lattice NBAC-colouring.

(Case (ii) holds): Let $C := (e_1, \ldots, e_n)$ be an unbalanced monochromatic circuit with $\gamma := \psi(C)$, where for each j we have $e_j = (v_j, v_{j+1}, \gamma_j)$ and $v_{n+1} = v_1$. Suppose C is red and $\gamma \notin \mathbb{Z}\alpha$, then

$$\nu(\gamma W) = \nu\left(-\sum_{j=1}^{n} W^{\gamma_j}_{v_j,v_{j+1}}\right) \geq \min\{\nu(W^{\gamma_j}_{v_j,v_{j+1}}) : 1 \leq j \leq n\} > 0,$$

contradicting that $\nu(\gamma W) \leq 0$. Likewise, if C is blue and $\gamma \notin \mathbb{Z}\beta$, then

$$\nu(\gamma Z) = \nu\left(-\sum_{j=1}^{n} Z^{\gamma_j}_{v_j,v_{j+1}}\right) \geq \min\{\nu(Z^{\gamma_j}_{v_j,v_{j+1}}) : 1 \leq j \leq n\} \geq 0,$$

contradicting that $\nu(\gamma Z) < 0$.

Now let $C := (e_1, \ldots, e_n)$ be an almost monochromatic circuit where for each j we have $e_j = (v_j, v_{j+1}, \gamma_j)$, $v_{n+1} = v_1$, and $\delta(e_n) \neq \delta(e_i)$ for all $i \in \{1, \ldots, n-1\}$. If C is almost red and

$\psi(C) = c\alpha$ for some $c \in \mathbb{Z}$, then,

$$\nu(W^{\gamma_n}_{v_1,v_n}) = \nu\left(\sum_{j=1}^{n-1} W^{\gamma_j}_{v_j,v_{j+1}} + c\alpha W\right) \geq \min\{\nu(\alpha W), \nu(W^{\gamma_j}_{v_j,v_{j+1}}) : 1 \leq j \leq n\} > 0,$$

contradicting that $\nu(W^{\gamma_n}_{v_1,v_n}) \leq 0$. Similarly, if C is almost blue and $\psi(C) = c\beta$ for some $c \in \mathbb{Z}$, then,

$$\nu(Z^{\gamma_n}_{v_1,v_n}) = \nu\left(\sum_{j=1}^{n-1} Z^{\gamma_j}_{v_j,v_{j+1}} + c\beta Z\right) \geq \min\{\nu(\beta Z), \nu(Z^{\gamma_j}_{v_j,v_{j+1}}) : 1 \leq j \leq n\} > 0,$$

contradicting that $\nu(Z^{\gamma_n}_{v_1,v_n}) \leq 0$.

It now follows that if G has no monochromatic circuits then δ is a type 1 flexible 2-lattice NBAC-colouring, and if G has a monochromatic circuit then δ is a type 2 flexible 2-lattice NBAC-colouring.

(Case (iii) holds): Let $C := (e_1, \ldots, e_n)$ be an unbalanced monochromatic circuit with $\gamma := \psi(C) \notin \mathbb{Z}\alpha$, where for each j we have $e_j = (v_j, v_{j+1}, \gamma_j)$ and $v_{n+1} = v_1$. Suppose C is red, then

$$\nu(\gamma W) = \nu\left(-\sum_{j=1}^{n} W^{\gamma_j}_{v_j,v_{j+1}}\right) \geq \min\{\nu(W^{\gamma_j}_{v_j,v_{j+1}}) : 1 \leq j \leq n\} > 0,$$

contradicting that $\nu(\gamma W) \leq 0$. Likewise, if C is blue, then

$$\nu(\gamma Z) = \nu\left(-\sum_{j=1}^{n} Z^{\gamma_j}_{v_j,v_{j+1}}\right) \geq \min\{\nu(Z^{\gamma_j}_{v_j,v_{j+1}}) : 1 \leq j \leq n\} \geq 0,$$

contradicting that $\nu(\gamma Z) < 0$.

Now let $C := (e_1, \ldots, e_n)$ be an almost monochromatic circuit where for each j we have $e_j = (v_j, v_{j+1}, \gamma_j)$, $v_{n+1} = v_1$, $\psi(C) := c\alpha$ for some $c \in \mathbb{Z}$, and $\delta(e_n) \neq \delta(e_i)$ for all $i \in \{1, \ldots, n-1\}$. If C is almost red, then,

$$\nu(W^{\gamma_n}_{v_n,v_1}) = \nu\left(-\sum_{j=1}^{n-1} W^{\gamma_j}_{v_j,v_{j+1}} + c\alpha W\right) \geq \min\{\nu(\alpha W), \nu(W^{\gamma_j}_{v_j,v_{j+1}}) : 1 \leq j \leq n\} > 0,$$

contradicting that $\nu(W^{\gamma_n}_{v_n,v_1}) \leq 0$. Similarly, if C is almost blue, then,

$$\nu(Z^{\gamma_n}_{v_n,v_1}) = \nu\left(-\sum_{j=1}^{n-1} Z^{\gamma_j}_{v_j,v_{j+1}} + c\alpha Z\right) \geq \min\{\nu(\alpha Z), \nu(Z^{\gamma_j}_{v_j,v_{j+1}}) : 1 \leq j \leq n\} > 0,$$

contradicting that $\nu(Z^{\gamma_n}_{v_n,v_1}) \leq 0$.

It now follows that if G has no monochromatic circuits then δ is a type 1 flexible 2-lattice NBAC-colouring, if G only has monochromatic circuits for a single colour then δ is a type 2 flexible 2-lattice NBAC-colouring, and if G has monochromatic circuits for both colours then δ is a type 3 flexible 2-lattice NBAC-colouring. □

We are now ready for our necessity lemma.

Lemma 6.5. *Let (G, p, L) be a full 2-periodic framework in $\mathbb{R}^2$. If (G, p, L) is flexible then one of the following holds:*

(i) G has an active type 1 flexible 2-lattice NBAC-colouring,
(ii) G has an active type 2 flexible 2-lattice NBAC-colouring,
(iii) G has an active type 3 flexible 2-lattice NBAC-colouring,
(iv) G has an active fixed lattice NBAC-colouring,

(v) $\operatorname{rank}(G) < 2$.
(vi) *G is disconnected.*

Proof. Suppose $\operatorname{rank}(G) = 2$ and G is connected. Choose any $\tilde{e} \in E(G)$, then by Lemma 3.14 (ii), there exists an algebraic curve $\mathcal{C} \subset \mathcal{V}_{\tilde{e}}(G, p, L)$. We now have three possibile outcomes:

(1) $\mathcal{C} \subset \mathcal{V}^f_{\tilde{e}}(G, p, L)$.
(2) G contains a pair of parallel edges (v, w, γ) and (v, w, γ') such that $\gamma - \gamma' = (\lambda_1, \lambda_2)$.
$$(\lambda_1 x_1 + \lambda_2 x_2)^2 + (\lambda_1 y_1 + \lambda_2 y_2)^2$$
is not constant on $\mathcal{C}$.
(3) Possibilities 1 and 2 do not hold.

(Possibility 1 holds): If $\mathcal{C} \subset \mathcal{V}^f_{\tilde{e}}(G, p, L)$ then by Lemma 4.3, G has an active fixed lattice NBAC-colouring.

(Possibility 2 holds): By Lemma 6.4, G has either an active type 1, type 2 or type 3 flexible 2-lattice NBAC-colouring.

(Possibility 3 holds): As $\mathcal{C} \not\subset \mathcal{V}^f_{\tilde{e}}(G, p, L)$, we may choose $\mu := (\mu_1, \mu_2) \in \mathbb{Z}^2$ such that
$$(\lambda_1 x_1 + \lambda_2 x_2)^2 + (\lambda_1 y_1 + \lambda_2 y_2)^2$$
is not constant. By Lemma 3.23, there exists a Henneberg 1 extension (G', p', L) of (G, p, L) at v_1 by $0, \lambda$ such that (G', p', L) is flexible and $\mathcal{V}^f_{\tilde{e}}(G', p', L) \neq \mathcal{V}_{\tilde{e}}(G', p', L)$. As Possibility 2 holds for (G', p', L), then by Lemma 6.4, G' has an active type k flexible 1-lattice NBAC-colouring δ' for some $k \in \{1, 2, 3\}$.

Suppose G' has a colouring δ' as described above. Let δ be the colouring of G with $\delta(e) := \delta'(e)$ for all $e \in E(G)$. We note that δ is an active type k' flexible 2-lattice NBAC-colouring for some $k' \in \{1, 2, 3\}$ if and only if δ' is not monochromatic on the subgraph G of G'. As $\operatorname{rank}(G) = 2$ and δ' is a type k flexible 2-lattice NBAC-colouring, δ' is not monochromatic on G, thus G has an active type k' flexible 2-lattice NBAC-colouring for some $k' \in \{1, 2, 3\}$. $\square$

6.2. **Constructing flexible frameworks: Low rank graphs.** Our first construction lemma is the simplest, as essentially the framework is not connected.

Lemma 6.6. *Let G be a $\mathbb{Z}^2$-gain graph. If $\operatorname{rank}(G) < 2$ then there exists a full placement-lattice (p, L) of G in $\mathbb{R}^2$ such that (G, p, L) is flexible.*

Proof. If $\operatorname{rank}(G) = 0$ then this holds by Lemma 4.7, so we may suppose $\operatorname{rank}(G) = 1$, i.e. $\operatorname{span}(G) = \mathbb{Z}\alpha$ for some non-zero $\alpha \in \mathbb{Z}^2$. By Proposition 2.8, we may assume every edge of G has gain in $\mathbb{Z}\alpha$. Choose any injective map p, any full lattice L, and any element $\beta \in \mathbb{Z}^2$ that is linearly independent of α. We may now define the fixed lattice flex (p_t, L_t) for $t \in [0, 2\pi]$, where $p_t = p$ and
$$L_t.\alpha := L.\alpha, \qquad L_t.\beta := (1 + t)L.\beta.$$
$\square$

6.3. **Constructing flexible frameworks: Type 1 flexible 2-lattice NBAC-colourings.** We remember that a type 1 flexible 2-lattice NBAC-colouring is a NBAC-colouring δ where all monochromatic circuits are balanced.

Lemma 6.7. *Let G be a $\mathbb{Z}^2$-gain graph with a type 1 flexible 2-lattice NBAC-colouring. Then there exists $G' \approx G$ such that each blue edge has trivial gain and no red edge has trivial gain.*

Proof. The proof follows a similiar method as Lemma 5.4. $\square$

Lemma 6.8. *Let H be a balanced $\mathbb{Z}^2$-gain graph with no multiple edges and no loops. Then there exists a placement q of H in $\mathbb{Z}^2$ such that for all $(v, w, \gamma) \in E(H)$, $q(w) - q(v) = 2\gamma$.*

Proof. The proof follows the same method as Lemma 5.5. $\square$

We are now ready for our construction lemma for type 1 flexible 2-lattice NBAC-colourings. We note that it is essentially the same as the construction given in Lemma 5.6.

Lemma 6.9. *Let G be a $\mathbb{Z}^2$-gain graph with a type 1 flexible 2-lattice NBAC-colouring δ. Then there exists a full placement-lattice (p, L) of G in $\mathbb{R}^2$ such that (G, p, L) is a flexible full 2-periodic framework.*

Proof. By Lemma 6.7, we may assume all blue edges of G have trivial gain and all red edges have non-trivial gain. Let $R_1, \ldots, R_n$ be the red components of G and define E_j to be the set of edges (v, w, γ) in G^δ_{red} with $v, w \in R_j$. By Lemma 6.8, for each R_j there exists a placement q_j in $\mathbb{R}^2$ where $q_j(w) - q_j(v) = 2\gamma$ for all $(v, w, \gamma) \in E_j$. Choose a set of points $\{x_j : 1 \leq j \leq n\} \subset \mathbb{R}^2$ such that for any blue edge $(v, w, 0) \in E(G)$ with $v \in R_j$, $w \in R_k$ and $j \neq k$, we have $q_j(v) + x_j \neq q_k(w) + x_k$. We now define for each $t \in [0, 2\pi]$ the full placement-lattice (p_t, L_t) of G in $\mathbb{R}^2$, with

$$L_t.(1, 0) := (-2 + \cos t, \sin t), \qquad L_t.(0, 1) := (\sin t, -2 - \cos t)$$

and

$$p_t(v) := q_j(v) + x_j$$

for $v \in R_j$. We shall denote $(p, L) := (p_0, L_0)$.

To see that (p, L) is a well-defined placement-lattice, choose any $e = (v, w, \gamma)$ with $\gamma = (\gamma_1, \gamma_2)$ and suppose that $p(v) = p(w) + L.\gamma$. First assume $\delta(e) = \text{red}$, then $\gamma \neq (0, 0)$ and $v, w \in R_j$. It follows that

$$-L.\gamma = q_j(w) - q_j(v) = 2\gamma.$$

However as $-L.\gamma = (\gamma_1, 3\gamma_2)$, then $-L.\gamma = 2\gamma$ if and only if $\gamma = (0, 0)$, contradicting that all red edges have non-trivial gain. Now assume $\delta(e) = \text{blue}$, then $\gamma = (0, 0)$. By our choice of the set $\{x_j : 1 \leq j \leq n\}$, we must have $v, w \in R_j$; further, as $\gamma = (0, 0)$ then $q_j(v) = q_j(w)$. Let $(e_1, \ldots, e_{n-1})$ be a red path from w to v with $e_j = (v_j, v_{j+1}, \gamma_j) \in E_j$, $v_1 = w$ and $v_n = v$. Since $q_j(v) = q_j(w)$, $\sum_{j=1}^{n-1} \gamma_j = 0$, however we note that $(e_1, \ldots, e_{n-1}, e)$ is a balanced almost red circuit, contradicting that δ is a type 1 flexible 2-lattice NBAC-colouring.

Choose any edge $e = (v, w, \gamma)$ with $\gamma = (\gamma_1, \gamma_2)$. If $\delta(e) = \text{blue}$ then $\gamma = 0$. As $p_t = p$ then for each $t \in [0, 2\pi]$,

$$\|p_t(v) - p_t(w) - L_t.\gamma\|^2 = \|p(v) - p(v) - L.\gamma\|^2.$$

If $\delta(e) = \text{red}$ then $v, w \in R_j$, thus for each $t \in [0, 2\pi]$,

$$\begin{aligned} \|p_t(v) - p_t(w) - L_t.\gamma\|^2 &= (\gamma_1 \cos t + \gamma_2 \sin t)^2 + (\gamma_1 \sin t - \gamma_2 \cos t)^2 \\ &= \gamma_1^2 + \gamma_2^2 \end{aligned}$$

It follows that (p_t, L_t) is a flex of (G, p, L) as required. We refer the reader to Figure 12 for an example of the construction. □

6.4. **Constructing flexible frameworks: Type 2 flexible 2-lattice NBAC-colourings.** We remember that a type 2 flexible 2-lattice NBAC-colouring is a NBAC-colouring δ where there exists $\alpha, \beta \in \mathbb{Z}^2$ such that

- either α, β are linearly independent or exactly one of α, β is equal to $(0, 0)$,
- $\operatorname{span}(G^\delta_{\text{red}})$ is a non-trivial subgroups of $\mathbb{Z}\alpha$ or $\alpha = (0, 0)$,
- $\operatorname{span}(G^\delta_{\text{blue}})$ is a non-trivial subgroups of $\mathbb{Z}\beta$ or $\beta = (0, 0)$,
- there are no almost red circuits with gain in $\mathbb{Z}\alpha$, and
- there are no almost blue circuits with gain in $\mathbb{Z}\beta$,

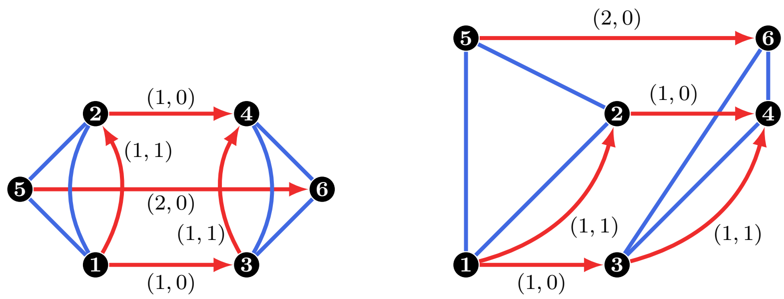

FIGURE 12. (Left): A $\mathbb{Z}^2$-gain graph with a type 1 flexible 2-lattice NBAC-colouring. (Right): The constructed full 2-periodic framework in $\mathbb{R}^2$.

Lemma 6.10. *Let G be a $\mathbb{Z}^2$-gain graph and δ a type 2 flexible 2-lattice NBAC-colouring of G with α, β as described previously. Then there exists $G' \approx G$ such that each red edge has gain $a\alpha + b\beta$ for some $a, b \in \mathbb{Z}$ with $a \neq 0$, and each blue edge has gain $c\beta$ for some $c \in \mathbb{Z}$.*

Proof. As $\operatorname{span}(G^{\delta}_{\text{blue}}) = \mathbb{Z}\beta$ is balanced, by Proposition 2.8, we may suppose all blue edges of G have gain in $\mathbb{Z}\beta$. Let $B_1, \ldots, B_n$ be the blue components of G and choose $N \in \mathbb{N}$ such that $N > |a|$ for all $(v, w, \gamma) \in E(G)$ with $\gamma = a\alpha + b\beta$. We now define the gain equivalent graph

$$G' := \left(\prod_{i=1}^{n} \prod_{v \in B_i} \phi_v^{iN\alpha + iN\beta} \right) (G).$$

We first note that any blue edge of G' will have gain in $\mathbb{Z}\beta$ since both of its ends will lie in the same blue component. Choose a red edge $(v, w, \gamma) \in E(G)$ with $\gamma = a\alpha + b\beta$ and suppose $v \in B_i$ and $w \in B_j$. We note that

$$\begin{aligned} \left(\prod_{i=1}^{n} \prod_{v \in B_i} \phi_v^{iN\alpha + iN\beta} \right) (v, w, \gamma) &= \phi_v^{iN\alpha + iN\beta} \circ \phi_w^{jN\alpha + jN\beta} (v, w, \gamma) \\ &= (v, w, (N(i-j) + a)\alpha + (N(i-j) + b)\beta). \end{aligned}$$

As $N > |a|$ and $i - j \in \mathbb{Z}$ then$(N(i-j) + a)\alpha = 0$ if and only if $a = 0$ and $i = j$. If this holds, then as $v, w \in B_i$, we can define an almost blue circuit from v to v with red edge $(v, w, b\beta)$ and gain in $\mathbb{Z}\beta$ (as every blue edge has gain in $\mathbb{Z}\beta$), contradicting that δ is a type 2 flexible 2-lattice NBAC-colouring. It now follows that $a \neq 0$ as required. $\square$

Lemma 6.11. *Let $\alpha, \beta \in \mathbb{Z}^2$ be linearly independent and let H be a $\mathbb{Z}^2$-gain graph where* $\operatorname{span}(H) \leq \mathbb{Z}\alpha$. *Then there exists a placement q of H in $\mathbb{Z}^2$ such that for all $(v, w, a\alpha + b\beta) \in E(H)$, $q(v) - q(w) = b$.*

Proof. Define the $\mathbb{Z}$-gain graph H' with vertex set $V(H') := V(H)$ and edge set

$$E(H') := \{(v, w, b\beta) : (v, w, a\alpha + b\beta) \in E(H)\};$$

we delete any loops with trivial gain that may arrive, and note that multiple edges may become a single edge. By Lemma 5.5, we may define a placement q' of H' such that $q'(v) - q'(w) = -2b$ for all $(v, w, b\beta) \in E(H')$. We now define q to be the placement of H where $q(v) := -\frac{1}{2}q'(v)$. $\square$

We are now ready for our construction lemma regarding type 2 flexible 2-lattice NBAC-colourings.

Lemma 6.12. *Let G be a $\mathbb{Z}^2$-gain graph with a type 2 flexible 2-lattice NBAC-colouring δ. Then there exists a full placement-lattice (p, L) of G in $\mathbb{R}^2$ such that (G, p, L) is a flexible full 2-periodic framework.*

Proof. Without loss of generality we may assume $\operatorname{span}(G^\delta_{\text{red}}) = \mathbb{Z}\alpha$ and $\operatorname{span}(G^\delta_{\text{blue}}) = \mathbb{Z}\beta$, with $\alpha \neq (0,0)$. If $\beta = (0,0)$ then δ is a fixed-lattice NBAC-colouring and the result holds by Lemma 4.6, thus we may assume α, β are linearly independent.

By Lemma 6.10, we may assume all red edges have gain $a\alpha + b\beta$ for some $a, b \in \mathbb{Z}$ with $a \neq 0$, and all blue edges have gain $c\beta$ for some $c \in \mathbb{Z}$. Let $R_1, \dots, R_n$ be the red components of G and define E_j to be the set of edges (v, w, γ) in G^δ_{red} with $v, w \in R_j$. By Lemma 6.11, for each R_j there exists a placement q_j in $\mathbb{R}$ where $q_j(v) - q_j(w) = b$ for all $(v, w, \gamma) \in E_j$ with $\gamma = a\alpha + b\beta$. We now define for each $t \in [0, 2\pi]$ the full placement-lattice (p_t, L_t) of G in $\mathbb{R}^2$, with

$$L_t.\alpha := (\cos t, \sin t), \qquad L_t.\beta := (1, 0)$$

and

$$p_t(v) := (q_j(v), j)$$

for $v \in R_j$. We shall denote $(p, L) := (p_0, L_0)$.

To see that (p, L) is a well-defined placement-lattice, choose any $e = (v, w, \gamma)$ and suppose that $p(v) = p(w) + L.\gamma$. Suppose $\delta(e) = \text{red}$, then $\gamma = a\alpha + b\beta$ for some $a, b \in \mathbb{Z} \setminus \{0\}$ and $v, w \in R_j$. We note

$$p(v) = (q_j(v), j) = (q_j(w) + b, j + a) = p(w) + L.\gamma,$$

which implies $a = 0$, a contradiction. Now suppose $\delta(e) = \text{blue}$, then $\gamma = c\beta$ for some $c \in \mathbb{Z}$. If $v \in R_j$ and $w \in R_k$ then

$$p(v) = (q_j(v), j) = (q_k(w) + c, k) = p(w) + L.\gamma,$$

thus $j = k$. Let $P := (e_1, \dots, e_{n-1})$ be a red path from w to v with $e_i = (v_i, v_{i+1}, \gamma_i) \in E_j$, $v_1 = w$, $v_n = v$, and $\gamma_i = a_i\alpha + c_i\beta$. Define $C := (e_1, \dots, e_{n-1}, e)$. As

$$c = q_j(v) - q_j(w) = \sum_{i=1}^{n-1} (q_j(v_{i+1}) - q_j(v_i)) = -\sum_{i=1}^{n-1} c_i,$$

then $\psi(C) = a\alpha$ for some $a \in \mathbb{Z}$. We note that the circuit $(e_1, \dots, e_{n-1}, e)$ is an almost red circuit with gain $a\alpha$, contradicting that δ is a type 2 flexible 2-lattice NBAC-colouring.

Choose any edge $e = (v, w, \gamma)$ with $\gamma = a\alpha + b\beta$. If $\delta(e) = \text{blue}$ then $a = 0$. As $p_t = p$ and $L_t.\beta = (1, 0)$ then $\|p_t(v) - p_t(w) - L_t.\gamma\|^2$ is constant. If $\delta(e) = \text{red}$ then $v, w \in R_j$, thus for each $t \in [0, 2\pi]$,

$$\|p_t(v) - p_t(w) - L_t.\gamma\|^2 = (q_j(v) - q_j(w) - b - a\cos t)^2 + (a \sin t)^2 = a^2.$$

It follows that (p_t, L_t) is a flex of (G, p, L) as required. We refer the reader to Figure 13 for an example of the construction. $\square$

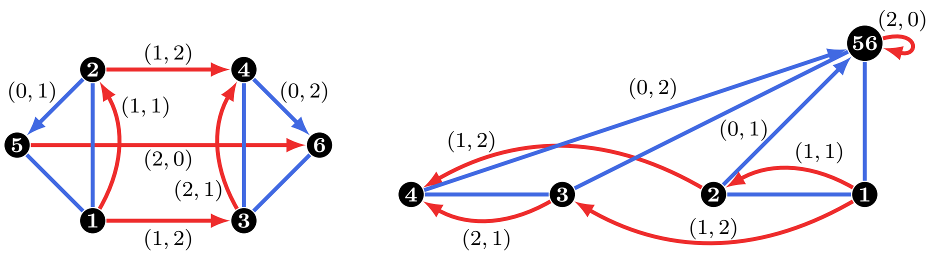

FIGURE 13. (Left): A $\mathbb{Z}^2$-gain graph with a type 2 flexible 2-lattice NBAC-colouring ($\alpha = (1, 0)$, $\beta = (0, 1)$). (Right): The constructed full 2-periodic framework in $\mathbb{R}^2$.

6.5. **Conjectures regarding type 3 flexible 2-lattice NBAC-colourings.** We remember that a NBAC-colouring δ of a $\mathbb{Z}^2$-gain graph G is a type 3 flexible 2-lattice NBAC-colouring if there exists $\alpha \in \mathbb{Z}^2 \setminus \{(0,0)\}$ such that

- $\operatorname{span}(G^\delta_{\text{red}})$ and $\operatorname{span}(G^\delta_{\text{red}})$ are non-trivial subgroups of $\mathbb{Z}\alpha$, and
- there are no almost monochromatic circuits with gain in $\mathbb{Z}\alpha$.

It is an open question to whether a graph has a flexible placement of a $\mathbb{Z}^2$-gain graph in $\mathbb{R}^2$ if it has a type 3 flexible 2-lattice NBAC-colouring. As this is the case for all other types of flexible 2-lattice NBAC-colourings, we would conjecture the following.

Conjecture 6.13. *Let G be a $\mathbb{Z}^2$-gain graph with type 3 flexible 2-lattice NBAC-colouring. Then there exists a full placement-lattice (p, L) of G in $\mathbb{R}^2$ such that (G, p, L) is a flexible full 2-periodic framework.*

All examples of $\mathbb{Z}^2$-gain graphs with a type 3 flexible 2-lattice NBAC-colouring discovered so far will also have either a type 1 or type 2 flexible 2-lattice NBAC-colouring, a fixed lattice NBAC-colouring, or have a low rank. Due to this, we conjecture the following.

Conjecture 6.14. *Let G be a $\mathbb{Z}^2$-gain graph with type 3 flexible 2-lattice NBAC-colouring. Then G has either a type 1 or type 2 flexible 2-lattice NBAC-colouring, G has a fixed lattice NBAC-colouring, or* $\operatorname{rank}(G) < 2$.

If Conjecture 6.13 is true, then by Lemma 6.5, Lemma 6.9, Lemma 6.12, Lemma 4.7, and Lemma 6.6, we also have Conjecture 6.1 would be true. If Conjecture 6.14 is true, then we obtain the slightly stronger result.

Conjecture 6.15. *Let G be a connected $\mathbb{Z}^2$-gain graph. Then there exists a full placement-lattice (p, L) of G in $\mathbb{R}^2$ such that (G, p, L) is a flexible full 2-periodic framework if and only if either:*

(i) G has a type 1 flexible 2-lattice NBAC-colouring,
(ii) G has a type 2 flexible 2-lattice NBAC-colouring,
(iii) G has a fixed lattice NBAC-colouring,
(iv) $\operatorname{rank}(G) < 2$.

7. Special cases of 2-periodic frameworks

We shall now focus on 2-periodic frameworks with loops. With this added assumption, we can fully characterise whether a $\mathbb{Z}^2$-gain graph has flexible placement-lattice by observing the graph's NBAC-colourings.

7.1. 2-periodic frameworks with loops.

Lemma 7.1. *Let (G, p, L) be a k-periodic framework in $\mathbb{K}^d$ and suppose G has a loop (w, w, α). If G' is the $\mathbb{Z}^k$-gain graph with*

$$V(G') := V(G), \quad E(G') := E(G) \cup \{(v, v, c\alpha) : v \in V(G), c \in \mathbb{N}\},$$

then,

$$\mathcal{V}_{\mathbb{K}}(G', p, L) = \mathcal{V}_{\mathbb{K}}(G, p, L)$$

Proof. It is immediate that $\mathcal{V}_{\mathbb{K}}(G', p, L) \subset \mathcal{V}_{\mathbb{K}}(G, p, L)$. Choose any placement-lattice $(p', L') \in \mathcal{V}_{\mathbb{K}}(G, p, L)$. As $(w, w, \alpha) \in E(G)$ then

$$\|L'.\alpha\|^2 = \|p'(w) - p'(w) - L'.\alpha\|^2 = \|p(w) - p(w) - L.\alpha\|^2 = \|L.\alpha\|^2.$$

We note that for any $v \in V(G)$ and non-zero $c \in \mathbb{Z}$,

$$\|p'(v) - p'(v) - L'.c\alpha\|^2 = c^2\|L'.\alpha\|^2 = c^2\|L.\alpha\|^2 = \|p(v) - p(v) - L.c\alpha\|^2,$$

thus $(p', L') \in \mathcal{V}_{\mathbb{K}}(G', p, L)$ as required. $\square$

Lemma 7.2. *Let G be a connected $\mathbb{Z}^2$-gain graph and suppose that there exists some $\alpha \in \mathbb{Z}^2 \setminus \{(0,0)\}$ such that for every vertex $v \in V(G)$ and $c \in \mathbb{N}$, $(v, v, c\alpha) \in E(G)$. If δ is a NBAC-colouring of G then every loop with gain $c\alpha$ for some $c \in \mathbb{N}$ is the same colour.*

Proof. We first note that every loop at a vertex must have the same colour; indeed, suppose there exists a loop $(v, v, c\alpha)$ for some integer $n > 1$, where $\delta(v, v, n\alpha) \neq \delta(v, v, \alpha)$, then the circuit

$$(\overbrace{(v, v, -\alpha), \dots, (v, v, -\alpha)}^{n \text{ times}}, (v, v, n\alpha))$$

is an almost monochromatic balanced circuit, contradicting that δ is a NBAC-colouring. Suppose not all loops are the same colour, then as G is connected, there exists distinct vertices $v, w \in V(G)$ connected by an edge (v, w, γ) where $\delta(v, v, \alpha) \neq \delta(w, w, \alpha)$. Without loss of generality we may assume $\delta(v, v, \alpha) = \delta(v, w, \gamma)$, then the circuit

$$((v, w, \gamma), (w, w, \alpha), (w, v, -\gamma), (v, v, -\alpha))$$

is an almost monochromatic balanced circuit, contradicting that δ is a NBAC-colouring. □

Lemma 7.3. *Let G be a connected $\mathbb{Z}^2$-gain graph and suppose that there exists some $\alpha \in \mathbb{Z}^2 \setminus \{(0,0)\}$ such that for every vertex $v \in V(G)$ and $c \in \mathbb{N}$, $(v, v, c\alpha) \in E(G)$. Then there are no type 1 or type 3 flexible 2-lattice NBAC-colourings of G.*

Proof. Suppose there does exist type 1 or type 3 flexible 2-lattice NBAC-colouring δ of G. As one of G^δ_{red} or G^δ_{blue} must contain a loop and thus be unbalanced, δ is must be a type 3 flexible 2-lattice NBAC-colouring. By Lemma 7.3, every loop of the form $(v, v, c\gamma)$ has the same colour, and without loss of generality we shall assume they are all red. We note immediately that there cannot be any unbalanced blue circuits; indeed, if C was a blue circuit from v to v with gain $c\alpha$, then the circuit formed by C followed by $(v, v, -c\alpha)$ will be an almost blue balanced circuit. However this implies $\operatorname{rank}(G^\delta_{\text{blue}}) = 0$, contradicting that δ is a type 3 flexible 2-lattice NBAC-colouring. □

Lemma 7.4. *Let G be a connected $\mathbb{Z}^2$-gain graph with a loop. Then there are no active type 1 or type 3 flexible 2-lattice NBAC-colourings of G.*

Proof. Let $(w, w, \alpha) \in E(G)$. By Lemma 7.1, we may assume that for every vertex $v \in V(G)$ and $c \in \mathbb{N}$, we have $(v, v, c\alpha) \in E(G)$. The result now follows from Lemma 7.3. □

We may now prove a special case of Conjecture 6.13.

Theorem 7.5. *Let G be a connected $\mathbb{Z}^2$-gain graph with a loop. Then there exists a full placement-lattice (p, L) of G in $\mathbb{R}^2$ such that (G, p, L) is a flexible full 2-periodic framework if and only if either:*

(i) G has an active type 2 flexible 2-lattice NBAC-colouring,
(ii) G has an active fixed lattice NBAC-colouring,
(iii) $\operatorname{rank}(G) = 1$.

Proof. Suppose there exists a full placement-lattice (p, L) of G in $\mathbb{R}^2$ such that (G, p, L) is a flexible full 2-periodic framework. Since G contains a loop then $\operatorname{rank}(G) \geq 1$. By Lemma 6.5 and Lemma 7.4, either G has an active type 2 flexible 2-lattice NBAC-colouring, G has an active fixed lattice NBAC-colouring, or $\operatorname{rank}(G) = 1$.

Now suppose that either G has an active type 2 flexible 2-lattice NBAC-colouring, G has an active fixed lattice NBAC-colouring, or $\operatorname{rank}(G) = 1$. Then by Lemma 6.12, Lemma 4.7 or Lemma 6.6, there exists a full placement-lattice (p, L) of G in $\mathbb{R}^2$ such that (G, p, L) is a flexible full 2-periodic framework. □

7.2. **Scissor flexes.** We now define a special class of flex.

Definition 7.6. Let (p_t, L_t) a flex of a 2-periodic framework (G, p, L) in $\mathbb{R}^2$. If there exists linearly independent $\alpha, \beta \in \mathbb{Z}^2$ such that $\|L_t.\alpha\|^2$ and $\|L_t.\beta\|^2$ are constant but $(L_t.\alpha).(L_t.\beta)$ is not constant, then (p_t, L_t) is a *scissor* flex.

We shall show in Theorem 7.10 that graphs that have a placement-lattice with a scissor flex are exactly those that have a type 2 flexible 2-lattice NBAC-colouring. We first prove the following lemmas.

Lemma 7.7. *Let G be a connected $\mathbb{Z}^2$-gain graph and $\alpha, \beta \in \mathbb{Z}^2$ be linearly independent. Suppose that $(v, v, c\alpha), (v, v, c\beta) \in E(G)$ for all $v \in V(G)$ and $c \in \mathbb{N}$. If δ is a NAC-colouring of G then either:*

(i) All loops of G are the same colour.
(ii) All loops of G with gain in $\mathbb{Z}\alpha$ are red (respectively, blue), all loops of G with gain in $\mathbb{Z}\beta$ are blue (respectively, red), and G has no loops with gain in $\mathbb{Z}\alpha \cup \mathbb{Z}\beta$.

Proof. Without loss of generality, by Lemma 7.2 we have two possibilities:

(1) All loops with gain in $\mathbb{Z}\alpha \cup \mathbb{Z}\beta$ are red.
(2) All loops with gain in $\mathbb{Z}\alpha$ are red and all loops with gain in $\mathbb{Z}\beta$ are blue.

Suppose (1) holds. If G has no loops with gain $\gamma \notin \mathbb{Z}\alpha \cup \mathbb{Z}\beta$ then (i) holds. Suppose G has a loop $l := (v, v, \gamma)$ with $\gamma \notin \mathbb{Z}\alpha \cup \mathbb{Z}\beta$. Choose $a, b \in \mathbb{Z}$ such that $\gamma = a\alpha + b\beta$. If $\delta(l) =$ blue then we note that the circuit

$$((v, v, -a\alpha), (v, v, -a\beta), l)$$

is an almost red balanced circuit, contradicting that δ is a NBAC-colouring, thus $\delta(l) =$ red. As l was chosen arbitrarily, (i) holds.

Suppose (2) holds but G has a loop $l := (v, v, \gamma)$ with $\gamma \notin \mathbb{Z}\alpha \cup \mathbb{Z}\beta$. Choose $a, b \in \mathbb{Z}$ such that $\gamma = a\alpha + b\beta$. If $\delta(l) =$ blue then the circuit

$$C := ((v, v, -a\alpha), (v, v, -a\beta), l)$$

is an almost blue balanced circuit, while if $\delta(l) =$ red then C is an almost red balanced circuit. As both possibilities contradict that δ is a NBAC-colouring, then no such loop may exist, thus (ii) holds. □

Lemma 7.8. *Let G be a connected $\mathbb{Z}^2$-gain graph with loops $l_\alpha := (v, v, \alpha)$, $l_\beta := (v, v, \beta)$, where α and β are linearly independent. Then all active NBAC-colourings of G with $\delta(l_\alpha) \neq \delta(l_\beta)$ are type 2 flexible 2-lattice NBAC-colourings.*

Proof. Let δ be an active NBAC-colouring of G with $\delta(l_\alpha) \neq \delta(l_\beta)$. Without loss of generality, we may assume $\delta(l_\alpha) =$ red and $\delta(l_\beta) =$ blue. By Lemma 7.1, we may assume that $(v, v, c\gamma) \in E(G)$ for all $v \in V(G)$ and $c \in \mathbb{N}$. By Lemma 7.7, all loops with gain in $\mathbb{Z}\alpha$ are red, all loops with gain in $\mathbb{Z}\beta$ are blue, and there are no loops with gain $\gamma \notin \mathbb{Z}\alpha \cup \mathbb{Z}\beta$.

Suppose there exists a red circuit C from v to v with $\psi(C) = a\alpha + b\beta$. If $a, b \neq 0$, then the circuit formed from C followed by $(v, v, -a\alpha), (v, v, -a\beta)$ is an almost red balanced circuit, while if $a = 0, b \neq 0$, then the circuit formed from C followed by $(v, v, -a\beta)$ is an almost red balanced circuit. As both contradict that δ is a NBAC-colouring of G, then $\psi(C) \in \mathbb{Z}\alpha$. We similarly note that for any blue circuit C', $\psi(C') \in \mathbb{Z}\beta$.

Let $C := (e_1, \ldots, e_n)$ be an almost monochromatic circuit where for each j we have $e_j = (v_j, v_{j+1}, \gamma_j)$, $v_{n+1} = v_1$, and $\delta(e_n) \neq \delta(e_i)$ for all $i \in \{1, \ldots, n-1\}$. If C is almost red and $\psi(C) = c\alpha$ for some $k \in \mathbb{Z}$, then

$$(e_1, \ldots, e_n, (v_1, v_1, -c\alpha))$$

is an almost red balanced circuit, contradicting that δ is a NBAC-colouring. If C is almost blue and $\psi(C) = c\beta$ for some $k \in \mathbb{Z}$, then

$$(e_1, \dots, e_n, (v_1, v_1, -c\beta))$$

is an almost blue balanced circuit, contradicting that δ is a NBAC-colouring. It now follows that δ is a type 2 flexible 2-lattice as required. □

Lemma 7.9. *Let G be a connected $\mathbb{Z}^2$-gain graph with loops $l_\alpha := (v, v, \alpha)$, $l_\beta := (v, v, \beta)$, where α and β are linearly independent. If $\delta(l_\alpha) = \delta(l_\beta)$ for all active NBAC-colourings of G, then for any 2-periodic framework (G, p, L), $\mathcal{V}_{\tilde{e}}(G, p, L) = \mathcal{V}^f_{\tilde{e}}(G, p, L)$.*

Proof. Let (G, p, L) be a flexible 2-periodic framework and choose any $(p', L') \in \mathcal{V}_{\tilde{e}}(G, p, L)$. Since we have loops $l_\alpha, l_\beta \in E(G)$, it follows that $\|L'.\alpha\|^2 = \|L.\alpha\|^2$ and $\|L'.\beta\|^2 = \|L.\beta\|^2$. We note

$$\begin{aligned}(L.\alpha).(L.\beta) &= (p(v) - p(v) - L.\alpha).(p(v) - p(v) - L.\beta)\\ (L'.\alpha).(L'.\beta) &= (p'(v) - p'(v) - L'.\alpha).(p'(v) - p'(v) - L'.\beta),\end{aligned}$$

thus by Proposition 3.21, $(L.\alpha).(L.\beta) = (L'.\alpha).(L'.\beta)$. It now follows that $L' \sim L$ as required. □

Theorem 7.10. *Let G be a connected $\mathbb{Z}^2$-gain graph. Then the following are equivalent:*
(i) G has a type 2 flexible 2-lattice NBAC-colouring.
(ii) There exists a full 2-periodic (G, p, L) framework in $\mathbb{R}^2$ with a scissor flex.

Proof. (i) $\Rightarrow$ (ii): This follows as the construction defined in Lemma 6.12 has a scissor flex.

(ii) $\Rightarrow$ (i): Let (p_t, L_t) be a scissor flex of (G, p, L). Define G' to be the $\mathbb{Z}^2$-gain graph formed from G by adding the loops $l_\alpha := (v, v, \alpha)$ and $l_\beta := (v, v, \beta)$ for some $v \in V(G)$. We note that (p_t, L_t) is a flex of (G', p, L) also. As $\operatorname{rank}(G) = 2$, then by Lemma 6.5, G has a NBAC-colouring.

If all NBAC-colourings of G have $\delta(l_\alpha) = \delta(l_\beta)$, then by Lemma 7.9, (G', p, L) is fixed lattice flexible, a contradiction. It follows that G has a NBAC-colouring δ with $\delta(l_\alpha) \neq \delta(l_\beta)$. By Lemma 7.8, δ is a type 2 flexible 2-lattice NBAC-colouring as required. □

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