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A dynamical characterization of the small world phase

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Abstract

Small-world (SW) networks have been identified in many different fields. Topological coefficients like the clustering coefficient and the characteristic path length have been used in the past for a qualitative characterization of these networks. Here a dynamical approach is used to characterize the small-world phenomenon. Using the Watts–Strogatz β-model, a coupled map dynamical system is defined on the network. Entrance to and exit from the SW phase are related to the behavior of the ergodic invariants of the dynamics.

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1. Introduction

Networks are prevalent in all domains of life and science. Social, economic and political networks are the backbone of human society. The internet is a network. The metabolic processes of living beings are a network with the substrates as nodes that are linked together whenever they participate in the same biochemical reaction. Protein–protein as well as gene expression and regulation are biological networks, etc.

Regular lattices and random graphs [1] have been studied for a long time. More recently [2,3] small-world networks became the object of growing attention and were identified in many different fields. They seem to be the underlying structure for some important phenomena like the rapid spread of diseases [4], social networks, cooperative behavior between competing agents [5], problem solving organizations and communication networks.

Topologically, small-world (SW) networks are identified by the values of two statistical properties:

- the *clustering coefficient* (CC) that measures the average probability for two agents, having a common neighbor, to be themselves connected and

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– the characteristic *path length* (PL), that is, the average length of the shortest path connecting each pair of agents.

Regular lattices have long path lengths and high clustering, whereas random graphs have short path lengths but low clustering. SW networks exhibit short PL's and, at the same time, high CC's.

In many model networks, the simultaneous occurrence of high CC and low PL is observed over an interval, between order and randomness, which is called the SW phase. However, this phenomenon can only be defined as a phase, in the statistical mechanics sense, if order parameters are found to characterize the regular-to-SW and the SW-to-random phase transitions. Alternately, the SW region might simply be a crossover phenomenon between regular and random graphs [6].

Further information on the SW phenomenon has been obtained in the past from the study of several quantities. Farkas et al. [7] studied the spectral density of the adjacency matrix, with increasing randomness, concluding that, in spite of the blurring of singularities, a consistently high value of the third moment implies the existence of a large number of triangles in the SW network. Monasson [8], on the other hand, studied the spectral properties of the Laplacian operator, that characterizes the time evolution of a diffusive field and localization properties on the graph.

In this Letter a dynamical systems approach is used to characterize the small-world phenomenon. Using the Watts–Strogatz β-model [3], we study a coupled map system on the network, with interactions defined by the network connections. The SW phase is related to the behavior of the ergodic invariants of the dynamics. Entrance to the SW phase is related to the Lyapunov spectrum and exit from the SW phase corresponds to the region where "entropy" and "conditional exponents entropy" [9,10] split apart.

2. The dynamical model

Consider a β-family of models, each one with N agents on a circle and periodic boundary conditions. For $\beta = 0$, each agent in the model is connected to its $2v$ nearest neighbors. For $\beta \neq 0$, the network structure is obtained by looking at each one of the connections of the $\beta = 0$ structure and, with probability β, replacing this connection by a new random one [3].

On each one of the β-networks, a dynamical system is defined, with a map at each node and convex-coupling interactions defined by the network connections

$$x_i(t+1) = \sum_{j=1}^{N} W_{ij} f\bigl(x_j(t)\bigr), \tag{1}$$

where

$$W_{ij} = \begin{cases} 1 - \frac{n_v(i)}{2v} c & \text{if } i = j, \\ \frac{c}{2v} & \text{if } i \neq j \text{ and } i \text{ is connected to } j, \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

$n_v(i)$ is the number of agents connected to i and c is a control parameter.

For the agent dynamics we choose

$$f(x) = \alpha x \mod 1. \tag{3}$$

Typically $\alpha = 2$.

For the $\beta = 0$ network, each agent has exactly $2v$ neighbors and the Lyapunov exponents are [10]

$$\lambda_0(k) = \log\left\{\alpha\left(1 - c + \frac{c}{v}\sum_{j=1}^{v}\cos(j\theta_k)\right)\right\}, \tag{4}$$

with $\theta_k = \frac{2\pi k}{N}$, $k = 0, \ldots, N-1$. In the $N \to \infty$ limit, the Lyapunov spectrum is a continuous smooth function, as illustrated in the upper plot of Fig. 1. As we will shortly see, the random rewiring of the network induces shifts on the Lyapunov spectrum. For simplicity c is chosen in such a way that, for $\beta = 0$, the lowest Lyapunov exponent is zero. As β increases, the matrices of the tangent map cease to be regularly organized, the Lyapunov spectrum develops gaps and some of the exponents become negative. This is illustrated in the lower plot of Fig. 1 for $N = 800$ and $2v = 6$.

It is also the appearance of random long range connections that is responsible for the reduction of the path length in SW networks. Therefore it is natural to consider the modifications in the Lyapunov spectrum as the dynamical signature of the onset of the SW phase. Of particular dynamical significance is the shift of part of the spectrum towards negative values. That is, in this model, the randomness arising from the rewiring leads to an effective reduction on the number

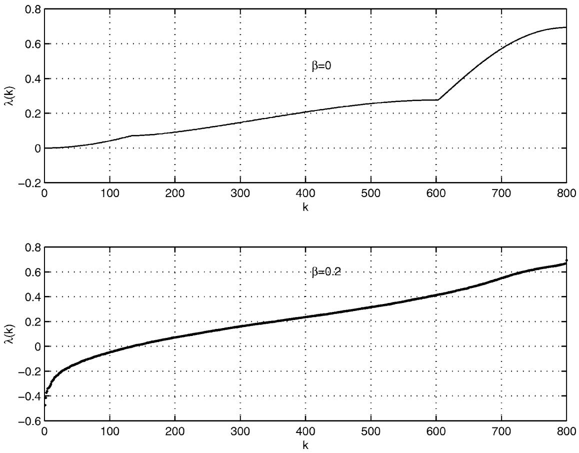

Fig. 1. Lyapunov spectrum for $\beta = 0$ and for a typical network at $\beta = 0.2$ ($N = 800$, $2v = 6$).

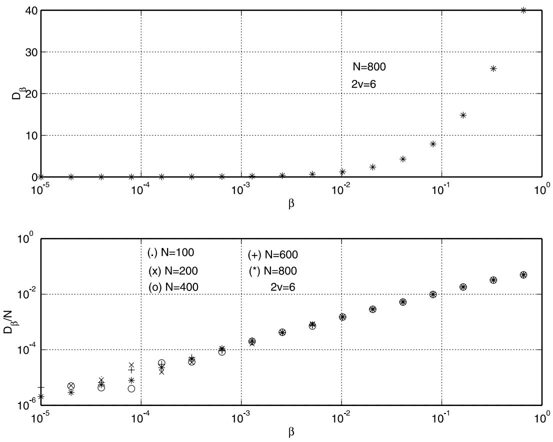

Fig. 2. The D_β parameter ("SW order parameter") averaged over 100 sample networks ($N = 100, \ldots, 800$, $2v = 6$).

of dynamical degrees of freedom. We define D_β

$$D_\beta = -\sum_{\lambda_i<0} \lambda_i \tag{5}$$

to quantify this effect.[1] To characterize the modifications of the Lyapunov spectrum, another possibility would be to measure the singular part of the spectrum associated to the gaps. However, the natural intervals in the spectrum, that arise from the finiteness of N, make this measurement less reliable.

In the upper plot of Fig. 2 we show the average values of D_β taken over 100 different samples for each β (with $N = 800$ and 6 as the average degree of the network). A good fit to all the data shown in the log–log lower plot of Fig. 2 is

$$D_\beta = cN(\beta - \beta_{c_1})^{\eta_1}, \tag{6}$$

with $\beta_{c_1} \leqslant 10^{-5}$ and $\eta_1 = 1.01 \pm 0.06$.

In practice it is only after $\beta \simeq 10^{-3}$ that small-world effects (and D_β values) become appreciable. Nevertheless, the fact that the data is consistent with $\beta_{c_1} = 0$ implies that, using $\frac{1}{N} D_\beta$ as an order parameter for the small-world phase, this phase starts at $\beta = 0^+$, the regular phase being only the isolated point $\beta = 0$. This is consistent with the analysis in Ref. [11].

Barthélémy and Amaral [6] have studied the average path length L for this model as a function of the network size N. They find L to be a scaling function of N/N^*, N^* being a crossover size, function of the degree of disorder ($N^* \sim \beta^{-2/3}$). This would imply the small-world (SW) effect to be not a phase transition, but a crossover phenomenon. An alternative point of view would be that SW is indeed a phase but that L is not the appropriate order parameter. With D_β we find no evidence for a crossover. Notice that in Fig. 2 part of the data consistent with Eq. (6) is obtained for network sizes below N^* (as determined by the authors of Ref. [6]).

To characterize the exit from the SW phase, we use the notion of *conditional Lyapunov exponents*. They were introduced by Pecora and Carroll in their study of the synchronization of chaotic systems [12]. Like the Lyapunov exponents, the conditional exponents are well-defined ergodic invariants [9]. The idea is that the conditions that in Oseledec's theorem insure the existence of the Lyapunov exponents also establish the existence of characteristic exponents formed by subblocks of the tangent map matrix. For details on the role of the conditional exponents as ergodic invariants characterizing self-organization in multi-agent systems, we refer to [10].

Here, for each agent i, we consider a subblock of dimension $d_i \times d_i$ formed by himself and those that are connected to it. The positive conditional exponents $\lambda^*_\beta(j)$ associated to each subblock are computed

[1] For other c values we would define

$$D_\beta = -\sum\nolimits_{(\lambda_i - \lambda_{\min}(\beta=0))<0} (\lambda_i - \lambda_{\min}(\beta = 0)).$$

and a dimension-weighed sum is performed over all subblocks. This gives a version of what elsewhere [9,10] has been called a *conditional exponents entropy*.

$$h_\beta^* = \sum_{i=1}^{N} \left(\frac{1}{d_i} \sum_{\lambda_\beta^* > 0} \lambda_\beta^*(j) \right). \tag{7}$$

Subtracting h_β^* from the sum of the positive Lyapunov exponents, $h_\beta = \sum_{\lambda_\beta > 0} \lambda_\beta(j)$, we define the coefficient

$$C_\beta = \left| \frac{h_0^* - h_0}{h_\beta^* - h_\beta} \right|, \tag{8}$$

which is also an ergodic invariant.

This coefficient has the following dynamical interpretation: the Lyapunov exponents measure the rate of information production or, from an alternative point of view, they define the dynamical freedom of the system, in the sense that they control the amount of change that is needed today to have an effect on the future. In this sense the larger a Lyapunov exponent is, the freer the system is in that particular direction, because a very small change in the present state will induce a large change in the future. The conditional exponents have a similar interpretation concerning the dynamics as seen from the point of view of each agent and his neighborhood [10]. However, the actual information production rate is given by the sum of the positive Lyapunov exponents, not by the sum of the conditional exponents. Therefore, the quantity $h_\beta^* - h_\beta$ is a measure of apparent dynamical freedom (or apparent rate of information production). As self-organization in a system concerns the dynamical relation of the whole to its parts, this quantity may also be looked at as a measure of dynamical self-organization.

C_β involves the ratio of differences between local and global rates of entropy production. Notice however that, whereas in the numerator neighborhoods are local in the Jacobian matrix, in the denominator, because of the random rewiring, neighborhoods involve very different sites. Therefore one should not expect C_β to be a simple function.

In Fig. 3 we show the average values of C_β taken over 100 different samples for each β (with 6 as the average degree of the network and $N =$

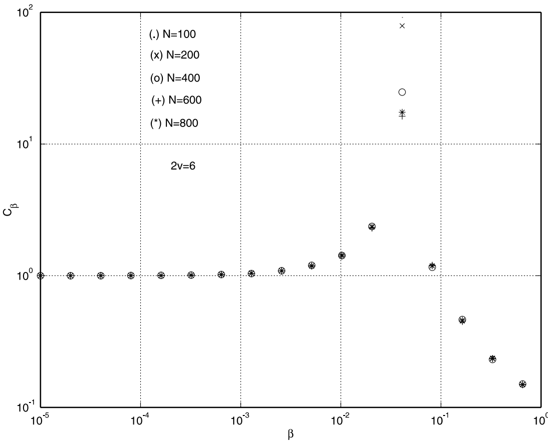

Fig. 3. The C_β parameter (“SW exit”) averaged over 100 sample networks ($N = 100, \ldots, 800$, $2v = 6$).

$100, 200, 400, 600, 800$). Notice the N-independence[2] of C_β which follows from the fact that, in Eq. (8) it is defined as a ratio of two quantities with the same N-dependence. For small β values the difference between the entropy and the conditional exponents entropy is a small quantity, that may be easily computed from the network parameters. It means that each agent may have exact information on the global behavior from observation of his own neighborhood. When β increases the difference changes sign and becomes very large, meaning that the neighborhood information has ceased to provide reliable information on the global dynamics of the network. This is the dynamical correlate of the decreasing cluster properties and allows us to define the transition at the divergence point β_{c_2} of C_β. We find

$$\beta_{c_2} \simeq 0.04. \tag{9}$$

Near the transition region

$$C_\beta \sim |\beta - \beta_{c_2}|^{-\eta_2},$$

with $\eta_2 \simeq 1.14$ below the transition and $\eta_2 \simeq 0.93$ above it.

[2] The apparent N-dependence near β_{c_2} is due to numerical imprecisions near the point where $h_\beta^* - h_\beta$ changes sign.

3. Remarks and conclusions

(1) The ergodic invariants (Lyapunov spectrum and conditional exponents) provide a link between the topological properties of SW networks and the dynamical behavior of a coupled map system modelled on the network. In addition, the power laws obeyed by these invariants provide a framework to identify the SW phenomenon as a phase in the statistical mechanics sense.

(2) Coupled map behavior, evolution of a diffusive field [8] and spectrum of the adjacency matrix [7, 13] supply complementary information on the SW phenomenon. It is therefore conceivable that quantities obtained from these other approaches might also be used to construct order parameters characterizing the SW phase.

(3) A direct relation seems to exist between the topological properties of a network and the dynamical behavior of dynamical systems living on that network. However, this relation is only circumstantial and it would be interesting to establish it in a more rigorous basis.

(4) We have also computed the C_β and D_β coefficients for other neighborhood structures, ranging from $2\nu = 4$ to $2\nu = 10$. The conclusion was that both C_β and D_β are robust dynamical characterizations of the SW phenomenon. On the range that was explored, the transition point of C_β was found to be 2ν-independent. As for D_β, although it has a weak dependence on ν for large β, for small β its behavior is always consistent with a transition at $\beta \approx 0$ and slope ≈ 1.

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