Dynamics of a stochastic fractional nonlocal reaction-diffusion model driven by additive noise

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October 7, 2022

Abstract. In this paper, we are concerned with the long-time behavior of stochastic fractional nonlocal reaction-diffusion equations driven by additive noise. We use the techniques of random dynamical systems to transform the stochastic model into a random one. To deal with the new nonlocal term appeared in the transformed equation, we first use a generalization of Peano's theorem to prove the existence of local solutions, and then adopt the Galerkin method to prove existence and uniqueness of weak solutions. Next, the existence of pullback attractors for the equation and its associated Wong-Zakai approximation equation driven by colored noise are shown, respectively. Furthermore, we establish the upper semi-continuity of random attractors of the Wong-Zakai approximation equation as $\delta \to 0^+$.

Keywords and phrases: Stochastic fractional nonlocal reaction-diffusion equation, additive noise, random attractor, colored noise, upper semi-continuity.

1 Introduction

Nonlocal problems modeled by partial differential equations have been extensively studied in [1, 2, 6, 7, 8, 14, 17, 22, 23, 38, 45] and the references therein. Chipot et al. [13] studied the behavior of a large class of nonlocal nonlinear elliptic problems as follows

$$\begin{cases} -a\left(\int_D u\right)\Delta u + \lambda u = f, & \text{in } D, \\ \partial_n u + \gamma\left(\int_{D'} u\right) = 0, & \text{on } \partial D, \end{cases}$$

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where D is a bounded open subset of $\mathbb{R}^N$, the boundary ∂D is Lipschitz, $D' \subset D$, $\lambda > 0$, the functions a and γ belong to $C\left(\mathbb{R};\mathbb{R}^+\right)$, $f \in L^2(D)$ and $\partial_n u$ is the normal derivative of u.

Instead of considering the nonlocal term $a\left(\int_D u\right)$, Chipot et al. [15, 16] extended this term to a more general nonlocal operator $a(l(u))$ where the functional $l \in \mathcal{L}\left(L^2(D);\mathbb{R}\right)$, i.e. for some $g \in L^2(D)$

$$l(u) = l_g(u) = \int_D g(x)u(x)dx.$$

Then the following nonlocal parabolic equation

$$\frac{\partial u}{\partial t} - a(l(u))\Delta u = f$$

received much attention. More precisely, the case in which f is independent of u was considered in [11, 12], where existence and uniqueness of solutions, as well as the long time dynamics, are established under some suitable assumptions. Furthermore, as the nonlocal operator is not acting globally in the whole domain in [36, 37] but contained in a ball centered on each position point, radial solutions, bifurcation analysis, branch of solutions and their stability are studied. When f depends on u in a semilinear form, making use of a fixed point theorem, the existence and uniqueness of a weak solution were proved in [33] for a semilinear problem with nonlocal diffusion where the domain has smooth boundary and the term f is Lipschitz continuous. In addition, the existence and uniqueness of periodic solutions were also analyzed. Later, the existence and regularity of pullback attractors were studied for a deterministic non-autonomous parabolic equation with nonlocal diffusion and additional non-autonomous terms (see [8]). Based on [8], time delays were taken into account in non-autonomous nonlocal partial differential equations where the time-dependent term $h(t, u_t)$ involves delays, one can see [45] for more details.

It is worth noting that random attractors play an important role in the study of dynamical behavior of stochastic partial differential equations. One can refer to [3, 4, 5, 9, 10, 18, 19, 27] and the references therein. Applying the theory of random dynamical systems, stochastic nonlocal partial differential equations with linear noise (multiplicative noise and additive noise) were investigated in [42], where the well-posedness and long-time behavior of solutions were exploited. Moreover, the well-posedness and asymptotic behavior of a class of stochastic nonlocal partial differential equations driven by nonlinear white noise were studied in [44], as well as the random non-autonomous problem driven by colored noise. As special cases, Wong-Zakai approximation models with additive and multiplicative noise possess random attractors which converge upper-semicontinuously to the respective random attractors of the stochastic equations driven by standard Brownian motions. We also remark that more results can be found in [21, 24, 26, 39] for colored noise and [32, 34, 40, 51] for Wong-Zakai approximations.

Notice that the Laplacian operator in these aforementioned papers is the standard one. When $\gamma \in (0, 1)$, the operator $(-\Delta)^\gamma$ is called the fractional Laplacian. The relationship between the standard Laplacian and the fractional Laplacian follows from the properties that $\lim\limits_{\gamma\to 1^-}(-\Delta)^\gamma u = -\Delta u$ and $\lim\limits_{\gamma\to 0^+}(-\Delta)^\gamma u = u$ (see [35, Proposition 4.4]). As far as we are aware, great attention has been devoted to the study of fractional partial differential equations, see, e.g., [25, 28, 30, 31, 41, 46, 49, 50] and the references therein. In fact, there are different definitions about fractional Laplacian operator, one being called integral fractional operator and the other spectral fractional operator. More details on fractional Laplacian can be seen in [35]. Very recently, [43] dealt with stochastic nonlocal reaction-diffusion equations driven by multiplicative noise which focused on the integral fractional operator. Since the fractional Laplacian $(-\Delta)^\gamma$ leads to not enough dissipativity, the regularity of solutions is proved by a priori estimates instead of obtaining energy estimates directly.

Motivated by these papers above, in this work we consider the following non-autonomous stochastic fractional nonlocal reaction-diffusion equation driven by additive white noise,

$$\begin{cases} \dfrac{\partial u}{\partial t} + a(l(u))(-\Delta)^\gamma u = f(u) + h(t) + \phi \dfrac{dW}{dt}, & \text{in } \mathcal{O} \times (\tau, +\infty), \\ u = 0, & \text{on } \partial\mathcal{O} \times (\tau, +\infty), \\ u(x,\tau) = u_\tau(x), & \text{in } \mathcal{O}, \end{cases} \tag{1.1}$$

where $\mathcal{O}$ is a smooth bounded domain of $\mathbb{R}^n, \tau \in \mathbb{R}$, $l \in \mathcal{L}\left(L^2(\mathcal{O}); \mathbb{R}\right)$, $h \in L^2_{loc}\left(\mathbb{R}; L^2(\mathcal{O})\right)$, $\phi \in H^\gamma(\mathcal{O}) \cap H^2(\mathcal{O})$ and $a \in C\left(\mathbb{R}; \mathbb{R}^+\right)$ satisfies that there exist two positive constants m and M such that

$$m \le a(s) \le M, \quad \forall s \in \mathbb{R}, \tag{1.2}$$

and its corresponding approximation by colored noise given by

$$\begin{cases} \dfrac{\partial u_\delta}{\partial t} + a(l(u_\delta))(-\Delta)^\gamma u_\delta = f(u_\delta) + h(t) + \phi \zeta_\delta(\theta_t \omega), & \text{in } \mathcal{O} \times (\tau, +\infty), \\ u_\delta = 0, & \text{on } \partial\mathcal{O} \times (\tau, +\infty), \\ u_\delta(x,\tau) = u_{\delta,\tau}(x), & \text{in } \mathcal{O}, \end{cases} \tag{1.3}$$

where the process $\zeta_\delta(\theta_t\omega)$ is the so-called colored noise (see Section 5 for more details).

Our main purposes are to prove the existence and uniqueness of weak solutions of both problems, as well as the existence and the upper semi-continuity of random attractors of problem (1.3) to (1.1) when δ goes to zero. Unlike the case of stochastic fractional nonlocal reaction-diffusion equation driven by multiplicative white noise, there appears a new nonlocal term in the process of transforming the stochastic equation (1.1) into a random one by means of the Ornstein-Uhlenbeck transformation. Thus, the existence and uniqueness of solutions become much more difficult to be proved. In this case, we present a more detailed proof of the well-posedness of the solutions. We first need to apply a generalization of Peano's theorem to prove the existence of local solutions, and then use the well-known Galerkin method to prove existence and uniqueness of weak solutions. In what follows, we establish the existence of pullback attractors of problem (1.1) and its associated problem driven by colored noise (1.3), respectively. Finally, we give the upper semi-continuity of attractors.

The structure of the paper is as follows. In Section 2, we first recall the basic concepts on random dynamical systems. Next, we present some ergodic properties of the Ornstein-Uhlenbeck process and the fractional Laplacian operator, respectively. And then we transform the fractional stochastic nonlocal problem into a random one via the Ornstein-Uhlenbeck process. Section 3 is devoted to proving the well-posedness of problem (1.1). In Section 4, we show the existence of pullback attractors of problem (1.1) by obtaining the existence of random absorbing set and the asymptotic compactness of the random dynamical system which is generated by solutions to problem (1.1). Similar to Section 4, the existence of pullback attractors of problem driven by colored noise associated with (1.1) is proved in Section 5. In the last section, the convergence of solutions and attractors of problem (1.1) are discussed as $\delta \to 0^+$.

2 Preliminaries

In this section, we will recall some basic concepts related to non-autonomous random dynamical systems and properties of Ornstein-Uhlenbeck processes which will be used throughout this paper. One can refer to [3, 48] for more details. Besides, the concept and some properties of integral fractional Laplacian operator are shown as follows.

2.1 Non-autonomous random dynamical systems

Let $(X, \|\cdot\|_X)$ be a complete separable metric space, and $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ an ergodic metric dynamical system, where $\Omega = C_0(\mathbb{R};\mathbb{R}) = \{\omega \in C(\mathbb{R};\mathbb{R}) : \omega(0) = 0\}$, $\mathcal{F}$ is the Borel-σ-algebra induced by the compact-open topology of Ω, $\mathbb{P}$ is the Wiener measure on $(\Omega, \mathcal{F})$ and $\theta = \{\theta_t\}_{t\in\mathbb{R}}$ is a measurable flow on Ω, which is defined by

$$\theta : (\mathbb{R}\times\Omega, \mathcal{F}\otimes\mathcal{B}(\mathbb{R})) \to (\Omega, \mathcal{F}), \quad \theta_0 = \mathrm{id}_\Omega, \quad \theta_{t+s} = \theta_t\circ\theta_s,\ t, s \in \mathbb{R}. \tag{2.1}$$

The *Wiener shift* operators which form the flow θ

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R}, \omega \in \Omega$$

leave the Wiener measure $\mathbb{P}$ invariant. More precisely, $\mathbb{P}$ is ergodic with respect to θ.

Since the above probability space is canonical we have for a Wiener process and its shift operator

$$W(t,\omega) = \omega(t), \quad W(t,\theta_s\omega) = \omega(t+s) - \omega(s) = W(t+s,\omega) - W(s,\omega).$$

It is important to note that the measurability in (2.1) is not true if we replace $\mathcal{F}$ by its completion, see Appendix A3 of [3].

Definition 2.1. *A continuous non-autonomous random dynamical system on X over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is a $\left(\mathcal{B}\left(\mathbb{R}^+\right)\times\mathcal{F}\times\mathcal{B}(X), \mathcal{B}(X)\right)$-measurable mapping:*

$$\varphi : \mathbb{R}^+\times\mathbb{R}\times\Omega\times X \to X, (\cdot,\tau,\cdot,\cdot) \mapsto \varphi(\cdot,\tau,\cdot,\cdot)$$

such that φ satisfies:
(1) $\varphi(0,\tau,\omega,\cdot)$ is identity on X;
(2) $\varphi(t+s,\tau,\omega,\cdot) = \varphi(t,\tau+s,\theta_s\omega,\varphi(s,\tau,\omega,\cdot))$ for all $s,t \geq 0$;
(3) $\varphi(t,\tau,\omega,\cdot) : X \to X$ is continuous for all $t \geq 0$.

Definition 2.2. *A set-valued mapping $D(\tau, \cdot) : \Omega \to 2^X$, $\omega \mapsto D(\tau, \omega)$ is said to be a random set, if for every $\tau \in \mathbb{R}$, the mapping $\omega \mapsto d(u, D(\tau, \omega))$ is measurable for any $u \in X$, moreover, $\{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is also closed for each $\omega \in \Omega$.*

Definition 2.3. *A random set $\{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is called tempered, if for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\beta > 0$*

$$\lim_{t\to+\infty} \mathrm{e}^{-\beta t} d\left(D\left(\tau - t, \theta_{-t}\omega\right)\right) = 0,$$

where $d(D) = \sup\{\|b\|_X : b \in D\}$.

Definition 2.4. *A random set $\{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is said to be a random pullback absorbing set if for any tempered random set $\{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$, every $\tau \in \mathbb{R}$, $\omega \in \Omega$, there exists t_0 such that for each $t \geq t_0$,*

$$\varphi\left(t, \tau - t, \theta_{-t}\omega, D\left(\tau - t, \theta_{-t}\omega\right)\right) \subset B(\tau, \omega).$$

Definition 2.5. *Let $\mathcal{D}$ be a collection of random subset of X, φ is said to be $\mathcal{D}$-pullback asymptotically compact in X, if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$, $\{\varphi(t_n, \tau - t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{+\infty}$ has a convergent subsequence in X whenever $t_n \to +\infty$, and $x_n \in B(\tau - t_n, \theta_{-t_n}\omega)$ with $\{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$.*

Definition 2.6. *A family $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is called a $\mathcal{D}$-pullback attractor for φ if for every $\tau \in \mathbb{R}$, $\omega \in \Omega$,*
(1) $\mathcal{A}(\tau, \omega)$ is compact in X and for every $\tau \in \mathbb{R}$, $\omega \mapsto d(X, \mathcal{A}(\tau, \omega))$ is measurable;
(2) $\mathcal{A}$ is invariant: $\varphi(t, \tau, \omega, \mathcal{A}(\tau, \omega)) = \mathcal{A}(t + \tau, \theta_t\omega)$, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$;
(3) $\mathcal{A}$ attracts all member of $\mathcal{D}$: for every $B \in \mathcal{D}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, we have

$$\lim_{t\to\infty} d_X\left(\varphi\left(t, \tau - t, \theta_{-t}\omega, B\left(\tau - t, \theta_{-t}\omega\right)\right), \mathcal{A}(\tau, \omega)\right) = 0,$$

where d_X is Hausdorff semi-distance in X.

The following result is a sufficient condition ensuring existence and uniqueness of random attractors for non-autonomous random dynamical system.

Theorem 2.1. *([48, Theorem 2.23]) Let φ be a continuous non-autonomous random dynamical system on X over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$, if there exists a closed random tempered absorbing set $\{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ of φ and φ is asymptotically compact in X, then the $\mathcal{D}$-pullback attractor $\mathcal{A}$ is unique and is given by,*

$$\mathcal{A}(\tau, \omega) = \bigcap_{r\geq 0} \overline{\bigcup_{t\geq r} \varphi\left(t, \tau - t, \theta_{-t}\omega, B\left(\tau - t, \theta_{-t}\omega\right)\right)}, \quad \tau \in \mathbb{R}, \omega \in \Omega.$$

2.2 Ornstein-Uhlenbeck process

Consider the following linear one-dimensional stochastic differential equation

$$dZ + Zdt = dW \tag{2.2}$$

with initial value

$$Z(t_0) = Z_0.$$

Then

$$\begin{aligned} Z\left(t, t_0, Z_0\right) =& e^{-(t-t_0)} Z_0 + \int_{t_0}^{t} e^{-(t-s)} dW(s) \\ =& e^{-(t-t_0)} Z_0 + W(t) - e^{-(t-t_0)} W(t_0) - \int_{t_0}^{t} e^{-(t-s)} W(s) ds \end{aligned}$$

is the unique solution of (2.2).

Let $z\left(\theta_t\omega\right) = \lim_{t_0\to-\infty} Z\left(t, t_0, Z_0\right) = W(t) - \int_{-\infty}^{t} e^{-(t-s)} W(s, \omega) ds$, then $Z(t, \omega) = z\left(\theta_t\omega\right)$ is a stationary solution of (2.2). Moreover, the random variable $z\left(\omega\right)$ satisfies the following properties:

Lemma 2.1. *(See [9])* (1) *There exists a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant set Ω_1 of full measure such that*

$$\begin{aligned} &\lim_{t\to\pm\infty} \frac{\left|z\left(\theta_t\omega\right)\right|}{|t|} = 0, \\ &\lim_{t\to\pm\infty} \frac{1}{t} \int_0^t z\left(\theta_r\omega\right) dr = 0, \\ &\lim_{t\to\pm\infty} \frac{1}{t} \int_0^t \left|z\left(\theta_r\omega\right)\right| dr = \mathbb{E}|z| < +\infty. \end{aligned}$$

(2) *The mapping $t \mapsto z\left(\theta_t\omega\right)$ is continuous.*

2.3 Fractional settings

Let $\mathcal{S}$ be the Schwartz space of rapidly decaying C^∞ functions in $\mathbb{R}^n$ and $0<\gamma<1$. For every $u\in\mathcal{S}$, the fractional Laplacian operator $(-\Delta)^\gamma$ at the point x is defined by

$$(-\Delta)^\gamma u(x)=C(n,\gamma)P.V.\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+2\gamma}}dy,\quad x\in\mathbb{R}^n, \tag{2.3}$$

where $P.V.$ is a commonly used abbreviation for "in the principal value sense" and

$$C(n,\gamma)=\Big(\int_{\mathbb{R}^n}\frac{1-\cos\xi_1}{|\xi|^{n+2\gamma}}d\xi\Big)^{-1}\quad\text{with}\quad \xi=\{\xi_1,\xi_2,\cdots,\xi_n\}\in\mathbb{R}^n.$$

By a standard change of variable, (2.3) is equivalent to

$$(-\Delta)^\gamma u(x)=-\frac{C(n,\gamma)}{2}\int_{\mathbb{R}^n}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2\gamma}}dy,\quad x\in\mathbb{R}^n.$$

For any $0<\gamma<1$, the fractional Sobolev space $W^{\gamma,2}(\mathbb{R}^n):=H^\gamma(\mathbb{R}^n)$ is defined by

$$H^\gamma(\mathbb{R}^n)=\left\{u\in L^2(\mathbb{R}^n):\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2\gamma}}dxdy<+\infty\right\},$$

equipped with the norm

$$\|u\|_{H^\gamma(\mathbb{R}^n)}=\left(\int_{\mathbb{R}^n}|u(x)|^2dx+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2\gamma}}dxdy\right)^{\frac{1}{2}}$$

and inner product

$$(u,v)_{H^\gamma(\mathbb{R}^n)}=\int_{\mathbb{R}^n}u(x)v(x)dx+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2\gamma}}dxdy,\quad\forall u,v\in H^\gamma(\mathbb{R}^n).$$

From now on, we denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and the inner product of $L^2(\mathbb{R}^n)$, respectively. Moreover, we denote by $\|\cdot\|_{\dot{H}^\gamma(\mathbb{R}^n)}$ the Gagliardo semi-norm of $H^\gamma(\mathbb{R}^n)$ which is defined by

$$\|u\|^2_{\dot{H}^\gamma(\mathbb{R}^n)}=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2\gamma}}dxdy,\quad u\in H^\gamma(\mathbb{R}^n).$$

Thus, $\|u\|^2_{H^\gamma(\mathbb{R}^n)}=\|u\|^2+\|u\|^2_{\dot{H}^\gamma(\mathbb{R}^n)}=\|u\|^2+\dfrac{2}{C(n,\gamma)}\|(-\Delta)^\gamma u\|^2$ for all $u\in H^\gamma(\mathbb{R}^n)$.

Since $(-\Delta)^\gamma$ is a nonlocal operator, we here see the homogeneous Dirichlet boundary (1.1) as $u=0$ on $\mathbb{R}^n\backslash\mathcal{O}$ instead of $u=0$ only on $\partial\mathcal{O}$. From this perspective, we define two spaces $V=\{u\in H^\gamma(\mathbb{R}^n):u=0$ a.e. on $\mathbb{R}^n\backslash\mathcal{O}\}$ and $H=\{u\in L^2(\mathbb{R}^n):u=0$ a.e. on $\mathbb{R}^n\backslash\mathcal{O}\}$. Furthermore, let $b:V\times V\to\mathbb{R}$ be a bilinear form given by, for $v_1,v_2\in V$,

$$b(v_1,v_2)=\mu(v_1,v_2)+\frac{C(n,\gamma)}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(v_1(x)-v_1(y))(v_2(x)-v_2(y))}{|x-y|^{n+2\gamma}}dxdy,$$

where μ is a constant chosen later. For convenience, we associate an operator $A:V\to V^*$ with b such that

$$(A(v_1),v_2)_{V^*,V}=b(v_1,v_2),\qquad\text{for all } v_1,v_2\in V,$$

where V^* is the dual space of V and $(\cdot,\cdot)_{V^*,V}$ is the duality paring of V^* and V.

Since A is injective and surjective, the inverse $A^{-1}:V^*\to V$ is well-defined. On the other hand, the embedding $V\hookrightarrow H$ is compact and $H=H^*\subset V^*$ yield that $A^{-1}:H\to V\subset H$ is a symmetric compact operator. Then, by means of the Hilbert-Schmidt theorem, A has a family of eigenfunctions $\{w_j\}_{j=1}^{+\infty}$ which forms an orthonormal basis of H. Moreover, if λ_j denotes the eigenvalue of operator A corresponding to w_j, i.e.,

$$Aw_j = \lambda_j w_j, \quad j = 1, 2, \cdots,$$

then λ_j satisfies

$$0 < \mu < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \to +\infty \quad \text{as} \quad j \to +\infty.$$

2.4 Setting of the problem

Let $f \in C(\mathbb{R})$ and assume that there exist positive constants C_f, κ, β_1 and $\beta_2 > 1$ such that

$$(f(s) - f(r))(s - r) \leq C_f(s - r)^2, \quad \forall s, r \in \mathbb{R}, \tag{2.4}$$

and

$$-\kappa - \beta_1|s|^2 \leq f(s)s \leq \kappa - \beta_2|s|^2, \quad \forall s \in \mathbb{R}. \tag{2.5}$$

Now we consider its associated Nemytskii operator $\tilde{f} : L^2(\mathcal{O}) \to L^2(\mathcal{O})$ which is defined as $\tilde{f}(u)(x) = f(u(x))$ for all $u \in L^2(\mathcal{O})$ and $x \in \mathcal{O}$. By simplicity, we identify f with $\tilde{f}$ when no confusion is possible.

Next, we fix a number μ such that $0 < \mu < \min\left\{\frac{\beta_1}{m}, \frac{\beta_2}{M}, \frac{C(n,\gamma)}{2}\right\}$, and define the operator $F : L^2(\mathcal{O}) \to L^2(\mathcal{O})$ by

$$F(u) = a(l(u))\mu u + f(u),$$

which is given by

$$F(u)(x) := a(l(u))\mu u(x) + f(u(x)), \quad x \in \mathcal{O}.$$

It is easy to check that F is continuous in $L^2(\mathcal{O})$. In what follows, we will write $F(u(x))$ instead of $F(u)(x)$, although this notation does not match the typical one for the Nemytskii operator associated to a real valued function.

By (2.5) and the Young inequality, we have, for all $u \in L^2(\mathcal{O})$ and $x \in \mathcal{O}$,

$$-\kappa - (\beta_1 - m\mu)|u(x)|^2 \leq F(u(x))u(x) \leq \kappa - (\beta_2 - M\mu)|u(x)|^2. \tag{2.6}$$

From (2.6), we deduce that there exists $\alpha > 0$ such that

$$|F(u(x))| \leq \alpha(|u(x)| + 1). \tag{2.7}$$

Then, problem (1.1) can be written as

$$\begin{cases} \dfrac{\partial u}{\partial t} + a(l(u))(-\Delta)^\gamma u + a(l(u))\mu u = F(u) + h(t) + \phi\dfrac{dW}{dt}, & x \in \mathcal{O}, t > \tau, \\ u(x,t) = 0, & x \in \mathbb{R}^n \backslash \mathcal{O}, t > \tau, \\ u(x,\tau) = u_\tau(x), & x \in \mathcal{O}. \end{cases} \tag{2.8}$$

To study the dynamics of problem (2.8), we first transform the stochastic fractional nonlocal differential equation (2.8) into a random one by doing a change of variable. To start off, we denote by $u(\cdot) := u(\cdot, \tau, \omega, u_\tau)$ the solution to problem (2.8), we do the change of variable $v(t) = u(t) - \phi z(\theta_t\omega)$ with

$v(\tau)=u(\tau)-\phi z\left(\theta_\tau \omega\right)$, then we obtain

$$
\begin{cases}\dfrac{\partial v}{\partial t}+a(l(v)+z\left(\theta_t \omega\right) l(\phi))((-\Delta)^\gamma+\mu) v+a(l(v)+z\left(\theta_t \omega\right) l(\phi)) z\left(\theta_t \omega\right)\left((-\Delta)^\gamma+\mu\right) \phi \\ =F(v+\phi z\left(\theta_t \omega\right))+h(t)+\phi z\left(\theta_t \omega\right), & x \in \mathcal{O}, t>\tau, \\ v(x, t)=0, & x \in \mathbb{R}^n \backslash \mathcal{O}, t>\tau, \\ v(x, \tau)=u_\tau(x)-\phi z\left(\theta_\tau \omega\right):=v_\tau(x), & x \in \mathcal{O}.\end{cases} \tag{2.9}
$$

Notice that this change of variable is, in reality, a homeomorphism and therefore a conjugation (see [3] for more details).

3 Well-posedness of problem (2.8)

In this section, we are interested in proving the well-posedness of weak solution of problem (2.8). To this end, we first introduce the definition of a weak solution of problem (2.9), and then prove the existence and uniqueness of weak solution of problem (2.9). For that, we need the following assumption to obtain the existence of local solution of problem (2.9).

(H_1) For any $t \in[\tau, T]$, the mapping $t \rightarrow h(t)$ is measurable.

Definition 3.7. *Given $\phi \in V$ and the initial value $v_\tau \in H$. A weak solution to equation (2.9) is a function $v(\cdot):=v\left(\cdot, \tau, \omega, v_\tau\right)$ that belongs to $L^2([\tau,+\infty) ; V) \cap L^{\infty}([\tau,+\infty) ; H)$, such that for every $\zeta \in V$,*

$$
\begin{aligned}
& \frac{d}{d t}(v, \zeta)+a\left(l(v)+z\left(\theta_t \omega\right) l(\phi)\right)\left(\frac{C(n, \gamma)}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v(x)-v(y))(\zeta(x)-\zeta(y))}{|x-y|^{n+2 \gamma}} d x d y+\mu(v, \zeta)\right) \\
& \quad+a\left(l(v)+z\left(\theta_t \omega\right) l(\phi)\right) z\left(\theta_t \omega\right)\left(\frac{C(n, \gamma)}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\phi(x)-\phi(y))(\zeta(x)-\zeta(y))}{|x-y|^{n+2 \gamma}} d x d y+\mu(\phi, \zeta)\right) \\
& =\int_{\mathcal{O}} F\left(v+\phi z\left(\theta_t \omega\right)\right) \zeta(x) d x+\int_{\mathcal{O}} h(t) \zeta(x) d x+\int_{\mathcal{O}} z\left(\theta_t \omega\right) \phi(x) \zeta(x) d x,
\end{aligned} \tag{3.1}
$$

where the above equation must be understood in the sense of distribution on $(\tau,+\infty)$.

Theorem 3.1. *Suppose (H_1) holds, $a \in C\left(\mathbb{R} ; \mathbb{R}^{+}\right)$ is locally Lipschitz and satisfies (1.2), $f \in C(\mathbb{R})$ fulfills (2.4)-(2.5), which implies $F: L^2\left(\mathbb{R}^n\right) \rightarrow L^2\left(\mathbb{R}^n\right)$ satisfies (2.6)-(2.7). In addition, $h \in L_{loc}^2(\mathbb{R} ; H)$ and $l \in \mathcal{L}\left(L^2(\mathcal{O}) ; \mathbb{R}\right)$. Then, for every $\tau \in \mathbb{R}, \omega \in \Omega$ and each initial datum $v_\tau \in H$, there exists a unique weak solution $v\left(t, \tau, \omega, v_\tau\right)$ to problem (2.9) in the sense of Definition 3.7.*

Proof. **Step 1: Existence of local solution.** Using spectral theory, there exists a sequence $\left\{w_i\right\}_{i \geq 1}$ which is a Hilbert basis of H composed by the eigenfunctions of A. Firstly, we consider the function $v_n(t):=v_n\left(t, \tau, \omega, v_\tau\right)=\sum_{j=1}^n \varphi_{n j}(t) w_j$ for all $n \geq 1$, which satisfies the following system

$$
\begin{cases}\dfrac{d}{d t}\left(v_n(t), w_j\right)+a\left(l\left(v_n(t)\right)+z\left(\theta_t \omega\right) l(\phi)\right)\left(A v_n(t), w_j\right)_{V^*, V}+a\left(l\left(v_n(t)\right)+z\left(\theta_t \omega\right) l(\phi)\right) z\left(\theta_t \omega\right)\left(A \phi, w_j\right)_{V^*, V} \\ =\left(F\left(v_n(t)+\phi z\left(\theta_t \omega\right)\right), w_j\right)+\left(h(t), w_j\right)+z\left(\theta_t \omega\right)\left(\phi, w_j\right), \\ \left(v_n(\tau), w_j\right)=\left(v_\tau, w_j\right), \quad j=1,2, \cdots, n.\end{cases} \tag{3.2}
$$

Notice that the above equation is a Cauchy problem for the following ordinary differential system in $\mathbb{R}^n$,

$$
\begin{aligned}
\varphi_{n j}^{\prime}(t)+\lambda_j a\left(l\left(v_n(t)\right)+z\left(\theta_t \omega\right) l(\phi)\right) \varphi_{n j}(t)+a\left(l\left(v_n(t)\right)+z\left(\theta_t \omega\right) l(\phi)\right) z\left(\theta_t \omega\right)\left(A \phi, w_j\right)_{V^*, V} & \\
=\left(F\left(v_n(t)+\phi z\left(\theta_t \omega\right)\right), w_j\right)+\left(h(t), w_j\right)+z\left(\theta_t \omega\right)\left(\phi, w_j\right), \quad j=1,2, \cdots, n, &
\end{aligned} \tag{3.3}
$$

where $t \geq \tau$, λ_j is the eigenvalue associated to the eigenfunction w_j, and the vector $(\varphi_{n1}, \varphi_{n2}, \cdots, \varphi_{nn})$ is the unknown variable. Define

$$
\begin{aligned}
g: \quad & \mathcal{T} \to \mathbb{R}^n \\
(t,x) \mapsto & \Bigg(-\lambda_1 \xi(x) x_1 - \xi(x) z(\theta_t \omega)(A\phi, w_1)_{V^*,V} + \left(F\left(\sum_{i=1}^n x_i w_i + \phi z(\theta_t \omega) \right), w_1 \right) + (h(t), w_1) + z(\theta_t \omega)(\phi, w_1), \\
& \dots, \\
& -\lambda_n \xi(x) x_n - \xi(x) z(\theta_t \omega)(A\phi, w_n)_{V^*,V} + \left(F\left(\sum_{i=1}^n x_i w_i + \phi z(\theta_t \omega) \right), w_n \right) + (h(t), w_n) + z(\theta_t \omega)(\phi, w_n) \Bigg),
\end{aligned}
$$

where

$$\mathcal{T} = \left\{ (t,x) \in [\tau, T] \times \mathbb{R}^n : \tau \leq t \leq T, \left| x - \left((v_\tau, w_1), \dots, (v_\tau, w_n) \right) \right| \leq d \right\}$$

for any fixed $d \in \mathbb{R}^+$ and

$$x = (x_1, \dots, x_n) \mapsto \xi(x) = a\left(l\left(\sum_{i=1}^n x_i w_i \right) + z(\theta_t \omega) l(\phi) \right).$$

Next, we need to prove that g is a Caratheodory function.
Firstly, let x be fixed. Following the continuity of z and f, as well as assumption (H_1), the function

$$g_j(\cdot, x) = -\lambda_j \xi(x) x_j - \xi(x) z(\theta.\omega)(A\phi, w_j)_{V^*,V} + \left(F\left(\sum_{i=1}^n x_i w_i + \phi z(\theta.\omega) \right), w_j \right) + (h(\cdot), w_j) + z(\theta.\omega)(\phi, w_j)$$

is measurable. Therefore, the function $g(\cdot, x)$ is measurable.

Secondly, we need to verify that the function $g(t, \cdot)$ is continuous a.e. $t \in [\tau, T]$. In fact, the functions ξ and $x \in \mathbb{R}^n \mapsto \left(F\left(\sum_{i=1}^n x_i w_i + \phi z(\theta_t \omega) \right), w_j \right)$ are continuous, then we can obtain

$$g_j(t, x) = -\lambda_j \xi(x) x_j - \xi(x) z(\theta_t \omega)(A\phi, w_j)_{V^*,V} + \left(F\left(\sum_{i=1}^n x_i w_i + \phi z(\theta_t \omega) \right), w_j \right) + (h(t), w_j) + z(\theta_t \omega)(\phi, w_j)$$

is a continuous function with respect to x.

Thirdly, we are going to prove that for each $(t, x) \in \mathcal{T}$, there exist a neighborhood $V(t, x)$ and a Lebesgue measurable function $Q(\cdot) \in L^1(\tau, T)$ such that for any $(s, y) \in V(t, x)$,

$$|g(s, y)| \leq Q(s).$$

From the definition of $\mathcal{T}$, there exists a constant $C_{\mathcal{T}}$ such that

$$|x| \leq d + |((v_\tau, w_1), \dots, (v_\tau, w_n))| \leq C_{\mathcal{T}}.$$

By (2.7), we have

$$\left| \left(F\left(\sum_{i=1}^n x_i w_i + \phi z(\theta_t \omega) \right), w_j \right) \right| \leq \alpha C_{\mathcal{T}} \left(\sum_{i=1}^n \|w_i\| \right) \|w_j\| + \alpha \left(|\mathcal{O}|^{1/2} + |z(\theta_t \omega)| \|\phi\| \right) \|w_j\|. \quad (3.4)$$

Observe that taking into account the expression of ξ and the continuity of the function a in the compact interval $I' := \left[-\|l\| C_{\mathcal{T}} \sum_{i=1}^n \|w_i\| - |z(\theta_t \omega)| \|l\| \|\phi\|, \|l\| C_{\mathcal{T}} \sum_{i=1}^n \|w_i\| + |z(\theta_t \omega)| \|l\| \|\phi\| \right]$, there exists a constant $M' > 0$ such that

$$\xi(x) \leq M', \quad \forall x \in \mathbb{R}^n : |x| \leq C_{\mathcal{T}}.$$

Then

$$\begin{aligned}&|\xi(x)z(\theta_t\omega)(A\phi,w_j)_{V^*,V}|\\ \le&|\xi(x)z(\theta_t\omega)[\mu(\phi,w_j)+((-\Delta)^\gamma\phi,w_j)]|\\ \le&\frac{M'}{2}|z(\theta_t\omega)|\|\phi\|_V^2+\frac{M'(C(n,\gamma)^2+2C(n,\gamma))}{16}|z(\theta_t\omega)|\|w_j\|^2.\end{aligned}\tag{3.5}$$

On the other hand, it holds

$$|(h(t),w_j)|\le\frac{1}{2}\|h(t)\|^2+\frac{1}{2}\|w_j\|^2\tag{3.6}$$

and

$$|z(\theta_t\omega)(\phi,w_j)|\le\frac{1}{2}|z(\theta_t\omega)|^2\|\phi\|^2+\frac{1}{2}\|w_j\|^2.\tag{3.7}$$

Then, bearing this in mind, we deduce from (3.4)-(3.7) that

$$\begin{aligned}|g_j(t,x)|\le&\lambda_jM'C_{\mathcal{T}}+(\frac{1}{2}+\frac{M'}{2})|z(\theta_t\omega)|^2\|\phi\|_V^2+(\frac{\mu^2}{2}+\frac{M'(C(n,\gamma)^2+2C(n,\gamma))}{16})\|w_j\|^2\\ &+\alpha\left(|\mathcal{O}|^{1/2}+|z(\theta_t\omega)|\|\phi\|\right)\|w_j\|+\alpha C_{\mathcal{T}}\left(\sum_{i=1}^n\|w_i\|\right)\|w_j\|+\|w_j\|^2+\frac{1}{2}\|h(t)\|^2\\ \triangleq&Q(t),\end{aligned}$$

where $Q(\cdot)\in L^1(\tau,T)$. As a consequence, there exists a local solution to (3.3).

Step 2: Uniqueness of local solution. Assume that there exist two solutions φ_n^1,φ_n^2 of the ordinary differential system (3.3) in (τ,t_1) and (τ,t_2) respectively. Then, it holds

$$\begin{cases}(\varphi_{nj}^1(t)-\varphi_{nj}^2(t))'=g_j(t,\varphi_n^1(t))-g_j(t,\varphi_n^2(t)), & t\in[\tau,\min\{t_1,t_2\}],\\ (\varphi_{nj}^1-\varphi_{nj}^2)(\tau), & j=1,2,\cdots,n,\end{cases}\tag{3.8}$$

where $g_j\left(t,\varphi_n^i(t)\right)=-\lambda_ja\left(l\left(\varphi_n^i(t)\right)+z(\theta_t\omega)l(\phi)\right)\varphi_{nj}^i(t)-a\left(l\left(\varphi_n^i(t)\right)+z(\theta_t\omega)l(\phi)\right)z(\theta_t\omega)(A\phi,w_j)_{V^*,V}$ $+\left(F\left(\varphi_n^i(t)+\phi z(\theta_t\omega)\right),w_j\right)+(h(t),w_j)+z(\theta_t\omega)(\phi,w_j)$ for $i=1,2$.

Then, multiplying (3.8) by $\varphi_{nj}^1-\varphi_{nj}^2$ and summing from $j=1$ to n, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|v_n^1(t)-v_n^2(t)\right\|^2+a\left(l\left(v_n^1(t)\right)+z(\theta_t\omega)l(\phi)\right)b(v_n^1(t)-v_n^2(t),v_n^1(t)-v_n^2(t))\\ \le&\lambda_n|a\left(l\left(v_n^1(t)+z(\theta_t\omega)l(\phi)\right)\right)-a\left(l\left(v_n^2(t)+z(\theta_t\omega)l(\phi)\right)\right)||\left(v_n^2(t),v_n^1(t)-v_n^2(t)\right)|\\ &+|a\left(l\left(v_n^1(t)+z(\theta_t\omega)l(\phi)\right)\right)-a\left(l\left(v_n^2(t)\right)+z(\theta_t\omega)l(\phi)\right)||\left(A\phi,v_n^1(t)-v_n^2(t)\right)_{V^*,V}|\\ &+\left|\left(F\left(v_n^1(t)+\phi z(\theta_t\omega)\right)-F\left(v_n^2(t)+\phi z(\theta_t\omega)\right),v_n^1(t)-v_n^2(t)\right)\right|.\end{aligned}$$

Since the function a is locally Lipschitz, for any bounded interval $[-R,R]$ of $\mathbb{R}$, there exists a positive constant $L_a(R)$ such that

$$|a(x)-a(y)|\le L_a(R)|x-y|\quad\forall x,y\in[-R,R].$$

Note that $v_n^1,v_n^2\in C([\tau,\min\{t_1,t_2\}];H)$, it fulfils that there exists a constant K such that for all $t\in[\tau,\min\{t_1,t_2\}]$

$$\|v_n^1(t)\|\vee\|v_n^2(t)\|\le K.$$

In addition, taking into account that $l\in\mathcal{L}(L^2(\mathcal{O});\mathbb{R})$, it follows that $\left\{l\left(v_n^i(t)\right)+z(\theta_t\omega)l(\phi)\right\}_{t\in[\tau,\min\{t_1,t_2\}]}\in$

$[-R,R]$ for $i=1,2$, for some $R>0$. Hence, by the Young inequality, we deduce

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|v_n^1(t)-v_n^2(t)\right\|^2+m\Big(\frac{C(n,\gamma)}{2}\|v_n^1-v_n^2\|^2_{\dot{H}^\gamma(\mathbb{R}^n)}+\mu\|v_n^1(t)-v_n^2(t)\|^2\Big)\\
\le&\lambda_n|a\left(l\left(v_n^1(t)\right)+z\left(\theta_t\omega\right)l(\phi)\right)-a\left(l\left(v_n^2(t)\right)+z\left(\theta_t\omega\right)l(\phi)\right)||\left(v_n^2,v_n^1-v_n^2\right)|\\
&+|a\left(l\left(v_n^1(t)\right)+z\left(\theta_t\omega\right)l(\phi)\right)-a\left(l\left(v_n^2(t)\right)+z\left(\theta_t\omega\right)l(\phi)\right)||\left(A\phi,v_n^1-v_n^2\right)_{V^*,V}|\\
&+|(F\left(v_n^1(t)+\phi z\left(\theta_t\omega\right)\right)-F\left(v_n^2(t)+\phi z\left(\theta_t\omega\right)\right),v_n^1(t)-v_n^2(t))|\\
\le&\lambda_nL_a(R)\|l\|\|v_n^2(t)\|\|v_n^1(t)-v_n^2(t)\|^2+\lambda_nL_a(R)\|l\|\|\phi\|\|v_n^1(t)-v_n^2(t)\|^2\\
&+\mu L_a(R)\|l\|(\|v_n^2\|_{C([\tau,\min\{t_1,t_2\}];H)}+|z\left(\theta_t\omega\right)|\|\phi\|_{L^\infty(\mathcal{O})})\|v_n^1(t)-v_n^2(t)\|^2\\
&+M\mu\|v_n^1(t)-v_n^2(t)\|^2+C_f\|v_n^1(t)-v_n^2(t)\|^2\\
\le&\big[\lambda_nL_a(R)\|l\|(K+\|\phi\|)+\mu(K+|z\left(\theta_t\omega\right)|\|\phi\|_{L^\infty(\mathcal{O})})L_a(R)\|l\|+M\mu+C_f\big]\|v_n^1(t)-v_n^2(t)\|^2.
\end{aligned}
$$

Then, using the Gronwall Lemma, we obtain the uniqueness of local solution.

Step 3: Existence of weak solution. Multiplying by φ_{nj} in (3.3) and then summing from $j=1$ to n, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|v_n(t)\right\|^2+m(\frac{C(n,\gamma)}{2}\|v_n(t)\|^2_{\dot{H}^\gamma(\mathbb{R}^n)}+\mu\|v_n(t)\|^2)\\
\le&-a\left(l\left(v_n(t)\right)+z\left(\theta_t\omega\right)l(\phi)\right)z\left(\theta_t\omega\right)\left(A\phi,v_n(t)\right)_{V^*,V}+\left(F\left(v_n(t)+\phi z\left(\theta_t\omega\right)\right),v_n(t)\right)\\
&+\left(h(t),v_n(t)\right)+z\left(\theta_t\omega\right)\left(\phi,v_n(t)\right),\qquad \text{a.e. } t\in[0,t_n),
\end{aligned}
\tag{3.9}
$$

where $[0,t_n)$ is the interval of existence of maximal solution. The first term on the right-hand side of (3.9) can be bounded by

$$
\begin{aligned}
&-a\left(l\left(v_n(t)\right)+z\left(\theta_t\omega\right)l(\phi)\right)z\left(\theta_t\omega\right)\left(A\phi,v_n(t)\right)_{V^*,V}\\
\le&\frac{M^2C(n,\gamma)}{2m}|z\left(\theta_t\omega\right)|^2\|\phi\|_V^2+\frac{\mu m}{4}\|v_n(t)\|^2+\frac{mC(n,\gamma)}{8}\|v_n(t)\|^2_{\dot{H}^\gamma(\mathbb{R}^n)}.
\end{aligned}
\tag{3.10}
$$

By (2.7) and the Young inequality, the second term on the right-hand side of (3.9) can be bounded by

$$
\begin{aligned}
\left(F\left(v_n(t)+\phi z\left(\theta_t\omega\right)\right),v_n(t)\right)\le&\alpha\int_{\mathcal{O}}(|v_n(t)+\phi z\left(\theta_t\omega\right)|+1)v_n(t)dx\\
\le&(\alpha+\frac{\mu m}{4})\|v_n(t)\|^2+\frac{2\alpha^2}{\mu m}|z\left(\theta_t\omega\right)|^2\|\phi\|^2+\frac{2\alpha^2}{\mu m}|\mathcal{O}|.
\end{aligned}
\tag{3.11}
$$

By the Young inequality, the last two terms on the right-hand side of (3.9) can be bounded by

$$
(h(t),v_n(t))\le\frac{2}{\mu m}\|h(t)\|^2+\frac{\mu m}{8}\|v_n(t)\|^2
\tag{3.12}
$$

and

$$
z\left(\theta_t\omega\right)\left(\phi,v_n(t)\right)\le\frac{2}{\mu m}|z\left(\theta_t\omega\right)|^2\|\phi\|^2+\frac{\mu m}{8}\|v_n(t)\|^2.
\tag{3.13}
$$

(3.9)-(3.13) imply

$$
\begin{aligned}
&\frac{d}{dt}\left\|v_n(t)\right\|^2+\frac{\mu m}{2}\left\|v_n(t)\right\|_V^2\\
\le&2\alpha\left\|v_n(t)\right\|^2+[\frac{M^2C(n,\gamma)}{m}+\frac{4\alpha^2+4}{\mu m}]\left|z\left(\theta_t\omega\right)\right|^2\left\|\phi\right\|_V^2+\frac{4}{\mu m}\|h(t)\|^2+\frac{4\alpha^2|\mathcal{O}|}{\mu m}.
\end{aligned}
\tag{3.14}
$$

Integrating (3.14) between τ and t with $\tau\le t<t_n$, we have

$$
\begin{aligned}
&\left\|v_n(t)\right\|^2+\frac{\mu m}{2}\int_\tau^t\left\|v_n(r)\right\|_V^2dr\\
\le&2\alpha\int_\tau^t\left\|v_n(r)\right\|^2dr+\left\|v_n(\tau)\right\|^2+\|\phi\|_V^2\int_\tau^t\left|z\left(\theta_r\omega\right)\right|^2dr+\frac{4}{\mu m}\int_\tau^t\|h(r)\|^2dr+\frac{4\alpha^2|\mathcal{O}|}{\mu m}(t-\tau).
\end{aligned}
$$

By the Gronwall lemma, we infer that $\{v_n(\cdot)\}_{n=1}^{+\infty}$ is well defined on $[\tau, t_n)$. Actually, for all $T>\tau$ and for almost all $\omega \in \Omega$, it is bounded in $L^{\infty}((\tau, T); H) \cap L^2((\tau, T); V)$. Since

$$a\left(l\left(v_n(t)\right)+z\left(\theta_t \omega\right) l(\phi)\right) \leq M, \quad \forall t \in(\tau, T), \quad \forall n \geq 1 .$$

There exists a positive constant C such that

$$\int_\tau^T\left|a\left(l\left(v_n(t)\right)+z\left(\theta_t \omega\right) l(\phi)\right)\right|\left\|A v_n(t)\right\|_{V^*}^2 d t \leq C \int_\tau^T\left\|v_n(t)\right\|_V^2 d t,$$

and

$$\int_\tau^T\left|a\left(l\left(v_n(t)\right)+z\left(\theta_t \omega\right) l(\phi)\right)\right|\|A \phi\|_{V^*}^2 d t \leq C\|\phi\|_V^2(T-\tau) .$$

Note that $\phi \in V$, we deduce that the sequences $\left\{a\left(l\left(v_n(t)\right)+z\left(\theta_t \omega\right) l(\phi)\right) A v_n(t)\right\}_{n=1}^{+\infty}$ and $\left\{a\left(l\left(v_n(t)\right)+z\left(\theta_t \omega\right) l(\phi)\right) A \phi\right\}_{n=1}^{+\infty}$ are bounded in $L^2\left((\tau, T) ; V^*\right)$. On the other hand, we have

$$\begin{aligned}
& \int_\tau^T \int_{\mathcal{O}}\left|F\left(v_n(t)+z\left(\theta_t \omega\right) \phi\right)\right|^2 d x d t \\
\leq & 2 \alpha^2 \int_\tau^T \int_{\mathcal{O}}\left(1+\left|v_n(t)+\phi z\left(\theta_t \omega\right)\right|^2\right) d x d t \\
\leq & 4 \alpha^2 \int_\tau^T\left|v_n(t)\right|^2 d t+4 \alpha^2|\phi|^2 \int_\tau^T\left|z\left(\theta_t \omega\right)\right|^2 d t+2 \alpha^2(T-\tau)|\mathcal{O}| .
\end{aligned}$$

Since $\{v_n\}_{n=1}^{+\infty}$ is bounded in $L^{\infty}((\tau, T); H)$, $z(\theta.\omega)$ is continuous in (τ, T) and $\phi \in V$, we have $\{F(v_n+\phi z(\theta.\omega))\}_{n=1}^{+\infty}$ is bounded in $L^2((\tau, T); H)$.

To prove the sequence $\{v_n'\}_{n=1}^{+\infty}$ is bounded in $L^2((\tau, T); V^*)$, we define the projector $P_n: V^* \rightarrow V^*$ as in the proof of [45, Theorem 2.7]. Then by the above estimates, we derive

$$\begin{aligned}
& \int_\tau^T\left\|v_n^{\prime}(t)\right\|_{V^*}^2 d t \\
\leq & \int_\tau^T \|-a\left(l\left(v_n(t)\right)+z\left(\theta_t \omega\right) l(\phi)\right)\left(A v_n(t)+A \phi\right)+P_n F\left(v_n(t)+\phi z\left(\theta_t \omega\right)\right)+P_n h(t)+\phi z\left(\theta_t \omega\right) \|_{V^*}^2 d t \\
\leq & \left(C+4 \alpha^2\right) \int_\tau^T\left\|v_n(t)\right\|_V^2 d t+\left(1+4 \alpha^2\right)\|\phi\|^2 \int_\tau^T\left|z\left(\theta_t \omega\right)\right|^2 d t+C\|\phi\|_V^2(T-\tau)+2 \alpha^2(T-\tau)|\mathcal{O}|+\|h(t)\|^2,
\end{aligned}$$

which means $\{v_n'\}_{n=1}^{+\infty}$ is bounded in $L^2((\tau, T); V^*)$.

From compactness arguments and the Aubin-Lions lemma, there exist a subsequence of $\{v_n\}_{n=1}^{+\infty}$ (relabeled the same), $\tilde{v} \in H$, $v \in L^{\infty}((\tau, T); H) \cap L^2((\tau, T); V)$, $\tilde{F} \in L^2((\tau, T); H)$ and $\chi \in L^2((\tau, T); V)$ such that for all $T>\tau$

$$\begin{cases}
v_n \rightarrow v \text { weakly-star in } L^{\infty}((\tau, T) ; H), \\
v_n \rightarrow v \text { weakly in } L^2((\tau, T) ; V), \\
\left(v_n\right)^{\prime} \rightarrow v^{\prime} \text { weakly in } L^2\left((\tau, T) ; V^*\right), \\
v_n \rightarrow v \text { strongly in } L^2((\tau, T) ; H), \\
v_n(x, t) \rightarrow v(x, t), \text { a.e. }(x, t) \in \mathcal{O} \times(\tau, T), \\
v_n(t) \rightarrow v(t) \text { strongly in } H, \text { a.e. } t \in(\tau, T), \\
F\left(v_n+\phi z(\theta . \omega)\right) \rightarrow \tilde{F} \text { weakly in } L^2((\tau, T) ; H), \\
a\left(l\left(v_n\right)+z(\theta . \omega) l(\phi)\right) v_n \rightarrow \chi \text { weakly in } L^2((\tau, T) ; V), \\
v_n(T) \rightarrow \tilde{v} \text { weakly in } H .
\end{cases} \tag{3.15}$$

Next, we take the limit for each term to prove the existence of weak solutions to problem (2.9). Firstly, we will prove that $\chi = a\left(l\left(v\right) + z\left(\theta.\omega\right)l(\phi)\right)v$. As $a \in C\left(\mathbb{R};\mathbb{R}^+\right), l \in \mathcal{L}(L^2(\mathcal{O});\mathbb{R})$ and (3.15) holds, we have

$$a\left(l\left(v_n\right) + z\left(\theta.\omega\right)l(\phi)\right) \to a\left(l\left(v\right) + z\left(\theta.\omega\right)l(\phi)\right).$$

Therefore,

$$a\left(l\left(v_n\right) + z\left(\theta.\omega\right)l(\phi)\right)v_n \to a\left(l\left(v\right) + z\left(\theta.\omega\right)l(\phi)\right)v.$$

From [29, Lemma 1.3] and the fact that $\{a\left(l\left(v_n\right) + z\left(\theta.\omega\right)l(\phi)\right)v_n\}_{n=1}^{+\infty}$ is bounded in $L^2\left((\tau,T);V\right)$, we have $\chi = a\left(l\left(v\right) + z\left(\theta.\omega\right)l(\phi)\right)v$.

Secondly, we need to check that $\tilde{F} = F\left(v + \phi z\left(\theta.\omega\right)\right)$. Since F is continuous and $v_n(\cdot)$ converges to $v(\cdot)$ strongly in H, a.e. $t \in (\tau,T)$, we deduce that there exists a subsequence (not relabeled) such that

$$F\left(v_n(t) + \phi z\left(\theta_t\omega\right)\right) \to F\left(v(t) + \phi z\left(\theta_t\omega\right)\right), \text{ a.e. } (x,t) \in \mathcal{O}\times(\tau,T).$$

In addition, $\{F\left(v_n + \phi z\left(\theta.\omega\right)\right)\}_{n=1}^{+\infty}$ is bounded in $L^2\left((\tau,T);H\right)$. Applying [29, Lemma 1.3], it follows that $\tilde{F} = F\left(v + \phi z\left(\theta.\omega\right)\right)$.

To prove that v is a weak solution to (2.9), it remains to check that $v(\tau) = v_\tau$ and $v(T) = \tilde{v}$. For any $\varphi \in V$, multiplying by φ in (3.2), integrating between τ and T and then taking the limit when $n \to +\infty$, we deduce that

$$\begin{aligned}
&\left(\tilde{v}, w_j\right)\varphi(T) - \left(v_\tau, w_j\right)\varphi(\tau) - \int_\tau^T \varphi'(t)(v(t), w_j)dt \\
&+ \int_\tau^T a\left(l\left(v(t)\right) + z\left(\theta_t\omega\right)l(\phi)\right)b\left(v(t), w_j\right)\varphi(t)dt \\
&+ \int_\tau^T a\left(l\left(v(t)\right) + z\left(\theta_t\omega\right)l(\phi)\right)z\left(\theta_t\omega\right)\left(A\phi, w_j\right)_{V^*,V}\varphi(t)dt \\
=& \int_\tau^T \left(F\left(v(t) + \phi z\left(\theta_t\omega\right)\right), w_j\right)\varphi(t)dt + \int_\tau^T (h(t), w_j)\varphi(t)dt + \int_\tau^T z\left(\theta_t\omega\right)\left(\phi, w_j\right)\varphi(t)dt.
\end{aligned} \tag{3.16}$$

On the other hand, multiplying by φ in (3.1) and then integrating between τ and T, we have

$$\begin{aligned}
&\left(v(T), w_j\right)\varphi(T) - \left(v(\tau), w_j\right)\varphi(\tau) - \int_\tau^T \varphi'(t)(v(t), w_j)dt \\
&+ \int_\tau^T a\left(l\left(v(t)\right) + z\left(\theta_t\omega\right)l(\phi)\right)b\left(v(t), w_j\right)\varphi(t)dt \\
&+ \int_\tau^T a\left(l\left(v(t)\right) + z\left(\theta_t\omega\right)l(\phi)\right)z\left(\theta_t\omega\right)\left(A\phi, w_j\right)_{V^*,V}\varphi(t)dt \\
=& \int_\tau^T \left(F\left(v(t) + \phi z\left(\theta_t\omega\right)\right), w_j\right)\phi(t)dt + \int_\tau^T (h(t), w_j)\varphi(t)dt + \int_\tau^T z\left(\theta_t\omega\right)\left(\phi, w_j\right)\varphi(t)dt.
\end{aligned} \tag{3.17}$$

Comparing (3.16) with (3.17), we find $\left(\tilde{v}, w_j\right)\varphi(T) - \left(v_\tau, w_j\right)\varphi(\tau) = \left(v(T), w_j\right)\varphi(T) - \left(v(\tau), w_j\right)\varphi(\tau)$. Since φ is arbitrary and $\{\omega_j\}_{j=1}^{+\infty}$ is an orthonormal basis of H, we obtain that $v_\tau = v(\tau)$ and $v(T) = \tilde{v}$ in H.

Step 4: Uniqueness and continuity with respect to initial data. Assume that there exist two weak solutions, $v_1\left(\cdot,\tau,\omega,v_\tau^1\right)$ and $v_2\left(\cdot,\tau,\omega,v_\tau^2\right)$ to equation (2.9). Then we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|v_1(t)-v_2(t)\right\|^2+a\left(l\left(v_1(t)\right)+z\left(\theta_t\omega\right)l(\phi)\right)b(v_1(t)-v_2(t),v_1(t)-v_2(t))\\
\leq&\left|\left[a\left(l\left(v_2(t)\right)+z\left(\theta_t\omega\right)l(\phi)\right)-a\left(l\left(v_1(t)\right)+z\left(\theta_t\omega\right)l(\phi)\right)\right]\left(Av_2(t),v_1(t)-v_2(t)\right)_{V^*,V}\right|\\
&+\left|\left[a\left(l\left(v_1(t)+z\left(\theta_t\omega\right)l(\phi)\right)\right)-a\left(l\left(v_2(t)\right)+z\left(\theta_t\omega\right)l(\phi)\right)\right]\left(A\phi,v_1(t)-v_2(t)\right)_{V^*,V}\right|\\
&+\Big(F\left(v_1(t)+\phi z\left(\theta_t\omega\right)\right)-F\left(v_2(t)+\phi z\left(\theta_t(\omega)\right)\right),v_1(t)-v_2(t)\Big).
\end{aligned}$$

It is easy to prove the uniqueness and continuity with respect to initial data in H owing to the same reasons as those in the proof of uniqueness of local solution. □

By the relationship between u and v, we can easily obtain that (2.8) possesses a weak solution. We now define a mapping $\Phi:\mathbb{R}^+\times\mathbb{R}\times\Omega\times H\to H$ such that, for all $t\in\mathbb{R}^+$, $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $u_\tau\in H$,

$$\Phi\left(t,\tau,\omega,u_\tau\right)=u\left(t+\tau,\tau,\theta_{-\tau}\omega,u_\tau\right)=v\left(t+\tau,\tau,\theta_{-\tau}\omega,v_\tau\right)+\phi z(\theta_t\omega).$$

Then Φ is a continuous cocycle on H over $\left(\Omega,\mathcal{F},\mathbb{P},\{\theta_t\}_{t\in\mathbb{R}}\right)$.

4 Existence of random attractors to problem (2.8)

This section is devoted to uniform estimates of solutions for the fractional stochastic nonlocal reaction-diffusion equations (2.8). Next, we introduce further hypothesis on h to complete the uniform estimates.
(H_2) Assume that for every $\tau\in\mathbb{R}$,

$$\int_{-\infty}^{0}e^{\frac{\mu m}{2}s}\|h(s+\tau)\|^2ds<+\infty, \tag{4.1}$$

and for every $\beta>0$,

$$\lim_{t\to+\infty}e^{-\beta t}\int_{-\infty}^{0}e^{\frac{\mu m}{2}s}\|h(s-t)\|^2ds=0. \tag{4.2}$$

Lemma 4.1. *Suppose the conditions of Theorem 3.1 and (H_2) hold. Then for every $\sigma\in\mathbb{R}$, $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $B=\{B(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$, there exists $T=T\left(\tau,\omega,B,\sigma,m,\mu\right)\geq 0$ such that for all $t\geq T$, the solution u of problem (2.8) satisfies*

$$\begin{aligned}
&\left\|u\left(\sigma,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}\right)\right\|^2\\
\leq&1+2C_1\int_{-\infty}^{\sigma-\tau}e^{\frac{\mu m}{2}(s+\tau-\sigma)}\left|z\left(\theta_s\omega\right)\right|^2\|\phi\|^2ds\\
&+\frac{8\mu M^2}{m}\int_{-\infty}^{\sigma-\tau}e^{\frac{\mu m}{2}(s+\tau-\sigma)}\left|z\left(\theta_s\omega\right)\right|^2\|(-\Delta)^\gamma\phi\|^2ds\\
&+\frac{4}{\mu m}\int_{-\infty}^{\sigma-\tau}e^{\frac{\mu m}{2}(s+\tau-\sigma)}\|h(s+\tau)\|^2ds+\frac{4C_2}{\mu m}+2|z\left(\theta_{\sigma-\tau}\omega\right)|^2\|\phi\|^2,
\end{aligned}$$

where $u_{\tau-t}\in B(\tau-t,\,\theta_{-t}\omega)$.

Proof. Taking the inner product of (2.9) with $v(t)=v\left(t,\tau,\omega,v_\tau\right)$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|v(t)\right\|^2+m\Big(\frac{C(n,\gamma)}{2}\|v(t)\|^2_{\dot{H}^\gamma(\mathbb{R}^n)}+\mu\|v(t)\|^2\Big)\\
\leq&-a\left(l\left(v(t)\right)+z\left(\theta_t\omega\right)l(\phi)\right)z\left(\theta_t\omega\right)\left(A\phi,v(t)\right)_{V^*,V}+\left(F\left(v(t)+\phi z\left(\theta_t\omega\right)\right),v(t)\right)\\
&+\left(h(t),v(t)\right)+z\left(\theta_t\omega\right)\left(\phi,v(t)\right).
\end{aligned}\tag{4.3}$$

The first term on the right-hand side of (4.3) can be bounded by

$$\begin{aligned}
&-a\left(l\left(v(t)\right)+z\left(\theta_t\omega\right)l(\phi)\right)z\left(\theta_t\omega\right)\left(A\phi,v(t)\right)_{V^*,V}\\
\le&\mu M|z\left(\theta_t\omega\right)|(\phi,v(t))+M|z\left(\theta_t\omega\right)|((-\Delta)^\gamma\phi,v(t))\\
\le&\frac{\mu m}{4}\|v(t)\|^2+\frac{2\mu M^2}{m}|z\left(\theta_t\omega\right)|^2\|\phi\|^2+\frac{2\mu M^2}{m}|z\left(\theta_t\omega\right)|^2\|(-\Delta)^\gamma\phi\|^2.
\end{aligned} \tag{4.4}$$

By (2.6), (2.7) and the Young inequality, the second term on the right-hand side of (4.3) can be bounded by

$$\begin{aligned}
&\left(F\left(v(t)+\phi z\left(\theta_t\omega\right)\right),v(t)\right)\\
=&\int_{\mathcal{O}}F\left(v(t)+\phi z\left(\theta_t\omega\right)\right)\left(v(t)+\phi z\left(\theta_t\omega\right)\right)dx-z\left(\theta_t\omega\right)\int_{\mathcal{O}}F\left(v(t)+\phi z\left(\theta_t\omega\right)\right)\phi dx\\
\le&-\left(\beta_2-M\mu\right)\int_{\mathcal{O}}|v(t)+\phi z\left(\theta_t\omega\right)|^2dx+\kappa|\mathcal{O}|+\alpha|z\left(\theta_t\omega\right)|\int_{\mathcal{O}}(|v(t)+\phi z\left(\theta_t\omega\right)|+1)\phi dx\\
\le&-\left(\beta_2-M\mu\right)\int_{\mathcal{O}}|v(t)+\phi z\left(\theta_t\omega\right)|^2dx+\kappa|\mathcal{O}|+\frac{\beta_2-M\mu}{2}\int_{\mathcal{O}}|v(t)+\phi z\left(\theta_t\omega\right)|^2dx\\
&+\frac{\beta_2-M\mu}{2}|\mathcal{O}|+\frac{\alpha^2|z\left(\theta_t\omega\right)|^2}{\beta_2-M\mu}\|\phi\|^2\\
=&-\frac{\beta_2-M\mu}{2}\int_{\mathcal{O}}|v(t)+\phi z\left(\theta_t\omega\right)|^2dx+(\kappa+\frac{\beta_2-M\mu}{2})|\mathcal{O}|+\frac{\alpha^2}{\beta_2-M\mu}|z\left(\theta_t\omega\right)|^2\|\phi\|^2.
\end{aligned} \tag{4.5}$$

By the Young inequality, the last two terms on the right-hand side of (4.3) can be estimated as

$$(h(t),v(t))\le\frac{1}{\mu m}\|h(t)\|^2+\frac{\mu m}{4}\|v(t)\|^2 \tag{4.6}$$

and

$$z\left(\theta_t\omega\right)\left(\phi,v(t)\right)\le\frac{1}{\mu m}|z\left(\theta_t\omega\right)|^2\|\phi\|^2+\frac{\mu m}{4}\|v(t)\|^2. \tag{4.7}$$

Together with (4.3)-(4.7), we obtain

$$\begin{aligned}
&\frac{d}{dt}\|v(t)\|^2+\frac{\mu m}{2}\|v(t)\|^2+mC(n,\gamma)\|v(t)\|^2_{\dot{H}^\gamma(\mathbb{R}^n)}\\
\le&\Big[\frac{4\mu M^2}{m}+\frac{2}{\mu m}+\frac{2\alpha^2}{\beta_2-M\mu}\Big]|z\left(\theta_t\omega\right)|^2\|\phi\|^2+\frac{4\mu M^2}{m}|z\left(\theta_t\omega\right)|^2\|(-\Delta)^\gamma\phi\|^2\\
&+\frac{2}{\mu m}\|h(t)\|^2+(2\kappa+\beta_2-M\mu)|\mathcal{O}|\\
:=&C_1|z\left(\theta_t\omega\right)|^2\|\phi\|^2+\frac{4\mu M^2}{m}|z\left(\theta_t\omega\right)|^2\|(-\Delta)^\gamma\phi\|^2+\frac{2}{\mu m}\|h(t)\|^2+C_2,
\end{aligned} \tag{4.8}$$

where $C_1=\dfrac{4\mu M^2}{m}+\dfrac{2}{\mu m}+\dfrac{2\alpha^2}{\beta_2-M\mu}$ and $C_2=(2\kappa+\beta_2-M\mu)|\mathcal{O}|$.
Multiplying (4.8) by $e^{\frac{\mu m}{2}t}$ and then integrating over $(\tau-t,\sigma)$ with $\sigma\ge\tau-t$, we have for every $\omega\in\Omega$,

$$\begin{aligned}
&\|v\left(\sigma,\tau-t,\omega,v_{\tau-t}\right)\|^2+mC(n,\gamma)\int_{\tau-t}^{\sigma}e^{\frac{\mu m}{2}(s-\sigma)}\|v(s,\tau-t,\omega,v_{\tau-t})\|^2_{\dot{H}^\gamma(\mathbb{R}^n)}ds\\
\le&e^{\frac{\mu m}{2}(\tau-t-\sigma)}\|v_{\tau-t}\|^2+C_1\int_{\tau-t}^{\sigma}e^{\frac{\mu m}{2}(s-\sigma)}|z\left(\theta_s\omega\right)|^2\|\phi\|^2ds\\
&+\frac{4\mu M^2}{m}\int_{\tau-t}^{\sigma}e^{\frac{\mu m}{2}(s-\sigma)}|z\left(\theta_s\omega\right)|^2\|(-\Delta)^\gamma\phi\|^2ds\\
&+\frac{2}{\mu m}\int_{\tau-t}^{\sigma}e^{\frac{\mu m}{2}(s-\sigma)}\|h(s)\|^2ds+\frac{2C_2}{\mu m}(1-e^{\frac{\mu m}{2}(\tau-t-\sigma)}).
\end{aligned}$$

Replacing ω by $\theta_{-\tau}\omega$ and then changing variables, we obtain the following result

$$\begin{aligned}
&\|v(\sigma,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2+mC(n,\gamma)\int_{\tau-t}^{\sigma}e^{\frac{\mu m}{2}(s-\sigma)}\|v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|_{\dot{H}^\gamma(\mathbb{R}^n)}^2ds\\
\le & e^{\frac{\mu m}{2}(\tau-t-\sigma)}\|v_{\tau-t}\|^2+C_1\int_{-t}^{\sigma-\tau}e^{\frac{\mu m}{2}(s+\tau-\sigma)}|z(\theta_s\omega)|^2\|\phi\|^2ds\\
&+\frac{4\mu M^2}{m}\int_{-t}^{\sigma-\tau}e^{\frac{\mu m}{2}(s+\tau-\sigma)}|z(\theta_s\omega)|^2\|(-\Delta)^\gamma\phi\|^2ds\\
&+\frac{2}{\mu m}\int_{-t}^{\sigma-\tau}e^{\frac{\mu m}{2}(s+\tau-\sigma)}\|h(s+\tau)\|^2ds+\frac{2C_2}{\mu m}(1-e^{\frac{\mu m}{2}(\tau-t-\sigma)}).
\end{aligned}$$

Since $v(t,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})=u(t,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})-\phi z(\theta_{t-\tau}\omega)$ with $v_{\tau-t}=u_{\tau-t}-\phi z(\theta_{-t}\omega)$, we have

$$\begin{aligned}
&\|u(\sigma,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2\\
\le & 2e^{\frac{\mu m}{2}(\tau-t-\sigma)}\|u_{\tau-t}-\phi z(\theta_{-t}\omega)\|^2+2C_1\int_{-t}^{\sigma-\tau}e^{\frac{\mu m}{2}(s+\tau-\sigma)}|z(\theta_s\omega)|^2\|\phi\|^2ds\\
&+\frac{8\mu M^2}{m}\int_{-t}^{\sigma-\tau}e^{\frac{\mu m}{2}(s+\tau-\sigma)}|z(\theta_s\omega)|^2\|(-\Delta)^\gamma\phi\|^2ds\\
&+\frac{4}{\mu m}\int_{-t}^{\sigma-\tau}e^{\frac{\mu m}{2}(s+\tau-\sigma)}\|h(s+\tau)\|^2ds+\frac{4C_2}{\mu m}(1-e^{\frac{\mu m}{2}(\tau-t-\sigma)})+2|z(\theta_{\sigma-\tau}\omega)|^2\|\phi\|^2\\
\le & 4e^{\frac{\mu m}{2}(\tau-t-\sigma)}\|u_{\tau-t}\|^2+4e^{\frac{\mu m}{2}(\tau-t-\sigma)}\|\phi\|^2|z(\theta_{-t}\omega)|^2\\
&+2C_1\int_{-t}^{\sigma-\tau}e^{\frac{\mu m}{2}(s+\tau-\sigma)}|z(\theta_s\omega)|^2\|\phi\|^2ds\\
&+\frac{8\mu M^2}{m}\int_{-t}^{\sigma-\tau}e^{\frac{\mu m}{2}(s+\tau-\sigma)}|z(\theta_s\omega)|^2\|(-\Delta)^\gamma\phi\|^2ds\\
&+\frac{4}{\mu m}\int_{-t}^{\sigma-\tau}e^{\frac{\mu m}{2}(s+\tau-\sigma)}\|h(s+\tau)\|^2ds+\frac{4C_2}{\mu m}(1-e^{\frac{\mu m}{2}(\tau-t-\sigma)})+2|z(\theta_{\sigma-\tau}\omega)|^2\|\phi\|^2.
\end{aligned}\tag{4.9}$$

Note that $u_{\tau-t}\in B(\tau-t,\theta_{-t}\omega)$ and $B=\{B(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}$ is tempered, there exists $T_1=T_1(\tau,\omega,\sigma,B,m,\mu)>0$ such that for all $t\ge T_1$,

$$4e^{\frac{\mu m}{2}(\tau-t-\sigma)}\|u_{\tau-t}\|^2\le\frac{1}{2}.\tag{4.10}$$

By Lemma 2.1, there exists $T_2=T_2(\omega,m,\mu)\ge T_1$ such that for all $t\ge T_2$,

$$4e^{\frac{\mu m}{2}(\tau-t-\sigma)}\|\phi\|^2|z(\theta_{-t}\omega)|^2\le\frac{1}{2}.\tag{4.11}$$

Since $\phi\in V$, we deduce that

$$\frac{8\mu M^2}{m}\int_{-t}^{\sigma-\tau}e^{\frac{\mu m}{2}(s+\tau-\sigma)}|z(\theta_s\omega)|^2\|(-\Delta)^\gamma\phi\|^2ds<+\infty.\tag{4.12}$$

On the other hand, (4.1) and the continuity of $z(\theta.\omega)$ indicate that

$$2C_1\int_{-t}^{\sigma-\tau}e^{\frac{\mu m}{2}(s+\tau-\sigma)}|z(\theta_s\omega)|^2\|\phi\|^2ds+\frac{4}{\mu m}\int_{-t}^{\sigma-\tau}e^{\frac{\mu m}{2}(s+\tau-\sigma)}\|h(s+\tau)\|^2ds<+\infty.\tag{4.13}$$

By (4.9)-(4.13), we have for all $t \geq T_2$,

$$
\begin{aligned}
&\|u(\sigma,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2 \\
\leq & 1+2C_1\int_{-t}^{\sigma-\tau} e^{\frac{\mu m}{2}(s+\tau-\sigma)}\left|z(\theta_s\omega)\right|^2\|\phi\|^2 ds \\
&+\frac{8\mu M^2}{m}\int_{-t}^{\sigma-\tau} e^{\frac{\mu m}{2}(s+\tau-\sigma)}\left|z(\theta_s\omega)\right|^2\|(-\Delta)^\gamma\phi\|^2 ds \\
&+\frac{4}{\mu m}\int_{-t}^{\sigma-\tau} e^{\frac{\mu m}{2}(s+\tau-\sigma)}\|h(s+\tau)\|^2 ds+\frac{4C_2}{\mu m}+2|z(\theta_{\sigma-\tau}\omega)|^2\|\phi\|^2.
\end{aligned}
$$

Therefore, the proof is complete. □

Based on Lemma 4.1, we will prove that (2.8) has a $\mathcal{D}$-pullback absorbing set in H.

Lemma 4.2. *Suppose the conditions of Theorem 3.1 and (H_2) hold. Then the continuous cocycle Φ has a $\mathcal{D}$-pullback absorbing set $B=\{B(\tau,\omega): \tau\in\mathbb{R}, \omega\in\Omega\}\in\mathcal{D}$, which is given by*

$$
B(\tau,\omega)=\left\{u\in H: \|u\|^2\leq R(\tau,\omega)\right\},
$$

where

$$
\begin{aligned}
R(\tau,\omega)=&1+2C_1\int_{-\infty}^{0} e^{\frac{\mu m}{2}s}\left|z(\theta_s\omega)\right|^2\|\phi\|^2 ds+\frac{8\mu M^2}{m}\int_{-\infty}^{0} e^{\frac{\mu m}{2}s}\left|z(\theta_s\omega)\right|^2\|(-\Delta)^\gamma\phi\|^2 ds \\
&+\frac{4}{\mu m}\int_{-\infty}^{0} e^{\frac{\mu m}{2}s}\|h(s+\tau)\|^2 ds+\frac{4C_2}{\mu m}+2|z(\omega)|^2\|\phi\|^2.
\end{aligned}
$$

Proof. As a special case of Lemma 4.1 with $\sigma=\tau$, there exists $T=T(\tau,\omega,B,m,\mu,\alpha)>0$ such that for all $t\geq T$,

$$
u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\in B(\tau,\omega).
$$

Now we only need to check that R is tempered, i.e., for any $\beta>0$,

$$
\lim_{t\to+\infty} e^{-\beta t}R(\tau-t,\theta_{-t}\omega)=0. \tag{4.14}
$$

In fact,

$$
\begin{aligned}
&e^{-\beta t}R(\tau-t,\theta_{-t}\omega) \\
=&e^{-\beta t}+2C_1e^{-\beta t}\int_{-\infty}^{0} e^{\frac{\mu m}{2}s}\left|z(\theta_{s-t}\omega)\right|^2\|\phi\|^2 ds \\
&+\frac{8\mu M^2}{m}e^{-\beta t}\int_{-\infty}^{0} e^{\frac{\mu m}{2}s}\left|z(\theta_{s-t}\omega)\right|^2\|(-\Delta)^\gamma\phi\|^2 ds \\
&+\frac{4}{\mu m}e^{-\beta t}\int_{-\infty}^{0} e^{\frac{\mu m}{2}s}\|h(s+\tau)\|^2 ds+\frac{4C_2}{\mu m}e^{-\beta t}+2e^{-\beta t}|z(\theta_{-t}\omega)|^2\|\phi\|^2.
\end{aligned}
$$

Combining Lemma 2.1 with (4.2), one can easily check that $e^{-\beta t}R(\tau-t,\theta_{-t}\omega)\to 0$ as $t\to+\infty$. □

Lemma 4.3. *Suppose the conditions of Lemma 4.1 hold. Then the continuous cocycle Φ is asymptotically compact in H.*

Proof. By Lemma 4.1, there exist $T=T(\tau,\omega,B,m,\mu,\alpha)>0$ and $C=C(\tau,\omega,m,\mu,\alpha)>0$ such that, for all $t\geq T$ and $u_\tau\in B(\tau-t,\theta_{-t}\omega)$,

$$
\|u(\tau-1,\tau-t,\theta_{-\tau}\omega,u_\tau)\|\leq C. \tag{4.15}
$$

When $t_k \to +\infty$ $(k \to +\infty)$ and $u_{\tau,k} \in B(\tau - t_k, \theta_{-t_k}\omega)$, there is $K_1 = K_1(\tau, \omega, B) > 0$ such that for all $k \geq K_1$,

$$\|u(\tau - 1, \tau - t_k, \theta_{-\tau}\omega, u_{\tau,k})\| \leq C.$$

This shows that

$$\{u(\tau - 1, \tau - t_k, \theta_{-\tau}\omega, u_{\tau,k})\} \text{ is bounded in } H. \tag{4.16}$$

Follows from [43, Lemma 3.3], there exists $\widehat{u} \in L^2(\tau - 1, \tau; H)$ such that, up to a subsequence, and for all $n \in \mathbb{N}$ as $k \to +\infty$

$$\begin{aligned} u(\cdot, \tau - t_k, \theta_{-\tau}\omega, u_{\tau,k}) =& u(\cdot, \tau - 1, \theta_{-\tau}\omega, u(\tau - 1, \tau - t_k, \theta_{-\tau}\omega, u_{\tau,k})) \\ \to& \widehat{u}(\cdot) \text{ in } L^2([\tau - 1, \tau]; H). \end{aligned} \tag{4.17}$$

By choosing a further subsequence (which we do not relabel), it follows from (4.17) that

$$u(s, \tau - t_k, \theta_{-\tau}\omega, u_{\tau,k}) \to \widehat{u}(s) \text{ in } H \text{ for almost all } s \in [\tau - 1, \tau], \tag{4.18}$$

which means the sequence $\{u(s, \tau - t_k, \theta_{-\tau}\omega, u_{\tau,k})\}$ has a convergent subsequence in H. The proof is complete. □

By Lemma 4.2, Lemma 4.3 and Theorem 2.1, we deduce the existence of $\mathcal{D}$-pullback attractors for Φ as stated in the following result.

Theorem 4.1. *Suppose the conditions of Lemma 4.1 hold. Then the continuous cocycle Φ generated by problem (2.8) has a unique $\mathcal{D}$-pullback random attractor $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ in H.*

5 Existence of random attractors to fractional random nonlocal PDEs driven by colored noise

In this section, we discuss the following equation driven by colored noise as an approximation to fractional stochastic nonlocal differential equation (1.1),

$$\begin{cases} \dfrac{\partial u_\delta}{\partial t} + a(l(u_\delta))(-\Delta)^\gamma u_\delta = f(u_\delta) + h(t) + \phi\zeta_\delta(\theta_t\omega), & \text{in } \mathcal{O} \times (\tau, +\infty), \\ u_\delta = 0, & \text{on } \partial\mathcal{O} \times (\tau, +\infty), \\ u_\delta(x, \tau) = u_{\delta,\tau}(x), & \text{in } \mathcal{O}, \end{cases} \tag{5.1}$$

where the process $\zeta_\delta(\theta_t\omega)$ is the so-called colored noise. For convenience, we rewrite (5.1) in the following form, similarly as we did with (2.8):

$$\begin{cases} \dfrac{\partial u_\delta}{\partial t} + a(l(u_\delta))(-\Delta)^\gamma u_\delta + a(l(u_\delta))\mu u_\delta = F(u_\delta) + h(t) + \phi\zeta_\delta(\theta_t\omega), & x \in \mathcal{O}, t > \tau, \\ u_\delta(x, t) = 0, & x \in \mathbb{R}^n \backslash \mathcal{O}, t > \tau, \\ u_\delta(x, \tau) = u_{\delta,\tau}(x), & x \in \mathcal{O}. \end{cases} \tag{5.2}$$

Let us recall some properties of the colored noise which will be useful in our subsequent analysis. Although it can be found in several published works, we prefer to present them here to make our paper more readable.

For every $\omega \in \Omega$ and $\delta > 0$, y_δ satisfies

$$\frac{dy_\delta}{dt} = -y_\delta + \zeta_\delta(\theta_t\omega),$$

which possesses a special solution given by

$$y_\delta(\theta_t\omega) = e^{-t}\int_{-\infty}^{t} e^s \zeta_\delta(\theta_s\omega)ds.$$

And it has the following properties:

Lemma 5.1. *[20, Lemma 3.2]* (1) *The mapping $t \mapsto y_\delta(\theta_t\omega)$ is continuous. For every $\omega \in \Omega$ and $0 < \delta \leq \frac{1}{2}$,*

$$\lim_{t\to\pm\infty} \frac{|y_\delta(\theta_t\omega)|}{|t|} = 0,$$

and

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t y_\delta(\theta_s\omega)\,\mathrm{d}s = 0.$$

(2) *Let $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $T > 0$. Then for every $\varepsilon > 0$, there exists $\delta_0 = \delta_0(\tau, \omega, T, \varepsilon) > 0$ such that for all $0 < \delta < \delta_0$ and $t \in [\tau, \tau + T]$,*

$$|y_\delta(\theta_t\omega) - z(\theta_t\omega)| < \varepsilon.$$

Performing the change of variable $v_\delta(t) = u_\delta(t) - \phi y_\delta(\theta_t\omega)$ with $v_\delta(\tau) = u_\delta(\tau) - \phi y_\delta(\theta_\tau\omega)$, we obtain

$$\begin{cases} \dfrac{\partial v_\delta}{\partial t} + a(l(v_\delta) + y_\delta(\theta_t\omega)\, l(\phi))((-\Delta)^\gamma + \mu)v_\delta + a(l(v_\delta) + y_\delta(\theta_t\omega)\, l(\phi))y_\delta(\theta_t\omega)\, ((-\Delta)^\gamma + \mu)\phi \\ = F(v_\delta + \phi y_\delta(\theta_t\omega)) + h(t) + \phi y_\delta(\theta_t\omega), & x \in \mathcal{O}, t > \tau, \\ v_\delta(x,t) = 0, & x \in \mathbb{R}^n \backslash \mathcal{O}, t > \tau, \\ v_\delta(x,\tau) = u_{\delta,\tau}(x) - \phi y_\delta(\theta_\tau\omega) := v_{\delta,\tau}(x), & x \in \mathcal{O}. \end{cases} \tag{5.3}$$

Since (5.3) is a random equation, similar to Theorem 3.1 in the previous section, one can prove that problem (5.3) has a unique solution $v_\delta(\cdot) := v_\delta(\cdot, \tau, \omega, v_\tau) \in L^2([\tau, +\infty); V) \cap L^\infty([\tau, +\infty); H)$.

We now define a cocycle $\Phi_\delta : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times H \to H$ such that, for all $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $u_{\delta,\tau} \in H$,

$$\Phi_\delta(t, \tau, \omega, u_{\delta,\tau}) = u_\delta(t + \tau, \tau, \theta_{-\tau}\omega, u_{\delta,\tau}) = v_\delta(t + \tau, \tau, \theta_{-\tau}\omega, v_{\delta,\tau}) + \phi y_\delta(\theta_t\omega).$$

Lemma 5.2. *Suppose the conditions of Theorem 3.1 and (H_2) hold. Then, for every $\sigma \in \mathbb{R}$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, the continuous cocycle Φ_δ has a $\mathcal{D}$-pullback absorbing set $B_\delta = \{B_\delta(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, which is given by*

$$B_\delta(\tau, \omega) = \left\{u_\delta \in H : \|u_\delta\|^2 \leq R_\delta(\tau, \omega)\right\},$$

where

$$\begin{aligned} R_\delta(\tau,\omega) =& 1 + 2C_1\int_{-\infty}^0 e^{\frac{\mu m}{2}s}\, |y_\delta(\theta_s\omega)|^2\, \|\phi\|^2 ds + \frac{8\mu M^2}{m}\int_{-\infty}^0 e^{\frac{\mu m}{2}s}\, |y_\delta(\theta_s\omega)|^2\, \|(-\Delta)^\gamma\phi\|^2 ds \\ &+ \frac{4}{\mu m}\int_{-\infty}^0 e^{\frac{\mu m}{2}s}\|h(s+\tau)\|^2 ds + \frac{4C_2}{\mu m} + 2|y_\delta(\omega)|^2\|\phi\|^2. \end{aligned}$$

Proof. Taking the inner product of (5.3) with $v_\delta(t) = v_\delta(t, \tau, \omega, v_{\delta,\tau})$, and following (1.2), (2.7) and the Young inequality, we obtain

$$\begin{aligned} &\frac{d}{dt}\|v_\delta(t)\|^2 + \frac{\mu m}{2}\|v_\delta(t)\|^2 + mC(n,\gamma)\|v_\delta(t)\|^2_{\dot{H}^\gamma(\mathbb{R}^n)} \\ \leq& C_1|y_\delta(\theta_t\omega)|^2\|\phi\|^2 + \frac{4\mu M^2}{m}|y_\delta(\theta_t\omega)|^2\|(-\Delta)^\gamma\phi\|^2 + \frac{2}{\mu m}\|h(t)\|^2 + C_2, \end{aligned} \tag{5.4}$$

where C_1 and C_2 are showed as Lemma 4.1.
Multiplying (5.4) by $e^{\frac{\mu m}{2}t}$ and integrating over $(\tau-t,\sigma)$ with $\sigma\geq\tau-t$, we have for every $\omega\in\Omega$,

$$\begin{aligned}&\|v_\delta(\sigma,\tau-t,\omega,v_{\delta,\tau-t})\|^2+mC(n,\gamma)\int_{\tau-t}^{\sigma}e^{(\frac{\mu m}{2}-2\alpha)(s-\sigma)}\|v_\delta(s,\tau-t,\omega,v_{\delta,\tau-t})\|^2_{\dot H^\gamma(\mathbb{R}^n)}ds\\ \leq&e^{\frac{\mu m}{2}(\tau-t-\sigma)}\|v_{\delta,\tau-t}\|^2+C_1\int_{\tau-t}^{\sigma}e^{\frac{\mu m}{2}(s-\sigma)}\left|y_\delta(\theta_s\omega)\right|^2\|\phi\|^2ds\\ &+\frac{4\mu M^2}{m}\int_{\tau-t}^{\sigma}e^{\frac{\mu m}{2}(s-\sigma)}\left|y_\delta(\theta_s\omega)\right|^2\|(-\Delta)^\gamma\phi\|^2ds\\ &+\frac{2}{\mu m}\int_{\tau-t}^{\sigma}e^{\frac{\mu m}{2}(s-\sigma)}\|h(s)\|^2ds+\frac{2C_2}{\mu m}(1-e^{\frac{\mu m}{2}(\tau-t-\sigma)}).\end{aligned}$$

Replacing ω by $\theta_{-\tau}\omega$ and then changing variables appropriately, we obtain the following result

$$\begin{aligned}&\|v_\delta(\sigma,\tau-t,\theta_{-\tau}\omega,v_{\delta,\tau-t})\|^2+mC(n,\gamma)\int_{\tau-t}^{\sigma}e^{\frac{\mu m}{2}(s-\sigma)}\|v_\delta(s,\tau-t,\theta_{-\tau}\omega,v_{\delta,\tau-t})\|^2_{\dot H^\gamma(\mathbb{R}^n)}ds\\ \leq&e^{\frac{\mu m}{2}(\tau-t-\sigma)}\|v_{\delta,\tau-t}\|^2+C_1\int_{-t}^{\sigma-\tau}e^{\frac{\mu m}{2}(s+\tau-\sigma)}\left|y_\delta(\theta_s\omega)\right|^2\|\phi\|^2ds\\ &+\frac{4\mu M^2}{m}\int_{-t}^{\sigma-\tau}e^{\frac{\mu m}{2}(s+\tau-\sigma)}\left|y_\delta(\theta_s\omega)\right|^2\|(-\Delta)^\gamma\phi\|^2ds\\ &+\frac{2}{\mu m}\int_{-t}^{\sigma-\tau}e^{\frac{\mu m}{2}(s+\tau-\sigma)}\|h(s+\tau)\|^2ds+\frac{2C_2}{\mu m}(1-e^{\frac{\mu m}{2}(\tau-t-\sigma)}).\end{aligned}$$

Since $v_\delta(t,\tau-t,\theta_{-\tau}\omega,v_{\delta,\tau-t})=u_\delta(t,\tau-t,\theta_{-\tau}\omega,u_{\delta,\tau-t})-\phi y_\delta(\theta_{t-\tau}\omega)$ with $v_{\delta,\tau-t}=u_{\delta,\tau-t}-\phi y_\delta(\theta_{-t}\omega)$, we have

$$\begin{aligned}&\|u_\delta(\sigma,\tau-t,\theta_{-\tau}\omega,u_{\delta,\tau-t})\|^2\\ \leq&2e^{\frac{\mu m}{2}(\tau-t-\sigma)}\|u_{\delta,\tau-t}-\phi y_\delta(\theta_{-t}\omega)\|^2+2C_1\int_{-t}^{\sigma-\tau}e^{\frac{\mu m}{2}(s+\tau-\sigma)}\left|y_\delta(\theta_s\omega)\right|^2\|\phi\|^2ds\\ &+\frac{8\mu M^2}{m}\int_{-t}^{\sigma-\tau}e^{\frac{\mu m}{2}(s+\tau-\sigma)}\left|y_\delta(\theta_s\omega)\right|^2\|(-\Delta)^\gamma\phi\|^2ds\\ &+\frac{4}{\mu m}\int_{-t}^{\sigma-\tau}e^{\frac{\mu m}{2}(s+\tau-\sigma)}\|h(s+\tau)\|^2ds+\frac{4C_2}{\mu m}(1-e^{\frac{\mu m}{2}(\tau-t-\sigma)})+2\|\phi y_\delta(\theta_{\sigma-\tau}\omega)\|^2\\ \leq&4e^{\frac{\mu m}{2}(\tau-t-\sigma)}\|u_{\delta,\tau-t}\|^2+4e^{\frac{\mu m}{2}(\tau-t-\sigma)}\|\phi\|^2|y_\delta(\theta_{-t}\omega)|^2\\ &+2C_1\int_{-t}^{\sigma-\tau}e^{\frac{\mu m}{2}(s+\tau-\sigma)}\left|y_\delta(\theta_s\omega)\right|^2\|\phi\|^2ds\\ &+\frac{8\mu M^2}{m}\int_{-t}^{\sigma-\tau}e^{\frac{\mu m}{2}(s+\tau-\sigma)}\left|y_\delta(\theta_s\omega)\right|^2\|(-\Delta)^\gamma\phi\|^2ds\\ &+\frac{4}{\mu m}\int_{-t}^{\sigma-\tau}e^{\frac{\mu m}{2}(s+\tau-\sigma)}\|h(s+\tau)\|^2ds+\frac{4C_2}{\mu m}(1-e^{\frac{\mu m}{2}(\tau-t-\sigma)})+2\|\phi y_\delta(\theta_{\sigma-\tau}\omega)\|^2.\end{aligned}$$

Lemma 5.1 and inequality (4.1) imply that there exists $T=T(\tau,\omega,B,\sigma,m,\mu)\geq0$ such that for all $t\geq T$,

$$\begin{aligned}&\|u_\delta(\sigma,\tau-t,\theta_{-\tau}\omega,u_{\delta,\tau-t})\|^2\\ \leq&1+2C_1\int_{-t}^{\sigma-\tau}e^{\frac{\mu m}{2}(s+\tau-\sigma)}\left|y_\delta(\theta_s\omega)\right|^2\|\phi\|^2ds\\ &+\frac{8\mu M^2}{m}\int_{-t}^{\sigma-\tau}e^{\frac{\mu m}{2}(s+\tau-\sigma)}\left|y_\delta(\theta_s\omega)\right|^2\|(-\Delta)^\gamma\phi\|^2ds\\ &+\frac{4}{\mu m}\int_{-t}^{\sigma-\tau}e^{\frac{\mu m}{2}(s+\tau-\sigma)}\|h(s+\tau)\|^2ds+\frac{4C_2}{\mu m}+2\|\phi y_\delta(\theta_{\sigma-\tau}\omega)\|^2.\end{aligned}$$

On the other hand, combining Lemma 5.1 with inequality (4.2), one can easily check that R_δ is tempered (we omit the details here). $\square$

Similar to Lemma 4.3, we have the following result on Φ_δ.

Lemma 5.3. *Suppose the conditions of Lemma 5.2 hold. Then the continuous cocycle Φ_δ is asymptotically compact in H.*

By Lemma 5.2 and Lemma 5.3, we deduce the following result about existence of $\mathcal{D}$-pullback attractors for Φ_δ.

Theorem 5.1. *Suppose the conditions of Lemma 5.2 hold. Then the continuous cocycle Φ_δ of problem (5.2) has a unique $\mathcal{D}$-pullback random attractor $\mathcal{A}_\delta = \{\mathcal{A}_\delta(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ in H.*

6 Upper semi-continuity of random attractors

In this section, we will establish the convergence of solutions and the upper semi-continuity of random attractors of (5.2) as $\delta \to 0^+$.

Lemma 6.1. *Suppose the conditions of Theorem 3.1 hold. If u_δ and u are the solutions of problems (5.2) and (2.8), respectively, then for all $\tau \in \mathbb{R}$, $\omega \in \Omega$, $\varepsilon > 0$ and $T > 0$, there exist $\delta_0 = \delta_0(\tau, \omega, T, \varepsilon)$ and $C = C(\tau, \omega, T, \mu, m, M, C_f)$ such that for all $0 < \delta < \delta_0$ and $t \in [\tau, \tau + T]$,*

$$\|u_\delta(t, \tau, \omega, u_{\delta,\tau}) - u(t, \tau, \omega, u_\tau)\|^2 \le e^{C(t-\tau)}\|u_{\delta,\tau} - u_\tau\|^2 + C\varepsilon. \tag{6.1}$$

Proof. By (5.3) and (2.9), taking the inner product with $v_\delta(t) - v(t)$ we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v_\delta(t) - v(t)\|^2 + a\left(l\left(v(t)\right) + z\left(\theta_t\omega\right) l(\phi)\right) b\Big(v_\delta(t) - v(t), v_\delta(t) - v(t)\Big) \\
\le&\left|\Big(\left[a\left(l\left(v_\delta(t)\right) + y_\delta\left(\theta_t\omega\right) l(\phi)\right) - a\left(l\left(v(t)\right) + z\left(\theta_t\omega\right) l(\phi)\right)\right] A v_\delta(t), v_\delta(t) - v(t)\Big)\right| \\
&+\left|\Big(\left[a\left(l\left(v_\delta(t) + y_\delta\left(\theta_t\omega\right) l(\phi)\right)\right) - a\left(l\left(v(t)\right) + z\left(\theta_t\omega\right) l(\phi)\right)\right] A\phi, v_\delta(t) - v(t)\Big)\right| \\
&+\Big(F\left(v_\delta(t) + \phi y_\delta\left(\theta_t\omega\right)\right) - F\left(v(t) + \phi z\left(\theta_t\omega\right)\right), v_\delta(t) - v(t)\Big) \\
&+\Big(y_\delta\left(\theta_t\omega\right) - z\left(\theta_t\omega\right)\Big)\Big(\phi, v_\delta(t) - v(t)\Big).
\end{aligned} \tag{6.2}$$

Since a is locally Lipschitz and $l \in \mathcal{L}(L^2(\mathcal{O}); \mathbb{R})$, the first term on the right-hand side of (6.2) can be bounded by

$$\begin{aligned}
&\left|\Big(\left[a\left(l\left(v_\delta(t)\right) + y_\delta\left(\theta_t\omega\right) l(\phi)\right) - a\left(l\left(v(t)\right) + z\left(\theta_t\omega\right) l(\phi)\right)\right] A v_\delta(t), v_\delta(t) - v(t)\Big)\right| \\
\le& L_a(R)\|l\|\Big[\|v_\delta(t) - v(t)\| + \|\phi\||y_\delta\left(\theta_t\omega\right) - z\left(\theta_t\omega\right)|\Big]\Big((-\Delta)^\gamma v_\delta(t) + \mu v_\delta(t), v_\delta(t) - v(t)\Big) \\
\le& L_a(R)\|l\|\Big[\|v_\delta(t) - v(t)\| + \|\phi\||y_\delta\left(\theta_t\omega\right) - z\left(\theta_t\omega\right)|\Big] \\
&\times\Big(\frac{C(n,\gamma)}{2}\|v_\delta(t)\|_{\dot{H}^\gamma(\mathbb{R}^n)}\|v_\delta(t) - v(t)\| + \mu\|v_\delta(t)\|\|v_\delta(t) - v(t)\|\Big) \\
\le& \frac{C(n,\gamma)}{2}L_a^2(R)\|l\|^2\Big(\|v_\delta(t)\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|v_\delta(t)\|\Big) \\
&\times\Big(\|v_\delta(t) - v(t)\|^2 + \|v_\delta(t) - v(t)\|\|\phi\||y_\delta\left(\theta_t\omega\right) - z\left(\theta_t\omega\right)|\Big) \\
\le& \frac{C(n,\gamma)}{2}L_a^2(R)\|l\|^2\Big(\|v_\delta(t)\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|v_\delta(t)\|\Big)\Big(\frac{3}{2}\|v_\delta(t) - v(t)\|^2 + \frac{1}{2}\|\phi\|^2|y_\delta\left(\theta_t\omega\right) - z\left(\theta_t\omega\right)|^2\Big).
\end{aligned} \tag{6.3}$$

Similarly, for the second term on the right-hand side of (6.2), we have

$$
\begin{aligned}
&|([a\left(l\left(v_\delta(t)+y_\delta\left(\theta_t\omega\right)l(\phi)\right)\right)-a\left(l\left(v(t)\right)+z\left(\theta_t\omega\right)l(\phi)\right)]\,A\phi,v_\delta(t)-v(t))| \\
\leq& L_a(R)\|l\|\Big[\|v_\delta(t)-v(t)\|+\|\phi\||y_\delta\left(\theta_t\omega\right)-z\left(\theta_t\omega\right)|\Big]\Big((-\Delta)^\gamma\phi+\mu\phi,v_\delta(t)-v(t)\Big) \\
\leq& L_a(R)\|l\|\Big[\|v_\delta(t)-v(t)\|+\|\phi\||y_\delta\left(\theta_t\omega\right)-z\left(\theta_t\omega\right)|\Big] \\
&\times\Big(\frac{C(n,\gamma)}{2}\|\phi\|_{\dot{H}^\gamma(\mathbb{R}^n)}\|v_\delta(t)-v(t)\|+\mu\|\phi\|\|v_\delta(t)-v(t)\|\Big) \\
\leq& \frac{C(n,\gamma)}{2}L_a^2(R)\|l\|^2\Big(\|\phi\|_{\dot{H}^\gamma(\mathbb{R}^n)}+\|\phi\|\Big) \\
&\times\Big(\|v_\delta(t)-v(t)\|^2+\|v_\delta(t)-v(t)\|\|\phi\||y_\delta\left(\theta_t\omega\right)-z\left(\theta_t\omega\right)|\Big) \\
\leq& \frac{C(n,\gamma)}{2}L_a^2(R)\|l\|^2\Big(\|\phi\|_{\dot{H}^\gamma(\mathbb{R}^n)}+\|\phi\|\Big)\Big(\frac{3}{2}\|v_\delta(t)-v(t)\|^2+\frac{1}{2}\|\phi\|^2|y_\delta\left(\theta_t\omega\right)-z\left(\theta_t\omega\right)|^2\Big).
\end{aligned}
\tag{6.4}
$$

In terms of $f\in C(\mathbb{R})$, the locally Lipschitz continuity of a and $l\in\mathcal{L}(L^2(\mathcal{O});\mathbb{R})$, for the third term on the right-hand side of (6.2), we obtain

$$
\begin{aligned}
&\Big(F\left(v_\delta(t)+\phi y_\delta\left(\theta_t\omega\right)\right)-F\left(v(t)+\phi z\left(\theta_t\omega\right)\right),v_\delta(t)-v(t)\Big) \\
=&\Big(F\left(v_\delta(t)+\phi y_\delta\left(\theta_t\omega\right)\right)-F\left(v(t)+\phi y_\delta\left(\theta_t\omega\right)\right),v_\delta(t)-v(t)\Big) \\
&+\Big(F\left(v(t)+\phi y_\delta\left(\theta_t\omega\right)\right)-F\left(v(t)+\phi z\left(\theta_t\omega\right)\right),v_\delta(t)-v(t)\Big) \\
\leq&\mu M\|v_\delta(t)-v(t)\|^2+\mu L_a(R)\|l\|(\|v\|_{C([\tau,\tau+T];H)}+|y_\delta\left(\theta_t\omega\right)|\|\phi\|_{L^\infty(\mathcal{O})})\|v_\delta(t)-v(t)\|^2 \\
&+\mu M|y_\delta\left(\theta_t\omega\right)-z\left(\theta_t\omega\right)|\|\phi\|\|v_\delta(t)-v(t)\| \\
&+\mu L_a(R)\|l\||y_\delta\left(\theta_t\omega\right)-z\left(\theta_t\omega\right)|\|\phi\|\Big[\|v\|_{C([\tau,\tau+T];H)}+|z\left(\theta_t\omega\right)|\|\phi\|_{L^\infty(\mathcal{O})}\Big]\|v_\delta(t)-v(t)\| \\
&+C_f\|v_\delta(t)-v(t)\|^2+C_f|y_\delta\left(\theta_t\omega\right)-z\left(\theta_t\omega\right)|\|\phi\|\|v_\delta(t)-v(t)\| \\
\leq&\Big(\mu M+\mu L_a(R)\|l\|(\|v\|_{C([\tau,\tau+T];H)}+|y_\delta\left(\theta_t\omega\right)|\|\phi\|_{L^\infty(\mathcal{O})})+\frac{3}{2}+C_f\Big)\|v_\delta(t)-v(t)\|^2 \\
&+\frac{1}{2}|y_\delta\left(\theta_t\omega\right)-z\left(\theta_t\omega\right)|^2\|\phi\|^2\Big(\mu^2M^2+\mu^2L_a^2(R)\|l\|^2\Big[\|v\|_{C([\tau,\tau+T];H)}+|z\left(\theta_t\omega\right)|\|\phi\|_{L^\infty(\mathcal{O})}\Big]^2\Big).
\end{aligned}
\tag{6.5}
$$

Applying the Young inequality to the last term on the right-hand side of (6.2), we have

$$
\begin{aligned}
\Big(y_\delta\left(\theta_t\omega\right)-z\left(\theta_t\omega\right)\Big)\Big(\phi,v_\delta(t)-v(t)\Big)\leq&|y_\delta\left(\theta_t\omega\right)-z\left(\theta_t\omega\right)|\|\phi\|\|v_\delta(t)-v(t)\| \\
\leq&\frac{1}{2}\|v_\delta(t)-v(t)\|^2+\frac{1}{2}|y_\delta\left(\theta_t\omega\right)-z\left(\theta_t\omega\right)|^2\|\phi\|^2.
\end{aligned}
\tag{6.6}
$$

Combining (6.2)-(6.6), we have the following estimation

$$
\begin{aligned}
&\frac{d}{dt}\|v_\delta(t)-v(t)\|^2 \\
\leq&\Big(\frac{3C(n,\gamma)}{2}L_a^2(R)\|l\|^2\Big(\|v_\delta(t)\|_{\dot{H}^\gamma(\mathbb{R}^n)}+\|v_\delta(t)\|+\|\phi\|_{\dot{H}^\gamma(\mathbb{R}^n)}+\|\phi\|\Big) \\
&+2\Big(\mu M+\mu L_a(R)\|l\|(\|v\|_{C([\tau,\tau+T];H)}+|y_\delta\left(\theta_t\omega\right)|\|\phi\|_{L^\infty(\mathcal{O})})+\frac{3}{2}+C_f\Big)+1\Big)\|v_\delta(t)-v(t)\|^2 \\
&+|y_\delta\left(\theta_t\omega\right)-z\left(\theta_t\omega\right)|^2\|\phi\|^2\Big(\frac{C(n,\gamma)}{2}L_a^2(R)\|l\|^2\Big(\|v_\delta(t)\|_{\dot{H}^\gamma(\mathbb{R}^n)}+\|v_\delta(t)\|+\|\phi\|_{\dot{H}^\gamma(\mathbb{R}^n)}+\|\phi\|\Big) \\
&+\mu^2M^2+\mu^2L_a^2(R)\|l\|^2\Big[\|v\|_{C([\tau,\tau+T];H)}+|z\left(\theta_t\omega\right)|\|\phi\|_{L^\infty(\mathcal{O})}\Big]^2+1\Big).
\end{aligned}
\tag{6.7}
$$

By Lemma 5.1, for every $\varepsilon>0$, there exists $\delta_0=\delta_0(\tau,\omega,T,\varepsilon)>0$ such that, for all $0<\delta<\delta_0$ and $t\in[\tau,\tau+T]$,

$$|y_\delta\left(\theta_t\omega\right)-z(\theta_t\omega)|<\varepsilon.$$

Note that $v \in L^2([\tau, \tau+T]; V) \cap C([\tau, \tau+T]; H)$ and the mapping $t \mapsto z(\theta_t \omega)$ is continuous, we derive

$$\frac{d}{dt}\|v_\delta(t) - v(t)\|^2 \le C\|v_\delta(t) - v(t)\|^2 + C\varepsilon.$$

The Gronwall lemma implies that

$$\|v_\delta(t, \tau, \omega, v_{\delta,\tau}) - v(t, \tau, \omega, v_\tau)\|^2 \le e^{C(t-\tau)}\|v_{\delta,\tau} - v_\tau\|^2 + C\varepsilon.$$

By the definition of u, it follows that

$$\|u_\delta(t, \tau, \omega, u_{\delta,\tau}) - u(t, \tau, \omega, u_\tau)\|^2 \le e^{C(t-\tau)}\|u_{\delta,\tau} - u_\tau\|^2 + \left[|y_\delta(\theta_\tau\omega) - z(\theta_\tau\omega)|^2 + |y_\delta(\theta_t\omega) - z(\theta_t\omega)|^2\right]\|\phi\|^2 + C\varepsilon,$$

which together with Lemma 5.1 implies that

$$\|u_\delta(t, \tau, \omega, u_{\delta,\tau}) - u(t, \tau, \omega, u_\tau)\|^2 \le e^{C(t-\tau)}\|u_{\delta,\tau} - u_\tau\|^2 + C\varepsilon.$$

□

As a straightforward consequence of Lemma 6.1, we obtain the following convergence of solution.

Lemma 6.2. *Suppose the conditions of Lemma 6.1 hold and $\delta_n \to 0^+$. Let u_{δ_n} and u be the solutions of problem (5.2) and (2.8) with initial data $u_{\delta_n,\tau}$ and u_τ, respectively. If $u_{\delta_n,\tau} \to u_\tau$ in H as $\delta_n \to 0^+$, then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t > \tau$,*

$$u_{\delta_n}(t, \tau, \omega, u_{\delta_n,\tau}) \to u(t, \tau, \omega, u_\tau) \text{ in } H \text{ as } \delta_n \to 0^+.$$

For the attractor $\mathcal{A}_\delta$ of Φ_δ, we have the following compactness.

Lemma 6.3. *Suppose the conditions of Theorem 3.1 and (H_2) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, if $\delta_n \to 0^+$ and $u_{\delta_n} \in \mathcal{A}_{\delta_n}(\tau, \omega)$, the sequence $\{u_{\delta_n}\}_{n=1}^{+\infty}$ is precompact in H.*

Proof. Since $u_{\delta_n} \in \mathcal{A}_{\delta_n}(\tau, \omega)$, by the invariance of $\mathcal{A}_{\delta_n}$, there exists $\tilde{u}_{\delta_n} \in \mathcal{A}_{\delta_n}(\tau - 1, \theta_{-1}\omega)$ such that

$$u_{\delta_n} = \Phi_{\delta_n}(1, \tau - 1, \theta_{-1}\omega, \tilde{u}_{\delta_n}) = u_{\delta_n}(1, \tau - 1, \theta_{-\tau}\omega, \tilde{u}_{\delta_n}). \tag{6.8}$$

By Lemma 5.2, there exists $N_1 = N_1(\tau, \omega, \alpha) \geqslant 1$ such that, for all $n \geqslant N$,

$$\begin{aligned} R_{\delta_n}(\tau - 1, \theta_{-1}\omega) =& 1 + 2C_1 \int_{-\infty}^0 e^{\frac{\mu m}{2}s} |y_\delta(\theta_{s-1}\omega)|^2 \|\phi\|^2 ds + \frac{8\mu M^2}{m}\int_{-\infty}^0 e^{\frac{\mu m}{2}s} |y_\delta(\theta_{s-1}\omega)|^2 \|(-\Delta)^\gamma\phi\|^2 ds \\ &+ \frac{4}{\mu m}\int_{-\infty}^0 e^{\frac{\mu m}{2}s}\|h(s + \tau - 1)\|^2 ds + \frac{4C_2}{\mu m} + 2|y_\delta(\theta_{-1}\omega)|^2\|\phi\|^2, \end{aligned}$$

which together with the fact $\tilde{u}_{\delta_n} \in \mathcal{A}_{\delta_n}(\tau - 1, \theta_{-1}\omega) \subseteq B_{\delta_n}(\tau - 1, \theta_{-1}\omega)$ implies that for all $n \ge N_1$,

$$\begin{aligned} \|\tilde{u}_{\delta_n}\|^2 \le& 1 + 2C_1 \int_{-\infty}^0 e^{\frac{\mu m}{2}s} |y_\delta(\theta_{s-1}\omega)|^2 \|\phi\|^2 ds + \frac{8\mu M^2}{m}\int_{-\infty}^0 e^{\frac{\mu m}{2}s} |y_\delta(\theta_{s-1}\omega)|^2 \|(-\Delta)^\gamma\phi\|^2 ds \\ &+ \frac{4}{\mu m}\int_{-\infty}^0 e^{\frac{\mu m}{2}s}\|h(s + \tau - 1)\|^2 ds + \frac{4C_2}{\mu m} + 2|y_\delta(\theta_{-1}\omega)|^2\|\phi\|^2. \end{aligned}$$

By definition, we know that $\{\tilde{v}_{\delta_n}\}_{n=1}^\infty$ is bounded in H. Therefore, there exists $\bar{v} \in L^2([\tau - 1, \tau]; H)$ such that, up to a subsequence,

$$v_{\delta_n}(\cdot, \tau - 1, \theta_{-\tau}\omega, \tilde{v}_{\delta_n}) \to \bar{v}(\cdot) \text{ in } L^2([\tau - 1, \tau]; H),$$

which implies, up to a subsequence,

$$v_{\delta n}(s,\tau-1,\theta_{-\tau}\omega,\tilde{v}_n)\to\bar{v}(s)\ \text{in}\ H\quad a.e.\ \text{for}\ s\in(\tau-1,\tau).$$

We then have

$$u_{\delta_n}(s,\tau,\theta_{-\tau}\omega,\tilde{u}_{\delta_n})\to\bar{v}(s)+\phi y_\delta(\theta_s\omega)\ \text{in}\ H\quad a.e.\,\text{for}\ s\in(\tau-1,\tau). \tag{6.9}$$

Since $\delta_n\to 0$, by Lemma 6.2 and (6.9), we have

$$u_{\delta_n}(\tau,s,\theta_{-\tau}\omega,u_{\delta_n}(s,\tau-1,\theta_{-\tau}\omega,\tilde{u}_{\delta_n}))\to u(\tau,s,\theta_{-\tau}\omega,\bar{v}(s)+\phi y_\delta(\theta_s\omega))\ \text{in}\ H.$$

Note that

$$u_{\delta_n}(\tau,s,\theta_{-\tau}\omega,u_{\delta_n}(s,\tau-1,\theta_{-\tau}\omega,\tilde{u}_{\delta_n}))=u_{\delta n}(\tau,\tau-1,\theta_{-\tau}\omega,\tilde{u}_{\delta_n}).$$

Thus, we have

$$u_{\delta_n}(\tau,\tau-1,\theta_{-\tau}\omega,\tilde{u}_{\delta_n})\to u(\tau,s,\theta_{-\tau}\omega,\bar{v}(s)+\phi y_\delta(\theta_s\omega))\ \text{in}\ H,$$

which along with (6.8) completes the proof. □

Now we present the upper semi-continuity of random attractors as $\delta\to 0^+$.

Theorem 6.1. *Suppose the conditions of Theorem 3.1 and (H_2) hold. Then, for all $\tau\in\mathbb{R}$, $\omega\in\Omega$,*

$$\lim_{\delta\to 0^+} d_H(\mathcal{A}_\delta(\tau,\omega),\mathcal{A}(\tau,\omega))=0.$$

Proof. Let $\delta_n\to 0$ and $u_{\delta_n,\tau}\to u_\tau$ in H. We derive from Lemma 5.1 that, for all $\tau\in\mathbb{R},\omega\in\Omega$,

$$\lim_{\delta\to 0}\|B_\delta(\tau,\omega)\|=\|B(\tau,\omega)\|. \tag{6.10}$$

Then by (6.10), Lemmas 6.2 and 6.3, we prove this theorem from Theorem 3.1 in [47]. □

Acknowledgments

The work of the authors was partially supported by the NNSF of China (11471190,11971260), the Spanish Ministerio de Ciencia, Innovación y Universidades (MCIU), Agencia Estatal de Investigación (AEI), Fondo Europeo de Desarrollo Regional (FEDER) under the project PID2021-122991NB-C21 and the Junta de Andalucía and FEDER under the project P18-FR-4509.

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