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A MIXED NONLINEAR TIME-FRACTIONAL RAYLEIGH-STOKES EQUATION

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Abstract. This paper investigates a nonlinear time-fractional Rayleigh-Stokes equation with mixed nonlinearities containing a power-type function, a logarithmic function and an inverse time-forcing term. Applying Lagrange's mean value theorem and the compactness of the Sobolev embeddings, we estimate the complex Lipschitz property of mixed nonlinearity. We investigate the local well-posed results (local existence, regularity estimate, continuation) of the solutions in Hilbert scales space. Moreover, the global existence theory affiliated to the finite-time blow-up is considered.

1. **Introduction and statement of the main results.** In recent years, fractional PDEs have received considerable attention for their ability to capture long-term correlations, for example, material properties and memory effects. For practical problems and many important modern physics models, it is more suitable to examine the models with fractional derivative rather than the classical models, as the physical model considers the memory effects [15,32], the non-local effect (power-law memory) in time and space [17, 18, 23, 25, 42, 43], especially the engineering problems [5, 30]. With many related applications, new ideas and different methods have been formulated and developed to consider several types of fractional PDEs. Fractional PDEs and fractional stochastic PDEs have been studied by T. Caraballo et al. [10,12,13,33–35]. Some of the impressive and selected works in recent times can be mentioned, for example, J. Nieto et al. [14, 26], D. O'Regan et al. [19, 28] and many special works of other authors.

In the present paper, for $\Omega \subset \mathbb{R}^N (N \leq 4)$ with a smooth boundary $\partial\Omega$, we study the nonlinear time fractional Rayleigh-Stokes equation ($(\text{FRSE})_t$ for short) on a

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bounded domain by considering the following problem

$$\begin{cases} \partial_t u - a\partial_t^\alpha \Delta u - \Delta u = \frac{1}{r_\beta(t)}|u|^{p-2}u \log \langle u \rangle_b, & \text{in } \Omega \times (0,+\infty), \\ u = 0, \quad \text{on } (\mathbb{R}^N \setminus \Omega) \times (0,+\infty), \\ u = u_0(x), \quad \text{in } \Omega \times \{0\}, \end{cases} \tag{$\mathbb{P}$}$$

where $a, b > 0$, $0 < \alpha < 1$ and $p > 2$ are given constants; $\langle u \rangle_b = b + |u|$ and Δ denotes the Laplacian operator. The function $r_\beta^{-1}(t)$, $t > 0$, stands for an inverse time-forcing term, and in this paper, we choose $r_\beta(t) := t^\beta$ for some $0 < \beta < \alpha < 1$. Denote by $u(x,t), x \in \Omega, t > 0$ the unknown function and by $u_0(x) = u(x,0)$ the initial data. The notation ∂_t indicates the partial derivative with respect to t and ∂_t^α is the Riemann-Liouville (R-L) time fractional derivative of order $\alpha \in (0,1)$

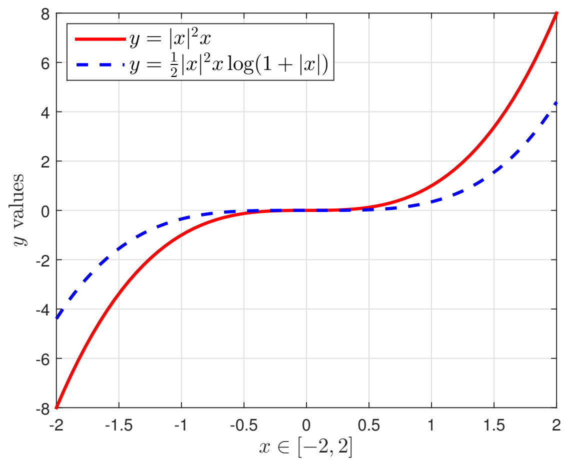
$$\partial_t^\alpha v(x,t) = \frac{1}{\Gamma(1-\alpha)}\partial_t\Big(\int_0^t \frac{v(x,s)}{(t-s)^\alpha} ds\Big),$$

where $\Gamma(\cdot)$ is the Gamma function, for more details see [21, 29].

Fluid mechanics is an important area of mathematical physics [2, 3]. In non-Newtonian fluid models, the problems of $(\mathrm{FRSE})_t$ are used to describe their dynamical behavior, see [31]. In ($\mathbb{P}$), the time fractional R-L derivative $\partial_t^\alpha (0 < \alpha < 1)$ describes the elasticity of flow. More detailed applications of $(\mathrm{FRSE})_t$ can be found in [11, 20, 31] and references therein. C. Xue et al. [39] investigated $(\mathrm{FRSE})_t$ for a generalized second-grade fluid in a porous half-space with a heated flat plate. By applying the Fourier sine transform and the fractional Laplace transform, the authors obtained the exact solutions of the velocity and temperature fields. E. Bazhlekova et al. [11] solved the linear equation of Problem ($\mathbb{P}$) for a generalized second-grade fluid. They obtained some results on the Sobolev regularity and the optimal error estimates. Recently, D. Lan [22] analyzed some sufficient conditions to ensure the global solvability, and asymptotic behavior of the solutions of ($\mathbb{P}$). For the inverse problem of ($\mathbb{P}$), i.e., if we replaced the initial condition $u_0(x)$ in ($\mathbb{P}$) by the final value problem $u_T(x) = u(x,T)$ for $T > 0$, T.B. Ngoc et al. [27] studied the results on the local well-posedness (existence, regularity) of the mild solutions. N.H. Tuan et al. [36] studied the inverse Problem of $(\mathrm{FRSE})_t$ and the authors showed that this problem is ill-posed according to Hadamard's definition. To regularize the unstable solution, they used the trigonometric method in nonparametric regression associated with Fourier truncated expansion method.

To study nonlinear problems of $(\mathrm{FRSE})_t$ of the type ($\mathbb{P}$), we know that the difficulty comes not only from the analysis of the fractional derivative but also from the nonlinear terms. Indeed, the first challenge is, for the classical PDEs, most of its results are on weak solutions. However, the fractional derivatives are nonlocal operators, so we are not able anymore to apply the methods on the weak solutions, like the compact method, Galerkin method, and so on. Hence, we have to find a solution in the sense of mild solution. The limited estimates of solution operators in the formula of the mild solution are also tough tasks, which is the reason why the results on the mild solution in case of fractional PDEs are limited. The second challenge is to deal with nonlinear functions of the form $\frac{1}{t^\beta}|u|^{p-2}u \log \langle u \rangle_b$, for $0 < \beta < \alpha < 1$, $p > 2$ and $b > 0$. The logarithmic functions exhibit slow cumulative nonlinearity, thus providing a different kind of description of a dynamic process. The study of logarithmic nonlinearity has a long history in physics because it occurs naturally in inflationary cosmology, quantum mechanics and nuclear physics [9]. PDEs with logarithmic nonlinear sources have attracted the interest of many authors with many attractive results, for instance, [7, 8, 37, 38, 40] and the references therein. There are many results about power-type nonlinearities in the form $u|u|^k$, for $k > 0$, see [1, 4–6, 16, 37] and references therein. We know that these functions grow fast, so we shall use the logarithmic functions to control their sudden increase.

Example 1.1. To see more clearly we consider an example for two real-valued functions $y = |x|^2 x$ and $y = \frac{1}{2}|x|^2 x \log(1 + |x|)$, we have their graphs as follows



Clearly, we realize the importance of logarithmic functions in controlling the high nonlinearity of power-type nonlinear functions. To achieve the behavior analysis of the solutions in spatial space $\mathbb{H}^{2q}(\Omega)$ (see (3) for the definition of $\mathbb{H}^{2q}(\Omega)$ space), we have the challenges from estimating Lipschitz property of the mixed nonlinear source functions $F(u) = |u|^{p-2} u \log \langle u \rangle_b$, for $0 < \beta < \alpha < 1$, $p > 2$ and $b > 0$ (locally Lipschitz property). To estimate the Lipschitz property of the function $F(u)$, we rely on Theorem 2.3 and Lagrange's mean value theorem. We always have to make sure that the expression of the logarithmic function is positive. Therefore, we use the positive translation for the function $|u|$ by a constant $b > 0$. Moreover, the inverse time-forcing term $t^{-\beta}$ for $t > 0$, $0 < \beta < \alpha < 1$ in the right hand side of ($\mathbb{P}$) also creates a difficulty as they cause a singular integral. We have to resort to calculus through Euler's integral and the Gamma functions. In fact, the solution operators of our problem contain the kernels $S_\alpha(t)$ and $s^{-\beta} S_\alpha(t-s)$, for $0 < s < t$, (see (6)). They bring some difficulties in estimating and analyzing the solutions. Indeed, we shall write solution (6) as a fixed-point equation

$$
\begin{aligned}
&u(t) = \mathscr{Y} u(t), \\
&\mathscr{Y} u(t) = S_\alpha(t) u_0 + \underbrace{\int_0^t s^{-\beta} S_\alpha(t-s) F(u)(s) ds}_{=:\mathcal{J}_{\alpha,\beta} u(t)} .
\end{aligned}
\tag{1}
$$

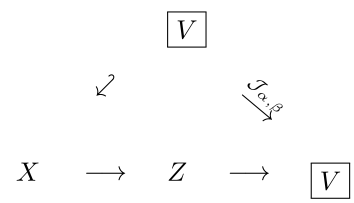
X, Z, V denote spaces that need to be chosen appropriately. For $u \in X$ then $F(u) \in Z$ and $\mathcal{J}_{\alpha,\beta} u \in V$, i.e., $\mathcal{J}_{\alpha,\beta} : X \to V$, it would be difficult for us to apply the fixed point theorem. Let us analyze in more detail through the following diagram

$$\begin{array}{ccccccc} \mathcal{J}_{\alpha,\beta}: & X & \overset{(step\ 2)}{\longrightarrow} & Z & \overset{(step\ 1)}{\longrightarrow} & V \\ & u & \longmapsto & F(u) & \longmapsto & \mathcal{J}_{\alpha,\beta}u \end{array} \tag{Space-estimate}$$

Choosing the appropriate spaces X, Z, V is a real challenge. First, we estimate $\|\mathcal{J}_{\alpha,\beta}u\|_V$, it causes the term $\|F(u)\|_Z$ to appear. Next, estimating $\|F(u)\|_Z$ makes the norms $\|u\|_X$. Final, to apply the fixed point theorems we need $\mathcal{J}_{\alpha,\beta}: V \to V$, that is, we shall find a space V satisfying the following embedding $X \hookleftarrow V$. To make it easier to visualize the picture of the above settings, for the operator

$$\mathcal{J}_{\alpha,\beta}: V \hookrightarrow X \to Z \to V$$

and we summarize it in the following schema



Remark 1.2. In this paper, for $-\frac{N}{4} < q < \frac{N}{4}$, we choose $V \equiv \mathbb{H}^{2q}(\Omega)$, using Theorem 2.2 we infer that $Z \equiv L^2(\Omega)$. Using Sobolev embeddings Theorem 2.5 we can deduce that

$$\mathbb{H}^{2q}(\Omega) \equiv V \hookrightarrow X \equiv L^{(\cdot)}(\Omega),$$

with the indexes $(\cdot)$ depending on N, p, q. In addition, for choosing $V \equiv \mathbb{H}^{2q}(\Omega)$, we need to estimate the term $\|\mathcal{J}_{\alpha,\beta}u(t)\|_{\mathbb{H}^{2q}}$ (which contains forcing term $t^{-\beta}$ and operator $S_\alpha(t)$, $t > 0$). By using Theorem 2.2 to estimate operator $\mathcal{J}_{\alpha,\beta}$ in the norm of $\mathbb{H}^{2q}(\Omega)$, this produces the Euler singular integrals and the calculations for these integrals are available (related the Beta functions, see Theorem 2.4). So, it is interesting to study this model with the mixed nonlinearity in the form $\frac{1}{t^\beta}|u|^{p-2}u \log \langle u \rangle_b$, for $p > 2, b > 0, \beta \in (0,1)$.

Next, we shall present the main results of this paper. The first result is about the local in time existence of mild solutions to Problem ($\mathbb{P}$) in $C([0,T];\mathbb{H}^{2q}(\Omega))$ for $q \in (0,1)$. From now on, we shall write $C_{[0,T]}\mathbb{H}^{2q}_x = C([0,T];\mathbb{H}^{2q}(\Omega))$ for short.

Theorem 1.3 (Local existence-regularity)**.** *Let $N \leq 4, 0 < \beta < \alpha < 1, p > 2$, q satisfy the condition*

$$0 \leq \max\left\{ \frac{(p+p'-2)N}{4(p+p'-1)}, \left(1 - \frac{p-2}{(p+p'-2)(p-1)}\right)\frac{N}{4} \right\} \leq q < \frac{N}{4},$$

for $p' > \max\{0, \frac{5}{2} - p\}$, and let $u_0 \in \mathbb{H}^{2q+2}(\Omega)$. Then, there exists a positive time $T \equiv T(u_0)$ such that Problem ($\mathbb{P}$) admits a mild solution (unique) which belongs to $C_{[0,T]}\mathbb{H}^{2q}_x$. Moreover, for $\alpha \in (\frac{1}{2}, 1)$, $0 < \beta < 2\alpha - 1$ and if $u_0 \in \mathbb{H}^{2q'}(\Omega)$ for $q - q' \leq 1$, we have the following regularity estimate (decay estimate)

$$\|u(t)\|_{\mathbb{H}^{2q}(\Omega)} \lesssim \frac{\|u_0\|_{\mathbb{H}^{2q'}(\Omega)}}{\lambda_1^{1-q+q'}} t^{\alpha-1}, \quad \textit{for } t \in (0,T],\ q' \geq q-1, \tag{2}$$

where $\lambda_1 > 0$ is the first eigenvalue of the minus Laplace operator.

Remark 1.4. In Theorem 1.3, it is obvious that $q < 1$, from (2) this means that q' can be a negative number or zero or positive. That is why we want to consider the regularity of the solution with the minimum possible conditions on the initial condition. Specifically, if $q' = q - 1 < 0$, $u_0 \in \mathbb{H}^{-(2-2q)}(\Omega)$ then we have

$$t^{1-\alpha}\|u(t)\|_{\mathbb{H}^{2q}(\Omega)} \lesssim \|u_0\|_{\mathbb{H}^{-(2-2q)}(\Omega)}, \quad t \in (0,T].$$

If $q' = 0$, $u_0 \in L^2(\Omega)$ then we have

$$t^{1-\alpha}\|u(t)\|_{\mathbb{H}^{2q}(\Omega)} \lesssim \frac{\|u_0\|_{L^2(\Omega)}}{\lambda_1^{1-q}}, \quad t \in (0,T].$$

If $q' = q + 1$, $u_0 \in \mathbb{H}^{2q+2}(\Omega)$ then

$$t^{1-\alpha}\|u(t)\|_{\mathbb{H}^{2q}(\Omega)} \lesssim \frac{\|u_0\|_{\mathbb{H}^{2q+2}(\Omega)}}{\lambda_1^{2}}, \quad t \in (0,T].$$

We know that Problem ($\mathbb{P}$) possesses a unique mild solution for $0 \le t \le T$. Our second main result addresses whether this solution can be stretched over $[0, T_+]$ for $T_+ > T$. First, we give the following definition.

Definition 1.5. Let $q \in (0,1)$ and let $u \in C_{[0,T]}\mathbb{H}^{2q}_x$ be a mild solution to ($\mathbb{P}$). A function u_+ is called a *continuation* of u iff $u_+ \in C_{[0,T_+]}\mathbb{H}^{2q}_x$ is also a solution to ($\mathbb{P}$) for $T_+ > T$ and $u_+(t) = u(t)$, $\forall t \in [0,T]$.

Theorem 1.6 (Continuation)**.** *Assume conditions in Theorem 1.3 are fulfilled. If $u_0 \in \mathbb{H}^{2q+2}(\Omega)$ then the unique solution of ($\mathbb{P}$) on the interval $[0,T]$ can be stretched to the time-interval $[0,T_+]$, for some $T_+ > T$.*

The following theorem is about blow-up alternative (i.e., either blow-up in finite-time or global existence). Let us explain this in the following definitions.

Definition 1.7 (*Maximal existence time*)**.** Let $u(\cdot)$ be a mild solution to ($\mathbb{P}$). We define the time $T_{\max} > 0$ as the maximal-time existence of $u(\cdot)$ by the following:

(i) $T_{\max} = \infty$ iff the mild solution $u(t)$ exists $\forall t \in [0,\infty)$.
(ii) $T_{\max} = T < \infty$ iff $u(t)$ exists $\forall t \in [0,T)$, but existence of solution is not reached at $t = T$.

Definition 1.8 (*Finite-time blow-up*)**.** Let $u(\cdot)$ be a mild solution to ($\mathbb{P}$). Then $u(\cdot)$ blows up in finite-time iff $T_{\max} < \infty$ and the following limit occurs

$$\lim_{t \uparrow T_{\max}^-} \|u(t)\|_{\mathbb{H}^{2q}(\Omega)} = \infty.$$

Theorem 1.9 (Blow-up alternative)**.** *Assume conditions in Theorem 1.3 hold. If $u_0 \in \mathbb{H}^{2q+2}(\Omega)$, then there is a maximal-time $T_{\max} > 0$ so that Problem ($\mathbb{P}$) possesses a unique solution $u \in C_{[0,T_{\max}]}\mathbb{H}^{2q}_x$. Therefore, either $T_{\max} = \infty$ (i.e global existence) or finite-time blow-up occurs with*

$$\limsup_{t \uparrow T_{\max}^- < \infty} \|u(t)\|_{\mathbb{H}^{2q}(\Omega)} = \infty.$$

Remark 1.10. In above results, we set $\beta < \alpha \in (0,1)$ and we achieved the local/global well-posedness theory in $C([0,T);\mathbb{H}^{2q}(\Omega))$ (here, T is finite or $T = \infty$). However, if we put $(0,1) \ni \beta > \alpha$, we must need the time-weighted Banach space to ensure that our results follow.

2. Notations and preliminaries.

2.1. **Notations.** For positive numbers Y, Z, we shall write $Y \lesssim_k Z$ if there is a constant $C(k) > 0$ such that $Y \le C(k)Z$. Writing $Y \simeq Z$ implies $Y = CZ$, for some constant $C > 0$. For $n \in \mathbb{N}^*$, $\{\xi_n\}_{n\in\mathbb{N}^*}$ a family of eigenvector, we have $-\Delta\xi_n(x) = \lambda_n\xi_n(x)$, for $x \in \Omega$ and $\xi_n(x) = 0$, for $x \in \partial\Omega$, and λ_n, $n \in \mathbb{N}^*$ are eigenvalues satisfying

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le ... \le \lambda_n \le ... \uparrow \infty.$$

Denote by $\|\cdot\|_B$ the norm in B (here B be a Banach space), $\|\cdot\|_q$ will denote the norm in L^q-Banach spaces and $\|\cdot\|$ is the usual norm in L^2-Hilbert space. For $T > 0$, $1 \le q \le \infty$, let $L^q(0,T;B)$ be the Banach space of measurable functions $v : (0,T) \to B$ such that $\|v\|_{L^q(0,T;B)} < \infty$, with

$$\|v\|_{L^q(0,T;B)} := \begin{cases} \left(\displaystyle\int_0^T \|v(t)\|_B^q \, dt\right)^{\frac{1}{q}}, & \text{if } q < \infty, \\ \operatorname*{ess\,sup}_{t\in(0,T)} \|v(t)\|_B, & \text{if } q = \infty. \end{cases}$$

$C([0,T];B)$ denotes the space of continuous functions $[0,T] \to B$ with the norm

$$\|v\|_{C([0,T];B)} = \sup_{t\in[0,T]} \|v(t)\|_B < \infty.$$

Note that if the interval $(0,T)$ is bounded then this space is a Banach space with the norm of $L^\infty(0,T;B)$.

For any $q \ge 0$, we define the Hilbert scales space

$$\mathbb{H}^q(\Omega) = \Bigg\{ L^2(\Omega) \ni v = \sum_{n\in\mathbb{N}^*} \xi_n \langle v, \xi_n\rangle_{L^2(\Omega)} : \\ \|v\|_{\mathbb{H}^q(\Omega)} = \left(\sum_{n\in\mathbb{N}^*} \left|\langle v, \xi_n\rangle_{L^2(\Omega)}\right|^2 \lambda_n^q\right)^{\frac{1}{2}} < \infty \Bigg\}, \tag{3}$$

where $\langle\cdot,\cdot\rangle_{L^2(\Omega)}$ is an inner product in $L^2(\Omega)$. Obviously, we have $\mathbb{H}^0(\Omega) = L^2(\Omega)$. Let $\mathbb{H}^{-q}(\Omega)$ denote the Hilbert space with respect to the norm (e.g. see [24])

$$\|v\|_{\mathbb{H}^{-q}(\Omega)} = \left(\sum_{n\in\mathbb{N}^*} \left|{}_{-q}\langle v, \xi_n\rangle_q \lambda_n^{-q}\right|^2\right)^{\frac{1}{2}}, \tag{4}$$

for $v \in \mathbb{H}^{-q}(\Omega)$ and where ${}_{-q}\langle\cdot,\cdot\rangle_q$ is the dual product between $\mathbb{H}^{-q}(\Omega)$ and $\mathbb{H}^q(\Omega)$. We note that

$${}_{-q}\langle v_1, v_2\rangle_q = \langle v_1, v_2\rangle_{L^2(\Omega)}, \quad \text{for } v_1 \in L^2(\Omega),\ v_2 \in \mathbb{H}^q(\Omega).$$

Remark 2.1. For $q \ge 0$, from the definition of the space $\mathbb{H}^q(\Omega)$, we observe that

$$\|v\| \le \frac{1}{\lambda_1^{\frac{q}{2}}} \|v\|_{\mathbb{H}^q(\Omega)}, \quad \text{and} \quad \|v\|_{\mathbb{H}^q(\Omega)} \le \frac{1}{\lambda_1^{\frac{l}{2}}} \|v\|_{\mathbb{H}^{q+l}(\Omega)}.$$

Moreover, if $0 < q < m$, we have $\mathbb{H}^m(\Omega) \subseteq \mathbb{H}^q(\Omega)$.

2.2. **Preliminaries.** Suppose that Problem ($\mathbb{P}$) possesses a solution u in the following Fourier series form $u(x,t) = \sum_{n\in\mathbb{N}^*} \xi_n(x) \left\langle u(\cdot,t), \xi_n \right\rangle_{L^2(\Omega)}$. Thanks to Bazhlekova et al. [11], the solution of ($\mathbb{P}$) can be converted to the integral equation as follows

$$\left\langle u(\cdot,t), \xi_n \right\rangle_{L^2(\Omega)} = S_{\alpha,n}(t) \left\langle u_0, \xi_n \right\rangle_{L^2(\Omega)} + \int_0^t s^{-\beta} S_{\alpha,n}(t-s) \left\langle F(u)(s), \xi_n \right\rangle_{L^2(\Omega)} ds,$$

where $F(u) = |u|^{p-2} u \log \left\langle u \right\rangle_b$, for $b > 0$, $p > 2$ and $S_{\alpha,n}(t)$ is given by

$$S_{\alpha,n}(t) = \int_0^\infty e^{-zt} H(n,\alpha,z) dz, \tag{5}$$

$$H(n,\alpha,z) = \frac{a}{\pi} \frac{\lambda_n \sin(\alpha\pi) z^\alpha}{(-z + \lambda_n a z^\alpha \cos\alpha\pi + \lambda_n)^2 + (\lambda_n a z^\alpha \sin\alpha\pi)^2},$$

and the Laplace transform of $S_n(\alpha,t)$ is given by

$$\mathscr{L}\left(S_{\alpha,n}(t)\right) = \frac{1}{t + a\lambda_n t^\alpha + \lambda_n}.$$

This implies that

$$u(t) = S_\alpha(t) u_0 + \int_0^t s^{-\beta} S_\alpha(t-s) F(u)(s) ds, \tag{6}$$

whereupon, we defined $S_\alpha(t)v = \sum_{n\in\mathbb{N}^*} S_{\alpha,n}(t) \left\langle v, \xi_n \right\rangle_{L^2(\Omega)} \xi_n(x)$.

The following lemmas will be needed throughout our paper.

Lemma 2.2 (Time-decay of operators)**.** *For $0 < \alpha, q < 1$, the following estimate holds*

$$\|S_\alpha(t)\|_{\mathbb{L}(L^2(\Omega), \mathbb{H}^{2q}(\Omega))} \le \frac{C_1(a,\alpha)}{\lambda_1^{1-q}} t^{\alpha-1}, \quad t \ge 0,$$

where $C_1(a,\alpha) = 1 + \frac{\Gamma(1-\alpha)}{a\pi\sin(\alpha\pi)}$.

Proof. Note that

$$\begin{aligned} H(n,\alpha,z) &= \frac{a}{\pi} \frac{\lambda_n \sin(\alpha\pi) z^\alpha}{(-z + \lambda_n a z^\alpha \cos\alpha\pi + \lambda_n)^2 + (\lambda_n a z^\alpha \sin\alpha\pi)^2} \\ &\le \frac{a}{\pi} \frac{\lambda_n \sin(\alpha\pi) z^\alpha}{(\lambda_n a z^\alpha \sin\alpha\pi)^2} = \frac{1}{a\pi\sin(\alpha\pi)} \frac{z^{-\alpha}}{\lambda_n}. \end{aligned}$$

The estimation of the term $S_{\alpha,n}(t)$ for $n \in \mathbb{N}^*, t > 0$ is bounded by

$$\begin{aligned} S_{\alpha,n}(t) &\le \frac{1}{a\pi\sin(\alpha\pi)} \frac{1}{\lambda_n} \int_0^\infty e^{-zt} z^{-\alpha} dz \\ &= \frac{1}{a\pi\sin(\alpha\pi)} \frac{t^{\alpha-1}}{\lambda_n} \int_0^\infty e^{-zt} (zt)^{-\alpha} d(zt) \\ &\le \frac{1}{a\pi\sin(\alpha\pi)} \frac{t^{\alpha-1}}{\lambda_n} \int_0^\infty e^{-\xi} \xi^{-\alpha} d\xi = \frac{\Gamma(1-\alpha)}{a\pi\sin(\alpha\pi)} \frac{t^{\alpha-1}}{\lambda_n}, \end{aligned}$$

where we have transformed $\xi = zt$ and $\int_0^\infty e^{-\xi} \xi^{-\alpha} d\xi = \Gamma(1-\alpha)$. Hence

$$S_{\alpha,n}(t) \frac{\lambda_n}{t^{\alpha-1}} \le \frac{\Gamma(1-\alpha)}{a\pi\sin(\alpha\pi)}. \tag{7}$$

From [11, Theorem 2.2] it follows that $S_{\alpha,n}(t) \le 1$ for all $(\alpha, n, t) \in ((0,1), \mathbb{N}^*, \mathbb{R}^+)$ and from (7) we have that

$$S_{\alpha,n}(t)\left(1+\frac{\lambda_n}{t^{\alpha-1}}\right) \le 1+\frac{\Gamma(1-\alpha)}{a\pi\sin(\alpha\pi)}.$$

This implies that

$$S_{\alpha,n}(t) \le \frac{C_1(a,\alpha)}{1+\frac{\lambda_n}{t^{\alpha-1}}} = \frac{C_1(a,\alpha)}{1+\lambda_n t^{1-\alpha}}, \quad \text{where } C_1(a,\alpha) = 1+\frac{\Gamma(1-\alpha)}{a\pi\sin(\alpha\pi)}. \tag{8}$$

Thus, for $v \in L^2(\Omega)$, one obtains that

$$\begin{aligned}
\|S_\alpha(t)v\|^2_{\mathbb{H}^{2q}(\Omega)} &= \left\| \sum_{n\in\mathbb{N}^*} S_{\alpha,n}(t) \langle v, \xi_n\rangle_{L^2(\Omega)} \xi_n \right\|^2_{\mathbb{H}^{2q}(\Omega)} \\
&= \sum_{n\in\mathbb{N}^*} |S_{\alpha,n}(t)|^2 \langle v, \xi_n\rangle^2_{L^2(\Omega)} \lambda_n^{2q} \\
&\le \sum_{n\in\mathbb{N}^*} \left(\frac{C_1(a,\alpha)}{1+\lambda_n t^{1-\alpha}}\right)^2 \langle v, \xi_n\rangle^2_{L^2(\Omega)} \lambda_n^{2q} \\
&\le \left(\frac{C_1(a,\alpha)}{\lambda_1^{1-q}} t^{\alpha-1}\right)^2 \|v\|^2_{L^2(\Omega)}, \quad \text{for } 0<q<1.
\end{aligned}$$

This finishes the proof. ∎

Lemma 2.3 (Regularity property of mixed nonlinearity)**.** *For $p' > 0$, there is a positive contant E such that the real function*

$$f(y) = |y|^p \log\langle y\rangle_b, \quad \textit{for } b, p > 0,$$

satisfies

$$f(y) \le E + |y|^{p+p'}. \tag{9}$$

Proof. Since $\lim_{|y|\uparrow+\infty} \frac{\log\langle y\rangle_b}{|y|^{p'}} = 0$, there exists $y_0 > 0$ such that

$$\frac{\log\langle y\rangle_b}{|y|^{p'}} < 1, \quad \text{for all } |y| > y_0.$$

Thus, we conclude that

$$f(y) \le |y|^{p+p'}, \quad \text{for all } |y| > y_0.$$

Since $p > 0$, then $|f(y)| \le E$, for some $E > 0$ and for all $|y| \le y_0$. Thus,

$$f(y) \le E + |y|^{p+p'}.$$

The proof is complete. ∎

Lemma 2.4 (Euler integral, see [29])**.** *Let $\sigma > 0$, consider the function $g_\sigma : \mathbb{R} \to [0,\infty)$ given by*

$$g_\sigma(t) = \begin{cases} \frac{1}{\Gamma(\sigma)} t^{\sigma-1}, & \textit{if } t > 0, \\ 0, & \textit{if } t \le 0, \end{cases}$$

where $\Gamma(\sigma)$ be the gamma function. Then, for $\sigma_1, \sigma_2 > 0$, we have the following related convolution

$$g_{\sigma_1} * g_{\sigma_2}(t) = g_{\sigma_1+\sigma_2}(t).$$

More generally, for $a, b \in \mathbb{R}$, we have

$$\int_a^b (b-s)^{\sigma_1-1}(s-a)^{\sigma_2-1}ds = (b-a)^{\sigma_1+\sigma_2-1}B(\sigma_1,\sigma_2), \tag{10}$$

where $B(\sigma_1,\sigma_2)$ is the Beta function defined by $B(\sigma_1,\sigma_2) = \frac{\Gamma(\sigma_1)\Gamma(\sigma_2)}{\Gamma(\sigma_1+\sigma_2)}$.

Proof. For $\sigma_1, \sigma_2 > 0$, we observe that for $t > 0$

$$\begin{aligned} g_{\sigma_1} * g_{\sigma_2}(t) &= \frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)} \int_0^t (t-s)^{\sigma_1-1}s^{\sigma_2-1}ds \\ &= \frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)} t^{\sigma_1+\sigma_2-1} \int_0^1 (1-s)^{\sigma_1-1}s^{\sigma_2-1}ds \\ &= \frac{B(\sigma_1,\sigma_2)}{\Gamma(\sigma_1)\Gamma(\sigma_2)} t^{\sigma_1+\sigma_2-1} = \frac{1}{\Gamma(\sigma_1+\sigma_2)} t^{\sigma_1+\sigma_2-1} = g_{\sigma_1+\sigma_2}(t), \end{aligned}$$

where we recall that the beta function $B(\sigma_1,\sigma_2) = \int_0^1 (1-s)^{\sigma_1-1}s^{\sigma_2-1}ds$ and $B(\sigma_1,\sigma_2) = \frac{\Gamma(\sigma_1)\Gamma(\sigma_2)}{\Gamma(\sigma_1+\sigma_2)}$. The proof of (10) is similar. ∎

Lemma 2.5 (Sobolev embedding, see [1, 25])**.** *For $\Omega \subset \mathbb{R}^N$, $N \ge 1, q \in [0, \frac{N}{2})$ and $1 \le p < \infty$, the following Sobolev embedding holds*

$$\mathbb{H}^q(\Omega) \hookrightarrow L^p(\Omega), \quad \text{for } 0 \le q < \frac{N}{2},\ p \le \frac{2N}{N-2q}. \tag{SE}$$

Remark 2.6. In this paper, we use the following Sobolev embeddings $\mathbb{H}^{2q}(\Omega) \hookrightarrow L^{(\cdot)}(\Omega)$ with the indexes $(\cdot)$ are dependent on N, p, q. From Theorem 2.5, we infer that $q < \frac{N}{4}$. Theorem 2.2 requires a constant $q < 1$. This is the reason that we consider the Problem ($\mathbb{P}$) with dimensions ≤ 4.

3. Proof of main results.

We are now in a position to prove the main results of this paper.

3.1. Proof of Theorem 1.3.

For $N \le 4, p > 2$, $0 < \beta < \alpha < 1$, we pick

$$0 \le \max\left\{ \frac{(p+p'-2)N}{4(p+p'-1)}; \left(1 - \frac{p-2}{(p+p'-2)(p-1)}\right)\frac{N}{4} \right\} \le q < \frac{N}{4}, \tag{11}$$

for $p' > \max\{0, \frac{5}{2} - p\}$. Writing $C_{[0,T]}\mathbb{H}_x^{2q} = C([0,T];\mathbb{H}^{2q}(\Omega))$ for short, we rewrite (6) as a fixed-point equation

$$\begin{aligned} u(t) &= \mathscr{Y}u(t), \\ \mathscr{Y}u(t) &:= S_\alpha(t)u_0 + \int_0^t s^{-\beta}S_\alpha(t-s)F(u)(s)ds. \end{aligned} \tag{12}$$

• **Step 1**: *Local existence.* First, we assert that $\mathscr{Y}u$ is well-defined for $u \in C_{[0,T]}\mathbb{H}_x^{2q}$ and the map $\mathscr{Y} : C_{[0,T]}\mathbb{H}_x^{2q} \to C_{[0,T]}\mathbb{H}_x^{2q}$. Now, we need to find a closed ball $V_R \subset C_{[0,T]}\mathbb{H}_x^{2q}$ such that $\mathscr{Y} : V_R \to V_R$ is a contraction mapping as $T > 0$ is sufficiently small. Let $T > 0$ and $R > 0$ to be chosen later, we set the following space

$$V_R := \left\{ u \in C_{[0,T]}\mathbb{H}_x^{2q} : u(x,0) = u_0(x),\ \text{and } \|u-u_0\|_{C_{[0,T]}\mathbb{H}_x^{2q}} \le R \right\}, \tag{13}$$

accompanied by the distance $d(u,v) = \|u-v\|_{C_{[0,T]}\mathbb{H}^{2q}_x}$. We write the mapping $\mathscr{Y}$ in (12) as

$$\mathscr{Y}u(t) := S_\alpha(t)u_0 + \mathcal{J}_{\alpha,\beta}u(t), \quad \mathcal{J}_{\alpha,\beta}u(t) = \int_0^t s^{-\beta} S_\alpha(t-s)F(u)(s)ds. \tag{14}$$

Our goal is to prove that the mapping $\mathscr{Y} : V_R \to V_R$ is invariant and a contraction.

○ *Claim 1*: *If $u_0 \in \mathbb{H}^{2q+2}(\Omega)$, then $\mathscr{Y}$ is V_R-invariant.* Indeed, from $S_{\alpha,n}(t) \le 1$ for all $(\alpha, n, t) \in ((0,1), \mathbb{N}^*, \mathbb{R}^+)$, we derive from Theorem 2.1 that

$$\begin{aligned}\|S_\alpha(t)u_0 - u_0\|_{\mathbb{H}^{2q}(\Omega)} &= \left(\sum_{n\in\mathbb{N}^*} |S_{\alpha,n}(t)-1|^2 \langle u_0, \xi_n\rangle^2_{L^2(\Omega)} \lambda_n^{2q}\right)^{\frac{1}{2}} \\ &\le \left(\sum_{n\in\mathbb{N}^*} \langle u_0, \xi_n\rangle^2_{L^2(\Omega)} \lambda_n^{2q}\right)^{\frac{1}{2}} \\ &= \|u_0\|_{\mathbb{H}^{2q}(\Omega)} \le \frac{1}{\lambda_1}\|u_0\|_{\mathbb{H}^{2q+2}(\Omega)}, \quad \forall t \in (0,T], \end{aligned} \tag{15}$$

Remark 3.1. Actually, in (15) we only need $u_0 \in \mathbb{H}^{2q}(\Omega)$ for the existence of solutions, but for these solutions to be extended (see Theorem 1.6) we require $u_0 \in \mathbb{H}^{2q+2}(\Omega)$.

Next, thanks to Theorem 2.2, we obtain that

$$\begin{aligned}\|\mathcal{J}_{\alpha,\beta}u(t)\|_{\mathbb{H}^{2q}(\Omega)} &= \left\|\int_0^t s^{-\beta} S_\alpha(t-s)F(u)(s)ds\right\|_{\mathbb{H}^{2q}(\Omega)} \\ &\le \int_0^t s^{-\beta}\|S_\alpha(t-s)F(u)(s)\|_{\mathbb{H}^{2q}(\Omega)}ds \\ &\le \frac{C_1(a,\alpha)}{\lambda_1^{1-q}}\int_0^t s^{-\beta}(t-s)^{\alpha-1}\|F(u)(s)\|_{L^2(\Omega)}ds, \quad t\in(0,T]. \end{aligned} \tag{16}$$

Using Theorem 2.3, we obtain that

$$\begin{aligned}&\int_\Omega |F(u)|^2 dx \le \int_\Omega \left(|u|^{p-1}\log\langle u\rangle_b\right)^2 dx \\ &\le 2\int_\Omega \left(E^2 + |u|^{2p+2p'-2}\right)dx \le 2\left(E^2\int_\Omega 1dx + \int_\Omega |u|^{2p+2p'-2}dx\right) \\ &\le 2\left(E^2\mathrm{Vol}(\Omega) + \|u\|^{2p+2p'-2}_{L^{2p+2p'-2}(\Omega)}\right) \le 2\left(E\sqrt{\mathrm{Vol}(\Omega)} + \|u\|^{p+p'-1}_{L^{2p+2p'-2}(\Omega)}\right)^2,\end{aligned}$$

where the following elementary inequalities have been used: $a^2+b^2 \le (a+b)^2 \le 2a^2+2b^2$, for $a,b>0$. From the above inequality, we conclude that

$$\|F(u)\|_{L^2(\Omega)} \le \sqrt{2}\left(E\sqrt{\mathrm{Vol}(\Omega)} + \|u\|^{p+p'-1}_{L^{2p+2p'-2}(\Omega)}\right).$$

From (11) we have $q \in [0, \frac{N}{4})$ and $q \ge \frac{(p+p'-2)N}{4(p+p'-1)} \Longrightarrow 2p+2p'-2 \le \frac{2N}{N-4q}$, we imply from (SE) in Theorem 2.5 that $\mathbb{H}^{2q}(\Omega) \hookrightarrow L^{2p+2p'-2}(\Omega)$ and one obtains that

$$\|F(u)\|_{L^2(\Omega)} \lesssim_{N,q,p,p'} \sqrt{2}\left(E\sqrt{\mathrm{Vol}(\Omega)} + \|u\|^{p+p'-1}_{\mathbb{H}^{2q}(\Omega)}\right). \tag{17}$$

The right hand side of (16) becomes

$\underline{\text{The (RHS) of (16)}}$

$$\begin{aligned}
&\lesssim_{N,q,p,p'} \frac{\sqrt{2}C_1(a,\alpha)}{\lambda_1^{1-q}} \int_0^t s^{-\beta}(t-s)^{\alpha-1}\left(E\sqrt{\mathrm{Vol}(\Omega)} + \|u(s)\|_{\mathbb{H}^{2q}(\Omega)}^{p+p'-1}\right) ds \\
&\lesssim_{N,q,p,p'} \frac{\sqrt{2}C_1(a,\alpha)}{\lambda_1^{1-q}} \left(E\sqrt{\mathrm{Vol}(\Omega)} + \|u\|_{C_{[0,T]}\mathbb{H}_x^{2q}}^{p+p'-1}\right) \int_0^t s^{-\beta}(t-s)^{\alpha-1} ds \\
&\lesssim_{N,q,p,p'} \frac{\sqrt{2}C_1(a,\alpha)}{\lambda_1^{1-q}} \left(E\sqrt{\mathrm{Vol}(\Omega)} + \left(R + \lambda_1^{-1}\|u_0\|_{\mathbb{H}^{2q+2}(\Omega)}\right)^{p+p'-1}\right) \\
&\times \int_0^t s^{-\beta}(t-s)^{\alpha-1} ds, \qquad (18)
\end{aligned}$$

where from Theorem 2.1 one has $u_0 \in \mathbb{H}^{2q+2}(\Omega) \subseteq \mathbb{H}^{2q}(\Omega)$ and from (13) we infer that

$$\|u\|_{C_{[0,T]}\mathbb{H}_x^{2q}} \le R + \|u_0\|_{\mathbb{H}^{2q}(\Omega)} \le R + \frac{1}{\lambda_1}\|u_0\|_{\mathbb{H}^{2q+2}(\Omega)}, \text{ for } u \in V_R, t \in [0,T]. \quad (19)$$

Now we use Theorem 2.4 to compute the following integral

$$\begin{aligned}
\int_0^t (t-s)^{\alpha-1}s^{-\beta} ds &= \int_0^t (t-s)^{\alpha-1}s^{(1-\beta)-1} ds = \Gamma(\alpha)\Gamma(1-\beta)[g_\alpha * g_{1-\beta}(t)] \\
&= \Gamma(\alpha)\Gamma(1-\beta)g_{\alpha+1-\beta}(t) = \frac{\Gamma(\alpha)\Gamma(1-\beta)}{\Gamma(\alpha+1-\beta)}t^{\alpha-\beta} \\
&= B(\alpha, 1-\beta)t^{\alpha-\beta}, \quad \text{with } 0<\beta<\alpha<1. \qquad (20)
\end{aligned}$$

From (16), (18) and (20), for all $t \in [0,T]$ one obtains

$$\begin{aligned}
&\|\mathcal{J}_{\alpha,\beta}u(t)\|_{\mathbb{H}^{2q}(\Omega)} \\
&\lesssim_{N,q,p,p'} \frac{\sqrt{2}C_1(a,\alpha)}{\lambda_1^{1-q}} \left(E\sqrt{\mathrm{Vol}(\Omega)} + \left(R + \lambda_1^{-1}\|u_0\|_{\mathbb{H}^{2q+2}(\Omega)}\right)^{p+p'-1}\right) \\
&\times B(\alpha, 1-\beta)t^{\alpha-\beta}. \qquad (21)
\end{aligned}$$

Hence, from (15) and (21), for all $t \in [0,T]$, it follows that

$$\begin{aligned}
&\|\mathscr{Y}u - u_0\|_{C_{[0,T]}\mathbb{H}_x^{2q}} \le \|S_\alpha(\cdot)u_0 - u_0\|_{C_{[0,T]}\mathbb{H}_x^{2q}} + \|\mathcal{J}_{\alpha,\beta}u\|_{C_{[0,T]}\mathbb{H}_x^{2q}} \\
&\lesssim_{N,q,p,p'} \frac{1}{\lambda_1}\|u_0\|_{\mathbb{H}^{2q+2}(\Omega)} \\
&+ \frac{\sqrt{2}C_1(a,\alpha)}{\lambda_1^{1-q}} \left(E\sqrt{\mathrm{Vol}(\Omega)} + \left(R + \lambda_1^{-1}\|u_0\|_{\mathbb{H}^{2q+2}(\Omega)}\right)^{p+p'-1}\right) B(\alpha, 1-\beta)t^{\alpha-\beta}. \qquad (22)
\end{aligned}$$

Therefore we see that if $C(N,q,p,p')\frac{1}{\lambda_1}\|u_0\|_{\mathbb{H}^{2q+2}(\Omega)} = \frac{R}{2}$, and

$$\begin{aligned}
R &\ge 2\sqrt{2}C(N,q,p,p')\frac{C_1(a,\alpha)}{\lambda_1^{1-q}} \\
&\times \left(E\sqrt{\mathrm{Vol}(\Omega)} + \left(R + \lambda_1^{-1}\|u_0\|_{\mathbb{H}^{2q+2}(\Omega)}\right)^{p+p'-1}\right) B(\alpha, 1-\beta)T^{\alpha-\beta},
\end{aligned}$$

then $\mathscr{Y}$ is an invariant mapping in V_R.

∘ Claim 2: *$\mathscr{Y} : V_R \to V_R$ be the contraction map.* Let $u_1, u_2 \in V_R$, similar to (16) and using Theorem 2.2, for all $t \in [0,T]$, we have

$$\|\mathscr{Y}u_1(t) - \mathscr{Y}u_2(t)\|_{\mathbb{H}^{2q}(\Omega)}$$

$$\leq \frac{C_1(a,\alpha)}{\lambda_1^{1-q}} \int_0^t s^{-\beta}(t-s)^{\alpha-1} \|F(u_1)(s) - F(u_2)(s)\|_{L^2(\Omega)} ds. \tag{23}$$

As a consequence of the Lagrange mean value theorem, for $0 \leq \omega \leq 1$ and $d_\omega(x,y) = \omega x + (1-\omega)y$, we have

$$\begin{aligned} |F(u_1) - F(u_2)| &= \left|\dot{F}(d_\omega(u_1,u_2))(u_1-u_2)\right| \leq \left|\dot{F}(d_\omega(u_1,u_2))\right| |u_1-u_2| \\ &\leq [1 + (p-1)\log \langle d_\omega(u_1,u_2)\rangle_b] \, |d_\omega(u_1,u_2)|^{p-2} |u_1-u_2| \\ &\leq |d_\omega(u_1,u_2)|^{p-2} |u_1-u_2| \\ &\quad + (p-1)\log \langle d_\omega(u_1,u_2)\rangle_b \, |d_\omega(u_1,u_2)|^{p-2} |u_1-u_2|, \end{aligned}$$

where for $F(s) = |s|^{p-2} s \log \langle s\rangle_b$ we imply that $\dot{F}(s) \leq [1+(p-1)\log\langle s\rangle_b] \, |s|^{p-2}$ with $\dot{F}(s) = \frac{dF}{ds}$. By virtue of Theorem 2.3, we observe that

$$\log \langle d_\omega(u_1,u_2)\rangle_b \, |d_\omega(u_1,u_2)|^{p-2} \leq E + |d_\omega(u_1,u_2)|^{p+p'-2}, \quad \text{for some } E > 0,$$

and from the fact that $|d_\omega(u_1,u_2)| \leq |u_1+u_2|$ for all $\omega \in [0,1]$ we derive

$$\begin{aligned} |F(u_1) - F(u_2)| &\leq |d_\omega(u_1,u_2)|^{p-2} |u_1-u_2| \\ &\quad + (p-1)\left(E + |d_\omega(u_1,u_2)|^{p+p'-2}\right) |u_1-u_2| \\ &\leq |u_1+u_2|^{p-2} |u_1-u_2| \\ &\quad + (p-1)E\,|u_1-u_2| + (p-1)\,|u_1+u_2|^{p+p'-2} |u_1-u_2|. \end{aligned} \tag{24}$$

We then use Hölder's inequality to deduce

$$\begin{aligned} &\int_\Omega \left(|u_1+u_2|^{p-2} |u_1-u_2|\right)^2 dx \\ &= \int_\Omega |u_1+u_2|^{2p-4} |u_1-u_2|^2 dx \\ &\leq \left(\int_\Omega |u_1+u_2|^{2p-2} dx\right)^{\frac{p-2}{p-1}} \left(\int_\Omega |u_1-u_2|^{2p-2} dx\right)^{\frac{1}{p-1}} \\ &\leq 2^{2p-4} \left(\int_\Omega \sum_{j=1,2} |u_j|^{2p-2} dx\right)^{\frac{p-2}{p-1}} \left(\int_\Omega |u_1-u_2|^{2p-2} dx\right)^{\frac{1}{p-1}} \\ &\leq \left(2^{p-2}\left(\sum_{j=1,2} \|u_j\|_{L^{2p-2}(\Omega)}\right)^{p-2}\right)^2 \|u_1-u_2\|^2_{L^{2p-2}(\Omega)}, \end{aligned} \tag{25}$$

where we have used $(c+d)^z \leq 2^z(c^z+d^z)$ and $c^z + d^z \leq (c+d)^z$ for $z > 1$ (here, $z = 2p-2$, for $p > 2$). Similarly, $p' > \max\{0, \frac{5}{2} - p\}$ implies that $p' > \frac{5}{2} - p \Rightarrow 2p + 2p' - 4 > 1 \Rightarrow \frac{(2p+2p'-4)(p-1)}{p-2} > 1$ for $p > 2$ and we shall use these basic inequalities above with $z = \frac{(2p+2p'-4)(p-1)}{p-2} > 1$ to obtain the following estimate

$$\begin{aligned} &\int_\Omega \left(|u_1+u_2|^{p+p'-2} |u_1-u_2|\right)^2 dx = \int_\Omega |u_1+u_2|^{2p+2p'-4} |u_1-u_2|^2 dx \\ &\leq \left(\int_\Omega |u_1+u_2|^{\frac{(2p+2p'-4)(p-1)}{p-2}} dx\right)^{\frac{p-2}{p-1}} \left(\int_\Omega |u_1-u_2|^{2p-2} dx\right)^{\frac{1}{p-1}} \end{aligned}$$

$$\le 2^{2p+2p'-4}\left(\int_\Omega \sum_{j=1,2} |u_j|^{\frac{(2p+2p'-4)(p-1)}{p-2}}\,dx\right)^{\frac{p-2}{p-1}}\left(\int_\Omega |u_1-u_2|^{2p-2}\,dx\right)^{\frac{1}{p-1}}$$

$$\le \left(2^{p+p'-2}\left(\sum_{j=1,2}\|u_j\|_{L^{\frac{(2p+2p'-4)(p-1)}{p-2}}(\Omega)}\right)^{p+p'-2}\right)^2 \|u_1-u_2\|^2_{L^{2p-2}(\Omega)}. \tag{26}$$

Therefore, by combining (24) - (26), we obtain

$$\begin{aligned}&\|F(u_1)-F(u_2)\|_{L^2(\Omega)}\\ &\le (p-1)E\,\|u_1-u_2\|_{L^2(\Omega)} + 2^{p-2}\left(\sum_{j=1,2}\|u_j\|_{L^{2p-2}(\Omega)}\right)^{p-2}\|u_1-u_2\|_{L^{2p-2}(\Omega)}\\ &\quad + 2^{p+p'-2}(p-1)\left(\sum_{j=1,2}\|u_j\|_{L^{\frac{(2p+2p'-4)(p-1)}{p-2}}(\Omega)}\right)^{p+p'-2}\|u_1-u_2\|_{L^{2p-2}(\Omega)}.\end{aligned} \tag{27}$$

From (11), we have that

$$\begin{cases} q\in[0,\frac{N}{4}) \Longrightarrow 2\le \frac{2N}{N-4q},\\ q\ge \frac{p+p'-2}{p+p'-1}\frac{N}{4} \Rightarrow 2p+2p'-2\le\frac{2N}{N-4q} \Longrightarrow 2p-2\le\frac{2N}{N-4q},\\ q\ge (1-\frac{p-2}{(p-1)(p+p'-2)})\frac{N}{4} \Longrightarrow \frac{(2p+2p'-4)(p-1)}{p-2}\le\frac{2N}{N-4q},\end{cases}$$

and using the Sobolev embedding (SE) in Theorem 2.5, one obtains

$$\mathbb{H}^{2q}(\Omega)\hookrightarrow L^2(\Omega),\quad \mathbb{H}^{2q}(\Omega)\hookrightarrow L^{2p-2}(\Omega),\quad \mathbb{H}^{2q}(\Omega)\hookrightarrow L^{\frac{(2p+2p'-4)(p-1)}{p-2}}(\Omega). \tag{28}$$

From (27), we have that

$$\begin{aligned}&\|F(u_1)-F(u_2)\|_{L^2(\Omega)}\\ &\lesssim_{N,q,p,p'} (p-1)E\,\|u_1-u_2\|_{\mathbb{H}^{2q}(\Omega)}\\ &\quad + 2^{p-2}\left(\sum_{j=1,2}\|u_j\|_{\mathbb{H}^{2q}(\Omega)}\right)^{p-2}\|u_1-u_2\|_{\mathbb{H}^{2q}(\Omega)}\\ &\quad + 2^{p+p'-2}(p-1)\left(\sum_{j=1,2}\|u_j\|_{\mathbb{H}^{2q}(\Omega)}\right)^{p+p'-2}\|u_1-u_2\|_{\mathbb{H}^{2q}(\Omega)}\\ &= K(\sum_{j=1,2}\|u_j\|_{\mathbb{H}^{2q}(\Omega)})\,\|u_1-u_2\|_{\mathbb{H}^{2q}(\Omega)},\end{aligned} \tag{29}$$

where $K(z)$ is defined by

$$K(z) = (p-1)E + 2^{p-2}z^{p-2} + 2^{p+p'-2}(p-1)z^{p+p'-2}. \tag{30}$$

From (23) and (29), one has that

$$\underline{\text{The (RHS) of (23)}}$$

$$\lesssim_{N,q,p,p'} \frac{C_1(a,\alpha)}{\lambda_1^{1-q}}\int_0^t s^{-\beta}(t-s)^{\alpha-1}K(\sum_{j=1,2}\|u_j(s)\|_{\mathbb{H}^{2q}(\Omega)})\,\|u_1(s)-u_2(s)\|_{\mathbb{H}^{2q}(\Omega)}\,ds.$$

By an argument analogous to (19) for any $u_j \in V_R$, $j=1,2$ and by choosing $R>0$ such that

$$\|u_j\|_{C_{[0,T]}\mathbb{H}^{2q}_x} \le R + \|u_0\|_{\mathbb{H}^{2q}(\Omega)} \le R + \frac{1}{\lambda_1}\|u_0\|_{\mathbb{H}^{2q+2}(\Omega)}, \quad t\in[0,T], j=1,2,$$

and similarly for $v \in V_R$, then we have

$$\cdots \lesssim_{N,q,p,p'} \frac{C_1(a,\alpha)K(2R+2\lambda_1^{-1}\|u_0\|_{\mathbb{H}^{2q+2}(\Omega)})}{\lambda_1^{1-q}} \times \int_0^t s^{-\beta}(t-s)^{\alpha-1}\|u_1(s)-u_2(s)\|_{\mathbb{H}^{2q}(\Omega)}\,ds.$$

Using (20), we obtain

$$\begin{aligned}&\underline{\text{The (RHS) of (23)}}\\ &\lesssim_{N,q,p,p'} \frac{C_1(a,\alpha)K(2R+2\lambda_1^{-1}\|u_0\|_{\mathbb{H}^{2q+2}(\Omega)})}{\lambda_1^{1-q}}\|u_1-u_2\|_{C_{[0,T]}\mathbb{H}^{2q}_x}\int_0^t s^{-\beta}(t-s)^{\alpha-1}ds\\ &\lesssim_{N,q,p,p'} \frac{C_1(a,\alpha)K(2R+2\lambda_1^{-1}\|u_0\|_{\mathbb{H}^{2q+2}(\Omega)})}{\lambda_1^{1-q}}\|u_1-u_2\|_{C_{[0,T]}\mathbb{H}^{2q}_x}B(\alpha,1-\beta)t^{\alpha-\beta}.\end{aligned} \tag{31}$$

Introducing (31) into (23), it follows for $0<\beta<\alpha<1, t\in[0,T]$, that

$$\|\mathscr{Y}u_1(t)-\mathscr{Y}u_2(t)\|_{\mathbb{H}^{2q}(\Omega)} \le \frac{C(N,q,p,p')C_1(a,\alpha)K(2R+2\lambda_1^{-1}\|u_0\|_{\mathbb{H}^{2q+2}(\Omega)})}{\lambda_1^{1-q}} \times B(\alpha,1-\beta)T^{\alpha-\beta}\|u_1-u_2\|_{C_{[0,T]}\mathbb{H}^{2q}_x}.$$

Then, the following estimate holds

$$\|\mathscr{Y}u_1-\mathscr{Y}u_2\|_{C_{[0,T]}\mathbb{H}^{2q}_x} \le \frac{C(N,q,p,p')C_1(a,\alpha)K(2R+2\lambda_1^{-1}\|u_0\|_{\mathbb{H}^{2q+2}(\Omega)})}{\lambda_1^{1-q}} \times B(\alpha,1-\beta)T^{\alpha-\beta}\|u_1-u_2\|_{C_{[0,T]}\mathbb{H}^{2q}_x}. \tag{32}$$

Choosing now $T>0$, small enough, such that

$$\frac{C(N,q,p,p')C_1(a,\alpha)K(2R+2\lambda_1^{-1}\|u_0\|_{\mathbb{H}^{2q+2}(\Omega)})}{\lambda_1^{1-q}}B(\alpha,1-\beta)T^{\alpha-\beta}<1,$$

it follows that $\mathscr{Y}$ is a contraction mapping on V_R. Hence, by virtue of the contraction mapping principle we conclude that $\mathscr{Y}$ possesses a unique fixed point u in V_R.
• **Step 2**: *Regularity estimate.* First, we establish the following lemma that will play a crucial role in our proofs.

Lemma 3.2 (Weakly singular Gronwall's inequality, see [41, Theorem 3.8])**.** *Let a,b be non-negative constants and $c>0$. Let $0<\sigma_1,\sigma_2<1$, $\sigma_3\ge 0$ with $\sigma_1+\sigma_2+\sigma_3<1$. Suppose that $t^{\sigma_1}u(t)\in L^\infty_+(0,T]$ and u satisfies the inequality*

$$u(t)\le at^{-\sigma_1}+b+c\int_0^t (t-s)^{-\sigma_2}s^{-\sigma_3}u(s)ds, \quad \text{for a.e.} \quad t\in(0,T].$$

Then, there exists an explicit constant $\tilde{C}=C(b,c,\sigma_1,\sigma_2,\sigma_3,T)$ such that

$$u(t)\le a\tilde{C}t^{-\sigma_1}, \quad \text{for a.e.} \quad t\in(0,T].$$

From Eq. (6), we rewrite the solution as

$$u(t) = S_\alpha(t)u_0 + \mathcal{J}_{\alpha,\beta}u(t), \quad \mathcal{J}_{\alpha,\beta}u(t) = \int_0^t s^{-\beta} S_\alpha(t-s)F(u)(s)ds.$$

Thanks to Theorem 2.2 and Theorem 2.1, we have

$$\begin{aligned} \|S_\alpha(t)u_0\|_{\mathbb{H}^{2q}(\Omega)} &\le \frac{C_1(a,\alpha)}{\lambda_1^{1-q}} t^{\alpha-1} \|u_0\|_{L^2(\Omega)} \\ &\le \frac{C_1(a,\alpha)}{\lambda_1^{1-q+q'}} \|u_0\|_{\mathbb{H}^{2q'}(\Omega)} t^{\alpha-1} \\ &= \frac{\mathcal{A}_1}{\lambda_1^{1-q+q'}} \|u_0\|_{\mathbb{H}^{2q'}(\Omega)} t^{\alpha-1}, \quad \forall t \in (0,T], \end{aligned} \tag{33}$$

where $\mathcal{A}_1 = C_1(a,\alpha) > 0$, $q \in (0,1)$ and $q' \ge q-1$. It follows readily from (16), (17) and (20) that

$$\begin{aligned} \|\mathcal{J}_{\alpha,\beta}u(t)\|_{\mathbb{H}^{2q}(\Omega)} &\le \frac{\sqrt{2}C_1(a,\alpha)}{\lambda_1^{1-q}} \int_0^t s^{-\beta}(t-s)^{\alpha-1}\left(E\sqrt{\mathrm{Vol}(\Omega)} + \|u(s)\|_{\mathbb{H}^{2q}(\Omega)}^{p+p'-1}\right) ds \\ &\le \frac{\sqrt{2}C_1(a,\alpha)E\sqrt{\mathrm{Vol}(\Omega)}}{\lambda_1^{1-q}} \int_0^t s^{-\beta}(t-s)^{\alpha-1} ds \\ &\quad + \frac{\sqrt{2}C_1(a,\alpha)}{\lambda_1^{1-q}} \|u\|_{L^\infty(0,T;\mathbb{H}^{2q}(\Omega))}^{p+p'-2} \int_0^t s^{-\beta}(t-s)^{\alpha-1} \|u(s)\|_{\mathbb{H}^{2q}(\Omega)} ds \\ &\le \mathcal{A}_2 + \mathcal{A}_3 \int_0^t s^{-\beta}(t-s)^{\alpha-1} \|u(s)\|_{\mathbb{H}^{2q}(\Omega)} ds, \end{aligned} \tag{34}$$

where $\mathcal{A}_2 = \frac{\sqrt{2}rE\sqrt{\mathrm{Vol}(\Omega)}\mathcal{A}_1 B(\alpha,1-\beta)T^{\alpha-\beta}}{\lambda_1^{1-q}} > 0, \mathcal{A}_3 = \frac{\sqrt{2}M^{p+p'-2}\mathcal{A}_1}{\lambda_1^{1-q}} > 0$ for $q \in (0,1), p > 2$ and the constant $p' > \max\{0, \frac{5}{2} - p\}$. A combination of (33) and (34) implies

$$\|u(t)\|_{\mathbb{H}^{2q}(\Omega)} \le \frac{\mathcal{A}_1}{\lambda_1^{1-q+q'}} \|u_0\|_{\mathbb{H}^{2q'}(\Omega)} t^{\alpha-1} + \mathcal{A}_2 + \mathcal{A}_3 \int_0^t s^{-\beta}(t-s)^{\alpha-1} \|u(s)\|_{\mathbb{H}^{2q}(\Omega)} ds.$$

Notice that, for $0 < \beta < 2\alpha - 1$ and $\sigma_1 = \sigma_2 = 1 - \alpha, \sigma_3 = \beta$ in Theorem 3.2, we deduce that $\sigma_1 + \sigma_2 + \sigma_3 = 2 - 2\alpha + \beta < 1$. We invoke Theorem 3.2 to obtain that

$$\|u(t)\|_{\mathbb{H}^{2q}(\Omega)} \le \tilde{C} \frac{\|u_0\|_{\mathbb{H}^{2q'}(\Omega)}}{\lambda_1^{1-q+q'}} t^{\alpha-1}, \quad t \in (0,T],\ 0 < q < 1, q' \ge q-1,$$

where $\tilde{C} = C(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \alpha, \beta, T)$ is an explicit constant. This completes the proof of Theorem 1.3. ∎

3.2. **Proof of Theorem 1.6.** Since we already know that the local mild solution of ($\mathbb{P}$) does exist, we next prove this solution has a continuation to a bigger interval of existence. Let

$$u : [0,T] \to \mathbb{H}^{2q}(\Omega) \text{ be the unique solution of Problem } (\mathbb{P}),$$

with $T > 0$ the time defined in Theorem 1.3. Fix $R > 0$, for $T < T_+ \equiv T_+(R)$, and $u_0 \in \mathbb{H}^{2q+2}(\Omega)$, $u_T = u(\cdot,T) \in \mathbb{H}^{2q}(\Omega)$ we need to prove that there exists $u_+ : [0,T_+] \to \mathbb{H}^{2q}(\Omega)$, which is a mild solution of Problem ($\mathbb{P}$) extending the previous solution $u(\cdot)$. This should be well dealt with $p > 2$, $q \in (0,1)$ satisfies

(11), $0 < \beta < \alpha < 1$, there exist a constant p' such that $p' > \max\{0, \frac{5}{2} - p\}$ and $0 < \gamma < \alpha - \beta$, and

$$\max_{i=1,\dots,4}(P_i) \le \frac{R}{4}, \tag{35}$$

with

$$P_1 := \frac{C_2(N,\Omega,T,a)}{1-\alpha} T_+^{1-\alpha} \|u_0\|_{\mathbb{H}^{2q+2}(\Omega)}, \tag{36}$$

$$\begin{aligned} P_2 := &\frac{\sqrt{2}C(N,q,p,p')C_1(a,\alpha)}{\lambda_1^{1-q}} \left(E\sqrt{\mathrm{Vol}(\Omega)} + \left(R + \|u_T\|_{\mathbb{H}^{2q}(\Omega)}\right)^{p+p'-1}\right) \\ &\times B(\alpha, 1-\beta) T_+^{\alpha-\beta}, \end{aligned} \tag{37}$$

$$\begin{aligned} P_37 := &\frac{\sqrt{2}C(N,q,p,p')C_3(a,\pi,\alpha,\gamma)}{\lambda_1^{1-q}} \left(E\sqrt{\mathrm{Vol}(\Omega)} + \left(R + \|u_T\|_{\mathbb{H}^{2q}(\Omega)}\right)^{p+p'-1}\right) \\ &\times B(\alpha-\gamma, 1-\beta) T_+^{\alpha-\beta}, \end{aligned} \tag{38}$$

$$P_4 := \frac{C(N,q,p,p')C_1(a,\alpha)K(2R + 2\|u_T\|_{\mathbb{H}^{2q}(\Omega)})}{\lambda_1^{1-q}} B(\alpha, 1-\beta) T_+^{\alpha-\beta}, \tag{39}$$

where $K(z)$ is defined in (30). For $T_+ > T > 0$ and $R > 0$, let us define

$$V_R^+ := \left\{ u_+ \in C_{[0,T_+]}\mathbb{H}_x^{2q} : \left| \begin{array}{ll} u_+(t) = u(t), & \forall t \in [0,T], \\ \|u_+ - u\|_{C_{[T,T_+]}\mathbb{H}_x^{2q}} \le R, & \forall t \in [T, T_+], \end{array} \right. \right\} \tag{40}$$

where we denoted $C_{[0,T_+]}\mathbb{H}_x^{2q} := C([0,T_+]; \mathbb{H}^{2q}(\Omega))$ and $C_{[T,T_+]}\mathbb{H}_x^{2q} := C([T,T_+]; \mathbb{H}^{2q}(\Omega))$ for short.

• **Step 1**: *We prove that* $\mathscr{Y} : V_R^+ \to V_R^+$ *where* $V_R^+ \subset C_{[0,T_+]}\mathbb{H}_x^{2q}$ *is the ball* (40). Indeed, for $\mathscr{Y}$ defined as in (14), let $u_+ \in V_R^+$, we consider the following two cases.

∗ If $t \in [0,T]$, then in view of the Claim 2 of Theorem 1.3, we know that Problem ($\mathbb{P}$) admits a unique solution and

$$u_+(t) = u(t), \forall t \in [0,T] \text{ implies } \|\mathscr{Y}u_+ - u_T\|_{C_{[0,T]}\mathbb{H}_x^{2q}} = 0 \text{ in } V_R^+.$$

∗ For $t \in [T, T_+]$, one has

$$\begin{aligned} \|\mathscr{Y}u_+(t) - u_T\|_{\mathbb{H}^{2q}(\Omega)} &\le \|(S_\alpha(t) - S_\alpha(T))\, u_0\|_{\mathbb{H}^{2q}(\Omega)} \\ &+ \left\| \int_T^t s^{-\beta} S_\alpha(t-s) F(u_+)(s) ds \right\|_{\mathbb{H}^{2q}(\Omega)} \\ &+ \left\| \int_0^T \left(s^{-\beta} S_\alpha(t-s) - s^{-\beta} S_\alpha(T-s)\right) F(u_+)(s) ds \right\|_{\mathbb{H}^{2q}(\Omega)} \\ &=: \sum_{i=1}^3 \|A_i(t)\|_{\mathbb{H}^{2q}(\Omega)}, \quad \text{(respectively)}. \end{aligned} \tag{41}$$

To estimate the term $\|A_1(t)\|_{\mathbb{H}^{2q}(\Omega)}$ in (41), applying [11, Theorem 2.1], we know that

$$\sum_{n\in\mathbb{N}^*} \left|\dot{S}_{\alpha,n}(t)\right|^2 \langle v, \xi_n \rangle_{L^2(\Omega)}^2 \le C_2^2(N,\Omega,T,a) t^{-2\alpha} \|v\|_{\mathbb{H}^2(\Omega)}^2, \quad \text{for } v \in \mathbb{H}^2(\Omega), \tag{42}$$

where $\dot{S}_{\alpha,n}(t) = \frac{dS_{\alpha,n}(t)}{dt}$ and $C_2(N,\Omega,T,a)$ is a constant depending on Ω, N, T, a. For $T \le t \le T_+$, we have

$$\begin{aligned}
\|A_1(t)\|_{\mathbb{H}^{2q}(\Omega)} &= \left\| \sum_{n\in\mathbb{N}^*} \left(S_{\alpha,n}(t) - S_{\alpha,n}(T)\right) \langle u_0, \xi_n\rangle_{L^2(\Omega)} \xi_n \right\|_{\mathbb{H}^{2q}(\Omega)} \\
&= \left(\sum_{n\in\mathbb{N}^*} \left(\int_T^t \dot{S}_{\alpha,n}(z)dz \right)^2 \langle u_0, \xi_n\rangle^2_{L^2(\Omega)} \lambda_n^{2q} \right)^{\frac{1}{2}} \\
&\le C_2(N,\Omega,T,a) \|u_0\|_{\mathbb{H}^{2q+2}(\Omega)} \int_T^t z^{-\alpha} dz \\
&= C_2(N,\Omega,T,a) \|u_0\|_{\mathbb{H}^{2q+2}(\Omega)} \frac{t^{1-\alpha} - T^{1-\alpha}}{1-\alpha} \\
&\le C_2(N,\Omega,T,a) \|u_0\|_{\mathbb{H}^{2q+2}(\Omega)} \frac{(t-T)^{1-\alpha}}{1-\alpha}, \quad \text{for } 0<\alpha<1.
\end{aligned} \tag{43}$$

Hence, we obtain

$$\begin{aligned}
\|A_1(t)\|_{\mathbb{H}^{2q}(\Omega)} &\le C_2(N,\Omega,T,a) \frac{(t-T)^{1-\alpha}}{1-\alpha} \|u_0\|_{\mathbb{H}^{2q+2}(\Omega)} \\
&\le \frac{C_2(N,\Omega,T,a)}{1-\alpha} T_+^{1-\alpha} \|u_0\|_{\mathbb{H}^{2q+2}(\Omega)}, \quad t\in[T,T_+], \alpha\in(0,1).
\end{aligned}$$

By using (35) and (36), the following estimate holds for all $t \in [0, T_+]$

$$\|A_1\|_{C_{[0,T_+]}\mathbb{H}_x^{2q}} \le \frac{C_2(N,\Omega,T,a)}{1-\alpha} T_+^{1-\alpha} \|u_0\|_{\mathbb{H}^{2q+2}(\Omega)} = P_1 \le \frac{R}{4}. \tag{44}$$

Similar to (16), we have that

$$\|A_2(t)\|_{\mathbb{H}^{2q}(\Omega)} \le \frac{C_1(a,\alpha)}{\lambda_1^{1-q}} \int_T^t s^{-\beta}(t-s)^{\alpha-1} \|F(u)(s)\|_{L^2(\Omega)} ds. \tag{45}$$

Using (17) and arguing as in (18) and from (40) we have that for $T \le t \le T_+, u \in V_R^+$

$$\|u\|_{C_{[T,T_+]}\mathbb{H}_x^{2q}} \le R + \|u_T\|_{\mathbb{H}^{2q}(\Omega)}, \quad u_T(x) = u(x,T), x\in\Omega, \tag{46}$$

and we obtain for $t \in [T, T_+]$

$$\begin{aligned}
&\underline{\text{The (RHS) of (45)}} \\
&\lesssim_{N,q,p,p'} \frac{\sqrt{2}C_1(a,\alpha)}{\lambda_1^{1-q}} \int_T^t s^{-\beta}(t-s)^{\alpha-1} \left(E\sqrt{\mathrm{Vol}(\Omega)} + \|u(s)\|^{p+p'-1}_{\mathbb{H}^{2q}(\Omega)} \right) ds \\
&\lesssim_{N,q,p,p'} \frac{\sqrt{2}C_1(a,\alpha)}{\lambda_1^{1-q}} \left(E\sqrt{\mathrm{Vol}(\Omega)} + \|u\|^{p+p'-1}_{C_{[T,T_+]}\mathbb{H}_x^{2q}} \right) \int_T^t s^{-\beta}(t-s)^{\alpha-1} ds \\
&\lesssim_{N,q,p,p'} \frac{\sqrt{2}C_1(a,\alpha)}{\lambda_1^{1-q}} \left(E\sqrt{\mathrm{Vol}(\Omega)} + \left(R + \|u_T\|_{\mathbb{H}^{2q}(\Omega)}\right)^{p+p'-1} \right) \\
&\quad \times \int_T^t (t-s)^{\alpha-1}(s-T)^{-\beta} ds,
\end{aligned}$$

where for $0 \le T \le s \le t \le T_+$, we have used $s^{-\beta} \le (s-T)^{-\beta}$ for $\beta > 0$. Thus, from (10) for $T \le t \le T_+$, we have the following bound

$$\|A_2(t)\|_{\mathbb{H}^{2q}(\Omega)}$$

$$\leq \frac{\sqrt{2}C(N,q,p,p')C_1(a,\alpha)}{\lambda_1^{1-q}}\left(E\sqrt{\mathrm{Vol}(\Omega)}+\left(R+\|u_T\|_{\mathbb{H}^{2q}(\Omega)}\right)^{p+p'-1}\right)$$
$$\times B(\alpha,1-\beta)T_+^{\alpha-\beta}=P_2. \tag{47}$$

Using (35) and (37), we infer that for all $t\in[0,T]$

$$\|A_2\|_{C_{[0,T_+]}\mathbb{H}^{2q}_x}\leq P_2\leq\frac{R}{4}. \tag{48}$$

To estimate the third term in (41), an upper bound of $S_{\alpha,n}(t)-S_{\alpha,n}(t')$ must be performed. From (5), for $0\leq t'\leq t$, we derive

$$|S_{\alpha,n}(t)-S_{\alpha,n}(t')|=\int_0^\infty\left|e^{-zt'}-e^{-zt}\right|H(n,\alpha,z)dz$$
$$\leq\frac{1}{a\pi\sin(\alpha\pi)}\frac{1}{\lambda_n}\int_0^\infty\left(e^{-zt'}-e^{-zt}\right)z^{-\alpha}dz. \tag{49}$$

By $0<\gamma<\alpha-\beta$, using the inequality $1-e^{-w}\leq w$, for $w\geq 0$, we observe

$$\begin{aligned}(1-e^{-w})^{\frac{1}{\gamma}}&=(1-e^{-w})(1-e^{-w})^{\frac{1}{\gamma}-1}\\&\leq 1-e^{-w}\\&\leq w.\end{aligned}$$

This implies that

$$1-e^{-w}\leq w^\gamma,\quad 0<\gamma<1.$$

Hence, we obtain that

$$\int_0^\infty\left(e^{-zt'}-e^{-zt}\right)z^{-\alpha}dz\leq(t-t')^\gamma\int_0^\infty e^{-zt'}z^{\gamma-\alpha}dz$$
$$=\frac{(t-t')^\gamma}{(t')^{1+\gamma-\alpha}}\int_0^\infty e^{-y}y^{\gamma-\alpha}dy=\mathit{\Gamma}(1+\gamma-\alpha)\frac{(t-t')^\gamma}{(t')^{1+\gamma-\alpha}}, \tag{50}$$

by setting $y=zt$. Combining (49) and (50) we show that

$$|S_{\alpha,n}(t)-S_{\alpha,n}(t')|\leq\frac{\mathit{\Gamma}(1+\gamma-\alpha)}{a\pi\sin(\alpha\pi)}\frac{1}{\lambda_n}\frac{(t-t')^\gamma}{(t')^{1+\gamma-\alpha}}$$
$$=\frac{C_3(a,\pi,\alpha,\gamma)}{\lambda_n}\frac{(t-t')^\gamma}{(t')^{1+\gamma-\alpha}},\quad\text{with } C_3(a,\pi,\alpha,\gamma)=\frac{\mathit{\Gamma}(1+\gamma-\alpha)}{a\pi\sin(\alpha\pi)}.$$

Hence, we deduce that for $q\in(0,1), 0<\gamma<\alpha-\beta$ and for all $t\in[T,T_+]$

$$\|A_3(t)\|_{\mathbb{H}^{2q}(\Omega)}$$
$$\leq\int_0^T\left\|\sum_{n\in\mathbb{N}^*}\left(s^{-\beta}S_{\alpha,n}(t-s)-s^{-\beta}S_{\alpha,n}(T-s)\right)\langle F(u_+)(s),\xi_n\rangle_{L^2(\Omega)}\xi_n\right\|_{\mathbb{H}^{2q}(\Omega)}ds$$
$$\leq\int_0^T\left(\sum_{n\in\mathbb{N}^*}s^{-\beta}\left|S_{\alpha,n}(t-s)-S_{\alpha,n}(T-s)\right|^2\langle F(u_+)(s),\xi_n\rangle^2_{L^2(\Omega)}\lambda_n^{2q}\right)^{\frac{1}{2}}ds$$
$$\leq\frac{C_3(a,\pi,\alpha,\gamma)}{\lambda_1^{1-q}}(t-T)^\gamma\int_0^T s^{-\beta}(T-s)^{\alpha-\gamma-1}\left(\sum_{n\in\mathbb{N}^*}\langle F(u_+)(s),\xi_n\rangle^2_{L^2(\Omega)}\right)^{\frac{1}{2}}ds$$
$$\leq\frac{C_3(a,\pi,\alpha,\gamma)}{\lambda_1^{1-q}}(t-T)^\gamma\int_0^T s^{-\beta}(T-s)^{\alpha-\gamma-1}\|F(u_+)(s)\|_{L^2(\Omega)}ds. \tag{51}$$

Using (17) and (46),

$$
\begin{aligned}
&\underline{\text{The (RHS) of (51)}}\\
&\lesssim_{N,q,p,p'} \frac{\sqrt{2}C_3(a,\pi,\alpha,\gamma)}{\lambda_1^{1-q}}(t-T)^{\gamma}\\
&\quad\times \int_0^T s^{-\beta}(T-s)^{\alpha-\gamma-1}\left(E\sqrt{\mathrm{Vol}(\Omega)}+\|u(s)\|_{\mathbb{H}^{2q}(\Omega)}^{p+p'-1}\right)ds\\
&\lesssim_{N,q,p,p'} \frac{\sqrt{2}C_3(a,\pi,\alpha,\gamma)}{\lambda_1^{1-q}}\left(E\sqrt{\mathrm{Vol}(\Omega)}+\|u\|_{C_{[T,T_+]}\mathbb{H}_x^{2q}}^{p+p'-1}\right)T_+^{\gamma}\\
&\quad\times \int_0^T s^{-\beta}(T-s)^{\alpha-\gamma-1}ds\\
&\lesssim_{N,q,p,p'} \frac{\sqrt{2}C_3(a,\pi,\alpha,\gamma)}{\lambda_1^{1-q}}\left(E\sqrt{\mathrm{Vol}(\Omega)}+\left(R+\|u_T\|_{\mathbb{H}^{2q}(\Omega)}\right)^{p+p'-1}\right)T_+^{\gamma}\\
&\quad\times \int_0^T (T-s)^{\alpha-\gamma-1}s^{-\beta}ds.
\end{aligned}
$$

By applying now (10),

$$
\begin{aligned}
&\|A_3(t)\|_{\mathbb{H}^{2q}(\Omega)}\\
&\le \frac{\sqrt{2}C(N,q,p,p')C_3(a,\pi,\alpha,\gamma)}{\lambda_1^{1-q}}\left(E\sqrt{\mathrm{Vol}(\Omega)}+\left(R+\|u_T\|_{\mathbb{H}^{2q}(\Omega)}\right)^{p+p'-1}\right)T_+^{\gamma}\\
&\quad\times B(\alpha-\gamma,1-\beta)T^{\alpha-\gamma-\beta}.
\end{aligned}
$$

Hence, for all $t\in[T,T_+]$, $0<\gamma<\alpha-\beta$, one obtains

$$
\begin{aligned}
&\|A_3\|_{C_{[T,T_+]}\mathbb{H}_x^{2q}}\\
&\le \frac{\sqrt{2}C(N,q,p,p')C_3(a,\pi,\alpha,\gamma)}{\lambda_1^{1-q}}\left(E\sqrt{\mathrm{Vol}(\Omega)}+\left(R+\|u_T\|_{\mathbb{H}^{2q}(\Omega)}\right)^{p+p'-1}\right)\\
&\quad\times B(\alpha-\gamma,1-\beta)T_+^{\alpha-\beta}=P_3.
\end{aligned}
$$

From (35) and (38), we deduce for all $t\in[0,T_+]$

$$
\|A_3\|_{C_{[0,T_+]}\mathbb{H}_x^{2q}}\le P_3\le \frac{R}{4}. \tag{52}
$$

Combining the results in (44), (48), (52), we conclude that for all $t\in[0,T_+]$

$$
\|\mathscr{Y}u_+-u_T\|_{C_{[0,T_+]}\mathbb{H}_x^{2q}}\le \frac{R}{4}+\frac{R}{4}+\frac{R}{4}=\frac{3R}{4}<R.
$$

We have proved that $\mathscr{Y}: V_R^+\mapsto V_R^+$ is an invariant mapping.
• **Step 2**: *We shall prove that $\mathscr{Y}$ is a contraction mapping on V_R^+.* Let $w,v\in V_R^+$, one has

$$
\mathscr{Y}w(t)-\mathscr{Y}v(t)=\int_0^t s^{-\beta}S_\alpha(t-s)\left(F(w)(s)-F(v)(s)\right)ds,\quad 0\le t\le T_+, \tag{53}
$$

and we observe that $\mathscr{Y}w(t)-\mathscr{Y}v(t)=0$, $\forall t\in[0,T]$. Next, for $t\in[T,T_+]$, in the same way as in the proof of Claim 2 in Step 1 of Theorem 1.3, one obtains

$$
\|\mathscr{Y}w(t)-\mathscr{Y}v(t)\|_{\mathbb{H}^{2q}(\Omega)}
$$

$$\leq \frac{C(N,q,p,p')C_1(a,\alpha)K(2R+2\|u_T\|_{\mathbb{H}^{2q}(\Omega)})}{\lambda_1^{1-q}} B(\alpha,1-\beta)T_+^{\alpha-\beta}\|w-v\|_{C_{[0,T_+]}\mathbb{H}_x^{2q}},$$

where $K(z)=(p-1)E+2^{p-2}z^{p-2}+2^{p+p'-2}(p-1)z^{p+p'-2}$ is defined in Theorem 1.3. Hence, from (39), we deduce that

$$\begin{aligned}\|\mathscr{Y}w-\mathscr{Y}v\|_{C_{[0,T_+]}\mathbb{H}_x^{2q}} &\leq \frac{C(N,q,p,p')C_1(a,\alpha)K(2R+2\|u_T\|_{\mathbb{H}^{2q}(\Omega)})}{\lambda_1^{1-q}}\\ &\quad\times B(\alpha,1-\beta)T_+^{\alpha-\beta}\|w-v\|_{C_{[0,T_+]}\mathbb{H}_x^{2q}}\\ &= P_4\|w-v\|_{\mathscr{C}^{\alpha s}_{[0,T_+]}\mathbb{H}_x^{2q}}.\end{aligned}$$

From (35), we have

$$\|\mathscr{Y}w-\mathscr{Y}v\|_{C_{[0,T_+]}\mathbb{H}_x^{2q}} \leq \frac{R}{4}\|w-v\|_{C_{[0,T_+]}\mathbb{H}_x^{2q}}. \tag{54}$$

There is no loss of generality in assuming $0\leq R<4$, therefore, $\mathscr{Y}$ is a $\frac{R}{4}$-contraction. Consequently, we infer that $\mathscr{Y}$ admits a unique fixed point u_+ of $\mathscr{Y}$ in V_R^+, i.e., u_+ is a continuation of u, provided we choose $R<4$. The proof of Theorem 1.6 is finished. ∎

3.3. **Proof of Theorem 1.9.** As an immediate consequence of Theorem 1.6, we guarantee the existence of a maximal time. Next, we prove the results which are on the global existence or non-continuation by a finite-time blow-up. Let $u_0\in\mathbb{H}^{2q+2}(\Omega)$ and define

$$T_{\max} := \sup\{T>0 \text{ such that Problem } (\mathbb{P}) \text{ possesses a solution on } [0,T]\}.$$

Suppose that $T_{\max}<\infty$, and

$$\sup_{t\in[0,T_{\max})}\|u(t)\|_{\mathbb{H}^{2q}(\Omega)}\leq R,$$

for some $R>0$. And assume that there is a sequence $\{t_m\}_{m\in\mathbb{N}}\subset[0,T_{\max})$ such that $t_m\uparrow T_{\max}$. Our purpose is to prove that $\{u(\cdot,t_m)\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{H}^{2q}(\Omega)$. In fact, set $\varepsilon>0$, fix $N_\varepsilon\in\mathbb{N}$ satisfying $0<t_{m_1}<t_{m_2}<T_{\max}$ for $m_1,m_2>N_\varepsilon$, then we have

$$\begin{aligned}&\|u(\cdot,t_{m_2})-u(\cdot,t_{m_1})\|_{\mathbb{H}^{2q}(\Omega)}\\ &\leq \|(S_\alpha(t_{m_2})-S_\alpha(t_{m_1}))u_0\|_{\mathbb{H}^{2q}(\Omega)}\\ &\quad+\left\|\int_{t_{m_1}}^{t_{m_2}} s^{-\beta}S_\alpha(t_{m_2}-s)F(u)(s)ds\right\|_{\mathbb{H}^{2q}(\Omega)}\\ &\quad+\left\|\int_0^{t_{m_1}}\left(s^{-\beta}S_\alpha(t_{m_1}-s)-s^{-\beta}S_\alpha(T_{\max}-s)\right)F(u)(s)ds\right\|_{\mathbb{H}^{2q}(\Omega)}\\ &\quad+\left\|\int_0^{t_{m_2}}\left(s^{-\beta}S_\alpha(T_{\max}-s)-s^{-\beta}S_\alpha(t_{m_2}-s)\right)F(u)(s)ds\right\|_{\mathbb{H}^{2q}(\Omega)}\\ &=:\sum_{i=4}^{7}\|A_i\|_{\mathbb{H}^{2q}(\Omega)}.\end{aligned}$$

Similarly to (43), we have

$$\|A_4\|_{\mathbb{H}^{2q}(\Omega)}=\|(S_\alpha(t_{m_2})-S_\alpha(t_{m_1}))u_0\|_{\mathbb{H}^{2q}(\Omega)}$$

$$\leq \frac{C_2(N,\Omega,T,a)}{1-\alpha} \|u_0\|_{\mathbb{H}^{2q+2}(\Omega)} |t_{m_1} - t_{m_2}|^{1-\alpha}, \quad \text{for } \alpha \in (0,1).$$

In the same way as (16), (17) and using (10), we derive

$$\begin{aligned}
&\|A_5\|_{\mathbb{H}^{2q}(\Omega)} \\
&\leq \int_{t_{m_1}}^{t_{m_2}} \left\| s^{-\beta} S_\alpha(t_{m_2}-s) F(u)(s) \right\|_{\mathbb{H}^{2q}(\Omega)} ds \\
&\leq \frac{C_1(a,\alpha)}{\lambda_1^{1-q}} \int_{t_{m_1}}^{t_{m_2}} s^{-\beta}(t_{m_2}-s)^{\alpha-1} \|F(u)(s)\|_{L^2(\Omega)} ds \\
&\lesssim_{N,q,p,p'} \frac{\sqrt{2}C_1(a,\alpha)}{\lambda_1^{1-q}} \int_{t_{m_1}}^{t_{m_2}} s^{-\beta}(t_{m_2}-s)^{\alpha-1} \left(E\sqrt{\mathrm{Vol}(\Omega)} + \|u(s)\|_{\mathbb{H}^{2q}(\Omega)}^{p+p'-1} \right) ds \\
&\lesssim_{N,q,p,p'} \frac{\sqrt{2}C_1(a,\alpha)}{\lambda_1^{1-q}} \left(E\sqrt{\mathrm{Vol}(\Omega)} + R^{p+p'-1} \right) \int_{t_{m_1}}^{t_{m_2}} (t_{m_2}-s)^{\alpha-1}(s-t_{m_1})^{-\beta} ds \\
&\leq \frac{\sqrt{2}C(N,q,p,p')C_1(a,\alpha)}{\lambda_1^{1-q}} \left(E\sqrt{\mathrm{Vol}(\Omega)} + R^{p+p'-1} \right) B(\alpha, 1-\beta) \left| t_{m_2} - t_{m_1} \right|^{\alpha-\beta},
\end{aligned}$$

where for $0 \leq t_{m_1} \leq s \leq t_{m_2} < T_{\max}$, we have used $s^{-\beta} \leq (s-t_{m_1})^{-\beta}$. Similar to (51) and using (17), one has

$$\begin{aligned}
&\|A_6\|_{\mathbb{H}^{2q}(\Omega)} \\
&\leq \int_0^{t_{m_1}} \left\| \left(s^{-\beta} S_\alpha(t_{m_1}-s) - s^{-\beta} S_\alpha(T_{\max}-s) \right) F(u)(s) \right\|_{\mathbb{H}^{2q}(\Omega)} ds \\
&\leq \frac{C_3(a,\pi,\alpha,\gamma)}{\lambda_1^{1-q}} |T_{\max} - t_{m_1}|^\gamma \int_0^{t_{m_1}} s^{-\beta} \left(t_{m_1} - s \right)^{\alpha-\gamma-1} \|F(u)(s)\|_{L^2(\Omega)} ds \\
&\lesssim_{N,q,p,p'} \frac{C_3(a,\pi,\alpha,\gamma)}{\lambda_1^{1-q}} |T_{\max} - t_{m_1}|^\gamma \\
&\quad \times \int_0^{t_{m_1}} s^{-\beta} \left(t_{m_1} - s \right)^{\alpha-\gamma-1} \left(E\sqrt{\mathrm{Vol}(\Omega)} + \|u(s)\|_{\mathbb{H}^{2q}(\Omega)}^{p+p'-1} \right) ds \\
&\lesssim_{N,q,p,p'} \frac{C_3(a,\pi,\alpha,\gamma)}{\lambda_1^{1-q}} \left(E\sqrt{\mathrm{Vol}(\Omega)} + R^{p+p'-1} \right) |T_{\max} - t_{m_1}|^\gamma \\
&\quad \times \int_0^{t_{m_1}} \left(t_{m_1} - s \right)^{\alpha-\gamma-1} s^{-\beta} ds \\
&\leq \frac{C(N,q,p,p')C_3(a,\pi,\alpha,\gamma)}{\lambda_1^{1-q}} \left(E\sqrt{\mathrm{Vol}(\Omega)} + R^{p+p'-1} \right) \\
&\quad \times B(\alpha-\gamma, 1-\beta) T_{\max}^{\alpha-\gamma-\beta} |T_{\max} - t_{m_1}|^\gamma,
\end{aligned}$$

for $0 < \beta < \alpha < 1$ and $0 < \gamma < \alpha - \beta$. Similarly,

$$\begin{aligned}
\|A_7\|_{\mathbb{H}^{2q}(\Omega)} &\leq \int_0^{t_{m_2}} \left\| \left(s^{-\beta} S_\alpha(T_{\max}-s) - s^{-\beta} S_\alpha(t_{m_2}-s) \right) F(u)(s) \right\|_{\mathbb{H}^{2q}(\Omega)} ds \\
&\leq \frac{C(N,q,p,p')C_2(a,\pi,\alpha,\gamma)}{\lambda_1^{1-q}} \left(E\sqrt{\mathrm{Vol}(\Omega)} + R^{p+p'-1} \right) \\
&\quad \times B(\alpha-\gamma, 1-\beta) T_{\max}^{\alpha-\gamma-\beta} |T_{\max} - t_{m_2}|^\gamma.
\end{aligned}$$

Thus, since $\{t_m\}_{m\in\mathbb{N}^*}$ is a convergent sequence, we can pick $N_\varepsilon \in \mathbb{N}^*$ large enough for $m_2 \geq m_1 \geq N_\varepsilon$ such that $|t_{m_2} - t_{m_1}|$ is small enough and $t_{m_1}, t_{m_2} \uparrow T_{\max}$ to ensure

$$\begin{cases} \frac{C_2(N,\Omega,T,a)}{1-\alpha} \|u_0\|_{\mathbb{H}^{2q+2}(\Omega)} |t_{m_1} - t_{m_2}|^{1-\alpha} < \frac{\varepsilon}{4}; \\ \frac{\sqrt{2}C(N,q,p,p')C_1(a,\alpha)}{\lambda_1^{1-q}} \left(E\sqrt{\mathrm{Vol}(\Omega)} + R^{p+p'-1}\right) B(\alpha, 1-\beta) |t_{m_2} - t_{m_1}|^{\alpha-\beta} < \frac{\varepsilon}{4}; \\ \frac{C(N,q,p,p')C_3(a,\pi,\alpha,\gamma)}{\lambda_1^{1-q}} \left(E\sqrt{\mathrm{Vol}(\Omega)} + R^{p+p'-1}\right) \\ \quad \times B(\alpha-\gamma, 1-\beta) T_{\max}^{\alpha-\gamma-\beta} |T_{\max} - t_{m_1}|^{\gamma} < \frac{\varepsilon}{4}; \\ \frac{C(N,q,p,p')C_3(a,\pi,\alpha,\gamma)}{\lambda_1^{1-q}} \left(E\sqrt{\mathrm{Vol}(\Omega)} + R^{p+p'-1}\right) \\ \quad \times B(\alpha-\gamma, 1-\beta) T_{\max}^{\alpha-\gamma-\beta} |T_{\max} - t_{m_2}|^{\gamma} < \frac{\varepsilon}{4}, \end{cases}$$

for $0 < \beta < \alpha < 1$ and $0 < \gamma < \alpha - \beta$. Thus, for any $\varepsilon > 0$, there is $N_\varepsilon \in \mathbb{N}$ such that

$$\|u(\cdot, t_{m_2}) - u(\cdot, t_{m_1})\|_{\mathbb{H}^{2q}(\Omega)} < \varepsilon, \quad \text{for } m_1, m_2 \geq N_\varepsilon,$$

which means that $\{u(\cdot, t_m)\}_{m\in\mathbb{N}} \in \mathbb{H}^{2q}(\Omega)$ is a Cauchy sequence. Arguing by contradiction, we suppose that $\{t_m\}_{m\in\mathbb{N}^*}$ is arbitrary and the following limit exists

$$\lim_{t\uparrow T_{\max}^-} \|u(\cdot, t)\|_{\mathbb{H}^{2q}(\Omega)} < \infty.$$

We can invoke Theorem 1.6 to infer that the solution can be stretched to a larger existence-time interval (u may be extended beyond $T_{\max}$), and this contradicts the definition of $T_{\max}$. Therefore, either $T_{\max} = \infty$ or if $T_{\max} < \infty$ then

$$\lim_{t\uparrow T_{\max}^-} \|u(\cdot, t)\|_{\mathbb{H}^{2q}(\Omega)} = \infty.$$

The proof of Theorem 1.9 is finished. ∎

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