Quickest Detection Problems for Ornstein–Uhlenbeck Processes

DOI:
10.1287/moor.2021.0186

To appear in *Math. Oper. Res.*

Quickest Detection Problems for Ornstein-Uhlenbeck Processes

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Consider an Ornstein-Uhlenbeck process that initially reverts to zero at a known mean-reversion rate β_0 and then after some random/unobservable time this mean-reversion rate is changed to β_1 . Assuming that the process is observed in real time, the problem is to detect when exactly this change occurs as accurately as possible. We solve this problem in the most uncertain scenario when the random/unobservable time is (i) exponentially distributed and (ii) independent from the process prior to the change of its mean-reversion rate. The solution is expressed in terms of a stopping time that minimises the probability of a false early detection and the expected delay of a missed late detection. Allowing for both positive and negative values of β_0 and β_1 (including zero), the problem and its solution embed many intuitive and practically interesting cases. For example, the detection of a mean-reverting process changing to a simple Brownian motion ($\beta_0 > 0$ and $\beta_1 = 0$) and vice versa ($\beta_0 = 0$ and $\beta_1 > 0$) finds a natural application to *pairs trading* in finance. The formulation also allows for the detection of a *transient* process becoming *recurrent* ($\beta_0 < 0$ and $\beta_1 \geq 0$) as well as a *recurrent* process becoming *transient* ($\beta_0 \geq 0$ and $\beta_1 < 0$). The resulting optimal stopping problem is inherently two-dimensional (due to a state-dependent signal-to-noise ratio) and various properties of its solution are established. In particular, we find the somewhat surprising fact that the optimal stopping boundary is an *increasing* function of the modulus of the observed process for *all* values of β_0 and β_1 .

1. Introduction

The Ornstein-Uhlenbeck (OU) process (cf. [43]) is one of the fundamental building blocks of modern stochastic modelling, due in part to its ability to incorporate mean-reverting and stationary effects into underlying uncertainty processes. In each application, such mean reversion is often caused by some underlying physical or economic force and so detecting the existence (or absence) of mean-reverting dynamics is of great importance in detecting the existence (or absence) of these underlying forces.

Imagine an OU process that initially reverts to zero at a known mean-reversion rate β_0 and then after some random/unobservable time θ , the underlying (unobservable) mean-reverting force is removed, resulting in simple Brownian kinematics. The problem is to detect the time θ at which the force is removed as accurately as possible (neither too early nor too late). The main objective of the present paper is to derive the solution to this problem when θ is assumed to be (i) exponentially distributed and (ii) independent from the initial dynamics of the OU process. Our motivating application for this problem is described in Section 2 below.

Mathematics Subject Classification 2020. Primary 60G40, 60J60, 60H30. Secondary 35K67, 45G10, 62C10.

Key words and phrases: Quickest detection, Brownian motion, Ornstein-Uhlenbeck process, optimal stopping, signal-to-noise ratio, Bernoulli equation, parabolic partial differential equation, free-boundary problem, smooth fit, nonlinear Fredholm integral equation, pairs trading, risk management.

Denoting the position of the OU process by X we study the problem above by embedding it into the more general setting where an OU process X changes its initial mean-reversion rate $\beta_0 \in I\!R$ to a new mean-reversion rate $\beta_1 \in I\!R$ at a random/unobservable time θ. The error to be minimised over all stopping times τ of X is expressed as the linear combination of the probability of the *false alarm* $\mathsf{P}_\pi(\tau < \theta)$ and the expected *detection delay* $\mathsf{E}_\pi(\tau - \theta)^+$ where $\pi \in [0,1]$ denotes the probability that θ has already occurred at time 0. This problem formulation of quickest detection dates back to [36] and has been extensively studied to date (see [39] for financial applications of interest for the present paper and [40] for a general overview). The linear combination represents the Lagrangian and once the optimal stopping problem has been solved in this form it will also lead to the solution of the constrained problems where an upper bound is imposed on either the probability of the false alarm or the expected detection delay respectively.

A canonical example is the standard Brownian motion in one dimension with one *constant* drift changing to another. This problem has also been solved in finite horizon (see [16] and the references therein). Books [37, Section 4.4] and [32, Section 22] contain expositions of these results and provide further details and references. The *signal-to-noise ratio* (defined as the difference between the new drift and the old drift divided by the diffusion coefficient) in all these problems is *constant* so that the resulting optimal stopping problem for the *posterior probability ratio* process $\varPhi$ of θ given X is *one-dimensional.* A more general problem formulation for diffusion processes X in one dimension when one *non-constant* drift changes to another has been considered in [17]. A specific problem of this kind when X is a Bessel process has been solved in [20]. The signal-to-noise ratio in these problems is not constant and the resulting optimal stopping problem for $\varPhi$ coupled with X (to make it Markovian) is *two-dimensional.* The infinitesimal generator of the Markov/diffusion process $(\varPhi, X)$ in these problems is of *parabolic* type. The problem studied in the present paper belongs to the latter class of problems and we build our exposition on findings from [17] and [20]. A plain comparison with previous results and arguments indicates the following differences/novelties.

Firstly, the paper [20] studies only a change from a recurrent to a transient process. For a long time it has been recognised in the statistical community however that the converse is also of considerable interest. In the present paper we allow for both positive and negative values of β_0 and β_1 (including zero) and thus allow for the detection of both a *transient* process becoming *recurrent* ($\beta_0 < 0 \leq \beta_1$) and a *recurrent* process becoming *transient* ($\beta_0 \geq 0 > \beta_1$). The problem and its solution embed many intuitive and practically interesting cases. For example, the detection of a mean-reverting process changing to a simple Brownian motion ($\beta_0 > 0$ and $\beta_1 = 0$) and vice versa ($\beta_0 = 0$ and $\beta_1 > 0$) finds a natural application to *pairs trading* in finance. Secondly, the resulting optimal stopping problem is inherently two-dimensional (due to a state-dependent signal-to-noise ratio) and various properties of its solution are established. In particular, we find the somewhat surprising fact that the optimal stopping boundary is an *increasing* function of the modulus of the observed process for *all* values of β_0 and β_1. The concluding arguments used in the proof differ from the arguments for monotonicity of the optimal stopping boundary used in [1] and [20]. Moreover, the proof of monotonicity verifies the conjecture in this particular case stated in [17, Lemma 4.1] that the optimal stopping boundary is increasing whenever the (square of) the signal-to-noise ratio is increasing. Thirdly, the infinitesimal generator $I\!L_{\varPhi,X}$ of the Markov/diffusion process $(\varPhi, X)$ is of parabolic type. It is therefore possible to reduce $I\!L_{\varPhi,X}$ and $(\varPhi, X)$ to their canonical forms $I\!L_{U,\varPhi}$ and $(U, \varPhi)$

where the process U is of bounded variation (see [20, Section 6] for details). Focusing on (U, X) in this paper instead we show that the exponential of U solves the *Bernoulli equation* and hence U can be expressed in closed form as a path-dependent functional of X. This fact holds not only for the OU process but for any diffusion process X in one dimension when one drift changes to another. The appearance of the Bernoulli equation in this context and deriving a closed-form expression for U are both novel in the literature to our knowledge. Fourthly, solutions in the case of Brownian motion and Bessel processes are known, however, to our knowledge this paper is the first to solve the quickest detection problem for the OU process.

The exposition of the material is organised as follows. In Section 2 we discuss our motivating application to pairs trading and also provide hints on broader financial applications (detection of the birth and bursting of price bubbles). In Section 3 we formulate the optimal stopping problem and list ten canonical cases of interest for mean-reversion rates and levels (the zero-th case of equal mean-reversion rates is one-dimensional and its solution is known). In the present paper we focus on the ten canonical cases and leave the other cases (including their ramifications) for future studies. In Section 4 we recall that a change of measure simplifies the setting that makes the subsequent analysis possible. In Section 5 we reduce the underlying stochastic process to its canonical form and show that the resulting bounded variation process solves the Bernoulli equation and thus can be determined explicitly. The optimal stopping problem is formulated in Lagrange form and in Section 6 we disclose its Mayer formulation (see [32, Section 5] for the terminology). We could determine the Mayer function explicitly only when $\beta_0 \neq -\lambda/2$ and $\beta_1 = 0$ and it appears that simple/explicit forms are not available in the remaining cases. This stands in contrast with the results from [20, Section 5] where the Mayer function is available explicitly in all cases. The absence of explicit Mayer functions introduces difficulties in arguments at places and in the remaining sections we show how to overcome them (e.g. the existence of an optimal stopping time in the following section is established by enlarging the state space from dimension two to dimension four). In Section 7 we establish the existence of an optimal stopping time and derive basic properties of the optimal stopping boundary. In Section 8 we disclose the free-boundary problem which stands in one-to-one correspondence with the optimal stopping problem and show that the value function and the optimal stopping boundary solve the free-boundary problem uniquely. In Section 9 we show that the optimal stopping boundary can be characterised as the unique solution to a nonlinear Fredholm equation. This equation can be used to find the optimal stopping boundary numerically (using Picard iteration).

2. Motivating application

Our motivating application for the problem studied is drawn from the field of statistical arbitrage in finance, and in particular from the risk management of so-called *pairs trading* strategies. Pairs trading is a popular (and historically very profitable) trading strategy commonly employed by hedge funds and investment banks. These strategies were reportedly pioneered at Morgan Stanley in the 1980s and they have been extremely popular ever since (see [44]). In such trades, a trader can construct profitable strategies based on the assumption that two tradeable assets are co-integrated, meaning that an (appropriately scaled) price difference/spread is assumed to be stationary and will revert to zero (and hence can be modelled as an OU process).

When the prices of the two co-integrated securities temporarily diverge, a pairs trader would short the outperforming stock and long the underperforming one, betting that the spread between the two would eventually converge. (Such a strategy can be considered market neutral, allowing traders to profit from almost any market condition, contributing to its popularity. For example, in a severe market downtown, the strategy would experience a loss on the long position, but an offsetting gain on the short position, resulting in a loss close to zero in spite of the large market move.) However, should the assumed convergence never happen, a large loss will usually be realized upon liquidation, resulting in a significant negative skew in the trader's profit distribution, with frequent small gains but infrequent large losses. A notable pairs trader was Long-Term Capital Management, a hedge fund management firm that collapsed in the late 1990s with losses in excess of US$4.6 billion, due, in no small part, to the unraveling of their highly leveraged long/short sovereign bond positions (see [26]).

In this paper, we assume that a change in the underlying OU spread process can be modelled as a 'disorder' and the role of quickest detection in this setting is as a risk management tool to detect when the spread process (with a prior belief of mean reversion) becomes uncoupled and changes from OU to Brownian motion (BM), a nonstationary process. This risk may be interpreted as *model risk* and it should be quantified and managed the same as other risks that an investor/trader faces. The current standard practice to account for such risk is to adopt a stop-loss approach (see, for example, [10], [25], [23] and [24]) in which the mean-reverting model is believed up to a predetermined loss level, at which time a trader should abandon the model and liquidate their positions at a loss. However, to our knowledge, there exists no theory as to where to place such stop-loss limits. Further, the commonly used threshold strategies are perhaps too naive, since a large deviation in the spread may simply be a *very good* trading opportunity and *not* a sign of a model breakdown. This paper provides a more rigorous approach to model-risk management in which the entire spread dynamics (not just excursions) are considered to detect potential structural changes. Naturally, the quickest detection of any change in the underlying price dynamics would no doubt help the trader to mitigate such model risk, allowing them to *cut their losses* sooner and ultimately to *improve* their risk-adjusted trading performance.

Despite its popularity in practice, the academic literature on pairs trading has been surprisingly scarce, with the majority of the existing papers focusing solely on the statistical performance of various pairs trading strategies. For example, [5], [7], [18] and [19], all find a significant abnormal return from simple pairs trading strategies. There are also a number of books considering the applied aspects of pairs trading, for example, see [9], [44], and [45]. Of the limited theoretical work in the area, the most notable are [11], who were the first to propose that the spread be modelled as a mean-reverting process, and [10] who modelled the spread as an OU process and formulated various optimal stopping problems associated with pairs trading. More recently, [2], [3], [21], [23], [41] and [42] have also modelled the spread as an OU process, however none of these papers have considered the problem of detecting a change in the underlying OU process. Finally, we refer the reader to the recent article [22] for a much more comprehensive review of the current pairs trading literature.

To solve the specific problem described above we embed it into a more general setting of a changing OU process, where we allow the mean-reversion rate (and level) to change from one specified value to another (not necessarily zero) value. As such, the analysis in the present paper may find applications in other areas outside of our motivating application. For example, the detection of a change from a negative to a positive (or zero) mean-reversion rate could help detect the *bursting* of an asset price bubble, since the OU process with a negative mean-reversion rate may be used to model bubble-like behaviour in asset price dynamics (see [33] and [34]). The converse problem (detecting a change from positive to negative rate) could also be applied to the detection of the *birth* of such bubbles.

3. Formulation of the problem

In this section we formulate the quickest detection problem under consideration.

1. We consider the Bayesian formulation of the problem where it is assumed that one observes a sample path of an Ornstein-Uhlenbeck process X whose rate of mean reversion $\beta_0 \in I\!R$ changes to $\beta_1 \in I\!R$ (along with a change of the mean-reverting level $x_0 \in I\!R$ to $x_1 \in I\!R$) at some random/unobservable time θ taking value 0 with probability $\pi \in [0,1]$ and being exponentially distributed with parameter $\lambda > 0$ given that $\theta > 0$. The problem is to detect the unknown time θ as accurately as possible (neither too early nor too late).

2. Standard arguments imply that the previous setting can be realised on a probability space $(\Omega, \mathcal{F}, \mathsf{P}_\pi)$ with the probability measure P_π decomposed as follows

$$\mathsf{P}_\pi = \pi \mathsf{P}^0 + (1-\pi) \int_0^\infty \lambda e^{-\lambda t} \mathsf{P}^t \, dt \tag{3.1}$$

for $\pi \in [0,1]$ where P^t is the probability measure under which the observed process X undergoes the change at time $t \in [0,\infty)$. The unobservable time θ is a non-negative random variable satisfying $\mathsf{P}_\pi(\theta = 0) = \pi$ and $\mathsf{P}_\pi(\theta > t \,|\, \theta > 0) = e^{-\lambda t}$ for $t > 0$. Thus $\mathsf{P}^t(X \in \cdot\,)$ $= \mathsf{P}_\pi(X \in \cdot \mid \theta = t)$ is the probability law of an Ornstein-Uhlenbeck process whose rate of mean reversion β_0 changes to β_1 (and whose mean-reverting level x_0 changes to x_1) at time $t > 0$. To remain consistent with this notation we also denote by P^∞ the probability measure under which the observed process X undergoes no change in its dynamics. Thus $\mathsf{P}^\infty(X \in \cdot\,) = \mathsf{P}_\pi(X \in \cdot \,|\, \theta = \infty)$ is the probability law of an Ornstein-Uhlenbeck process with mean-reversion rate β_0 and mean-reverting level x_0 at all times.

3. The observed process X solves the stochastic differential equation

$$dX_t = \big[\mu_0(X_t) + I(t \geq \theta)\big(\mu_1(X_t) - \mu_0(X_t)\big)\big]\, dt + \sigma(X_t)\, dB_t \tag{3.2}$$

driven by a standard Brownian motion B under P_π where we set

$$\mu_0(x) = \beta_0(x_0 - x) \quad \& \quad \mu_1(x) = \beta_1(x_1 - x) \quad \& \quad \sigma(x) = \sigma > 0 \tag{3.3}$$

for $x \in I\!R$. The unobservable time θ and the driving Brownian motion B are assumed to be independent under P_π for $\pi \in [0,1]$.

4. Embedded within this formulation are multiple cases that may find application in different contexts (both within pairs trading and beyond). We now list and briefly discuss these cases upon noting that our primary interest will be in Cases 1-10 below.

Case 0: $\beta_0 = \beta_1 \neq 0$ (along with $x_0 \neq x_1$). This corresponds to a mean-reverting process (OU) changing to another mean-reverting process (OU) with the same mean-reversion rate but with a different mean-reverting level at time θ . Note that ρ from (3.7) below equals the constant $\beta_0(x_1-x_0)/\sigma$ in this case so that the problem is reducible to a one-dimensional optimal stopping problem which can be solved explicitly (cf. [32, Subsection 22.1]).

Case 1: $\beta_0 > 0$ and $\beta_1 = 0$ (along with $x_0 = x_1 = 0$ without loss of generality). This corresponds to a stationary process (OU) becoming non-stationary (BM) at time θ .

Case 2: $\beta_0 = 0$ and $\beta_1 > 0$ (along with $x_0 = x_1 = 0$ without loss of generality). This corresponds to a non-stationary process (BM) becoming stationary (OU) at time θ .

Case 3: $\beta_0 < 0$ and $\beta_1 = 0$ (along with $x_0 = x_1 = 0$ without loss of generality). This corresponds to a transient process (OU with negative mean-reversion rate) becoming recurrent (BM) at time θ .

Case 4: $\beta_0 = 0$ and $\beta_1 < 0$ (along with $x_0 = x_1 = 0$ without loss of generality). This corresponds to a recurrent process (BM) becoming transient (OU with negative mean-reversion rate) at time θ .

Case 5: $0 < \beta_0 < \beta_1$ (along with $x_0 = x_1 = 0$). This corresponds to a stationary process (OU) becoming 'more' stationary (OU with a larger mean-reversion rate but to the same mean-reverting level) at time θ .

Case 6: $0 < \beta_1 < \beta_0$ (along with $x_0 = x_1 = 0$). This corresponds to a stationary process (OU) becoming 'less' stationary (OU with a smaller mean-reversion rate but to the same mean-reverting level) at time θ .

Case 7: $\beta_1 < 0 < \beta_0$ (along with $x_0 = x_1 = 0$). This corresponds to a stationary process (OU) becoming transient (OU with negative mean-reversion rate) at time θ .

Case 8: $\beta_1 < \beta_0 < 0$ (along with $x_0 = x_1 = 0$). This corresponds to a transient process (OU with negative mean-reversion rate) becoming 'more' transient (OU with a more negative mean-reversion rate but to the same mean-reverting level) at time θ .

Case 9: $\beta_0 < \beta_1 < 0$ (along with $x_0 = x_1 = 0$). This corresponds to a transient process (OU with negative mean-reversion rate) becoming 'less' transient (OU with a less negative mean-reversion rate but to the same mean-reverting level) at time θ .

Case 10: $\beta_0 < 0 < \beta_1$ (along with $x_0 = x_1 = 0$). This corresponds to a transient process (OU with negative mean-reversion rate) becoming stationary (OU but to the same mean-reverting level) at time θ .

In Cases 1-4 we can set $x_0 = x_1 = 0$ without loss of generality since if either β_0 or β_1 is zero, then there exists only one mean-reverting level and this can be set to zero by an appropriate translation of X without loss of generality. In Cases 5-10 however, when both β_0 and β_1 are non-zero, there are two mean-reverting levels and the possibility for a change in either of them. In all these cases an appropriate translation in X would allow us to set $x_0 = 0$ (with $x_1 \neq 0$) without loss of generality.

5. Being based upon continuous observations of X , the problem is to find a stopping time τ_* of X (i.e. a stopping time with respect to the natural filtration $\mathcal{F}_t^X = \sigma(X_s \,|\, 0 \le s \le t)$ of X for $t \ge 0$) that is 'as close as possible' to the unknown time θ . More precisely, the

problem consists of computing the value function

$$V(\pi) = \inf_{\tau} \left[\mathsf{P}_\pi(\tau < \theta) + c\,\mathsf{E}_\pi(\tau - \theta)^+ \right] \tag{3.4}$$

and finding the optimal stopping time τ_* at which the infimum in (3.4) is attained for $\pi \in [0,1]$ and $c > 0$ given and fixed. Note in (3.4) that $\mathsf{P}_\pi(\tau < \theta)$ is the probability of the *false alarm* and $\mathsf{E}_\pi(\tau - \theta)^+$ is the expected *detection delay* associated with a stopping time τ of X for $\pi \in [0,1]$.

6. To tackle the optimal stopping problem (3.4) we consider the *posterior probability distribution* process $\Pi = (\Pi_t)_{t \geq 0}$ of θ given X that is defined by

$$\Pi_t = \mathsf{P}_\pi(\theta \leq t \,|\, \mathcal{F}_t^X) \tag{3.5}$$

for $t \geq 0$. The right-hand side of (3.4) can thus be rewritten to read

$$V(\pi) = \inf_{\tau} \mathsf{E}_\pi \left(1 - \Pi_\tau + c \int_0^\tau \Pi_t \, dt \right) \tag{3.6}$$

for $\pi \in [0,1]$. If the *signal-to-noise ratio* defined by

$$\rho(x) = \frac{\mu_1(x) - \mu_0(x)}{\sigma(x)} \tag{3.7}$$

is constant for $x \in I\!R$, then Π is known to be a one-dimensional Markov (diffusion) process so that the optimal stopping problem (3.6) can be tackled using established techniques both in infinite and finite horizon (see [32, Section 22]). Note that this is no longer the case in the setting of the present problem since from (3.3) we see that

$$\rho(x) = \delta x + \gamma \neq \text{constant} \tag{3.8}$$

for $x \in I\!R$ where we have set $\delta = (\beta_0 - \beta_1)/\sigma$ and $\gamma = (\beta_1 x_1 - \beta_0 x_0)/\sigma$.

7. To connect the process Π to the observed process X we consider the *likelihood ratio* process $L = (L_t)_{t \geq 0}$ defined by

$$L_t = \frac{d\mathsf{P}_t^0}{d\mathsf{P}_t^\infty} \tag{3.9}$$

where P_t^0 and P_t^∞ denote the restrictions of the probability measures P^0 and P^∞ to $\mathcal{F}_t^X$ for $t \geq 0$. By the Girsanov theorem one finds that

$$L_t = \exp\left(\int_0^t \frac{\mu_1(X_s) - \mu_0(X_s)}{\sigma^2(X_s)} \, dX_s - \frac{1}{2} \int_0^t \frac{\mu_1^2(X_s) - \mu_0^2(X_s)}{\sigma^2(X_s)} \, ds \right) \tag{3.10}$$

for $t \geq 0$. A direct calculation based on (3.1) shows that the *posterior probability ratio* process $\Phi = (\Phi_t)_{t \geq 0}$ of θ given X that is defined by

$$\Phi_t = \frac{\Pi_t}{1 - \Pi_t} \tag{3.11}$$

can be expressed in terms of L (and hence X as well) as follows

$$\Phi_t = e^{\lambda t} L_t \left(\Phi_0 + \lambda \int_0^t \frac{ds}{e^{\lambda s} L_s} \right) \tag{3.12}$$

for $t \geq 0$ where $\Phi_0 = \pi/(1-\pi)$.

8. To derive stochastic differential equations for the posterior processes Π and Φ one may apply Itô's formula in (3.10) to find that

$$dL_t = \frac{\mu_1(X_t) - \mu_0(X_t)}{\sigma^2(X_t)} L_t \, [dX_t - \mu_0(X_t)\, dt] \tag{3.13}$$

with $L_0 = 1$. Further applications of Itô's formula in (3.11) and (3.12) then show that

$$d\Pi_t = \lambda(1-\Pi_t)\, dt + \rho(X_t)\Pi_t(1-\Pi_t)\, d\bar{B}_t \tag{3.14}$$

$$d\Phi_t = \left[\lambda(1+\Phi_t) + \rho^2(X_t) \frac{\Phi_t^2}{1+\Phi_t} \right] dt + \rho(X_t)\, \Phi_t \, d\bar{B}_t \tag{3.15}$$

upon noting that X solves

$$dX_t = \big[\mu_0(X_t) + \Pi_t \big(\mu_1(X_t) - \mu_0(X_t) \big) \big]\, dt + \sigma(X_t)\, d\bar{B}_t \tag{3.16}$$

where $\bar{B} = (\bar{B}_t)_{t \geq 0}$ is the *innovation process* defined by

$$\bar{B}_t = \int_0^t \frac{dX_s}{\sigma(X_s)} - \int_0^t \left[\frac{\mu_0(X_s)}{\sigma(X_s)} + \Pi_s \frac{\mu_1(X_s) - \mu_0(X_s)}{\sigma(X_s)} \right] ds \tag{3.17}$$

for $t \geq 0$ from where we see by Lévy's characterisation theorem that $\bar{B}$ is a standard Brownian motion with respect to $(\mathcal{F}_t^X)_{t \geq 0}$ under P_π for $\pi \in [0,1]$.

4. Measure change

In this section we recall that changing the measure P_π for $\pi \in [0,1]$ to P^∞ in the optimal stopping problems (3.4) and (3.6) above provides crucial simplifications of the setting which makes the subsequent analysis possible.

1. The result of Lemma 1 in [20] identifies the Radon-Nikodym derivative corresponding to the measure change from P_π to P^∞ to be

$$\frac{d\mathsf{P}_{\pi,\tau}}{d\mathsf{P}_\tau^\infty} = e^{-\lambda \tau} \frac{1-\pi}{1-\Pi_\tau} \tag{4.1}$$

for all stopping times τ of X and all $\pi \in [0,1)$, where P_τ^∞ and $\mathsf{P}_{\pi,\tau}$ denote the restrictions of measures P^∞ and P_π to $\mathcal{F}_\tau^X$ for $\pi \in [0,1)$ respectively. From (3.15) and (3.17) we see that the stochastic differential equations for (Φ, X) under the measure P^∞ simplify as follows

$$d\Phi_t = \lambda(1+\Phi_t)\, dt + \rho(X_t)\Phi_t \, dB_t \tag{4.2}$$

$$dX_t = \mu_0(X_t)\, dt + \sigma(X_t)\, dB_t \tag{4.3}$$

where (4.3) follows directly from (3.2) upon recalling that θ formally equals ∞ under P^∞. Further, note from (3.13) that the stochastic differential equation for L under P^∞ reads

$$dL_t = \rho(X_t)\, L_t\, dB_t \tag{4.4}$$

from where we see that

$$L_t = \exp\left(\int_0^t \rho(X_s)\, dB_s - \frac{1}{2}\int_0^t \rho^2(X_s)\, ds\right) \tag{4.5}$$

for $t \geq 0$. Finally, the following general facts are helpful in the subsequent analysis of the optimal stopping problems (3.4) and (3.6) above.

Proposition 1. *The process* $(e^{\lambda t}(1-\Pi_t))_{t\geq 0}$ *is a continuous martingale under* P_π *such that* $e^{\lambda t}(1-\Pi_t) \to 0$ *with* P_π*-probability one as* $t \to \infty$ *for* $\pi \in [0,1]$ *given and fixed. The process* $(e^{-\lambda t}(1+\Phi_t))_{t\geq 0}$ *is a continuous martingale under* P^∞ *such that* $e^{-\lambda t}(1+\Phi_t) \to 0$ *with* P^∞*-probability one as* $t \to \infty$.

Proof. Using (4.1) we recognise $e^{\lambda t}(1-\Pi_t)$ and $e^{-\lambda t}(1+\Phi_t)$ as constant multiples of the Radon-Nikodym derivatives $d\mathsf{P}_t^\infty/d\mathsf{P}_{\pi,t}$ and $d\mathsf{P}_{\pi,t}/d\mathsf{P}_t^\infty$ and hence the two processes are martingales under P_π and P^∞ respectively for $t \geq 0$ whenever $\pi \in [0,1]$ is given and fixed. The two convergence relations follow from the fact that the probability measures P_π and P^∞ are singular for $\pi \in [0,1]$ (cf. Theorem 2 in [38, p. 527]). This completes the proof. □

2. The optimal stopping problem (3.6) admits a transparent reformulation under the measure P^∞ in terms of the process Φ solving (4.2)+(4.3). Indeed, recalling that Φ starts at $\Phi_0 = \pi/(1-\pi)$ and indicating this dependence on the initial point by a superscript to Φ, Proposition 2 in [20] shows that the value function V from (3.6) satisfies the identity

$$V(\pi) = (1-\pi)\left[1 + c\hat{V}(\pi)\right] \tag{4.6}$$

where the value function $\hat{V}$ is given by

$$\hat{V}(\pi) = \inf_\tau \mathsf{E}^\infty\left[\int_0^\tau e^{-\lambda t}\left(\Phi_t^{\pi/(1-\pi)} - \frac{\lambda}{c}\right) dt\right] \tag{4.7}$$

for $\pi \in [0,1)$ and the infimum in (4.7) is taken over all stopping times τ of X.

3. Since ρ in (3.7) is not constant, then to tackle the resulting optimal stopping problem (4.7) for the Markov process (Φ, X) solving (4.2)+(4.3), we will enable (Φ, X) to start at any point (φ, x) in $[0,\infty)\times I\!R$ under the probability measure $\mathsf{P}_{\varphi,x}^\infty$ so that the optimal stopping problem (4.7) extends as follows

$$\hat{V}(\varphi, x) = \inf_\tau \mathsf{E}_{\varphi,x}^\infty\left[\int_0^\tau e^{-\lambda t}\left(\Phi_t - \frac{\lambda}{c}\right) dt\right] \tag{4.8}$$

for $(\varphi, x) \in [0,\infty)\times I\!R$ with $\mathsf{P}_{\varphi,x}^\infty\big((\Phi_0, X_0) = (\varphi, x)\big) = 1$ where the infimum in (4.8) is taken over all stopping times τ of (Φ, X). In this way we have reduced the initial quickest detection problem (3.4) to the optimal stopping problem (4.8) for the strong Markov process (Φ, X) solving the system of stochastic differential equations

$$d\Phi_t = \lambda(1+\Phi_t)\,dt + (\delta X_t+\gamma)\,\Phi_t\,dB_t \tag{4.9}$$

$$dX_t = \beta_0(x_0-X_t)\,dt + \sigma\,dB_t \tag{4.10}$$

under the measure $\mathsf{P}^\infty_{\varphi,x}$ with $(\varphi,x) \in [0,\infty)\times I\!R$. Note that this optimal stopping problem is inherently/fully two-dimensional.

5. Reduction to the canonical process

Recall that we have reduced the initial quickest detection problem (3.4) to the optimal stopping problem (4.8) for the strong Markov process (Φ, X) solving (4.9)+(4.10). In the sequel we study Cases 1-10 in detail (stated following (3.3) above) where $x_0 = x_1 = 0$ and hence $\gamma = 0$ so that the equations (4.9)+(4.10) read as follows

$$d\Phi_t = \lambda(1+\Phi_t)\,dt + \delta X_t \Phi_t\,dB_t \tag{5.1}$$

$$dX_t = -\beta_0 X_t\,dt + dB_t \tag{5.2}$$

under the measure $\mathsf{P}^\infty_{\varphi,x}$ with $(\varphi,x) \in [0,\infty)\times I\!R$ where we have also set $\sigma = 1$ without loss of generality (as one can replace X by X/σ otherwise). A key difficulty with this system is that X enters the diffusion coefficient of the stochastic differential equation (5.1). This makes the applicability of available comparison theorems for (Φ, X) more challenging. To tackle the issue, note that since the stochastic differential equations (5.1)+(5.2) are driven by the same Brownian motion, we know that the resulting infinitesimal generator equation must be of parabolic type. It follows therefore that reducing this equation to its canonical form by means of a diffeomorphic transformation (to be found) replaces the process (Φ, X) by the process (U, X) where U is a process of bounded variation. We refer to (U, X) as the *canonical process* in this setting. A similar reduction from (Φ, X) to (U, Φ) has been carried out in [20, Section 5] and we will see below that the choice of (U, X) has certain advantages in comparison with (U, Φ) since the stochastic differential equation for X remains fully decoupled from the stochastic differential equation for U in the system. Moreover, we will see that, quite remarkably, the stochastic/ordinary differential equation for U can be identified with a *Bernoulli* differential equation which is known to be solvable in a closed form. Passing finally to the process $(V, Z) := (e^{-U}, X^2)$ will enable us to exploit the symmetry of X around zero by shrinking the state space $I\!R\times I\!R$ of (U, X) to its essential component $(0,\infty)\times[0,\infty)$ forming the state space of (V, Z). Interestingly, in this context we also observe that Z is a Feller branching diffusion process (see [13] and [29]). The resulting canonical optimal stopping problem (5.31) corresponding to (4.8) will be used in Sections 7-9 below.

1. *Reduction from* (Φ, X) *to* (U, X). From (5.1)+(5.2) we see that the infinitesimal generator of the strong Markov process (Φ, X) is given by

$$I\!L_{\Phi,X} = \lambda(1+\varphi)\partial_\varphi - \beta_0 x\partial_x + \delta\varphi x\partial_{\varphi x} + \frac{1}{2}\delta^2\varphi^2 x^2\partial_{\varphi\varphi} + \frac{1}{2}\partial_{xx}. \tag{5.3}$$

To reduce this equation to its canonical form let us name the coefficients in (5.3) by setting

$$a(\varphi,x) = \frac{1}{2}\delta^2\varphi^2 x^2 \quad \& \quad 2b(\varphi,x) = \delta\varphi x \quad \& \quad c(\varphi,x) = \frac{1}{2} \tag{5.4}$$

for $(\varphi, x) \in [0,\infty)\times I\!R$ given and fixed. Then

$$b^2(\varphi,x) - a(\varphi,x)c(\varphi,x) = \frac{1}{4}\delta^2\varphi^2x^2 - \frac{1}{4}\delta^2\varphi^2x^2 = 0 \tag{5.5}$$

showing that the equation for $I\!L_{\Phi,X}$ in (5.3) is *parabolic*. Moreover, its unique family of characteristic curves (obtained by letting u below be constant) is given by

$$\frac{dx}{d\varphi} = \frac{b(\varphi,x)}{a(\varphi,x)} = \frac{1}{\delta\varphi x} \Leftrightarrow \delta x dx = \frac{d\varphi}{\varphi} \Leftrightarrow \frac{\delta}{2}x^2 = \log\varphi + u. \tag{5.6}$$

Setting $\xi(\varphi,x) = u = \delta x^2/2 - \log\varphi$ and $\eta(\varphi,x) = x$ we see that the Jacobian $J = \partial(\xi,\eta)/\partial(\varphi,x) = \xi_\varphi\eta_x - \xi_x\eta_\varphi = \xi_\varphi = -1/\varphi \notin \{0,\infty\}$ when $\varphi \neq 0$ as needed (for the inverse function theorem). It follows that the coefficients $A(u,x)$ and $B(u,x)$ associated with ∂_{uu} and ∂_{ux} in the resulting equation are zero and the coefficient $C(u,x)$ associated with ∂_{xx} remains the same as in (5.3). To find the coefficients associated with ∂_u and ∂_x we may note that the contributing terms are calculated as follows

$$\partial_\varphi = \xi_\varphi\partial_\xi + \eta_\varphi\partial_\eta = -\tfrac{1}{\varphi}\partial_\xi \quad \& \quad \partial_x = \xi_x\partial_\xi + \eta_x\partial_\eta = \delta x\partial_\xi + \partial_\eta \tag{5.7}$$

$$\partial_{\varphi\varphi} = \xi_{\varphi\varphi}\partial_\xi + \eta_{\varphi\varphi}\partial_\eta + \xi_\varphi^2\partial_{\xi\xi} + 2\xi_\varphi\eta_\varphi\partial_{\xi\eta} + \eta_\varphi^2\partial_{\eta\eta} = \tfrac{1}{\varphi^2}\partial_\xi + \tfrac{1}{\varphi^2}\partial_{\xi\xi} \tag{5.8}$$

$$\partial_{xx} = \xi_{xx}\partial_\xi + \eta_{xx}\partial_\eta + \xi_x^2\partial_{\xi\xi} + 2\xi_x\eta_x\partial_{\xi\eta} + \eta_x^2\partial_{\eta\eta} = \delta\partial_\xi + \delta^2x^2\partial_{\xi\xi} + 2\delta x\partial_{\xi\eta} + \partial_{\eta\eta} \tag{5.9}$$

$$\partial_{\varphi x} = \xi_{\varphi x}\partial_\xi + \eta_{\varphi x}\partial_\eta + \xi_\varphi\xi_x\partial_{\xi\xi} + (\xi_x\eta_\varphi + \xi_\varphi\eta_x)\,\partial_{\xi\eta} + \eta_\varphi\eta_x\partial_{\eta\eta} = -\tfrac{\delta x}{\varphi}\partial_{\xi\xi} - \tfrac{1}{\varphi}\partial_{\xi\eta}. \tag{5.10}$$

Inserting these expressions into the equation (5.3) we find that its canonical form is given by

$$I\!L_{U,X} = \Big[\frac{\delta}{2}\big(1 - \kappa x^2\big) - \lambda\big(1 + e^{u - \frac{\delta}{2}x^2}\big)\Big]\partial_u - \beta_0 x\,\partial_x + \frac{1}{2}\partial_{xx} \tag{5.11}$$

where we set $\kappa = \beta_0 + \beta_1$ upon recalling that $\delta = \beta_0 - \beta_1$. Setting

$$U_t = \frac{\delta}{2}X_t^2 - \log\Phi_t \tag{5.12}$$

for $t \geq 0$ it follows therefore that the canonical process (U,X) solves the following system of stochastic differential equations

$$dU_t = \Big[\frac{\delta}{2}\big(1 - \kappa X_t^2\big) - \lambda\big(1 + e^{U_t - \frac{\delta}{2}X_t^2}\big)\Big]dt \tag{5.13}$$

$$dX_t = -\beta_0 X_t\,dt + dB_t \tag{5.14}$$

under $\mathsf{P}_{u,x}^\infty$ with $\mathsf{P}_{u,x}^\infty\big((U_0,X_0) = (u,x)\big) = 1$ for $(u,x) \in I\!R\times I\!R$. Recalling known sufficient conditions (see e.g. [35, pp. 166-173]) we see that the system (5.13)+(5.14) has a unique weak solution and hence by the well-known result (see e.g. [35, pp. 158-163]) we can conclude that (U,X) is a (time-homogeneous) strong Markov process under $\mathsf{P}_{u,x}^\infty$ for $(u,x) \in I\!R\times I\!R$ having the infinitesimal generator given by (5.11) above. From (5.12) we moreover see that (U,X) is a strong solution to (5.13)+(5.14) that is pathwise unique (since strong existence plus uniqueness in law imply pathwise uniqueness [see (3.24) in [12] and the references therein]).

The arguments for reducing the process (Φ, X) to its canonical form (U, X) presented above are analytic. Proposition 4 in [20] presents probabilistic arguments for the same reduction and on closer inspection one can verify that the results are mutually consistent. It is now possible to reformulate the optimal stopping problem (4.8) in terms of the process (U, X) and perform its analysis upon noting that U is of bounded variation (with no diffusion part) and hence known comparison theorems for the system (5.13)+(5.14) are available. In this context it is also useful to recall that the OU equation (5.14) has a unique strong solution given by

$$X_t^x = e^{-\beta_0 t}\left(x + \int_0^t e^{\beta_0 s}\, dB_s\right) \tag{5.15}$$

for $t \geq 0$ under P^∞ where the initial point $x \in I\!R$ is visible/explicit (and the expression (5.15) defines a Markovian flow). We now disclose the remarkable fact that equation (5.13) is also solvable in a closed form.

2. *Bernoulli equation.* Note that equation (5.13) can be written as

$$\frac{dU_t}{dt} = f(X_t) - g(X_t)e^{U_t} \tag{5.16}$$

where the functions f and g are defined as

$$f(x) = \frac{\delta}{2}(1-\kappa x^2)-\lambda \quad \& \quad g(x) = \lambda e^{-\frac{\delta}{2}x^2} \tag{5.17}$$

for $x \in I\!R$. Setting $R_t := e^{U_t}$ we see that (5.16) transforms into the Bernoulli equation

$$\frac{dR_t}{dt} = f(X_t)R_t - g(X_t)R_t^2 \tag{5.18}$$

having a quadratic nonlinearity in the final term. The well-known substitution $S_t := 1/R_t$ transforms (5.18) into a first-order linear equation which after being solved in a closed form yields the expression

$$U_t = \int_0^t f(X_s)\, ds - \log\left(\int_0^t e^{\int_0^s f(X_r)\, dr} g(X_s)\, ds + e^{-U_0}\right) \tag{5.19}$$

as the general solution to the equation (5.16) for $t \geq 0$.

Remark 2. Note that the appearance of the Bernoulli equation in this setting is not unique to the OU process (4.10). For general diffusions X of the form (3.2) (with differentiable μ_0 and μ_1) we observe that $U = (U_t)_{t \geq 0}$ defined by

$$U_t = \int_0^{X_t} \frac{\mu_1(y)-\mu_0(y)}{\sigma^2(y)}\, dy - \log \Phi_t \tag{5.20}$$

for $t \geq 0$ also solves (5.16) and thus has the same structure as (5.19) but with

$$f(x) = \frac{\sigma^2(x)}{2}\left(\frac{\mu_1-\mu_0}{\sigma}\right)'(x) + \left(\frac{\mu_1^2-\mu_0^2}{2\sigma}\right)(x) - \lambda \tag{5.21}$$

$$g(x) = \lambda \exp\Big(-\int_0^x \frac{\mu_1(y)-\mu_0(y)}{\sigma^2(y)}\, dy \Big) \tag{5.22}$$

for $x \in I\!R$ (cf. [20, Proposition 4]).

3. *Passage from* (U,X) *to* (V,Z) . Setting

$$V_t = e^{-U_t} \quad \& \quad Z_t = X_t^2 \tag{5.23}$$

we see that (5.19) can be rewritten in a more familiar form as follows

$$V_t = e^{-\int_0^t f(\sqrt{Z_s})\, ds} \bigg[V_0 + \int_0^t e^{\int_0^s f(\sqrt{Z_r})\, dr} g(\sqrt{Z_s})\, ds \bigg] \tag{5.24}$$

for $t \ge 0$. By Itô's formula we find that

$$Z_t = Z_0 + \int_0^t (1-2\beta_0 Z_s)\, ds + 2\int_0^t \sqrt{Z_s}\, d\hat{B}_s \tag{5.25}$$

where $\hat{B}_t = \int_0^t \text{sign}(X_s)\, dB_s$ for $t \ge 0$ is a standard Brownian motion by Lévy's characterisation theorem. From (5.25) we see that Z is a Feller branching diffusion process with values in $[0,\infty)$ having 0 as an instantaneously reflecting boundary point (cf. [13] and [29]). From (5.15) we find that

$$Z_t^z = e^{-2\beta_0 t} \Big(\sqrt{z} + \int_0^t e^{-\beta_0 s}\, dB_s \Big)^2 \tag{5.26}$$

under P^∞ where the initial point $z \in [0,\infty)$ is visible/explicit (and the expression (5.26) defines a Markovian flow). From (5.13) and (5.25) we find that the process (V,Z) solves the following system of stochastic differential equations

$$dV_t = \Big[\lambda - \frac{\delta}{2}(1-\kappa Z_t) V_t + \lambda e^{-\frac{\delta}{2} Z_t} \Big] dt \tag{5.27}$$

$$dZ_t = (1-2\beta_0 Z_t)\, dt + 2\sqrt{Z_t}\, d\hat{B}_t \tag{5.28}$$

under $\mathsf{P}_{v,z}^\infty$ with $\mathsf{P}_{v,z}^\infty\big((V_0,Z_0)=(v,z)\big)=1$ for $(v,z) \in (0,\infty)\times[0,\infty)$. For the same reasons as following (5.13)+(5.14) we can conclude that (V,Z) is a (time-homogeneous) strong Markov process under $\mathsf{P}_{v,z}^\infty$ for $(v,z) \in (0,\infty)\times[0,\infty)$ having the infinitesimal generator given by

$$I\!L_{V,Z} = \Big[\lambda - \frac{\delta}{2}(1-\kappa z) v + \lambda e^{-\frac{\delta}{2} z} \Big] \partial_v + (1-2\beta_0 z)\partial_z + 2z\partial_{zz} \tag{5.29}$$

where we recall that $\delta = \beta_0 - \beta_1$ and $\kappa = \beta_0 + \beta_1$. Moreover, since the system (5.27)+(5.28) has a unique weak solution (meaning that the martingale problem is well posed) we know that the explicit solutions (5.24)+(5.26) define a Markovian flow of the initial point $(V_0,Z_0)=(v,z)$ in $(0,\infty)\times[0,\infty)$ (i.e. the coordinate process on the canonical space is strong Markov with respect to the law of (V^v,Z^z) under P^∞ for $(v,z) \in (0,\infty)\times[0,\infty)$). This is an important fact which makes some parts of the subsequent analysis possible.

Proposition 3. *The value function* $\hat{V}$ *from* (4.8) *satisfies the identity*

$$\hat{V}(\varphi,x) = \check{V}\Big(\varphi e^{-\frac{\delta}{2}x^2}, x^2\Big) \tag{5.30}$$

for $(\varphi,x)\in[0,\infty)\times I\!R$ *where the value function* $\check{V}$ *is defined by*

$$\check{V}(v,z) = \inf_\tau \mathsf{E}^\infty_{v,z}\bigg[\int_0^\tau e^{-\lambda t}\Big(V_t e^{\frac{\delta}{2}Z_t} - \frac{\lambda}{c}\Big)\,dt\bigg] \tag{5.31}$$

for $(v,z)\in(0,\infty)\times[0,\infty)$ *and the infimum is taken over all stopping times* τ *of the strong Markov process* (V,Z) *solving* (5.27)+(5.28) *and being explicitly given by* (5.24)+(5.26) *above.*

Proof. Combining (5.12) and (5.23) we see that $v=e^{-u}=\varphi e^{-(\delta/2)x^2}$ and $z=x^2$ so that the claims follow from the facts derived following (5.23) above. □

6. Mayer formulation

The optimal stopping problem (4.8) is Lagrange formulated. In this section we derive its Mayer reformulation when $\beta_0\neq-\lambda/2$ and $\beta_1=0$. Such an explicit reformulation does not appear to be possible in the remaining cases.

Proposition 4. *If we have* $\beta_0\neq-\lambda/2$ *and* $\beta_1=0$ *, then the value function* $\hat{V}$ *from* (4.8) *can be expressed as*

$$\hat{V}(\varphi,x) = \inf_\tau \mathsf{E}^\infty_{\varphi,x}\Big[e^{-\lambda\tau}\hat{M}(\Phi_\tau,X_\tau)\Big] - \hat{M}(\varphi,x) \tag{6.1}$$

for $(\varphi,x)\in[0,\infty)\times I\!R$ *, where the function* $\hat{M}$ *is given by*

$$\hat{M}(\varphi,x) = \varphi x^2 + \big(x^2+\tfrac{1}{\lambda}\big)/\big(1+\tfrac{2\beta_0}{\lambda}\big) + \tfrac{1}{c} \tag{6.2}$$

for $(\varphi,x)\in[0,\infty)\times I\!R$ *and the infimum in* (6.1) *is taken over all stopping times* τ *of* (Φ,X) *.*

Proof. Recalling that the infinitesimal generator $I\!L_{\Phi,X}$ of (Φ,X) is given by (5.3) above, it is easily verified that we have

$$\big(I\!L_{\Phi,X}\hat{M} - \lambda\hat{M}\big)(\varphi,x) = \varphi - \tfrac{\lambda}{c} \tag{6.3}$$

for $(\varphi,x)\in[0,\infty)\times I\!R$ when $\beta_0\neq-\lambda/2$ and $\beta_1=0$ as assumed throughout. Hence by Itô's formula and the optional sampling theorem we find that

$$\begin{aligned}\mathsf{E}^\infty_{\varphi,x}\Big[e^{-\lambda\tau}\hat{M}(\Phi_\tau,X_\tau)\Big] &= \hat{M}(\varphi,x) + \mathsf{E}^\infty_{\varphi,x}\bigg[\int_0^\tau e^{-\lambda t}\big(I\!L_{\Phi,X}\hat{M}-\lambda\hat{M}\big)(\Phi_s,X_s)\,ds\bigg] \\ &= \hat{M}(\varphi,x) + \mathsf{E}^\infty_{\varphi,x}\bigg[\int_0^\tau e^{-\lambda t}\Big(\Phi_t - \frac{\lambda}{c}\Big)\,dt\bigg]\end{aligned} \tag{6.4}$$

for $(\varphi,x)\in[0,\infty)\times I\!R$ and all (bounded) stopping times τ of (Φ,X). Taking the infimum over all such stopping times on both sides of (6.4) we see that (6.1) holds as claimed. □

The optimal stopping problem (5.31) can be Mayer reformulated similarly and we will omit further details. The absence of explicit Mayer reformulations in the remaining cases forces us to analyse the optimal stopping problem (4.8) in its Lagrange form in what follows.

7. Properties of the optimal stopping boundary

In this section we establish the existence of an optimal stopping time in (4.8) and derive basic properties of the optimal stopping boundary.

1. Looking at (4.8) we may conclude that the (candidate) continuation and stopping sets in this problem need to be defined as follows

$$C = \{ (\varphi, x) \in [0,\infty)\times I\!R \mid \hat{V}(\varphi, x) < 0 \} \tag{7.1}$$

$$D = \{ (\varphi, x) \in [0,\infty)\times I\!R \mid \hat{V}(\varphi, x) = 0 \} \tag{7.2}$$

respectively. In the absence of a Mayer reformulation of the Lagrange formulated problem (4.8) as discussed in Section 6 above, we embed the two-dimensional Markov process (Φ, X) into the four-dimensional Markov process (T, Φ, X, I) where $T_t = t$ and $I_t = \int_0^t e^{-\lambda T_s}(\Phi_s - \lambda/c)\, ds$ for $t \geq 0$. Recalling (5.15) and the fact that the right-hand side in (3.12) defines a Markovian functional of the initial point, we see that (T, Φ, X, I) can be realised as a Markovian flow $(t+s, \Phi_s^{\varphi,x}, X_s^x, i+I_s^{t,\varphi,x})$ for $s \geq t$ with $(t, \varphi, x, i) \in [0,\infty)^2 \times I\!R^2$ (superscripts throughout indicate the initial points of Markov processes). From this Markovian flow representation we see that the expectation in (4.8) defines a continuous function of the initial point for every (bounded) stopping time τ given and fixed. It follows therefore that the value function $\bar{V}$ associated with (T, Φ, X, I) by setting $\bar{V}(t, \varphi, x, i) = \inf_\tau \mathsf{E}^\infty[i + I_\tau^{t,\varphi,x}]$ for $(t, \varphi, x, i) \in [0,\infty)^2 \times I\!R^2$ is upper semicontinuous. Noting that the optimal stopping problem of $\bar{V}$ is Mayer formulated, and its loss function $\bar{M}(t, \varphi, x, i) = i$ for $(t, \varphi, x, i) \in [0,\infty)^2 \times I\!R^2$ is continuous and hence lower semicontinuous, it follows by [32, Corollary 2.9] that the first entry time of (T, Φ, X, I) into the closed set $\bar{D} = \{ \bar{V} = \bar{M} \}$ is optimal whenever finite almost surely. Noting that $\bar{V}(t, \varphi, x, i) = i + e^{-\lambda t}\hat{V}(\varphi, x)$ for $(t, \varphi, x, i) \in [0,\infty)^2 \times I\!R^2$ this shows that the first entry time of the process (Φ, X) into the closed set D defined by

$$\tau_D = \inf \{ t \geq 0 \mid (\Phi_t, X_t) \in D \} \tag{7.3}$$

is optimal in (4.8) whenever $\mathsf{P}_{\varphi,x}^\infty(\tau_D < \infty) = 1$ for all $(\varphi, x) \in [0,\infty)\times I\!R$. In the sequel we will establish this and other properties of τ_D by analysing the boundary of D.

2. Since the integrand in (4.8) is strictly negative for $\varphi < \lambda/c$ it is clear that this region is contained in C (otherwise the first exit times of (Φ, X) from sufficiently small neighbourhoods would violate stopping at once). When the process (Φ, X) belongs to the set where $\varphi \geq \lambda/c$ then the incentive to continue (as opposed to stop) is measured by the ability of (Φ, X) to return to the favourable region where $\varphi < \lambda/c$ without too much loss. We now show that this incentive is not endless.

Proposition 5. *The stopping set D is non-empty.*

Proof. Suppose that D is empty. Then

$$\hat{V}(\varphi, x) = \mathsf{E}_{\varphi,x}^\infty\Big[\int_0^\infty e^{-\lambda t}\Big(\Phi_t - \frac{\lambda}{c}\Big)\, dt\Big] = \Big[\int_0^\infty e^{-\lambda t}\Big(\mathsf{E}_{\varphi,x}^\infty(\Phi_t) - \frac{\lambda}{c}\Big)\, dt\Big] \tag{7.4}$$

for $(\varphi, x) \in [0,\infty)\times I\!R$ given and fixed. From (5.1) we find that

$$\mathsf{E}_{\varphi,x}^\infty(\Phi_t) = \varphi + \lambda \int_0^t \big(1 + \mathsf{E}_{\varphi,x}^\infty(\Phi_s)\big)\, ds \tag{7.5}$$

for $t \geq 0$. Setting $m(t) = \mathsf{E}^\infty_{\varphi,x}(\Phi_t)$ this shows that $m'(t) = \lambda(1+m(t))$ for $t \geq 0$ with $m(0) = \varphi$. Solving this initial value problem we get

$$\mathsf{E}^\infty_{\varphi,x}(\Phi_t) = (1+\varphi)e^{\lambda t} - 1 \tag{7.6}$$

for $t \geq 0$. Inserting (7.6) into (7.4) we obtain $\hat{V}(\varphi,x) = \infty$. This contradicts the fact that $\hat{V}$ is non-positive and hence we can conclude that D is non-empty as claimed. □

3. The previous proof shows that $\mathsf{P}^\infty_{\varphi,x}(\tau_D < \infty) > 0$ for all $(\varphi,x) \in [0,\infty) \times I\!R$. We can strengthen this conclusion to $\mathsf{P}^\infty_{\varphi,x}(\tau_D < \infty) = 1$ for all $(\varphi,x) \in [0,\infty) \times I\!R$ by recalling from (4.6) that the optimal stopping problem (4.8) is equivalent to the optimal stopping problem (3.6) so that τ_D from (7.3) redefined by means of (3.11) above is optimal in (3.6). Noting from Proposition 1 that $\Pi_t \rightarrow 1$ with P_π-probability one as $t \rightarrow \infty$, it follows from the structure of (3.6) (i.e. its integral term) that $\mathsf{P}_\pi(\tau_D < \infty) = 1$ for $\pi \in [0,1]$. Since the set $\{\tau_D < \infty\}$ belongs to $\mathcal{F}^X_{\tau_D}$ and by (4.1) above the probability measures P_π and P^∞ restricted to $\mathcal{F}^X_{\tau_D}$ are equivalent (i.e. $\mathsf{P}_\pi(F) = 0$ if and only if $\mathsf{P}^\infty(F) = 0$ for $F \in \mathcal{F}^X_{\tau_D}$) for $\pi \in [0,1)$, it follows that $\mathsf{P}^\infty_{\varphi,x}(\tau_D < \infty) = 1$ for all $(\varphi,x) \in [0,\infty) \times I\!R$ as claimed.

4. In view of the previous facts define the (least) boundary between C and D by setting

$$b(x) = \inf\{\, \varphi \geq 0 \mid (\varphi,x) \in D\,\} \tag{7.7}$$

for every $x \in I\!R$ given and fixed. Clearly $b(x) \geq \lambda/c$ and the infimum in (7.7) is attained for every $x \in I\!R$ since D is closed. From (7.14) below we see that

$$b(x) = b(-x) \tag{7.8}$$

for all $x \in I\!R$. For this reason we can often focus our attention to b restricted on $[0,\infty)$ in what follows. Moreover, from (3.12) we see that $\varphi \mapsto \Phi^{\varphi,x}$ is increasing on $[0,\infty)$ and hence from the structure of (4.8) it is clear that

$$\varphi \mapsto \hat{V}(\varphi,x) \text{ is increasing on } [0,\infty) \tag{7.9}$$

for every $x \in I\!R$ given and fixed. Hence if $(\varphi_1,x) \in D$ and $\varphi_2 \geq \varphi_1$ then $(\varphi_2,x) \in D$ because $0 = \hat{V}(\varphi_1,x) \leq \hat{V}(\varphi_2,x) \leq 0$ so that $\hat{V}(\varphi_2,x) = 0$ implying the claim. From this implication we see that b from (7.7) defines the (entire) boundary between C and D so that

$$C = \{\,(\varphi,x) \in [0,\infty) \times I\!R \mid \varphi < b(x)\,\} \tag{7.10}$$

$$D = \{\,(\varphi,x) \in [0,\infty) \times I\!R \mid \varphi \geq b(x)\,\} \tag{7.11}$$

where the function b is finite valued as it will be formally verified below. Finally, from (3.12) we also see that $\varphi \mapsto \Phi^{\varphi,x}$ is a linear function of $\varphi \in [0,\infty)$ and hence from the structure of (4.8) it is easily seen that

$$\varphi \mapsto \hat{V}(\varphi,x) \text{ is concave on } [0,\infty) \tag{7.12}$$

for every $x \in I\!R$ given and fixed. This fact will now enable us to derive a counterpart of (7.9) in the other variable.

Proposition 6. *We have*

$$x \mapsto \hat{V}(\varphi, x) \text{ is increasing on } (-\infty, 0] \text{ and decreasing on } [0, \infty) \tag{7.13}$$

for every $\varphi \in [0, \infty)$ *given and fixed.*

Proof. By symmetry of X in (5.2) around zero combined with its placement in (5.1) it is evident that the mapping $x \mapsto \hat{V}(\varphi, x)$ is even on $I\!R$, i.e. we have

$$\hat{V}(\varphi, x) = \hat{V}(\varphi, -x) \tag{7.14}$$

for all $x \in I\!R$. This is because (Φ, X) & $(\Phi, -X)$ are equally distributed as is seen by rewriting the system (5.1)+(5.2) with $-X$ & $-B$ in place of X & B and using the uniqueness in law for the system. The values in (7.14) coincide because they are equivalently taken in (4.8) as the infima over the first entry times of (Φ, X) & $(\Phi, -X)$ into closed sets, and for each entry time into a closed set, the two integrals appearing in (4.8) are equally distributed because (Φ, X) & $(\Phi, -X)$ are so. It follows from (7.14) that it is enough to show that $x \mapsto \hat{V}(\varphi, x)$ is decreasing on $[0, \infty)$ for every $\varphi \in [0, \infty)$ given and fixed. We will divide the proof of this fact into five parts.

1. Recalling that $Z = X^2$ we see from (5.1) and (5.28) that

$$d\Phi_t = \lambda(1+\Phi_t)\, dt + \delta \sqrt{Z_t}\, \Phi_t\, d\hat{B}_t \tag{7.15}$$

$$dZ_t = (1-2\beta_0 Z_t)\, dt + 2\sqrt{Z_t}\, d\hat{B}_t \tag{7.16}$$

where $\hat{B}_t = \int_0^t \text{sign}(X_s)\, dB_s$ for $t \geq 0$ is a standard Brownian motion. By symmetry of X in (5.2) around zero combined with its placement in (5.1) and the structure of the integrand in (4.8) it is clear that we can consider (4.8) as an optimal stopping problem for the strong Markov process (Φ, Z) solving (7.15)+(7.16) with the value function $\tilde{V}$ satisfying

$$\tilde{V}(\varphi, z) = \hat{V}(\varphi, \sqrt{z}) \tag{7.17}$$

for $(\varphi, z) \in [0, \infty) \times [0, \infty)$. Thus to show that $x \mapsto \hat{V}(\varphi, x)$ is decreasing on $[0, \infty)$ it is enough to show that

$$z \mapsto \tilde{V}(\varphi, z) \text{ is decreasing on } [0, \infty) \tag{7.18}$$

for every $\varphi \in [0, \infty)$ given and fixed.

2. Let $\tilde{C}$ and $\tilde{D}$ denote the continuation and stopping sets for $\tilde{V}$. Setting

$$\tilde{b}(z) = b(\sqrt{z}) \tag{7.19}$$

for $z \in [0, \infty)$ we see from (7.10)+(7.11) combined with (7.8) that

$$\tilde{C} = \{ (\varphi, z) \in [0, \infty) \times [0, \infty) \mid \varphi < \tilde{b}(z) \} \tag{7.20}$$

$$\tilde{D} = \{ (\varphi, z) \in [0, \infty) \times [0, \infty) \mid \varphi \geq \tilde{b}(z) \}. \tag{7.21}$$

Setting $L(\varphi) = \varphi - \lambda/c$ for $\varphi \in [0,\infty)$ standard Markovian results of optimal stopping (cf. [32, Subsection 7.2]) imply that $\tilde{V}$ and $\tilde{b}$ solve the free-boundary problem

$$\mathbb{L}_{\Phi,Z}\tilde{V} - \lambda\tilde{V} = -L \text{ in } \tilde{C} \tag{7.22}$$

$$\tilde{V} = 0 \text{ at } \partial\tilde{C} \text{ (instantaneous stopping)} \tag{7.23}$$

$$\tilde{V}_\varphi = \tilde{V}_z = 0 \text{ at } \partial_r\tilde{D} \text{ (smooth fit)} \tag{7.24}$$

where $\mathbb{L}_{\Phi,Z}$ is the infinitesimal generator of (Φ, Z) given by

$$\mathbb{L}_{\Phi,Z} = \lambda(1+\varphi)\partial_\varphi + (1-2\beta_0 z)\partial_z + 2\delta\varphi z\partial_{\varphi z} + \frac{1}{2}\delta^2\varphi^2 z\partial_{\varphi\varphi} + 2z\partial_{zz} \tag{7.25}$$

and $\partial_r\tilde{D}$ denotes the set of boundary points of $\tilde{C}$ that are (probabilistically) regular for $\tilde{D}$ (see [6, Subsection 2.3] for more details). It is a matter of routine to verify that $\mathbb{L}_{U,X}$ from (5.11) satisfies Hörmander's condition (4.41) in [31]. Indeed, in the notation of that paper, we see that $\mathbb{L}_{U,X} = D_0 + D_1^2$ with $D_0 = a\partial_u - \beta_0 x\partial_x \sim [a; -\beta_0 x]$ and $D_1 = (1/\sqrt{2})\partial_x \sim [0; 1/\sqrt{2}]$ where the function $a = a(u,x)$ can be read off explicitly from (5.11) above. A direct calculation shows that $[D_1, D_0] = (\partial_x a/\sqrt{2})\partial_u - (\beta_0/\sqrt{2})\partial_x \sim [\partial_x a/\sqrt{2}; -\beta_0/\sqrt{2}]$ and $[D_1,[D_1,D_0]] = (\partial_x^2 a/(\sqrt{2})^2)\partial_u \sim [\partial_x^2 a/(\sqrt{2})^2; 0]$. Continuing by induction we find that $[D_1,[D_1,\ldots,[D_1,D_0]\ldots]] = (\partial_x^n a/(\sqrt{2})^n)\partial_u \sim [\partial_x^n a/(\sqrt{2})^n; 0]$ for all $n \geq 2$. Hence we see that Hörmander's condition $\dim Lie(D_0, D_1) = 2$ holds if $\partial_x^n a \neq 0$ for some $n \geq 0$ (with $\partial_x^0 a := a$) both at any given point, which is the case since otherwise (by Taylor expansion since a is analytic) we would be able to derive the false conclusion that $a(u_0, x) = 0$ for all x belonging to an open interval containing $x_0 \in \mathbb{R}$ with $u_0 \in \mathbb{R}$ given and fixed. Recalling (5.12) it follows therefore by Corollary 8 in [31] that $\hat{V}$ from (4.8) belongs to C^∞ on C. From (7.17) we thus see that $\tilde{V}$ belongs to C^∞ on $\tilde{C}$ off the boundary $z = 0$ (at which the process (Φ, Z) spends zero time relative to Lebesgue measure with probability one). Note that the smooth fit condition (7.24) can be derived using the results and methods of [6, Section 4] since the initial point of the Markovian flow (3.12)+(5.26) for (Φ, Z) is explicitly visible (see the proofs of Propositions 13 and 14 in [20] for more details).

3. By (7.25) we see that (7.22) reads

$$\lambda(1+\varphi)\tilde{V}_\varphi + (1-2\beta_0 z)\tilde{V}_z + 2\delta\varphi z\tilde{V}_{\varphi z} + \frac{1}{2}\delta^2\varphi^2 z\tilde{V}_{\varphi\varphi} + 2z\tilde{V}_{zz} - \lambda\tilde{V} = -L \tag{7.26}$$

in $\tilde{C}$. Differentiating both sides of (7.26) with respect to z and setting

$$\tilde{U} := \tilde{V}_z \tag{7.27}$$

we find that $\tilde{U}$ solves

$$(\lambda+(\lambda+2\delta)\varphi)\tilde{U}_\varphi + (3-2\beta_0 z)\tilde{U}_z + 2\delta\varphi z\tilde{U}_{\varphi z} + \frac{1}{2}\delta^2\varphi^2 z\tilde{U}_{\varphi\varphi} + 2z\tilde{U}_{zz} - (2\beta_0+\lambda)\tilde{U} = -\frac{1}{2}\delta^2\varphi^2\tilde{V}_{\varphi\varphi} \tag{7.28}$$

in $\tilde{C}$. Setting

$$\mathbb{L}_{\tilde{\Phi},\tilde{Z}} = (\lambda+(\lambda+2\delta)\varphi)\partial_\varphi + (3-2\beta_0 z)\partial_z + 2\delta\varphi z\partial_{\varphi z} + \frac{1}{2}\delta^2\varphi^2 z\partial_{\varphi\varphi} + 2z\partial_{zz} \tag{7.29}$$

we see that (7.28) can be rewritten as follows

$$\mathbb{L}_{\tilde{\Phi},\tilde{Z}}\tilde{U} - r\tilde{U} = -H \;\text{ in }\; \tilde{C} \tag{7.30}$$

where we set $r = 2\beta_0 + \lambda$ and

$$H = \frac{1}{2}\delta^2\varphi^2\tilde{V}_{\varphi\varphi} \le 0 \tag{7.31}$$

in $\tilde{C}$ by (7.12) and (7.17). Standard arguments (see e.g. [35, pp 158-163 & 166-173]) show that $\mathbb{L}_{\tilde{\Phi},\tilde{Z}}$ is the infinitesimal generator of the strong Markov process $(\tilde{\Phi}, \tilde{Z})$ which can be characterised as a unique weak solution to the system of stochastic differential equations

$$d\tilde{\Phi}_t = (\lambda + (\lambda + 2\delta)\tilde{\Phi}_t)\, dt + \delta\sqrt{\tilde{Z}_t}\,\tilde{\Phi}_t\, d\tilde{B}_t \tag{7.32}$$

$$d\tilde{Z}_t = (3 - 2\beta_0\tilde{Z}_t)\, dt + 2\sqrt{\tilde{Z}_t}\, d\tilde{B}_t \tag{7.33}$$

under a probability measure $\tilde{\mathsf{P}}_{\varphi,z}$ such that $\tilde{\mathsf{P}}_{\varphi,z}\big((\tilde{\Phi}_0, \tilde{Z}_0) = (\varphi, z)\big) = 1$ for $(\varphi, z) \in [0,\infty) \times [0,\infty)$ where $\tilde{B}$ is a standard Brownian motion. Note that $\tilde{Z}$ is a Feller branching diffusion process with values in $[0,\infty)$ having 0 as an entrance boundary point (cf. [13] and [29]).

4. The previous conclusions suggest to consider the stopping time

$$\sigma_{\tilde{D}^0} = \inf\{\, t \ge 0 \mid (\tilde{\Phi}_t, \tilde{Z}_t) \in \tilde{D}^0 \,\} \tag{7.34}$$

where $\tilde{D}^0$ denotes the interior of $\tilde{D}$. Then it is well known (cf. [4, Theorem 11.4, p. 62]) that $(\tilde{\Phi}_{\sigma_{\tilde{D}^0}}, \tilde{Z}_{\sigma_{\tilde{D}^0}})$ belongs to the set $\partial_r\tilde{D}^0$ of boundary points of $\tilde{C}$ that are (probabilistically) regular for $\tilde{D}^0$. Since $\partial_r\tilde{D}^0$ is contained in the set $\partial_r\tilde{D}$ of boundary points of $\tilde{C}$ that are (probabilistically) regular for $\tilde{D}$ it follows that $(\tilde{\Phi}_{\sigma_{\tilde{D}^0}}, \tilde{Z}_{\sigma_{\tilde{D}^0}})$ belongs to the set $\partial_r\tilde{D}$ and hence by the second part of (7.24) upon recalling (7.27) we can conclude that

$$\tilde{U}(\tilde{\Phi}_{\sigma_{\tilde{D}^0}}, \tilde{Z}_{\sigma_{\tilde{D}^0}}) = 0 \tag{7.35}$$

with $\tilde{\mathsf{P}}_{\varphi,z}$-probability one for any $(\varphi, z) \in [0,\infty) \times [0,\infty)$ given and fixed. Suppose that $\tilde{U}(\varphi, z) > 0$ for some $(\varphi, z) \in \tilde{C}$ and consider the stopping time

$$\nu = \inf\{\, t \ge 0 \mid (\tilde{\Phi}_t, \tilde{Z}_t) \in N \,\} \tag{7.36}$$

where N denotes the set of all points in the closure of $\tilde{C}$ at which $\tilde{U}$ equals zero. Then $\nu \le \sigma_{\tilde{D}^0}$ and by Itô's formula and the optional sampling theorem we find that

$$\tilde{U}(\varphi, z) = \tilde{\mathsf{E}}_{\varphi,z}\Big[e^{-r(\nu\wedge\tau_n)}\tilde{U}(\tilde{\Phi}_{\nu\wedge\tau_n}, \tilde{Z}_{\nu\wedge\tau_n})\Big] + \tilde{\mathsf{E}}_{\varphi,z}\Big[\int_0^{\nu\wedge\tau_n} e^{-rt} H(\tilde{\Phi}_t, \tilde{Z}_t)\, dt\Big] \tag{7.37}$$

for $n \ge 1$ where we use (7.30) above and $(\tau_n)_{n\ge1}$ is a localising sequence of stopping times for the continuous local martingale arising from Itô's formula. Since $\tilde{U}(\tilde{\Phi}_\nu, \tilde{Z}_\nu) = 0$ with $\tilde{U}(\tilde{\Phi}_t, \tilde{Z}_t) \ge 0$ for $t \in [0,\nu]$ we see by Fatou's lemma that

$$0 = \tilde{\mathsf{E}}_{\varphi,z}\Big[e^{-r\nu}\tilde{U}(\tilde{\Phi}_\nu, \tilde{Z}_\nu)\Big] \ge \limsup_{n\to\infty} \tilde{\mathsf{E}}_{\varphi,z}\Big[e^{-r(\nu\wedge\tau_n)}\tilde{U}(\tilde{\Phi}_{\nu\wedge\tau_n}, \tilde{Z}_{\nu\wedge\tau_n})\Big] \tag{7.38}$$

at least when $\tilde{U}$ is bounded on $\tilde{C} \setminus N$ and $r > 0$ i.e. $\beta_0 > -\lambda/2$. Letting $n \to \infty$ in (7.37) and using (7.38) we find by the monotone convergence theorem that

$$\tilde{U}(\varphi, z) \le \tilde{\mathsf{E}}_{\varphi,z}\bigg[\int_0^\nu e^{-rt} H(\tilde{\Phi}_t, \tilde{Z}_t)\, dt\bigg] \le 0 \tag{7.39}$$

where in the final inequality we use (7.31) above. Since $\tilde{U}(\varphi, x) > 0$ this is a contradiction and hence $\tilde{U}(\varphi, x) \le 0$ for all $(\varphi, z) \in \tilde{C}$. Recalling (7.27) this shows that (7.18) is satisfied as claimed and the proof is complete in this case.

5. The general case can be reduced to the case of bounded $\tilde{U}$ by approximating the optimal stopping problem (4.8) with a sequence of optimal stopping problems having bounded continuation sets C_n which approximate the continuation set C alongside the pointwise convergence of the approximating value functions $\hat{V}^n$ to the value function $\hat{V}$ as $n \to \infty$. For instance, this can be achieved using the same arguments as above by reflecting $Z = X^2$ downwards at a given level $n \ge 1$ while keeping the remaining probabilistic characteristics of (Φ, X) unchanged. Indeed, recalling (7.17) and approximating $\tilde{V}^n$ and $\tilde{V}$ by taking their infima over all stopping times $\tau \le \tau_n$ instead, where τ_n denotes the first entry time of Z into $[n, \infty)$, we see that the resulting/approximating function $\tilde{V}_n$ is the same for both $\tilde{V}^n$ and $\tilde{V}$ because (Φ, Z) remains unchanged on $[0, \tau_n]$ for $n \ge 1$. Moreover, noting that the 'negative' integrand $e^{-\lambda t}(\lambda/c)$ in $\tilde{V}^n$ and $\tilde{V}$ integrates to a finite value $1/c$ over all $t \in [0, \infty)$, it is easily verified using the monotone convergence theorem with $\tau_n \uparrow \infty$ as $n \to \infty$ that $\tilde{V}_n - R_n \le \tilde{V}^n \le \tilde{V}_n$ with $\tilde{V}_n \to \tilde{V}$ and $R_n \to 0$ pointwise as $n \to \infty$. Letting $n \to \infty$ in the previous two inequalities we thus see that $\tilde{V}^n \to \tilde{V}$ pointwise as claimed. Applying then the first part of the proof above when $\beta_0 > -\lambda/2$ i.e. $r > 0$ to the approximating value function $\tilde{V}^n$ of $\tilde{V}$ from (7.17) upon using that $\tilde{C}_n$ and therefore $\tilde{U}^n$ as well are bounded (because the vertical component $[0, n]$ of the state space is bounded while the 'negative' integrand in $\tilde{V}^n$ globally integrates to a finite value as pointed out above), we can conclude that each $z \mapsto \tilde{V}^n(\varphi, z)$ is decreasing for $n \ge 1$ and $\varphi \in [0, \infty)$. Hence passing to the pointwise limit as $n \to \infty$ we obtain that $z \mapsto \tilde{V}(\varphi, z)$ is decreasing as claimed for $\varphi \in [0, \infty)$. The case $\beta_0 \le -\lambda/2$ can be reduced to the case $r > 0$ by replacing X with $\tilde{X} := F(X)$ where F is a strictly increasing C^∞ solution to $I\!\!L_X F = 0$. This has the effect of setting the initial drift $\tilde{\beta}_0$ of the observed diffusion process $\tilde{X}$ equal to 0 , so that $r = 2\tilde{\beta}_0 + \lambda = \lambda > 0$, which makes the arguments above applicable to $\tilde{X}$ in place of X . This completes the proof. □

5. We can now derive the following important consequence of Proposition 6 about the boundary b between the sets C and D .

Corollary 7. *The mapping* $x \mapsto b(x)$ *is decreasing on* $(-\infty, 0]$ *and increasing on* $[0, \infty)$.

Proof. If $(\varphi, x) \in D$ then by (7.13) we have $0 = \hat{V}(\varphi, x) \le \hat{V}(\varphi, y) \le 0$ whenever $x \le y \le 0$ or $0 \le y \le x$ so that $(\varphi, y) \in D$ in both cases. These two conclusions combined with (7.7) above imply the two claims on b and the proof is complete. □

Despite being finite at one point at least, and hence at all larger points in $(-\infty, 0]$ or smaller points in $[0, \infty)$ as well, the boundary b could in principle still take infinite values. We now show that this is not the case.

Proposition 8. *We have* $b(x) < \infty$ *for all* $x \in I\!R$.

Proof. Since b is even it is enough to show that $b(x) < \infty$ for all $x \in [0,\infty)$. Note that $b(0) < \infty$ since otherwise by Corollary 7 the stopping set D would be empty contradicting the conclusion of Proposition 5 above. Suppose that $b(x_0) = \infty$ for some $x_0 > 0$. Pick two arbitrary points $x_1 < x_2$ in (x_0,∞) . Note that $b(x) = \infty$ for all $x \ge x_0$ by Corollary 7 so that $[0,\infty)\times(x_0,x_2) \subseteq C$. Hence considering the stopping time

$$\sigma = \inf\{\, t \ge 0 \mid X_t \notin (x_0,x_2)\,\} \tag{7.40}$$

we can conclude that $\sigma \le \tau_D$ almost surely with respect to P_{φ,x_1} for every $\varphi \in [0,\infty)$. It follows therefore that

$$\begin{aligned} \hat{V}(\varphi,x_1) &= \mathsf{E}^\infty_{\varphi,x_1}\Big[\int_0^{\tau_D} e^{-\lambda t}\Big(\varPhi_t - \frac{\lambda}{c}\Big)\,dt\Big] \ge \mathsf{E}^\infty_{\varphi,x_1}\Big[\int_0^{\tau_D} e^{-\lambda t}\varPhi_t\,dt\Big] - \frac{1}{c} \\ &\ge \mathsf{E}^\infty_{\varphi,x_1}\Big[\int_0^{\sigma} e^{-\lambda t}\varPhi_t\,dt\Big] - \frac{1}{c} = \mathsf{E}^\infty\Big[\int_0^{\sigma^{(x_1)}} e^{-\lambda t}\varPhi_t^{\varphi,x_1}\,dt\Big] - \frac{1}{c} \\ &\ge \varphi\,\mathsf{E}^\infty\Big[\int_0^{\sigma^{(x_1)}} L_t^{(x_1)}\,dt\Big] - \frac{1}{c} \end{aligned} \tag{7.41}$$

for all $\varphi \in [0,\infty)$ where in the final inequality we use (3.12). We write $\sigma^{(x_1)}$ and $L_t^{(x_1)}$ in (7.41) to indicate that both σ from (7.40) and L_t from (4.5) depend on x_1 noting also that neither of them depends on $\varphi \in [0,\infty)$. Since $\sigma^{(x_1)} > 0$ and $L_t^{(x_1)} > 0$ for all $t \ge 0$ we see that the final expectation in (7.41) is strictly positive. Choosing $\varphi \in [0,\infty)$ sufficiently large we can therefore make $\hat{V}(\varphi,x_1)$ strictly positive which contradicts the fact that $\hat{V}$ is non-negative. This shows that $b(x_0)$ cannot be infinite and the proof is complete. □

Monotonicity of b implies its (probabilistic) regularity for D , which in turn provides a smooth fit of $\hat{V}$ at b , yielding continuity of b . This can be formalised as follows.

Proposition 9. *The mapping* $x \mapsto b(x)$ *is continuous on* $I\!R$.

Proof. To establish that

$$\hat{V} \text{ is continuous on } [0,\infty)\times I\!R \tag{7.42}$$

we see from (7.14) that it is enough to show that $\hat{V}$ is continuous on $[0,\infty)\times[0,\infty)$. For this, recall from (5.14) that each starting point x of X in $(0,\infty)$ is (probabilistically) regular for both $[0,x)$ and (x,∞) . Since the sample path $t \mapsto U_t$ satisfying (5.13) is C^1 we see from (5.12) that each starting point φ of $\varPhi$ in $(0,\infty)$ is (probabilistically) regular for both $[0,\varphi)$ and (φ,∞) . Moreover, to account for a correlation between $\varPhi$ and X via the same B , we see from (5.12) when $\delta < 0$ that each starting point (φ,x) of $(\varPhi,X)$ in $(0,\infty)\times(0,\infty)$ is (probabilistically) regular for the quadrant $[\varphi,\infty)\times[0,x]$. If $\delta > 0$ then passing to $-\delta < 0$ and $-X$ in (5.1)+(5.2) we see that the same arguments (with a modified U in (5.12) obtained by replacing δ by $-\delta$ in (5.13) above) imply the same regularity conclusion for the quadrant. Finally, invoking (5.25) and using the same arguments as above we see that the same regularity conclusion extends to the starting points $(0,x)$ and $(\varphi,0)$ with $x \ge 0$ and $\varphi \ge 0$ as well. Recalling then the result of Corollary 7 that $x \mapsto b(x)$ is increasing on $[0,\infty)$ this shows that each starting point of (Φ, X) belonging to ∂C is (probabilistically) regular for D. Since the initial points of the Markovian flows (3.12) and (5.15) for Φ and X are explicitly visible, the latter regularity fact enables us to verify that (either or) both sections of $\hat{V}$ are locally (uniformly) continuous on $[0,\infty)$. This implies that (7.42) holds as claimed. Making use of these facts we can conclude by Theorem 8 and Remark 9 in [6] that

$$\hat{V}_\varphi \;\&\; \hat{V}_x \text{ are continuous on } [0,\infty) \times I\!R \tag{7.43}$$

(see the proofs of Propositions 13 and 14 in [20] for similar arguments). In particular, this shows that $\hat{V}$ satisfies the smooth fit condition (in the perpendicular direction) at ∂C and hence we can conclude by the result of Theorem 3 in [30] that $x \mapsto b(x)$ is continuous on $I\!R$ as claimed. □

8. Free-boundary problem

In this section we derive a free-boundary problem that stands in one-to-one correspondence with the optimal stopping problem (4.8). Using the results derived in the previous sections we show that the value function $\hat{V}$ from (4.8) and the optimal stopping boundary b from (7.10)+(7.11) solve the free-boundary problem. This establishes the existence of a solution to the free-boundary problem. Its uniqueness in a natural class of functions will follow from a more general uniqueness result that will be established in Section 9 below. This will also yield an explicit integral representation of the value function $\hat{V}$ expressed in terms of the optimal stopping boundary b.

1. Consider the optimal stopping problem (4.8) where the Markov process (Φ, X) solves the system of stochastic differential equations (5.1)+(5.2) driven by a standard Brownian motion B under the probability measure P^∞. Recall that the infinitesimal generator of (Φ, X) is the second-order parabolic differential operator $I\!L_{\Phi,X}$ given in (5.3) above. Recalling the arguments leading to (7.22)-(7.24) and relying on other properties of $\hat{V}$ and b derived above, we are naturally led to formulate the following free-boundary problem for finding $\hat{V}$ and b:

$$I\!L_{\Phi,X}\hat{V} - \lambda\hat{V} = -L \quad \text{in } C \tag{8.1}$$

$$\hat{V} = 0 \quad \text{on } D \quad \text{(instantaneous stopping)} \tag{8.2}$$

$$\hat{V}_\varphi = \hat{V}_x = 0 \quad \text{at } \partial C \quad \text{(smooth fit)} \tag{8.3}$$

where $L(\varphi) = \varphi - \lambda/c$ for $\varphi \in [0,\infty)$, C is the (continuation) set from (7.10) above, D is the (stopping) set from (7.11) above, and $\partial C = \{ (\varphi, x) \in [0,\infty) \times I\!R \mid \varphi = b(x) \}$ is the (optimal stopping) boundary between the sets C and D.

2. To formulate the existence and uniqueness result for the free-boundary problem (8.1)-(8.3), we let $\mathcal{C}$ denote the class of functions (U, a) such that

(8.4) U belongs to $C^1(\bar{C}_a) \cap C^2(C_a)$ and is continuous & bounded on $[0,\infty) \times I\!R$

(8.5) a is continuous on $I\!R$, decreasing on $(-\infty, 0]$, increasing on $[0,\infty)$ and satisfies

$$a(x) \geq \lambda/c \text{ for } x \in I\!R$$

where we set $C_a = \{ (\varphi, x) \in [0,\infty) \times I\!R \mid \varphi < a(x) \}$ and $\bar{C}_a = \{ (\varphi, x) \in [0,\infty) \times I\!R \mid \varphi \leq a(x) \}$ denotes the closure of C_a.

Theorem 10. *The free-boundary problem* (8.1)-(8.3) *has a unique solution* $(\hat{V}, b)$ *in the class* $\mathcal{C}$ *where* $\hat{V}$ *is given in* (4.8) *and* b *is given in* (7.10)+(7.11) *above.*

Proof. We first show that the pair $(\hat{V}, b)$ belongs to the class $\mathcal{C}$ and solves the free-boundary problem (8.1)-(8.3). For this, note that the optimal stopping problem (4.8) is Lagrange formulated so that standard arguments (see e.g. the final paragraph of Section 2 in [6] and recall the regularity facts on $\hat{V}$ stated following (7.25) above) imply that $\hat{V}$ belongs to $C^2(C)$ and satisfies (8.1). From (7.42) we know that $\hat{V}$ is continuous on $[0,\infty) \times I\!R$ and from (4.8) we readily find that

$$-\frac{1}{c} \leq \hat{V}(\varphi, x) \leq 0 \tag{8.6}$$

for all $(\varphi, x) \in [0,\infty) \times I\!R$. Clearly $\hat{V}$ satisfies (8.2) and from (7.43) above we see that $\hat{V}$ satisfies (8.3) and belongs to $C^1(\bar{C})$ as required in (8.4) above. The fact that b satisfies (8.5) was established in Corollary 7 and Proposition 9 above. This shows that $(\hat{V}, b)$ belongs to $\mathcal{C}$ and solves (8.1)-(8.3) as claimed. To derive uniqueness of the solution we will first see in the next section that any solution (U, a) to (8.1)-(8.3) from the class $\mathcal{C}$ admits an explicit integral representation for U expressed in terms of a, which in turn solves a nonlinear Fredholm integral equation, and we will see that this equation cannot have other solutions satisfying the required properties. From these facts we can conclude that the free-boundary problem (8.1)-(8.3) cannot have other solutions in the class $\mathcal{C}$ as claimed. □

9. Nonlinear integral equation

In this section we show that the optimal stopping boundary b from (7.10)+(7.11) can be characterised as the unique solution to a nonlinear Fredholm integral equation. This also yields an explicit integral representation of the value function $\hat{V}$ from (4.8) expressed in terms of the optimal stopping boundary b. As a consequence of the existence and uniqueness result for the nonlinear Fredholm integral equation we also obtain uniqueness of the solution to the free-boundary problem (8.1)-(8.3) as explained in the proof of Theorem 10 above. Finally, collecting the results derived throughout the paper we conclude our exposition by disclosing the solution to the initial problem.

1. To formulate the theorem below, let p denote the transition density function of the (time-homogeneous) Markov process $(\varPhi, X)$ under P^∞ in the sense that

$$\mathsf{P}^\infty_{\varphi,x}(\varPhi_t \leq \psi, X_t \leq y) = \int_0^\psi \int_{-\infty}^y p(t; \varphi, x; \eta, z)\, d\eta\, dz \tag{9.1}$$

for $t > 0$ with (φ, x) and (ψ, y) in $[0,\infty) \times I\!R$. The function p is characterised as the unique non-negative solution to the Kolmogorov backward equation

$$p_t(t; \varphi, x; \eta, z) = I\!L_{\varPhi,X}(p)(t; \varphi, x; \eta, z) \tag{9.2}$$

$$(9.3) \qquad p(0+;\varphi,x;\eta,z) = \delta_{(\varphi,x)}(\eta,z) \ \text{(weakly)}$$

satisfying $\int_0^\infty \int_{-\infty}^\infty p(t;\varphi,x;\eta,z)\,d\eta\,dz = 1$ for $t>0$ with (φ,x) and (η,z) in $[0,\infty)\times I\!R$ (cf. [14]) where we recall that $I\!L_{\Phi,X}$ is given in (5.3) above and $\delta_{(\varphi,x)}$ denotes the Dirac measure at (φ,x) . The initial value problem (9.2)+(9.3) can be used to determine p .

Having p we can then evaluate the expression of interest appearing in the statement of the theorem below as follows

$$(9.4) \qquad \int_0^\infty e^{-\lambda t}\,\mathsf{E}_{\varphi,x}^\infty\big[L(\Phi_t) I(\Phi_t < b(X_t))\big]\,dt = \int_0^\infty \int_{-\infty}^\infty \int_0^{b(y)} e^{-\lambda t} L(\psi)\, p(t;\varphi,x;\psi,y)\,d\psi\,dy\,dt$$

for (φ,x) in $[0,\infty)\times I\!R$ where we recall that $L(\varphi)=\varphi-\lambda/c$ for $\varphi\in[0,\infty)$ and b is given in (7.10)+(7.11) above. Recall that we assume throughout that $\gamma=0$ (due to $x_0=x_1=0$) and $\sigma=1$ so that (Φ,X) from (4.8) solves (5.1)+(5.2) above.

Theorem 11 (Existence and uniqueness). *The optimal stopping boundary b in the problem (4.8) can be characterised as the unique solution to the nonlinear integral equation*

$$(9.5) \qquad \int_0^\infty \int_{-\infty}^\infty \int_0^{b(y)} e^{-\lambda t}\big(\psi - \tfrac{\lambda}{c}\big)\, p(t;b(x),x;\psi,y)\,d\psi\,dy\,dt = 0$$

in the class of continuous functions $b: I\!R \rightarrow I\!R$ which are decreasing on $(-\infty,0]$, increasing on $[0,\infty)$ and satisfy $b(x)\ge \lambda/c$ for $x\in I\!R$. The value function $\hat{V}$ in the problem (4.8) admits the following representation

$$(9.6) \qquad \hat{V}(\varphi,x) = \int_0^\infty \int_{-\infty}^\infty \int_0^{b(y)} e^{-\lambda t}\big(\psi - \tfrac{\lambda}{c}\big)\, p(t;\varphi,x;\psi,y)\,d\psi\,dy\,dt$$

for (φ,x) in $[0,\infty)\times I\!R$. The optimal stopping time in the problem (4.8) is given by

$$(9.7) \qquad \tau_b = \inf\{\, t\ge 0 \mid \Phi_t \ge b(X_t)\,\}$$

under $\mathsf{P}_{\varphi,x}^\infty$ with (φ,x) in $[0,\infty)\times I\!R$ given and fixed (see Figures 1 and 2).

Proof. 1. *Existence.* We first show that the optimal stopping boundary b in problem (4.8) solves the nonlinear integral equation (9.5). Recalling that b satisfies (8.5), as established in Theorem 10 above, this will prove the existence of a solution to (9.5).

For this, we first show that Itô's formula is applicable to $\hat{V}$ composed with (Φ,X) . In view of the bijective C^2 transformation (5.12) combined with (7.14) above, for this it is enough to show that Itô's formula is applicable to $\check{V}$ composed with (V,Z) . The arguments developed in the first part of the proof of Theorem 19 in [20] show that the local time-space formula from [28, Theorem 2.1] is applicable to $\check{V}$ composed with (V,Z) , and because of the smooth fit conditions $\check{V}_v=\check{V}_z=0$ at the optimal stopping boundary for $\check{V}$, this formula reduces to classic Itô's formula. It follows therefore that Itô's formula is applicable to $\hat{V}$ composed with (Φ,X) as claimed.

Integrating by parts first and then applying Itô's formula we get

$$(9.8) \qquad e^{-\lambda t}\hat{V}(\Phi_t,X_t) = \hat{V}(\varphi,x) + \int_0^t e^{-\lambda s}\big(I\!L_{\Phi,X}\hat{V} - \lambda\hat{V}\big)(\Phi_s,X_s)\, I(\Phi_s \ne b(X_s))\,ds$$

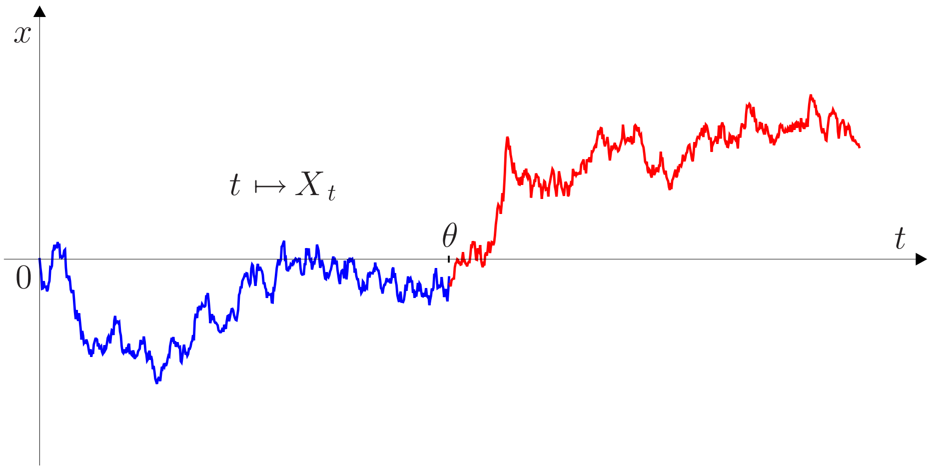

Figure 1. Simulated sample path of the Ornstein-Uhlenbeck process X solving (3.2)+(3.3) with $\lambda = 1$, $\beta_0 = 1$, $\beta_1 = 0$, $x_0 = 0$, $x_1 = 0$ and $\sigma = 1$.

$$+ \int_0^t e^{-\lambda s} \Big(\delta X_s \Phi_s \hat{V}_\varphi(\Phi_s, X_s) - \hat{V}_x(\Phi_s, X_s) \Big)\, dB_s$$
$$= \hat{V}(\varphi, x) - \int_0^t e^{-\lambda s} L(\Phi_s)\, I(\Phi_s < b(X_s))\, ds + M_t$$

under $\mathsf{P}^\infty_{\varphi,x}$ with $(\varphi, x) \in [0,\infty) \times I\!R$ given and fixed, where in the second equality we make use of (8.1) and (8.2), and $M_t = \int_0^t e^{-\lambda s}(\delta X_s \Phi_s \hat{V}_\varphi(\Phi_s, X_s) - \hat{V}_x(\Phi_s, X_s))\, dB_s$ is a continuous local martingale for $t \geq 0$. Choosing a localisation sequence $(\tau_n)_{n \geq 1}$ of stopping times for M and taking $\mathsf{E}^\infty_{\varphi,x}$ on both sides of (9.8) with τ_n in place of t we find by the optional sampling theorem that

$$\mathsf{E}^\infty_{\varphi,x}\big[e^{-\lambda \tau_n} \hat{V}(\Phi_{\tau_n}, X_{\tau_n})\big] = \hat{V}(\varphi, x) - \mathsf{E}^\infty_{\varphi,x}\bigg[\int_0^{\tau_n} e^{-\lambda s} L(\Phi_s)\, I(\Phi_s < b(X_s))\, ds\bigg] \tag{9.9}$$

for $n \geq 1$. From (8.6) we see that the left-hand side in (9.9) tends to zero as $n \to \infty$. Moreover, recalling that $L(\varphi) = \varphi - \lambda/c$ for $\varphi \in [0,\infty)$ we see that the integral in (9.9) equals the difference of two integrals with positive integrands where the expected value of the second integral converges to a finite value. Letting $n \to \infty$ in (9.9) and using the monotone convergence theorem we can therefore conclude that

$$\hat{V}(\varphi, x) = \int_0^\infty e^{-\lambda s}\, \mathsf{E}^\infty_{\varphi,x}\big[L(\Phi_s)\, I(\Phi_s < b(X_s))\big]\, ds \tag{9.10}$$

for $(\varphi, x) \in [0,\infty) \times I\!R$. Combining this expression with (9.4) we see that (9.6) holds as claimed. Setting $\varphi = b(x)$ in (9.6) for $x \in I\!R$ given and fixed, and using that $\hat{V}(b(x), x) = 0$, we see that b solves (9.5) and this completes the proof of existence.

2. *Uniqueness*. To show that b is a unique solution to the equation (9.5) in the specified class of functions, one can adopt the four-step procedure from the proof of uniqueness given in [8, Theorem 4.1] extending and further refining the original uniqueness arguments from [27,

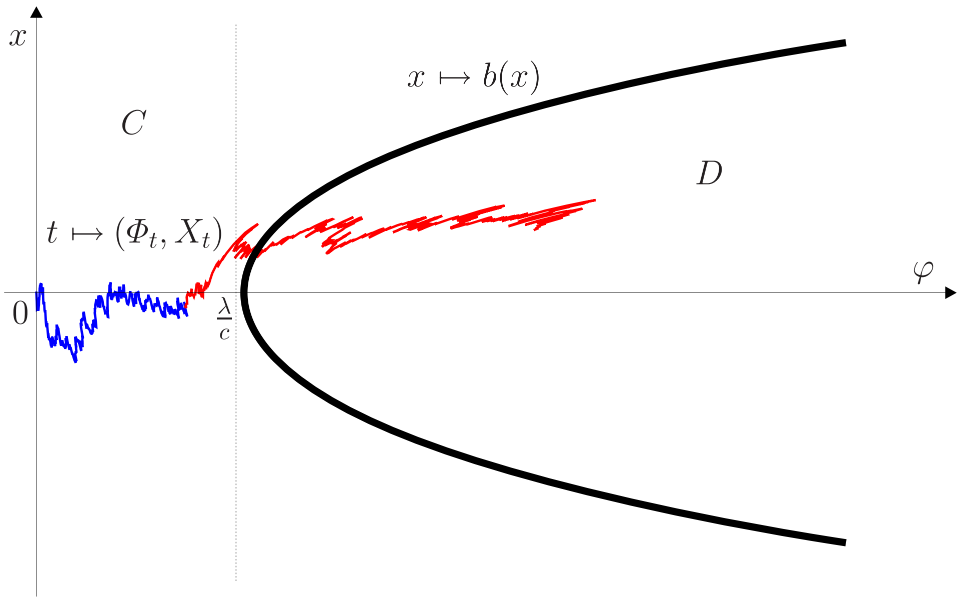

Figure 2. Kinematics of the process (Φ, X) associated with the sample path from Figure 1 and location of the optimal stopping boundary b when $\lambda = 1$ and $c = 1/2$.

Theorem 3.1]. Given that the present setting creates no additional difficulties we will omit further details of this verification (for fuller details see the uniqueness proof of Theorem 19 in [20]). This completes the proof. $\square$

The nonlinear Fredholm integral equation (9.5) can be used to find the optimal stopping boundary b numerically (using Picard iteration). Note that the integral over the entire real line in (9.4)-(9.6) equals twice the integral over the positive real line thanks to symmetry.

2. Collecting the results derived throughout the paper we now disclose the solution to the initial problem when $x_0 = x_1 = 0$ and $\sigma = 1$ as it can be assumed in Cases 1-10 (Section 3) above without loss of generality.

Corollary 12. *For any initial point* $x \in I\!R$ *of the process* X *solving* (3.2)+(3.3) *with* $x_0 = x_1 = 0$ *and* $\sigma = 1$ *, the value function of the initial problem* (3.4) *is given by*

$$V(\pi) = (1-\pi)\Big[1+c\hat{V}\big(\tfrac{\pi}{1-\pi}, x\big)\Big] \tag{9.11}$$

for $\pi \in [0,1]$ *where the function* $\hat{V}$ *is given by* (9.6) *above. The optimal stopping time in the initial problem* (3.4) *is given by*

$$\tau_* = \inf\Big\{\, t \geq 0 \;\Big|\; e^{(\lambda-\frac{\delta}{2})t+\frac{\delta}{2}(X_t^2-x^2+\kappa\int_0^t X_s^2\,ds)} \times \Big(\frac{\pi}{1-\pi} + \lambda\int_0^t e^{-(\lambda-\frac{\delta}{2})s-\frac{\delta}{2}(X_s^2-x^2+\kappa\int_0^s X_r^2\,dr)}\Big) \geq b(X_t)\,\Big\} \tag{9.12}$$

where $\delta = \beta_0 - \beta_1$, $\kappa = \beta_0 + \beta_1$ *and* b *is a unique solution to the integral equation* (9.5) *in* *the class of continuous functions* $b : I\!R \rightarrow I\!R$ *which are decreasing on* $(-\infty, 0]$ *, increasing on* $[0, \infty)$ *and satisfy* $b(x) \geq \lambda/c$ *for* $x \in I\!R$ *.*

Proof. The identity (9.11) follows by combining (4.6)+(4.7) with the result of Theorem 11 above. The explicit form (9.12) follows from (9.7) in Theorem 11 combined with (3.12) and (4.5) where the stochastic integral can be expressed by means of a deterministic integral using Itô's formula as in (5.25) above. This completes the proof. □

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