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Detecting the Presence of a Random Drift in Brownian Motion

P. Johnson, J. L. Pedersen, G. Peskir & C. Zucca

Consider a standard Brownian motion in one dimension, having either a zero drift, or a non-zero drift that is randomly distributed according to a known probability law. Following the motion in real time, the problem is to detect as soon as possible and with minimal probabilities of the wrong terminal decisions, whether a non-zero drift is present in the observed motion. We solve this problem for a class of admissible laws in the Bayesian formulation, under any prior probability of the non-zero drift being present in the motion, when the passage of time is penalised linearly.

1. Introduction

Imagine the motion of a Brownian particle in one dimension, having either a zero drift, or a non-zero drift μ that is randomly distributed according to a known probability law. Given that the position X of the Brownian particle is being observed in real time, the problem is to detect as soon as possible and with minimal probabilities of the wrong terminal decisions, whether a non-zero drift is present in the observed motion. The purpose of the present paper is to derive the solution to this problem in the Bayesian formulation, under any prior probability of the non-zero drift being present in the motion, when the passage of time is penalised linearly.

The loss to be minimised over sequential decision rules is expressed as a linear combination of the expected running time and the probabilities of the wrong terminal decisions. This problem formulation of sequential testing dates back to [26] (see [27], [15], [30]) and has been extensively studied to date (see [11] and the references therein). The linear combination represents the Lagrangian and once the optimisation problem has been solved in this form it will also lead to the solution of the constrained problem where upper bounds are imposed on the probabilities of the wrong terminal decisions. The constrained problem itself will not be considered in the present paper as this extension is somewhat lengthy and more routine.

Standard arguments show that the initial optimisation problem can be reduced to an optimal stopping problem for the posterior probability process Π of the drift being non-zero given X . A classic example of X is obtained when the non-zero drift μ is deterministic (see [16] and [23]). This problem has also been solved in finite horizon (see [10]). Books [24, Section 4.2] and [20, Section 21] contain expositions of these results and provide further details and references.

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Signal-to-noise ratio (i.e. the non-zero drift divided by the diffusion coefficient) is both constant and deterministic in these problems. Sequential testing problems for X in one dimension where the deterministic signal-to-noise ratio is not constant were studied more recently in [11] and [13]. In these problems Π is no longer Markovian, however, the process (Π, X) is a two-dimensional Markov/diffusion process. We will see below that the Markov/diffusion process (Π, X) is no longer time-homogeneous when the constant signal-to-noise ratio is random.

Another classic example of X is obtained when the non-zero drift μ takes either of the two specified values with strictly positive probabilities (see [25] for a discrete time analogue). One can then ask which of the three possible values for the drift (the third one being zero) is present in the observed process X . This problem has been studied more recently in [32] (see also [5] for a Poisson process analogue and [9] for a closely related problem in three dimensions). The Markov/diffusion process Π is two-dimensional in this case. Allowing further values for the non-zero drift μ to enter the scene, and asking the analogous question, increases the dimension of Π further and makes the problem closer to intractable. The problem under consideration in the present paper may be viewed as a finite/infinite-dimensional analogue of the latter problem (when the range of $\mu \neq 0$ is finite/infinite) where the complicated question of detecting the exact value of the drift μ is replaced by the simpler question of detecting whether a non-zero drift μ is present at all.

A closely related problem of detecting the sign of a random drift μ dates back to [3]. The wrong terminal decisions in that problem formulation are multiplied by the modulus of μ to account for its size. The probability law of μ in this problem formulation can also be one-sided (i.e. concentrated on either \mathbb{R}_- or \mathbb{R}_+) in which case the opposite sign is assigned to the zero value of μ . When μ is normally distributed this problem has been solved in [31]. The paper [8] studies the analogous problem for a general probability law of μ in the canonical formulation where the wrong terminal decisions are no longer multiplied by the modulus of μ . The method of proof in [8] makes an essential use of the innovation process associated with X .

In the present paper we abandon the innovation process and apply a measure change instead. Among other things this enables us to settle the uniqueness question which was left open in [8] (see Remark 13 below). We focus on the laws of μ for which Π itself is a time-inhomogeneous Markov process. A simple time change then reduces the posterior probability ratio process $\Phi := \Pi/(1-\Pi)$ to a standard Brownian motion process whose initial points are thus expressible explicitly. Exploiting the latter fact we show that the optimal stopping boundaries for Φ are monotone functions of time whenever the ratio between the first and second spatial derivative of the likelihood ratio function is a monotone function of space. The monotone optimal stopping boundaries then make the optimal stopping problem tractable using established techniques. We show that the sufficient condition on the ratio between the first and second spatial derivative is satisfied whenever the probability law of μ is either (i) one-sided (i.e. concentrated on either \mathbb{R}_- or \mathbb{R}_+) or (ii) two-point spatially symmetric (i.e. concentrated on $-m$ and m for m in $(0, \infty)$) among other possibilities. More general two-sided probability laws of μ can fail to satisfy the sufficient condition and their closer examination is left for future research.

2. Formulation of the problem

In this section we formulate the sequential testing problem under consideration. The initial formulation of the problem will be reevaluated under a change of measure in the next section.

1. We consider a Bayesian formulation of the problem where it is assumed that one observes a sample path of the standard Brownian motion X , having either a zero drift with the prior probability $1-\pi$, or a non-zero random drift μ with the prior probability π in $[0, 1]$. The problem is to detect, as soon as possible and with minimal probabilities of the wrong terminal decisions, whether a non-zero drift is present in the observed motion. This problem belongs to the class of sequential testing problems as discussed in Section 1 above.

2. Standard arguments imply that the previous setting can be realised on a probability space $(\Omega, \mathcal{F}, \mathbb{P}_\pi)$ with the probability measure \mathbb{P}_π decomposed as follows

$$(2.1) \quad \mathbb{P}_\pi = (1-\pi)\mathbb{P}_0 + \pi\mathbb{P}_1$$

for $\pi \in [0, 1]$, where \mathbb{P}_0 is the probability measure under which the observed process X has a zero drift, and \mathbb{P}_1 is the probability measure under which the observed process X has a non-zero random drift μ . This can be formally achieved by introducing an unobservable random variable μ taking value zero with probability $1-\pi$ and taking non-zero values with probabilities determined by πF_μ , where $F_\mu : \mathbb{R} \rightarrow \mathbb{R}$ is a probability distribution function that is continuous at 0, and assuming that X solves the stochastic differential equation

$$(2.2) \quad dX_t = \mu dt + dB_t$$

driven by a standard Brownian motion B that is independent from μ under \mathbb{P}_π for $\pi \in [0, 1]$. Note that without loss of generality we may assume that X starts at zero in (2.2).

3. Being based upon the continued observation of X , the problem is to test sequentially the hypotheses $H_0 : \mu = 0$ and $H_1 : \mu \neq 0$ with a minimal loss. For this, we are given a sequential decision rule (τ, d_τ) , where τ is a stopping time of X (i.e. a stopping time with respect to the natural filtration $\mathcal{F}_t^X = \sigma(X_s | 0 \leq s \leq t)$ of X for $t \geq 0$), and d_τ is an \mathcal{F}_τ^X -measurable random variable taking values 0 and 1. After stopping the observation of X at time τ , the terminal decision function d_τ takes value i if and only if the hypothesis H_i is to be accepted for $i = 0, 1$. With constants $a > 0$ and $b > 0$ given and fixed, the problem then becomes to compute the risk function

$$(2.3) \quad V(\pi) = \inf_{(\tau, d_\tau)} \mathbb{E}_\pi[\tau + aI(d_\tau = 0, \mu \neq 0) + bI(d_\tau = 1, \mu = 0)]$$

for $\pi \in [0, 1]$ and find the optimal decision rule $(\tau_*, d_{\tau_*}^*)$ at which the infimum in (2.3) is attained. Note that $\mathbb{E}_\pi(\tau)$ in (2.3) is the expected waiting time until the terminal decision is made, and $\mathbb{P}_\pi(d_\tau = 0, \mu \neq 0)$ and $\mathbb{P}_\pi(d_\tau = 1, \mu = 0)$ in (2.3) are probabilities of the wrong terminal decisions respectively.

4. To tackle the sequential testing problem (2.3) we consider the *posterior probability process* $\Pi = (\Pi_t)_{t \geq 0}$ of H_1 given X that is defined by

$$(2.4) \quad \Pi_t = \mathbb{P}_\pi(\mu \neq 0 | \mathcal{F}_t^X)$$

for $t \geq 0$. Noting that $\mathbb{P}_\pi(d_\tau = 0, \mu \neq 0) = \mathbb{E}_\pi[(1-d_\tau)\Pi_\tau]$ and $\mathbb{P}_\pi(d_\tau = 1, \mu = 0) = \mathbb{E}_\pi[d_\tau(1-\Pi_\tau)]$, and defining $\tilde{d}_\tau = I(a\Pi_\tau \geq b(1-\Pi_\tau))$ for any given (τ, d_τ) , it is easily seen that the initial problem (2.3) is equivalent to the optimal stopping problem

$$(2.5) \quad V(\pi) = \inf_{\tau} \mathbb{E}_\pi[\tau + M(\Pi_\tau)]$$

where the infimum is taken over all stopping times τ of X and $M(\pi) = a\pi \wedge b(1-\pi)$ for $\pi \in [0, 1]$. Letting τ_* denote the optimal stopping time in (2.5), and setting $c = b/(a+b)$, these arguments also show that the optimal decision function in (2.3) is given by $d_{\tau_*}^* = 0$ if $\Pi_{\tau_*} < c$ and $d_{\tau_*}^* = 1$ if $\Pi_{\tau_*} \geq c$. Thus to solve the initial problem (2.3) it is sufficient to solve the optimal stopping problem (2.5) and this is what we do in the sequel.

3. Measure change

In this section we show that changing the probability measure \mathbb{P}_π to \mathbb{P}_0 for $\pi \in (0, 1)$ provides important simplifications of the setting which make the subsequent analysis of the optimal stopping problem (2.5) more transparent. The change of measure argument is presented in Lemma 1 below. This is then followed by a reformulation of the optimal stopping problem (2.5) under the new probability measure \mathbb{P}_0 in Proposition 2 below.

1. To connect the process Π to the observed process X , we see from (2.2) that the *Kallianpur-Striebel formula* (see Theorem 3 and its Corollary in [14]) yields

$$(3.1) \quad \mathbb{E}_\pi(G(\mu) \mid \mathcal{F}_t^X) = \frac{(1-\pi)G(0) + \pi \int_{-\infty}^{\infty} G(m) e^{mX_t - \frac{m^2}{2}t} F_\mu(dm)}{(1-\pi) + \pi \int_{-\infty}^{\infty} e^{mX_t - \frac{m^2}{2}t} F_\mu(dm)}$$

whenever $G : \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function such that $\mathbb{E}_\pi|G(\mu)| < \infty$ for $\pi \in [0, 1]$. Taking $G = 1_{\mathbb{R} \setminus \{0\}}$ in (3.1) we see from (2.4) that

$$(3.2) \quad \Pi_t = \frac{\pi \int_{-\infty}^{\infty} e^{mX_t - \frac{m^2}{2}t} F_\mu(dm)}{(1-\pi) + \pi \int_{-\infty}^{\infty} e^{mX_t - \frac{m^2}{2}t} F_\mu(dm)}$$

for $t \geq 0$ and $\pi \in [0, 1]$. Embedded in the right-hand side of (3.2) we recognise the *likelihood ratio process* $L = (L_t)_{t \geq 0}$ given by

$$(3.3) \quad L_t := \frac{d\mathbb{P}_{1,t}}{d\mathbb{P}_{0,t}} = \ell(t, X_t) = \int_{-\infty}^{\infty} e^{mX_t - \frac{m^2}{2}t} F_\mu(dm)$$

where $\mathbb{P}_{0,t}$ and $\mathbb{P}_{1,t}$ denote the restrictions of the probability measures \mathbb{P}_0 and \mathbb{P}_1 to \mathcal{F}_t^X for $t \geq 0$, and the function $\ell : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$(3.4) \quad \ell(t, x) = \int_{-\infty}^{\infty} e^{mx - \frac{m^2}{2}t} F_\mu(dm)$$

for $(t, x) \in (0, \infty) \times \mathbb{R}$ with $\ell(0, 0) = 1$. Note that ℓ is a general non-negative solution to the *backward heat equation*, i.e. we have

$$(3.5) \quad \ell_t + \frac{1}{2}\ell_{xx} = 0$$

on $[0, \infty) \times \mathbb{R}$ (cf. [29], [21]). From (3.2) and (3.3) we thus see that

$$(3.6) \quad \Pi_t = \frac{\frac{\pi}{1-\pi} L_t}{1 + \frac{\pi}{1-\pi} L_t} = \frac{\Phi_t}{1 + \Phi_t}$$

where $\Phi = (\Phi_t)_{t \geq 0}$ is the *posterior probability ratio process* given by

$$(3.7) \quad \Phi_t := \frac{\Pi_t}{1 - \Pi_t} = \Phi_0 L_t$$

for $t \geq 0$ with $\Phi_0 = \pi/(1-\pi)$ for $\pi \in [0, 1)$.

2. To derive a stochastic differential equation for the process Φ , we may apply Itô's formula in (3.3) and use (3.5) to find that

$$(3.8) \quad d\Phi_t = \Phi_0 \ell_x(t, X_t) dX_t = \frac{\ell_x(t, X_t)}{\ell(t, X_t)} \Phi_t dX_t$$

with $\Phi_0 = \pi/(1-\pi)$ for $\pi \in [0, 1)$. From (3.1) and (3.4) above we recognise the ratio on the right-hand side of (3.8) as the *mean-square predictor* of μ given X under \mathbb{P}_1 defined by

$$(3.9) \quad \mathbb{E}_1(\mu | \mathcal{F}_t^X) =: \hat{\mu}(t, X_t)$$

for $t \geq 0$ where the function $\hat{\mu} : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$(3.10) \quad \hat{\mu}(t, x) = \frac{\ell_x(t, x)}{\ell(t, x)} = \frac{\int_{-\infty}^{\infty} m e^{mx - \frac{m^2}{2}t} F_{\mu}(dm)}{\int_{-\infty}^{\infty} e^{mx - \frac{m^2}{2}t} F_{\mu}(dm)}$$

for $(t, x) \in [0, \infty) \times \mathbb{R}$. Recalling that ℓ solves the backward heat equation (3.5) we see that $\hat{\mu}$ coincides with the Hopf-Cole solution (cf. [12], [4]) to the *backward Burgers equation*

$$(3.11) \quad \hat{\mu}_t + \hat{\mu} \hat{\mu}_x + \frac{1}{2} \hat{\mu}_{xx} = 0$$

on $[0, \infty) \times \mathbb{R}$ (cf. [1], [2]). Using (3.9)+(3.10) in (3.8) we see that Φ solves the following stochastic differential equation

$$(3.12) \quad d\Phi_t = \hat{\mu}(t, X_t) \Phi_t dX_t$$

with $\Phi_0 = \pi/(1-\pi)$ for $\pi \in [0, 1)$.

3. Recalling that the *innovation process* defined by

$$(3.13) \quad \bar{B}_t := X_t - \int_0^t \mathbb{E}_{\pi}(\mu | \mathcal{F}_s^X) ds$$

is a standard Brownian motion under \mathbb{P}_{π} (which is easily verified using Lévy's characterisation theorem), substituting (3.13) in (3.8) and making use of (3.1), we see that the resulting (time-dependent) system of stochastic differential equations for Φ and X driven by \bar{B} has a unique

weak solution (cf. [22, pp 166–173]). Hence we can conclude that $((t, \Phi_t, X_t))_{t \geq 0}$ is a (strong) Markov process under \mathbb{P}_π for $\pi \in [0, 1)$ (cf. [22, pp 158–163]). We will see below that although $((t, \Phi_t))_{t \geq 0}$ can be a Markov process on its own for a large class of distribution functions F_μ , this is not true in general without some knowledge of X . The probability measure \mathbb{P}_0 appears to be especially appealing in this context because X itself is a standard Brownian motion (with no drift) under \mathbb{P}_0 . This motivates us to change the probability measure \mathbb{P}_π to \mathbb{P}_0 for $\pi \in (0, 1)$ in the setting of the optimal stopping problem (2.5).

4. In the sequel we let $\mathbb{P}_{\pi, \tau}$ denote the restriction of the measure \mathbb{P}_π to \mathcal{F}_τ^X for $\pi \in [0, 1]$ where τ is a stopping time of X .

Lemma 1. *The following identity holds*

$$(3.14) \quad \frac{d\mathbb{P}_{\pi, \tau}}{d\mathbb{P}_{0, \tau}} = \frac{1 - \pi}{1 - \Pi_\tau}$$

for all stopping times τ of X and all $\pi \in [0, 1)$.

Proof. A standard rule for the Radon-Nikodym derivatives based on (2.1) gives

$$(3.15) \quad \begin{aligned} 1 - \Pi_\tau &= \mathbb{P}_\pi(\mu=0 | \mathcal{F}_\tau^X) \\ &= (1 - \pi) \mathbb{P}_0(\mu=0 | \mathcal{F}_\tau^X) \frac{d\mathbb{P}_{0, \tau}}{d\mathbb{P}_{\pi, \tau}} + \pi \mathbb{P}_1(\mu=0 | \mathcal{F}_\tau^X) \frac{d\mathbb{P}_{1, \tau}}{d\mathbb{P}_{\pi, \tau}} \\ &= (1 - \pi) \frac{d\mathbb{P}_{0, \tau}}{d\mathbb{P}_{\pi, \tau}} \end{aligned}$$

for all τ and π as above due to $\mathbb{P}_0(\mu=0) = 1$ and $\mathbb{P}_1(\mu=0) = 0$. This shows that (3.14) is satisfied as claimed. \square

Similarly to (3.14) and (3.15) we find that

$$(3.16) \quad \frac{d\mathbb{P}_{\pi, \tau}}{d\mathbb{P}_{1, \tau}} = \frac{\pi}{\Pi_\tau}$$

which together with (3.14) implies that

$$(3.17) \quad L_\tau := \frac{d\mathbb{P}_{1, \tau}}{d\mathbb{P}_{0, \tau}} = \frac{1 - \pi}{\pi} \frac{\Pi_\tau}{1 - \Pi_\tau}$$

for all stopping times τ of X and all $\pi \in [0, 1)$. Note that the second identity in (3.17) is equivalent to the first identity in (3.6) above.

5. We now show that the optimal stopping problem (2.5) admits a transparent reformulation under the measure \mathbb{P}_0 in terms of the process Φ defined by (3.7) and solving (3.12) above. Recall that Φ starts at $\Phi_0 = \pi/(1 - \pi)$ and this dependence on the initial point will be indicated by a superscript to Φ when needed.

Proposition 2. *The value function V from (2.5) satisfies the identity*

$$(3.18) \quad V(\pi) = (1 - \pi) \hat{V}(\pi)$$

where the value function \hat{V} is given by

$$(3.19) \quad \hat{V}(\pi) = \inf_{\tau} \mathbf{E}_0 \left[\tau (1 + \Phi_{\tau}^{\pi/(1-\pi)}) + \hat{M}(\Phi_{\tau}^{\pi/(1-\pi)}) \right]$$

for $\pi \in [0, 1)$ with $\hat{M}(\varphi) = a\varphi \wedge b$ for $\varphi \in \mathbb{R}_+$ and the infimum in (3.19) is taken over all stopping times τ of X .

Proof. With π and τ as above, we find by (3.14) in Lemma 1 that

$$(3.20) \quad \mathbf{E}_{\pi} [\tau + M(\Pi_{\tau})] = (1-\pi) \mathbf{E}_0 \left[\frac{\tau}{1-\Pi_{\tau}} + \frac{M(\Pi_{\tau})}{1-\Pi_{\tau}} \right] = (1-\pi) \mathbf{E}_0 [\tau(1+\Phi_{\tau}) + \hat{M}(\Phi_{\tau})]$$

where in the final equality we use (3.7) above. Taking the infimum over all τ on both sides of (3.20), we obtain (3.19) as claimed and the proof is complete. \square

4. Admissible laws

In the previous section we have reduced the initial sequential testing problem (2.3) to the optimal stopping problem (3.19). In this section we will describe the class of admissible probability laws of μ for which the latter optimal stopping problem is solvable using a simple time change technique.

1. To tackle the optimal stopping problem (3.19) we need to enable the underlying Markov process to start at arbitrary points in its state space under \mathbf{P}_0 . We could consider (3.19) as an optimal stopping problem for the two-dimensional Markov process $((t, X_t))_{t \geq 0}$ upon recalling that X coincides with the standard Brownian motion B under \mathbf{P}_0 but this would only give a solution for a single prior π in $[0, 1)$. Instead, to exploit the natural grouping of the time and space variables, we will consider (3.19) as an optimal stopping problem for the three-dimensional Markov process $((t, \Phi_t, X_t))_{t \geq 0}$ which is reducible to a two-dimensional Markov process $((t, \Phi_t))_{t \geq 0}$ under \mathbf{P}_0 when the probability law of μ satisfies additional conditions. For this, recall from (3.3)+(3.7) that

$$(4.1) \quad \Phi_t = \Phi_0 \ell(t, X_t)$$

for $t \geq 0$ where X is a standard Brownian motion under \mathbf{P}_0 . Noting that

$$(4.2) \quad x \mapsto \ell(t, x) \text{ is strictly convex on } \mathbb{R}$$

for all $t \geq 0$, it is well known (cf. [28, p. 516]) that Φ of the form (4.1) defines a (time-inhomogeneous) Markov process if and only if either

$$(4.3) \quad x \mapsto \ell(t, x) \text{ is strictly increasing or decreasing on } \mathbb{R}$$

for all $t \geq 0$, or the following identity holds

$$(4.4) \quad \ell(t, x) = k(t, |x - z|)$$

for all $(t, x) \in [0, \infty) \times \mathbb{R}_+$ with some continuous function $k : [0, \infty) \times \mathbb{R}_+ \rightarrow \mathbb{R}$ and $z \in \mathbb{R}$ such that $y \mapsto k(t, y)$ is strictly increasing on \mathbb{R}_+ for all $t \geq 0$. From (4.2) we see that z

in (4.4) is the (unique) point at which the infimum of $x \mapsto \ell(t, x)$ is attained on \mathbb{R} and the function $x \mapsto \ell(t, x)$ is symmetric around z for every $t \geq 0$ given and fixed. Note that (4.3) is satisfied if and only if the probability law of μ is one-sided (i.e. concentrated on either \mathbb{R}_- or \mathbb{R}_+) and (4.4) is satisfied (with $z = 0$) if the probability law of μ is two-sided and symmetric (around zero). More generally, it is easily verified that (4.4) is satisfied if the probability density function f_μ of μ is expressible as $f_\mu(m) = e^{-zm}g_\mu(m)$ for $m \in \mathbb{R}$ and some $z \in \mathbb{R}$ where $g_\mu : \mathbb{R} \rightarrow \mathbb{R}$ is an even function. For example, this is true if $\mu \sim N(\nu, \sigma^2)$ with $\nu \in \mathbb{R}$ and $\sigma^2 > 0$. Thus, if either (4.3) or (4.4) holds, then $((t, \Phi_t))_{t \geq 0}$ is a two-dimensional Markov process under \mathbb{P}_0 and the optimal stopping problem (3.19) extends as follows

$$(4.5) \quad \hat{V}(t, \varphi) = \inf_{\tau} \mathbf{E}_{t, \varphi}^0 [\tau(1 + \Phi_{t+\tau}) + \hat{M}(\Phi_{t+\tau})]$$

for $(t, \varphi) \in [0, \infty) \times \mathbb{R}_+$ with $\mathbf{P}_{t, \varphi}^0(\Phi_t = \varphi) = 1$, where the infimum in (4.5) is taken over all stopping times τ of Φ , and we move 0 from the subscript to a superscript for notational reasons. In this way we have reduced the initial sequential testing problem (2.3) to the optimal stopping problem (4.5) for the time-inhomogeneous Markov process Φ defined in (3.7) and solving (3.12) under $\mathbf{P}_{t, \varphi}^0$ for $(t, \varphi) \in [0, \infty) \times \mathbb{R}_+$.

2. In addition to either (4.3) or (4.4) being satisfied, we will also assume throughout that one of the following two conditions is satisfied

$$(4.6) \quad x \mapsto \frac{\ell_x(t, x)}{\ell_{xx}(t, x)} \text{ is decreasing on } \mathbb{R}$$

$$(4.7) \quad x \mapsto \frac{\ell_x(t, x)}{\ell_{xx}(t, x)} \text{ is increasing on } \mathbb{R}$$

for all $t \geq 0$. We now show that these conditions are equivalent to the fact that the diffusion coefficient squared in the stochastic differential equation (3.12) for Φ is a monotone function of time. This in turn will imply that the optimal stopping boundaries in (4.5) are monotone functions of time as it will be shown below.

Let us focus on the first identity in (3.8), which is equivalent to (3.12), and let us assume that (4.3) is satisfied. Note that focusing on the second identity in (3.8) instead, or assuming that (4.4) is satisfied, would lead to exactly the same conclusions and only the notation would be somewhat more complicated. Note that if Φ solves (3.8)/(3.12) with $\Phi_0 = 1$ then $\Phi^\varphi := \varphi\Phi$ solves (3.8)/(3.12) with $\Phi_0 = \varphi$ for $\varphi \in \mathbb{R}_+$. Hence there is no loss of generality in assuming that $\Phi_0 = 1$ in what follows. Denoting the inverse function of $x \mapsto \ell(t, x)$ by $\varphi \mapsto \ell^{-1}(t, \varphi)$ for $t \geq 0$ given and fixed, we see that the diffusion coefficient in the stochastic differential equation (3.8) is given by

$$(4.8) \quad \sigma(t, \varphi) := \ell_x(t, \ell^{-1}(t, \varphi))$$

for $(t, \varphi) \in [0, \infty) \times \mathbb{R}_+$. Note that when (4.4) is satisfied then $\varphi \mapsto \ell^{-1}(t, \varphi)$ denotes the inverse function of $x \mapsto \ell(t, x) = k(t, x - z)$ for $x \geq z$. Since the arguments used in the proofs are analogous we will only focus on the case when (4.3) is satisfied in the sequel.

Proposition 3. *The mapping $t \mapsto \sigma^2(t, \varphi)$ is decreasing or increasing on $[0, \infty)$ for every $\varphi \in \mathbb{R}_+$ if and only if (4.6) or (4.7) holds respectively.*

Proof. Omitting the function arguments for simplicity we see from (4.8) that

$$(4.9) \quad (\sigma^2)_t = 2\sigma(\ell_{xt} + \ell_{xx}(\ell^{-1})_t).$$

Since $\ell(t, \ell^{-1}(t, \varphi)) = \varphi$ we see by differentiating with respect to t that $\ell_t + \ell_x(\ell^{-1})_t = 0$ from where we find that $(\ell^{-1})_t = -\ell_t/\ell_x$ upon recalling that $\ell_x > 0$ (strictly above z when (4.4) holds). Inserting this expression back into (4.9) we obtain

$$(4.10) \quad (\sigma^2)_t = 2\sigma(\ell_x \ell_{xt} - \ell_{xx} \ell_t)/\ell_x.$$

Noticing that the right-hand side of (4.10) has a familiar differential form, this shows that

$$(4.11) \quad \text{sign}((\sigma^2)_t) = -\text{sign}((\ell_x/\ell_t)_x) = \text{sign}((\ell_x/\ell_{xx})_x)$$

where in the final identity we use that $\ell_t = -\frac{1}{2}\ell_{xx}$ by (3.5) above. Both equivalence claims now follow directly from (4.11) and the proof is complete. \square

Definition 4 (Admissible laws). The probability law of a random drift μ is said to be *admissible*, if (i) either of the conditions (4.3) and (4.4) holds and (ii) either of the conditions (4.6) and (4.7) holds.

If the probability law associated with the probability distribution function F_μ of a random drift μ is admissible, we will also say that F_μ or μ itself is admissible. The following proposition shows that the family of admissible laws is sufficiently large to be of theoretical and practical interest.

Proposition 5.

(4.12) *If μ is one-sided (i.e. concentrated on either \mathbb{R}_- or \mathbb{R}_+), then μ is admissible.*

(4.13) *If μ takes values $-m$ and m with probabilities $1-p$ and p respectively for some $m > 0$ and $p \in (0, 1)$, then μ is admissible.*

Proof. (4.12): If μ is one-sided, then clearly (4.3) is satisfied. Moreover, we claim that (4.6) holds when μ is one-sided. For this, with $(t, x) \in [0, \infty) \times \mathbb{R}$ given and fixed, note that

$$(4.14) \quad Q(A) = \frac{\int_{-\infty}^{\infty} 1_A(m) e^{mx - \frac{m^2}{2}t} F_\mu(dm)}{\int_{-\infty}^{\infty} e^{mx - \frac{m^2}{2}t} F_\mu(dm)} \quad (A \in \mathcal{B}(\mathbb{R}))$$

defines a probability measure on the Borel σ -algebra of \mathbb{R} . Note also that the identity mapping M defined by $M(m) = m$ for $m \in \mathbb{R}$ is a random variable on the probability space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), Q)$ and we will denote the expectation of M^p with respect to Q by $\mathbf{E}M^p$ for $p > 0$. From (4.10) and (4.11) we see that

$$(4.15) \quad \text{sign}((\ell_x/\ell_{xx})_x) = \text{sign}(\ell_x \ell_{xt} - \ell_{xx} \ell_t)$$

where the function argument is omitted for simplicity. Differentiating under the integral sign in (3.4), which is justifiable by standard means, we find that

$$(4.16) \quad \ell_x \ell_{xt} - \ell_{xx} \ell_t = -\frac{1}{2} \mathbf{E} M \mathbf{E} M^3 + \frac{1}{2} (\mathbf{E} M^2)^2.$$

From (4.15) and (4.16) we see that establishing (4.6) is equivalent to showing that

$$(4.17) \quad (\mathbf{E} M^2)^2 \leq \mathbf{E} M \mathbf{E} M^3.$$

To show that (4.17) holds, let us first assume that μ is concentrated on \mathbb{R}_+ . Then $M \geq 0$ and we can therefore define a probability measure \mathbf{P}_M by setting

$$(4.18) \quad \mathbf{P}_M(A) = \frac{1}{\mathbf{E} M} \int_A M d\mathbf{P}$$

for $A \in \mathcal{F}$. Hence by Jensen's inequality we find that

$$(4.19) \quad \frac{\mathbf{E} M^2}{\mathbf{E} M} = \mathbf{E}_M(M) \leq (\mathbf{E}_M(M^2))^{1/2} = \left(\frac{1}{\mathbf{E} M} \mathbf{E} M^3 \right)^{1/2}$$

where \mathbf{E}_M denotes the expectation with respect to \mathbf{P}_M . Squaring both sides in (4.19) we see that (4.17) holds and hence we can conclude that (4.6) is satisfied as claimed. If μ is concentrated on \mathbb{R}_- then $M \leq 0$ and replacing M by $-M$ in (4.17) with the inequality unchanged, we see that the first part of the proof above when $-M \geq 0$ implies that (4.6) is satisfied in this case as well.

(4.13): Firstly, we claim that (4.4) is satisfied in this case. Indeed, using (3.4) we find that

$$(4.20) \quad \ell(t, x) = e^{-\frac{m^2}{2}t} [(1-p)e^{-mx} + pe^{mx}]$$

for $(t, x) \in [0, \infty) \times \mathbb{R}$. It is easily verified that the function $x \mapsto (1-p)e^{-mx} + pe^{mx}$ attains its infimum at $z = (1/2m) \log((1-p)/p)$ on \mathbb{R} and that $\ell(t, z+x) = \ell(t, z-x)$ for all $x \in \mathbb{R}_+$ with $t \geq 0$ given and fixed. Setting $k(t, x) := \ell(t, z+x)$ for $x \in \mathbb{R}_+$ with $t \geq 0$ given and fixed, this shows that (4.4) holds as claimed. Secondly, we claim that (4.7) holds in this case. For this, differentiating in (4.20) we find that

$$(4.21) \quad \begin{aligned} \frac{\ell_x(t, x)}{\ell_{xx}(t, x)} &= \frac{-(1-p)m e^{-mx} + pm e^{mx}}{(1-p)m^2 e^{-mx} + pm^2 e^{mx}} \\ &= \frac{1}{m} \left(\frac{pe^{2mx} - (1-p)}{pe^{2mx} + (1-p)} \right) = \frac{1}{m} \left(1 - \frac{2(1-p)}{pe^{2mx} + (1-p)} \right) \end{aligned}$$

which clearly is increasing in x belonging to \mathbb{R} for $t \geq 0$ so that (4.7) holds as claimed and the proof is complete. \square

5. Lagrange formulation

The optimal stopping problem (4.5) is Mayer formulated. In this section we derive its Lagrange reformulation (see [20, Section 6] for the terminology) which is helpful in the subsequent analysis of the problem.

Proposition 6. *The value function \hat{V} from (4.5) can be expressed as*

$$(5.1) \quad \hat{V}(t, \varphi) = \inf_{\tau} \mathbf{E}_{t, \varphi}^0 \left[\int_0^{\tau} (1 + \Phi_{t+s}) ds - \frac{a}{2} \ell_{\tau}^{b/a}(\Phi) \right] + \hat{M}(\varphi)$$

for $(t, \varphi) \in [0, \infty) \times \mathbb{R}_+$ where $\ell_{\tau}^{b/a}(\Phi)$ is the local time of Φ at b/a and τ given by

$$(5.2) \quad \ell_{\tau}^{b/a}(\Phi) = \mathbf{P}\text{-}\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^{\tau} I\left(\frac{b}{a} - \varepsilon \leq \Phi_{t+s} \leq \frac{b}{a} + \varepsilon\right) d\langle \Phi, \Phi \rangle_s$$

and the infimum in (5.1) is taken over all stopping times τ of Φ .

Proof. Integration by parts gives

$$(5.3) \quad s\Phi_{t+s} = \int_0^s \Phi_{t+r} dr + \int_0^s r d\Phi_{t+r}$$

for $s \geq 0$ where the final term defines a continuous local martingale under $\mathbf{P}_{t, \varphi}^0$ in view of (3.12) above for $(t, \varphi) \in [0, \infty) \times \mathbb{R}_+$ given and fixed. Making use of a localising sequence of stopping times for this local martingale if needed, and applying the optional sampling theorem, we find from (5.3) that

$$(5.4) \quad \mathbf{E}_{t, \varphi}^0 [\tau \Phi_{t+\tau}] = \mathbf{E}_{t, \varphi}^0 \left[\int_0^{\tau} \Phi_{t+s} ds \right]$$

for any (bounded) stopping time τ of Φ . Moreover, noting that $\varphi \mapsto \hat{M}(\varphi) = a\varphi \wedge b$ is a concave function on \mathbb{R}_+ with $\hat{M}'(d\varphi) = -a\delta_{b/a}(d\varphi)$ where $\delta_{b/a}$ is the Dirac measure at b/a , we find by the Itô-Tanaka formula using (3.12) that

$$(5.5) \quad \begin{aligned} \hat{M}(\Phi_{t+s}) &= \hat{M}(\varphi) + \int_0^s \hat{M}'_{\pm}(\Phi_{t+r}) d\Phi_{t+r} + \frac{1}{2} \int_{-\infty}^{\infty} \ell_s^{\psi}(\Phi) \hat{M}'(d\psi) \\ &= \hat{M}(\varphi) + \int_0^s \hat{M}'_{\pm}(\Phi_{t+r}) \hat{\mu}(t+r, X_{t+r}) \Phi_{t+r} dX_{t+r} - \frac{a}{2} \ell_s^{b/a}(\Phi) \end{aligned}$$

for $s \geq 0$ where the second term on the right-hand side defines a continuous local martingale under $\mathbf{P}_{t, \varphi}^0$. Making use of a localising sequence of stopping times for this local martingale if needed, and applying the optional sampling theorem, we find from (5.5) that

$$(5.6) \quad \mathbf{E}_{t, \varphi}^0 [\hat{M}(\Phi_{t+\tau})] = \hat{M}(\varphi) - \frac{a}{2} \mathbf{E}_{t, \varphi}^0 [\ell_{\tau}^{b/a}(\Phi)]$$

for any (bounded) stopping time τ of Φ . Inserting the right-hand sides of (5.4) and (5.6) into (4.5) we obtain (5.1) as claimed and the proof is complete. \square

The Lagrange reformulation (5.1) of the optimal stopping problem (4.5) reveals the underlying rationale for continuing vs stopping in a clearer manner. Indeed, recalling that the local time process $s \mapsto \ell_{t+s}^{b/a}(\Phi)$ strictly increases only when Φ_{t+s} is at b/a , and that $\ell_{t+s}^{b/a}(\Phi) \sim \sqrt{s}$ is strictly larger than $\int_0^s (1 + \Phi_{t+r}) dr \sim s$ for small s , we see from (5.1) that it should never be optimal to stop at $\varphi = b/a$ and the incentive for stopping should increase the further away Φ_{t+s} gets from b/a for $s \geq 0$. We will see in the next section that these informal conjectures can be formalised and this will give a proof of the fact the straight line $\{(t, \varphi) \in [0, \infty) \times \mathbb{R}_+ \mid \varphi = b/a\}$ is contained in the continuation set of the optimal stopping problem (4.5).

6. Properties of the optimal stopping boundaries

In this section we establish the existence of an optimal stopping time in (4.5) and derive basic properties of the optimal stopping boundaries for admissible laws (cf. Definition 4).

1. Looking at (4.5) we may conclude that the (candidate) continuation and stopping sets in this problem need to be defined as follows

$$(6.1) \quad C = \{ (t, \varphi) \in [0, \infty) \times \mathbb{R}_+ \mid \hat{V}(t, \varphi) < \hat{M}(\varphi) \}$$

$$(6.2) \quad D = \{ (t, \varphi) \in [0, \infty) \times \mathbb{R}_+ \mid \hat{V}(t, \varphi) = \hat{M}(\varphi) \}$$

respectively. The process Φ can be realised (semi-explicitly) as a stochastic flow under \mathbb{P}_0 using (3.3) and (3.7) above. This gives

$$(6.3) \quad \Phi_{t+s}^{t,\varphi} = \frac{\pi}{1-\pi} \int_{-\infty}^{\infty} e^{mx - \frac{m^2}{2}t} e^{mX_s - \frac{m^2}{2}s} F_{\mu}(dm)$$

for $s \geq 0$ so that $\mathbb{P}_0(\Phi_t^{t,\varphi} = \varphi) = 1$ where

$$(6.4) \quad \varphi = \frac{\pi}{1-\pi} \int_{-\infty}^{\infty} e^{mx - \frac{m^2}{2}t} F_{\mu}(dm)$$

for $x \in \mathbb{R}$ standing in one-to-one correspondence with $\varphi \in \mathbb{R}_+$ when $t \geq 0$ is fixed (with $x \geq z$ when (4.4) holds). Recall that $\pi \in [0, 1]$ is a prior probability (cf. Section 2) and X is a standard Brownian motion under \mathbb{P}_0 . Yet another (semi-explicit) stochastic flow realisation of Φ under \mathbb{P}_0 can be obtained by solving the stochastic differential equation (3.12). One can verify by Itô's formula that this leads to the following expression

$$(6.5) \quad \Phi_{t+s}^{t,\varphi} = \varphi \exp \left(\int_0^s \hat{\mu}(t+r, x+X_r) dX_r - \frac{1}{2} \int_0^s \hat{\mu}^2(t+r, x+X_r) dr \right)$$

for $s \geq 0$ so that $\mathbb{P}_0(\Phi_t^{t,\varphi} = \varphi) = 1$ where $\varphi \in \mathbb{R}_+$ is given by (6.4) above for $x \in \mathbb{R}$ and $t \geq 0$. From (6.5) we see that the time-space flow of Φ is continuous as a (Markovian) functional of the initial point (t, φ) in $[0, \infty) \times \mathbb{R}_+$. This implies that the expectation in (4.5) defines a continuous function of the initial point (t, φ) in $[0, \infty) \times \mathbb{R}_+$ for every (bounded) stopping time τ given and fixed. Taking the infimum over all (bounded) stopping times τ we can thus conclude that the value function \hat{V} is upper semicontinuous on $[0, \infty) \times \mathbb{R}_+$. The loss function $(t, \varphi) \mapsto t(1+\varphi) + \hat{M}(t, \varphi)$ in (4.5) is continuous on $[0, \infty) \times \mathbb{R}_+$ and hence lower semicontinuous as well. It follows therefore by [20, Corollary 2.9] that the first entry time of the time-space process $((t+s, \Phi_{t+s}))_{s \geq 0}$ into the closed set D defined by

$$(6.6) \quad \tau_D = \inf \{ s \geq 0 \mid (t+s, \Phi_{t+s}) \in D \}$$

is optimal in (4.5) whenever $\mathbb{P}_{t,\varphi}^0(\tau_D < \infty) = 1$ for all $(t, \varphi) \in [0, \infty) \times \mathbb{R}_+$. In the sequel we will establish this and other properties of τ_D by analysing the boundary of D .

2. We first show that the horizontal line $\varphi = b/a$ is contained in C . Motivated by the Lagrange reformulation (5.1) of (4.5) we now give a proof of this fact based on the local time as discussed following the proof of Proposition 6 above.

Lemma 7. *The set $\{(t, \varphi) \in [0, \infty) \times \mathbb{R}_+ \mid \varphi = b/a\}$ is contained in the continuation set C of the optimal stopping problem (4.5).*

Proof. From (5.1) we see that

$$(6.7) \quad \hat{V}(t, \varphi) = \inf_{\tau} \mathbf{E}_0 \left[\int_0^{\tau} (1 + \Phi_{t+s}^{t, \varphi}) ds - \frac{a}{2} \ell_{\tau}^{b/a}(\Phi^{t, \varphi}) \right] + \hat{M}(\varphi)$$

for $(t, \varphi) \in [0, \infty) \times \mathbb{R}_+$ where the infimum is taken over all stopping times τ of Φ . By the Itô-Tanaka formula we find using (3.12) that

$$(6.8) \quad |\Phi_{t+s}^{t, b/a} - b/a| = M_s + \ell_s^{b/a}(\Phi^{t, b/a})$$

for $s \geq 0$ where $(M_s)_{s \geq 0}$ is a continuous martingale. Recalling (3.3) and (3.7) we see that the left-hand side in (6.8) equals

$$(6.9) \quad \begin{aligned} |\Phi_{t+s}^{t, b/a} - b/a| &= \left| \frac{\pi}{1-\pi} \int_{-\infty}^{\infty} e^{mx - \frac{m^2}{2}t} e^{mX_s - \frac{m^2}{2}s} F_{\mu}(dm) - \frac{b}{a} \right| \\ &= \frac{\pi}{1-\pi} \left| \int_{-\infty}^{\infty} e^{mx - \frac{m^2}{2}t} (e^{mX_s - \frac{m^2}{2}s} - 1) F_{\mu}(dm) \right| \\ &= \frac{\pi}{1-\pi} \left| \int_{-\infty}^{\infty} e^{mx - \frac{m^2}{2}t} \left(\sum_{n=1}^{\infty} \frac{(mX_s - \frac{m^2}{2}s)^n}{n!} \right) F_{\mu}(dm) \right| \\ &= \frac{\pi}{1-\pi} \left| \int_{-\infty}^{\infty} e^{mx - \frac{m^2}{2}t} (mX_s - \frac{m^2}{2}s) \left(\sum_{n=1}^{\infty} \frac{(mX_s - \frac{m^2}{2}s)^{n-1}}{n!} \right) F_{\mu}(dm) \right| \end{aligned}$$

for $s \geq 0$ where $x \in \mathbb{R}$ is chosen so that (6.4) above holds with $\varphi = b/a$ as needed. Taking \mathbf{E}_0 on both sides of (6.8)+(6.9) and using that $X_s \sim \sqrt{s} X_1$ under \mathbf{P}_0 we get

$$(6.10) \quad \begin{aligned} \mathbf{E}_0[\ell_s^{b/a}(\Phi^{t, b/a})] \\ = \frac{\pi}{1-\pi} \mathbf{E}_0 \left| \int_{-\infty}^{\infty} e^{mx - \frac{m^2}{2}t} \sqrt{s} (mX_1 - \frac{m^2}{2}\sqrt{s}) \left(\sum_{n=1}^{\infty} \frac{(m\sqrt{s}X_1 - \frac{m^2}{2}s)^{n-1}}{n!} \right) F_{\mu}(dm) \right| \end{aligned}$$

for $s \geq 0$. Dividing both sides of (6.10) by \sqrt{s} and letting $s \downarrow 0$ this shows that

$$(6.11) \quad \lim_{s \downarrow 0} \frac{1}{\sqrt{s}} \mathbf{E}_0[\ell_s^{b/a}(\Phi^{t, b/a})] = \frac{\pi}{1-\pi} \left| \int_{-\infty}^{\infty} m e^{mx - \frac{m^2}{2}t} F_{\mu}(dm) \right| \mathbf{E}_0|X_1| \in (0, \infty).$$

It means that $\mathbf{E}_0[\ell_s^{b/a}(\Phi^{t, b/a})] \sim \sqrt{s}$ as $s \downarrow 0$. On the other hand, it is clear from (6.3) and (6.4) that $\mathbf{E}_0[\int_0^s (1 + \Phi_{t+r}^{t, b/a}) dr] = \int_0^s (1 + b/a) dr = (1 + b/a)s \sim s$ as $s \downarrow 0$. Combining these two facts it is evident that the expectation in (6.7) is strictly negative when $\varphi = b/a$ if $\tau = s$ is taken sufficiently small in $(0, \infty)$. This shows that each point $(t, b/a)$ belongs to C for $t \geq 0$ and the proof is complete. \square

3. Moving from the vertical line $\varphi = b/a$ downwards and upwards let us formally define the (least) boundaries between C and D by setting

$$(6.12) \quad b_0(t) = \sup \{ \varphi \in [0, b/a] \mid (t, \varphi) \in D \} \ \& \ b_1(t) = \inf \{ \varphi \in [b/a, \infty) \mid (t, \varphi) \in D \}$$

for every $t \geq 0$ given and fixed. Clearly $b_0(t) < b/a < b_1(t)$ for all $t \geq 0$ and the supremum and infimum in (6.12) are attained since D is closed. We now show that admissible laws from Definition 4 above imply that b_0 and b_1 are monotone functions of time that separate C and D entirely. The key fact in this direction will be presented first.

Proposition 8. *The mapping $t \mapsto \hat{V}(t, \varphi)$ is increasing or decreasing on $[0, \infty)$ for every $\varphi \in \mathbb{R}_+$ if and only if (4.6) or (4.7) holds respectively.*

Proof. 1. Recall from (3.8) and (4.8) that Φ solves

$$(6.13) \quad d\Phi_t = \sigma(t, \Phi_t) dX_t$$

where X is a standard Brownian motion under \mathbb{P}_0 . Recall also from Proposition 3 that

$$(6.14) \quad t \mapsto \sigma^2(t, \varphi) \text{ is decreasing or increasing on } [0, \infty)$$

for every $\varphi \in \mathbb{R}_+$ if and only if (4.6) or (4.7) holds respectively. Passing to a stochastic flow of the process Φ we see that the optimal stopping problem (4.5) reads

$$(6.15) \quad \hat{V}(t, \varphi) = \inf_{\tau} \mathbb{E}_0[\tau(1 + \Phi_{t+\tau}^{t, \varphi}) + \hat{M}(\Phi_{t+\tau}^{t, \varphi})]$$

for $(t, \varphi) \in [0, \infty) \times \mathbb{R}_+$ where the infimum is taken over all stopping times τ of Φ .

2. Motivated by (6.13) consider the additive functional A defined by

$$(6.16) \quad A_s = A_s^{(t, \varphi)} := \int_0^s \sigma^2(t+r, \Phi_{t+r}^{t, \varphi}) dr$$

for $s \geq 0$ and $(t, \varphi) \in [0, \infty) \times \mathbb{R}_+$ given and fixed. Note that $s \mapsto A_s$ is continuous and strictly increasing with $A_0 = 0$ and $A_s \uparrow A_\infty$ as $s \uparrow \infty$. Hence the same properties hold for the inverse T of A defined by

$$(6.17) \quad T_s = T_s^{(t, \varphi)} := A_s^{-1}$$

for $s \in [0, A_\infty)$ with $T_s \uparrow \infty$ as $s \uparrow A_\infty$. Since A is adapted to $(\mathcal{F}_t^\Phi)_{t \geq 0}$ it follows that each T_s is a stopping time with respect to $(\mathcal{F}_t^\Phi)_{t \geq 0}$ so that T defines a time change relative to $(\mathcal{F}_t^\Phi)_{t \geq 0}$. Define the time-changed process $\hat{\Phi}$ by setting

$$(6.18) \quad \hat{\Phi}_s^\varphi = \hat{\Phi}_{t+s}^{t, \varphi} := \Phi_{t+T_s}^{t, \varphi}$$

for $s \in [0, A_\infty)$. Time changing the stochastic differential equation (6.13) we find that

$$(6.19) \quad \hat{\Phi}_s^\varphi = \varphi + \int_0^{T_s} \sigma(t+r, \Phi_{t+r}^{t, \varphi}) dX_r = \varphi + \int_0^s \sigma(t+T_r, \hat{\Phi}_{t+r}^{t, \varphi}) dX_{T_r} =: \varphi + \tilde{B}_s$$

for $\varphi \in \mathbb{R}_+$ where \tilde{B} is a continuous local martingale with $\langle \tilde{B}, \tilde{B} \rangle_s = \int_0^{T_s} \sigma^2(t+r, \Phi_{t+r}^{t, \varphi}) dr = A_{T_s} = s$ for $s \in [0, A_\infty)$. Hence by Lévy's characterisation theorem we can conclude that \tilde{B} is a standard Brownian motion (starting at zero). Note that by changing the initial standard Brownian motion in (6.13) we can achieve that \tilde{B} does not depend on t . It follows therefore

that $\hat{\Phi}^\varphi$ is a standard Brownian motion starting at φ in \mathbb{R}_+ . From (6.16) it is easily seen using (6.17) that

$$(6.20) \quad T_s = T_s^{(t,\varphi)} = \int_0^s \frac{dr}{\sigma^2(t+T_r, \Phi_{t+T_r}^{t,\varphi})} = \int_0^s \frac{dr}{\sigma^2(t+T_r^{(t,\varphi)}, \varphi + \tilde{B}_r)}$$

for $s \in [0, A_\infty)$. From (6.20) combined with (6.14) we see that

$$(6.21) \quad \frac{\partial}{\partial s} T_s^{(t_1,\varphi)} \underset{\geq}{\leq} \frac{\partial}{\partial s} T_s^{(t_2,\varphi)}$$

whenever $T_s^{(t_1,\varphi)} = T_s^{(t_2,\varphi)}$ for some s and $t_1 \leq t_2$ in $[0, \infty)$ (with the same \tilde{B} as above) if and only if (4.6) or (4.7) holds respectively. From (6.21) combined with $T_0^{(t,\varphi)} = 0$ we see that

$$(6.22) \quad t \mapsto T_s^{(t,\varphi)} \text{ is increasing or decreasing on } [0, \infty)$$

for every $s \in [0, A_\infty)$ if and only if (4.6) or (4.7) holds respectively. Returning to (6.15) and applying the time change from (6.17) hence we see that

$$(6.23) \quad \begin{aligned} \hat{V}(t_1, \varphi) &= \inf_{\tau} \mathbf{E}_0 [T_{A_\tau} (1 + \Phi_{t_1+T_{A_\tau}}^{t_1,\varphi}) + \hat{M}(\Phi_{t_1+T_{A_\tau}}^{t_1,\varphi})] \\ &= \inf_{\sigma} \mathbf{E}_0 [T_{\sigma}^{(t_1,\varphi)} (1 + \hat{\Phi}_{\sigma}^{\varphi}) + \hat{M}(\hat{\Phi}_{\sigma}^{\varphi})] \\ &\underset{\geq}{\leq} \inf_{\sigma} \mathbf{E}_0 [T_{\sigma}^{(t_2,\varphi)} (1 + \hat{\Phi}_{\sigma}^{\varphi}) + \hat{M}(\hat{\Phi}_{\sigma}^{\varphi})] \\ &= \inf_{\tau} \mathbf{E}_0 [T_{A_\tau} (1 + \Phi_{t_2+T_{A_\tau}}^{t_2,\varphi}) + \hat{M}(\Phi_{t_2+T_{A_\tau}}^{t_2,\varphi})] \\ &= \hat{V}(t_2, \varphi) \end{aligned}$$

for $t_1 \leq t_2$ in $[0, \infty)$ and $\varphi \in \mathbb{R}_+$ if and only if (4.6) or (4.7) holds respectively. This completes the proof of the initial equivalence claim. \square

Corollary 9. *We have $0 < b_0(t) < b/a < b_1(t) < \infty$ for all $t > 0$ with*

$$(6.24) \quad C = \{ (t, \varphi) \in [0, \infty) \times \mathbb{R}_+ \mid b_0(t) < \varphi < b_1(t) \}$$

$$(6.25) \quad D = \{ (t, \varphi) \in [0, \infty) \times \mathbb{R}_+ \mid 0 \leq \varphi \leq b_0(t) \text{ or } b_1(t) \leq \varphi < \infty \}.$$

Moreover, the following implications are satisfied (see Figure 1 below):

$$(6.26) \quad \text{If (4.6) holds then the mapping } t \mapsto b_0(t) \text{ is increasing and the mapping } t \mapsto b_1(t) \text{ is decreasing on } [0, \infty).$$

$$(6.27) \quad \text{If (4.7) holds then the mapping } t \mapsto b_0(t) \text{ is decreasing and the mapping } t \mapsto b_1(t) \text{ is increasing on } [0, \infty).$$

Proof. 1. From (6.5) and (6.15) we see that

$$(6.28) \quad \varphi \mapsto \hat{V}(t, \varphi) \text{ is increasing and concave on } [0, \infty)$$

for every $t \geq 0$ given and fixed. Concavity of $\varphi \mapsto \hat{V}(t, \varphi)$ combined with non-negativity and piecewise linearity of $\varphi \mapsto \hat{M}(\varphi)$ in (4.5) implies that if $(t, \varphi) \in D$ with $\varphi < b/a$ and $\varphi_1 < \varphi$

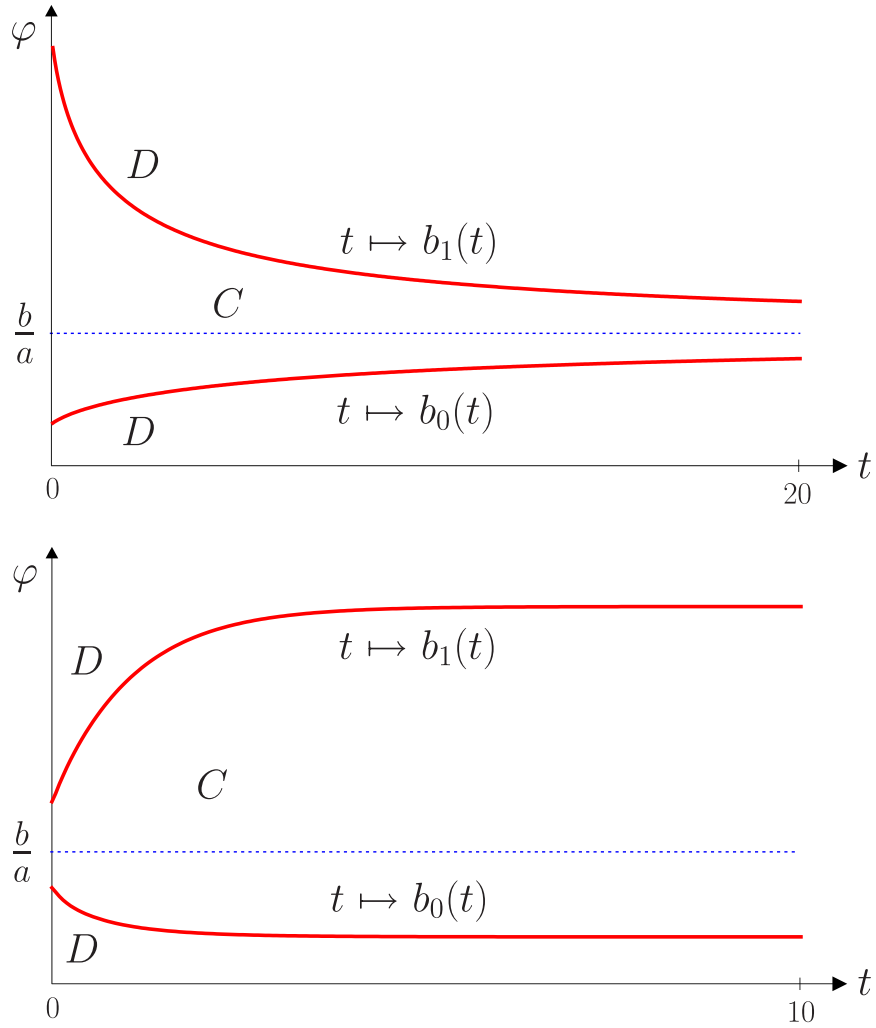


Figure 1. The optimal stopping boundaries b_0 and b_1 when (i) $\mu \sim Exp(1)$ and (4.6) holds along $a = b = 20$ (upper figure) and (ii) μ takes values ± 1 with probability $1/2$ and (4.7) holds along $a = b = 10$ (lower figure).

then $(t, \varphi_1) \in D$ as well as that if $(t, \varphi) \in D$ with $\varphi > b/a$ and $\varphi_2 > \varphi$ then $(t, \varphi_2) \in D$. This shows that b_0 and b_1 from (6.12) alone separate C and D fully and hence (6.24) and (6.25) are valid as claimed. Moreover, we know from Proposition 8 that

$$(6.29) \quad t \mapsto \hat{V}(t, \varphi) - \hat{M}(\varphi) \text{ is increasing or decreasing on } [0, \infty)$$

for every $\varphi \in \mathbb{R}_+$ if and only if (4.6) or (4.7) holds respectively. In the former case (i.e. when (4.6) holds) we see that if $(t_1, \varphi) \in D$ and $t_2 \geq t_1$ then $0 = \hat{V}(t_1, \varphi) - \hat{M}(\varphi) \leq \hat{V}(t_2, \varphi) - \hat{M}(\varphi) \leq 0$ so that $\hat{V}(t_2, \varphi) - \hat{M}(\varphi) = 0$ and hence $(t_2, \varphi) \in D$ as well. This shows that (6.26) holds as claimed. Similarly, in the latter case (i.e. when (4.7) holds) we see that if $(t_1, \varphi) \in D$ and $t_0 \leq t_1$ then $0 = \hat{V}(t_1, \varphi) - \hat{M}(\varphi) \leq \hat{V}(t_0, \varphi) - \hat{M}(\varphi) \leq 0$ so that $\hat{V}(t_0, \varphi) - \hat{M}(\varphi) = 0$ and hence $(t_0, \varphi) \in D$ as well. This shows that (6.27) holds as claimed.

2. We show that $b_0(t) > 0$ for all $t > 0$. Clearly, if b_0 is increasing then it is sufficient to disprove the existence of $t_0 > 0$ such that $[0, t_0] \times [0, b/a] \subseteq C$, and similarly, if b_0 is decreasing

then it is sufficient to disprove the existence of $t_1 > 0$ such that $(t_1, \infty) \times [0, b/a] \subseteq C$. Suppose first that $[0, t_0] \times [0, b/a] \subseteq C$ for some $t_0 > 0$. Consider the stopping time

$$(6.30) \quad \tau_{b/a}^{(0, \varphi)} = \inf \{ s \in [0, t_0] \mid \Phi_s^{0, \varphi} \geq b/a \}$$

under \mathbb{P}_0 for $\varphi \in (0, b/a)$ given and fixed. Then $\tau_{b/a}^{(0, \varphi)} \leq \tau_D^{(0, \varphi)}$ where $\tau_D := \tau_D^{(0, \varphi)} = \inf \{ s \geq 0 \mid \Phi_s^{0, \varphi} \in D \}$ is an optimal stopping time, so that from (6.15) we find that

$$(6.31) \quad \hat{V}(0, \varphi) = \mathbb{E}_0[\tau_D(1 + \Phi_{\tau_D}^{0, \varphi}) + \hat{M}(\Phi_{\tau_D}^{0, \varphi})] \geq \mathbb{E}_0(\tau_{b/a}^{(0, \varphi)}).$$

Letting $\varphi \downarrow 0$ and using that $\lim_{\varphi \downarrow 0} \tau_{b/a}^{(0, \varphi)} = t_0$ we see from (6.31) using Fatou's lemma that $\liminf_{\varphi \downarrow 0} \hat{V}(0, \varphi) \geq t_0 > 0$. It follows that taking $\varphi > 0$ sufficiently small we get $\hat{V}(0, \varphi) > a\varphi = \hat{M}(\varphi)$ which is a contradiction since $\hat{V} \leq \hat{M}$. This shows that $b_0(t) > 0$ for all $t > 0$ when b_0 is increasing as claimed. Next suppose that $(t_1, \infty) \times [0, b/a] \subseteq C$ for some $t_1 > 0$. Consider the same stopping time as in (6.30) with t_1 in place of 0 and $t_0 = \infty$ when $\varphi \in (0, b/a)$ is given and fixed. Using the same arguments as above we obtain the inequality (6.31) with t_1 in place of 0 for all $\varphi \in (0, b/a)$. Letting $\varphi \downarrow 0$ and using that $\tau_{b/a}^{(t_1, \varphi)} \rightarrow \infty$ we see from the previous inequality that $\hat{V}(t_1, \varphi) \rightarrow \infty$ which is a contradiction since $\hat{V} \leq \hat{M}$. This shows that $b_0(t) > 0$ for all $t > 0$ when b_0 is decreasing as claimed.

3. We show that $b_1(t) < \infty$ for all $t > 0$. Clearly, if b_1 is decreasing then it is sufficient to disprove the existence of $t_0 > 0$ such that $[0, t_0] \times [b/a, \infty) \subseteq C$, and similarly, if b_1 is increasing then it is sufficient to disprove the existence of $t_1 > 0$ such that $(t_1, \infty) \times [b/a, \infty) \subseteq C$. Suppose first that $[0, t_0] \times [b/a, \infty) \subseteq C$ for some $t_0 > 0$. Consider the stopping time

$$(6.32) \quad \tau_{b/a}^{(0, \varphi)} = \inf \{ s \in [0, t_0] \mid \Phi_s^{0, \varphi} \leq b/a \}$$

under \mathbb{P}_0 for $\varphi \in (b/a, \infty)$ given and fixed. Then $\tau_{b/a}^{(0, \varphi)} \leq \tau_D^{(0, \varphi)}$ where $\tau_D := \tau_D^{(0, \varphi)} = \inf \{ s \geq 0 \mid \Phi_s^{0, \varphi} \in D \}$ is an optimal stopping time, so that by (5.4) and (6.15) we find that

$$(6.33) \quad \hat{V}(0, \varphi) = \mathbb{E}_0 \left[\int_0^{\tau_D} (1 + \Phi_s^{0, \varphi}) ds + \hat{M}(\Phi_{\tau_D}^{0, \varphi}) \right] \geq \mathbb{E}_0 \left[\int_0^{\tau_{b/a}^{(0, \varphi)}} (1 + \varphi \Phi_s^{0, 1}) ds \right]$$

where in the final term we use (6.5) above. Letting $\varphi \rightarrow \infty$ and using that $\lim_{\varphi \uparrow \infty} \tau_{b/a}^{(0, \varphi)} = t_0$ we see from (6.33) using Fatou's lemma that $\hat{V}(0, \varphi) \rightarrow \infty$ which is a contradiction since $\hat{V} \leq \hat{M}$. This shows that $b_1(t) < \infty$ for all $t > 0$ when b_1 is decreasing as claimed. Next suppose that $(t_1, \infty) \times [b/a, \infty) \subseteq C$ for some $t_1 > 0$. Consider the same stopping time as in (6.32) with t_1 in place of 0 and $t_0 = \infty$ when $\varphi \in (b/a, \infty)$ is given and fixed. Using the same arguments as above we obtain the inequality (6.33) with t_1 in place of 0 for all $\varphi \in (b/a, \infty)$. Letting $\varphi \rightarrow \infty$ and using that $\tau_{b/a}^{(t_1, \varphi)} \rightarrow \infty$ we see from the previous inequality that $\hat{V}(t_1, \varphi) \rightarrow \infty$ which is a contradiction since $\hat{V} \leq \hat{M}$. This shows that $b_1(t) < \infty$ for all $t > 0$ when b_1 is increasing as claimed and the proof is complete. \square

7. Free-boundary problem

In this section we derive a free-boundary problem that stands in one-to-one correspondence with the optimal stopping problem (4.5). Using the results derived in the previous sections

we show that the value function \hat{V} from (4.5) and the optimal stopping boundaries b_0 & b_1 from (6.12) solve the free-boundary problem. This establishes the existence of a solution to the free-boundary problem. Its uniqueness in a natural class of functions will follow from a more general uniqueness that will be established in Section 8 below. This will also yield a double-integral representation of the value function \hat{V} expressed in terms of the optimal stopping boundaries b_0 & b_1 .

1. Consider the optimal stopping problem (4.5) where the (time-inhomogeneous) Markov process Φ solves the stochastic differential equation (6.13) where X is a standard Brownian motion under \mathbb{P}_0 . From (6.13) we see that the infinitesimal generator of Φ is given by

$$(7.1) \quad \mathbb{L}_\Phi = \frac{1}{2} \sigma^2(t, \varphi) \partial_{\varphi\varphi}$$

where $\sigma(t, \varphi)$ is given explicitly by (4.8) for $(t, \varphi) \in [0, \infty) \times \mathbb{R}_+$. Looking at (4.5), and relying on other properties of \hat{V} and b_0 & b_1 established above, we are naturally led to formulate the following free-boundary problem for finding \hat{V} and b_0 & b_1 :

$$(7.2) \quad \hat{V}_t + \mathbb{L}_\Phi \hat{V} = -L \text{ in } C$$

$$(7.3) \quad \hat{V}(t, \varphi) = \hat{M}(\varphi) \text{ for } (t, \varphi) \in D \text{ (instantaneous stopping)}$$

$$(7.4) \quad \hat{V}_\varphi(t, \varphi) = \hat{M}'(\varphi) \text{ for } \varphi = b_0(t) \text{ \& } \varphi = b_1(t) \text{ with } t > 0 \text{ (smooth fit)}$$

where we set $L(\varphi) = 1 + \varphi$ for $\varphi \in \mathbb{R}_+$ and the (continuation) set C and the (stopping) set D are given by (6.24) and (6.25) respectively. Clearly the global condition (7.3) can be replaced by the local condition $\hat{V}(t, \varphi) = \hat{M}(\varphi)$ for $\varphi = b_0(t)$ & $\varphi = b_1(t)$ with $t > 0$ so that the free-boundary problem (7.2)-(7.4) needs to be considered on the closure of C only (extending \hat{V} to the rest of D as \hat{M} being then evident).

2. To formulate the existence and uniqueness result for the free-boundary problem (7.2)-(7.4) we let \mathcal{C} denote the class of functions $(F; c_0, c_1)$ such that

$$(7.5) \quad F \text{ belongs to } C^1(\bar{C}_{c_0, c_1}) \cap C^2(C_{c_0, c_1}) \text{ and is bounded on } [0, \infty) \times \mathbb{R}_+$$

$$(7.6) \quad c_0 \text{ is continuous and increasing/decreasing on } (0, \infty) \text{ with } 0 < c_0 < b/a$$

$$(7.7) \quad c_1 \text{ is continuous and decreasing/increasing on } (0, \infty) \text{ with } b/a < c_1 < \infty$$

when (4.6)/(4.7) holds respectively in (7.6) and (7.7), where we set $C_{c_0, c_1} = \{(t, \varphi) \in [0, \infty) \times \mathbb{R}_+ \mid c_0(t) < \varphi < c_1(t)\}$ and $\bar{C}_{c_0, c_1} = \{(t, \varphi) \in [0, \infty) \times \mathbb{R}_+ \mid c_0(t) \leq \varphi \leq c_1(t)\}$.

Theorem 10. *The free-boundary problem (7.2)-(7.4) has a unique solution $(\hat{V}; b_0, b_1)$ in the class \mathcal{C} where \hat{V} is given by (4.5) and b_0 & b_1 are defined in (6.12).*

Proof. We first show that the triple $(\hat{V}; b_0, b_1)$ belongs to the class \mathcal{C} . For this, if (4.6) holds then by (6.26) we know that b_0 is increasing and b_1 is decreasing. Using the fact visible from (6.13) that Φ is a time-changed Brownian motion, hence we can conclude that each boundary point between C and D is probabilistically regular for the interior of D . Similarly, if (4.7) holds then by (6.27) we know that b_0 is decreasing and b_1 is increasing. In this case we can use arguments similar to those in the second part of Example 17 in [6] to show

that b_0 and b_1 are locally Lipschitz. Hence by the same time-change arguments as above we can again conclude that each boundary point between C and D is probabilistically regular for the interior of D . The method of proof outlined in Example 12 and Example 17 of [6] then enables us to infer the global C^1 regularity of the value function \hat{V} in the sense that

$$(7.8) \quad (t, \varphi) \mapsto \hat{V}_t(t, \varphi) \text{ is continuous on } [0, \infty) \times \mathbb{R}_+$$

$$(7.9) \quad (t, \varphi) \mapsto \hat{V}_\varphi(t, \varphi) \text{ is continuous on } [0, \infty) \times \mathbb{R}_+.$$

Moreover, since the (horizontal) smooth fit holds at each boundary point between C and D we can apply the general result of Theorem 3 in [19] and conclude that

$$(7.10) \quad t \mapsto b_0(t) \ \& \ t \mapsto b_1(t) \text{ are continuous on } [0, \infty).$$

Combined with properties derived in Section 6 above, this shows that the triple $(\hat{V}; b_0, b_1)$ belongs to the class \mathcal{C} as claimed. The Lagrange reformulation (5.1) of the optimal stopping problem (4.5) combined with standard arguments (see e.g. the final paragraph of Section 2 in [6]) shows that \hat{V} satisfies (7.2). This fact combined with (7.8) implies that

$$(7.11) \quad \hat{V}_{\varphi\varphi} \text{ admits a continuous extension from } C \text{ to } \bar{C}$$

where C is the continuation set and \bar{C} is its closure as defined following (7.7) above (omitting subscripts for simplicity). Clearly \hat{V} also satisfies (7.3) while (7.4) follows from the arguments yielding (7.8) and (7.9) above. This shows that the triple $(\hat{V}; b_0, b_1)$ is a solution to the free-boundary problem (7.2)-(7.4) in the class \mathcal{C} . To derive uniqueness of the solution we will first see in the next section that any solution $(F; c_0, c_1)$ to (7.2)-(7.4) in the class \mathcal{C} admits a closed double-integral representation for F in terms of c_0 and c_1 , which in turn solve a coupled system of nonlinear Volterra integral equations, and we will see that this system cannot have other solutions satisfying the required properties. From these facts we can conclude that the free-boundary problem (7.2)-(7.4) cannot have other solutions as claimed. \square

8. Nonlinear integral equations

In this section we show that the optimal stopping boundaries b_0 and b_1 from (6.25) can be characterised as the unique solution to a coupled system of nonlinear Volterra integral equations. This also yields a closed double-integral representation of the value function from (4.5) in terms of the optimal stopping boundaries b_0 and b_1 . As a consequence of the existence and uniqueness result for the coupled system of nonlinear Volterra integral equations we also obtain uniqueness of the solution to the free-boundary problem (7.2)-(7.4) as explained in the proof of Theorem 10 above. Finally, collecting the results derived throughout the paper we conclude our exposition by disclosing the solution to the initial problem.

1. Recalling that X is a standard Brownian motion under \mathbb{P}_0 we readily find from (4.1) that the probability density function of $\Phi_{t+s}^{t,\varphi}$ under \mathbb{P}_0 is given by

$$(8.1) \quad f(t+s, \psi; t, \varphi) = \frac{1}{\Phi_0 \sqrt{2\pi s}} \frac{1}{\ell_x(t+s, \ell^{-1}(t+s, \psi/\Phi_0))} \\ \times \exp\left(-\frac{(\ell^{-1}(t+s, \psi/\Phi_0) - \ell^{-1}(t, \varphi/\Phi_0))^2}{2s}\right)$$

for $0 \leq t < t+s$ and φ & ψ in \mathbb{R}_+ where the inverse function ℓ^{-1} is defined following (4.8) above. Having f we can evaluate the expression of interest in the theorem below as follows

$$(8.2) \quad K(t+s, \varphi_0, \varphi_1; t, \varphi) = \mathbb{E}_0 \left[L(\Phi_{t+s}^{t,\varphi}) I(\varphi_0 < \Phi_{t+s}^{t,\varphi} < \varphi_1) \right] = \int_{\varphi_0}^{\varphi_1} L(\psi) f(t+s, \psi; t, \varphi) d\psi$$

for $0 \leq t < t+s$ and φ & $\varphi_0 < \varphi_1$ in \mathbb{R}_+ where we recall that $L(\psi) = 1+\psi$ for $\psi \in \mathbb{R}_+$.

Theorem 11 (Existence and uniqueness). *The optimal stopping boundaries b_0 and b_1 in the problem (4.5) can be characterised as the unique solution to the coupled system of nonlinear Volterra integral equations*

$$(8.3) \quad ab_0(t) = \int_0^\infty K(t+s, b_0(t+s), b_1(t+s); t, b_0(t)) ds$$

$$(8.4) \quad b = \int_0^\infty K(t+s, b_0(t+s), b_1(t+s); t, b_1(t)) ds$$

in the class of continuous functions b_0 and b_1 on $[0, \infty)$ where $t \mapsto b_0(t)$ is increasing/decreasing and $t \mapsto b_1(t)$ is decreasing/increasing with $0 < b_0(t) < b/a < b_1(t) < \infty$ for $t > 0$ when (4.6)/(4.7) holds respectively. The value function \hat{V} in the problem (4.5) admits the following representation

$$(8.5) \quad \hat{V}(t, \varphi) = \int_0^\infty K(t+s, b_0(t+s), b_1(t+s); t, \varphi) ds$$

for $(t, \varphi) \in [0, \infty) \times \mathbb{R}_+$. The optimal stopping time in the problem (4.5) is given by

$$(8.6) \quad \tau_{b_0, b_1} = \inf \{ s \geq 0 \mid \Phi_{t+s}^{t,\varphi} \notin (b_0(t+s), b_1(t+s)) \}$$

under \mathbb{P}_0 with $(t, \varphi) \in [0, \infty) \times \mathbb{R}_+$ given and fixed (see Figure 1 above).

Proof. 1. *Existence.* We first show that the optimal stopping boundaries b_0 and b_1 in the problem (4.5) solve the system (8.3)+(8.4). Recalling that b_0 and b_1 satisfy the properties stated following (8.3)+(8.4) as established above, this will prove the existence of the solution to (8.3)+(8.4). For this, we will first note that Itô's formula is applicable to \hat{V} composed with $(t+s, \Phi_{t+s}^{t,\varphi})$ for $s \geq 0$ with $(t, \varphi) \in [0, \infty) \times \mathbb{R}_+$ given and fixed. Indeed, recalling that \hat{V} is $C^{1,2}$ on the closure of C and equals \hat{M} on D (which also is $C^{1,2}$ since the singularity line $\varphi = b/a$ of \hat{M} is contained in C as established in Lemma 7 above) we see that the local time-space formula from [18] is applicable to \hat{V} composed with $(t+s, \Phi_{t+s}^{t,\varphi})$ for $s \geq 0$ and moreover this formula reduces to Itô's formula due to the smooth fit condition (7.4). Using (7.1) and (7.2) this yields

$$(8.7) \quad \begin{aligned} \hat{V}(t+s, \Phi_{t+s}^{t,\varphi}) &= \hat{V}(t, \varphi) + \int_0^s (\hat{V}_t + \mathbb{L}_\Phi \hat{V})(t+r, \Phi_{t+r}^{t,\varphi}) dr + M_s \\ &= \hat{V}(t, \varphi) - \int_0^s L(\Phi_{t+r}^{t,\varphi}) I(b_0(t+r) < \Phi_{t+r}^{t,\varphi} < b_1(t+r)) dr + M_s \end{aligned}$$

where $M_s = \int_0^s \hat{V}_\varphi(t+r, \Phi_{t+r}^{t,\varphi}) \sigma(t+r, \Phi_{t+r}^{t,\varphi}) dX_r$ is a continuous local martingale for $s \geq 0$. Taking a localisation sequence of stopping times $(\tau_n)_{n \geq 1}$ for M , replacing s on both sides of (8.7) by $s \wedge \tau_n$, applying the optional sampling theorem and letting $n \rightarrow \infty$, we obtain

$$(8.8) \quad \mathbb{E}_0[\hat{V}(t+s, \Phi_{t+s}^{t,\varphi})] = \hat{V}(t, \varphi) - \mathbb{E}_0 \left[\int_0^s L(\Phi_{t+r}^{t,\varphi}) I(b_0(t+r) < \Phi_{t+r}^{t,\varphi} < b_1(t+r)) dr \right]$$

for $s \geq 0$. Letting $s \rightarrow \infty$ and noting that $0 \leq \hat{V}(t+s, \Phi_{t+s}^{t,\varphi}) \leq \hat{M}(\Phi_{t+s}^{t,\varphi}) = a\Phi_{t+s}^{t,\varphi} \wedge b \rightarrow 0$ \mathbb{P}_0 -a.s. by (8.11) below, we see that the dominated and monotone convergence theorems yield

$$(8.9) \quad \hat{V}(t, \varphi) = \mathbb{E}_0 \left[\int_0^\infty L(\Phi_{t+s}^{t,\varphi}) I(b_0(t+s) < \Phi_{t+s}^{t,\varphi} < b_1(t+s)) ds \right]$$

which establishes the representation (8.5) upon recalling (8.2) above. Recalling that $\hat{V}(t, b_0(t)) = \hat{M}(b_0(t)) = ab_0(t)$ and $\hat{V}(t, b_1(t)) = \hat{M}(b_1(t)) = b$ for all $t \geq 0$ we see that (8.5) implies (8.3) and (8.4) as claimed.

2. *Uniqueness.* To show that b_0 and b_1 are a unique solution to the system (8.3)+(8.4) one can adopt the four-step procedure from the proof of uniqueness given in [7, Theorem 4.1] extending and further refining the original arguments from [17, Theorem 3.1] in the case of a single boundary. Note that although the present horizon is infinite and any stopping time τ among the four stopping times used in the four-step procedure can take infinite values as well in certain situations, the optional sampling theorem is still applicable to $\tau \wedge n$ for $n \geq 1$ given and fixed, and then letting $n \rightarrow \infty$ one obtains the desired conclusions as when the horizon is finite. The key argument which makes this possible (in the first step) is obtained by noting that $0 \leq \hat{M} \leq b$ so that the monotone and dominated convergence theorems yield

$$(8.10) \quad \begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}_0[\hat{M}(\Phi_{t+\tau}^{t,\varphi}) I(\tau \leq n)] &= \mathbb{E}_0[\hat{M}(\Phi_{t+\tau}^{t,\varphi}) I(\tau < \infty)] = \lim_{n \rightarrow \infty} \mathbb{E}_0[\hat{M}(\Phi_{t+\tau \wedge n}^{t,\varphi}) I(\tau < \infty)] \\ &= \lim_{n \rightarrow \infty} \mathbb{E}_0[\hat{M}(\Phi_{t+\tau \wedge n}^{t,\varphi})] - \lim_{n \rightarrow \infty} \mathbb{E}_0[\hat{M}(\Phi_{t+n}^{t,\varphi}) I(\tau = \infty)] \\ &= \hat{M}(\varphi) \end{aligned}$$

for $(t, \varphi) \in [0, \infty) \times \mathbb{R}_+$, where in the final equality we use (5.6) given that $\Phi^{t,\varphi}$ spends no time at b/a up to $t+\tau$ (as in the first step), and more crucially the fact that $\Phi_{t+n}^{t,\varphi} \rightarrow 0$ \mathbb{P}_0 -a.s. as $n \rightarrow \infty$ due to (8.11) below combined with the fact that $\hat{M}(0) = 0$. Given that the present setting creates no additional difficulties we will omit further details of this verification and this completes the proof. \square

2. We single out the following important consequence of the measure change in Section 3 for establishing the representation (8.9) above.

Proposition 12. We have

$$(8.11) \quad \Phi_{t+s}^{t,\varphi} \rightarrow 0 \quad \mathbb{P}_0\text{-a.s.}$$

as $s \rightarrow \infty$ for $(t, \varphi) \in [0, \infty) \times \mathbb{R}_+$ given and fixed.

Proof. Recall from (6.3) and (6.13) that $(\Phi_{t+s}^{t,\varphi})_{s \geq 0}$ is a positive local martingale under \mathbb{P}_0 . Since a positive local martingale is a supermartingale and a positive supermartingale converges

almost surely, it is enough to show that the convergence relation (8.11) holds in \mathbb{P}_0 -probability. For this, note from (6.3) with (6.4) that

$$(8.12) \quad \begin{aligned} \mathbb{P}_0(\Phi_{t+s}^{t,\varphi} \geq \varepsilon) &= \mathbb{P}_0\left(\frac{\pi}{1-\pi} \int_{-\infty}^{\infty} e^{mx - \frac{m^2}{2}t} e^{mX_s - \frac{m^2}{2}s} F_\mu(dm) \geq \varepsilon\right) \\ &= \mathbb{P}_0\left(\int_{-\infty}^{\infty} M(m) e^{m\sqrt{s}X_1 - \frac{(m\sqrt{s})^2}{2}} F_\mu(dm) \geq \varepsilon\right) \end{aligned}$$

for $\varepsilon > 0$ due to $X_s \sim \sqrt{s}X_1$ for $s \geq 0$ because X is a standard Brownian motion under \mathbb{P}_0 and where we set $M(m) := (\pi/(1-\pi))e^{mx - (m^2/2)t}$ for $m \in \mathbb{R}$ with $x = 0$ if $t = 0$. Fix $\omega \in \Omega$ and note that

$$(8.13) \quad N_s(m) := M(m) e^{m\sqrt{s}X_1(\omega) - \frac{(m\sqrt{s})^2}{2}} \rightarrow 0$$

as $s \rightarrow \infty$ for every $m \in \mathbb{R}$. Moreover, we have

$$(8.14) \quad \sup_{s \geq 0} N_s(m) := M(m) \sup_{s \geq 0} e^{m\sqrt{s}X_1(\omega) - \frac{(m\sqrt{s})^2}{2}} \leq M(m) \sup_{s \geq 0} e^{s|X_1(\omega)| - \frac{s^2}{2}} =: M(m)Z(\omega)$$

for all $m \in \mathbb{R}$ where $Z(\omega)$ is a constant which does not depend on $m \in \mathbb{R}$. Since $m \mapsto M(m)Z(\omega)$ is integrable on the probability space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), F_\mu(dm))$ it follows by the dominated convergence theorem using (8.13) that

$$(8.15) \quad \lim_{s \rightarrow \infty} \int_{-\infty}^{\infty} M(m) e^{m\sqrt{s}X_1(\omega) - \frac{(m\sqrt{s})^2}{2}} F_\mu(dm) = 0$$

for all $\omega \in \Omega$. Using this fact in (8.12) we see that $\mathbb{P}_0(\Phi_{t+s}^{t,\varphi} \geq \varepsilon) \rightarrow 0$ as $s \rightarrow \infty$ for every $\varepsilon > 0$. Thus (8.11) holds in \mathbb{P}_0 -probability as sufficient and the proof is complete. \square

Remark 13. The question of uniqueness addressed in Theorem 11 above was left open in [8, Section 5]. The method of proof in [8] makes an essential use of the innovation process associated with X under the measure \mathbb{P}_π for $\pi \in [0, 1]$. Abandoning the innovation process and changing the measure \mathbb{P}_π to \mathbb{P}_0 for $\pi \in (0, 1)$ as done in Section 3 above yields the convergence relation (8.11) which in turn settles the question of uniqueness as explained in the second part of the proof above (cf. [8, Remark 5.4]).

Remark 14. The coupled system of nonlinear Volterra integral equations (8.3)+(8.4) can be used to find the optimal stopping boundaries b_0 and b_1 numerically. Note that the identity (8.8) can be used to produce a finite horizon approximation to the system obtained by replacing s with $T-t$ in (8.8) which yields the following extension of (8.9) above

$$(8.16) \quad \hat{V}(t, \varphi) = \mathbf{E}_0[\hat{M}(\Phi_T^{t,\varphi})] + \mathbf{E}_0\left[\int_0^{T-t} L(\Phi_{t+s}^{t,\varphi}) I(b_0(t+s) < \Phi_{t+s}^{t,\varphi} < b_1(t+s)) ds\right]$$

for $(t, \varphi) \in [0, T] \times \mathbb{R}_+$. Recalling that $\hat{V}(t, b_0(t)) = \hat{M}(b_0(t)) = ab_0(t)$ and $\hat{V}(t, b_1(t)) = \hat{M}(b_1(t)) = b$ for all $t \geq 0$ we see that (8.16) implies validity of the following extension of the system (8.3)+(8.4) above

$$(8.17) \quad ab_0(t) = \int_0^\infty \hat{M}(\psi) f(T, \psi; t, b_0(t)) d\psi + \int_0^{T-t} K(t+s, b_0(t+s), b_1(t+s); t, b_0(t)) ds$$

$$(8.18) \quad b = \int_0^\infty \hat{M}(\psi) f(T, \psi; t, b_1(t)) d\psi + \int_0^{T-t} K(t+s, b_0(t+s), b_1(t+s); t, b_1(t)) ds$$

for $t \in [0, T]$. Collecting the results derived throughout the paper we now disclose the solution to the initial problem for admissible laws (cf. Definition 4).

Corollary 15. *The value function in the initial problem (2.3) is given by*

$$(8.19) \quad V(\pi) = (1-\pi) \hat{V}\left(0, \frac{\pi}{1-\pi}\right)$$

for $\pi \in (0, 1)$ where the function \hat{V} is given by (8.5) above. The optimal stopping time in the initial problem (2.3) is given by

$$(8.20) \quad \tau_* = \inf \left\{ t \geq 0 \mid \frac{\pi}{1-\pi} \int_{-\infty}^\infty e^{mX_t - \frac{m^2}{2}t} F_\mu(dm) \notin (b_0(t), b_1(t)) \right\}$$

where b_0 and b_1 are a unique solution to the coupled system of nonlinear Volterra integral equations (8.3)+(8.4). The optimal decision function d_{τ_*} equals 0 or 1 and we conclude that a non-zero drift is not present or is present in the observed motion if the stopping in (8.20) happens at b_0 or b_1 respectively.

Proof. The identity (8.19) follows by combining (3.18)+(3.19) in Proposition 2 with (4.5) and the result of Theorem 11. The explicit form (8.20) follows from (8.6) in Theorem 11 combined with (3.4) and (4.1). The final claim on the optimal decision function follows from the general argument invoked following (2.5) above. This completes the proof. \square

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