






Symbolic Semantics for Probabilistic Programs

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Abstract. We present a new symbolic execution semantics of probabilistic programs that include observe statements and sampling from continuous distributions. Building on Kozen’s seminal work, this symbolic semantics consists of a countable collection of measurable functions, along with a partition of the state space. We use the new semantics to provide a full correctness proof of symbolic execution for probabilistic programs. We also implement this semantics in the tool `symProb`, and illustrate its use on examples.

1 Introduction

Probabilistic programming languages are designed to make probabilistic computations easier to express for a broader scientific community. They can be used to model behaviour based on data that carries uncertainty or randomness, as found in, e.g., robotics [30], machine learning [12, 23], statistics [11], and cryptography [14]. Besides traditional programming constructs, a key aspect of a probabilistic language is the ability to *sample* random values, in order to represent the uncertainty that occurs in the real world. It is essential that these programming languages have a rigorous foundation, such that correctness and safety properties can be guaranteed when designing and implementing tools for probabilistic program analysis, optimization, and compilation.

Probabilistic semantics was first studied by Kozen [19], using measure theory to relate denotational and operational semantics of imperative probabilistic programs. The *denotational* semantics represents states as probability measures over program variables and programs as measure transformers. It enables one to effectively reason about programs as a whole. The *operational* semantics, on the other hand, is a computation model describing step-by-step computation in terms of a measurable function. A correctness theorem for the operational semantics states that the produced sets of outputs are distributed correctly according to the measure transformer of the denotational semantics.

To enable more detailed reasoning about probabilistic programs, we introduce a new semantics, inspired by symbolic execution techniques. For non-probabilistic programs, symbolic execution [1, 18] has been very successful in program analysis techniques such as debugging, test generation, and verification (e.g., [3, 5–8, 13, 15, 16]). Its appeal arises from the fact that one symbolic execution is an abstraction of possibly infinitely many executions in the concrete state space, that all share a single execution path through the program. Symbolic execution interprets program variables symbolically, such that assignments update a symbolic substitution and conditional statements produce so-called

path conditions that must be true for the program’s input variables to execute each path. Thus, symbolic execution generates pairs consisting of a symbolic substitution and a path condition. If the Boolean path condition holds in some initial state, then running the program is the same as applying the corresponding substitution to the initial state.

We use the new semantics to provide a full correctness proof of symbolic execution of probabilistic programs with respect to a denotational semantics à la Kozen. The correctness proof is highly nontrivial: one semantics deals with the program’s symbolic execution traces, the other interprets programs as measure transformers. Our approach is to first prove a one-to-one correspondence between traces and the elements of our new symbolic semantics. The latter is equivalent to the operational semantics first introduced by Kozen [19], and hence to the denotational semantics. The full details of this proof are included in the appendices.

We consider a language with *observe* statements: in some interpretations of Bayesian inference, a programmer can express to have observed a value as a sample from some discrete or continuous distribution [28]. In others, including the imperative language that we consider, observe statements are given the semantics of asserting the truth of a provided Boolean formula. Observe statements were not studied in Kozen’s work, and an operational semantics for probabilistic programming with observe statements has not been formally defined in that setting. For our correctness proof, therefore, we must extend both denotational and operational semantics with an interpretation of *observe*, and state and prove the corresponding correctness theorem anew.

Intuitively, observe statements in an operational semantics can be modelled by *rejection sampling*. That is, any execution that violates the observed formula is aborted. The probability mass will then reside in execution traces where the observe condition holds, and the probability of the set of traces where it does not hold will be annihilated. We formalize this semantics and prove its correctness with respect to a denotational semantics that interprets observe b as a measure transformer that restricts measures to the set corresponding with the Boolean formula b . This is an interpretation based on Bayes’ theorem.

Contributions and Overview. We start the paper by introducing the language using an example program and discussing the challenges in the technical development (Section 2). The subsequent technical sections of the paper present the following contributions:

- An in-depth description of symbolic execution of probabilistic programs (Section 3), which supports discrete and continuous sampling as well as Bayesian inference through observing Boolean formulas of positive-measure sets. In symbolic execution, sampled values will be represented by symbolic variables.
- As a main contribution we provide a full correctness proof of symbolic execution of probabilistic programs (Section 4) by introducing a new *symbolic* semantics for imperative probabilistic programs and proving correspondence with established semantics (which we extend to support *observe* statements).
- To showcase symbolic execution of probabilistic programs, we have developed the tool `symProb` that performs bounded symbolic execution according to our new symbolic semantics (Section 5). At the end of a symbolic execution, `symProb` reports the path condition, the substitution mapping, and the set of Boolean formulas which correspond to the program’s observe statements. We report on running `symProb` on a series of examples to show how our semantics can be applied.

2 Probabilistic Programming

We introduce the language through an example and then present its grammar.

2.1 Example Program

Consider the probabilistic program on the right, modelling the gender distribution among people taller than 200 centimeters. In the Bernoulli sampling statement `bern(0.51)` for the variable `gender` we assume that 51% of the total population is male. Among men, height is normally distributed with mean 175 and variance 72, and among women with mean 161 and variance 50. The last line conditions the distribution on people taller than 200 centimeters.

```
gender ~ bern(0.51);
if (gender = 1) {
  height ~ norm(175,72);
} else {
  height ~ norm(161,50);
}
observe (height >= 200);
```

Symbolic execution has been very effective in analysis of non-probabilistic programs. The technique builds variable substitutions by analyzing assignments, and resolves conditional branching and iteration by use of nondeterminism. The conditionals are stored under variable substitution as a Boolean formula – called the *path condition* – that needs to be satisfied by the initial state for the corresponding substitution to be a representation of the program. Some problems arise when programs have discrete and continuous sample and observe statements, such as the above.

Consider for example a Bernoulli sampling statement $x_i \sim \text{bern}(e)$. One could introduce additional sets of symbolic variables for sampling statements, but it is practically infeasible to do so for each possible bias e , especially if one allows parameters to be arithmetic expressions. In our approach, we assume a finite number of *primitive* distributions (that is, unparameterized) and encode *parameterized* distributions using these primitives. A Bernoulli sampling statement $x_i \sim \text{bern}(e)$ is then encoded by use of uniform continuous sampling from the unit interval $[0, 1]$, written $x_i \sim \text{rnd}$, as follows:

```
 $x_i \sim \text{rnd}; \text{if } (x_i < t) \{x_i := 1\} \text{ else } \{x_i := 0\}$ 
```

A denotational semantics can be used to show soundness of such encodings.

For Gaussian distributed samples, the primitive distribution used is the *standard* Gaussian distribution, which has mean zero and variance one. To obtain a sample from a Gaussian distribution with mean e_1 and variance e_2 – both arithmetic expressions – a standard Gaussian sample is scaled (multiplied by $\sqrt{e_2}$) and translated (add e_1).

With these encodings, one trace through the example program above has final substitution σ and path condition ϕ , given by

$$\sigma = \{\text{gender} \mapsto 1, \text{height} \mapsto z_0 \sqrt{72} + 175\} \quad \phi \equiv y_0 < 0.51 \wedge 1 = 1$$

Here, y_0 is a symbolic variable that is uniformly distributed over the interval $[0, 1]$. As it turns out, the probability of the path condition as measured by the input measure is the *prior* probability 0.51 of the trace.

The obtained Boolean formula from *observe* statements in this symbolic execution is

$$\psi \equiv z_0 \cdot \sqrt{72} + 175 \geq 200,$$

where z_0 is a symbolic variable representing a standard Gaussian sample. Measuring the set represented by this Boolean formula with the input measure yields exactly the *likelihood* of the model we have in mind. To keep the prior and the likelihood separate, we collect Boolean conditions under *observe* statements in what we coin the *path observations*.

This paper formally describes the procedure above for general probabilistic programs, and proves correctness with respect to a denotational semantics. Building on work by Kozen [19], we choose to interpret substitutions from symbolic execution as measurable functions on the value space, and path conditions and path observations as measurable sets of values for the program variables. These can then be compared against Kozen’s denotational semantics, where program states are subprobability measures over the domain of values for the program variables, and a program p is a transformer of input to output measures $\llbracket p \rrbracket: \mathcal{M}(\mathbb{R}^n) \rightarrow \mathcal{M}(\mathbb{R}^n)$. The use of measure theory is unavoidable in order to handle continuous random variables. For example the semantics of `height \sim norm(175,72)`, given an input $\mu \in \mathcal{M}(\mathbb{R})$, is the measure:

$$\llbracket \text{height} \sim \text{norm}(175,72) \rrbracket(\mu) : A \mapsto \gamma_{175,72}(A)$$

Here, $\gamma_{175,72}$ denotes the Gaussian measure with expected value 175 and variance 72.

After defining the language formally in Section 2.2, we provide a detailed description of symbolic execution of probabilistic programs in Section 3. There, towards a correctness proof, we also introduce the *symbolic semantics*. This is a *big-step* semantics for symbolic execution—this is stated in Theorem 4. Section 4 is aimed at proving correctness with respect to a measure-transforming semantics. The correctness statement (Theorem 5) says that the sum of probabilities of all symbolic execution paths, as measured by the input measure, expressed using the final substitutions, path conditions, and path observations, is the same as the probability under the denotational semantics. The technical contribution is concluded with a proof of concept tool that implements symbolic execution and we perform some experiments with it (Section 5).

2.2 Language

To express programs like the above example, we consider a language that contains basic imperative constructs (assignments, sequential composition, conditionals, and loops) together with constructs to manipulate random variables (sampling) and observe statements that allow conditioning on an observation defined by a Boolean condition:

| | | |
|---|---|---|
| $\mathbb{E} \ni e ::= q \in \mathbb{Q}$ x_i $\text{op}(e_1, \dots, e_{\# \text{op}})$ | $\mathbb{B} \ni b ::= \text{False}$ True $e \diamond e$ $b \parallel b$ $b \&\& b$ $!b$ | $\mathbb{P} \ni p ::= \text{Skip}$ $x_i := e$ $x_i \sim \text{rnd}$ $\text{observe } b$ $p \ddagger p$ $\text{if } b \text{ } p \text{ else } p$ $\text{while } b \text{ } p$ |
| $x_i \in \mathcal{X}$ $\diamond \in \{<, \leq, =, ! =, \geq, >\}$ | | |

It defines *expressions* $e \in \mathbb{E}$ over program variables $x_i \in \mathcal{X}$ (where $i \in \mathbb{N}$) and constants $q \in \mathbb{Q}$ using operators op of arity $\# \text{op}$. Expressions $e \in \mathbb{E}$ are interpreted as functions

$\bar{e} : \mathbb{R}^n \rightarrow \mathbb{R}$ that compute the value of e given a valuation $v : \{0, 1, \dots, n-1\} \rightarrow \mathbb{R}$ of the program variables. Formally, op is a function $\overline{\text{op}} : \mathbb{R}^m \rightarrow \mathbb{R}$ where $m = \#\text{op}$, and

$$\bar{q}(v) = q, \quad \bar{x}_i(v) = v(i), \quad \overline{\text{op}(e_1, \dots, e_{\#\text{op}})}(v) = \overline{\text{op}}(\bar{e}_1(v), \dots, \bar{e}_{\#\text{op}}(v))$$

The grammar also defines *Boolean expressions* $b \in \mathbb{B}$ that can relate expressions and apply the standard Boolean operators. We slightly overload notation and interpret Boolean expressions b as subsets \bar{b} of \mathbb{R}^n where the formula holds. For example, $\overline{\text{True}} = \mathbb{R}^n$ and if $b = e_1 \diamond e_2$ then

$$\bar{b} = \{v \in \mathbb{R}^n \mid \bar{e}_1(v) \diamond \bar{e}_2(v)\}.$$

Boolean conjunction, disjunction, and negation are respectively interpreted as set intersection, union, and complement in \mathbb{R}^n .

The Boolean expressions are used in the *if* and *while* conditions and in observe statements of *program statements* $p \in \mathbb{P}$. Whenever some program $p \in \mathbb{P}$ is fixed, there is also a fixed amount of n variables.

Sampling statements $x_i \sim \text{rnd}$ draw uniform random samples from the unit interval $[0, 1]$. For presentation purposes, we sample from only one primitive distribution in the formal grammar, and we do so at the level of statements rather than in expressions. Expressions can be made probabilistic, however, by first sampling and then using the variable in a (Boolean) expression. We also stress again that the language can be extended to support sampling from a multitude of other primitive distributions, both discrete and continuous, without affecting any of the results in this paper.

With the extensions described in Section 2.1, the probabilistic program presented there is in the language generated by our grammar.

3 Symbolic Execution

The inductive rules that define a transition system implementing symbolic execution of probabilistic programs are presented in fig. 1. The *symbolic states* in this transition system are quintuples consisting of the following data: (i) a *program* p that is to be executed; (ii) a *substitution* σ of program variables to symbolic expressions (defined shortly), which captures past program behavior; (iii) a *sampling index*¹ $k \in \mathbb{N}$; (iv) the *path condition* as a precondition for this symbolic trace; and (v) the *path observation* as a means to bookkeep which states will be accepted by *observe* statements throughout execution. Progression of the system depends only on the program syntax (i). Below we provide a detailed description of the components (ii)-(v). The substitutions (ii) and path conditions (iv) follow established methods from symbolic execution [3]; the notion of path observation (v) is novel.

The program `Skip` cannot make a transition and represents the *terminated* program. The system is nondeterministic due to the pairs of rules for *if* and *while* statements. The system has infinite symbolic traces due to Rule `iter-T`. Customarily, $\xrightarrow{*}$ denotes the reflexive-transitive closure of \longrightarrow .

¹ One for each primitive distribution: we use one in this paper for presentation purposes. Two sampling indices are used in the tool `symProb`.

| | |
|---|--------|
| $\frac{}{(x_i := e, \sigma, k, \phi, \psi) \longrightarrow (\text{Skip}, \sigma[x_i \mapsto \sigma e], k, \phi, \psi)}$ | asgn |
| $\frac{}{(x_i \sim \text{rnd}, \sigma, k, \phi, \psi) \longrightarrow (\text{Skip}, \sigma[x_i \mapsto y_k], k + 1, \phi, \psi)}$ | smp1 |
| $\frac{}{(\text{observe } b, \sigma, k, \phi, \psi) \longrightarrow (\text{Skip}, \sigma, k, \phi, \psi \wedge \sigma b)}$ | obs |
| $\frac{}{(\text{Skip} \ ; \ p, \sigma, k, \phi, \psi) \longrightarrow (p, \sigma, k, \phi, \psi)}$ | seq-0 |
| $\frac{(p, \sigma, k, \phi, \psi) \longrightarrow (p', \sigma', k', \phi', \psi')}{(p \ ; \ q, \sigma, k, \phi, \psi) \longrightarrow (p' \ ; \ q, \sigma', k', \phi', \psi')}$ | seq-n |
| $\frac{}{(\text{if } b \ p_1 \ \text{else } p_2, \sigma, k, \phi, \psi) \longrightarrow (p_1, \sigma, k, \phi \wedge \sigma b, \psi)}$ | if-T |
| $\frac{}{(\text{if } b \ p_1 \ \text{else } p_2, \sigma, k, \phi, \psi) \longrightarrow (p_2, \sigma, k, \phi \wedge \sigma!b, \psi)}$ | if-F |
| $\frac{}{(\text{while } b \ p, \sigma, k, \phi, \psi) \longrightarrow (\text{Skip}, \sigma, k, \phi \wedge \sigma!b, \psi)}$ | iter-F |
| $\frac{}{(\text{while } b \ p, \sigma, k, \phi, \psi) \longrightarrow (p \ ; \ \text{while } b \ p, \sigma, k, \phi \wedge \sigma b, \psi)}$ | iter-T |

Fig. 1: Inductive transition rules for symbolic execution

3.1 Symbolic Substitutions

To capture the behavior of a symbolic trace, a *substitution* assigns to every program variable a *symbolic expression*. Symbolic expressions are generated by the following grammar (recall that $\#_{\text{op}}$ denotes the arity of op):

$$\mathbb{S}\mathbb{E} \ni \mathbb{E} ::= i \mid q \in \mathbb{Q} \mid x_i \in \mathcal{X} \mid y_k \in \mathcal{Y} \mid \text{op}(E, \dots, E_{\#_{\text{op}}})$$

$\mathbb{S}\mathbb{E}$ extends \mathbb{E} with the set $\mathcal{Y} = \{y_0, y_1, \dots\}$ as base cases, representing the random samples that may be drawn during execution. When the language samples from more than one primitive distribution, a set of symbolic variables $\mathcal{Y}_{\mathcal{D}}$ is used for every primitive distribution \mathcal{D} . For each such distribution \mathcal{D} , one will need a sampling index $k_{\mathcal{D}}$ in the transition system. We use $\mathcal{Z} = \{z_0, z_1, \dots\}$ for symbolic variables representing standard normal samples in our examples and in the tool.

Thus, formally, substitutions are maps $\sigma : \mathcal{X} \rightarrow \mathbb{S}\mathbb{E}$. We sometimes write $\sigma(i)$ in lieu of $\sigma(x_i)$. The *updated* substitution $\sigma[i \mapsto E]$ denotes the substitution σ' such that $\sigma'(j) = \sigma(j)$ for $j \neq i$ and $\sigma'(i) = E$. Any substitution σ inductively extends to expressions $e \in \mathbb{E}$ by $\sigma(\text{op}(e_1, \dots, e_{\#_{\text{op}}})) := \text{op}(\sigma(e_1), \dots, \sigma(e_{\#_{\text{op}}}))$.

If symbolic execution is to conform with a denotational semantics, the symbolic substitutions have to be interpreted as concrete state transformers. For this, first, symbolic *expressions* are interpreted as functions $\mathbb{R}^{n+\omega} \rightarrow \mathbb{R}$, where the first n arguments are

the values of the program variables, and the rest is an infinite stream of samples to be drawn. Here, one may use extra streams of sample spaces for any additional primitive distributions in the language. The mapping $|\cdot| : \mathbb{R}^{n+\omega} \rightarrow \mathbb{R}$ equips symbolic expressions with a formal interpretation as follows:

$$\begin{aligned} |q| : \rho \mapsto q, & & |x_i| : \rho \mapsto \rho_i, & & |y_k| : \rho \mapsto \rho_{n+k}, \\ |\text{op}(E_1, \dots, E_{\# \text{op}})| : \rho \mapsto \overline{\text{op}}(|E_1|(\rho), \dots, |E_{\# \text{op}}|(\rho)) \end{aligned}$$

The symbols y_k thus pick the k -th sample available in \mathbb{R}^ω . Symbolic expressions free of y_k are just program expressions, and their interpretations agree in the following way:

Lemma 1 (Substitution lemma for expressions). *For all $e \in \mathbb{E}$, $|e|(\rho) = \bar{e}(\rho|_n)$.*

This interpretation of $E \in \mathbb{SE}$ as a function $\mathbb{R}^{n+\omega} \rightarrow \mathbb{R}$ extends naturally to symbolic substitutions $\sigma \in \mathbb{SE}^n$ as functions $\mathbb{R}^{n+\omega} \rightarrow \mathbb{R}^n$, by mere point-wise application after substitution. However, since we want to compose substitutions (their interpretations) due to sequencing, the codomain must be extended from \mathbb{R}^n to $\mathbb{R}^{n+\omega}$:

Definition 2 (Interpretation of symbolic substitutions). *Let $\sigma : \mathcal{X} \rightarrow \mathbb{SE}$ be a substitution and $k \in \mathbb{N}$ a sampling index. The interpretation of σ at sampling index k , denoted $|\sigma|_k$, is the function $\mathbb{R}^{n+\omega} \rightarrow \mathbb{R}^{n+\omega}$ defined by*

$$|\sigma|_k : \rho \mapsto (|\sigma(x_0)|(\rho), \dots, |\sigma(x_{n-1})|(\rho), \rho_{n+k}, \rho_{n+k+1}, \rho_{n+k+2}, \dots)$$

The first n values of $|\sigma|_k(\rho)$ are just pointwise applications of $|\cdot|$. The remaining elements of $|\sigma|_k(\rho)$ are the same as those of ρ , but left-shifted k positions. This formalizes the idea that $|\sigma|_k$ has already drawn k samples.

3.2 Path Conditions and Path Observations

The Boolean formulas for the condition in *branching* and *iteration* statements are aggregated under substitution to build the *path condition* of a symbolic trace. The path condition represents the unique part of the input space that triggers the symbolic trace. Similarly, *observed* Boolean formulas under substitution make up the *path observation*, and represent the part of the input space that will lead to acceptance of *observe* statements in this trace.

Path conditions and observations are expressed as *symbolic Boolean expressions*:

$$\mathbb{SB} \ni B ::= \perp \mid \top \mid E \diamond E \mid B \vee B \mid B \wedge B \mid \neg B$$

Substitutions $\sigma : \mathcal{X} \rightarrow \mathbb{SE}$ extend through $\mathbb{E} \rightarrow \mathbb{SE}$ further to Booleans, as in $\mathbb{B} \rightarrow \mathbb{SB}$, in a completely trivial way. For example, $\sigma(e_1 \diamond e_2 \ \&\& \ \text{True}) = \sigma(e_1) \diamond \sigma(e_2) \wedge \top$.

Path conditions and path observations – their symbolic Boolean expressions – are interpreted as subsets of $\mathbb{R}^{n+\omega}$ where the formula is satisfied. The mapping $|\cdot|$ equips symbolic Boolean expressions with a formal interpretation as follows:

$$\begin{aligned} |\perp| &= \emptyset, & |B_1 \vee B_2| &= |B_1| \cup |B_2|, \\ |\top| &= \mathbb{R}^{n+\omega}, & |B_1 \wedge B_2| &= |B_1| \cap |B_2|, \\ |\neg B| &= \mathbb{R}^{n+\omega} \setminus |B|, & |E_1 \diamond E_2| &= \{\rho \in \mathbb{R}^{n+\omega} \mid |E_1|(\rho) \diamond |E_2|(\rho)\}. \end{aligned}$$

Here we overload notation of $|\cdot|$ to use it for both expressions and Boolean expressions.

3.3 Final Configurations

Let σ_0 denote the *initial* substitution $\{x_i \mapsto x_i\}_{x_i \in \mathcal{X}}$. We call the quadruple (σ, k, ϕ, ψ) the (symbolic) *configuration* of a state $(p, \sigma, k, \phi, \psi)$ and $(\sigma_0, 0, \top, \top)$ is the *initial configuration*. The configurations we are mostly interested in result from a finite symbolic execution trace starting from the initial configuration:

$$\Gamma_p := \{(\sigma, k, \phi, \psi) \mid (p, \sigma_0, 0, \top, \top) \xrightarrow{*} (\text{Skip}, \sigma, k, \phi, \psi)\}$$

is called the set of *final configurations* of p .

For a program $p \in \mathbb{P}$ and a configuration $(\sigma, k, \phi, \psi) \in \Gamma_p$, p transforms inputs $\rho \in |\phi| \cap |\psi|$ to the output $|\sigma|_k(\rho)$. That is, p behaves like $|\sigma|_k$ on $|\phi| \cap |\psi|$. Execution of p on inputs from $|\phi| \setminus |\psi|$ leads to an unsatisfied *observe* statement.

Example 3. Consider the example program in Section 2.1 (call it p) that models the gender distribution among people taller than 200 centimeters. Due to Bernoulli sampling, it contains an additional *if* statement hidden in the encoding. The program has thus *four* symbolic traces; their respective final configurations $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \Gamma_p$ are:

| Final | Substitution σ | k_y | k_z | Path condition ϕ | Path observation ψ |
|------------|---|-------|-------|---------------------------------|-------------------------------|
| γ_1 | $\{g \mapsto 1, h \mapsto z_0\sqrt{72} + 175\}$ | 1 | 1 | $y_0 < 0.51 \wedge 1 = 1$ | $z_0\sqrt{72} + 175 \geq 200$ |
| γ_2 | $\{g \mapsto 1, h \mapsto z_0\sqrt{50} + 161\}$ | 1 | 1 | $y_0 < 0.51 \wedge 1 \neq 1$ | $z_0\sqrt{50} + 161 \geq 200$ |
| γ_3 | $\{g \mapsto 0, h \mapsto z_0\sqrt{72} + 175\}$ | 1 | 1 | $y_0 \geq 0.51 \wedge 0 = 1$ | $z_0\sqrt{72} + 175 \geq 200$ |
| γ_4 | $\{g \mapsto 0, h \mapsto z_0\sqrt{50} + 161\}$ | 1 | 1 | $y_0 \geq 0.51 \wedge 0 \neq 1$ | $z_0\sqrt{50} + 161 \geq 200$ |

The final configurations γ_2 and γ_3 have unsatisfiable path conditions. Path conditions represent the *priors* in the model and path observations represent *likelihoods*.

3.4 Symbolic Semantics

Now we introduce symbolic semantics for probabilistic programs with the main purpose of proving correctness of symbolic execution, which we just described. The new semantics is a set of functions, and can in fact be considered a denotational semantics for symbolic execution of probabilistic programs. Defined directly on the syntax of programs, the semantics consists of the interpretations of all final symbolic configurations—this is Theorem 4 below.

For a triple (F, B, \mathcal{O}) in the definition below, F is the final substitution of a trace, B is the path condition, and \mathcal{O} is the path observation. Recall that $\mathcal{B}(X)$ is the Borel σ -algebra of X and let $\mathcal{B}(X, Y)$ denote the space of Borel measurable functions $X \rightarrow Y$. Then, for programs $p \in \mathbb{P}$ in n variables, the sets

$$\mathcal{F}_p \subset \mathcal{B}(\mathbb{R}^{n+\omega}, \mathbb{R}^{n+\omega}) \times \mathcal{B}(\mathbb{R}^{n+\omega}) \times \mathcal{B}(\mathbb{R}^{n+\omega})$$

are defined inductively on the structure of p as follows:

- For *inaction* Skip, the state remains unaltered and there is no restriction on the path condition or the path observation:

$$\mathcal{F}_{\text{skip}} := \{(\rho \mapsto \rho, \mathbb{R}^{n+\omega}, \mathbb{R}^{n+\omega})\}$$

- An *assignment* has no restriction on the precondition, but the state is updated according to the assignment:

$$\mathcal{F}_{x_i := e} := \{(\rho \mapsto \rho[i \mapsto \bar{e}(\rho|_n)], \mathbb{R}^{n+\omega}, \mathbb{R}^{n+\omega})\}$$

Only the first n values $\rho|_n$ of the state ρ are needed to evaluate e , and $\rho[i \mapsto a]$ denotes the state ρ' where $\rho'(j) = \rho(j)$ if $j \neq i$ and $\rho'(i) = a$.

- When *sampling* we also merely perform an appropriate state update:

$$\mathcal{F}_{x_i \sim \text{rnd}} := \{(\rho \mapsto \text{sample}_i(\rho), \mathbb{R}^{n+\omega}, \mathbb{R}^{n+\omega})\}$$

Here, $\text{sample}_i(\rho)$ is an updated stream ρ' that has drawn one sample:

$$\text{sample}_i(\rho) = (\rho_0, \dots, \rho_{i-1}, \rho_n, \rho_{i+1}, \dots, \rho_{n-1}, \rho_{n+1}, \rho_{n+2}, \dots)$$

- *Observing* a Boolean formula only updates the path observation:

$$\mathcal{F}_{\text{observe } b} := \{(\rho \mapsto \rho, \mathbb{R}^{n+\omega}, \bar{b} \times \mathbb{R}^\omega)\}$$

- When *sequencing* two programs p_1 and p_2 , range over all pairs of executions $(F_1, B_1, \mathcal{O}_1) \in \mathcal{F}_{p_1}$ and $(F_2, B_2, \mathcal{O}_2) \in \mathcal{F}_{p_2}$ and compose them. The first path condition B_1 should be satisfied and, after executing the first component F_1 , the second path condition B_2 should be satisfied (and sim. for the path observation):

$$\mathcal{F}_{p_1 \& p_2} := (F_2 \circ F_1, B_1 \cap F_1^{-1}[B_2], \mathcal{O}_1 \cap F_1^{-1}[\mathcal{O}_2]) \mid (F_i, B_i, \mathcal{O}_i) \in \mathcal{F}_{p_i}, i = 1, 2\}$$

- The two branches of an *if* statement are put together in a binary union of sets. The path conditions are updated accordingly ($-^c$ denotes complement):

$$\begin{aligned} \mathcal{F}_{\text{if } b \text{ p else } q} := & \{(F, B \cap (\bar{b} \times \mathbb{R}^\omega), \mathcal{O}) \mid (F, B, \mathcal{O}) \in \mathcal{F}_p\} \\ & \cup \{(F, B \cap (\bar{b}^c \times \mathbb{R}^\omega), \mathcal{O}) \mid (F, B, \mathcal{O}) \in \mathcal{F}_q\} \end{aligned}$$

- In a *while* statement, the union is over every possible number of iterations m . For $m = 0$, the behavior is that of Skip and the precondition is the negation of the Boolean formula. Every next number $m + 1$ of loop iterations takes all possible executions of m iterations, pre-composes all possible additional iterations, and updates the preconditions accordingly:

$$\mathcal{F}_{\text{while } b \text{ p}} := \bigcup_{m=0}^{\infty} \mathbb{F}_{b,p}^m \{(v \mapsto v, \bar{b}^c, \mathbb{R}^{n+\omega})\},$$

where $\mathbb{F}_{b,p}^m$ denotes m applications of the mapping $\mathbb{F}_{b,p}$ from $\mathcal{B}(\mathbb{R}^{n+\omega}, \mathbb{R}^{n+\omega}) \times \mathcal{B}(\mathbb{R}^{n+\omega}) \times \mathcal{B}(\mathbb{R}^{n+\omega})$ to itself that pre-composes an additional iteration of the loop:

$$\begin{aligned} \mathbb{F}_{b,p} : \mathcal{F} \mapsto & \{ (F \circ F_q, (\bar{b} \times \mathbb{R}^\omega) \cap B_q \cap F_q^{-1}[B], \mathcal{O}_q \cap F_q^{-1}[\mathcal{O}]) \\ & \mid (F, B, \mathcal{O}) \in \mathcal{F}, (F_q, B_q, \mathcal{O}_q) \in \mathcal{F}_q \} \end{aligned}$$

The sets \mathcal{F}_p form a big-step semantics for symbolic execution of probabilistic programs:

Theorem 4. *For any program $p \in \mathbb{P}$, there is a one-to-one correspondence between final configurations $(\sigma, k, \phi, \psi) \in \Gamma_p$ and triples $(F, B, \mathcal{O}) \in \mathcal{F}_p$ such that $F = |\sigma|_k$, $B = |\phi|$, and $\mathcal{O} = |\psi|$.*

In this correspondence, we consider all final configurations $(\sigma, k, \phi, \psi) \in \Gamma_p$ with unsatisfiable path condition, i.e., $|\phi| = \emptyset$, equivalent. Similarly, all triples (F, B, \mathcal{O}) for which $B = \emptyset$ are considered equivalent. The proof is in Appendix A.2.

4 Correctness

The symbolic execution engine described in the previous section is now proven correct with respect to a denotational semantics. Following Kozen [19], probabilistic programs are mappings of measures on the Borel measurable space of \mathbb{R}^n , where n is the number of program variables. Observe statements were not considered in his work. They have been studied [4] in this context of measure transformer semantics, but in the absence of unbounded loops.

To be fully precise, we need to recall some definitions from measure theory. A *measurable space* (X, Σ) is a set X equipped with a σ -algebra Σ , i.e., a set $\Sigma \subseteq \mathcal{P}(X)$ that (i) contains the emptyset, (ii) is closed under complements in X , and (iii) is closed under countable union. Elements of Σ are called *measurable sets*. The *Borel* σ -algebra $\mathcal{B}(X)$ is the one generated by the open sets of X . Whenever we say that functions or sets are measurable, we mean with respect to the Borel σ -algebras. A *(sub)probability measure* on (X, Σ) is a function $\mu : \Sigma \rightarrow [0, 1]$ such that $\mu(\emptyset) = 0$ and $\mu(\bigcup_{i \in I} A_i) = \sum_{i \in I} \mu(A_i)$ for any countable disjoint family of sets $(A_i)_{i \in I} \subseteq \Sigma$. We let $\mathcal{M}(X)$ denote the set of measures on the Borel σ -algebra of X ; this makes $\mathcal{M}(\mathbb{R}^n)$ the state space in the denotational semantics.

The denotational semantics of a program p is a function $\llbracket p \rrbracket : \mathcal{M}(\mathbb{R}^n) \rightarrow \mathcal{M}(\mathbb{R}^n)$, pushing a measure corresponding to the current program state forward, according to the statement being interpreted. The inductive definition is as follows:

- *Skip* does not change the given measure: $\llbracket \text{Skip} \rrbracket(\mu) = \mu$.
- For an *assignment*, let α_e^i be the function that updates the i -th value appropriately:

$$\alpha_e^i : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{i-1}, \bar{e}(x_1, \dots, x_n), x_{i+1}, \dots, x_n)$$

This function is measurable, so its preimages can be measured by μ . Thus, the semantics of assignments as *pushforward* measures $\llbracket x_i := e \rrbracket(\mu) := \mu \circ (\alpha_e^i)^{-1}$ is well-defined. Intuitively, the measure $\llbracket x_i := e \rrbracket(\mu)$ measures with μ the set of values in \mathbb{R}^n that lead to appropriately updated values for the i -th variable.

- *Sampling* updates the measure component of the corresponding variable with the distribution measure λ to be sampled from:

$$\llbracket x_i \sim \text{rnd} \rrbracket(\mu) : A_1 \times \dots \times A_n \mapsto \mu(A_1 \times \dots \times A_{i-1} \times \mathbb{R} \times A_{i+1} \times \dots \times A_n) \cdot \lambda(A_i).$$

For λ , we use the Lebesgue measure on the unit interval for uniform continuous sampling. Other measures can be used for other primitive sampling statements. The measure $\llbracket x_i \sim \text{rnd} \rrbracket(\mu)$ here is defined on the rectangles of \mathbb{R}^n , and by Carathéodory's extension theorem, defines a unique measure on all of \mathbb{R}^n .

- When *observing*, we restrict the measure to the observed measurable set. For measurable $B \subseteq \mathbb{R}^n$, let $e_B(\mu)$ denote the subprobability measure $A \mapsto \mu(A \cap B)$. Then $\llbracket \text{observe } b \rrbracket(\mu) := e_{\bar{b}}(\mu)$. Note that this measure is not normalized.
- *Sequencing* is just composition: $\llbracket p ; q \rrbracket = \llbracket q \rrbracket \circ \llbracket p \rrbracket$.
- *Branching* restricts the measure of one branch (resp. the other) to the measurable subset where the condition is true (resp. false). Formally,

$$\llbracket \text{if } b \text{ } p_1 \text{ else } p_2 \rrbracket(\mu) = (\llbracket p_1 \rrbracket \circ e_{\bar{b}})(\mu) + (\llbracket p_2 \rrbracket \circ e_{\bar{\bar{b}}})(\mu)$$

This sum of measures is setwise. The measure is first restricted by $e_{\bar{b}}$ (or $e_{\bar{\bar{b}}}$) to a subprobability measure over the part of the state space where the condition is true (or false) and then passed on to be transformed by $\llbracket p_1 \rrbracket$ (or $\llbracket p_2 \rrbracket$).

- Interpreting *iterations* as repeated unfolding of if statements, the infinite sum

$$\llbracket \text{while } b \text{ } p \rrbracket(\mu) = \left(\sum_{m=0}^{\infty} e_{\bar{\bar{b}}} \circ (\llbracket p \rrbracket \circ e_{\bar{b}})^m \right)(\mu)$$

describes the semantics of while loops.

Now $\llbracket p \rrbracket$ is a positive operator of norm at most one for all p defined by the grammar in Section 2.2. This means that any subprobability measure is mapped to a subprobability measure [2]. Without while and observe statements, this norm would be exactly one. Intuitively, this is because programs would then always terminate, and no probability mass would ever be lost either by diverging while loops or by conditional probability. The measure semantics is not normalized.

Recall that Γ_p is the set of final configurations of symbolic execution of a program p .

Theorem 5 (Correctness of Symbolic Execution of Probabilistic Programs). *Let $p \in \mathbb{P}$ be a program, $\mu \in \mathcal{M}(\mathbb{R}^n)$ a distribution measure over the input variables, and $A \subseteq \mathbb{R}^n$ a measurable set. Then*

$$\llbracket p \rrbracket(\mu)(A) = \sum_{(\sigma, k, \phi, \psi) \in \Gamma_p} (\mu \otimes \lambda^\omega)(|\sigma|_k^{-1}[A \times \mathbb{R}^\omega] \cap |\phi| \cap |\psi|)$$

A friendly reminder that the measure λ was some chosen measure implicitly used in the denotational semantics; λ^ω denotes the product measure of infinitely many copies of it.

Proof (Sketch). For every program p , there is a function f_p such that

- $(\mu \otimes \lambda^\omega)(f_p^{-1}[A \times \mathbb{R}^\omega]) = \llbracket p \rrbracket(\mu)(A)$, and
- $(\mu \otimes \lambda^\omega)(f_p^{-1}[A \times \mathbb{R}^\omega]) = \sum_{(F, B, \mathcal{O}) \in \mathcal{F}_p} (\mu \otimes \lambda^\omega)(F^{-1}[A \times \mathbb{R}^\omega] \cap B \cap \mathcal{O})$.

The function f_p is basically the operational semantics defined in [19], but extended to an observe construct that rejects unsatisfied observed formulas. The theorem is proven by chaining these two equalities and applying Theorem 4. See Appendix A.3 for the full proof.

| Case Study | Number of Paths | | Samples | Lines | Time (sec.) |
|--------------------|-----------------|-----------|---------|-------|-------------|
| | Actual | Discarded | | | |
| BurglarAlarm | 4 | 12 | 4 | 26 | 0.31 |
| DieCond | 20 | 0 | 20 | 17 | 2.15 |
| Grass | 28 | 36 | 6 | 21 | 0.80 |
| MurderMystery | 2 | 2 | 2 | 12 | 0.06 |
| NeighborAge | 4 | 4 | 4 | 14 | 0.10 |
| NeighborBothBias | 7 | 5 | 4 | 19 | 0.24 |
| NoisyOr | 256 | – | 8 | 37 | 2.33 |
| Piranha | 3 | 1 | 2 | 12 | 0.07 |
| Random Z2 Walk (1) | 16 | – | 4 | 14 | 0.19 |
| Random Z2 Walk (2) | 52 | – | 6 | 14 | 0.60 |
| Random Z2 Walk (4) | 712 | – | 10 | 14 | 8.37 |
| Random Z2 Walk (8) | 159,436 | – | 18 | 14 | 2167.90 |
| SecuritySynthesis | 1 | 0 | 8 | 14 | 0.04 |
| TrueSkillFigure9 | 1 | 0 | 9 | 20 | 0.03 |
| TwoCoins | 3 | 1 | 2 | 6 | 0.05 |

Table 1: Performance metrics for `symProb` on a series of case studies. For the “Random Z2 Walk (i)” cases, i refers to the number of times the main `while` loop was unrolled.

5 Implementation and Experiments

We have developed `symProb` as a prototype implementation of the symbolic execution technique presented in Section 3². `symProb` takes as input a probabilistic program written in the language described in Section 2.2 (up to some natural imperative-style syntactic additions such as keywords for *else* and delimiter use of parentheses and braces). `symProb` performs *bounded* symbolic execution, meaning that all while loops are unrolled a finite number of times, to ensure termination. Finite loops are fully unrolled while all other loops are unrolled a configurable, yet fixed, number of times. `symProb` reports the final configuration: the final substitution σ , the amount of samples, the path condition ϕ , and the path observation ψ . All symbolic configurations reported by `symProb` are expressed as uninterpreted symbolic expressions. `symProb` is written in Rust in around 2,000 lines of code and uses Z3 [22] for real numbers to determine branch satisfiability.

We have executed `symProb` on a series of examples sourced from PSI [10], R2 [24], “Fun” programs from Infer.NET [21], and Barthe et al. [2]. We summarize our findings in Table 1. All the experiments were done on a machine with 3.3GHz Intel Core i7-5820K and 32 GB of RAM, running Linux 6.3.2-arch1-1.

One feat of symbolic execution of probabilistic programs that `symProb` implements, is that paths with an unsatisfiable *observe* statement can be filtered out, since they have zero likelihood in the probabilistic model. `symProb` therefore *discards* such paths.

For each experiment, we report the following metrics: the number of traces explored, the number of traces which were discarded due to a failed observe statement (‘–’ if there were no observe statements in the program), the maximum number of random samples drawn, the number of lines of code, and the time `symProb` took to explore all paths.

² Source code for `symProb` and all experiments are available on Zenodo [31].

We make a few general observations about the results. Outside of the DieCond, Grass, NoisyOr, and Random Z2 Walk (4,8) examples, symProb terminates in under a second. While these are relatively small examples, they still showcase a range of path counts and number of random samples drawn. Additionally, of the case studies that had observe statements present, many paths were discarded due to reaching an unsatisfiable observe statement. The Grass case study, for example, which involves six Bernoulli samples about weather conditions and such, had 36 paths discarded. In Section 7, we discuss how we might utilize this information in future work to optimize the performance of probabilistic programs with observe statements.

6 Related Work

Symbolic transition systems as described in this work have recently been formalized in a non-probabilistic setting by de Boer and Bonsangue [3]. From that starting point, this work extends to sampling in probabilistic programming by incorporating symbolic random variables $\{y_0, y_1, \dots\}$ in the symbolic substitutions and path conditions. Furthermore, we introduced *path observations* to keep track of observe statements.

Denotational semantics for Bayesian inference is an active research area [9, 17, 27, 28]. Mostly, the focus has been on *discrete* probabilistic programs [17, 25], whereas we support sampling of both discrete *and* continuous distributions, thereby generalizing the probabilistic choice based on discrete sampling used in these works.

Staton [27, 28] describes *observe-from* statements in his denotational semantics as an encoding for a *score* construct used to attach a likelihood scoring to an execution trace. The construct observe e from \mathcal{D} , where \mathcal{D} is some distribution, is then sugar for score $f_{\mathcal{D}}(e)$, where $f_{\mathcal{D}}$ is the probability density function for the distribution \mathcal{D} . The score value in that semantics is therefore akin to the probability measure of our path observation. Staton’s work considers language semantics of a functional nature, where the difficulty mainly lies in handling higher-order functions. In an imperative language, this is generally not a major concern. This allows us to consider the more general construct of observing Boolean formulas (of positive measure), rather than observing fixed samples (which may have zero measure). To the best of our knowledge, our work provides the first semantics for Boolean observe statements as a forward state transformer (as proposed by Kozen [19]) for an imperative probabilistic programming language in the presence of unbounded loops.

Sampson et al. [26] used symbolic execution to transform probabilistic programs into a Bayesian network. Their semantics was aimed at verification of certain probabilistic assertions, however, and lacked a formal foundation in measure theory. Luckow et al. combined symbolic execution with model counting to analyze programs with discrete distributions and nondeterministic choice [20]. Their work used schedulers to reduce Markov Decision Processes to Markov Chains, in contrast to our work with continuous distributions and observe statements, founded in measure theory. Susag et al. [29] considered symbolic execution for programs with random sampling to automatically verify quantitative correctness properties over unknown inputs. While they also used symbolic random variables, they were more focused on verifying correctness properties of “real-world” programs (e.g., written in C++) that use random sampling. They solely considered discrete distributions and did not support observe statements.

Gehr et al. [10] developed PSI, *probabilistic symbolic inference*, a tool that enables programmers to perform posterior distribution, expectation, and assertion queries through symbolic inference. Their symbolic reasoning engine works on symbolic representations of probability distributions, whereas we have symbolic terms which are interpreted as *random variables* of which we do not know the distribution in principle. By default, PSI computes a symbolic representation of the joint posterior distribution represented by the given probabilistic program. In contrast, `symProb` explores all paths through a given probabilistic program and reports on the path condition, substitution map, and path observation of each path. We see these two tools as complementary, as inference is only one aspect of probabilistic programming.

7 Conclusion and Future Work

We have defined new symbolic semantics for imperative probabilistic programs supporting forward execution and conditioning (Bayesian inference), which we proved to be a big-step semantics for symbolic execution of probabilistic programs. To support Bayesian inference, we extended Kozen’s denotational semantics to include `observe` statements of positive-measure events. Significantly, the symbolic semantics thus theoretically supports implementation of a symbolic executor for imperative probabilistic programs with continuous domain variables. We introduced *path observations* to keep track of the part of the initial state space that lead to acceptance of observe statements for each symbolic trace. The symbolic transition system is implemented in our prototype tool `symProb`. Its effectiveness has been demonstrated on example programs, producing results with path conditions and path observations that correctly represent (prior) path probabilities and likelihoods of the symbolic traces, as expected in light of Theorem 5. Since path conditions represent priors and path observations represent likelihoods, we believe it to be beneficial to conceptually separate the two in a symbolic executor.

Interestingly, our semantics and its accompanying correctness results enable one to prove the correctness of certain program transformations. More generally, we believe the symbolic semantics has potential applications in model checking tools, deductive proof systems, optimizations, and reasoning about termination. For example, one can exploit this theory to commute sample and observe statements under appropriate substitutions. In complex probabilistic models, the placement of observe statements may then become an optimization problem. We plan to explore such program transformations both theoretically and experimentally in the future. Moreover, we intend to investigate the limiting behavior of our correctness result, which contains an infinite sum in the presence of unbounded loops. One naturally wonders how fast the sum approximates the true posterior of the program, and how different structures in a program influence the speed of this approximation.

A shortcoming of our semantics for *observe* statements is that zero-measure events cannot be observed in our language. Denotationally, the measure would then always yield zero; operationally, *almost all* simulations will be aborted. Zero-measure observed Boolean conditions may be included in future work, for example by considering *measure couplings* and *disintegrations* [9].

Data availability. The source code for symProb and the experiments we performed with it are available on Zenodo [31].

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A Proofs

A.1 Proof of Lemma 1

Lemma 1 (Substitution lemma for expressions). *For all $e \in \mathbb{E}$, $|e|(\rho) = \bar{e}(\rho|_n)$.*

For *Boolean* expressions, we moreover have the following:

Lemma 6 (Substitution lemma for Boolean expressions). *For all $b \in \mathbb{B}$, $|\sigma_0 b| = \bar{b} \times \mathbb{R}^\omega$.*

Proof. By induction on the structure of expressions $e \in \mathbb{E}$ and Boolean expressions $b \in \mathbb{B}$ as presented in Section 2.2. There are two base cases for e :

- For $e = x_i$, we have $|x_i|(\rho) = \rho_i = \overline{x_i}(\rho|_n)$.
- For $e = q$: $|q|(\rho) = q = \overline{q}(\rho|_n)$.

There is an induction step for every operator op of arity $\#_{\text{op}}$. Each induction step applies $\#_{\text{op}}$ induction hypotheses:

$$\begin{aligned} |\text{op}(e_1, \dots, e_{\#_{\text{op}}})|(\rho) &= \overline{\text{op}}(|e_1|(\rho), \dots, |e_{\#_{\text{op}}}|(\rho)) && \text{definition of } |\cdot| \\ &= \overline{\text{op}}(\overline{e_1}(\rho|_n), \dots, \overline{e_{\#_{\text{op}}}}(\rho|_n)) && \text{IH } \#_{\text{op}} \text{ times} \\ &= \text{op}(e_1, \dots, e_{\#_{\text{op}}})(\rho|_n) && \text{definition of } \overline{\cdot} \end{aligned}$$

We proceed now with structural induction on b . The base cases:

- $b = \text{True}$: $|\sigma_0(\text{True})| = |\top|$ if and only if $\rho \in (\overline{\text{True}} \times \mathbb{R}^\omega)$.
- $b = \text{False}$: $|\sigma_0(\text{False})| = |\perp| = \emptyset = \overline{\text{False}} \times \mathbb{R}^\omega$.
- $b = e_1 \diamond e_2$: $\rho \in |\sigma_0(e_1 \diamond e_2)|$ if and only if $|e_1|(\rho) \diamond |e_2|(\rho)$ (by def.) if and only if $\overline{e_1}(\rho|_n) \diamond \overline{e_2}(\rho|_n)$ (by Lemma 1) if and only if $\rho \in e_1 \diamond e_2 \times \mathbb{R}^\omega$ (by def.).
We thus conclude this case with the identity $|\sigma_0(e_1 \diamond e_2)| = \overline{e_1 \diamond e_2} \times \mathbb{R}^\omega$.
- $b = b_1 \parallel b_2$: simply apply IH twice to show that their respective binary unions are equal, and then conclude that

$$|\sigma_0(b_1 \parallel b_2)| = \overline{b_1 \parallel b_2} \times \mathbb{R}^\omega$$

- $b = b_1 \&\& b_2$ is analogous.
- $b = !b_1$ idem.

This completes the proof.

A.2 Proof of Theorem 4

Recall that $\Gamma_p = \{(\sigma, k, \phi, \psi) \mid (p, \sigma_0, 0, \top, \top) \xrightarrow{*} (\text{Skip}, \sigma, k, \phi, \psi)\}$.

Throughout all the proofs in this appendix, the fixed initial symbolic configuration $(\sigma_0, 0, \top, \top)$ is denoted by γ_0 and $\gamma, \gamma', \gamma'', \gamma_1, \gamma_2$ respectively abbreviate configurations (σ, k, ϕ, ψ) , $(\sigma', k', \phi', \psi')$, $(\sigma'', k'', \phi'', \psi'')$, $(\sigma_1, k_1, \phi_1, \psi_1)$, and $(\sigma_2, k_2, \phi_2, \psi_2)$.

Theorem 4. *For any program $p \in \mathbb{P}$, there is a one-to-one correspondence between final configurations $(\sigma, k, \phi, \psi) \in \Gamma_p$ and triples $(F, B, \mathcal{O}) \in \mathcal{F}_p$ such that $F = |\sigma|_k$, $B = |\phi|$, and $\mathcal{O} = |\psi|$.*

Proof. For every p , define the mapping

$$\Phi_p : \Gamma_p \rightarrow \mathcal{F}_p, \quad (\sigma, k, \phi, \psi) \mapsto (|\sigma|_k, |\phi|, |\psi|)$$

Proposition 7 says that this mapping is well-defined for all p , i.e., $(|\sigma|_k, |\phi|, |\psi|)$ is actually an element of \mathcal{F}_p . Indeed, take $q = \text{Skip}$ and use that $\mathcal{F}_{\text{Skip}}$ is the singleton $\{(\text{id}_{\mathbb{R}^{n+\omega}}, \mathbb{R}^{n+\omega}, \mathbb{R}^{n+\omega})\}$. Proposition 10 says that these mappings are all surjective.

For injectivity, there are some technical subtleties to be addressed. Consider the following example program p :

```
if(x<0){ if(1=0) x:=42 else x:=0 } else { if(1=0) x:=42 else x:=1 }
```

The program contains an *if* statement with a nested *if* statement in both branches. Symbolic execution of this program will thus yield *four* execution paths. *Two* of these paths have an unsatisfiable path condition due to the Boolean tests $1=0$. Furthermore, these two paths *both* transform the input variable x to the value 42. Hence, the big-step semantics, in contrast, has *three* elements:

$$\mathcal{F}_p = \{(x \mapsto 42, \emptyset, \mathbb{R}^{1+\omega}), (x \mapsto 0, \{x < 0\}, \mathbb{R}^{1+\omega}), (x \mapsto 1, \{x \geq 0\}, \mathbb{R}^{1+\omega})\}$$

where the two branches with unsatisfiable path conditions have “collapsed” to the first triple. Hence, technically speaking there is no bijection between Γ_p and \mathcal{F}_p for this edge case.

This problem could perhaps be solved by considering *disjoint* union of branching in the symbolic semantics, or *multisets*. But this would also require loop iterations and sequencing to be done in a disjoint union manner and would result in a lot of bookkeeping. To avoid this, we choose instead to identify all program executions (on the side of Γ_p as well as on the side of \mathcal{F}_p) that have unsatisfiable path conditions. This does not influence the well-definedness property, nor the established surjectivity.

By analysis of the rules in Figure 1, two traces $(p, \gamma_0) \xrightarrow{*} (\text{Skip}, \gamma)$ and $(p, \gamma_0) \xrightarrow{*} (\text{Skip}, \gamma')$ in Γ_p are different iff there is a symbolic Boolean expression B such that ϕ contains B and ϕ' contains $\neg B$ as a conjunct. This immediately entails $|\phi| \cap |\phi'| = \emptyset$. Then, either both are empty, hence ϕ and ϕ' are unsatisfiable and therefore considered equal, or their image under Φ_p is distinct. This shows injectivity of all mappings Φ_p .

Proposition 7 (Multi-step backward assimilation into symbolic semantics). *For all $p, q \in \mathbb{P}$, if $(p, \gamma_0) \xrightarrow{*} (q, \sigma, k, \phi, \psi)$ then for all $(F, B, \mathcal{O}) \in \mathcal{F}_q$:*

$$(F \circ |\sigma|_k, |\phi| \cap |\sigma|_k^{-1}[B], |\psi| \cap |\sigma|_k^{-1}[\mathcal{O}]) \in \mathcal{F}_p \quad (1)$$

Proof. The proof is by induction on the length of the transition chain $(p, \gamma_0) \xrightarrow{*} (S', \sigma, k, \phi, \psi)$, where, in the inductive step, we analyze the *first* rule that was used.

If $(p, \gamma_0) \xrightarrow{*} (q, \gamma)$ has length zero (from reflexivity) then we know that $q = p$ and $\gamma = \gamma_0$. With this, we may observe that $|\sigma|_k = |\sigma_0|_0 = \text{id}_{\mathbb{R}^{n+\omega}}$, and $|\phi| = |\psi| = |\top| = \mathbb{R}^{n+\omega}$. Now let $(F, B, \mathcal{O}) \in \mathcal{F}_q$. The result follows since the triple in (1) in this case is just (F, B, \mathcal{O}) , and $(F, B, \mathcal{O}) \in \mathcal{F}_q = \mathcal{F}_p$. For the inductive step let $(F, B, \mathcal{O}) \in \mathcal{F}_q$ be arbitrary and let

$$(p, \gamma_0) \longrightarrow (p', \gamma') \xrightarrow{*} (q, \gamma)$$

be a transition chain of length $\ell + 1$. We cannot immediately apply IH to the transition chain of length ℓ , since γ' may not be the initial configuration. Instead, we use Lemma 9 to obtain the chain

$$(p', \gamma_0) \xrightarrow{*} (q, \gamma'')$$

of length ℓ such that γ'' has the following properties:

- (i) $|\sigma|_k = |\sigma''|_{k''} \circ |\sigma'|_{k'}$;
- (ii) $|\phi| = |\phi'| \cap |\sigma'|_{k'}^{-1}[|\phi''|]$; and
- (iii) $|\psi| = |\psi'| \cap |\sigma'|_{k'}^{-1}[|\psi''|]$.

Now, by IH, since $(p', \gamma_0) \xrightarrow{*} (q, \gamma'')$ is a chain of length ℓ and $(F, B, \mathcal{O}) \in \mathcal{F}_q$:

$$(F', B', \mathcal{O}') := (F \circ |\sigma''|_{k''}, |\phi''| \cap |\sigma''|_{k''}^{-1}[B], |\psi''| \cap |\sigma''|_{k''}^{-1}[\mathcal{O}]) \in \mathcal{F}_{p'}$$

From the one-step Lemma 8, using the transition $(p, \gamma_0) \longrightarrow (p', \gamma')$, then

$$(F' \circ |\sigma'|_{k'}, |\sigma'|_{k'}^{-1}[B'] \cap |\phi'|, |\sigma'|_{k'}^{-1}[\mathcal{O}'] \cap |\psi'|) \in \mathcal{F}_p$$

We rewrite each of the three components appropriately as follows:

- (i) $F' \circ |\sigma'|_{k'} = F \circ |\sigma''|_{k''} \circ |\sigma'|_{k'} = F \circ |\sigma|_k$;
- (ii) $|\sigma'|_{k'}^{-1}[|\phi''| \cap |\sigma''|_{k''}^{-1}[B]] \cap |\phi'| = \underbrace{|\sigma'|_{k'}^{-1}[|\phi''|] \cap |\phi'|}_{|\phi|} \cap \underbrace{|\sigma'|_{k'}^{-1}[|\sigma''|_{k''}^{-1}[B]]}_{|\sigma|_k^{-1}[B]}$,

and this is just $|\phi| \cap |\sigma|_k^{-1}[B]$.

- (iii) Analogous to the previous item, $|\sigma'|_{k'}^{-1}[\mathcal{O}'] \cap |\psi'| = |\psi| \cap |\sigma|_k^{-1}[\mathcal{O}]$.

We have thus verified that

$$(F \circ |\sigma|_k, |\phi| \cap |\sigma|_k^{-1}[B], |\psi| \cap |\sigma|_k^{-1}[\mathcal{O}]) \in \mathcal{F}_p$$

and so the inductive step, and the whole proof with it, is finished.

Lemma 8 (One-step backward assimilation into symbolic semantics). *If $(p, \gamma_0) \longrightarrow (q, \gamma)$ and $(F, B, \mathcal{O}) \in \mathcal{F}_q$ then*

$$(F \circ |\sigma|_k, |\sigma|_k^{-1}[B] \cap |\phi|, |\sigma|_k^{-1}[\mathcal{O}] \cap |\psi|) \in \mathcal{F}_p \quad (2)$$

Proof. By induction on the height of the proof tree that justifies the transition rule $(p, \gamma_0) \longrightarrow (q, \gamma)$. All except Rule seq-n are base cases.

– For Rule asgn, we have the following transition from the initial configuration:

$$(x_i := e, \sigma_0, 0, \top, \top) \longrightarrow (\text{Skip}, \sigma_0[x_i \mapsto \sigma_0 e], 0, \top, \top)$$

By definition of $\mathcal{F}_{\text{skip}}$, which is the singleton $\{(\text{id}_{\mathbb{R}^{n+\omega}}, \mathbb{R}^{n+\omega}, \mathbb{R}^{n+\omega})\}$, it must be that $F = \text{id}_{\mathbb{R}^{n+\omega}}$, and $B = \mathcal{O} = \mathbb{R}^{n+\omega}$. Thus, since $\sigma_0 e = e$, with $\sigma = \sigma_0[i \mapsto e]$ and $k = 0$, we need to verify that

$$(\text{id}_{\mathbb{R}^{n+\omega}} \circ |\sigma|_k, |\sigma|_k^{-1}[\mathbb{R}^{n+\omega}] \cap |\top|, |\sigma|_k^{-1}[\mathbb{R}^{n+\omega}] \cap |\top|) \in \mathcal{F}_{x_i := e}$$

and recall that $\mathcal{F}_{x_i := e} = \{(\rho \mapsto \rho[i \mapsto \bar{e}(\rho)_n], \mathbb{R}^{n+\omega}, \mathbb{R}^{n+\omega})\}$. Since $|\sigma|_k$ is total and $|\top| = \mathbb{R}^{n+\omega}$, the only thing to show is that $|\sigma|_k : \rho \mapsto \rho[i \mapsto \bar{e}(\rho)_n]$. Let us look at $|\sigma|_k(\rho)$:

$$\begin{aligned} & (|\sigma(x_0)|(\rho), \dots, |\sigma(x_{n-1})|(\rho), \rho_n, \rho_{n+1}, \dots) && \text{def. of } |\sigma|_k \\ & = (|x_0|(\rho), \dots, |x_{i-1}|(\rho), |e|(\rho), \\ & \quad |x_{i+1}|(\rho), \dots, |x_{n-1}|(\rho), \rho_n, \rho_{n+1}, \dots) && \text{def. of } \sigma \\ & = (\rho_0, \dots, \rho_{i-1}, |e|(\rho), \rho_{i+1}, \dots, \rho_{n-1}, \rho_n, \rho_{n+1}, \dots) && \text{def. of } |\cdot| \\ & = (\rho_0, \dots, \rho_{i-1}, \bar{e}(\rho)_n, \rho_{i+1}, \dots, \rho_{n-1}, \rho_n, \rho_{n+1}, \dots) && \text{Lemma 1} \\ & = \rho[i \mapsto \bar{e}(\rho)_n] && \text{notation} \end{aligned}$$

This concludes the case.

- We continue with Rule **smpl**, for which we have the following transition:

$$(x_i \sim \text{rnd}, \gamma_0) \longrightarrow (\text{Skip}, \sigma_0[x_i \mapsto y_0], 1, \top, \top)$$

and by definition, we have $\mathcal{F}_{x_i \sim \text{rnd}} = \{(\rho \mapsto \text{sample}_i(\rho), \mathbb{R}^{n+\omega}, \mathbb{R}^{n+\omega})\}$ and $\mathcal{F}_{\text{skip}} = \{(\text{id}_{\mathbb{R}^{n+\omega}}, \mathbb{R}^{n+\omega}, \mathbb{R}^{n+\omega})\}$. Similar to the previous case, here it suffices to show that $|\sigma|_1(\rho) = \text{sample}_i(\rho)$ (where now $\sigma = \sigma_0[i \mapsto y_0]$):

$$\begin{aligned} |\sigma|_1(\rho) &= (|\sigma(x_0)|(\rho), \dots, |\sigma(x_{n-1})|(\rho), \rho_{n+1}, \rho_{n+2}, \dots) && \text{def. of } |\sigma|_k \\ &= (|x_0|(\rho), \dots, |x_{i-1}|(\rho), |y_0|(\rho), \\ &\quad |x_{i+1}|(\rho), \dots, |x_{n-1}|(\rho), \rho_{n+1}, \rho_{n+2}, \dots) && \text{def. of } \sigma \\ &= (\rho_0, \dots, \rho_{i-1}, \rho_n, \rho_{i+1}, \dots, \rho_{n-1}, \rho_{n+1}, \rho_{n+2}, \dots) && \text{def. of } |\cdot| \\ &= \text{sample}_i(\rho) && \text{notation} \end{aligned}$$

This case is now finished.

- Rule **obs**: the transition has the following shape:

$$(\text{observe } b, \gamma_0) \longrightarrow (\text{Skip}, \sigma_0, 0, \top, \top \wedge \sigma_0(b))$$

Again, $(F, B, \mathcal{O}) \in \mathcal{F}_{\text{skip}}$ means $F = \text{id}_{\mathbb{R}^{n+\omega}}$ and $B = \mathcal{O} = \mathbb{R}^{n+\omega}$. Note that $\mathcal{F}_{\text{observe } b} = \{(\text{id}_{\mathbb{R}^{n+\omega}}, \mathbb{R}^{n+\omega}, \bar{b} \times \mathbb{R}^\omega)\}$.

- (i) $F \circ |\sigma_0|_0 = \text{id}_{\mathbb{R}^{n+\omega}} \circ \text{id}_{\mathbb{R}^{n+\omega}} = \text{id}_{\mathbb{R}^{n+\omega}}$;
- (ii) $|\sigma_0|_0^{-1}[B] \cap |\phi| = |\sigma_0|_0^{-1}[\mathbb{R}^{n+\omega}] \cap |\top| = \mathbb{R}^{n+\omega} \cap \mathbb{R}^{n+\omega} = \mathbb{R}^{n+\omega}$; and
- (iii) $|\sigma_0|_0^{-1}[\mathcal{O}] \cap |\psi| = |\sigma_0|_0^{-1}[\mathbb{R}^{n+\omega}] \cap |\top \wedge \sigma_0(b)| = \mathbb{R}^{n+\omega} \cap |\top| \cap |\sigma_0(b)| = \bar{b} \times \mathbb{R}^\omega$.

Here, we used Lemma 6 in the last equality. This case is now finished.

- For Rule **seq-0**, we have the following transition:

$$(\text{Skip} \text{ ; } p, \gamma_0) \longrightarrow (p, \sigma_0, 0, \top, \top)$$

We recognize that if $(F, B, \mathcal{O}) \in \mathcal{F}_p$, then

$$(F \circ |\sigma_0|_0, |\sigma_0|_0^{-1}[B] \cap |\top|, |\sigma_0|_0^{-1}[\mathcal{O}] \cap |\psi|) = (F, B, \mathcal{O})$$

Now, since $(F, B, \mathcal{O}) \in \mathcal{F}_p$ and $(\text{id}_{\mathbb{R}^{n+\omega}}, \mathbb{R}^{n+\omega}, \mathbb{R}^{n+\omega}) \in \mathcal{F}_{\text{skip}}$, by definition, $\mathcal{F}_{\text{skip};p}$ also contains (F, B, \mathcal{O}) , so this case is done.

- Rule **seq-n**: the only inductive step. The derivation tree for the transition is the following:

$$\frac{(p, \gamma_0) \longrightarrow (p', \gamma)}{(p \text{ ; } q, \gamma_0) \longrightarrow (p' \text{ ; } q, \gamma)}$$

The induction hypothesis is the following: if $(p, \gamma_0) \longrightarrow (p', \gamma)$ and $(F, B, \mathcal{O}) \in \mathcal{F}_{p'}$ then $(F \circ |\sigma|_k, |\sigma|_k^{-1}[B] \cap |\phi|, |\sigma|_k^{-1}[\mathcal{O}] \cap |\psi|) \in \mathcal{F}_p$.

Now let $(F, B, \mathcal{O}) \in \mathcal{F}_{p';q}$. Then, by definition, there are $(F_1, B_1, \mathcal{O}_1) \in \mathcal{F}_{p'}$ and $(F_2, B_2, \mathcal{O}_2) \in \mathcal{F}_q$ such that (i) $F = F_2 \circ F_1$, (ii) $B = B_1 \cap F_1^{-1}[B_2]$, and (iii) $\mathcal{O} = \mathcal{O}_1 \cap F_1^{-1}[\mathcal{O}_2]$. We can apply the IH to $(F_1, B_1, \mathcal{O}_1) \in \mathcal{F}_{p'}$ with the transition $(p, \gamma_0) \longrightarrow (p', \gamma)$ to learn that

$$(F_1 \circ |\sigma|_k, |\sigma|_k^{-1}[B_1] \cap |\phi|, |\sigma|_k^{-1}[\mathcal{O}_1] \cap |\psi|) \in \mathcal{F}_p$$

But then, by definition, $\mathcal{F}_{p';q}$ contains (F', B', \mathcal{O}') , where

- (i) $F' = F_2 \circ (F_1 \circ |\sigma|_k) = F \circ |\sigma|_k$;
- (ii) $B' = |\sigma|_k^{-1}[B_1] \cap |\phi| \cap (F_1 \circ |\sigma|_k)^{-1}[B_2] = |\sigma|_k^{-1}[B] \cap |\phi|$; and
- (iii) $\mathcal{O}' = |\sigma|_k^{-1}[\mathcal{O}_1] \cap |\psi| \cap (F_1 \circ |\sigma|_k)^{-1}[\mathcal{O}_2] = |\sigma|_k^{-1}[\mathcal{O}] \cap |\psi|$.

This is exactly what we need to show in this case!

- Rule if-T: analogous to Rule if-F.
- For Rule if-F, we have the following shape of the transition:

$$(\text{if } b \ p_1 \ \text{else } p_2, \gamma_0) \longrightarrow (p_2, \sigma_0, 0, \top \wedge \sigma(!b), \top)$$

Now let $(F, B, \mathcal{O}) \in \mathcal{F}_{p_2}$. Using that $|\sigma_0|_0 = \text{id}_{\mathbb{R}^{n+\omega}}$, we recognize that

- (i) $F \circ |\sigma_0|_0 = F$;
- (ii) $|\sigma_0|_0^{-1}[B] \cap |\top \wedge \sigma_0(!b)| = B \cap (\bar{b}^c \times \mathbb{R}^\omega)$, by Lemma 6; and
- (iii) $|\sigma_0|_0^{-1}[\mathcal{O}] \cap |\top| = \mathcal{O}$.

Indeed, by definition, $(F, B \cap (\bar{b}^c \times \mathbb{R}^\omega), \mathcal{O}) \in \mathcal{F}_{\text{if } b \ p_1 \ \text{else } p_2}$, so the case is finished.

- Rule iter-F: we have the transition rule

$$(\text{while } b \ p, \gamma_0) \longrightarrow (\text{Skip}, \sigma_0, 0, \top \wedge \sigma_0(!b), \top)$$

and $(F, B, \mathcal{O}) \in \mathcal{F}_{\text{Skip}}$ means $F = \text{id}_{\mathbb{R}^{n+\omega}}$ and $B = \mathcal{O} = \mathbb{R}^{n+\omega}$. With these data, we recognize that

- (i) $F \circ |\sigma_0|_0 = \text{id}_{\mathbb{R}^{n+\omega}}$;
- (ii) $|\sigma_0|_0^{-1}[B] \cap |\top \wedge \sigma_0(!b)| = \bar{b}^c \times \mathbb{R}^\omega$ (Lemma 6); and
- (iii) $|\sigma_0|_0^{-1}[\mathcal{O}] \cap |\top| = \mathbb{R}^{n+\omega}$.

Recall the definition of $\mathcal{F}_{\text{while } b \ p}$:

$$\bigcup_{m=0}^{\infty} \mathbb{F}_{b,p}^m \{(\text{id}_{\mathbb{R}^{n+\omega}}, \bar{b}^c \times \mathbb{R}^\omega, \mathbb{R}^{n+\omega})\}$$

Consider $m = 0$ to finish the case.

- Rule iter-T: we have the transition rule in the following shape:

$$(\text{while } b \ p, \gamma_0) \longrightarrow (p \ ; \ \text{while } b \ p, \sigma_0, 0, \top \wedge \sigma_0 b, \top)$$

Consider $(F, B, \mathcal{O}) \in \mathcal{F}_{p \ ; \ \text{while } b \ p}$. By definition, then, there is $(F_p, B_p, \mathcal{O}_p) \in \mathcal{F}_p$ and $(F', B', \mathcal{O}') \in \mathcal{F}_{\text{while } b \ p}$ such that (i) $F = F' \circ F_p$, (ii) $B = B_p \cap F_p^{-1}[B']$, and (iii) $\mathcal{O} = \mathcal{O}_p \cap F_p^{-1}[\mathcal{O}']$. Furthermore, by definition of $\mathcal{F}_{\text{while } b \ p}$, there is $m \in \mathbb{N}$

- such that $(F', B', \mathcal{O}') \in \mathbb{F}_{b,p}^m \{(\text{id}_{\mathbb{R}^{n+\omega}}, \bar{b}^c \times \mathbb{R}^\omega, \mathbb{R}^{n+\omega})\}$. We now recognize that
- (i) $F \circ |\sigma_0|_0 = F = F' \circ F_p$;
 - (ii) $|\sigma_0|_0^{-1}[B] \cap |\top \wedge \sigma_0(b)| = B_p \cap F_p^{-1}[B'] \cap (\bar{b} \times \mathbb{R}^\omega)$ (Lemma 6); and
 - (iii) $|\sigma_0|_0^{-1}[\mathcal{O}] \cap |\top| = \mathcal{O} = \mathcal{O}_p \cap F_p^{-1}[\mathcal{O}']$.

Now, since $(F', B', \mathcal{O}') \in \mathbb{F}_{b,p}^m \{(\text{id}_{\mathbb{R}^{n+\omega}}, \bar{b}^c \times \mathbb{R}^\omega, \mathbb{R}^{n+\omega})\}$ and $(F_p, B_p, \mathcal{O}_p) \in \mathcal{F}_p$, it follows by definition of $\mathbb{F}_{b,p}$ that

$$(F' \circ F_p, (\bar{b} \times \mathbb{R}^\omega) \cap B_p \cap F_p^{-1}[B'], \mathcal{O}_p \cap F_p^{-1}[\mathcal{O}']) \in \mathbb{F}_{b,p}^{m+1} \{(\text{id}_{\mathbb{R}^{n+\omega}}, \bar{b}^c \times \mathbb{R}^\omega, \mathbb{R}^{n+\omega})\}$$

which is a subset of $\mathcal{F}_{\text{while } b \ p}$, so we are done.

Induction has finished for the small-step semantics \longrightarrow , and so the proof is done.

Lemma 9 (Canonical Symbolic Execution). *For all statements p and configurations γ , $(p, \gamma) \xrightarrow{*} (q, \gamma')$ if and only if $(p, \gamma_0) \xrightarrow{*} (q, \gamma_1)$ (of the same length) such that:*

- (i) $|\sigma'|_{k'} = |\sigma_1|_{k_1} \circ |\sigma|_k$,
- (ii) $k' = k_1 + k$,
- (iii) $|\phi'| = |\phi| \cap |\sigma|_k^{-1}[|\phi_1|]$, and
- (iv) $|\psi'| = |\psi| \cap |\sigma|_k^{-1}[|\psi_1|]$.

In case γ , γ' , and γ_1 satisfy properties Items (i) to (iv), we write $\gamma' \sim \gamma_1 \bullet \gamma$. We call $(p, \gamma_0) \xrightarrow{*} (q, \gamma_1)$ a canonical symbolic execution from p to q .

Proof. First some observations. If $\gamma' \sim \gamma_1 \bullet \gamma$ then $|\sigma'|_{k'}(\rho) = |\sigma_1|_{k_1}(|\sigma|_k(\rho))$ for all ρ , and we have the following four facts that we will use throughout the proof:

- (A) For all $\rho \in \mathbb{R}^{n+\omega}$ and $i \in \{0, \dots, n-1\}$, $|\sigma'(x_i)|(\rho) = |\sigma_1(x_i)|(|\sigma|_k(\rho))$;
- (B) For all $\rho \in \mathbb{R}^{n+\omega}$ and $\ell \in \mathbb{N}$,

$$|\sigma'|_{k'}(\rho)(n + \ell) = \rho_{k_1+k+n+\ell} = |\sigma_1|_{k_1}(\rho)(k + n + \ell) = |\sigma_1|_{k_1}(|\sigma|_k(\rho))(n + \ell)$$

- (C) For all expressions $e \in \mathbb{E}$ and all $\rho \in \mathbb{R}^{n+\omega}$:

$$|\sigma_1(e)|(\rho) = |\sigma'(e)|(|\sigma|_k(\rho))$$

- (D) For all Boolean expressions $b \in \mathbb{B}$ we have the following equality of sets:

$$|\sigma|_k^{-1}[|\sigma_1(b)|] = |\sigma'(b)|$$

Items (A) and (B) are element-wise equalities of $|\sigma'|_{k'}(\rho) = |\sigma_1|_{k_1}(|\sigma|_k(\rho))$. Item (C) is proven by induction on the structure of expressions:

- The base case x_i is Item (A).
- The case $e = q \in \mathbb{Q}$ is trivial.
- The inductive step with $e = \text{op}(e_1, \dots, e_{\#\text{op}})$ is as follows:

$$\begin{aligned} |\sigma_1(\text{op}(e_1, \dots, e_{\#\text{op}}))|(|\sigma|_k(\rho)) &= \overline{\text{op}}(|\sigma_1(e_1)|(|\sigma|_k(\rho)), \dots, |\sigma_1(e_{\#\text{op}})|(|\sigma|_k(\rho))) \\ &\stackrel{\text{IHs}}{=} \overline{\text{op}}(|\sigma'(e_1)|(\rho), \dots, |\sigma'(e_{\#\text{op}})|(\rho)) \\ &= |\sigma'(\text{op}(e_1, \dots, e_{\#\text{op}}))|(\rho) \end{aligned}$$

Item (D) is proven by induction on the structure of Boolean expressions:

- $b = \text{True}$: both sides equal $\mathbb{R}^{n+\omega}$ (since $|\sigma|_k$ is total).
- $b = \text{False}$: both sides equal \emptyset .
- $b = e_1 \diamond e_2$: write $\rho' := |\sigma|_k(\rho)$. Then

$$\begin{aligned} \rho \in |\sigma|_k^{-1}[|\sigma_1(e_1 \diamond e_2)|] &\iff \rho' \in |\sigma_1(e_1) \diamond \sigma_1(e_2)| \\ &\iff |\sigma_1(e_1)|(\rho') \diamond |\sigma_1(e_2)|(\rho') \\ &\iff |\sigma'(e_1)|(\rho) \diamond |\sigma'(e_2)|(\rho) \quad \text{Item (C)} \\ &\iff \rho \in |\sigma'(e_1 \diamond e_2)| \end{aligned}$$

– $b = b_1 \parallel b_2$: then

$$\rho \in |\sigma|_k^{-1}[|\sigma_1(b_1 \parallel b_2)|] \iff |\sigma|_k(\rho) \in |\sigma_1(b_1)| \cup |\sigma_1(b_2)|$$

and the two IHs will conclude this case.

- $b = b_1 \ \&\& \ b_2$, and
- $b = !b_1$ are analogous to the case for \parallel .

The proof of the lemma is by induction on the length of the transition chain, with an analysis of the *last* rule that was used. The base case (length of zero) is from the reflexive closure, which tells us that $\gamma_1 = \gamma_0$ for the canonical execution, and $\gamma' = \gamma$ for all other γ . With these data, Items (i) to (iv) are thus verified:

- (i) $|\sigma_1|_{k_1} \circ |\sigma|_k = |\sigma_0|_0 \circ |\sigma|_k = \text{id}_{\mathbb{R}^{n+\omega}} \circ |\sigma|_k = |\sigma|_k = |\sigma'|_{k'}$.
- (ii) $k_1 + k = 0 + k = k = k'$,
- (iii) $|\phi| \cap |\sigma|_k^{-1}[|\phi_1|] = |\phi'| \cap |\sigma|_k^{-1}[|\top|] = |\phi'| \cap |\sigma|_k^{-1}[\mathbb{R}^{n+\omega}] = |\phi'| \cap \mathbb{R}^{n+\omega} = |\phi'|$.
- (iv) Similar to the above item.

This finishes the base case for the induction. For the inductive step, we prove the statement for the pairs of transition chains

$$(p, \gamma_0) \xrightarrow{*} (q, \gamma_1) \longrightarrow (r, \gamma_2) \quad (p, \gamma) \xrightarrow{*} (q, \gamma') \longrightarrow (r, \gamma'')$$

of length $\ell + 1$, where IH gives us that $\gamma' \sim \gamma_1 \bullet \gamma$, and the goal is to show that $\gamma'' \sim \gamma_2 \bullet \gamma$. This in turn is done by induction on the height of the proof tree that justifies the two transitions

$$(q, \gamma_1) \longrightarrow (r, \gamma_2) \quad (q, \gamma') \longrightarrow (r, \gamma'')$$

So we continue with a case analysis of the rules in Figure 1 that may justify the outgoing transition from q —all except Rule **seq-n** are base cases:

- For Rule **asgn**, we have the following data:
 - (i) $\sigma_2 = \sigma_1[i \mapsto \sigma_1(e)]$ and $\sigma'' = \sigma'[i \mapsto \sigma'(e)]$;
 - (ii) $k_2 = k_1$ and $k'' = k'$ (so by IH: $k_2 + k = k_1 + k = k' = k''$);
 - (iii) $\phi_2 = \phi_1$ and $\phi'' = \phi'$. Then, by IH:

$$|\phi| \cap |\sigma|_k^{-1}[|\phi_2|] = |\phi| \cap |\sigma|_k^{-1}[|\phi_1|] = |\phi'| = |\phi''|$$

- (iv) Analogously to the above item we see that $|\psi| \cap |\sigma|_k^{-1}[|\psi_2|] = |\psi''|$.
The only non-trivial item we have to verify to conclude $\gamma'' \sim \gamma_2 \bullet \gamma$ is that

$$|\sigma''|_{k''} = |\sigma_2|_{k_2} \circ |\sigma|_k \tag{3}$$

Let $\rho \in \mathbb{R}^{n+\omega}$ be arbitrary. Then

$$\begin{aligned} |\sigma''|_{k''}(\rho) &= (|\sigma''(\mathbf{x}_0)|(\rho), \dots, |\sigma''(\mathbf{x}_{n-1})|(\rho), \rho_{n+k''}, \rho_{n+k''+1}, \dots) \\ &= (|\sigma'(\mathbf{x}_0)|(\rho), \dots, |\sigma'(e)|(\rho), \dots, \quad (i\text{-th var.}) \\ &\quad |\sigma'(\mathbf{x}_{n-1})|(\rho), \rho_{n+k'}, \rho_{n+k'+1}, \dots) \\ &\stackrel{\text{IH}}{=} (|\sigma_1(\mathbf{x}_0)|(\rho'), \dots, |\sigma'(e)|(\rho), \dots, \quad (i\text{-th var.}) \\ &\quad |\sigma_1(\mathbf{x}_{n-1})|(\rho'), \rho'_{n+k_1}, \rho'_{n+k_1+1}, \dots) \end{aligned} \tag{4}$$

Here, we have put $\rho' := |\sigma|_k(\rho)$ and used Items (A) and (B) for all elements except the i -th. On the other hand, we have, for the RHS of the goal (3):

$$\begin{aligned} |\sigma_2|_{k_2}(\rho') &= (|\sigma_2(x_0)|(\rho'), \dots, |\sigma_2(x_{n-1})|(\rho'), \rho'_{n+k_2}, \rho'_{n+k_2+1}, \dots) \\ &= (|\sigma_1(x_0)|(\rho'), \dots, |\sigma_1(e)|(\rho'), \dots \quad (i\text{-th var.}) \\ &\quad |\sigma_1(x_{n-1})|(\rho'), \rho'_{n+k_1}, \rho'_{n+k_1+1}, \dots) \end{aligned}$$

So we may identify this with (4) if indeed $|\sigma'(e)|(\rho) = |\sigma_1(e)|(\rho')$, and this follows from Item (C), using IH, i.e., $\gamma' \sim \gamma_1 \bullet \gamma$.

- We continue with Rule **smpl**, for which we have the following data:
 - (i) $\sigma_2 = \sigma_1[i \mapsto y_{k_1}]$ and $\sigma'' = \sigma'[i \mapsto y_{k'}]$;
 - (ii) $k_2 = k_1 + 1$ and $k'' = k' + 1$, from which we deduce

$$k_2 + k = k_1 + 1 + k = (k_1 + k) + 1 \stackrel{\text{IH}}{=} k' + 1 = k''$$

- (iii) and (iv): we have $\phi'' = \phi'$ and $\phi_2 = \phi_1$, and similarly for the path observations. Apply the same reasoning as in the case Rule **asgn** to observe that

$$|\phi''| = |\phi| \cap |\sigma|_k^{-1}[\phi_2] \quad \text{and} \quad |\psi''| = |\psi| \cap |\sigma|_k^{-1}[\psi_2]$$

Again, the only truly interesting item we have to verify is that $|\sigma''|_{k''} = |\sigma_2|_{k_2} \circ |\sigma|_k$. This is handled in exactly the same way as in the case for Rule **asgn** with the following exception: instead of having to show $|\sigma_1(e)|(\rho') = |\sigma'(e)|(\rho)$, we have to prove that $|y_{k_1}|(\rho') = |y_{k'}|(\rho)$ for all $\rho \in \mathbb{R}^{n+\omega}$. But this is easy:

$$|y_{k_1}|(\rho') = \rho'_{n+k_1} = \rho_{n+k_1+k} = \rho_{n+k'} = |y_{k'}|(\rho)$$

Thus, we concluded this case since $|\sigma''|_{k''} = |\sigma_2|_{k_2} \circ |\sigma|_k$ and so $\gamma'' \sim \gamma_2 \bullet \gamma$.

- We continue with Rule **obs**, for which we have the following data:
 - (i) $\sigma_2 = \sigma_1$ and $\sigma'' = \sigma'$, and thus (using (ii)):

$$|\sigma_2|_{k_2} \circ |\sigma|_k = |\sigma_1|_{k_1} \circ |\sigma|_k = |\sigma_1|_{k_1} \circ |\sigma|_k = |\sigma'|_{k'} = |\sigma''|_{k''}$$

- (ii) $k_2 = k_1$ and $k'' = k'$, from which we deduce $k_2 + k = k_1 + k \stackrel{\text{IH}}{=} k' = k''$.
- (iii) $\phi_2 = \phi_1$ and $\phi'' = \phi'$, so that $|\phi| \cap |\sigma|_k^{-1}[\phi_2] = |\phi''|$ by IH.
- (iv) $\psi_2 = \psi_1 \wedge \sigma_1(b)$ and $\psi'' = \psi' \wedge \sigma'(b)$.

In this case, the non-trivial item we have to verify is (iv), so we set out to prove that

$$|\psi| \cap |\sigma|_k^{-1}[\psi_2] = |\psi''| \tag{5}$$

For this, we first observe that

- $|\psi''| = |\psi'| \cap |\sigma'(b)|$;
- $|\psi_2| = |\psi_1| \cap |\sigma_1(b)|$; and
- by Item (D) and IH, we have $|\sigma|_k^{-1}[\sigma_1(b)] = |\sigma'(b)|$.
- The IH also tells us that $|\psi'| = |\psi| \cap |\sigma|_k^{-1}[\psi_1]$.

We can now make the following derivation:

$$\begin{aligned}
|\psi| \cap |\sigma|_k^{-1}[|\psi_2|] &= |\psi| \cap |\sigma|_k^{-1}[|\psi_1| \cap |\sigma_1(b)|] \\
&= |\psi| \cap |\sigma|_k^{-1}[|\psi_1|] \cap |\sigma|_k^{-1}[|\sigma_1(b)|] \\
&= |\psi'| \cap |\sigma|_k^{-1}[|\sigma_1(b)|] \\
&= |\psi'| \cap |\sigma'(b)| \\
&= |\psi''|
\end{aligned}$$

and we have thus concluded $\gamma'' \sim \gamma_2 \bullet \gamma$.

- In the case for Rule **seq-0**, where $r = \text{Skip} \circledast q$, there is nothing to prove, because $\gamma_2 = \gamma_1$, and $\gamma'' = \gamma'$. Thus, $\gamma'' \sim \gamma_2 \bullet \gamma$ directly by IH $\gamma' \sim \gamma_1 \bullet \gamma$.
- Rule **seq-n**: the only inductive step. Here $q = q_1 \circledast q_2$ and $r = q'_1 \circledast q_2$, and so

$$\frac{(q_1, \gamma_1) \longrightarrow (q'_1, \gamma_2)}{(q_1 \circledast q_2, \gamma_1) \longrightarrow (q'_1 \circledast q_2, \gamma_2)} \quad \frac{(q_1, \gamma') \longrightarrow (q'_1, \gamma'')}{(q_1 \circledast q_2, \gamma') \longrightarrow (q'_1 \circledast q_2, \gamma'')}$$

Here, we may assume that the transitions from q_1 to q'_1 are not by Rule **seq-n** or Rule **seq-0**, by observing that sequencing is associative, so we can evaluate right-associatively, without loss of generality. The analyses from all other cases for the transition from q_1 to q'_1 (and using the chain $(q_1, \gamma_0) \xrightarrow{*} (q_1, \gamma_0)$ of length $\ell = 0$) can now be repeated. This will establish that $(q_1, \gamma_0) \longrightarrow (q'_1, \gamma'_0)$ for some $\gamma'_0 = (\sigma'_0, k'_0, \phi'_0, \psi'_0)$ that satisfies (1) $\gamma_2 \sim \gamma'_0 \bullet \gamma_1$ and (2) $\gamma'' \sim \gamma'_0 \bullet \gamma'$. With this, we verify that $\gamma'' \sim \gamma_2 \bullet \gamma$.

(ii) $k_2 + k \stackrel{(1)}{=} k'_0 + k_1 + k \stackrel{\text{IH}}{=} k'_0 + k' \stackrel{(2)}{=} k''$;

(i) $|\sigma_2|_{k_2} \circ |\sigma|_k \stackrel{(1)}{=} |\sigma'_0|_{k'_0} \circ |\sigma_1|_{k_1} \circ |\sigma|_k \stackrel{\text{IH}}{=} |\sigma'_0|_{k'_0} \circ |\sigma'|_{k'} \stackrel{(2)}{=} |\sigma''|_{k''}$

(iii) We can make the following derivation of sets:

$$\begin{aligned}
|\phi| \cap |\sigma|_k^{-1}[|\phi_2|] &\stackrel{(1)}{=} |\phi| \cap |\sigma|_k^{-1}[|\phi_1| \cap |\sigma_1|_{k_1}^{-1}[|\phi'_0|]] \\
&\stackrel{\text{IH}}{=} |\phi'| \cap |\sigma|_k^{-1}[|\sigma_1|_{k_1}^{-1}[|\phi'_0|]] \\
&\stackrel{\text{IH}}{=} |\phi'| \cap |\sigma'|_{k'}^{-1}[|\phi'_0|] \\
&\stackrel{(2)}{=} |\phi''|
\end{aligned}$$

This concludes this item.

- (iv) The path observation ψ'' is done in exactly the same way as the path condition. Conclude that for every p and γ , $(p, \gamma_0) \xrightarrow{*} (r, \gamma_2)$ (of length $\ell + 1$) if and only if $(p, \gamma) \xrightarrow{*} (r, \gamma'')$ (of length $\ell + 1$) such that $\gamma'' \sim \gamma_2 \bullet \gamma$ in case the last transition rule of the chains was proven by Rule **seq-n**.
 - Rule **if-T**: looking at the rule, we know that
 - (i) and (ii): $k_2 = k_1$ and $k'' = k'$, and $\sigma_2 = \sigma_1$ and $\sigma'' = \sigma'$, so there is nothing to prove again.
 - (iii) the interesting case, where $\phi_2 = \phi_1 \wedge \sigma_1(b)$ and $\phi'' = \phi' \wedge \sigma'(b)$.
 - (iv) $\psi_2 = \psi_1$ and $\psi'' = \psi'$, so nothing to prove here.
- We check item (iii), and prove that

$$|\phi| \cap |\sigma|_k^{-1}[|\phi_2|] = |\phi''|$$

For this, we observe

- $|\phi''| = |\phi'| \cap |\sigma'(b)|$;
- $|\phi_2| = |\phi_1| \cap |\sigma_1(b)|$; and
- by Item (D) and IH, $|\sigma|_k^{-1}[|\sigma_1(b)|] = |\sigma'(b)|$.
- by IH, $|\phi'| = |\phi| \cap |\sigma|_k^{-1}[|\phi_1|]$.

With this, we derive:

$$\begin{aligned}
|\phi| \cap |\sigma|_k^{-1}[|\phi_2|] &= |\phi| \cap |\sigma|_k^{-1}[|\phi_1| \cap |\sigma_1(b)|] \\
&= |\phi| \cap |\sigma|_k^{-1}[|\phi_1|] \cap |\sigma|_k^{-1}[|\sigma_1(b)|] \\
&= |\phi'| \cap |\sigma|_k^{-1}[|\sigma_1(b)|] \\
&= |\phi'| \cap |\sigma'(b)| \\
&= |\phi''|
\end{aligned}$$

and we have thus concluded $\gamma'' \sim \gamma_2 \bullet \gamma$ in this case.

- Rule if-F is analogous to Rule if-T, where we just take complements.
- Rule iter-F: we have that
 - (i) and (ii) $k_2 = k_1$ and $k'' = k'$, and $\sigma_2 = \sigma_1$ and $\sigma'' = \sigma'$; nothing to prove,
 - (iii) $\phi_2 = \phi_1 \wedge \sigma_1(!b)$ and $\phi'' = \phi' \wedge \sigma'(!b)$; the interesting case, and
 - (iv) $\psi_2 = \psi_1$ and $\psi'' = \psi'$, so nothing to prove.
This case is again concluded analogously to Rule if-T.
- Rule iter-T: perhaps surprisingly, this case is exactly the same as Rule if-T. This is because we are doing forward induction on the length of the transition chain, so the symbolic execution is inherently finite. Indeed, it follows from the fact that $\gamma_2 = \gamma_1$ and $\gamma'' = \gamma'$, with the exception of the path conditions, that checking item (iii) is the only relevant task, and this is done exactly as in Rule if-T.

This finishes induction on the rules in Figure 1, and concludes the proof of Lemma 9.

Proposition 10 (Surjectivity of the Bijection). *If $(F, B, \mathcal{O}) \in \mathcal{F}_p$ then there is a configuration γ such that $(p, \gamma_0) \xrightarrow{*} (\text{Skip}, \gamma)$ and (i) $|\sigma|_k = F$, (ii) $|\phi| = B$, and (iii) $|\psi| = \mathcal{O}$.*

Proof. By induction on the structure of p .

- $p = \text{Skip}$: we have $(F, B, \mathcal{O}) = (\text{id}_{\mathbb{R}^{n+\omega}}, \mathbb{R}^{n+\omega}, \mathbb{R}^{n+\omega})$ and, indeed, $(\text{Skip}, \gamma_0) \xrightarrow{*} (\text{Skip}, \gamma_0)$ where γ_0 satisfies (i)-(iii);
- $p = x_i := e$: we have $(F, B, \mathcal{O}) = (\rho \mapsto \rho[i \mapsto \bar{e}(\rho|_n)], \mathbb{R}^{n+\omega}, \mathbb{R}^{n+\omega})$ and $(S, \gamma_0) \xrightarrow{*} (\text{Skip}, \gamma)$ with $\gamma = (\sigma_0[i \mapsto \sigma_0(e)], 0, \top, \top)$ and one verifies in exactly the same way as in the proof of Lemma 8 that $|\sigma|_k = F$ in this case.
- Idem for $p = x_i \sim \text{rnd}$, where $(F, B, \mathcal{O}) = (\rho \mapsto \text{sample}_i(\rho), \mathbb{R}^{n+\omega}, \mathbb{R}^{n+\omega})$ and $(S, \gamma_0) \xrightarrow{*} (\text{Skip}, \gamma)$ with $\gamma = (\sigma_0[i \mapsto y_0], 1, \top, \top)$.
- Now if $p = \text{observe } b$, then $(F, B, \mathcal{O}) = (\text{id}_{\mathbb{R}^{n+\omega}}, \mathbb{R}^{n+\omega}, \bar{b} \times \mathbb{R}^\omega)$ and $\gamma = (\sigma_0, 0, \top, \sigma_0 b)$. Again, checking that these data satisfy (i)-(iii) is done in the same way as in the proof of Lemma 8.
- Consider now the case where $p = p_1 \wp p_2$, where we have two IHs, namely one for all $(F_1, B_1, \mathcal{O}_1) \in \mathcal{F}_{p_1}$ and the other for all $(F_2, B_2, \mathcal{O}_2) \in \mathcal{F}_{p_2}$. So let $(F, B, \mathcal{O}) \in \mathcal{F}_p = \mathcal{F}_{p_1 \wp p_2}$. Then, by definition, there are $(F_1, B_1, \mathcal{O}_1) \in \mathcal{F}_{p_1}$ and $(F_2, B_2, \mathcal{O}_2) \in \mathcal{F}_{p_2}$, such that

- $F = F_2 \circ F_1$;
- $B = B_1 \cap F_1^{-1}[B_2]$; and
- $\mathcal{O} = \mathcal{O}_1 \cap F_1^{-1}[\mathcal{O}_2]$.

and by the two IHs, we obtain two canonical symbolic executions

$$(p_1, \gamma_0) \xrightarrow{*} (\text{Skip}, \gamma_1) \quad \text{and} \quad (p_2, \gamma_0) \xrightarrow{*} (\text{Skip}, \gamma_2), \quad (*)$$

By repeatedly applying Rule `seq-n` and finally Rule `seq-0`, we then also have

$$(p_1 \circledast p_2, \gamma_0) \xrightarrow{*} (\text{Skip} \circledast p_2, \gamma_1) \longrightarrow (p_2, \gamma_1) \xrightarrow{*} (\text{Skip}, \gamma)$$

for some γ . Then, using the canonical symbolic transition chain (*), and by Lemma 9, we have $\gamma \sim \gamma_2 \bullet \gamma_1$. We conclude this case by showing that properties (i)-(iii) are satisfied by exactly this γ :

- (i) $|\sigma|_k = |\sigma_2|_{k_2} \circ |\sigma_1|_{k_1} = F_2 \circ F_1 = F$;
- (ii) $|\phi| = |\phi_1| \cap |\sigma_1|_{k_1}^{-1}[\phi_2] = B_1 \cap F_1^{-1}[B_2] = B$; and
- (iii) $|\psi| = |\psi_1| \cap |\sigma_1|_{k_1}^{-1}[\psi_2] = \mathcal{O}_1 \cap F_1^{-1}[\mathcal{O}_2] = \mathcal{O}$.

And so the case for sequencing is finished.

– Now let $p = \text{if } b \text{ } p_1 \text{ else } p_2$, and $(F, B', \mathcal{O}) \in \mathcal{F}_p$. This can happen in either of two ways:

1. $(F, B, \mathcal{O}) \in \mathcal{F}_{p_1}$ and $B' = B \cap (\bar{b} \times \mathbb{R}^\omega)$. In this case, by the IH for p_1 , we have γ_1 such that $(p_1, \gamma_0) \xrightarrow{*} (\text{Skip}, \gamma_1)$ with the required properties (i)-(iii). This is the canonical execution, so we have $(\text{if } b \text{ } p_1 \text{ else } p_2, \gamma_0) \longrightarrow (p_1, \gamma') \xrightarrow{*} (\text{Skip}, \gamma'')$ for some γ' with $\gamma'' \sim \gamma_1 \bullet \gamma'$ by Lemma 9. But we know $\sigma' = \sigma_0$, $k' = 0$, $\phi' = \top \wedge \sigma_0 b$, and $\psi' = \top$. With this we verify (i)-(iii) for γ'' :
 - (i) $|\sigma''|_{k''} = |\sigma_1|_{k_1} \circ |\sigma'|_{k'} = |\sigma_1|_{k_1} \circ \text{id}_{\mathbb{R}^{n+\omega}} = |\sigma_1|_{k_1}$, and by IH for p_1 , this equals F .
 - (ii) $|\phi''| = |\phi'| \cap |\sigma'|_{k'}^{-1}[\phi_1] = |\top \wedge \sigma_0 b| \cap \text{id}_{\mathbb{R}^{n+\omega}}^{-1}[\phi_1]$, and by Lemma 6 and the IH that $|\phi_1| = B$, this is $(\bar{b} \times \mathbb{R}^\omega) \cap B = B'$.
 - (iii) $|\psi''| = |\psi'| \cap \text{id}_{\mathbb{R}^{n+\omega}}^{-1}[\top] = |\psi'| = \mathcal{O}$.

Thus, in the case that $(F, B, \mathcal{O}) \in \mathcal{F}_{p_1}$ and $B' = B \cap (\bar{b} \times \mathbb{R}^\omega)$, we have

$$(S, \gamma_0) \xrightarrow{*} (\text{Skip}, \gamma'')$$

where γ'' satisfies the properties (i)-(iii), so we are done.

2. The case that $(F, B, \mathcal{O}) \in \mathcal{F}_{p_2}$ and $B' = B \cap (\bar{b}^c \times \mathbb{R}^\omega)$ is analogous; just replace $\phi' = \top \wedge \sigma_0 b$ by $\phi' = \top \wedge \sigma_0 !b$.

We have thus finished the inductive step in case $p = \text{if } b \text{ } p_1 \text{ else } p_2$.

– We conclude the proof with iteration; let $p = \text{while } b \text{ } q$. Then, for $(F, B, \mathcal{O}) \in \mathcal{F}_p$ means there is m such that $(F, B, \mathcal{O}) \in \mathbb{F}_{b,q}^m(\mathbb{F}_0)$ (where $\mathbb{F}_0 = \{(\text{id}_{\mathbb{R}^{n+\omega}}, \bar{b}^c \times \mathbb{R}^\omega, \mathbb{R}^{n+\omega})\}$). We therefore proceed by induction on m that the following holds: for all $(F, B, \mathcal{O}) \in \mathbb{F}_{b,q}^m(\mathbb{F}_0)$, there is γ such that $(\text{while } b \text{ } q, \gamma_0) \xrightarrow{*} (\text{Skip}, \gamma)$.

The base case is when $m = 0$, in which case we have $(F, B, \mathcal{O}) = (\text{id}_{\mathbb{R}^{n+\omega}}, \bar{b}^c \times \mathbb{R}^\omega, \mathbb{R}^{n+\omega})$, and, indeed, $(\text{while } b \text{ } q, \gamma_0) \longrightarrow (\text{Skip}, \sigma_0, 0, \top \wedge \sigma_0 !b, \top)$. In this setting, it is routine to verify that properties (i)-(iii) hold.

We now prove the statement for $m + 1$. $(F, B, \mathcal{O}) \in \mathbb{F}_{b,q}^{m+1}(\mathbb{F}_0)$ means there are $(F_q, B_q, \mathcal{O}_q) \in \mathcal{F}_q$ and $(F_m, B_m, \mathcal{O}_m) \in \mathbb{F}_{b,q}^m(\mathbb{F}_0)$, such that

- $F = F_m \circ F_q$;
- $B = (\bar{b} \times \mathbb{R}^\omega) \cap B_q \cap F_q^{-1}[B_m]$; and
- $\mathcal{O} = \mathcal{O}_q \cap F_q^{-1}[\mathcal{O}_m]$.

Now apply the IH for the subterm q of p for $(F_q, B_q, \mathcal{O}_q) \in \mathcal{F}_q$, and for the case m in the induction on m for $(F_m, B_m, \mathcal{O}_m) \in \mathbb{F}_{b,q}^m(\mathbb{F}_0)$, to obtain *canonical* symbolic executions

$$(q, \gamma_0) \xrightarrow{*} (\text{Skip}, \gamma_q) \quad (*) \quad \text{and} \quad (\text{while } b \ q, \gamma_0) \xrightarrow{*} (\text{Skip}, \gamma_m) \quad (**)$$

where $\gamma_q = (\sigma_q, k_q, \phi_q, \psi_q)$ and $\gamma_m = (\sigma_m, k_m, \phi_m, \psi_m)$ with the properties that

- $|\sigma_q|_{k_q} = F_q$ and $|\sigma_m|_{k_m} = F_m$;
- $|\phi_q| = B_q$ and $|\phi_m| = B_m$; and
- $|\psi_q| = \mathcal{O}_q$ and $|\psi_m| = \mathcal{O}_m$.

Using Lemma 9 twice, we have an outgoing transition chain from $(\text{while } b \ q, \gamma_0)$ as follows:

$$\begin{aligned} (\text{while } b \ q, \gamma_0) &\longrightarrow (q \ ; \ \text{while } b \ q, \gamma'_0) && \text{Rule iter-T} \\ &\xrightarrow{*} (\text{Skip} \ ; \ \text{while } b \ q, \gamma') && \text{Rule seq-n and Lemma 9 with } (*) \\ &\longrightarrow (\text{while } b \ q, \gamma') && \text{Rule seq-0} \\ &\xrightarrow{*} (\text{Skip}, \gamma) && \text{Lemma 9 with } (**). \end{aligned}$$

where, $\gamma'_0 = (\sigma_0, 0, \top \wedge \sigma_0(b), \top)$, and (1) $\gamma' \sim \gamma_q \bullet \gamma'_0$ and (2) $\gamma \sim \gamma_m \bullet \gamma'$. We verify the properties (i)-(iii) for this γ :

- (i) $|\sigma|_k \stackrel{(2)}{=} |\sigma_m|_{k_m} \circ |\sigma'|_{k'} \stackrel{(1)}{=} |\sigma_m|_{k_m} \circ |\sigma_q|_{k_q} \circ |\sigma_0|_0 \stackrel{\text{IH}}{=} F_m \circ F_q \circ \text{id}_{\mathbb{R}^{n+\omega}} = F$;
- (ii) $|\phi| \stackrel{(2)}{=} |\phi'| \cap |\sigma'|_{k'}^{-1}[|\phi_m|] \stackrel{(1)}{=} |\top \wedge \sigma_0(b)| \cap |\sigma_0|_0^{-1}[|\phi_q|] \cap (|\sigma_q|_{k_q} \circ |\sigma_0|_0)^{-1}[B_m]$.
It is routine to equate this to $(\bar{b} \times \mathbb{R}^\omega) \cap B_q \cap F_q^{-1}[B_m] = B$.

(iii) Analogous to the above property

We have thus verified that there is γ such that $(\text{while } b \ T, \gamma_0) \xrightarrow{*} (\text{Skip}, \gamma)$ for $(F, B, \mathcal{O}) \in \mathbb{F}_{b,T}^{n+1}(\mathbb{F}_0)$ such that $|\sigma|_k = F$, $|\phi| = B$, and $|\psi| = \mathcal{O}$.

Induction on p has finished, and we finished the proof of Proposition 10. A notable consequence of this is that the mappings Φ_p defined in the proof of Theorem 4 are surjective.

Proposition 11 (Injectivity of the Bijection). *Let $p \in \mathbb{P}$. If $(p, \gamma_0) \xrightarrow{*} (\text{Skip}, \gamma_1)$ and $(p, \gamma_0) \xrightarrow{*} (\text{Skip}, \gamma_2)$ then either $\gamma_1 = \gamma_2$ or $|\phi_1| \cap |\phi_2| = \emptyset$.*

A.3 Proof of Theorem 5

We restate the theorem for convenience:

Theorem 5 (Correctness of Symbolic Execution of Probabilistic Programs). *Let $p \in \mathbb{P}$ be a program, $\mu \in \mathcal{M}(\mathbb{R}^n)$ a distribution measure over the input variables, and $A \subseteq \mathbb{R}^n$ a measurable set. Then*

$$\llbracket p \rrbracket(\mu)(A) = \sum_{(\sigma, k, \phi, \psi) \in \Gamma_p} (\mu \otimes \lambda^\omega)(|\sigma|_k^{-1}[A \times \mathbb{R}^\omega] \cap |\phi| \cap |\psi|)$$

Proof. For every program p , there is a function f_p such that

- (i) $(\mu \otimes \lambda^\omega)(f_p^{-1}[A \times \mathbb{R}^\omega]) = \llbracket p \rrbracket(\mu)(A)$, and
- (ii) $(\mu \otimes \lambda^\omega)(f_p^{-1}[A \times \mathbb{R}^\omega]) = \sum_{(F,B,\mathcal{O}) \in \mathcal{F}_p} (\mu \otimes \lambda^\omega)(F^{-1}[A \times \mathbb{R}^\omega] \cap B \cap \mathcal{O})$.

The function f_p is given in Definition 12 below. Item (i) is Lemma 13 (take $B = A \times \mathbb{R}^\omega$). Item (ii) follows from Lemma 14, which gives

$$f_p^{-1}[A \times \mathbb{R}^\omega] = \bigcup_{(F,B,\mathcal{O}) \in \mathcal{F}_p} F^{-1}[A \times \mathbb{R}^\omega] \cap B \cap \mathcal{O},$$

and measuring the union on the right-hand side is the same as summing all the individual measures – as in Item (ii) – because it is a disjoint union by Lemma 15. The theorem is proven by chaining the two equalities in Items (i) and (ii) and applying Theorem 4.

Q.E.D.

Definition 12 (Operational semantics [19], extended to support observe statements).

For programs $p \in \mathbb{P}$ in n variables, the partial functions $f_p : \mathbb{R}^{n+\omega} \rightarrow \mathbb{R}^{n+\omega} \cup \{*\}$ are defined inductively on the structure of p as:

$$f_p : \rho \mapsto \begin{cases} \rho & \text{if } p = \text{Skip} \\ \rho[i \mapsto \bar{e}(\rho|_n)] & \text{if } p = x_i := e \\ \text{sample}_i(\rho) & \text{if } p = x_i \sim \text{rnd} \\ \rho & \text{if } p = \text{observe } b \text{ and } \rho|_n \in \bar{b} \\ * & \text{if } p = \text{observe } b \text{ and } \rho|_n \notin \bar{b} \\ f_p(\rho) & \text{if } p = \text{if } b \text{ } p_1 \text{ else } p_2 \text{ and } \rho|_n \in \bar{b} \\ f_{p_2}(\rho) & \text{if } p = \text{if } b \text{ } p_1 \text{ else } p_2 \text{ and } \rho|_n \notin \bar{b} \\ f_q^m(\rho) & \text{if } p = \text{while } b \text{ } q, m := \min\{j \in \mathbb{N} \mid f_q^j(\rho)|_n \notin \bar{b}\} \\ (f_{p_2} \circ f_{p_1})(\rho) & \text{if } p = p_1 ; p_2 \end{cases}$$

where f^m denotes m -fold iterated applications of f (identity for $m = 0$).

In this definition, $*$ indicates an aborted execution due to an observe statement. The functions may be partial because loops might diverge: for certain $\rho \in \mathbb{R}^{n+\omega}$ there may not be $j \in \mathbb{N}$ for which $f_q^j(\rho) \notin \bar{b} \times \mathbb{R}^\omega$; we write $f_p(\rho) \uparrow$ if p diverges on input ρ . A failed observe yields the explicit *aborted* state $*$ (not the same as divergence!) to which the domain naturally extends.

The distribution of outputs through this operational semantics is the (possibly unnormalized) one defined by the denotational semantics:

Lemma 13 (Correctness of operational semantics [19], extended to support observe statements). Let $p \in \mathbb{P}$ be a program in n variables and $\mu \in \mathcal{M}(\mathbb{R}^n)$ a probability measure over the input variables. Then, for every $B \in \mathcal{B}(\mathbb{R}^{n+\omega})$,

$$(\llbracket p \rrbracket(\mu) \otimes \lambda^\omega)(B) = (\mu \otimes \lambda^\omega)(f_p^{-1}[B]) \quad (6)$$

Proof. For arbitrary $p \in \mathbb{P}$, define

$$\mathcal{H}_p := \{B \in \mathcal{B}(\mathbb{R}^{n+\omega}) \mid (\llbracket p \rrbracket(\mu) \otimes \lambda^\omega)(B) = (\mu \otimes \lambda^\omega)(f_p^{-1}[B]) \text{ for all } \mu \in \mathcal{M}(\mathbb{R}^n)\}$$

Assume now that \mathcal{H}_p contains all the cylinder sets for every p . It is not hard to see that every \mathcal{H}_p is also a Dynkin system:

- \mathcal{H}_p always contains the emptyset.
- If $B \in \mathcal{H}_p$, then

$$(\llbracket p \rrbracket(\mu) \otimes \lambda^\omega)(B^c) = (\llbracket p \rrbracket(\mu) \otimes \lambda^\omega)(\mathbb{R}^{n+\omega}) - (\llbracket p \rrbracket(\mu) \otimes \lambda^\omega)(B)$$

and, since $\mathbb{R}^{n+\omega} \in \mathcal{H}_p$ as a cylinder set, and $B \in \mathcal{H}_p$, this is equal to

$$(\mu \otimes \lambda^\omega)(f_p^{-1}[\mathbb{R}^{n+\omega}]) - (\mu \otimes \lambda^\omega)(f_p^{-1}[B]) = (\mu \otimes \lambda^\omega)(f_p^{-1}[B^c]),$$

so \mathcal{H}_p is always closed under complement.

- Both $(\llbracket p \rrbracket(\mu) \otimes \lambda^\omega)$ and $(\mu \otimes \lambda^\omega) \circ f_p^{-1}$ are measures, so \mathcal{H}_p is obviously closed under countable unions of pairwise disjoint sets.

By the π - λ -theorem, \mathcal{H}_p therefore contains all Borel sets.

It remains to show the assumption that \mathcal{H}_p contains the cylinder sets for every p . Thus, we show (6) by induction on the structure of p , where we may assume that B is a cylinder set, i.e., it can be written as

$$B = \underbrace{B_0 \times \cdots \times B_{n-1}}_{=: B^{(n)}} \times \underbrace{B_n \times B_{n+1} \times \cdots \times B_{n+p}}_{=: B^{(\omega)}} \times \mathbb{R}^\omega$$

for some $p \in \mathbb{N}$.

- For $p = \text{skip}$, we have $\llbracket p \rrbracket(\mu) = \mu$, so the LHS of (6) is just $(\mu \otimes \lambda^\omega)(B)$ for any measurable set B . Also $f_p^{-1}[B] = B$, so the RHS equals $(\mu \otimes \lambda^\omega)(B)$ as well.
- For $p = x_i := e$, for the LHS (6) we have $\llbracket p \rrbracket(\mu) = \mu \circ \alpha_e^i$, where

$$\alpha_e^i(x_0, \dots, x_{n-1}) = (x_0, \dots, x_{i-1}, \bar{e}(x_0, \dots, x_{n-1}), x_{i+1}, \dots, x_{n-1})$$

For this α_e^i , we can compute the following preimage:

$$(\alpha_e^i)^{-1}[B^{(n)}] = (B_0 \times \cdots \times B_{i-1} \times \mathbb{R} \times B_{i+1} \times \cdots \times B_{n-1}) \cap \bar{e}^{-1}[B^{(n)}] \quad (7)$$

Hence, the LHS of (6), which is $(\llbracket p \rrbracket(\mu) \otimes \lambda^\omega)(B^{(n)} \times B^{(\omega)})$, becomes the μ of this set (7), multiplied by $\lambda^\omega(B^{(\omega)})$. For the RHS, the preimage of B under $f_p : \rho \mapsto \rho[i \mapsto \bar{e}(\rho|_n)]$ is

$$f_p^{-1}[B] = \left((B_0 \times \cdots \times B_{i-1} \times \mathbb{R} \times B_{i+1} \times \cdots \times B_{n-1}) \cap \bar{e}^{-1}[B^{(n)}] \right) \times B^{(\omega)},$$

and measuring this with $(\mu \otimes \lambda^\omega)$ establishes this case.

- Now let $p = x_i \sim \text{rnd}$. We have, on the LHS of (6),

$$\llbracket p \rrbracket(\mu)(B_0 \times \cdots \times B_{n-1}) = \mu(B_0 \times \cdots \times B_{i-1} \times \mathbb{R} \times B_{i+1} \times \cdots \times B_{n-1}) \cdot \lambda(B_i)$$

and $(\llbracket p \rrbracket(\mu) \otimes \lambda^\omega)(B)$ is exactly this, multiplied by $\lambda^\omega(B^{(\omega)})$.

For the RHS, $f_p(\rho) = \text{sample}_i(\rho)$, where

$$\text{sample}_i(\rho) = (\rho_0, \dots, \rho_{i-1}, \rho_n, \rho_{i+1}, \dots, \rho_{n-1}, \rho_{n+1}, \rho_{n+2}, \dots)$$

so that

$$f_p^{-1}[B] = B_0 \times \cdots \times B_{i-1} \times \mathbb{R} \times B_{i+1} \times \cdots \times B_{n-1} \times B_i \times B^{(\omega)}$$

and measuring this with $(\mu \otimes \lambda^\omega)$ establishes the case.

- The last base case is $p = \text{observe } b$. We have, for the LHS,

$$(\llbracket p \rrbracket(\mu) \otimes \lambda^\omega)(B) = \mu(B^{(n)} \cap \bar{b}) \cdot \lambda^\omega(B^{(\omega)})$$

On the RHS, $f_p(\rho) = \rho$ if $\rho \in \bar{b} \times \mathbb{R}^\omega$ (and undefined otherwise), so $f_p^{-1}[B] = (B^{(n)} \cap \bar{b}) \times B^\omega$. Measuring this last set with $(\mu \otimes \lambda^\omega)$ establishes this case.

The base cases of the structure of p are established, and we continue with the induction steps. The induction hypotheses for substatements p' of p (these are p_1 and p_2 in the case for *if* and *sequencing*, and q in the case for *while*) are that $\mathcal{H}_{p'}$ contains all cylinder sets. By the reasoning above, the IH therefore entails that $\mathcal{H}_{p'}$ moreover contains all Borel sets.

- If $p = \text{if } b \text{ } p_1 \text{ else } p_2$, we have two induction hypotheses (IH), which we apply respectively using the measures $e_{\bar{b}}(\mu)$ and $e_{\overline{\bar{b}}}(\mu)$ in the following derivation:

$$\begin{aligned} & (\llbracket p \rrbracket(\mu) \otimes \lambda^\omega)(B) \\ &= (\llbracket p \rrbracket(\mu) \otimes \lambda^\omega)(B^{(n)} \times B^{(\omega)}) \\ &= \llbracket p \rrbracket(\mu)(B^{(n)}) \cdot \lambda^\omega(B^{(\omega)}) \\ &= \left(\llbracket p_1 \rrbracket(e_{\bar{b}}(\mu))(B^{(n)}) + \llbracket p_2 \rrbracket(e_{\overline{\bar{b}}}(\mu))(B^{(n)}) \right) \cdot \lambda^\omega(B^{(\omega)}) \\ &= \llbracket p_1 \rrbracket(e_{\bar{b}}(\mu))(B^{(n)}) \cdot \lambda^\omega(B^{(\omega)}) + \llbracket p_2 \rrbracket(e_{\overline{\bar{b}}}(\mu))(B^{(n)}) \cdot \lambda^\omega(B^{(\omega)}) \\ &= (\llbracket p_1 \rrbracket(e_{\bar{b}}(\mu)) \otimes \lambda^\omega)(B) + (\llbracket p_2 \rrbracket(e_{\overline{\bar{b}}}(\mu)) \otimes \lambda^\omega)(B) \\ &\stackrel{\text{IH}}{=} (e_{\bar{b}}(\mu) \otimes \lambda^\omega)(f_{p_1}^{-1}[B]) + (e_{\overline{\bar{b}}}(\mu) \otimes \lambda^\omega)(f_{p_2}^{-1}[B]) \\ &\stackrel{(*)}{=} (\mu \otimes \lambda^\omega)(f_{p_1}^{-1}[B] \cap (\bar{b} \times \mathbb{R}^\omega)) + (\mu \otimes \lambda^\omega)(f_{p_2}^{-1}[B] \cap (\overline{\bar{b}} \times \mathbb{R}^\omega)) \\ &= (\mu \otimes \lambda^\omega)\left(\left(f_{p_1}^{-1}[B] \cap (\bar{b} \times \mathbb{R}^\omega)\right) \cup \left(f_{p_2}^{-1}[B] \cap \overline{\bar{b}} \times \mathbb{R}^\omega\right)\right) \\ &= (\mu \otimes \lambda^\omega)(f_p^{-1}[B]) \end{aligned}$$

In the last equality we apply the definition of f_p . In the second to last equality, the sum of the measures is the measure of the union, because the sets are disjoint. Indeed, one is contained in $\bar{b} \times \mathbb{R}^\omega$ and the other in $\overline{\bar{b}} \times \mathbb{R}^\omega$. At (*), we use the fact that if (X, Σ_X, μ_X) and (Y, Σ_Y, μ_Y) are σ -finite measure spaces and $A \in \Sigma_X$, then $(e_A(\mu_X) \otimes \mu_Y)(E) = (\mu_X \otimes \mu_Y)(E \cap (A \times Y))$ for all $E \in \Sigma_X \otimes \Sigma_Y$. (Here, $\Sigma_X = \mathcal{B}(\mathbb{R}^n)$ and $\Sigma_Y = \mathcal{B}(\mathbb{R}^\omega)$.) This case is now finished.

– Now let $p = \text{while } b \text{ } q$. The LHS of (6) is

$$\begin{aligned}
(\llbracket p \rrbracket(\mu) \otimes \lambda^\omega)(B) &= \llbracket p \rrbracket(\mu)(B^{(n)}) \cdot \lambda^\omega(B^{(\omega)}) \\
&= \left(\sum_{m=0}^{\infty} e_{\bar{b}} \circ (\llbracket q \rrbracket \circ e_{\bar{b}})^m \right) (\mu)(B^{(n)}) \cdot \lambda^\omega(B^{(\omega)}) \\
&= \sum_{m=0}^{\infty} e_{\bar{b}} \left((\llbracket q \rrbracket \circ e_{\bar{b}})^m (\mu) \right) (B^{(n)}) \cdot \lambda^\omega(B^{(\omega)}) \\
&= \sum_{m=0}^{\infty} (e_{\bar{b}} \circ (\llbracket q \rrbracket \circ e_{\bar{b}})^m (\mu) \otimes \lambda^\omega)(B) \\
&= \sum_{m=0}^{\infty} ((\llbracket q \rrbracket \circ e_{\bar{b}})^m (\mu) \otimes \lambda^\omega)(B \cap (\bar{b}^c \times \mathbb{R}^\omega))
\end{aligned} \tag{8}$$

Looking at the RHS, we derive the following expression:

$$\begin{aligned}
f_p^{-1}[B] &= \{\rho \in \mathbb{R}^{n+\omega} \mid f_p(\rho) \in B\} \\
&= \{\rho \in \mathbb{R}^{n+\omega} \mid \exists m. f_q^m(\rho) \in B \cap (\bar{b}^c \times \mathbb{R}^\omega) \\
&\quad \wedge \forall j \in \{0, \dots, m-1\}. f_q^j(\rho) \in \bar{b} \times \mathbb{R}^\omega\} \\
&= \{\rho \in \mathbb{R}^{n+\omega} \mid \exists m. \rho \in f_q^{-m}[B \cap (\bar{b}^c \times \mathbb{R}^\omega)] \\
&\quad \wedge \forall j \in \{0, \dots, m-1\}. \rho \in f_q^{-j}[\bar{b} \times \mathbb{R}^\omega]\} \\
&= \underbrace{\bigcup_{m=0}^{\infty} \left\{ f_q^{-m}[B \cap (\bar{b}^c \times \mathbb{R}^\omega)] \cap \bigcap_{j=0}^{m-1} f_q^{-j}[\bar{b} \times \mathbb{R}^\omega] \right\}}_{=: U_m}
\end{aligned} \tag{9}$$

Here, $f_q^{-m}[A]$ denotes the m -th preimage, i.e., $f_q^{-m}[A] = A$ for $m = 0$ and

$$f_q^{-m}[A] = \underbrace{f_q^{-1}[f_q^{-1}[\dots[A]\dots]]}_{m \text{ times}}$$

The family of sets $(U_m)_{m \in \mathbb{N}}$ in (9) are pairwise disjoint. That is, if $m \neq m'$ then $U_m \cap U_{m'} = \emptyset$. Indeed, without loss of generality, $m' > m$, and then

$$\rho \in U_{m'} \implies \rho \in f_q^{-m}[\bar{b} \times \mathbb{R}^\omega] \implies \rho \notin f_q^{-m}[\bar{b}^c \times \mathbb{R}^\omega] \implies \rho \notin U_m$$

Hence, for the RHS, we have

$$(\mu \otimes \lambda^\omega)(f_p^{-1}[B]) = (\mu \otimes \lambda^\omega) \left(\bigcup_{m=0}^{\infty} U_m \right) = \sum_{m=0}^{\infty} (\mu \otimes \lambda^\omega)(U_m)$$

Comparing this with the derivation of the LHS (8), it will suffice to show the following holds for every m and every $C \in \mathcal{B}(\mathbb{R}^{n+\omega})$:

$$((\llbracket q \rrbracket \circ e_{\bar{b}})^m (\mu) \otimes \lambda^\omega)(C) = (\mu \otimes \lambda^\omega)(f_q^{-m}[C] \cap \bigcap_{j=0}^{m-1} f_q^{-j}[\bar{b} \times \mathbb{R}^\omega]) \tag{10}$$

Indeed, using $C = B \cap (\bar{b}^c \times \mathbb{R}^\omega)$ in (10) will establish the result for this case. The claim (10) can be proved by induction on m .

- The base case $m = 0$ is easy, since both sides are obviously $(\mu \otimes \lambda^\omega)(C)$.

- For the inductive step, we use the IH (*) for the substatement q in $p = \text{while } b \ q$, and another IH (**) for $m = \ell$. We prove the statement for $m = \ell + 1$.

$$\begin{aligned}
& ((\llbracket q \rrbracket \circ e_{\bar{b}})^{\ell+1}(\mu) \otimes \lambda^\omega)(C) \\
&= (\llbracket q \rrbracket (e_{\bar{b}}((\llbracket q \rrbracket \circ e_{\bar{b}})^\ell(\mu)))) \otimes \lambda^\omega(C) \\
&\stackrel{*}{=} (e_{\bar{b}}((\llbracket q \rrbracket \circ e_{\bar{b}})^\ell(\mu)) \otimes \lambda^\omega)(f_q^{-1}[C]) \\
&= ((\llbracket q \rrbracket \circ e_{\bar{b}})^\ell(\mu) \otimes \lambda^\omega)(f_q^{-1}[C] \cap (\bar{b} \times \mathbb{R}^\omega)) \\
&\stackrel{**}{=} (\mu \otimes \lambda^\omega)(f_q^{-\ell}[f_q^{-1}[C] \cap (\bar{b} \times \mathbb{R}^\omega)] \cap \bigcap_{j=0}^{\ell-1} f_q^{-j}[\bar{b} \times \mathbb{R}^\omega]) \\
&= (\mu \otimes \lambda^\omega)(f_q^{-(\ell+1)}[C] \cap \bigcap_{j=0}^{\ell} f_q^{-j}[\bar{b} \times \mathbb{R}^\omega])
\end{aligned}$$

The inductive proof shows that (10) holds for all m and every $C \in \mathcal{B}(\mathbb{R}^{n+\omega})$.

Conclude that $(\llbracket p \rrbracket(\mu) \otimes \lambda^\omega)(B) = (\mu \otimes \lambda^\omega)(f_p^{-1}[B])$ for all $B \in \mathcal{B}(\mathbb{R}^{n+\omega})$ and all $\mu \in \mathcal{M}(\mathbb{R}^n)$ in the case that $p = \text{while } b \ T$.

- For the case where $p = p_1 \ ; \ p_2$, we have

$$(\llbracket p \rrbracket(\mu) \otimes \lambda^\omega)(B) = (\llbracket p_2 \rrbracket \otimes \lambda^\omega)(\llbracket p_1 \rrbracket(\mu))(B)$$

Now we apply the IH for p_2 with $\llbracket p_1 \rrbracket(\mu)$ and then for p_1 with μ , to rewrite this further as

$$\dots = (\llbracket p_1 \rrbracket(\mu) \otimes \lambda^\omega)(f_{p_2}^{-1}[B]) = (\mu \otimes \lambda^\omega)(f_{p_1}^{-1}[f_{p_2}^{-1}[B]]) = (\mu \otimes \lambda^\omega)(f_p^{-1}[B])$$

Here, we again use the crucial fact that \mathcal{H}_{p_1} is a Dynkin systems that contains the cylinder sets, so they contain the Borel sets. In particular, \mathcal{H}_{p_1} contains $f_{p_2}^{-1}[B]$. This concludes the case for sequencing.

We have covered all cases for p , and the proof is done.

Lemma 14 (Correspondence between symbolic and concrete semantics). *Let $p \in \mathbb{P}$ be a program in n variables and $\rho \in \mathbb{R}^{n+\omega}$. Then $f_p(\rho) = \rho'$ for some $\rho' \in \mathbb{R}^{n+\omega}$ iff there is $(F, B, \mathcal{O}) \in \mathcal{F}_p$ such that $\rho \in B \cap \mathcal{O}$. In this case, $F(\rho) = \rho' = f_p(\rho)$.*

Proof. The proof is by induction on the structure of p . The base cases are trivial, because the symbolic semantics \mathcal{F}_p are the singletons $\{(F, B, \mathcal{O})\}$ where F is just f_p , B is the entire space, and so is \mathcal{O} , except in the case for observe, where it is the measurable set corresponding to the observed Boolean formula.

- If $p = \text{Skip}$ then $f_p = \text{id}_{\mathbb{R}^{n+\omega}}$ and $\mathcal{F}_p = \{(\text{id}_{\mathbb{R}^{n+\omega}}, \mathbb{R}^{n+\omega}, \mathbb{R}^{n+\omega})\}$.
- If $p = x_i := e$ then also $\mathcal{F}_p = \{(f_p, \mathbb{R}^{n+\omega}, \mathbb{R}^{n+\omega})\}$.
- Idem for $p = x_i \sim \text{rnd}$.
- If $p = \text{observe } b$ then f_p is the identity on \bar{b} and $f_p(\rho) = *$ in case $\rho \in \bar{b}^c$. Also, $(F, B, \mathcal{O}) \in \mathcal{F}_p$ iff $(F, B, \mathcal{O}) = (\text{id}_{\mathbb{R}^{n+\omega}}, \mathbb{R}^{n+\omega}, \bar{b})$. Thus, $f_p(\rho) = \rho$ iff $\rho \in \bar{b} = B \cap \mathcal{O}$ and in this case, $F(\rho) = \rho = f_p(\rho)$.

In the inductive steps, the ‘if’ and ‘only if’ parts are proven separately.

- **First sequencing:** let $p = p_1 \circledast p_2$.
 \implies : By definition, $f_p(\rho) = f_{p_2}(f_{p_1}(\rho))$. If $f_p(\rho) = \rho''$ for some $\rho'' \in \mathbb{R}^{n+\omega}$ then $f_{p_1}(\rho) = \rho'$ for some $\rho' \in \mathbb{R}^{n+\omega}$ and $f_{p_2}(\rho') = \rho''$. By IHs, then, there are $(F_1, B_1, \mathcal{O}_1) \in \mathcal{F}_{p_1}$ and $(F_2, B_2, \mathcal{O}_2) \in \mathcal{F}_{p_2}$ such that $\rho \in B_1 \cap \mathcal{O}_1$ and $\rho' \in B_2 \cap \mathcal{O}_2$. Moreover, $f_{p_1}(\rho) = F_1(\rho)$ and $f_{p_2}(\rho') = F_2(\rho')$. By definition of \mathcal{F}_p , $(F, B, \mathcal{O}) := (F_2 \circ F_1, B_1 \cap F_1^{-1}[B_2], \mathcal{O}_1 \cap F_1^{-1}[\mathcal{O}_2]) \in \mathcal{F}_p$. It can be straightforwardly verified that $\rho \in B \cap \mathcal{O}$ and $F(\rho) = f_p(\rho)$.
 \Leftarrow : Let $(F, B, \mathcal{O}) \in \mathcal{F}_p$. By definition, it is composed of two $(F_i, B_i, \mathcal{O}_i) \in \mathcal{F}_{p_i}$ ($i = 1, 2$) such that $F = F_2 \circ F_1$, $B = B_1 \cap F_1^{-1}[B_2]$, and $\mathcal{O} = \mathcal{O}_1 \cap F_1^{-1}[\mathcal{O}_2]$. For $\rho \in B$, we have $\rho \in B_1$ and $F_1(\rho) \in B_2$. If $\rho \in \mathcal{O}$ then $\rho \in B_1 \cap \mathcal{O}_1$ so $\rho' := F_1(\rho) \stackrel{\text{IH}}{=} f_{p_1}(\rho)$ for some $\rho' \in \mathbb{R}^{n+\omega}$. In addition, $\rho' \in B_2 \cap \mathcal{O}_2$, so $\rho'' := F_2(\rho') \stackrel{\text{IH}}{=} f_{p_2}(\rho')$ for some $\rho'' \in \mathbb{R}^{n+\omega}$. Hence, $\rho'' = f_{p_2}(f_{p_1}(\rho)) = f_p(\rho)$ and obviously $F(\rho) = \rho''$.
- **Now consider $p = \text{if } b \ p_1 \ \text{else } p_2$.**
 \implies : There are two cases to consider for the given ρ : $\rho \in \bar{b} \times \mathbb{R}^\omega$ or $\rho \notin \bar{b} \times \mathbb{R}^\omega$. We prove only the second; the first is analogous. In that case, $f_p(\rho) = f_{p_2}(\rho)$. If $f_{p_2}(\rho) = \rho'$ for some $\rho' \in \mathbb{R}^{n+\omega}$ then by IH there is $(F, B, \mathcal{O}) \in \mathcal{F}_{p_2}$ such that $\rho \in B \cap \mathcal{O}$ and $F(\rho) = \rho'$. By definition, $(F, B \cap (\bar{b}^\complement \times \mathbb{R}^\omega), \mathcal{O}) \in \mathcal{F}_p$ and, obviously, $\rho \in B \cap (\bar{b}^\complement \times \mathbb{R}^\omega) \cap \mathcal{O}$.
 \Leftarrow : Let $(F, B, \mathcal{O}) \in \mathcal{F}_p$. Then either (i) $B = B_1 \cap (\bar{b} \times \mathbb{R}^\omega)$ and $(F, B_1, \mathcal{O}) \in \mathcal{F}_{p_1}$, or (ii) $B = B_2 \cap (\bar{b}^\complement \times \mathbb{R}^\omega)$ and $(F, B_2, \mathcal{O}) \in \mathcal{F}_{p_2}$. We prove only case (i); case (ii) is analogous. Suppose $\rho \in B \cap \mathcal{O}$ then, since $\rho \in B_1 \cap \mathcal{O}$, by IH, $F(\rho) = f_{p_1}(\rho) = f_p(\rho)$, where the last equality uses $\rho \in \bar{b} \times \mathbb{R}^\omega$.
- **Finally, iteration:** $p = \text{while } b \ q$.
 \implies : If $f_p(\rho) = \rho'$ then there is $m \in \mathbb{N}$ such that $f_q^m(\rho) \in \bar{b}^\complement \times \mathbb{R}^\omega$ and for all $j < m$ in \mathbb{N} , $f_q^j(\rho) \in \bar{b} \times \mathbb{R}^\omega$. By induction on m , we show that for every such m and for every ρ , there is $(F, B, \mathcal{O}) \in \mathbb{F}_{b,q}^m(\mathbb{F}_0)$, where denotes $\{(\text{id}_{\mathbb{R}^{n+\omega}}, \bar{b}^\complement \times \mathbb{R}^\omega, \mathbb{R}^{n+\omega})\}$, such that $\rho \in B \cap \mathcal{O}$ and $F(\rho) = f_q^m(\rho)$. Verifying the base case $m = 0$ with $(F, B, \mathcal{O}) \in \mathbb{F}_0$ is routine.
Suppose now, for the inductive step, that $f_q^{m+1}(\rho) \in \bar{b}^\complement \times \mathbb{R}^\omega$ and for all $j < m + 1$ in \mathbb{N} , $f_q^j(\rho) \in \bar{b} \times \mathbb{R}^\omega$. Then, putting $\rho' := f_q(\rho)$ this just says $f_q^m(\rho') \in \bar{b}^\complement \times \mathbb{R}^\omega$ and for all $j < m$ in \mathbb{N} , $f_q^j(\rho') \in \bar{b} \times \mathbb{R}^\omega$.
 - by the IH on m , now, there is $(F', B', \mathcal{O}') \in \mathbb{F}_{b,q}^m(\mathbb{F}_0)$ such that $f_q(\rho) \in B' \cap \mathcal{O}'$ and $F'(f_q(\rho)) = f_q^m(f_q(\rho))$, and
 - by the IH on the subterm q , there is $(F_q, B_q, \mathcal{O}_q) \in \mathcal{F}_q$ such that $\rho \in B_q \cap \mathcal{O}_q$ and $F_q(\rho) = f_q(\rho)$, and

It is now mechanically verified that the element

$$(F, B, \mathcal{O}) := (F' \circ F_q, B_q \cap F_q^{-1}[B'] \cap (\bar{b} \times \mathbb{R}^\omega), \mathcal{O}_q \cap F_q^{-1}[\mathcal{O}']) \in \mathbb{F}_{b,q}^{m+1}(\mathbb{F}_0)$$

satisfies $\rho \in B \cap \mathcal{O}$ and $F(\rho) = f_q^{m+1}(\rho)$.

\Leftarrow : Conversely, for $(F, B, \mathcal{O}) \in \mathcal{F}_p$ we must have $m \in \mathbb{N}$ such that $(F, B, \mathcal{O}) \in \mathbb{F}_{b,q}^m(\mathbb{F}_0)$, and we proceed again by induction on m . For $m = 0$, if $\rho \in B \cap \mathcal{O}$, then

we know that $\rho \in \bar{b}^c \times \mathbb{R}^\omega$ and so, indeed $\min\{j \in \mathbb{N} \mid f_q^j(\rho) \in \bar{b}^c \times \mathbb{R}^\omega\} = 0$, so $f_p(\rho) = f_q^0(\rho) = \rho = F(\rho)$.

In the inductive step, let $(F, B, \mathcal{O}) \in \mathbb{F}_{b,q}^{m+1}(\mathbb{F}_0)$. Then, by definition of $\mathbb{F}_{b,q}$,

$$(F, B, \mathcal{O}) = (F' \circ F_q, B_q \cap F_q^{-1}[B'] \cap (\bar{b} \times \mathbb{R}^\omega), \mathcal{O}_q \cap F_q^{-1}[\mathcal{O}'])$$

for some $(F', B', \mathcal{O}') \in \mathbb{F}_{b,q}^m(\mathbb{F}_0)$ and $(F_q, B_q, \mathcal{O}_q) \in \mathcal{F}_q$. For all $\rho \in B \cap \mathcal{O}$, we also have $\rho \in B_q \cap \mathcal{O}_q$ and so $f_q(\rho) = F_q(\rho)$ by IH. Now $F_q(\rho) \in B' \cap \mathcal{O}'$ so by IH on m , we have $f_p(F_q(\rho)) = f_q^m(F_q(\rho))$ and $m = \min\{j \in \mathbb{N} \mid f_q^j(F_q(\rho)) \in \bar{b}^c \times \mathbb{R}^\omega\} = 0$. Since $F_q(\rho) = f_q(\rho)$ and also $\rho \in \bar{b} \times \mathbb{R}^\omega$, this means that

$$m + 1 = \min\{j \mid f_q^j(\rho) \in \bar{b}^c \times \mathbb{R}^\omega\}$$

and we conclude that $f_p(\rho) = f_q^{m+1}(\rho) = f_q^m(f_q(\rho)) = F'(F_q(\rho)) = F(\rho)$.

This finishes induction over the structure of p ; we are done with the proof of Lemma 14.

Lemma 15 (Disjoint Path Probabilities). *If $(F, B, \mathcal{O}), (F', B', \mathcal{O}') \in \mathcal{F}_p$ are two distinct elements, then $B \cap B' = \emptyset$.*

Proof. The proof is by induction on the structure of p . \mathcal{F}_p is a singleton for all base cases of p , so there there is nothing to prove there.

- Suppose $p = p_1 \circ p_2$ and let (F, B, \mathcal{O}) and $(F', B', \mathcal{O}') \in \mathcal{F}_p$. By definition of \mathcal{F}_p , there are $(F_1, B_1, \mathcal{O}_1), (F'_1, B'_1, \mathcal{O}'_1) \in \mathcal{F}_{p_1}$ and $(F_2, B_2, \mathcal{O}_2), (F'_2, B'_2, \mathcal{O}'_2) \in \mathcal{F}_{p_2}$ such that $F = F_2 \circ F_1$ and $F' = F'_2 \circ F'_1$; $B = B_1 \cap F_1^{-1}[B_2]$ and $B' = B'_1 \cap F'_1^{-1}[B'_2]$; and $\mathcal{O} = \mathcal{O}_1 \cap F_1^{-1}[\mathcal{O}_2]$ and $\mathcal{O}' = \mathcal{O}'_1 \cap F'_1^{-1}[\mathcal{O}'_2]$.
 1. If $(F_1, B_1, \mathcal{O}_1)$ and $(F'_1, B'_1, \mathcal{O}'_1)$ are distinct, B_1 and B'_1 are disjoint by IH on p_1 . Thus B and B' are disjoint, as $B \subseteq B_1$ and $B' \subseteq B'_1$.
 2. Otherwise, $F_1 = F'_1$ and $B_1 = B'_1$, but B_2 and B'_2 must be disjoint by IH on p_2 , since $(F_2, B_2, \mathcal{O}_2)$ and $(F'_2, B'_2, \mathcal{O}'_2)$ are distinct (otherwise (F, B, \mathcal{O}) and (F', B', \mathcal{O}') are equal). Then $F_1^{-1}[B_2]$ and $F_1^{-1}[B'_2]$ are disjoint (since $F_1 = F'_1$) and so $B \cap B' = \emptyset$, since $B \subseteq F_1^{-1}[B_2]$ and $B' \subseteq F_1^{-1}[B'_2]$.

We have concluded that $B \cap B' = \emptyset$ in both cases, thus finishing this step.

- For $p = \text{if } b \text{ } p_1 \text{ else } p_2$, let $(F_1, B'_1, \mathcal{O}_1), (F_2, B'_2, \mathcal{O}_2) \in \mathcal{F}_p$ be distinct. There are two possibilities for B'_1 :
 1. $B'_1 = B_1 \cap (\bar{b} \times \mathbb{R}^\omega)$ and $(F_1, B_1, \mathcal{O}_1) \in \mathcal{F}_{p_1}$. Two cases also for B'_2 :
 - (a) $B'_2 = B_2 \cap (\bar{b} \times \mathbb{R}^\omega)$ and $(F_2, B_2, \mathcal{O}_2) \in \mathcal{F}_{p_1}$. In this case, B_1 and B_2 are disjoint because of the IH. Since B'_1 and B'_2 are subsets of these (respectively), the required disjointness follows.
 - (b) $B'_2 = B_2 \cap (\bar{b}^c \times \mathbb{R}^\omega)$ and $(F_2, B_2, \mathcal{O}_2) \in \mathcal{F}_{p_2}$. Then B'_1 and B'_2 are disjoint because one is a subset of $\bar{b} \times \mathbb{R}^\omega$ and the other of $\bar{b}^c \times \mathbb{R}^\omega$.
 2. $B'_1 = B_1 \cap (\bar{b}^c \times \mathbb{R}^\omega)$ and $(F_1, B_1, \mathcal{O}_1) \in \mathcal{F}_{p_2}$. The rest of this case is a reasoning that is completely symmetrical to the above.

In all cases, $B'_1 \cap B'_2 = \emptyset$, so the inductive step is finished.

- Finally, the case where $p = \text{while } b \ q$. Let $(F_1, B_1, \mathcal{O}_1), (F_2, B_2, \mathcal{O}_2) \in \mathcal{F}_p$. Then there are m_1 and m_2 such that

$$(F_1, B_1, \mathcal{O}_1) \in \mathbb{F}_{b,q}^{m_1}(\mathbb{F}_0) \quad \text{and} \quad (F_2, B_2, \mathcal{O}_2) \in \mathbb{F}_{b,q}^{m_2}(\mathbb{F}_0),$$

where $\mathbb{F}_0 = \{(\text{id}_{\mathbb{R}^{n+\omega}}, \bar{b}^c \times \mathbb{R}^\omega, \mathbb{R}^{n+\omega})\}$. Proceed by a case analysis:

- If $m_1 = m_2 = 0$ then $(F_1, B_1, \mathcal{O}_1) = (F_2, B_2, \mathcal{O}_2) = (\text{id}_{\mathbb{R}^{n+\omega}}, \bar{b}^c \times \mathbb{R}^\omega, \mathbb{R}^{n+\omega})$, so there is nothing to prove.
- For the case $m_1 = m_2 > 0$, we proceed by induction on $m := m_1 = m_2$:
 - * The base case is $m = 1$. There are two $(F_{q,i}, B_{q,i}, \mathcal{O}_{q,i}) \in \mathcal{F}_q$, $i = 1, 2$ such that $F_i = F_{q,i}$, $B_i = B_{q,i} \cap F_{q,i}^{-1}[\bar{b}^c \times \mathbb{R}^\omega] \cap \bar{b} \times \mathbb{R}^\omega$, and $\mathcal{O}_i = \mathcal{O}_{q,i}$, by definition of $\mathbb{F}_{b,q}$. These two $(F_{q,i}, B_{q,i}, \mathcal{O}_{q,i})$, $i = 1, 2$ must be distinct, otherwise $(F_i, B_i, \mathcal{O}_i)$, $i = 1, 2$ are equal. Then the sets $B_{q,1}$ and $B_{q,2}$ are disjoint by the IH for the subterm q of p . A fortiori, B_1 and B_2 are disjoint.
 - * We now prove the statement for $m+1$. So let $(F_i, B_i, \mathcal{O}_i) \in \mathbb{F}_{b,q}^{m+1}(\mathbb{F}_0)$, $i = 1, 2$. Then there are, for $i = 1, 2$, $(F_{q,i}, B_{q,i}, \mathcal{O}_{q,i}) \in \mathcal{F}_q$ and $(F'_i, B'_i, \mathcal{O}'_i) \in \mathbb{F}_{b,q}^m(\mathbb{F}_0)$, such that
 - (i) $F_i = F'_i \circ F_{q,i}$;
 - (ii) $B_i = (\bar{b} \times \mathbb{R}^\omega) \cap B_{q,i} \cap F_{q,i}^{-1}[B'_i]$; and
 - (iii) $\mathcal{O}_i = \mathcal{O}_{q,i} \cap F_{q,i}^{-1}[\mathcal{O}'_i]$.

So we consider the following two cases:

1. $(F_{q,1}, B_{q,1}, \mathcal{O}_{q,1})$ and $(F_{q,2}, B_{q,2}, \mathcal{O}_{q,2})$ are distinct. By IH on q ,

$$B_{q,1} \cap B_{q,2} = \emptyset$$

2. Otherwise, if $(F_{q,1}, B_{q,1}, \mathcal{O}_{q,1}) = (F_{q,2}, B_{q,2}, \mathcal{O}_{q,2})$ then $(F'_1, B'_1, \mathcal{O}'_1)$ and $(F'_2, B'_2, \mathcal{O}'_2)$ must be distinct. But then by, by IH for m , we have $B'_1 \cap B'_2 = \emptyset$, and since $F_{q,1} = F_{q,2}$, we also have

$$F_{q,1}^{-1}[B'_1] \cap F_{q,2}^{-1}[B'_2] = \emptyset$$

In both cases B_1 and B_2 are disjoint, a fortiori.

Induction on m has finished.

- The last possibility is $m_2 \neq m_1 > 0$, or, w.l.o.g., $m_2 > m_1 > 0$. We will use the following claim; write $\mathbb{F}_0 = \{(\text{id}_{\mathbb{R}^{n+\omega}}, \bar{b}^c \times \mathbb{R}^\omega, \mathbb{R}^{n+\omega})\}$:

$$\forall m \in \mathbb{N}_{\geq 0}. \forall (F, B, \mathcal{O}) \in \mathbb{F}_{b,q}^m(\mathbb{F}_0). \exists (F_1, B_1, \mathcal{O}_1), \dots, (F_m, B_m, \mathcal{O}_m) \in \mathcal{F}_q \quad (*)$$

s.t.

- (i) $F = F_m \circ \dots \circ F_1$;
- (ii) $\forall j \in \{0, 1, \dots, m-1\}. B \subseteq F_1^{-1}[\dots F_j^{-1}[(\bar{b} \times \mathbb{R}^\omega) \cap B_{j+1}] \dots]$;
- (iii) $B \subseteq F^{-1}[\bar{b}^c \times \mathbb{R}^\omega]$.

Here, Item (ii) for $j = 0$ is just $B \subseteq (\bar{b} \times \mathbb{R}^\omega) \cap B_1$. We prove all three items by induction on m .

- * For $m = 1$ we know $(F, B, \mathcal{O}) = (F_q, B_q \cap (\bar{b} \times \mathbb{R}^\omega) \cap F_q^{-1}[\bar{b}^c \times \mathbb{R}^\omega], \mathcal{O}_q)$ for some $(F_q, B_q, \mathcal{O}_q) \in \mathcal{F}_q$. It is straightforwardly verified that by putting $(F_1, B_1, \mathcal{O}_1) := (F_q, B_q, \mathcal{O}_q)$, Items (i) to (iii) are satisfied.

* Now let $(F, B, \mathcal{O}) \in \mathbb{F}_{b,q}^{m+1}(\mathbb{F}_0)$. Then there are $(F_q, B_q, \mathcal{O}_q) \in \mathcal{F}_q$ and $(F', B', \mathcal{O}') \in \mathbb{F}_{b,q}^m(\mathbb{F}_0)$ such that $F = F' \circ F_q$, $B = B_q \cap (\bar{b} \times \mathbb{R}^\omega) \cap F_q^{-1}[B']$, and $\mathcal{O} = \mathcal{O}_q \cap F_q^{-1}[\mathcal{O}']$. By the induction hypothesis, then, there are $(F'_1, B'_1, \mathcal{O}'_1), \dots, (F'_m, B'_m, \mathcal{O}'_m) \in \mathcal{F}_q$ such that

- (i) $F' = F'_m \circ \dots \circ F'_1$;
- (ii) $B' \subseteq F'_1{}^{-1}[\dots F'_j{}^{-1}[(\bar{b} \times \mathbb{R}^\omega) \cap B'_{j+1}] \dots]$ for all $j \in \{0, 1, \dots, m-1\}$; and

(iii) $B' \subseteq F'^{-1}[\bar{b}^c \times \mathbb{R}^\omega]$.

Put $(F_1, B_1, \mathcal{O}_1) := (F_q, B_q, \mathcal{O}_q)$ and $(F_{j+1}, B_{j+1}, \mathcal{O}_{j+1}) := (F'_j, B'_j, \mathcal{O}'_j)$ for $1 \leq j \leq m$. Then every $(F_j, B_j, \mathcal{O}_j) \in \mathcal{F}_q$, and

- (i) $F = F' \circ F_q = F'_m \circ \dots \circ F'_1 \circ F_q = F_{m+1} \circ F_m \circ \dots \circ F_2 \circ F_1$;
- (ii) $B' \subseteq F_1{}^{-1}[\dots F_j{}^{-1}[(\bar{b} \times \mathbb{R}^\omega) \cap B'_{j+1}] \dots]$ for all $j \in \{0, 1, \dots, m-1\}$. And so, for all $j \in \{0, 1, \dots, m-1\}$:

$$\begin{aligned} B \subseteq F_q^{-1}[B'] &\subseteq F_1{}^{-1}[F_1{}^{-1}[\dots F_j{}^{-1}[(\bar{b} \times \mathbb{R}^\omega) \cap B'_{j+1}] \dots]] \\ &= F_1{}^{-1}[F_2{}^{-1}[\dots F_{j+1}{}^{-1}[(\bar{b} \times \mathbb{R}^\omega) \cap B_{j+2}] \dots]] \end{aligned}$$

This is to say that for all $j \in \{1, \dots, m\}$:

$$B \subseteq F_1{}^{-1}[\dots F_j{}^{-1}[(\bar{b} \times \mathbb{R}^\omega) \cap B_{j+1}] \dots]$$

The case $j = 0$ is trivially verified: $B \subseteq (\bar{b} \times \mathbb{R}^\omega) \cap B_q = (\bar{b} \times \mathbb{R}^\omega) \cap B_1$.

(iii) $B' \subseteq F'^{-1}[\bar{b}^c \times \mathbb{R}^\omega]$ so $B \subseteq F_q^{-1}[F'^{-1}[\bar{b}^c \times \mathbb{R}^\omega]] = F^{-1}[\bar{b}^c \times \mathbb{R}^\omega]$.

We have verified all three properties for $(F, B, \mathcal{O}) \in \mathbb{F}_{b,q}^{m+1}(\mathbb{F}_0)$, finished induction on m , and proved the claim (*), which we will use now.

Now let $(F, B, \mathcal{O}) \in \mathbb{F}_{b,q}^{m_1}(\mathbb{F}_0)$ and $(F', B', \mathcal{O}') \in \mathbb{F}_{b,q}^{m_2}(\mathbb{F}_0)$ (recall $m_2 > m_1 > 0$). Then by the claim (*) above, there are

$$\begin{aligned} &(F_1, B_1, \mathcal{O}_1), \dots, (F_{m_1}, B_{m_1}, \mathcal{O}_{m_1}), \\ &(F'_1, B'_1, \mathcal{O}'_1), \dots, (F'_{m_2}, B'_{m_2}, \mathcal{O}'_{m_2}) \in \mathcal{F}_q \end{aligned}$$

with the properties of Items (i) to (iii).

* Suppose first that

$$(F_1, B_1, \mathcal{O}_1) = (F'_1, B'_1, \mathcal{O}'_1), \dots, (F_{m_1}, B_{m_1}, \mathcal{O}_{m_1}) = (F'_{m_1}, B'_{m_1}, \mathcal{O}'_{m_1})$$

By property Item (iii), then,

$$B \supseteq F^{-1}[\bar{b}^c \times \mathbb{R}^\omega] = F_1{}^{-1}[\dots F_{m_1}{}^{-1}[\bar{b}^c \times \mathbb{R}^\omega] \dots]$$

On the other hand, by property Item (ii), since $m_1 \in \{0, \dots, m_2 - 1\}$,

$$B' \supseteq F_1{}^{-1}[\dots F_{m_1}{}^{-1}[\bar{b} \times \mathbb{R}^\omega] \dots]$$

and since $F_j = F'_j$ for all $j \in \{1, \dots, m_1\}$,

$$F_1{}^{-1}[\dots F_{m_1}{}^{-1}[\bar{b}^c \times \mathbb{R}^\omega] \dots] \cap F_1{}^{-1}[\dots F_{m_1}{}^{-1}[\bar{b} \times \mathbb{R}^\omega] \dots] = \emptyset$$

* Otherwise let M be the smallest integer in $\{1, \dots, m_1\}$ such that we have $(F_M, B_M, \mathcal{O}_M) \neq (F'_M, B'_M, \mathcal{O}'_M)$. Then

$$F_{M-1} \circ \dots \circ F_1 = F'_{M-1} \circ \dots \circ F'_1$$

By property Item (ii),

$$B \supseteq F_1^{-1}[\dots F_{M-1}^{-1}[B_M] \dots] \quad \text{and} \quad B' \supseteq F'_1{}^{-1}[\dots F'_{M-1}{}^{-1}[B'_M] \dots]$$

By IH on q , $B_M \cap B'_M = \emptyset$, so that now

$$F_1^{-1}[\dots F_{M-1}^{-1}[B_M] \dots] \cap F'_1{}^{-1}[\dots F'_{M-1}{}^{-1}[B'_M] \dots] = \emptyset$$

In both cases we conclude $B \cap B' = \emptyset$.

We have considered all possible values for m_1 and m_2 .

Inductive proof of Lemma 15 is now finished.