10-21-2023

Certain Invertible Operator-Block Matrices Induced by C*-Algebras and Scaled Hypercomplex Numbers

Daniel Alpay

Ilwoo Choo

D. Alpay, I. Cho *Res Math Sci(2023)10:47*
https://doi.org/10.1007/s40687-023-00411-0

Research in the Mathematical Sciences

RESEARCH

Certain invertible operator-block matrices induced by C^*-algebras and scaled hypercomplex numbers

Daniel Alpay[1*] and Ilwoo Cho[2]

*Correspondence: alpay@chapman.edu
[1]Department of Mathematics, Chapman University, 1 University Dr., Orange, CA 92866, USA
Full list of author information is available at the end of the article
D. Alpay thanks the Foster G. and Mary McGaw Professorship in Mathematical Sciences, which supported his research.

Abstract

The main purposes of this paper are (i) to enlarge scaled hypercomplex structures to operator-valued cases, where the operators are taken from a C^*-subalgebra of an operator algebra on a separable Hilbert space, (ii) to characterize the invertibility conditions on the operator-valued scaled-hypercomplex structures of (i), (iii) to study relations between the invertibility of scaled hypercomplex numbers, and that of operator-valued cases of (ii), and (iv) to confirm our invertibility of (ii) and (iii) are equivalent to the general invertibility of (2×2)-block operator matrices.

Keywords: Scaled hypercomplex numbers, Scaled hyperbolic numbers, Operator-hypercomplexes

Mathematics Subject Classification: 20G20, 46S10, 47S10

Contents

1 Introduction
2 Scaled hypercomplex numbers
2.1 Scaled hypercomplex rings
2.2 Invertibility on $\mathbb{H}_t$
2.3 The hypercomplex conjugate
3 Semi-normed spaces $\{(\mathbb{H}_t, \|.\|_t)\}_{t \in \mathbb{R}}$
4 Scaled operator-valued-hypercomplexes
5 Invertibility on $\mathcal{H}_2^t(\mathcal{A})$
5.1 The case where $t = 0$
5.2 The case where $t \neq 0$
5.3 Summary and discussion
6 The conjugation on a Unital C^*-Algebra $\mathcal{A}$
6.1 The case where $t = 0$
6.2 The case where $t \neq 0$
6.3 Summary and conclusion
7 The invertibility on $\mathbb{H}_t$ and on $\mathfrak{H}_2^t(\mathcal{A})$
8 The invertibility on (2×2)-Block Operators and That on $\mathfrak{H}_2^t(\mathcal{A})$
References

1 Introduction

In this paper, we extend the scaled hypercomplex structures $\mathbb{H}_t$ with a scale $t \in \mathbb{R}$ to the operator-valued structures by acting the operators of a C^*-subalgebra $\mathcal{A}$ of an operator algebra $B(H)$ on a separable Hilbert space H under certain bi-module actions of the Cartesian-product C^*-algebra $\mathcal{A}^2$ to $\mathbb{H}_t$ from the left and the right. Roughly speaking, we consider (2×2)-block operator matrices,

$$\begin{pmatrix} T_1 & tT_2 \\ T_2^* & T_1^* \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} T_1 & tT_2 \\ \overline{T_2} & \overline{T_1} \end{pmatrix},$$

where T_l^* are the usual adjoints of T_l in $\mathcal{A}$, and $\overline{T_l}$ are certain conjugates of T_l in $\mathcal{A}$, for all $l = 1, 2$, for any $t \in \mathbb{R}$. In particular, we are interested in inverses of such operators (if exist). Our main results not only provide the characterization of the invertibility on such operators, but also show the relations among the invertibility of the C^*-algebra of such operators, the invertibility on $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$, and that on

$$M_2(\mathcal{A}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A, B, C, D \in \mathcal{A} \right\}.$$

Throughout this paper, every vector $(a, b) \in \mathbb{C}^2$ is understood as hypercomplex numbers (a, b) induced by the complex numbers a and b. Under a suitable scaling in the real field $\mathbb{R}$, the set $\mathbb{C}^2$ of hypercomplex numbers forms a ring,

$$\mathbb{H}_t = \left(\mathbb{C}^2, +, \cdot_t\right),$$

where $(+)$ is the usual vector addition on $\mathbb{C}^2$, and $(\cdot_t)$ is the t-scaled vector-multiplication,

$$(a_1, b_1) \cdot_t (a_2, b_2) = \left(a_1 a_2 + t b_1 \overline{b_2},\ a_1 b_2 + b_1 \overline{a_2}\right),$$

on $\mathbb{C}^2$, where $\bar{z}$ are the conjugates of z in $\mathbb{C}$.

By the Hilbert-space representation $\left(\mathbb{C}^2, \pi_t\right)$ of $\mathbb{H}_t$ introduced in [3], we regard a hypercomplex number $h = (a, b) \in \mathbb{H}_t$ as a (2×2)-matrix,

$$\pi_t(h) \overset{\text{denote}}{=} [h]_t \overset{\text{def}}{=} \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} \text{ in } M_2(\mathbb{C}),$$

where $M_2(\mathbb{C})$ is the matricial algebra (or, the operator C^*-algebra $B\left(\mathbb{C}^2\right)$ acting on the Hilbert space $\mathbb{C}^2$) over $\mathbb{C}$, for $t \in \mathbb{R}$.

Remark and recall that the ring $\mathbb{H}_{-1}$ is the noncommutative field $\mathbb{H}$ of all quaternions (e.g., [6,22]), and the ring $\mathbb{H}_1$ is the ring of all split-quaternions (e.g., [1,2]). The algebra, analysis, spectra theory, operator theory, and free probability on $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$ are studied in [3]. The quaternions $\mathbb{H} = \mathbb{H}_{-1}$ and the split-quaternions $\mathbb{H}_1$ has been studied in various different fields in mathematics and applied science (e.g., [1,6,8,9,13–15,19,23,24,26]), as an extended algebraic structure of the complex field $\mathbb{C}$, or the hyperbolic numbers $\mathbb{D}$, which also motivates the construction and analysis on Clifford algebras (e.g., [4,7–10,14,16–18,20]). From the theories on the quaternions $\mathbb{H} = \mathbb{H}_{-1}$, we extend them to those on the scaled-hypercomplex rings $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$ in [3,5], generalizing the main results of [6].

Meanwhile, the invertibility on the algebra $M_2(\mathcal{A})$ of (2×2)-operator-block matrices is characterized under suitable invertibility assumptions on a unital C^*-subalgebra $\mathcal{A}$ of the operator algebra $B(H)$ on a separable Hilbert space H (e.g., see Chapter 3 of [1]). Our main results provide connections among the invertibility on $\mathbb{H}_t$, the invertibility on

$$\mathcal{H}_2^t(\mathcal{A}) = \left\{ \begin{pmatrix} T & tS \\ S^* & T^* \end{pmatrix} : T, S \in \mathcal{A} \right\},$$

that on

$$\mathfrak{H}_2^t(\mathcal{A}) = \left\{ \begin{pmatrix} T & tS \\ \overline{S} & \overline{T} \end{pmatrix} : T, S \in \mathcal{A} \right\},$$

and that on $M_2(\mathcal{A})$, by finding the invertibility characterizations on $\mathcal{H}_2^t(\mathcal{A})$ and on $\mathfrak{H}_2^t(\mathcal{A})$.

2 Scaled hypercomplex numbers

In this section, we review fundamental algebra, analysis, and operator theory on the scaled hypercomplex rings $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$. Let

$$\mathbb{C}^2 = \left\{ (a, b) : a, b \in \mathbb{C} \right\},$$

as the usual 2-dimensional Hilbert space over the complex field $\mathbb{C}$.

2.1 Scaled hypercomplex rings

Fix an arbitrarily scale t in the real field $\mathbb{R}$. On the Hilbert space $\mathbb{C}^2$, define the t-scaled vector-multiplication $(\cdot_t)$ by

$$(a_1, b_1) \cdot_t (a_2, b_2) \overset{\text{def}}{=} \left(a_1 a_2 + t b_1 \overline{b_2},\ a_1 b_2 + b_1 \overline{a_2} \right), \tag{2.1.1}$$

for $(a_l, b_l) \in \mathbb{C}^2$, for all $l = 1, 2$.

Proposition 1 *The algebraic structure* $\left(\mathbb{C}^2, +, \cdot_t\right)$ *forms a unital ring with its unity, or the* $(\cdot_t)$*-identity,* $(1, 0)$*, where* $(+)$ *is the usual vector addition on* $\mathbb{C}^2$*, and* $(\cdot_t)$ *is the vector multiplication* (2.1.1)*.*

Proof The pair $\left(\mathbb{C}^2, +\right)$ is an abelian group for $(+)$ with its $(+)$-identity $(0, 0)$. And the algebraic pair $\left(\mathbb{C}^{2\times}, \cdot_t\right)$ is a semigroup with its $(\cdot_t)$-identity $(1, 0)$ where $\mathbb{C}^{2\times} = \mathbb{C}^2 \setminus \{(0, 0)\}$. It is not difficult to check $(+)$ and $(\cdot_t)$ are distributed on $\mathbb{C}^2$ (e.g., see [2] for details). So, the algebraic triple $\left(\mathbb{C}^2, +, \cdot_t\right)$ forms a unital ring with its unity $(1, 0)$. □

Since $\mathbb{C}^2$ is a Hilbert space equipped with the usual-metric topology, one can understand these unital rings $\left\{\left(\mathbb{C}^2, +, \cdot_t\right)\right\}_{t\in\mathbb{R}}$ as topological rings.

Definition 2 For $t \in \mathbb{R}$, the ring $\mathbb{H}_t \overset{\text{denote}}{=} \left(\mathbb{C}^2, +, \cdot_t\right)$ is called the t-scaled hypercomplex ring.

For a fixed $t \in \mathbb{R}$, let $\mathbb{H}_t$ be the t-scaled hypercomplex ring. Define an injective map,

$$\pi_t : \mathbb{H}_t \to M_2(\mathbb{C}),$$

by

$$\pi_t((a, b)) = \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix},\ \forall (a, b) \in \mathbb{H}_t, \tag{2.1.2}$$

where $M_k(\mathbb{C})$ is the matricial algebra of all $(k \times k)$-matrices over $\mathbb{C}$, which is $*$-isomorphic to the operator algebra $B\left(\mathbb{C}^k\right)$ of all bounded linear operators on the Hilbert sace $\mathbb{C}^k$, for all $k \in \mathbb{N}$ (e.g., [8] and [9]). Such an injection π_t satisfies that

$$\pi_t(h_1 + h_2) = \pi_t(h_1) + \pi_t(h_2),$$

and

$$\pi_t(h_1 \cdot_t h_2) = \pi_t(h_1)\pi_t(h_2), \tag{2.1.3}$$

in $M_2(\mathbb{C})$ (e.g., see [3] for details).

Proposition 3 *The pair* $(\mathbb{C}^2, \pi_t)$ *forms an injective Hilbert-space representation of our* t*-scaled hypercomplex ring* $\mathbb{H}_t$*, where* π_t *is an action* (2.1.2)*.*

Proof The injection π_t of (2.1.2) is a ring-action of $\mathbb{H}_t$ acting on $\mathbb{C}^2$ by (2.1.3). Since $\mathbb{C}^2$ and $M_2(\mathbb{C})$ are finite-dimensional, the continuity of the ring-action π_t is guaranteed. □

By the above proposition, the realization,

$$\mathcal{H}_2^t \overset{\text{denote}}{=} \pi_t(\mathbb{H}_t) \overset{\text{def}}{=} \left\{ \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} \in M_2(\mathbb{C}) : (a, b) \in \mathbb{H}_t \right\}, \tag{2.1.4}$$

of $\mathbb{H}_t$ is well-determined in $M_2(\mathbb{C})$, in particular, by the injectivity of π_t. The realization $\mathcal{H}_2^t$ of (2.1.4) is called the t-scaled (hypercomplex-)realization of $\mathbb{H}_t$ (in $M_2(\mathbb{C})$) for $t \in \mathbb{R}$. For convenience, we denote the realization $\pi_t(h)$ of $h \in \mathbb{H}_t$ by $[h]_t$ in $\mathcal{H}_2^t$. By definition,

$$\mathbb{H}_t \overset{\text{T.R}}{=} \mathcal{H}_2^t \ \text{in} M_2(\mathbb{C}), \tag{2.1.5}$$

where "$\overset{\text{T.R}}{=}$" means "being topological-ring-isomorphic to." If $\mathbb{H}_t^\times \overset{\text{denote}}{=} \mathbb{H}_t \setminus \{(0,0)\}$, where $(0,0) \in \mathbb{H}_t$ is the $(+)$-identity, then, this set $\mathbb{H}_t^\times$ forms the maximal multiplicative monoid,

$$\mathbb{H}_t^\times \overset{\text{denote}}{=} \left(\mathbb{H}_t^\times, \cdot_t\right),$$

embedded in the ring $\mathbb{H}_t$, with its monoid-identity $(1,0)$, called the t-scaled hypercomplex monoid. By (2.1.5), the monoid $\mathbb{H}_t^\times$ is monoid-isomorphic to $\mathcal{H}_2^{t\times} \overset{\text{denote}}{=} \left(\mathcal{H}_2^{t\times}, \cdot\right)$ with its identity, $I_2 = [(1,0)]_t$, the (2×2)-identity matrix of $M_2(\mathbb{C})$, where $(\cdot)$ is the matricial multiplication, i.e.,

$$\mathbb{H}_t^\times = \left(\mathbb{H}_t^\times, \cdot_t\right) \overset{\text{Monoid}}{=} \left(\mathcal{H}_2^{t\times}, \cdot\right) = \mathcal{H}_2^{t\times},$$

where "$\overset{\text{Monoid}}{=}$" means "being monoid-isomorphic."

2.2 Invertibility on $\mathbb{H}_t$

For an arbitrarily fixed $t \in \mathbb{R}$, let $\mathbb{H}_t$ be the corresponding t-scaled hypercomplex ring, isomorphic to its t-scaled realization $\mathcal{H}_2^t$ by (2.1.5). Observe that, for any $(a,b) \in \mathbb{H}_t$, one has

$$\det([(a,b)]_t) = \det \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} = |a|^2 - t\left|b\right|^2. \tag{2.2.1}$$

where det is the determinant, and $|.|$ is the modulus on $\mathbb{C}$.

Lemma 4 *If* $(a,b) \in \mathbb{H}_t$*, then* $|a|^2 \neq t\left|b\right|^2$ *in* $\mathbb{C}$*, if and only if* (a,b) *is invertible in* $\mathbb{H}_t$ *with its inverse,*

$$(a,b)^{-1} = \left(\frac{\overline{a}}{|a|^2 - t\left|b\right|^2}, \frac{-b}{|a|^2 - t\left|b\right|^2} \right) \ \text{in } \mathbb{H}_t,$$

satisfying

$$\left[(a,b)^{-1}\right]_t = [(a,b)]_t^{-1} \ \text{ in } \mathcal{H}_2^t. \tag{2.2.2}$$

Proof The relation (2.2.2) holds whenever $\det([(a,b)]_t) \neq 0$. □

An algebraic structure $(X, +, \cdot)$ is said to be a noncommutative field, if it is a unital ring, where $(X^\times, \cdot)$ is a non-abelian group (e.g., [3,6]) with $X^\times = X \setminus \{0_X\}$, where 0_X is the $(+)$-identity.

Theorem 5 *We have that*

$$t < 0 \text{ in } \mathbb{R} \Longleftrightarrow \mathbb{H}_t \text{ is a noncommutative field.} \tag{2.2.3}$$

Proof $(\Rightarrow)$ By the above theorem, if $t < 0$ in $\mathbb{R}$, then every hypercomplex number (a, b) of the t-scaled hypercomplex monoid $\mathbb{H}_t^\times$ automatically satisfies the condition (2.2.2): $|a|^2 \neq t\left|b\right|^2$, because

$$|a|^2 > t\left|b\right|^2 \Longrightarrow |a|^2 \neq t\left|b\right|^2.$$

Thus, if $t < 0$, then every monoidal element $h \in \mathbb{H}_t^\times$ is invertible in $\mathbb{H}_t$, equivalently, the monoid $\mathbb{H}_t^\times$ is a group.
$(\Leftarrow)$ Assume that $t \geq 0$. First, let $t = 0$. If $(0, b) \in \mathbb{H}_0^\times$ (i.e., $b \neq 0$), then

$$\det([(0,b)]_0) = det\left(\begin{pmatrix} 0 & 0 \\ \overline{b} & 0 \end{pmatrix}\right) = 0,$$

implying that $[(0,b)]_0 \in \mathcal{H}_2^t$ is not invertible. Now, let $t > 0$. If $(a, b) \in \mathbb{H}_t^\times$, with $\left|b\right|^2 = \frac{|a|^2}{t}$ in $\mathbb{C}$, then

$$\det([(a,b)]_t) = |a|^2 - t\left|b\right|^2 = 0,$$

implying that (a, b) is not invertible in $\mathbb{H}_t$. So, if $t \geq 0$, then $\mathbb{H}_t$ is not a noncommutative field. □

By (2.2.3), the negative-scaled hypercomplex rings $\{\mathbb{H}_s\}_{s<0}$ are noncommutative fields, but, the non-negative-scaled hypercomplex rings $\{\mathbb{H}_t\}_{t\geq 0}$ cannot be noncommutative fields. So, for any scale $t \in \mathbb{R}$, the t-scaled hypercomplex ring $\mathbb{H}_t$ is decomposed by

$$\mathbb{H}_t = \mathbb{H}_t^{inv} \sqcup \mathbb{H}_t^{sing}$$

with

$$\mathbb{H}_t^{inv} = \left\{(a,b) : |a|^2 \neq t\left|b\right|^2\right\}, \tag{2.2.4}$$

and

$$\mathbb{H}_t^{sing} = \left\{(a,b) : |a|^2 = t\left|b\right|^2\right\},$$

where $\sqcup$ is the disjoint union. By (2.2.4), the t-scaled hypercomplex monoid $\mathbb{H}_t^\times$ is decomposed to be

$$\mathbb{H}_t^\times = \mathbb{H}_t^{inv} \sqcup \mathbb{H}_t^{\times sing},$$

with

$$\mathbb{H}_t^{\times sing} = \mathbb{H}_t^{sing} \setminus \{(0,0)\}. \tag{2.2.5}$$

Proposition 6 *The subset $\mathbb{H}_t^{inv}$ is a non-abelian group in the monoid $\mathbb{H}_t^\times$. Meanwhile, the subset $\mathbb{H}_t^{\times sing}$ is a semigroup in $\mathbb{H}_t^\times$ without identity.*

Proof Let $t \in \mathbb{R}$, and $\mathbb{H}_t^{\times}$, the t-scaled hypercomplex monoid, decomposed by (2.2.5). If $h_1, h_2 \in \mathbb{H}_t^{inv}$, then $h_1 \cdot_t h_2 \in \mathbb{H}_t^{inv}$, because

$$\det([h_1 \cdot_t h_2]_t) = \det([h_1]_t [h_2]_t) = \det([h_1]_t) \det([h_2]_t) \neq 0.$$

So, the algebraic pair $\left(\mathbb{H}_t^{inv}, \cdot_t\right)$ forms a group in the monoid $\mathbb{H}_t^{\times}$. Meanwhile, if $h_1, h_2 \in \mathbb{H}_t^{\times sing}$, then $h_1 \cdot_t h_2 \in \mathbb{H}_t^{\times sing}$, since

$$\det([h_1 \cdot_t h_2]_t) = \det([h_1]_t [h_2]_t) = \det([h_1]_t) \det([h_2]_t) = 0.$$

This operation $(\cdot_t)$ is associative on $\mathbb{H}_t^{\times sing}$, however, it does not have its identity $(1, 0)$ in $\mathbb{H}_t^{\times sing}$. Thus, the pair $\left(\mathbb{H}_t^{\times sing}, \cdot_t\right)$ forms a semigroup without identity in $\mathbb{H}_t^{\times}$. □

The block $\mathbb{H}_t^{inv}$ of (2.2.5) is called the group-part of $\mathbb{H}_t^{\times}$ (or, of $\mathbb{H}_t$), and the other algebraic block $\mathbb{H}_t^{\times sing}$ of (2.2.5) is called the semigroup-part of $\mathbb{H}_t^{\times}$ (or, of $\mathbb{H}_t$).

Corollary 7 *If $t < 0$ in $\mathbb{R}$, then $\mathbb{H}_t^{\times} = \mathbb{H}_t^{inv}$, and hence, $\mathbb{H}_t = \mathbb{H}_t^{inv} \cup \{(0, 0)\}$. Meanwhile, if $t \geq 0$ in $\mathbb{R}$, then $\mathbb{H}_t^{\times sing}$ is a non-empty properly embedded semigroup of $\mathbb{H}_t^{\times}$, without identity, satisfying the decomposition (2.2.4) of $\mathbb{H}_t$.*

Proof The proof is done by the above proposition. □

2.3 The hypercomplex conjugate

In this section, we consider certain adjoints on the scaled hypercomplex rings $\{\mathbb{H}_t\}_{t \in \mathbb{R}}$, motivated by the adjoints introduced in [4]. Fix an arbitrary scale $t \in \mathbb{R}$ and $\mathbb{H}_t$. Define a function,

$$(\dagger) : \mathbb{H}_t \to \mathbb{H}_t,$$

by

$$\dagger((a, b)) \overset{\text{denote}}{=} (a, b)^{\dagger} \overset{\text{def}}{=} (\overline{a}, -b), \ \forall (a, b) \in \mathbb{H}_t. \tag{2.3.1}$$

This function (2.3.1) satisfies that the injectivity,

$$h_1 = (a_1, b_1) \neq (a_2, b_2) = h_2 \text{ in } \mathbb{H}_t,$$

then

$$h_1^{\dagger} = (\overline{a_1}, -b_1) \neq (\overline{a_2}, -b_2) = h_2^{\dagger},$$

and the surjectivity in the sense that: for any $(a, b) \in \mathbb{H}_t$, there exists $(\overline{a}, -b) \in \mathbb{H}_t$, such that

$$(\overline{a}, -b)^{\dagger} = \left(\overline{\overline{a}}, -(-b)\right) = (a, b),$$

in $\mathbb{H}_t$. So, this function $(\dagger)$ of (2.3.1) is a bijection. Since $\mathbb{H}_t$ is topological-ring-isomorphic to its realization $\mathcal{H}_2^t$ by (2.1.5), one can define the bijection, also denoted by $(\dagger)$ on $\mathcal{H}_2^t$,

$$\dagger : \mathcal{H}_2^t \to \mathcal{H}_2^t,$$

definedby

$$\dagger([(a, b)]_t) \overset{\text{denote}}{=} [(a, b)]_t^{\dagger} \overset{\text{def}}{=} \left[(a, b)^{\dagger}\right]_t = [(\overline{a}, -b)]_t, \tag{2.3.2}$$

for all $(a, b) \in \mathbb{H}_t$. i.e., the bijection $\dagger$ of (2.3.2) on $\mathcal{H}_2^t$ is defined to be $\pi_2 \circ \dagger$ with the bijection $\dagger$ of (2.3.1) on $\mathbb{H}_t$.

Theorem 8 *The bijection J of (2.3.1) acting on $\mathbb{H}_t$ is an adjoint on $\mathbb{H}_t$ over $\mathbb{R}$ (or, a $\mathbb{R}$-adjoint on $\mathbb{H}_t$) in the sense that: for all $h_1, h_2 \in \mathbb{H}_t$,*

$$h_1^{\dagger\dagger} = \left(h_1^{\dagger}\right)^{\dagger} = h_1,$$
$$(h_1 + h_2)^{\dagger} = h_1^{\dagger} + h_2^{\dagger},$$
$$(h_1 \cdot_t h_2)^{\dagger} = h_2^{\dagger} \cdot_t h_1^{\dagger},$$

inadditionto

$$(r \cdot_t h)^{\dagger} = r \cdot_t h^{\dagger}, \tag{2.3.3}$$

for all $r \in \mathbb{R}$ and $h \in \mathbb{H}_t$.

Proof Since $\mathbb{H}_t$ is topological-ring-isomorphic to $\mathcal{H}_2^t$, it is sufficient to show that † is a $\mathbb{R}$-adjoint on $\mathcal{H}_2^t$ satisfying the conditions of (2.3.3). Observe that, for all $(a, b) \in \mathbb{H}_t$, we have

$$[(a,b)]_t^{\dagger\dagger} = \left[(a,b)^{\dagger}\right]_t^{\dagger} = [(\overline{a}, -b)]_t^{\dagger} = \left[\left(\overline{\overline{a}},\ -(-b)\right)\right]_t = [(a,b)]_t\,;$$

and, for any $(a_l, b_l) \in \mathbb{H}_t$, for $l = 1, 2$,

$$\begin{aligned}
&([(a_1,b_1)]_t + [(a_2,b_2)]_t)^{\dagger} = [(a_1+a_2, b_1+b_2)]_t^{\dagger} \\
&= \left[\left(\overline{a_1+a_2},\ -(b_1+b_2)\right)\right]_t = [(\overline{a_1}, -b_1)]_t + [(\overline{a_2}, -b_2)]_t \\
&= \left[(a_1,b_1)^{\dagger}\right]_t + \left[(a_2,b_2)^{\dagger}\right]_t = [(a_1,b_1)]_t^{\dagger} + [(a_2,b_2)]^{\dagger}\,;
\end{aligned}$$

and

$$\begin{aligned}
&([(a_1,b_1)]_t\,[(a_2,b_2)]_t)^{\dagger} = \begin{pmatrix} a_1a_2 + tb_1\overline{b_2} & t\,(a_1b_2 + b_1\overline{a_2}) \\ \overline{a_1b_2 + b_1\overline{a_2}} & \overline{a_1a_2 + tb_1\overline{b_2}} \end{pmatrix}^{\dagger} \\
&= \begin{pmatrix} \overline{a_1a_2 + tb_1\overline{b_2}} & t\,(-a_1b_2 - b_1\overline{a_2}) \\ \overline{-a_1b_2 - b_1\overline{a_2}} & a_1a_2 + tb_1\overline{b_2} \end{pmatrix} \\
&= \begin{pmatrix} \overline{a_2} & t\,(-b_2) \\ \overline{-b_2} & a_2 \end{pmatrix}\begin{pmatrix} \overline{a_1} & t\,(-b_1) \\ \overline{-b_1} & a_1 \end{pmatrix} \\
&= [(a_2,b_2)]_t^{\dagger}\,[(a_1,b_1)]_t^{\dagger}.
\end{aligned}$$

Moreover, if $r \in \mathbb{R}$ inducing $(r, 0) \in \mathbb{H}_t$ and $(a, b) \in \mathbb{H}_t$, then

$$\begin{aligned}
&([(r,0)]_t\,[(a,b)]_t)^{\dagger} = \begin{pmatrix} ra & trb \\ \overline{rb} & \overline{ra} \end{pmatrix}^{\dagger} = \begin{pmatrix} \overline{ra} & t\,(-rb) \\ \overline{-rb} & ra \end{pmatrix} \\
&= [(r,0)]_t\,[(\overline{a}, -b)]_t = [(r,0)]_t\,[(a,b)]_t^{\dagger}
\end{aligned}$$

Therefore, the bijection † of (2.3.2) is a $\mathbb{R}$-adjoint on $\mathcal{H}_2^t$. □

The above theorem shows that the bijection † of (2.3.1) is $\mathbb{R}$-adjoint on $\mathbb{H}_t$ by (2.3.3).

Definition 9 The bijection † of (2.3.1), or the bijection † of (2.3.2), is called the hypercomplex-conjugate on $\mathbb{H}_t$, respectively, on $\mathcal{H}_2^t$, for all $t \in \mathbb{R}$.

If $(a,b) \in \mathbb{H}_t$, then

$$[h]_t^{\dagger}\,[h]_t = \left[\left(|a|^2 - t\left|b\right|^2,\, 0\right)\right]_t = [h]_t\,[h]_t^{\dagger}, \tag{2.3.4}$$

for all $h = (a,b) \in \mathbb{H}_t$, for all $t \in \mathbb{R}$.

Proposition 10 *If $(a,b) \in \mathbb{H}^t$, then*

$$(a,b)^{\dagger} \cdot_t (a,b) = \left(|a|^2 - t\left|b\right|^2,\, 0\right) = (a,b) \cdot_t (a,b)^{\dagger}, \tag{2.3.5}$$

in $\mathbb{H}^t$, for all $t \in \mathbb{R}$. It implies that

$$\sigma_t\left((a,b)^{\dagger} \cdot_t (a,b)\right) = |a|^2 - t\left|b\right|^2 = \det\left([(a,b)]_t\right) = \sigma_t\left((a,b) \cdot_t (a,b)^{\dagger}\right), \tag{2.3.6}$$

for all $(a,b) \in \mathbb{H}_t$, for all $t \in \mathbb{R}$.

Proof The relation (2.3.5) is proven by (2.3.4). By (2.3.5), the first t-spectral-value relation of (2.3.6) is obtained, because

$$det\left([(a,b)]_t\right) = |a|^2 - t\left|b\right|^2,$$

for all $(a,b) \in \mathbb{H}_t$, for all $t \in \mathbb{R}$. □

3 Semi-normed spaces $\{(\mathbb{H}_t, \|.\|_t)\}_{t\in\mathbb{R}}$

Fix a scale $t \in \mathbb{R}$, and the corresponding t-scaled hypercomplex ring $\mathbb{H}_t$. We showed in Sect. 2.3 that, on $\mathbb{H}_t$, the hypercomplex-conjugate (†) is defined by

$$(a,b)^{\dagger} = (\overline{a},\, -b),\ \ \forall\, (a,b) \in \mathbb{H}_t,$$

as a $\mathbb{R}$-adjoint, inducing the $\mathbb{R}$-adjoint on the t-scaled realization $\mathcal{H}_2^t$,

$$[(a,b)]_t^{\dagger} = \left[(a,b)^{\dagger}\right]_t = [(\overline{a}, -b)]_t\,. \tag{3.1}$$

by (2.3.5) and (2.3.6), for all $(a,b) \in \mathbb{H}_t$. Since the t-scaled realization $\mathcal{H}_2^t$ is a sub-structure of $M_2(\mathbb{C})$, the normalized trace,

$$\tau = \frac{1}{2}tr \quad \text{on} \quad M_2(\mathbb{C}),$$

is restricted to $\tau \overset{\text{denote}}{=} \tau\mid_{\mathcal{H}_2^t}$ on $\mathcal{H}_2^t$, where tr is the usual trace on $M_2(\mathbb{C})$, i.e., for any $[(a,b)]_t \in \mathcal{H}_2^t$,

$$\tau\left([(a,b)]_t\right) = \frac{1}{2}tr\left(\begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix}\right) = \frac{1}{2}\left(a + \overline{a}\right),$$

equivalently,

$$\tau\left([(a,b)]_t\right) = \text{Re}\,(a),\ \ \forall\, (a,b) \in \mathbb{H}_t, \tag{3.2}$$

as a $\mathbb{R}$-linear functional satisfying the tracial property,

$$\tau\,(TS) = \tau\,(ST),\ \ \forall T, S \in \mathcal{H}_2^t.$$

By (3.1) and (3.2), without loss of generality, one can define a $\mathbb{R}$-trace τ on $\mathbb{H}_t$ by

$$\tau\left((a,b)\right) \overset{\text{def}}{=} \text{Re}\,(a),\ \ \forall\, (a,b) \in \mathbb{H}_t \tag{3.3}$$

Define now a form,

$$\langle , \rangle_t : \mathbb{H}_t \times \mathbb{H}_t \to \mathbb{R} \subset \mathbb{C},$$

by

$$\langle h_1, h_2 \rangle_t \overset{\text{def}}{=} \tau\left(h_1 \cdot_t h_2^{\dagger}\right),\ \ \forall h_1, h_2 \in \mathbb{H}_t, \tag{3.4}$$

where τ in (3.2) is in the sense of (3.3). Then, the form (3.4) satisfies that

$$\begin{aligned}
&\left\langle (a_1,b_1)+(a_2,b_2),\ (a_3,b_3)\right\rangle_t \\
&= \tau\left(\begin{pmatrix} a_1\overline{a_3}+a_2\overline{a_3}-t\left(b_1\overline{b_3}+b_2\overline{b_3}\right) & t\left(-a_1b_3-a_2b_3+a_3b_1+a_3b_2\right) \\ \overline{-a_1b_3-a_2b_3+a_3b_1+a_3b_2} & \overline{a_1\overline{a_3}+a_2\overline{a_3}-t\left(b_1\overline{b_3}+b_2\overline{b_3}\right)} \end{pmatrix}\right) \\
&= \text{Re}\left(a_1\overline{a_3}+a_2\overline{a_3}-t\left(b_1\overline{b_3}+b_2\overline{b_3}\right)\right) \\
&= \text{Re}\left(a_1\overline{a_3}-tb_1\overline{b_3}\right)+Re\left(a_2\overline{a_3}-tb_2\overline{b_3}\right) \\
&= \tau\left((a_1,b_1)\cdot_t(a_3,b_3)^{\dagger}\right)+\tau\left((a_2,b_2)\cdot_t(a_3,b_3)^{\dagger}\right) \\
&= \left\langle (a_1,b_1),(a_3,b_3)\right\rangle_t+\left\langle (a_2,b_2),(a_3,b_3)\right\rangle_t,
\end{aligned}$$

for all $(a_l,b_l)\in\mathbb{H}_t$, for $l=1,2,3$, i.e.,

$$\left\langle h_1+h_2,h_3\right\rangle_t=\left\langle h_1,h_3\right\rangle_t+\left\langle h_2,h_3\right\rangle_t, \tag{3.5}$$

similarly, one has

$$\left\langle h_1,h_2+h_3\right\rangle_t=\left\langle h_1,h_2\right\rangle_t+\left\langle h_1,h_3\right\rangle_t \tag{3.6}$$

for all $h_1,h_2,h_3\in\mathbb{H}_t$. Also, if $h_l=(a_l,b_l)\in\mathbb{H}_t$, for $l=1,2$, and $r\in\mathbb{R}$, then

$$\begin{aligned}
\left\langle rh_1,h_2\right\rangle_t &= \tau\left(((r,0)\cdot_t h_1)\cdot_t h_2^{\dagger}\right) \\
&= \tau\left(\begin{pmatrix} ra_1\overline{a_2}-trb_1\overline{b_2} & t\left(-ra_1b_2+ra_2b_1\right) \\ \overline{-ra_1b_2+ra_2b_1} & \overline{ra_1\overline{a_2}-trb_1\overline{b_2}} \end{pmatrix}\right) \\
&= \text{Re}\left(ra_1\overline{a_2}-trb_1\overline{b_2}\right)=rRe\left(a_1\overline{a_2}-tb_1\overline{b_2}\right) \\
&= r\tau\left((a_1,b_1)\cdot_t(a_2,b_2)^{\dagger}\right)=r\left\langle h_1,h_2\right\rangle_t,
\end{aligned}$$

i.e.,

$$\left\langle rh_1,h_2\right\rangle_t=r\left\langle h_1,h_2\right\rangle_t,\ \ \forall r\in\mathbb{R}\ \text{and} h_1,h_2\in\mathbb{H}_t. \tag{3.7}$$

similarly,

$$\left\langle h_1,rh_2\right\rangle_t=r\left\langle h_1,h_2\right\rangle_t,\ \ \forall r\in\mathbb{R}\ \text{and}\ h_1,h_2\in\mathbb{H}_t. \tag{3.8}$$

Lemma 11 *The form $\langle,\rangle_t$ of (3.6) is a well-defined bilinear form on $\mathbb{H}_t$ over $\mathbb{R}$.*

Proof It is shown by (3.5), (3.6), (3.7) and (3.8). □

By the above lemma, the t-scaled hypercomplex ring $\mathbb{H}_t$ is equipped with a well-defined bilinear form $\langle,\rangle_t$ of (3.4) over $\mathbb{R}$.

Lemma 12 *If $h_1,h_2\in\mathbb{H}_t$, then*

$$\left\langle h_1,h_2\right\rangle_t=\left\langle h_2,h_1\right\rangle_t\ in\mathbb{R} \tag{3.9}$$

Proof Let $h_l = (a_l, b_l) \in \mathbb{H}_t$, for $l = 1, 2$. Then

$$\langle h_1, h_2\rangle_t = \tau\left(\begin{pmatrix} a_1\overline{a_2} - tb_1\overline{b_2} & t\left(a_2b_1 - a_1b_2\right) \\ \overline{a_2b_1 - a_1b_2} & \overline{a_1\overline{a_2} - tb_1\overline{b_2}} \end{pmatrix}\right)$$
$$= \mathrm{Re}\left(a_1\overline{a_2} - tb_1\overline{b_2}\right),$$

and

$$\langle h_2, h_1\rangle_t = \tau\left(\begin{pmatrix} \overline{a_1}a_2 - t\overline{b_1}b_2 & t\left(a_1b_2 - a_2b_1\right) \\ \overline{a_1b_2 - a_2b_1} & \overline{\overline{a_1}a_2 - t\overline{b_1}b_2} \end{pmatrix}\right)$$
$$= \mathrm{Re}\left(\overline{a_1}a_2 - t\overline{b_1}b_2\right) = \mathrm{Re}\left(\overline{\overline{a_1}a_2 - t\overline{b_1}b_2}\right) = \overline{\langle h_1, h_2\rangle_t}. \tag{3.10}$$

Therefore, by (3.10),

$$\langle h_2, h_1\rangle_t = \overline{\langle h_1, h_2\rangle_t} = \langle h_1, h_2\rangle_t, \text{ in } \mathbb{R}.$$

□

By (3.9), the bilinear form $\langle , \rangle_t$ of (3.4) is symmetric.

Lemma 13 *If $h_1, h_2 \in \mathbb{H}_t$, then*

$$\left|\langle h_1, h_2\rangle_2\right|^2 \leq \left|\langle h_1, h_1\rangle_t\right|^2 \left|\langle h_2, h_2\rangle_t\right|^2 \tag{3.11}$$

where $|.|$ is the absolute value on $\mathbb{R}$.

Proof By (3.10), if $h_l = (a_l, b_l) \in \mathbb{H}_t$ for $l = 1, 2$, then one has

$$\left|\langle h_1, h_2\rangle_t\right| = \left|\mathrm{Re}\left(a_1\overline{a_2} - tb_1\overline{b_2}\right)\right|,$$

and hence,

$$\left|\langle h_l, h_l\rangle_t\right| = \left||a_l|^2 - t\left|b_l\right|^2\right|,$$

for $l = 1, 2$. Therefore, the inequality (3.11) holds. □

Observe now that, by (3.1) and (3.4), if $h = (a, b) \in \mathbb{H}_t$, then

$$\langle h, h\rangle_t = \tau\left((a, b) \cdot_t (a, b)^{\dagger}\right) = \mathrm{Re}\left(|a|^2 - t\left|b\right|^2\right),$$

implying that

$$\langle h, h\rangle_t = |a|^2 - t\left|b\right|^2 = \det\left([h]_t\right). \tag{3.12}$$

This formula (3.12) says that the bilinear form $\langle , \rangle_t$ of (3.4) is not positively defined in general.

Lemma 14 *Let $h = (a, b) \in \mathbb{H}_t$. If $\langle , \rangle_t$ is the bilinear form (3.4), then*

$$\langle h, h\rangle_t = 0 \Longleftrightarrow |a|^2 = t\left|b\right|^2 \Longleftrightarrow h \in \mathbb{H}_t^{sing}. \tag{3.13}$$

Proof The relation (3.13) is shown by (2.2.4) and (3.12). □

Now, let's consider the following concepts.

Definition 15 For a vector space X over $\mathbb{R}$, a form $\langle , \rangle : X \times X \to \mathbb{R}$ is a (definite) semi-inner product on X over $\mathbb{R}$, if (i) it is a bilinear form on X over $\mathbb{R}$, (ii)

$$\langle x_1, x_2 \rangle = \langle x_2, x_1 \rangle, \ \forall x_1, x_2 \in X,$$

and (iii) $\langle x, x \rangle \geq 0$, for all $x \in X$. If such a semi-inner product $\langle , \rangle$ satisfies an additional condition (iv)

$$\langle x, x \rangle = 0, \text{ if and only if } x = 0_X,$$

where 0_X is the zero vector of X, then it is called an inner product on X over $\mathbb{R}$. If $\langle , \rangle$ is a semi-inner product (or, an inner product) on the $\mathbb{R}$-vector space X, then the pair $(X, \langle , \rangle)$ is said to be a semi-inner product space (respectively, an inner product space) over $\mathbb{R}$ (in short, a $\mathbb{R}$-SIPS, respectively, a $\mathbb{R}$-IPS).

Every $\mathbb{R}$-IPS is automatically a $\mathbb{R}$-SIPS, but, not all $\mathbb{R}$-SIPSs are $\mathbb{R}$-IPSs.

Definition 16 For a vector space X over $\mathbb{R}$, a form $\langle , \rangle : X \times X \to \mathbb{R}$ is called an indefinite semi-inner product on X over $\mathbb{R}$, if (i) it is a bilinear form on X over $\mathbb{R}$, (ii)

$$\langle x_1, x_2 \rangle = \langle x_2, x_1 \rangle, \ \forall x_1, x_2 \in X,$$

and (iii) $\langle x, x \rangle \in \mathbb{R}$, for all $x \in X$. If such an indefinite semi-inner product $\langle , \rangle$ satisfies an additional condition (iv)

$$\langle x, x \rangle = 0, \text{ if and only if } x = 0_X,$$

then it is said to be an indefinite inner product on X over $\mathbb{R}$. If $\langle , \rangle$ is an indefinite semi-inner product (or, an indefinite inner product) on the $\mathbb{R}$-vector space X, then the pair $(X, \langle , \rangle)$ is called an indefinite-semi-inner product space (respectively, an indefinite-inner product space) over $\mathbb{R}$ (in short, a $\mathbb{R}$-ISIPS, respectively, $\mathbb{R}$-IIPS).

Depending on the scales, the scaled hypercomplex rings are regarded as certain vector spaces over $\mathbb{R}$, by the existence of the bilinear form $\langle , \rangle_t$.

Theorem 17 *Let $t \in \mathbb{R}$. Then*

$$t < 0 \Longrightarrow \langle , \rangle_t \text{ is an inner product on } \mathbb{H}_t, \tag{3.14}$$

meanwhile,

$$t \geq 0 \Longrightarrow \langle , \rangle_t \text{ is an indefinite semi-inner product on } \mathbb{H}_t. \tag{3.15}$$

Proof If $t < 0$ in $\mathbb{R}$, then $\mathbb{H}_t^{sing} = \{(0, 0)\}$, in $\mathbb{H}_t$, and hence,

$$\mathbb{H}_t = \mathbb{H}_t^{inv} \cup \{(0, 0)\}.$$

Thus, one has

$$\langle h, h \rangle_t = 0 \Longleftrightarrow h = (0, 0) \in \mathbb{H}_t,$$

whenever $t < 0$. Moreover, for any $h = (a, b) \in \mathbb{H}_t$, if $t < 0$, then

$$det\left([h]_t\right) = |a|^2 - t\left|b\right|^2 = \tau\left(\left[h \cdot_t h^\dagger\right]\right) = \langle h, h \rangle_t \geq 0.$$

Therefore, the statement (3.14) holds.

Assume now that $t \geq 0$ in $\mathbb{R}$. Then the semigroup-part $\mathbb{H}_t^{\times sing}$ is not empty in $\mathbb{H}_t$, and hence,

$$\mathbb{H}_t^{sing} \supset \{(0,0)\} \text{ in } \mathbb{H}_t,$$

and

$$det\left([(a,b)]_t\right) = |a|^2 - t\left|b\right|^2 \in \mathbb{R},$$

for $(a, b) \in \mathbb{H}_t$, in general. Thus, the statement (3.15) holds. □

The following corollary is an immediate consequence of the above theorem.

Corollary 18 *If $t < 0$, then the pair $(\mathbb{H}_t, \langle,\rangle_t)$ is a $\mathbb{R}$-IPS, meanwhile, if $t \geq 0$, then $(\mathbb{H}_t, \langle,\rangle_t)$ is a $\mathbb{R}$-ISIPS.*

Proof It is proven by (3.14) and (3.15). □

Recall that a pair $(X, \|.\|)$ of a vector space X over $\mathbb{R}$, and a map $\|.\| : X \to \mathbb{R}$ is called a semi-normed space, if $\|.\|$ is a semi-norm, in the sense that: (i) $\|x\| \geq 0$, for all $x \in X$, (ii) $\|rx\| = |r|\,\|x\|$, for all $r \in \mathbb{R}$ and $x \in X$, and (iii)

$$\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|, \ \forall x_1, x_2 \in X.$$

If the semi-norm $\|.\|$ satisfies an additional condition (iv)

$$\|x\| = 0 \Longleftrightarrow x = 0_X \text{ in } X,$$

then it called a norm on X. In such a case, the semi-normed space $(X, \|.\|)$ is called a normed space over $\mathbb{R}$.

Definition 19 If a pair $(X, \|.\|_X)$ of a vector space X over $\mathbb{R}$, and a semi-norm (respectively, a norm) $\|.\|$ on X, is complete under its semi-norm topology (respectively, norm topology) induced by $\|.\|$, then it is said to be a complete semi-normed space (respectively, a Banach space) over $\mathbb{R}$, in short, a complete $\mathbb{R}$-SNS (respectively, $\mathbb{R}$-Banach space).

Let $(\mathbb{H}_t, \langle,\rangle_t)$ be either a $\mathbb{R}$-IPS (if $t < 0$), or a $\mathbb{R}$-ISIPS (if $t \geq 0$). Define a function $\|.\|_t : \mathbb{H}_t \to \mathbb{R}$ by

$$\left\|h\right\|_t \overset{\text{def}}{=} \sqrt{\left|\langle h, h\rangle_t\right|} = \sqrt{\left|det\left([h]_t\right)\right|}, \ \forall h \in \mathbb{H}_t. \tag{3.16}$$

Theorem 20 *Let $t \in \mathbb{R}$, and $(\mathbb{H}_t, \langle,\rangle_t)$, either a $\mathbb{R}$-IPS (if $t < 0$), or a $\mathbb{R}$-ISIPS (if $t \geq 0$), and let $\|.\|_t$ be a function (3.18).*

$$t < 0 \Longrightarrow (\mathbb{H}_t, \|.\|_t)\, is a \mathbb{R} - Banach\ space, \tag{3.17}$$

meanwhile,

$$t \geq 0 \Longrightarrow (\mathbb{H}_t, \|.\|_t) \text{ is a complete } \mathbb{R}\text{-SNS}. \tag{3.18}$$

Proof By (3.14), if $t < 0$, then the pair $(\mathbb{H}_t, \langle,\rangle_t)$ forms a $\mathbb{R}$-IPS, inducing the norm $\|.\|_t$ of (3.16), canonically. So, if $t < 0$, then $(\mathbb{H}_t, \|.\|_t)$ forms a normed space over $\mathbb{R}$. The completeness of $(\mathbb{H}_t, \|.\|_t)$ is guaranteed by (3.11). So, the statement (3.17) holds.

Now, assume that $t \geq 0$, and $(\mathbb{H}_t, \langle,\rangle_t)$ is the corresponding $\mathbb{R}$-ISIPS. If $\|.\|_t$ is defined to be the function (3.16), then it is a semi-norm on $\mathbb{H}_t$ over $\mathbb{R}$. In particular,

$$\left\|(a,b)\right\|_t = 0 \Longleftrightarrow |a|^2 - t\left|b\right|^2 = 0 \Longleftrightarrow (a,b) \in \mathbb{H}_t^{sing},$$

in $\mathbb{H}_t$. So, this semi-norm $\|.\|_t$ cannot be a norm whenever $t \geq 0$. However, by (3.11), this semi-norm is complete on $\mathbb{H}_t$. Thus, the statement (3.18) holds. □

By the above two theorems, one obtain the following result.

Corollary 21 *Let $t \in \mathbb{R}$, and $(\mathbb{H}_t, \langle, \rangle_t)$, either a $\mathbb{R}$-IPS (if $t < 0$), or a $\mathbb{R}$-ISIPS (if $t \geq 0$).*

$$t < 0 \Longrightarrow (\mathbb{H}_t, \langle, \rangle_t) \text{ is a } \mathbb{R}\text{-Hilbert space,} \tag{3.19}$$

meanwhile,

$$t \geq 0 \Longrightarrow (\mathbb{H}_t, \langle, \rangle_t) \text{ is a complete } \mathbb{R}\text{-ISIPS.} \tag{3.20}$$

Proof The statement (3.19) (respectively, (3.20)) is proven by (3.14) and (3.17) (respectively, (3.15) and (3.18)). □

Since $\mathbb{H}_t$ is both a ring and a complete SNS, it forms a topological algebra "over $\mathbb{R}$," for all $t \in \mathbb{R}$.

Theorem 22 *Let $t \in \mathbb{R}$, and $\mathbb{H}_t$, the t-scaled hypercomplex ring.*

$$t < 0 \Longrightarrow \mathbb{H}_t \text{ is a } C^* - \text{algebra over } \mathbb{R} \text{ (or, } \mathbb{R} - C^* - \text{algebra)} \tag{3.21}$$

meanwhile,

$$t \geq 0 \Longrightarrow \mathbb{H}_t \text{ is a complete } \mathbb{R} - \text{semi} - \text{normed} * -\text{algebra.} \tag{3.22}$$

Proof By (3.19) and (3.20), the t-scaled hypercomplex ring $\mathbb{H}_t$ is a $\mathbb{R}$-vector space equipped with its complete semi-norm (or, norm if $t < 0$), for all $t \in \mathbb{R}$. It shows that $\mathbb{H}_t$ forms a complete semi-normed $*$-algebra over $\mathbb{R}$, equipped with the $\mathbb{R}$-adjoint ($\dagger$), the hypercomplex-conjugate, for all $t \in \mathbb{R}$. Especially, this complete semi-normed $*$-algebra $\mathbb{H}_t$ is acting on the $\mathbb{R}$-semi-normed space $(\mathbb{H}_t, \|.\|_t)$, by the action,

$$m : h \in \mathbb{H}_t \longmapsto m_h \in B\left((\mathbb{H}_t, \|.\|_t)\right),$$

where

$$m_h(\eta) = h \cdot_t \eta, \quad \forall \eta \in (\mathbb{H}_t, \|.\|_t), \tag{3.23}$$

satisfying

$$m_{h_1} m_{h_2} = m_{h_1 h_2}, \quad \forall h_1, h_2 \in \mathbb{H}_t,$$

and

$$m_h^* = m_{h^\dagger}, \quad \forall h \in \mathbb{H}_t,$$

and

$$\|m_h\| \overset{\text{def}}{=} \sup\left\{\left\|m_h(\eta)\right\|_t : \|\eta\|_t = 1\right\} = \left\|h\right\|_t, \ \forall h \in \mathbb{H}_t,$$

in the operator algebra $B_{\mathbb{R}}((\mathbb{H}_t, \|.\|_t))$ of all bounded $\mathbb{R}$-linear operators on the complete semi-normed space $(\mathbb{H}_t, \|.\|_t)$, where $\|.\|$ is the operator norm on $B_{\mathbb{R}}((\mathbb{H}_t, \|.\|_t))$. It shows that the function m of (3.23) is a continuous ring-action of $\mathbb{H}_t$ acting on $(\mathbb{H}_t, \|.\|_t)$. So,

$$m(\mathbb{H}_t) \overset{\text{def}}{=} \left\{m_h : h \in \mathbb{H}_t\right\}$$

forms the closed subalgebra of $B_{\mathbb{R}}\left((\mathbb{H}_t, \|.\|_t)\right)$, as a complete semi-normed $*$-algebra over $\mathbb{R}$. Clearly, there does exist the $*$-isomorphism,

$$\Psi_t : h \in \mathbb{H}_t \longmapsto m_h \in m\left(\mathbb{H}_t\right),$$

and hence, $\mathbb{H}_t$ is a complete $\mathbb{R}$-semi-normed $*$-algebra, for "all $t \in \mathbb{R}$." Therefore, the statement (3.21) holds.

In particular, if $t < 0$, then the operator algebra $B_{\mathbb{R}}\left((\mathbb{H}_t, \|.\|_t)\right)$ is on the $\mathbb{R}$-Hilbert space $(\mathbb{H}_t, \langle, \rangle_t)$, and hence, $\mathbb{H}_t \overset{*\text{-iso}}{=} m\left(\mathbb{H}_t\right)$ becomes a complete $\mathbb{R}$-Banach $*$-algebra acting on the $\mathbb{R}$-Hilbert space $(\mathbb{H}_t, \langle, \rangle_t)$, i.e., if $t < 0$, then $\mathbb{H}_t$ is ($*$-isomorphic to) a $\mathbb{R}$-C^*-algebra $(m\left(\mathbb{H}_t\right))$. Therefore, the statement (3.22) holds. □

Notation and Assumption. From below, the set $\mathbb{H}_t$ of all t-scaled hypercomplex numbers is understood to be the ring, or either the $\mathbb{R}$-Hilbert space (if $t < 0$) or the complete $\mathbb{R}$-ISIPS (if $t \geq 0$), or either the $\mathbb{R}$-C^*-algebra (if $t < 0$) or the complete $\mathbb{R}$-semi-normed $*$-algebra (if $t \geq 0$), case-by-case. And we call the set $\mathbb{H}_t$, the t-scaled hypercomplexes for $t \in \mathbb{R}$. □

Let $t \in \mathbb{R}$, and $\mathbb{H}_t$, the t-scaled hypercomplexes. Define a subset $\mathbb{D}_t$ of $\mathbb{H}_t$ by

$$\mathbb{D}_t \overset{\text{def}}{=} \left\{(x, y) \in \mathbb{H}_t : x, y \in \mathbb{R}\right\}, \tag{3.24}$$

realized to be

$$\mathcal{D}_2^t \overset{\text{def}}{=} \pi_t\left(\mathbb{D}_t\right) = \left\{\left[(x, y)\right]_t = \begin{pmatrix} x & ty \\ y & x \end{pmatrix} : (x, y) \in \mathbb{D}_t\right\}, \tag{3.25}$$

in $\mathcal{H}_2^t = \pi_t\left(\mathbb{H}_t\right)$. Then $\mathbb{D}_t$ is a sub-structure of $\mathbb{H}_t$, as a sub-ring algebraically, or, a closed subspace analytically, or a $*$-subalgebra over $\mathbb{R}$ operator-algebraically, case-by-case. By definition, one has the $\mathbb{R}$-adjoint on $\mathbb{D}_t$,

$$(x, y)^\dagger = (\overline{x}, -y) = (x, -y) \text{ in } \mathbb{D}_t,$$

because $x, y \in \mathbb{R}$.

Definition 23 The sub-structure $\mathbb{D}_t$ of (4.1) is called the t-scaled hyperbolics of the t-scaled hypercomplexes $\mathbb{H}_t$.

Note that, the (-1)-scaled hyperbolics $\mathbb{D}_{-1}$ is isomorphic to the complex field $\mathbb{C}$, and the 1-scaled hyperbolics $\mathbb{D}_1$ is isomorphic to the (classical) hyperbolic numbers,

$$\mathcal{D} = \left\{x + yj : j^2 = 1,\ x, y \in \mathbb{R}\right\}.$$

(e.g., see [4] in detail).

4 Scaled operator-valued-hypercomplexes

In this section, we extend our scaled hypercomplex numbers of $\{\mathbb{H}_t\}_{t \in \mathbb{R}}$ to operators under certain actions. Now, let $B\left(H\right)$ be the operator algebra of all bounded linear operators on a Hilbert space H, and $\mathcal{A}$ is a unital C^*-subalgebra of $B\left(H\right)$ with its unity $\mathbf{1} \in \mathcal{A}$, which is the identity operator on H.

Definition 24 Let $\mathcal{A}$ be a unital C^*-algebra in an operator algebra $B\left(H\right)$ on a Hilbert space H. Define the set $\mathcal{H}_2^t\left(\mathcal{A}\right)$ by

$$\mathcal{H}_2^t\left(\mathcal{A}\right) \overset{\text{def}}{=} \left\{\begin{pmatrix} T_1 & tT_2 \\ T_2^* & T_1^* \end{pmatrix} : T_1, T_2 \in \mathcal{A}\right\}, \tag{4.1}$$

where T_l^* are the adjoints of T_l in $\mathcal{A}$, for all $l = 1, 2$, equipped with the semi-norm $\|.\|_{(t)}$,

$$\left\| \begin{pmatrix} T_1 & tT_2 \\ T_2^* & T_1^* \end{pmatrix} \right\|_{(t)} \overset{\text{def}}{=} \|(\|T_1\|_{\mathcal{A}}, \|T_2\|_{\mathcal{A}})\|_t,$$

identified with

$$\left\| \begin{pmatrix} T_1 & tT_2 \\ T_2^* & T_1^* \end{pmatrix} \right\|_{(t)} = \sqrt{\left| \|T_1\|_{\mathcal{A}}^2 - t\,\|T_2\|_{\mathcal{A}}^2 \right|},$$

where $\|.\|_t$ is the semi-norm on $\mathbb{H}_t$ (for all $t \in \mathbb{R}$, in particular, the norm, if $t < 0$), and $\|.\|_{\mathcal{A}}$ is the C^*-norm on $\mathcal{A}$, inherited from the operator-norm on $B(H)$. We call $\mathcal{H}_2^t(\mathcal{A})$, the t-scaled $\mathcal{A}$(-valued)-hypercomplexes, for $t \in \mathbb{R}$, and all operator-block (2×2)-matrices of $\mathcal{H}_2^t(\mathcal{A})$ are said to be operator(-valued)-hypercomplexes.

By abusing notations, one may/can write each operator-hypercomplex $\begin{pmatrix} T_1 & tT_2 \\ T_2^* & T_1^* \end{pmatrix} \in \mathcal{H}_2^t(\mathcal{A})$ by $[(T_1, T_2)]_t$, for all $T_1, T_2 \in \mathcal{A}$.

Proposition 25 *Let $\mathcal{H}_2^t(\mathcal{A})$ be the t-scaled $\mathcal{A}$-hypercomplexes (4.1). Then*

$$\mathcal{H}_2^t(\mathcal{A}) \ \textit{ is a complete } \mathbb{R} - \textit{SNS}, \ \forall t \in \mathbb{R}, \tag{4.2}$$

In particular, if $t < 0$, then it is a $\mathbb{R}$-Banach space.

Proof Suppose $[(T_1, T_2)]_t, [(S_1, S_2)]_t \in \mathcal{H}_2^t(\mathcal{A})$, and $r_1, r_2 \in \mathbb{R}$. Then

$$r_1 [(T_1, T_2)]_t + r_2 [(S_1, S_2)]_t = \begin{pmatrix} r_1 T_1 + r_2 S_1 & t z_1 T_2 + t r_2 S_2 \\ \\ r_1 T_2^* + r_2 S_2^* & r_1 T_1^* + r_2 S_1^* \end{pmatrix},$$

identifies with

$$\begin{pmatrix} r_1 T_1 + r_2 S_1 & t\,(r_1 T_2 + r_2 S_2) \\ \\ (r_1 T_2 + r_2 S_2)^* & (r_1 T_1 + r_2 S_1)^* \end{pmatrix} = [(r_1 T_1 + r_2 S_1,\ r_1 T_2 + r_2 S_2)]_t,$$

contained in $\mathcal{H}_2^t(\mathcal{A})$. And hence, $\mathcal{H}_2^t(\mathcal{A})$ forms a $\mathbb{R}$-vector space. Since $\mathcal{A}$ is a C^*-algebra (and hence, it is complete over $\mathbb{R}$), and $\mathbb{H}_t$ is a complete $\mathbb{R}$-semi-normed $*$-algebra, this $\mathbb{R}$-vector space $\mathcal{H}_2^t(\mathcal{A})$ forms a complete $\mathbb{R}$-SNS, for any $t \in \mathbb{R}$.

Recall and remark that, if $t < 0$, then $\mathbb{H}_t$ is a $\mathbb{R}$-Banach $*$-algebra, and hence, in such a case, the semi-norm $\|.\|_{(t)}$ on $\mathcal{H}_2^t(\mathcal{A})$ becomes a norm, by definition. Thus, if $t < 0$, then $\mathcal{H}_2^t(\mathcal{A})$ becomes a complete $\mathbb{R}$-NS, equivalently, a $\mathbb{R}$-Banach space. □

The above proposition provides a structure theorem for the t-scaled $\mathcal{A}$-hypercomplexes $\mathcal{H}_2^t(\mathcal{A})$, characterized to be a complete $\mathbb{R}$-SNS, by (4.2). Then can it be a $\mathbb{R}$-algebra in a usual sense? The answer is negative. Observe that, if $[(T_1, T_2)]_t, [(S_1, S_2)]_t \in \mathcal{H}_2^t(\mathcal{A})$, then

$$([(T_1, T_2)]_t)([(S_1, S_2)]_t) = \begin{pmatrix} T_1 & tT_2 \\ T_2^* & T_1^* \end{pmatrix} \begin{pmatrix} S_1 & tS_2 \\ S_2^* & S_1^* \end{pmatrix},$$

identical to

$$\begin{pmatrix} T_1 S_1 + t T_2 S_2^* & t\left(T_1 S_2 + T_2 S_1^*\right) \\ \\ T_2^* S_1 + T_1^* S_2^* & t T_2^* S_2 + T_1^* S_1^* \end{pmatrix} \notin \mathcal{H}_2^t(\mathcal{A}), \tag{4.3}$$

in general, since

$$\left(T_1S_1 + tT_2S_2^*\right)^* = S_1^*T_1^* + tS_2T_2^* \neq T_1^*S_1^* + tT_2^*S_2,$$

or

$$\left(T_1S_2 + T_2S_1^*\right)^* = S_2^*T_1^* + S_1T_2^* \neq T_2^*S_1 + T_1^*S_2^*, \tag{4.4}$$

in $\mathcal{A}$, in general. i.e.,

$$\left([(T_1, T_2)]_t\right)\left([(S_1, S_2)]_t\right) \notin \mathcal{H}_2^t(\mathcal{A}),$$

in general, under the usual block-operator multiplication.

Theorem 26 *The C^*-algebra $\mathcal{A}$ is commutative in the sense that: $TS = ST$, for all $T, S \in \mathcal{A}$, if and only if the t-scaled $\mathcal{A}$-hypercomplexes $\mathcal{H}_2^t(\mathcal{A})$ is a $\mathbb{R}$-semi-normed $*$-algebra, for all $t \in \mathbb{R}$. i.e., for all $t \in \mathbb{R}$,*

$$\mathcal{A} : commutative \iff \mathcal{H}_2^t(\mathcal{A}) : complete\, \mathbb{R} - semi - normed * -algebra. \tag{4.5}$$

In particular, if $t < 0$, then $\mathcal{H}_2^t(\mathcal{A})$ is a $\mathbb{R}$-Banach $*$-algebra in the characterization (4.5).

Proof By (4.2), the t-scaled $\mathcal{A}$-hypercomplexes $\mathcal{H}_2^t(\mathcal{A})$ is a complete $\mathbb{R}$-SNS, for all $t \in \mathbb{R}$. Fix an arbitrary scale $t \in \mathbb{R}$. Assume that the C^*-algebra $\mathcal{A}$ is commutative. Then the vector-multiplication (4.3) is well-defined on $\mathcal{H}_2^t(\mathcal{A})$, i.e., the usual (2×2)-block-operator multiplication is closed on $\mathcal{H}_2^t(\mathcal{A})$, because the non-equalities in (4.4) become the equalities under the commutativity of $\mathcal{A}$. Therefore, equipped with this well-defined vector-multiplication (4.3), the complete $\mathbb{R}$-SNS $\mathcal{H}_2^t(\mathcal{A})$ forms a complete $\mathbb{R}$-semi-normed algebra. Define now a bijection $(\dagger)$ on $\mathcal{H}_2^t(\mathcal{A})$ by

$$\dagger\left([(T_1, T_2)]_t\right) \overset{\text{denote}}{=} [(T_1, T_2)]_t^\dagger \overset{\text{def}}{=} \left[\left(T_1^*, -T_2\right)\right]_t,$$

i.e.,

$$\begin{pmatrix} T_1 & tT_2 \\ T_2^* & T_1^* \end{pmatrix}^\dagger = \begin{pmatrix} T_1^* & t(-T_2) \\ -T_2^* & T_1 \end{pmatrix}, \text{ in } \mathcal{H}_2^t(\mathcal{A}), \tag{4.6}$$

like the $\mathbb{R}$-adjoint $(\dagger)$ on $\mathbb{H}_t$. Then

$$[(T_1, T_2)]_t^{\dagger\dagger} = \left[\left(T_1^*, -T_2\right)\right]_t^\dagger = \left[\left(T_1^{**}, -(-T_2)\right)\right]_t = [(T_1, T_2)]_t,$$

and

$$\left(r\,[(T_1, T_2)]_t\right)^\dagger = [(rT_1, rT_2)]_t^\dagger = \left[\left(rT_1^* r(-T_2)\right)\right]_t = r\,[(T_1, T_2)]_t^\dagger,$$

for all $[(T_1, T_2)]_t \in \mathcal{H}_2^t(\mathcal{A})$ and $r \in \mathbb{R}$, and

$$\begin{aligned} &\left([(T_1, T_2)]_t + [(S_1, S_2)]_t\right)^\dagger = [(T_1 + S_1, T_2 + S_2)]_t^\dagger \\ &\quad = \left[\left((T_1 + S_1)^*, -(T_2 + S_2)\right)\right]_t = \left[\left(T_1^* + S_1^*\right), -T_2 - S_2\right]_t \\ &\quad = \left[\left(T_1^*, -T_2\right)\right]_t + \left[\left(S_1^*, -S_2\right)\right]_t = [(T_1, T_2)]_t^\dagger + [(S_1, S_2)]_t^\dagger \end{aligned}$$

and

$$\left([(T_1, T_2)]_t\,[(S_1, S_2)]_t\right)^\dagger = \begin{pmatrix} T_1S_1 + tT_2S_2^* & t\left(T_1S_2 + T_2S_1^*\right) \\ \left(T_1S_2 + T_2S_1^*\right)^* & \left(T_1S_1 + tT_2S_2^*\right)^* \end{pmatrix}^\dagger$$

by (4.3) and (4.4), under the commutativity of $\mathcal{A}$

$$= \begin{pmatrix} \left(T_1S_1 + tT_2S_2^*\right)^* & t\left(-T_1S_2 - T_2S_1^*\right) \\ \\ \left(-T_1S_2 - T_2S_1^*\right)^* & T_1S_1 + tT_2S_2^* \end{pmatrix}$$

$$= \begin{pmatrix} S_1^*T_1^* + tS_2T_2^* & t\left(-T_1S_2 - T_2S_1^*\right) \\ \\ \left(-T_1S_2 - T_2S_1^*\right)^* & T_1S_1 + tT_2S_2^* \end{pmatrix}$$

$$= \begin{pmatrix} S_1^* & t\left(-S_2\right) \\ -S_2^* & S_1 \end{pmatrix} \begin{pmatrix} T_1^* & t\left(-T_2\right) \\ -T_2^* & T_1 \end{pmatrix}$$

by the commutativity of $\mathcal{A}$

$$= \left[(S_1, S_2)\right]_t^{\dagger} \left[(T_1, T_2)\right]_t^{\dagger},$$

in $\mathcal{H}_2^t(\mathcal{A})$, for all $[(T_1, T_2)]_t, [(S_1, S_2)]_t \in \mathcal{H}_2^t(\mathcal{A})$. Therefore, the bijection (†) of (4.6) is a well-defined $\mathbb{R}$-adjoint on $\mathcal{H}_2^t(\mathcal{A})$. So, this complete $\mathbb{R}$-semi-normed algebra $\mathcal{H}_2^t(\mathcal{A})$ forms a complete $\mathbb{R}$-semi-normed $*$-algebra if $\mathcal{A}$ is a commutative C^*-algebra.

Conversely, assume that $\mathcal{A}$ is a noncommutative C^*-algebra. Then, by (4.3) and (4.4), the complete $\mathbb{R}$-SNS $\mathcal{H}_2^t(\mathcal{A})$ cannot be a $\mathbb{R}$-algebra.

Therefore, the characterization (4.5) holds true.

Now, take $t < 0$ in $\mathbb{R}$. Then, by (4.5), one has that $\mathcal{A}$ is commutative, if and only if the t-scaled $\mathcal{A}$-hypercomplexes $\mathcal{H}_2^t(\mathcal{A})$ is a complete $\mathbb{R}$-semi-normed $*$-algebra. However, if $t < 0$, then, under the commutativity of $\mathcal{A}$, $\mathcal{H}_2^t(\mathcal{A})$ becomes a $\mathbb{R}$-Banach space, and hence, it forms a $\mathbb{R}$-Banach $*$-algebra. i.e., if $t < 0$ in $\mathbb{R}$, then $\mathcal{A}$ is commutative, if and only if $\mathcal{H}_2^t(\mathcal{A})$ is a $\mathbb{R}$-Banach $*$-algebra. □

The above theorem proves that the complete $\mathbb{R}$-SNSs $\left\{\mathcal{H}_2^t(\mathcal{A})\right\}_{t\in\mathbb{R}}$ can be the complete $\mathbb{R}$-semi-normed $*$-algebras equipped with the $\mathbb{R}$-adjoint (4.6), if and only if $\mathcal{A}$ is a commutative C^*-algebra by (4.5). Without the commutativity on $\mathcal{A}$, the complete $\mathbb{R}$-SNSs $\left\{\mathcal{H}_2^t(\mathcal{A})\right\}_{t\in\mathbb{R}}$ cannot be $\mathbb{R}$-algebras in a usual sense.

Remark 4.1. Suppose $T \in B(H)$ is a self-adjoint operator on a Hilbert space H. Then the C^*-subalgebra $\mathcal{A}_T = C^*(\{T\})$ of $B(H)$ generated by T is a commutative C^*-algebra, $*$-isomorphic to the C^*-algebra $C(\text{spec}(T))$ of all continuous functions on the compact set $\text{spec}(T)$, the spectrum of T, in $\mathbb{C}$. And hence, such commutative C^*-subalgebras do exist in $B(H)$. More generally, if $T_1, \ldots, T_N \in B(H)$ are self-adjoint, and mutually commuting from each other in the sense that:

$$T_l^* = T_l \text{ in } B(H), \ \forall l = 1, \ldots, N,$$

and

$$T_{l_1}T_{l_2} = T_{l_2}T_{l_1}, \ \forall l_1, l_2 \in \{1, \ldots, N\},$$

in $B(H)$, for $N \in \mathbb{N} \cup \{\infty\}$, then the C^*-subalgebra $\mathcal{A}_{T_1,\ldots,T_N} = C^*(\{T_1, \ldots, T_N\})$ of $B(H)$ forms a commutative C^*-algebra. (e.g., see [11,12]). □

Now, let $\mathcal{A}^2 = \mathcal{A} \times \mathcal{A}$ be the Cartesian-product C^*-algebra of two copies of $\mathcal{A}$'s, consisting of the operator-pairs of $\mathcal{A}$. Define a morphism α of $\mathcal{A}^2$ on the t-scaled hypercomplexes $\mathbb{H}_t$ by

$$\alpha(T_1, T_2)(a, b) = [(aT_1, bT_2)]_t = \begin{pmatrix} aT_1 & t(bT_2) \\ \overline{b}T_2^* & \overline{a}T_1^* \end{pmatrix}, \tag{4.7}$$

identical to

$$\begin{pmatrix} aT_1 & t\,(bT_2) \\ (bT_2)^* & (aT_1)^* \end{pmatrix} \text{ contained in } \mathcal{H}_2^t\,(\mathcal{A}),$$

for all $(a, b) \in \mathbb{H}_t$. This morphism α of (4.7) satisfies that

$$\alpha\,(z_1T_1 + z_2T_2, T_3) = z_1\alpha\,(T_1, T_3) + z_2\alpha\,(T_2, T_3),$$

and

$$\alpha\,(T_1, z_1T_2 + z_2T_3) = z_1\alpha\,(T_1, T_2) + z_2\alpha\,(T_1, T_3), \tag{4.8}$$

on $\mathbb{H}_t$, whose images are in $\mathcal{H}_2^t\,(\mathcal{A})$, for all $z_1, z_2 \in \mathbb{C}$ and $T_1, T_2, T_3 \in \mathcal{A}$.

Theorem 27 *Let $\mathcal{A}^2 = \mathcal{A} \times \mathcal{A}$ be the Cartesian-product C^*-algebra of the fixed unital C^*-algebra $\mathcal{A}$, and α, the morphism (4.7) from $\mathcal{A}^2$ to the t-scaled $\mathcal{A}$-hypercomplexes $\mathcal{H}_2^t\,(\mathcal{A})$. Then α is a well-defined continuous bi-module action of $\mathcal{A}^2$ acting on the t-scaled hypercomplexes $\mathbb{H}_t$ realized in $\mathcal{H}_2^t\,(\mathcal{A})$. i.e.,*

$$\alpha \text{ is a bi-module action of } \mathcal{A}^2 \text{ acting on } \mathbb{H}_t \text{ realized in } \mathcal{H}_2^t\,(\mathcal{A}). \tag{4.9}$$

Proof By the definition (4.7) of the morphism α, every image $\alpha\,(T_1, T_2)$ of α is a well-defined function from $\mathbb{H}_t$ into $\mathcal{H}_2^t\,(\mathcal{A})$, because,

$$\alpha\,(T_1, T_2)\,(a, b) = [(aT_1, bT_2)]_t\,,$$

in $\mathcal{H}_2^t\,(\mathcal{A})$, since $aT_1, bT_2 \in \mathcal{A}$, for all $(a, b) \in \mathbb{H}_t$, and $(T_1, T_2) \in \mathcal{A}^2$. i.e.,

$$\alpha\,(T_1, T_2) \in B_{\mathbb{R}}\left(\mathbb{H}_t, \mathcal{H}_2^t\,(\mathcal{A})\right), \ \forall\,(T_1, T_2) \in \mathcal{A}^2,$$

where $B_{\mathbb{R}}\left(\mathbb{H}_t, \mathcal{H}_2^t\,(\mathcal{A})\right)$ is the operator space of all bounded $\mathbb{R}$-linear transformations from the complete $\mathbb{R}$-SNS $\mathbb{H}_t$ to the complete $\mathbb{R}$-SNS $\mathcal{H}_2^t\,(\mathcal{A})$. Indeed, for any $r_1, r_2 \in \mathbb{R}$ and $(a_1, b_1), (a_2, b_2) \in \mathbb{H}_t$, one has that

$$\begin{aligned}
&\alpha\,(T_1, T_2)\,(r_1\,(a_1, b_1) + r_2\,(a_2, b_2)) \\
&\quad = \alpha\,(T_1, T_2)\,((r_1a_1 + r_2a_2,\ r_1b_1 + r_2b_2)) \\
&\quad = \begin{pmatrix} (r_1a_1 + r_2a_2)\,T_1 & t\,(r_1b_1 + r_2b_2)\,T_2 \\ \\ \left(r_1\overline{b_1} + r_2\overline{b_2}\right)T_2^* & (r_1\overline{a_1} + r_2\overline{a_2})\,T_1^* \end{pmatrix} \\
&\quad = r_1 \begin{pmatrix} a_1T_1 & tb_1T_2 \\ \\ \overline{b_1}T_2^* & \overline{a_1}T_1^* \end{pmatrix} + r_2 \begin{pmatrix} a_2T_1 & tb_2T_2 \\ \\ \overline{b_2}T_2^* & \overline{a_2}T_1^* \end{pmatrix} \\
&\quad = r_1\alpha\,(T_1, T_2)\,(a_1, b_1) + r_2\alpha\,(T_1, T_2)\,(a_2, b_2)
\end{aligned} \tag{4.10}$$

satisfying

$$\left\|\alpha\,(T_1, T_2)\,(a, b)\right\|_{(t)} = \sqrt{\left|\|aT_1\|_{\mathcal{A}}^2 - t\,\|bT_2\|_{\mathcal{A}}^2\right|},$$

identical to

$$\left\|\left(\|aT_1\|_{\mathcal{A}},\ \|bT_2\|_{\mathcal{A}}\right)\right\|_t < \infty,$$

implying that

$$\|\alpha\,(T_1, T_2)\| = \sup\left\{\left\|\alpha\,(T_1, T_2)\,(h)\right\|_{(t)} : \|h\|_t = 1\right\} < \infty, \tag{4.11}$$

for all $(T_1, T_2) \in \mathcal{A}^2$, where $\|.\|$ of (4.11) is the operator-norm on $B_{\mathbb{R}}\left(\mathbb{H}_t, \mathcal{H}_2^t(\mathcal{A})\right)$. Therefore,

$$\alpha(T_1, T_2) \in B_{\mathbb{R}}\left(\mathbb{H}_t, \mathcal{H}_2^t(\mathcal{A})\right), \ \forall (T_1, T_2) \in \mathcal{A}^2, \tag{4.12}$$

by (4.10) and (4.11).

Therefore, by the relation (4.8) and (4.12), the map α of (4.7) is a bi-module action of $\mathcal{A}^2$ acting on $\mathbb{H}_t$ realized in $\mathcal{H}_2^t(\mathcal{A})$, i.e., the relation (4.9) holds. □

The above theorem shows that, indeed, our scaled $\mathcal{A}$-hypercomplexes $\left\{\mathcal{H}_2^t(\mathcal{A})\right\}_{t\in\mathbb{R}}$ are well-defined, as the images of the bi-module action α of $\mathcal{A}^2$ acting on the t-scaled hypercomplexes $\mathbb{H}_t$, where $\alpha(T_1, T_2) \in B_{\mathbb{R}}\left(\mathbb{H}_t, \mathcal{H}_2^t(\mathcal{A})\right)$ by (4.9). It also illustrate the relation between $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$ and $\left\{\mathcal{H}_2^t(\mathcal{A})\right\}_{t\in\mathbb{R}}$, as complete $\mathbb{R}$-SNSs.

Corollary 28 *As a complete $\mathbb{R}$-SNS, the t-scaled $\mathcal{A}$-hypercomplexes $\mathcal{H}_2^t(\mathcal{A})$ is isomorphic to the bi-module $\alpha\left(\mathcal{A}^2\right)(\mathbb{H}_t)$.*

$$\mathcal{H}_2^t(\mathcal{A}) \overset{\text{iso}}{=}_{\mathcal{A}} (\mathbb{H}_t)_{\mathcal{A}} \overset{\text{denote}}{=} \alpha\left(\mathcal{A}^2\right)(\mathbb{H}_t). \tag{4.13}$$

Proof The key idea of the proof is that, for any $z \in \mathbb{C}$ and $T \in \mathcal{A}$, the scalar-product $zT \in \mathcal{A}$. Define a function $\Omega : _{\mathcal{A}}(\mathbb{H}_t)_{\mathcal{A}} \to \mathcal{H}_2^t(\mathcal{A})$ by

$$\Omega\left(\alpha(T_1, T_2)(a, b)\right) = [(aT_1, bT_2)]_t,$$

for all $(T_1, T_2) \in \mathcal{A}^2$ and $(a, b) \in \mathbb{H}_t$. Then this well-defined function Ω is injective, since if

$$\alpha(T_1, T_2)(a_1, b_1) \neq \alpha(S_1, S_2)(a_2, b_2),$$

in ${}_{\mathcal{A}}(\mathbb{H}_t)_{\mathcal{A}}$, then

$$[(a_1T_1, b_1T_2)]_t \neq [(a_2S_1, b_2S_2)]_t,$$

in $\mathcal{H}_2^t(\mathcal{A})$, by (4.7). Moreover, it is surjective, since, for any $[(T_1, T_2)]_t \in \mathcal{H}_2^t(\mathcal{A})$, there exists $(a, b) \in \mathbb{H}_t$, with $(a, b) \in \mathbb{H}_t$ with $a, b \in \mathbb{C} \setminus \{0\}$, such that

$$[(T_1, T_2)]_t = \alpha\left(\frac{1}{a}T_1, \frac{1}{b}T_2\right)(a, b).$$

Therefore, this function Ω is bijective from ${}_{\mathcal{A}}(\mathbb{H}_t)_{\mathcal{A}}$ onto $\mathcal{H}_2^t(\mathcal{A})$. Moreover, for any $r_1, r_2 \in \mathbb{R}$ and

$$\beta_l \overset{\text{denote}}{=} \alpha\left(T_{1,l}, T_{2,l}\right)(a_l, b_l) \in_{\mathcal{A}} (\mathbb{H}_t)_A, \text{ for } l = 1, 2,$$

one has

$$\Omega(r_1\beta_1 + r_2\beta_2) = \left[(r_1a_1T_{1,1} + r_2a_2T_{1,2},\ r_1b_1T_{1,2} + r_2b_2T_{2,2})\right]_t,$$

identical to

$$\Omega(r_1\beta_1 + r_2\beta_2) = r_1\Omega(\beta_1) + r_2\Omega(\beta_2),$$

in $\mathcal{H}_2^t(\mathcal{A})$. So, this bijection Ω is a $\mathbb{R}$-linear transformation, and hence, it is a $\mathbb{R}$-vector-space-isomrophism from ${}_{\mathcal{A}}(\mathbb{H}_t)_{\mathcal{A}}$ onto $\mathcal{H}_2^t(\mathcal{A})$. By the completeness, this isomorphism Ω is bounded. Therefore, the isomorphic relation (4.13) holds. □

The above corollary illustrates again that our scaled $\mathcal{A}$-hypercomplexes $\left\{\mathcal{H}_2^t(\mathcal{A})\right\}_{t\in\mathbb{R}}$ are well-defined as complete $\mathbb{R}$-SNSs. It also shows the connections between $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$ and $\left\{\mathcal{H}_2^t(\mathcal{A})\right\}_{t\in\mathbb{R}}$.

5 Invertibility on $\mathcal{H}_2^t(\mathcal{A})$

In this section, we study the invertibility on the t-scaled $\mathcal{A}$-hypercomplexes $\mathcal{H}_2^t(\mathcal{A})$, where $\mathcal{A}$ is a unital C^*-algebra in the operator algebra $B(H)$ on a Hilbert space H. First of all, to consider the invertibility on the complete $\mathbb{R}$-SNS $\mathcal{H}_2^t(\mathcal{A})$, we need a well-defined vector-multiplication on it. i.e., we need to understand $\mathcal{H}_2^t(\mathcal{A})$ as a $\mathbb{R}$-algebra. So, we restrict our interests to the cases where $\mathcal{A}$ is a commutative C^*-algebra by (4.5), and hence, understand $\mathcal{H}_2^t(\mathcal{A})$ as a complete $\mathbb{R}$-semi-normed $*$-algebra. Then the vector-multiplication,

$$\begin{pmatrix} T_1 & tT_2 \\ T_2^* & T_1^* \end{pmatrix}\begin{pmatrix} S_1 & tS_2 \\ S_2^* & S_1^* \end{pmatrix} = \begin{pmatrix} T_1S_1 + tT_2S_2^* & t\left(T_1S_2 + T_2S_1^*\right) \\ \left(T_1S_2 + T_2S_1^*\right)^* & \left(T_1S_2 + tT_2S_2^*\right)^* \end{pmatrix}, \tag{5.0.1}$$

is well-defined on $\mathcal{H}_2^t(\mathcal{A})$, for all $[(T_1, T_2)]_t$, $[(S_1, S_2)]_t \in \mathcal{H}_2^t(\mathcal{A})$. Note again that the commutativity on a fixed C^*-algebra $\mathcal{A}$ allows us to have the above multiplications "on $\mathcal{H}_2^t(\mathcal{A})$," by (4.5). i.e., $\mathcal{H}_2^t(\mathcal{A})$ becomes a complete $\mathbb{R}$-semi-normed $*$-algebra, equipped with its $\mathbb{R}$-adjoint $(\dagger)$,

$$\begin{pmatrix} T & tS \\ S^* & T^* \end{pmatrix}^\dagger = \begin{pmatrix} T^* & t(-S) \\ -S^* & T \end{pmatrix}, \tag{5.0.2}$$

for all $T, S \in \mathcal{A}$. Consider the case where

$$\begin{pmatrix} T_1 & tT_2 \\ T_2^* & T_1^* \end{pmatrix}\begin{pmatrix} S_1 & tS_2 \\ S_2^* & S_1^* \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} = \begin{pmatrix} S_1 & tS_2 \\ S_2^* & S_1^* \end{pmatrix}\begin{pmatrix} T_1 & tT_2 \\ T_2^* & T_1^* \end{pmatrix}, \tag{5.0.3}$$

where $\mathbf{1}$ is the unity (or, the identity operator), and $\mathbf{0}$ is the zero operator of $\mathcal{A}$, equivalently,

$$\begin{pmatrix} T_1S_1 + tT_2S_2^* & t\left(T_1S_2 + T_2S_1^*\right) \\ \left(T_1S_2 + T_2S_1^*\right)^* & \left(T_1S_2 + tT_2S_2^*\right)^* \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix},$$

and

$$\begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} S_1T_1 + tS_2T_2^* & t\left(S_1T_2 + S_2T_1^*\right) \\ \left(S_1T_2 + S_2T_1^*\right)^* & \left(S_1T_1 + tS_2T_2^*\right)^* \end{pmatrix}, \tag{5.0.4}$$

by (5.0.3). The equalities of (5.0.4) is equivalent to

$$T_1S_1 + tT_2S_2^* = \mathbf{1} = S_1T_1 + tS_2T_2^*,$$

and

$$T_1S_2 + T_2S_1^* = \mathbf{0} = S_1T_2 + S_2T_1^*, \tag{5.0.5}$$

in $\mathcal{A}$, if and only if

$$T_1S_1 + tT_2S_2^* = \mathbf{1} = T_1S_1 + tT_2^*S_2,$$

and

$$T_1S_2 + T_2S_1^* = \mathbf{0} = T_2S_1 + T_1^*S_2, \tag{5.0.6}$$

by (5.0.5) and the commutativity on $\mathcal{A}$.

Definition 29 Let $\mathcal{A}$ be a commutative unital C^*-algebra with its unity $\mathbf{1}$, and let $\mathcal{H}_2^t(\mathcal{A})$, the corresponding t-scaled $\mathcal{A}$-hypercomplexes. An element $\eta \in \mathcal{H}_2^t(\mathcal{A})$ is invertible "in" $\mathcal{H}_2^t(\mathcal{A})$, if there exists a unique element, denoted by η^{-1}, in $\mathcal{H}_2^t(\mathcal{A})$, such that

$$\eta\eta^{-1} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} = \eta^{-1}\eta, \text{ in } \mathcal{H}_2^t(\mathcal{A}),$$

where $\mathbf{0}$ is the zero element of $\mathcal{A}$.

By the above definition, one obtains the following result.

Proposition 30 *Suppose $\mathcal{A}$ is a commutative unital C^*-algebra with its unity $\mathbf{1}$, and $[(T_1, T_2)]_t \in \mathcal{H}_2^t(\mathcal{A})$. Then $[(T_1, T_2)]_t$ is invertible in $\mathcal{H}_2^t(\mathcal{A})$, if and only if there exists a unique element $[(S_1, S_2)]_t \in \mathcal{H}_2^t(\mathcal{A})$, such that*

$$T_1S_1 + tT_2S_2^* = \mathbf{1} = T_1S_1 + tT_2^*S_2,$$

and

$$T_1S_2 + T_2S_1^* = \mathbf{0} = T_2S_1 + T_1^*S_2, \tag{5.0.7}$$

in $\mathcal{A}$, for all scales $t \in \mathbb{R}$.

Proof The invertibility characterization (5.0.7) of the inverse $[(S_1, S_2)]_t = [(T_1, T_2)]_t^{-1}$ in $\mathcal{H}_2^t(\mathcal{A})$ is obtained by (5.0.6), under the commutativity of $\mathcal{A}$. □

Motivated by (5.0.7), we consider two different cases where $t \neq 0$, and where $t = 0$.

5.1 The case where $t = 0$

In this section, we let $\mathcal{A}$ be a fixed "commutative" unital C^*-algebra with its unity $\mathbf{1}$, and $\mathcal{H}_2^0(\mathcal{A})$, the 0-scaled $\mathcal{A}$-hypercomplexes,

$$\mathcal{H}_2^0(\mathcal{A}) = \left\{ \begin{pmatrix} T & \mathbf{0} \\ S^* & T^* \end{pmatrix} : T, S \in \mathcal{A} \right\},$$

equipped with the usual block-operator-matrix addition, and the multiplication (5.0.1), and the adjoint (5.0.2). By (5.0.7), an element $[(T_1, T_2)]_0$ is invertible in $\mathcal{H}_2^0(\mathcal{A})$, if and only if there exists a unique element $[(S_1, S_2)]_t \in \mathcal{H}_2^0(\mathcal{A})$, such that

$$T_1S_1 + 0 \cdot T_2S_2^* = \mathbf{1} = T_1S_1 + 0 \cdot T_2^*S_2,$$

and

$$T_1S_2 + T_2S_1^* = \mathbf{0} = T_2S_1 + T_1^*S_2, \tag{5.1.1}$$

in $\mathcal{A}$, and hence,

$$T_1S_1 = \mathbf{1}, \text{ and } T_1S_2 + T_2S_1^* = \mathbf{0} = T_2S_1 + T_1^*S_2, \tag{5.1.2}$$

in $\mathcal{A}$, by (5.1.1).

Observe the first equality $T_1S_1 = \mathbf{1}$ in (5.1.2). By the commutativity of $\mathcal{A}$, this equality is in fact identified with

$$T_1S_1 = \mathbf{1} = S_1T_1, \text{ in } \mathcal{A},$$

implying that T_1 is invertible in $\mathcal{A}$, with its inverse $T_1^{-1} = S_1$ in $\mathcal{A}$, i.e.,

$$S_1 = T_1^{-1}, \text{ in } \mathcal{A}, \tag{5.1.3}$$

where $T_1^{-1} \in \mathcal{A}$ means the inverse of $T_1 \in \mathcal{A}$, as the inverse operator of the operator algebra $B(H)$ (containing $\mathcal{A}$). And, by (5.1.2) and (5.1.3),

$$T_1S_2 = -T_2S_1^* = -T_2\left(T_1^{-1}\right)^*,$$

and hence,

$$S_2 = -T_1^{-1}T_2\left(T_1^{-1}\right)^*, \text{ in } \mathcal{A}. \tag{5.1.4}$$

Theorem 31 *An element* $[(T_1, T_2)]_0$ *is invertible in* $\mathcal{H}_2^0(\mathcal{A})$*, with its inverse* $[(S_1, S_2)]_0 \in \mathcal{H}_2^0(\mathcal{A})$*, if and only if*

$$T_1 \text{ is invertible with its inverse } S_1 = T_1^{-1}, \text{ in } \mathcal{A},$$

and

$$[(S_1, S_2)]_0 = \left[\left(T_1^{-1}, -T_1^{-1}T_2\left(T_1^{-1}\right)^*\right)\right]_0 \in \mathcal{H}_2^0(\mathcal{A}). \tag{5.1.5}$$

Proof An element $[(T_1, T_2)]_0$ is invertible in $\mathcal{H}_2^0(\mathcal{A})$ with its inverse $[(S_1, S_2)]_t \in \mathcal{H}_2^0(\mathcal{A})$, if and only if the relation (5.1.2) holds, if and only if T_1 is invertible in $\mathcal{A}$ with $S_1 = T_1^{-1}$ in $\mathcal{A}$, by (5.1.3), and

$$S_2 = -T_1^{-1}T_2\left(T_1^{-1}\right)^*, \text{ in } \mathcal{A},$$

by (5.1.4). Therefore, $[(T_1, T_2)]_0$ is invertible in $\mathcal{H}_2^0(\mathcal{A})$, if and only if the relation (5.1.5) holds. □

Observe that

$$\begin{aligned}
&([(T_1, T_2)]_0)\left(\left[\left(T_1^{-1}, -T_1^{-1}T_2\left(T_1^{-1}\right)^*\right)\right]_0\right) \\
&= \begin{pmatrix} T_1 & \mathbf{0} \\ T_2^* & T_1^* \end{pmatrix} \begin{pmatrix} T_1^{-1} & \mathbf{0} \\ -\left(T_1^{-1}\right)^* T_2^* T_1^{-1} & \left(T_1^{-1}\right)^* \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ T_2^* T_1^{-1} - T_1^*\left(T_1^{-1}\right)^* T_2^* T_1^{-1} & \mathbf{1} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ T_2^* T_1^{-1} - \left(T_1^{-1}T_1\right)^* T_2^* T_1^{-1} & \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix},
\end{aligned}$$

and, similarly,

$$\left(\left[\left(T_1^{-1}, -T_1^{-1}T_2\left(T_1^{-1}\right)^*\right)\right]_0\right)([(T_1, T_2)]_0) = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix},$$

in $\mathcal{H}_2^0(\mathcal{A})$, confirming the invertibility characterization (5.1.5) on $\mathcal{H}_2^0(\mathcal{A})$.

5.2 The case where $t \neq 0$

In Sect. 5.1, we characterize the invertibility on the 0-scaled $\mathcal{A}$-hypercomplexes $\mathcal{H}_2^0(\mathcal{A})$ by (5.1.5), where $\mathcal{A}$ is a commutative unital C^*-subalgebra of the operator algebra $B(H)$ on a Hilbert space H. As in Sect. 5.1, we fix a "commutative" unital C^*-algebra $\mathcal{A}$, and let $\mathcal{H}_2^t(\mathcal{A})$ be the corresponding t-scaled $\mathcal{A}$-hypercomplexes, where $t \neq 0$. Throughout this section, we automatically assume that any fixed scale $t \in \mathbb{R}$ is non-zero. Recall that, by (5.0.7), an element $[(T_1, T_2)]_t$ is invertible in $\mathcal{H}_2^t(\mathcal{A})$ with its inverse $[(S_1, S_2)]_t \in \mathcal{H}_2^t(\mathcal{A})$, if and only if

$$T_1S_1 + tT_2S_2^* = \mathbf{1} = T_1S_1 + tT_2^*S_2,$$

and

$$T_1S_2 + T_2S_1^* = \mathbf{0} = T_2S_1 + T_1^*S_2, \tag{5.2.1}$$

in $\mathcal{A}$. Since $t \neq 0$ in $\mathbb{R}$, the invertibility condition (5.2.1) is equivalent to

$$tT_2S_2^* = \mathbf{1} - T_1S_1 = tT_2^*S_2,$$

and

$$T_1S_2 + T_2S_1^* = \mathbf{0} = T_2S_1 + T_1^*S_2,$$

if and only if

$$T_2S_2^* = \frac{1}{t}\left(\mathbf{1} - T_1S_1\right) = T_2^*S_2 = \left(T_2S_2^*\right)^*,$$

and

$$T_1S_2 = -T_2S_1^*, \text{ and } T_2S_1 = -T_1^*S_2, \tag{5.2.2}$$

in $\mathcal{A}$. Suppose T_1 and T_2 are invertible in the commutative C^*-algebra $\mathcal{A}$. Then their adjoints T_1^* and T_2^* are invertible, too, with $\left(T_l^*\right)^{-1} = \left(T_l^{-1}\right)^*$ in $\mathcal{A}$, for all $l = 1, 2$. So, if T_1 and T_2 are invertible, then the invertibility condition (5.2.2) of $[(T_1, T_2)]_t \in \mathcal{H}_2^t(\mathcal{A})$ is equivalent to

$$S_2 = \frac{1}{t}\left(T_2^*\right)^{-1}\left(\mathbf{1} - T_1S_1\right),$$

respectively,

$$S_1 = -T_2^{-1}T_1^*S_2, \text{ in } \mathcal{A}, \tag{5.2.3}$$

implying that

$$S_2 = \frac{1}{t}\left(T_2^*\right)^{-1}\left(\mathbf{1} - T_1S_1\right) = \frac{1}{t}\left(T_2^*\right)^{-1}\left(\mathbf{1} - T_1\left(-T_2^{-1}T_1^*S_2\right)\right),$$

$\Longleftrightarrow$

$$S_2 = \frac{1}{t}\left(T_2^*\right)^{-1}\left(\mathbf{1} + T_1T_2^{-1}T_1^*S_2\right),$$

$\Longleftrightarrow$

$$S_2 = \frac{1}{t}\left(T_2^*\right)^{-1} + \frac{1}{t}\left(T_2^*\right)^{-1}T_1T_2^{-1}T_1^*S_2,$$

$\Longleftrightarrow$

$$\left(\mathbf{1} - \frac{1}{t}\left(T_2^*\right)^{-1}T_1T_2^{-1}T_1^*\right)S_2 = \frac{1}{t}\left(T_2^*\right)^{-1}, \text{ in } \mathcal{A}.$$

So, if $\mathbf{1} - \frac{1}{t}\left(T_2^*\right)^{-1}T_1T_2^{-1}T_1^*$ is invertible in $\mathcal{A}$, then

$$S_2 = \frac{1}{t}\left(T_2^*\right)^{-1}\left(\mathbf{1} - \frac{1}{t}\left(T_2^*\right)^{-1}T_1T_2^{-1}T_1^*\right)^{-1},$$

$\Longleftrightarrow$

$$S_2 = \frac{1}{t}\left(\left(\mathbf{1} - \frac{1}{t}\left(T_2^*\right)^{-1}T_1T_2^{-1}T_1^*\right)\left(T_2^*\right)\right)^{-1},$$

since $(TS)^{-1} = S^{-1}T^{-1} = T^{-1}S^{-1}$ in $\mathcal{A}$ (by the commutativity of $\mathcal{A}$), for all $T, S \in \mathcal{A}$, implying that

$$S_2 = \frac{1}{t}\left(T_2^* - \frac{1}{t}T_2^*\left(T_2^*\right)^{-1}T_1T_2^{-1}T_1^*\right)^{-1},$$

and hence,

$$S_2 = \frac{1}{t}\left(T_2^* - \frac{1}{t}T_1T_2^{-1}T_1^*\right)^{-1}, \text{ in } \mathcal{A} \tag{5.2.4}$$

and, under the same hypotheses,

$$S_1 = -T_2^{-1}T_1^*S_2, \quad \text{by} \quad (5.2.3),$$

$\Longleftrightarrow$

$$S_1 = -T_2^{-1}T_1^*\left(\frac{1}{t}\left(T_2^* - \frac{1}{t}T_1T_2^{-1}T_1^*\right)^{-1}\right),$$

by (5.2.4), if and only if

$$S_1 = -\frac{1}{t}T_2^{-1}T_1^*\left(T_2^* - \frac{1}{t}T_1T_2^{-1}T_1^*\right)^{-1}. \tag{5.2.5}$$

Theorem 32 *Assume that T_1, T_2 and $\mathbf{1} - \frac{1}{t}\left(T_2^*\right)^{-1} T_1T_2^{-1}T_1^*$ are invertible in $\mathcal{A}$. Then an element $[(T_1, T_2)]_t$ is invertible with its inverse $[(S_1, S_2)]_t$ in $\mathcal{H}_2^t(\mathcal{A})$, if and only if*

$$S_1 = -\frac{1}{t}T_2^{-1}T_1^*\left(T_2^* - \frac{1}{t}T_1T_2^{-1}T_1^*\right)^{-1},$$

and

$$S_2 = \frac{1}{t}\left(T_2^* - \frac{1}{t}T_1T_2^{-1}T_1^*\right)^{-1}. \tag{5.2.6}$$

Proof Suppose T_1, T_2 and $\mathbf{1} - \frac{1}{t}\left(T_2^*\right)^{-1} T_1T_2^{-1}T_1^*$ are invertible in the commutative C^*-algebra $\mathcal{A}$. The invertibility condition (5.2.6) is shown by (5.2.4) and (5.2.5) on $\mathcal{H}_2^t(\mathcal{A})$. □

The above theorem provides a partial characterization (5.2.6) of the invertibility on the non-zero-scaled $\mathcal{A}$-hypercomplexes $\mathcal{H}_2^t(\mathcal{A})$ of a commutative C^*-algebra $\mathcal{A}$, under certain invertibility assumptions on $\mathcal{A}$.

5.3 Summary and discussion

Let $\mathcal{H}_2^t(\mathcal{A})$ be the t-scaled $\mathcal{A}$-hypercomplexes of a "commutative" unital C^*-algebra $\mathcal{A}$, for all $t \in \mathbb{R}$. The main results of this section are summarized by the following corollary.

Corollary 33 *If $t = 0$ in $\mathbb{R}$, then an element $[(T_1, T_2)]_0$ is invertible in $\mathcal{H}_2^0(\mathcal{A})$ with its inverse $[(S_1, S_2)]_0 \in \mathcal{H}_2^t(\mathcal{A})$, if and only if*

T_1 is invertible in $\mathcal{A}$, with its inverse T_1^{-1},

and

$$[(S_1, S_2)]_0 = \left[\left(T_1^{-1}, -T_1^{-1}T_2\left(T_1^{-1}\right)^*\right)\right]_0 \in \mathcal{H}_2^0(\mathcal{A}). \tag{5.3.1}$$

Meanwhile, if $t \neq 0$ in $\mathbb{R}$, then an element $[(T_1, T_2)]_t$ is invertible with its inverse $[(S_1, S_2)]_t$ in $\mathcal{H}_2^t(\mathcal{A})$, if and only if

$$T_2S_2^* = \frac{1}{t}(\mathbf{1} - T_1S_1) = T_2^*S_2 = \left(T_2S_2^*\right)^*,$$

and

$$T_1S_2 = -T_2S_1^*, \text{ and } T_2S_1 = -T_1^*S_2, \tag{5.3.2}$$

in $\mathcal{A}$. So, as a special case, if $t \neq 0$, and T_1, T_2 and $\mathbf{1} - \frac{1}{t}\left(T_2^*\right)^{-1} T_1 T_2^{-1} T_1^*$ are invertible in $\mathcal{A}$, then the invertibility (5.3.2) is equivalent to

$$S_1 = -\frac{1}{t} T_2^{-1} T_1^* \left(T_2^* - \frac{1}{t} T_1 T_2^{-1} T_1^* \right)^{-1},$$

and

$$S_2 = \frac{1}{t} \left(T_2^* - \frac{1}{t} T_1 T_2^{-1} T_1^* \right), \text{ in } \mathcal{A}. \tag{5.3.3}$$

Proof The invertibility characterization (5.3.1) on $\mathcal{H}_2^0(\mathcal{A})$ is proven by (5.1.5). The invertibility characterization (5.3.2) on $\left\{\mathcal{H}_2^t(\mathcal{A})\right\}_{t\in\mathbb{R}\setminus\{0\}}$ is shown by (5.0.7), or (5.2.2). The proof of the special case (5.3.3) of (5.3.2) is done by (5.2.6). □

The above invertibility conditions on $\left\{\mathcal{H}_2^t(\mathcal{A})\right\}_{t\in\mathbb{R}}$ are interesting themselves. However, it is true that the commutativity assumption on a fixed unital C^*-algebra $\mathcal{A}$ is strong, but it is needed by (4.5). So, to avoid such a "strong" condition, we consider a new type adjoint-like structure on a unital C^*-algebra $\mathcal{A}$, motivated by (4.3) and (4.4). See Sect. 6 below.

6 The conjugation on a Unital C^*-Algebra $\mathcal{A}$

In this section, let $\mathcal{A}$ be a unital C^*-subalgebra of the operator algebra $B(H)$ on a separable (finite, or infinite dimensional) Hilbert space H, which is not necessarily commutative, where the dimension of H, which is the cardinality of the orthonormal basis of H, is $N \in \mathbb{N} \cup \{\infty\}$ (by the separability of H), i.e., $\dim_{\mathbb{C}} H = N$. Note that every element $T \in \mathcal{A}$ is realized to be a $(N \times N)$-matrix on H, i.e.,

$$T = \left[z_{ij}\right]_{N\times N} = \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1N} \\ z_{21} & z_{22} & \cdots & z_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ z_{N1} & z_{N2} & \cdots & z_{NN} \end{pmatrix},$$

where $N \in \mathbb{N} \cup \{\infty\}$ (e.g., see [11,12]). We now define the conjugate $(\overline{\bullet})$ on $\mathcal{A}$ by

$$\overline{T} = \overline{\left[z_{ij}\right]_{N\times N}} \overset{\text{def}}{=} \left[\overline{z_{ij}}\right]_{N\times N}, \ \forall T = \left[z_{ij}\right]_{N\times N} \in \mathcal{A}, \tag{6.0.1}$$

where $\overline{z_{ij}}$ are the usual conjugates of z_{ij} in $\mathbb{C}$. Then this conjugation on $\mathcal{A}$ satisfies that

$$\overline{\overline{T}} = \overline{\left[\overline{z_{ij}}\right]_{N\times N}} = \left[\overline{\overline{z_{ij}}}\right]_{N\times N} = \left[z_{ij}\right]_{N\times N} = T,$$

for all $T \in \mathcal{A}$; and

$$\overline{zT} = \overline{z\left[z_{ij}\right]_{N\times N}} = \overline{\left[zz_{ij}\right]_{N\times N}} = \left[\overline{zz_{ij}}\right]_{N\times N} = \overline{z}\overline{\left[z_{ij}\right]_{N\times N}} = \overline{z}\overline{T},$$

for all $z \in \mathbb{C}$ and $T \in \mathcal{A}$; and

$$\overline{T+S} = \overline{\left[z_{ij}\right]_{N\times N} + \left[w_{ij}\right]_{N\times N}} = \left[\overline{z_{ij}} + \overline{w_{ij}}\right]_{N\times N} = \overline{T} + \overline{S},$$

for all $T, S \in \mathcal{A}$; and

$$\overline{TS} = \overline{\left[z_{ij}\right]_{N\times N}\left[w_{ij}\right]_{N\times N}} = \overline{\left[\sum_{k=1}^{N} z_{ik} w_{kj}\right]_{N\times N}} = \left[\sum_{k=1}^{N} \overline{z_{ik} w_{kj}}\right]_{N\times N}$$

$$= \left[\sum_{k=1}^{N} \left(\overline{z_{ik}}\right)\left(\overline{w_{kj}}\right)\right]_{N\times N} = \left[\overline{z_{ij}}\right]_{N\times N}\left[\overline{w_{ij}}\right]_{N\times N} = \left(\overline{T}\right)\left(\overline{S}\right),$$

in $\mathcal{A}$. So, this conjugation (6.0.1) on a unital C^*-subalgebra $\mathcal{A}$ of $B(H)$ on a separable Hilbert space H is acting "like" an adjoint, but

$$\overline{TS} = \overline{T}\,\overline{S}, \text{ in } \mathcal{A}, \ \forall T, S \in \mathcal{A},$$

different from the usual adjoint $(*)$ on $\mathcal{A}$.

Proposition 34 *The conjugation* (6.0.1) *on a unital C^*-subalgebra $\mathcal{A}$ of $B(H)$ satisfies that*

$$\overline{\overline{T}} = T, \text{ and } \overline{zT} = \overline{z}\overline{T},$$

for all $T \in \mathcal{A}$ and $z \in \mathbb{C}$, and

$$\overline{T+S} = \overline{T} + \overline{S}, \text{ and } \overline{TS} = \overline{T}\,\overline{S},$$

for all $T, S \in \mathcal{A}$.

Proof The proof is done by the very above paragraph. □

Now, just like the scaled $\mathcal{A}$-hypercomplexes $\{\mathcal{H}_2^t(\mathcal{A})\}_{t\in\mathbb{R}}$ of (4.1), we define a following structure.

Definition 35 Let $\mathcal{A}$ be a unital C^*-subalgebra of the operator algebra $B(H)$ on a separable Hilbert space H. For any fixed $t \in \mathbb{R}$, define a $\mathbb{R}$-vector space,

$$\mathfrak{H}_2^t(\mathcal{A}) \overset{\text{def}}{=} \left\{ \begin{pmatrix} T_1 & tT_2 \\ \overline{T_2} & \overline{T_1} \end{pmatrix} : T_1, T_2 \in \mathcal{A} \right\},$$

of (2×2)-operator-block matrices, where $\overline{T}$ means the conjugate (6.0.1) of T in $\mathcal{A}$, equipped with the semi-norm,

$$\left\| \begin{pmatrix} T_1 & tT_2 \\ \overline{T_2} & \overline{T_1} \end{pmatrix} \right\|_{(t)} \overset{\text{def}}{=} \|(\|T_1\|_{\mathcal{A}}, \|T_2\|_{\mathcal{A}})\|_t .$$

We call the $\mathbb{R}$-SNS $\mathfrak{H}_2^t(\mathcal{A})$, the t-(scaled-)conjugate $\mathcal{A}$-hypercomplexes.

Note that, by the completeness of the C^*-norm $\|.\|_{\mathcal{A}}$ on $\mathcal{A}$, and the completeness of $\|.\|_t$ on the t-scaled hypercomplexes $\mathbb{H}_t$, the norm $\|.\|_{(t)}$ on the t-conjugate $\mathcal{A}$-hypercomplexes $\mathfrak{H}_2^t(\mathcal{A})$ is complete, i.e., $\mathfrak{H}_2^t(\mathcal{A})$ forms a complete semi-normed space, as a topological space. Just like Sect. 5, if there are no confusions, then we denote $\begin{pmatrix} T_1 & tT_2 \\ \overline{T_2} & \overline{T_1} \end{pmatrix}$ by $[(T_1, T_2)]_t$, i.e.,

$$\mathfrak{H}_2^t(\mathcal{A}) = \left\{ [(T_1, T_2)]_t : T_1, T_2 \in \mathcal{A} \right\}.$$

Observe that, if $[(T_1, T_2)]_t, [(S_1, S_2)]_t \in \mathfrak{H}_2^t(\mathcal{A})$ and $r_1, r_2 \in \mathbb{R}$, then

$$r_1\,[(T_1, T_2)]_t + r_2\,[(S_1, S_2)]_t = \begin{pmatrix} r_1 T_1 & t\,(r_1 T_2) \\ r_1\overline{T_2} & r_1\overline{T_1} \end{pmatrix} + \begin{pmatrix} r_2 S_1 & t\,(r_2 S_2) \\ r_2\overline{S_2} & r_2\overline{S_1} \end{pmatrix}$$

$$= \begin{pmatrix} r_1 T_1 + r_2 S_1 & t\,(r_1 T_2 + r_2 S_2) \\ \overline{r_1 T_2} + \overline{r_2 S_2} & \overline{r_1 T_1} + \overline{r_2 S_1} \end{pmatrix}$$

$$= \begin{pmatrix} r_1 T_1 + r_2 S_1 & t\,(r_1 T_2 + r_2 S_2) \\ \overline{r_1 T_2 + r_2 S_2} & \overline{r_1 T_1 + r_2 S_1} \end{pmatrix} \tag{6.0.2}$$

by (6.0.1), and hence, it is contained in $\mathfrak{H}_2^t(\mathcal{A})$, where $(+)$ is the usual block-operator-matrix addition, and

$$[(T_1, T_2)]_t [(S_1, S_2)]_t = \begin{pmatrix} T_1 & tT_2 \\ \overline{T_2} & \overline{T_1} \end{pmatrix} \begin{pmatrix} S_1 & tS_2 \\ \overline{S_2} & \overline{S_1} \end{pmatrix}$$

$$= \begin{pmatrix} T_1S_1 + tT_2\overline{S_2} & t\left(T_1S_2 + T_2\overline{S_1}\right) \\ \\ \overline{T_2}S_1 + \overline{T_1}\overline{S_2} & t\overline{T_2}S_2 + \overline{T_1}\overline{S_1} \end{pmatrix}$$

$$= \begin{pmatrix} T_1S_1 + tT_2\overline{S_2} & t\left(T_1S_2 + T_2\overline{S_1}\right) \\ \\ \overline{T_1S_2 + T_2\overline{S_1}} & \overline{T_1S_1 + tT_2\overline{S_2}} \end{pmatrix}, \tag{6.0.3}$$

by (6.0.1), showing that the product is also contained in $\mathfrak{H}_2^t(\mathcal{A})$, where $(\cdot)$ is the usual block-operator-matrix multiplication.

Theorem 36 *The t-conjugate $\mathcal{A}$-hypercomplexes $\mathfrak{H}_2^t(\mathcal{A})$ of a unital C^*-subalgebra $\mathcal{A}$ of the operator algebra $B(H)$ on a separable Hilbert space H is a complete $\mathbb{R}$-semi-normed algebra. i.e.,*

$$\mathfrak{H}_2^t(\mathcal{A}) \ \ \textit{is a complete } \mathbb{R} - \textit{semi} - \textit{normed algebra.} \tag{6.0.4}$$

Proof The t-conjugate $\mathcal{A}$-hypercomplexes $\mathfrak{H}_2^t(\mathcal{A})$ is a $\mathbb{R}$-vector space because the usual operator-block-matrix addition is closed on it by (6.0.2). So, as a complete semi-normed space, it forms a complete semi-normed $\mathbb{R}$-vector space. This $\mathbb{R}$-vector space $\mathfrak{H}_2^t(\mathcal{A})$ becomes an $\mathbb{R}$-algebra since the usual operator-block-matrix multiplication is closed on it by (6.0.3). Therefore, it is a complete $\mathbb{R}$-semi-normed algebra, i.e., the structure theorem (6.0.4) holds. □

By (6.0.4), we understand our t-conjugate $\mathcal{A}$-hypercomplexes $\mathfrak{H}_2^t(\mathcal{A})$ as a complete $\mathbb{R}$-semi-normed algebra. So, interestingly, without the commutativity assumption on a fixed unital C^*-subalgebra $\mathcal{A}$ of $B(H)$, one can consider the invertibility on $\mathfrak{H}_2^t(\mathcal{A})$, similar to, but different from Sect. 5.

Recall that the t-hypercomplex $\mathcal{A}$-hypercomplexes $\mathcal{H}_2^t(\mathcal{A})$ is isomorphic to the $\mathcal{A}$-$\mathcal{A}$ bimodule ${}_{\mathcal{A}}(\mathbb{H}_t)_{\mathcal{A}} = \alpha\left(\mathcal{A}^2\right)(\mathbb{H}_t)$, for any scale $t \in \mathbb{R}$, by (4.13). By the very construction of t-conjugate $\mathcal{A}$-hypercomplexes $\mathfrak{H}_2^t(\mathcal{A})$, we have a similar structure theorem like (4.13). Let $\mathcal{A}^2 = \mathcal{A} \times \mathcal{A}$ be the Cartesian-product C^*-algebra of two copies of a given unital C^*-algebra $\mathcal{A}$ (which is not necessarily commutative). And define an action,

$$\beta : \mathcal{A}^2 \to B_{\mathbb{R}}\left(\mathbb{H}_t, \mathfrak{H}_2^t(\mathcal{A})\right),$$

by

$$\beta(T_1, T_2)(a, b) = [aT_1, bT_2]_t = \begin{pmatrix} aT_1 & t(bT_2) \\ \overline{bT_2} & \overline{aT_1} \end{pmatrix},$$

in $\mathfrak{H}_2^t(\mathcal{A})$, for all $(T_1, T_2) \in \mathcal{A}^2$, and $(a, b) \in \mathbb{H}_t$. i.e., $\beta(T_1, T_2) \in B_{\mathbb{R}}\left(\mathcal{A}^2, \mathfrak{H}_2^t(\mathcal{A})\right)$, where $B_{\mathbb{R}}\left(\mathcal{A}^2, \mathfrak{H}_2^t(\mathcal{A})\right)$ is the operator space of all bounded $\mathbb{R}$-linear transformations from $\mathcal{A}^2$ into $\mathfrak{H}_2^t(\mathcal{A})$ over $\mathbb{R}$. Then, similar to the proof of (4.9), the morphism β is a well-defined bounded bi-module action from $\mathcal{A}^2$ acting on our t-conjugate $\mathcal{A}$-hypercomplexes $\mathfrak{H}_2^t(\mathcal{A})$. So, similar to (4.13), we obtain the following result.

Theorem 37 *The t-conjugate $\mathcal{A}$-hypercomplexes $\mathfrak{H}_2^t(\mathcal{A})$ of a unital C^*-algebra $\mathcal{A}$ is isomorphic to the $\mathcal{A}$-$\mathcal{A}$ bimodule $\beta\left(\mathcal{A}^2\right)(\mathbb{H}_t)$, i.e.,*

$$\mathfrak{H}_2^t(\mathcal{A}) \overset{\text{iso}}{=}_{\mathcal{A}} (\mathbb{H}_t)_{\mathcal{A}} \overset{\text{denote}}{=} \beta\left(\mathcal{A}^2\right)(\mathbb{H}_t).$$

Proof The proof is similar to that of (4.13). □

The above theorem shows a relation between the scaled hypercomplexes $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$ and scaled-conjugate $\mathcal{A}$-hypercomplexes $\left\{\mathfrak{H}_2^t(\mathcal{A})\right\}_{t\in\mathbb{R}}$. The difference between (4.13) and the above theorem is that a t-scaled $\mathcal{A}$-hypercomplexes $\mathcal{H}_2^t(\mathcal{A})$ is a $\mathbb{R}$-semi-normed "vector space" as a bimodule $_{\mathcal{A}}(\mathbb{H}_t)_{\mathcal{A}}$, meanwhile, a t-conjugate $\mathcal{A}$-hypercomplexes $\mathfrak{H}_2^t(\mathcal{A})$ is a $\mathbb{R}$-semi-normed "algebra" as a bimodule $_{\mathcal{A}}(\mathbb{H}_t)_{\mathcal{A}}$.

Definition 38 Let $\mathfrak{H}_2^t(\mathcal{A})$ be the t-conjugate $\mathcal{A}$-hypercomplexes for a scale $t \in \mathbb{R}$. An element $\eta \in \mathfrak{H}_2^t(\mathcal{A})$ is invertible "in $\mathfrak{H}_2^t(\mathcal{A})$" with its inverse $\eta^{-1} \in \mathfrak{H}_2^t(\mathcal{A})$, if

$$\eta\eta^{-1} = [(\mathbf{1},\mathbf{0})]_t = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} = \eta^{-1}\eta,$$

where $\mathbf{1}$ is the identity operator of $\mathcal{A}$, and $\mathbf{0}$ is the zero operator of $\mathcal{A}$, in $B(H)$.

Suppose $\eta = [(T_1, T_2)]_t$ is invertible in the t-conjugate $\mathcal{A}$-hypercomplexes $\mathfrak{H}_2^t(\mathcal{A})$ is invertible with its inverse $\eta^{-1} = [(S_1, S_2)]_t \in \mathfrak{H}_2^t(\mathcal{A})$. Then

$$\eta\eta^{-1} = \begin{pmatrix} T_1S_1 + tT_2\overline{S_2} & t\left(T_1S_2 + T_2\overline{S_1}\right) \\ \overline{T_1S_2 + T_2\overline{S_1}} & \overline{T_1S_1 + tT_2\overline{S_2}} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix},$$

and

$$\eta^{-1}\eta = \begin{pmatrix} S_1T_1 + tS_2\overline{T_2} & t\left(S_1T_2 + S_2\overline{T_1}\right) \\ \overline{S_2T_2 + S_2\overline{T_1}} & \overline{S_1T_1 + tS_2\overline{T_2}} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix},$$

if and only if

$$T_1S_1 + tT_2\overline{S_2} = \mathbf{1} = S_1T_1 + tS_2\overline{T_2},$$

and

$$T_1S_2 + T_2\overline{S_1} = \mathbf{0} = S_1T_2 + S_2\overline{T_1}, \text{ in } \mathcal{A}. \tag{6.0.5}$$

Proposition 39 *An element $[(T_1, T_2)]_t$ is invertible with its inverse $[(S_1, S_2)]_t$ in the t-conjugate $\mathcal{A}$-hypercomplexes $\mathfrak{H}_2^t(\mathcal{A})$, if and only if*

$$T_1S_1 + tT_2\overline{S_2} = \mathbf{1} = S_1T_1 + tS_2\overline{T_2},$$

and

$$T_1S_2 + T_2\overline{S_1} = \mathbf{0} = S_1T_2 + S_2\overline{T_1}, \text{ in } \mathcal{A}. \tag{6.0.6}$$

Proof The invertibility condition (6.0.6) on $\mathfrak{H}_2^t(\mathcal{A})$ is shown by (6.0.5). □

6.1 The case where $t = 0$

In this section, we fix the zero scale in $\mathbb{R}$, and the corresponding 0-conjugate $\mathcal{A}$-hypercomplexes $\mathfrak{H}_2^0(\mathcal{A})$ of a unital C^*-subalgebra $\mathcal{A}$ of the operator algebra $B(H)$ on a separable Hilbert space H, i.e.,

$$\mathfrak{H}_2^0(\mathcal{A}) = \left\{ [(T,S)]_0 = \begin{pmatrix} T & \mathbf{0} \\ \overline{S} & \overline{T} \end{pmatrix} : T, S \in \mathcal{A} \right\},$$

as a complete $\mathbb{R}$-semi-normed algebra by (6.0.4). By (6.0.6), an element $[(T_1, T_1)]_0$ is invertible with its inverse $[(S_1, S_2)]_0$ in $\mathfrak{H}_2^0(\mathcal{A})$, if and only if

$$T_1 S_1 = \mathbf{1} = S_1 T_1,$$

and

$$T_1 S_2 + T_2 \overline{S_1} = \mathbf{0} = S_1 T_2 + S_2 \overline{T_1}, \text{ in } \mathcal{A}. \tag{6.1.1}$$

The first formula of (6.1.1) implies that T_1 is invertible in $\mathcal{A}$, with its inverse $T_1^{-1} = S_1$. So, the invertibility condition (6.1.1) is equivalent to

$$S_1 = T_1^{-1}, \text{ and } T_1 S_2 + T_2 \overline{S_1} = \mathbf{0} = S_1 T_2 + S_2 \overline{T_1}, \tag{6.1.2}$$

in $\mathcal{A}$, if and only if

$$S_1 = T_1^{-1}, \; T_1 S_2 = -T_2 \overline{T_1^{-1}}, \text{ and } S_2 \overline{T_1} = -T_1^{-1} T_2,$$

if and only if

$$S_1 = T_1^{-1}, \; S_2 = -T_1^{-1} T_2 \overline{T_1^{-1}}, \text{ in } \mathcal{A}, \tag{6.1.3}$$

by (6.1.2).

Theorem 40 *An element* $[(T_1, T_2)]_0$ *is invertible in* $\mathfrak{H}_2^0(\mathcal{A})$, *if and only if*

T_1 *is invertible with its inverse* T_1^{-1} in $\mathcal{A}$,

and

$$[(T_1, T_2)]_0^{-1} = \left[\left(T_1^{-1}, \; -T_1^{-1} T_2 \overline{T_1^{-1}} \right) \right]_0 \in \mathfrak{H}_2^0(\mathcal{A}). \tag{6.1.4}$$

Proof The proof of the invertibility condition (6.1.4) on $\mathfrak{H}_2^t(\mathcal{A})$ is done by (6.1.3). □

The above theorem shows that $[(T, S)]_0$ is invertible in $\mathfrak{H}_2^0(\mathcal{A})$, if and only if there exists the inverse,

$$[(T, S)]_0^{-1} = \left[\left(T^{-1}, \; -T^{-1} S \overline{T^{-1}} \right) \right]_0 \in \mathfrak{H}_2^0(\mathcal{A}),$$

by (6.1.4). It shows that if T is not invertible in $\mathcal{A}$, then $[(T, S)]_0$ cannot be invertible in $\mathfrak{H}_2^0(\mathcal{A})$. So, all elements $[(T, S)]_0$ are not invertible in $\mathfrak{H}_2^0(\mathcal{A})$, whenever T is not invertible in $\mathcal{A}$.

6.2 The case where $t \neq 0$

In Sect. 6.1, we characterize the invertibility condition on the 0-conjugate $\mathcal{A}$-hypercomplexes $\mathfrak{H}_2^0(\mathcal{A})$ of a unital C^*-subalgebra $\mathcal{A}$ of the operator algebra $B(H)$ on a separable Hilbert space H, by (6.1.4). In this section, we fix a non-zero scale $t \in \mathbb{R} \setminus \{0\}$, and study the invertibility on the corresponding t-conjugate $\mathcal{A}$-hypercomplexes $\mathfrak{H}_2^t(\mathcal{A})$. Throughout this section, any given scale t is automatically assumed to be non-zero in $\mathbb{R}$.

By (6.0.6), an element $[(T_1, T_2)]_t$ is invertible with its inverse $[(S_1, S_2)]_t$ in $\mathfrak{H}_2^t(\mathcal{A})$, if and only if

$$T_1 S_1 + t T_2 \overline{S_2} = \mathbf{1} = S_1 T_1 + t S_2 \overline{T_2},$$

and

$$T_1 S_2 + T_2 \overline{S_1} = \mathbf{0} = S_1 T_2 + S_2 \overline{T_1}, \text{ in } \mathcal{A}. \tag{6.2.1}$$

This condition (6.2.1) is equivalent to

$$T_2\overline{S_2} = \frac{1}{t}\left(\mathbf{1} - T_1S_1\right), \;\; S_2\overline{T_2} = \frac{1}{t}\left(\mathbf{1} - S_1T_1\right),$$

and

$$T_2\overline{S_1} = -T_1S_2, \;\; S_2\overline{T_1} = -S_1T_2, \text{ in } \mathcal{A}. \tag{6.2.2}$$

Note that, an operator T is invertible in $\mathcal{A}$, if and only if the conjugate $\overline{T}$ is also invertible in $\mathcal{A}$, because

$$\overline{T^{-1}}\,\overline{T} = \overline{T^{-1}T} = \overline{\mathbf{1}} = \mathbf{1} = \overline{\mathbf{1}} = \overline{TT^{-1}} = \overline{T}\,\overline{T^{-1}},$$

implyingthat

$$\overline{T}\,\overline{T^{-1}} = \mathbf{1} = \overline{T^{-1}}\,\overline{T}, \Longleftrightarrow \left(\overline{T}\right)^{-1} = \overline{T^{-1}}, \text{ in } \mathcal{A}. \tag{6.2.3}$$

Assume that T_2 (and hence, $\overline{T_2}$) is invertible in $\mathcal{A}$ (by (6.2.3)).

Take the second equality of the first line of (6.2.2), and the second equality of the second line of (6.2.2). Then we obtain that

$$S_2 = \frac{1}{t}\left(\mathbf{1} - S_1T_1\right)\overline{T_2^{-1}}, \;\; \text{by } (6.2.3),$$

and

$$S_1 = -S_2\overline{T_1}T_2^{-1}, \;\; \text{in } \mathcal{A}. \tag{6.2.4}$$

From the second formula of (6.2.4), one has

$$S_1 = -\left(\frac{1}{t}\left(\mathbf{1} - S_1T_1\right)\overline{T_2^{-1}}\right)\overline{T_1}T_2^{-1},$$

$\Longleftrightarrow$

$$S_1 = -\left(\frac{1}{t}\overline{T_2^{-1}} - \frac{1}{t}S_1T_1\overline{T_2^{-1}}\right)\overline{T_1}T_2^{-1},$$

$\Longleftrightarrow$

$$S_1 = -\frac{1}{t}\overline{T_2^{-1}}\,\overline{T_1}T_2^{-1} + \frac{1}{t}S_1T_1\overline{T_2^{-1}}\,\overline{T_1}T_2^{-1},$$

implying that

$$S_1 - \frac{1}{t}S_1T_1\overline{T_2^{-1}}\,\overline{T_1}T_2^{-1} = -\frac{1}{t}\overline{T_2^{-1}}\,\overline{T_1}T_2^{-1},$$

$\Longleftrightarrow$

$$S_1\left(\mathbf{1} - \frac{1}{t}T_1\overline{T_2^{-1}}\,\overline{T_1}T_2^{-1}\right) = -\frac{1}{t}\overline{T_2^{-1}}\,\overline{T_1}T_2^{-1}, \text{ in } \mathcal{A}. \tag{6.2.5}$$

Now, assume that

$$\mathbf{1} - \frac{1}{t}T_1\overline{T_2^{-1}}\,\overline{T_1}T_2^{-1} \text{ is invertible in } \mathcal{A}.$$

Then, by (6.2.5), we have that

$$S_1 = -\frac{1}{t}\overline{T_2^{-1}}\,\overline{T_1}T_2^{-1}\left(\mathbf{1} - \frac{1}{t}T_1\overline{T_2^{-1}}\,\overline{T_1}T_2^{-1}\right)^{-1}. \tag{6.2.6}$$

Therefore, by (6.2.4) and (6.2.6),

$$S_2 = \frac{1}{t}\left(\mathbf{1} - S_1T_1\right)\overline{T_2^{-1}},$$

and

$$S_1 = -\frac{1}{t}\overline{T_2^{-1}}\,\overline{T_1}T_2^{-1}\left(\mathbf{1} - \frac{1}{t}T_1\overline{T_2^{-1}}\,\overline{T_1}T_2^{-1}\right)^{-1},$$

implying that

$$S_2 = \frac{1}{t}\left(\mathbf{1} - \left(-\frac{1}{t}\overline{T_2^{-1}}\,\overline{T_1}T_2^{-1}\left(\mathbf{1} - \frac{1}{t}T_1\overline{T_2^{-1}}\,\overline{T_1}T_2^{-1}\right)^{-1}\right)T_1\right)\overline{T_2^{-1}}, \tag{6.2.7}$$

in $\mathcal{A}$.

Theorem 41 *Suppose T_2 and $\mathbf{1} - \frac{1}{t} T_1 \overline{T_2^{-1}} \overline{T_1} T_2^{-1}$ are invertible in a unital C^*-algebra $\mathcal{A}$. An element $[(T_1, T_2)]_t$ is invertible in the t-conjugate $\mathcal{A}$-hypercomplexes $\mathfrak{H}_2^t(\mathcal{A})$ with its inverse $[(S_1, S_2)]_t \in \mathfrak{H}_2^t(\mathcal{A})$, if and only if*

$$S_1 = -\frac{1}{t} \overline{T_2^{-1}} \overline{T_1} T_2^{-1} \left(\mathbf{1} - \frac{1}{t} T_1 \overline{T_2^{-1}} \overline{T_1} T_2^{-1} \right)^{-1},$$

and

$$S_2 = \frac{1}{t} \left(\mathbf{1} - \left(-\frac{1}{t} \overline{T_2^{-1}} \overline{T_1} T_2^{-1} \left(\mathbf{1} - \frac{1}{t} T_1 \overline{T_2^{-1}} \overline{T_1} T_2^{-1} \right)^{-1} \right) T_1 \right) \overline{T_2^{-1}}, \tag{6.2.8}$$

in $\mathcal{A}$.

Proof By (6.2.2), an element $[(T_1, T_2)]_t$ is invertible with its inverse $[(S_1, S_2)]_t$ in $\mathfrak{H}_2^t(\mathcal{A})$, if and only if

$$T_2 \overline{S_2} = \frac{1}{t} (\mathbf{1} - T_1 S_1), \;\; S_2 \overline{T_2} = \frac{1}{t} (\mathbf{1} - S_1 T_1),$$

and

$$T_2 \overline{S_1} = -T_1 S_2, \;\; S_2 \overline{T_1} = -S_1 T_2, \text{ in } \mathcal{A}.$$

Under the assumption that

$$T_2 \text{ and } \mathbf{1} - \frac{1}{t} T_1 \overline{T_2^{-1}} \overline{T_1} T_2^{-1} \text{ are invertible in } \mathcal{A},$$

we have

$$S_1 = -\frac{1}{t} \overline{T_2^{-1}} \overline{T_1} T_2^{-1} \left(\mathbf{1} - \frac{1}{t} T_1 \overline{T_2^{-1}} \overline{T_1} T_2^{-1} \right)^{-1},$$

and

$$S_2 = \frac{1}{t} \left(\mathbf{1} - \left(-\frac{1}{t} \overline{T_2^{-1}} \overline{T_1} T_2^{-1} \left(\mathbf{1} - \frac{1}{t} T_1 \overline{T_2^{-1}} \overline{T_1} T_2^{-1} \right)^{-1} \right) T_1 \right) \overline{T_2^{-1}},$$

in $\mathcal{A}$, by (6.2.6) and (6.2.7), respectively. Therefore, the invertibility condition (6.2.8) is obtained under hypothesis. □

The above theorem partially characterizes the invertibility (6.0.6), or (6.2.1) on the t-conjugate $\mathcal{A}$-hypercomplexes $\mathfrak{H}_2^t(\mathcal{A})$ under certain invertibility assumption on $\mathcal{A}$ by (6.2.8).

6.3 Summary and conclusion

In this section, we summarize the main results of Sects. 6.1 and 6.2. Let $\mathcal{A}$ be a unital C^*-subalgebra of the operator algebra $B(H)$ on a separable Hilbert space H, and let $t \in \mathbb{R}$, and $\mathfrak{H}_2^t(\mathcal{A})$, the corresponding t-conjugate $\mathcal{A}$-hypercomplexes.

Corollary 42 *If $t = 0$ in $\mathbb{R}$, then an element $[(T_1, T_2)]_0$ is invertible with its inverse $[(S_1, S_2)]_0$ in $\mathfrak{H}_2^0(\mathcal{A})$, if and only if*

T_1 is invertible in $\mathcal{A}$,

and

$$[(S_1, S_2)]_0 = \left[\left(T_1^{-1}, \; -T_1^{-1} T_2 \overline{T_1^{-1}} \right) \right]_0 \in \mathfrak{H}_2^0(\mathcal{A}). \tag{6.3.1}$$

Meanwhile, if $t \neq 0$ *in* $\mathbb{R}$*, then* $[(T_1, T_2)]_t$ *is invertible with its inverse* $[(S_1, S_2)]_t$ *in* $\mathfrak{H}_2^t(\mathcal{A})$*, if and only if*

$$T_2\overline{S_2} = \frac{1}{t}(\mathbf{1} - T_1S_1), \ S_2\overline{T_2} = \frac{1}{t}(\mathbf{1} - S_1T_1),$$

and

$$T_2\overline{S_1} = -T_1S_2, \ S_2\overline{T_1} = -S_1T_2, \text{ in } \mathcal{A}. \tag{6.3.2}$$

In particular, if T_2 *and* $\mathbf{1} - \frac{1}{t}T_1\overline{T_2^{-1}}\,\overline{T_1}T_2^{-1}$ *are invertible in* $\mathcal{A}$*, then*

$$S_1 = -\frac{1}{t}\overline{T_2^{-1}}\,\overline{T_1}T_2^{-1}\left(\mathbf{1} - \frac{1}{t}T_1\overline{T_2^{-1}}\,\overline{T_1}T_2^{-1}\right)^{-1},$$

and

$$S_2 = \frac{1}{t}\left(\mathbf{1} - \left(-\frac{1}{t}\overline{T_2^{-1}}\,\overline{T_1}T_2^{-1}\left(\mathbf{1} - \frac{1}{t}T_1\overline{T_2^{-1}}\,\overline{T_1}T_2^{-1}\right)^{-1}\right)T_1\right)\overline{T_2^{-1}}, \tag{6.3.3}$$

in $\mathcal{A}$.

Proof The invertibility (6.3.1) on $\mathfrak{H}_2^0(\mathcal{A})$ is shown by (6.1.4). The invertibility characterization (6.3.2) on $\left\{\mathfrak{H}_2^t(\mathcal{A})\right\}_{t\in\mathbb{R}\setminus\{0\}}$ holds by (6.2.2). The special case (6.3.3) of (6.3.2) is proven by (6.2.8). □

The above corollary provides the invertibility characterization on the scaled-conjugate $\mathcal{A}$-hypercomplexes $\left\{\mathfrak{H}_2^t(\mathcal{A})\right\}_{t\in\mathbb{R}}$ of a unital C^*-subalgebra $\mathcal{A}$ of the operator algebra $B(H)$ on a separable Hilbert space H.

7 The invertibility on $\mathbb{H}_t$ and on $\mathfrak{H}_2^t(\mathcal{A})$

In this section, we briefly consider the relation between the invertibility on the t-scaled hypercomplexes $\mathbb{H}_t$ and that on the t-conjugate $\mathcal{A}$-hypercomplexes $\mathfrak{H}_2^t(\mathcal{A})$ of a unital C^*-subalgebra $\mathcal{A}$ of the operator algebra $B(H)$ on a separable Hilbert space H, for $t \in \mathbb{R}$. Remember that, for any scale $t \in \mathbb{R}$, a t-scaled hypercomplex number $(a, b) \in \mathbb{H}_t$ is invertible, if and only if $(a, b) \in \mathbb{H}_t^{inv}$, if and only if

$$\det([(a, b)]_t) = |a|^2 - t\left|b\right|^2 \neq 0,$$

where $[(a, b)]_t = \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} \in \mathcal{H}_2^t$, by (2.2.2). Also, recall that the t-conjugate $\mathcal{A}$-hypercomplexes $\mathfrak{H}_2^t(\mathcal{A})$ is isomorphic to the $\mathcal{A}$-$\mathcal{A}$ bimodule,

$$_{\mathcal{A}}(\mathbb{H}_t)_{\mathcal{A}} \overset{\text{denote}}{=} \beta\left(\mathcal{A}^2\right)(\mathbb{H}_t),$$

as a $\mathbb{R}$-SNS, where

$$\beta(T_1, T_2)(a, b) = [(aT_1, bT_2)]_t = \begin{pmatrix} aT_1 & t(bT_2) \\ \overline{bT_2} & \overline{aT_1} \end{pmatrix},$$

in $\mathfrak{H}_2^t(\mathcal{A})$, for all $(T_1, T_2) \in \mathcal{A}^2$ and $(a, b) \in \mathbb{H}_t$.

Note that, for any arbitrary $[(T_1, T_2)]_t \in \mathfrak{H}_2^t(\mathcal{A})$, there exists at least one $(a, b) \in \mathbb{H}_t$, with $a, b \in \mathbb{C} \setminus \{0\}$, such that

$$[(T_1, T_2)]_t = \left[\left(a\left(\frac{1}{a}T_1\right), b\left(\frac{1}{b}T_2\right)\right)\right]_t = \beta\left(\frac{1}{a}T_1, \frac{1}{b}T_2\right)(a, b),$$

in $\mathfrak{H}_2^t(\mathcal{A})$. For instance,

$$[(\mathbf{0}, \mathbf{0})]_t = \beta(\mathbf{0}, \mathbf{0})(1, 1),$$

$$[(T, \mathbf{0})]_t = \beta(T, \mathbf{0})(1, 1),$$

and

$$[(T, S)]_t = \beta\left(\frac{1}{i}T, \frac{1}{i}S\right)(i, i),$$

etc.. Now, let $(a, b) \in \mathbb{H}_t^{inv}$ with its inverse,

$$(a, b)^{-1} = \left(\frac{\overline{a}}{|a|^2 - t\,|b|^2}, \frac{-b}{|a|^2 - t\,|b|^2}\right) \in \mathbb{H}_t,$$

by (2.2.2). Assume that the operators T_1 and T_2 are invertible in the C^*-algebra $\mathcal{A}$, with their inverses T_1^{-1} and T_2^{-1}, respectively. For $(a, b) \in \mathbb{H}_t^{inv}$, consider the element,

$$\mathbf{T} = \beta(T_1, T_2)(a, b) = [(aT_1, bT_2)]_t,$$

and

$$\mathbf{S} = \beta\left(T_1^{-1}, \overline{T_2^{-1}}\right)\left((a, b)^{-1}\right) = \left[\left(\frac{\overline{a}T_1^{-1}}{|a|^2 - t\,|b|^2}, \frac{-b\overline{T_2^{-1}}}{|a|^2 - t\,|b|^2}\right)\right]_t, \tag{7.1}$$

in the t-conjugate $\mathcal{A}$-hypercomplexes $\mathfrak{H}_2^t(\mathcal{A})$. Remark that the conjugate $\overline{T_2}$ is also invertible in $\mathcal{A}$, with its inverse $\overline{T_2}^{-1} = \overline{T_2^{-1}}$ in $\mathcal{A}$, and hence, the element $\mathbf{S}$ of (7.1) is well-determined in $\mathfrak{H}_2^t(\mathcal{A})$. Observe that

$$\mathbf{TS} = ([(aT_1, bT_2)]_t)\left(\left[\left(\frac{\overline{a}T_1^{-1}}{|a|^2 - t\,|b|^2}, \frac{-b\overline{T_2^{-1}}}{|a|^2 - t\,|b|^2}\right)\right]_t\right)$$

$$= \begin{pmatrix} aT_1 & tbT_2 \\ \overline{b}\overline{T_2} & \overline{a}\overline{T_1} \end{pmatrix} \begin{pmatrix} \frac{\overline{a}T_1^{-1}}{|a|^2-t|b|^2} & \frac{-tb\overline{T_2^{-1}}}{|a|^2-t|b|^2} \\ \frac{-\overline{b}T_2^{-1}}{|a|^2-t|b|^2} & \frac{a\overline{T_1^{-1}}}{|a|^2-t|b|^2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{|a|^2\mathbf{1}-t|b|^2\mathbf{1}}{|a|^2-t|b|^2} & t\frac{-abT_1\overline{T_2^{-1}}+abT_2\overline{T_1^{-1}}}{|a|^2-t|b|^2} \\ \overline{\frac{-abT_1\overline{T_2^{-1}}+abT_2\overline{T_1^{-1}}}{|a|^2-t|b|^2}} & \frac{|a|^2\mathbf{1}-t|b|^2\mathbf{1}}{|a|^2-t|b|^2} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{1} & t\frac{-abT_1\overline{T_2^{-1}}+abT_2\overline{T_1^{-1}}}{|a|^2-t|b|^2} \\ \overline{\frac{-abT_1\overline{T_2^{-1}}+abT_2\overline{T_1^{-1}}}{|a|^2-t|b|^2}} & \mathbf{1} \end{pmatrix}, \tag{7.2}$$

and, similarly, we have

$$\mathbf{ST} = \begin{pmatrix} \mathbf{1} & t\frac{-abT_1\overline{T_2^{-1}}+abT_2\overline{T_1^{-1}}}{|a|^2-t|b|^2} \\ \overline{\frac{-abT_1\overline{T_2^{-1}}+abT_2\overline{T_1^{-1}}}{|a|^2-t|b|^2}} & \mathbf{1} \end{pmatrix}, \tag{7.3}$$

in $\mathfrak{H}_2^t(\mathcal{A})$, i.e.,

$$\mathbf{TS} = \left[\left(\mathbf{1}, \frac{-abT_1\overline{T_2^{-1}} + abT_2\overline{T_1^{-1}}}{|a|^2 - t\,|b|^2}\right)\right]_t = \mathbf{ST}, \tag{7.4}$$

in $\mathfrak{H}_2^t(\mathcal{A})$ by (7.2) and (7.3), whenever $\mathbf{T}$ and $\mathbf{S}$ are in the sense of (7.1).

Define a subset $\mathcal{A}^{inv}$ of $\mathcal{A}$ by the set of all invertible operators of $\mathcal{A}$, i.e.,

$$\mathcal{A}^{inv} \overset{\text{def}}{=} \left\{T \in \mathcal{A} : \exists T^{-1} \text{ in } \mathcal{A}\right\}.$$

Lemma 43 *Let $(a, b) \in \mathbb{H}_t^{inv}$ in $\mathbb{H}_t$ for $t \in \mathbb{R}$, where*

$$a \neq 0 \text{ and } b \neq 0, \text{ in } \mathbb{C}.$$

If $T \in \mathcal{A}^{inv}$ in the fixed C^*-algebra $\mathcal{A}$, then $[(aT_1, bT_2)]_t$ is invertible in the t-conjugate $\mathcal{A}$-hypercomplexes $\mathfrak{H}_2^t(\mathcal{A})$, with its inverse,

$$\left[\left(aT, b\overline{T}\right)\right]_t^{-1} = \left[\left(\frac{\overline{a}T^{-1}}{|a|^2 - t\,|b|^2}, \frac{-b\overline{T}}{|a|^2 - t\,|b|^2}\right)\right]_t,$$

satisfying

$$\left(\beta\left(T, \overline{T}\right)(a, b)\right)^{-1} = \beta\left(T^{-1}, \overline{T}\right)\left((a, b)^{-1}\right), \tag{7.5}$$

in $\mathfrak{H}_2^t(\mathcal{A})$, where

$$(a, b)^{-1} = \left(\frac{\overline{a}}{|a|^2 - t\,|b|^2}, \frac{-b}{|a|^2 - t\,|b|^2}\right) \in \mathbb{H}_t.$$

Proof Let $(a, b) \in \mathbb{H}_t^{inv}$ be an invertible t-scaled hypercomplex number of $\mathbb{H}_t$, where

$$a,\ b \in \mathbb{C} \setminus \{0\}, \text{ in } \mathbb{C},$$

with its inverse,

$$(a, b)^{-1} = \left(\frac{\overline{a}}{|a|^2 - t\,|b|^2}, \frac{-b}{|a|^2 - t\,|b|^2}\right) \in \mathbb{H}_t.$$

For the invertible operators $T_1, T_2 \in \mathfrak{H}_2^t(\mathcal{A})$, if we let

$$\mathbf{T} = \beta(T_1, T_2)(a, b) = [(aT_1, bT_2)]_t,$$

and

$$\mathbf{S} = \beta\left(T_1^{-1}, \overline{T_2^{-1}}\right)\left((a, b)^{-1}\right) = \left[\left(\frac{\overline{a}T_1^{-1}}{|a|^2 - t\,|b|^2}, \frac{-b\overline{T_2^{-1}}}{|a|^2 - t\,|b|^2}\right)\right]_t,$$

in $\mathfrak{H}_2^t(\mathcal{A})$, then

$$\mathbf{TS} = \left[\left(\mathbf{1}, \frac{-abT_1\overline{T_2^{-1}} + abT_2\overline{T_1^{-1}}}{|a|^2 - t\,|b|^2}\right)\right]_t = \mathbf{ST},$$

in $\mathfrak{H}_2^t(\mathcal{A})$, by (7.3) and (7.4). It shows that

$$T_1\overline{T_2^{-1}} = \mathbf{1} = T_2\overline{T_1^{-1}} \text{ in } \mathcal{A},$$

if and only if

$$\mathbf{TS} = [(\mathbf{1}, \mathbf{0})]_t = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} = \mathbf{ST},$$

in $\mathfrak{H}_2^t(\mathcal{A})$. Equivalently,

$$T_2^{-1} = \overline{T_1} \text{ in } \mathcal{A} \Longleftrightarrow \mathbf{TS} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} = \mathbf{ST} \text{ in } \mathfrak{H}_2^t(\mathcal{A}),$$

if and only if

$$T_2^{-1} = \overline{T_1} \text{ in } \mathcal{A} \Longleftrightarrow \mathbf{T}^{-1} = \mathbf{S} \text{ in } \mathfrak{H}_2^t(\mathcal{A}). \tag{7.6}$$

By (7.6), if $(a, b) \in \mathbb{H}_t^{inv}$ and $T \in \mathcal{A}^{inv}$, with $T^{-1} = \overline{T}$ in $\mathcal{A}$, then the element $\left[\left(aT, b\overline{T}\right)\right]_t$ is invertible in $\mathfrak{H}_2^t(\mathcal{A})$, with

$$\left[\left(aT, b\overline{T}\right)\right]_t^{-1} = \left[\left(\frac{\overline{a}T^{-1}}{|a|^2 - t\,|b|^2}, \frac{-b\overline{T}}{|a|^2 - t\,|b|^2}\right)\right]_t .$$

Therefore, the invertibility condition (7.5) holds on $\mathfrak{H}_2^t(\mathcal{A})$. □

Similar to (7.5), one can get the following result.

Lemma 44 *If* $(a, b) = (0, b) \in \mathbb{H}_t^{inv}$ *is an invertible t-scaled hypercomplex number of* $\mathbb{H}_t$*, where*

$$a = 0 \text{ and } b \neq 0, \text{ in } \mathbb{C},$$

then

$$\left[\left(0 \cdot T, \, b\overline{T}\right)\right]_t^{-1} = \left[\left(\mathbf{0}, b\overline{T}\right)\right]_t^{-1} = \left[\left(\mathbf{0}, \frac{bT^{-1}}{t\,|b|^2}\right)\right]_t \in \mathfrak{H}_2^t(\mathcal{A}),$$

equivalently,

$$\left(\beta\left(T, \overline{T}\right)(0, b)\right)^{-1} = \beta\left(\mathbf{0}, T^{-1}\right)\left((0, b)^{-1}\right), \text{ in } \mathfrak{H}_2^t(\mathcal{A}). \tag{7.7}$$

Proof If $(a, b) \in \mathbb{H}_t^{inv}$ is invertible in $\mathbb{H}_t$, with $a = 0$ and $b \neq 0$ in $\mathbb{C}$, then

$$(a, b)^{-1} = (0, b)^{-1} = \left(\frac{\overline{0}}{|0|^2 - t\,|b|^2}, \frac{-b}{|0|^2 - t\,|b|^2}\right) = \left(0, \frac{b}{t\,|b|^2}\right),$$

in $\mathbb{H}_t$. If $T \in \mathcal{A}^{inv}$ in $\mathcal{A}$, then

$$\left[\left(0 \cdot T, b\overline{T}\right)\right]_t^{-1} = \left[\left(\mathbf{0}, b\overline{T}\right)\right]_t^{-1} = \left[\left(\mathbf{0}, \frac{bT^{-1}}{t\,|b|^2}\right)\right]_t,$$

in $\mathfrak{H}_2^t(\mathcal{A})$. Thus, the invertibility (7.7) holds. Indeed,

$$\left[\left(\mathbf{0}, b\overline{T}\right)\right]_t \left[\left(\mathbf{0}, \frac{bT^{-1}}{t\,|b|^2}\right)\right]_t = \begin{pmatrix} \mathbf{0} & tb\overline{T} \\ \overline{b}T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \frac{bT^{-1}}{|b|^2} \\ \frac{\overline{bT^{-1}}}{t|b|^2} & \mathbf{0} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \frac{bT^{-1}}{|b|^2} \\ \frac{\overline{bT^{-1}}}{t|b|^2} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{0} & tb\overline{T} \\ \overline{b}T & \mathbf{0} \end{pmatrix}$$

$$= \left[\left(\mathbf{0}, \frac{bT^{-1}}{t\,|b|^2}\right)\right]_t \left[\left(\mathbf{0}, b\overline{T}\right)\right]_t,$$

in the t-conjugate $\mathcal{A}$-hypercomplexes $\mathfrak{H}_2^t(\mathcal{A})$. □

Just like (7.7), we obtain the following result.

Lemma 45 *If* $(a, b) = (a, 0) \in \mathbb{H}_t^{inv}$ *is an invertible t-scaled hypercomplex number of* $\mathbb{H}_t$*, where*

$$a \neq 0 \text{ and } b = 0, \text{ in } \mathbb{C},$$

then

$$\left[\left(aT,\ 0\cdot\overline{T}\right)\right]_t^{-1} = \left[(aT,\mathbf{0})\right]_t^{-1} = \left[\left(\frac{\overline{a}T^{-1}}{|a|^2},\ \mathbf{0}\right)\right]_t \in \mathfrak{H}_2^t\left(\mathcal{A}\right),$$

equivalently,

$$\left(\beta\left(T,\overline{T}\right)(0,b)\right)^{-1} = \beta\left(T^{-1},\mathbf{0}\right)\left((a,0)^{-1}\right),\ \text{in}\ \mathfrak{H}_2^t\left(\mathcal{A}\right). \tag{7.8}$$

Proof If $(a,b)\in\mathbb{H}_t^{inv}$ is invertible in $\mathbb{H}_t$, with $a\neq 0$ and $b=0$ in $\mathbb{C}$, then

$$(a,b)^{-1} = (a,0)^{-1} = \left(\frac{\overline{a}}{|a|^2 - t\,|0|^2},\ \frac{-0}{|a|^2 - t\,|0|^2}\right) = \left(\frac{\overline{a}}{|a|^2},\ 0\right),$$

in $\mathbb{H}_t$. If $T\in\mathcal{A}^{inv}$ in $\mathcal{A}$, then

$$\left[\left(aT,\ 0\cdot\overline{T}\right)\right]_t^{-1} = \left[(aT,\mathbf{0})\right]_t^{-1} = \left[\left(\frac{\overline{a}T^{-1}}{|a|^2},\ \mathbf{0}\right)\right]_t,$$

in $\mathfrak{H}_2^t\left(\mathcal{A}\right)$. Thus, the invertibility (7.8) holds. Indeed,

$$\left[(aT,\mathbf{0})\right]_t\left[\left(\frac{\overline{a}T^{-1}}{|a|^2},\ \mathbf{0}\right)\right]_t = \begin{pmatrix} aT & \mathbf{0} \\ \mathbf{0} & a\overline{T} \end{pmatrix}\begin{pmatrix} \frac{\overline{a}T^{-1}}{|a|^2} & \mathbf{0} \\ \mathbf{0} & \frac{a\overline{T^{-1}}}{|a|^2} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} = \begin{pmatrix} \frac{\overline{a}T^{-1}}{|a|^2} & \mathbf{0} \\ \mathbf{0} & \frac{a\overline{T^{-1}}}{|a|^2} \end{pmatrix}\begin{pmatrix} aT & \mathbf{0} \\ \mathbf{0} & a\overline{T} \end{pmatrix}$$

$$= \left[\left(\frac{\overline{a}T^{-1}}{|a|^2},\ \mathbf{0}\right)\right]_t \left[(aT,\mathbf{0})\right]_t,$$

in the t-conjugate $\mathcal{A}$-hypercomplexes $\mathfrak{H}_2^t\left(\mathcal{A}\right)$. □

By summarizing the above main results (7.5), (7.7) and (7.8), one has the following theorem.

Theorem 46 *Let* $(a,b)\in\mathbb{H}_t^{inv}$ *be invertible in the t-scaled hypercomplexes* $\mathbb{H}_t$*, with its inverse,*

$$(a,b)^{-1} = \left(\frac{\overline{a}}{|a|^2 - t\left|b\right|^2},\ \frac{-b}{|a|^2 - t\left|b\right|^2}\right)\in\mathbb{H}_t,$$

and let $T\in\mathcal{A}^{inv}$ *in* $\mathcal{A}$*. If* $a,b\in\mathbb{C}\setminus\{0\}$*, then*

$$\left(\beta\left(T,\overline{T}\right)(a,b)\right)^{-1} = \beta\left(T^{-1},\overline{T}\right)\left((a,b)^{-1}\right) \tag{7.9}$$

and if $a=0$ *and* $b\neq 0$ *in* $\mathbb{C}$*, then*

$$\left(\beta\left(T,\overline{T}\right)(a,b)\right)^{-1} = \beta\left(\mathbf{0},T^{-1}\right)\left((a,b)^{-1}\right) \tag{7.10}$$

and if $a\neq 0$ *and* $b=0$ *in* $\mathbb{C}$*, then*

$$\left(\beta\left(T,\overline{T}\right)(a,b)\right)^{-1} = \beta\left(T^{-1},\mathbf{0}\right)\left((a,b)^{-1}\right),\ \text{in}\ \mathfrak{H}_2^t\left(\mathcal{A}\right) \tag{7.11}$$

Proof The invertibility conditions (7.9), (7.10) and (7.11) are shown by (7.5), (7.7) and (7.8), respectively. □

The above theorem shows a relation among the invertibility on the scaled hypercomplexes $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$, the invertibility on $\mathcal{A}$, and that on the scaled-conjugate $\mathcal{A}$-hypercomplexes $\left\{\mathfrak{H}_2^t\left(\mathcal{A}\right)\right\}_{t\in\mathbb{R}}$ by (7.9), (7.10) and (7.11), where $\mathcal{A}$ is a unital C^*-subalgebra of the operator algebra $B\left(H\right)$ on a separable Hilbert space H.

8 The invertibility on (2 × 2)-Block Operators and That on $\mathfrak{H}_2^t(\mathcal{A})$

In this section, we confirm that our invertibility on the t-conjugate $\mathcal{A}$-hypercomplexes $\mathfrak{H}_2^t(\mathcal{A})$ is determined by the invertibility on the (2×2)-block operators $M_2(\mathcal{A})$ in the canonical sense of Chapter 3 of [1] over a unital C^*-subalgebra of the operator algebra $B(H)$ on a separable Hilbert space H, where

$$M_2(\mathcal{A}) = \left\{ \left[A_{ij}\right]_{2\times 2} \overset{\text{denote}}{=} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} : A_{11}, A_{12}, A_{21}, A_{22} \in \mathcal{A} \right\}.$$

The following proposition is know (e.g., see [1]).

Proposition 47 *Suppose operators A, D, $A - BD^{-1}C \in \mathcal{A}^{inv}$ are invertible in $\mathcal{A}$. Then a (2×2)-block operator $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_2(\mathcal{A})$ is invertible with its inverse $\begin{pmatrix} U & V \\ W & Z \end{pmatrix}$, if and only if*

$$U = \left(A - BD^{-1}C\right)^{-1},$$
$$V = -\left(A - BD^{-1}C\right)^{-1} BD^{-1},$$
$$W = -D^{-1}C\left(A - BD^{-1}C\right)^{-1},$$

and

$$Z = D^{-1} + D^{-1}C\left(A - BD^{-1}C\right)^{-1} BD^{-1}, \tag{8.1}$$

in $\mathcal{A}$. i.e., under the hypothesis, the inverse $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}$ is

$$\begin{pmatrix} \left(A - BD^{-1}C\right)^{-1} & -\left(A - BD^{-1}C\right)^{-1} BD^{-1} \\ -D^{-1}C\left(A - BD^{-1}C\right)^{-1} & D^{-1} + D^{-1}C\left(A - BD^{-1}C\right)^{-1} BD^{-1} \end{pmatrix},$$

in $M_2(\mathcal{A})$.

Proof See e.g., the formula (3.2.8) in Chapter 3 of [1]. □

With respect to (8.1), we consider a connection between the invertibility on the (2×2)-block-operator algebra $M_2(\mathcal{A})$ over $\mathcal{A}$, and that on the t-conjugate $\mathcal{A}$-hypercomplexes $\mathfrak{H}_2^t(\mathcal{A})$, for an arbitrary scale $t \in \mathbb{R}$. Recall that, by (6.1.4),

$$[(T_1, T_2)]_0^{-1} = \left[\left(T_1^{-1}, \; -T_1^{-1} T_2 \overline{T_1^{-1}}\right)\right]_0 \in \mathfrak{H}_2^0(\mathcal{A}) \tag{8.2}$$

and, by (6.2.8), if $[(T_1, T_2)]_t$ is invertible in $\mathfrak{H}_2^t(\mathcal{A})$ with its inverse $[(S_1, S_2)]_t \in \mathfrak{H}_2^t(\mathcal{A})$, then

$$S_1 = -\frac{1}{t}\overline{T_2^{-1}}\,\overline{T_1} T_2^{-1} \left(\mathbf{1} - \frac{1}{t} T_1 \overline{T_2^{-1}}\,\overline{T_1} T_2^{-1}\right)^{-1},$$

and

$$S_2 = \frac{1}{t}\left(\mathbf{1} - \left(-\frac{1}{t}\overline{T_2^{-1}}\,\overline{T_1} T_2^{-1} \left(\mathbf{1} - \frac{1}{t} T_1 \overline{T_2^{-1}}\,\overline{T_1} T_2^{-1}\right)^{-1}\right) T_1\right) \overline{T_2^{-1}}, \tag{8.3}$$

in $\mathcal{A}$, under suitable invertibility assumptions.

Assumption and Notation 8.1. (From below, **AN 8.1**) In the rest of this section, if we express "a certain formula holds under suitable invertibility assumptions," then it means that "if we write the inverse notation A^{-1} for an operator A in a fixed C^*-algebra $\mathcal{A}$, then it automatically assumed that A is invertible in $\mathcal{A}$ with its inverse $A^{-1} \in \mathcal{A}$." For instance, as in the above paragraph, "the formulas (8.3) holds under suitable invertibility assumptions" means that "the formula (8.3) holds by assuming that

$$T_1,\ T_2, \text{ and } \mathbf{1} - \frac{1}{t} T_1 \overline{T_2^{-1}}\, \overline{T_1} T_2^{-1}$$

are invertible in $\mathcal{A}$." □

Now, assume that $t = 0$ in $\mathbb{R}$, and let

$$A = T_1,\ B = 0 \cdot T_2 = \mathbf{0},\ C = \overline{T_2}, \text{ and } D = \overline{T_1},\ \ in\ \mathcal{A} \tag{8.4}$$

Then, if

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \overset{\text{let}}{=} \begin{pmatrix} T_1 & \mathbf{0} \\ \overline{T_2} & \overline{T_1} \end{pmatrix} = [(T_1, T_2)]_0 \in \mathfrak{H}_2^t(\mathcal{A}),$$

is invertible "in $M_2(\mathcal{A})$," then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} U & V \\ W & Z \end{pmatrix} \in M_2(\mathcal{A}),$$

where U, V, W and Z satisfy (8.1) under suitable invertibility assumption on $\mathcal{A}$, and hence,

$$U = \left(T_1 - \mathbf{0}\overline{T_1}^{-1}\overline{T_2}\right)^{-1} = T_1^{-1},$$

$$V = -\left(T_1 - \mathbf{0}\overline{T_1}^{-1}\overline{T_2}\right)^{-1} \mathbf{0}\overline{T_1}^{-1} = \mathbf{0},$$

$$W = -\overline{T_1}^{-1}\overline{T_2}\left(T_1 - \mathbf{0}\overline{T_1}^{-1}\overline{T_2}\right)^{-1} = -\overline{T_1}^{-1}\overline{T_2}T_1^{-1},$$

and

$$Z = \overline{T_1}^{-1} + \overline{T_1}^{-1}\overline{T_2}\left(T_1 - \mathbf{0}\overline{T_1}^{-1}\overline{T_2}\right)^{-1}\mathbf{0}\overline{T_1}^{-1} = \overline{T_1}^{-1}, \tag{8.5}$$

in $\mathcal{A}$, by (8.1) and (8.4).

Theorem 48 *Under suitable invertibility assumptions (in the sense of* **AN 8.1***), the invertibility on the* 0*-conjugate* $\mathcal{A}$*-hypercomplexes* $\mathfrak{H}_2^0(\mathcal{A})$ *and the invertibility on the algebra* $M_2(\mathcal{A})$ *are equivalent, i.e.,*

$$\textit{The invertibility on } \mathfrak{H}_2^0(\mathcal{A}) \overset{equi}{=} \textit{ the invertibility on } M_2(\mathcal{A}) \tag{8.6}$$

Proof Under suitable invertibility assumptions, by the invertibility (8.1) on $M_2(\mathcal{A})$, if

$$[(T_1, T_2)]_0 = \begin{pmatrix} T_1 & \mathbf{0} \\ \overline{T_2} & \overline{T_1} \end{pmatrix} \in \mathfrak{H}_2^0(\mathcal{A}) \subset M_2(\mathcal{A})$$

is invertible "in $M_2(\mathcal{A})$," then

$$[(T_1, T_2)]_0^{-1} = \begin{pmatrix} T_1^{-1} & \mathbf{0} \\ -\overline{T_1^{-1}}\overline{T_2}T_1^{-1} & \overline{T_1^{-1}} \end{pmatrix} \overset{\text{denote}}{=} \mathbf{U} \in M_2(\mathcal{A}),$$

by (8.5), because

$$\overline{A}^{-1} = \overline{A^{-1}}, \text{ in } \mathcal{A}, \text{ if } A \text{ is invertible in } \mathcal{A}.$$

It shows that the inverse $\mathbf{U} \in M_2(\mathcal{A})$ of $[(T_1, T_2)]_0 \in \mathfrak{H}_2^0(\mathcal{A})$ is identified with

$$\mathbf{U} = \begin{pmatrix} T_1^{-1} & 0 \cdot \left(-T_1^{-1} T_2 \overline{T_1^{-1}}\right) \\ \overline{-T_1^{-1} T_2 \overline{T_1^{-1}}} & \overline{T_1^{-1}} \end{pmatrix} \in M_2(\mathcal{A}),$$

contained "in $\mathfrak{H}_2^0(\mathcal{A})$," as

$$\mathbf{U} = \left[\left(T_1^{-1}, -T_1^{-1} T_2 \overline{T_1^{-1}}\right)\right]_0 \in \mathfrak{H}_2^0(\mathcal{A}).$$

Therefore, the invertibility on $M_2(\mathcal{A})$ implies the invertibility on $\mathfrak{H}_2^0(\mathcal{A})$ under suitable invertibility assumptions.

Since $\mathfrak{H}_2^0(\mathcal{A})$ is a subalgebra of $M_2(\mathcal{A})$ by definition, the invertibility on $\mathfrak{H}_2^0(\mathcal{A})$ implies that on $M_2(\mathcal{A})$. Therefore, the equivalence (8.6) holds. □

The above theorem shows that the invertibility on the 0-conjugate $\mathcal{A}$-hypercomplexes $\mathfrak{H}_2^0(\mathcal{A})$ and that on $M_2(\mathcal{A})$ are equivalent under suitable invertibility assumptions by (8.6).

Now, assume that $t \neq 0$ in $\mathbb{R}$, and let

$$A = T_1,\ B = tT_2,\ C = \overline{T_2},\ \text{and}\ D = \overline{T_1},\ \ in\ \mathcal{A} \tag{8.7}$$

Then, if

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \overset{\text{let}}{=} \begin{pmatrix} T_1 & tT_2 \\ \overline{T_2} & \overline{T_1} \end{pmatrix} = [(T_1, T_2)]_t \in \mathfrak{H}_2^t(\mathcal{A}),$$

is invertible "in $M_2(\mathcal{A})$," then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} U & V \\ W & Z \end{pmatrix} \in M_2(\mathcal{A}),$$

where U, V, W and Z satisfy (8.1);

$$U = \left(A - BD^{-1}C\right)^{-1},$$
$$V = -\left(A - BD^{-1}C\right)^{-1} BD^{-1},$$
$$W = -D^{-1}C\left(A - BD^{-1}C\right)^{-1},$$

and

$$Z = D^{-1} + D^{-1}C\left(A - BD^{-1}C\right)^{-1} BD^{-1},$$

in $\mathcal{A}$, equivalently,

$$U = \left(T_1 - tT_2\overline{T_1}^{-1}\overline{T_2}\right)^{-1},$$
$$V = -\left(T_1 - tT_2\overline{T_1}^{-1}\overline{T_2}\right)^{-1}(tT_2)\,\overline{T_1}^{-1},$$
$$W = -\overline{T_1}^{-1}\overline{T_2}\left(T_1 - tT_2\overline{T_1}^{-1}\overline{T_2}\right)^{-1},$$

and

$$Z = \overline{T_1}^{-1} + \overline{T_2}\left(T_1 - tT_2\overline{T_1}^{-1}\overline{T_2}\right)^{-1}(tT_2)\,\overline{T_1}^{-1}, \tag{8.8}$$

in $\mathcal{A}$, under suitable invertibility assumptions in $\mathcal{A}$, by (8.1) and (8.7). Recall that, by (8.3), under suitable invertibility assumptions, if $[(T_1, T_2)]_t$ is invertible with its inverse $[(S_1, S_2)]_t$ "in $\mathfrak{H}_2^t(\mathcal{A})$," then

$$S_1 = -\frac{1}{t}\overline{T_2^{-1}}\,\overline{T_1}T_2^{-1}\left(\mathbf{1} - \frac{1}{t}T_1\overline{T_2^{-1}}\,\overline{T_1}T_2^{-1}\right)^{-1},$$

and

$$S_2 = \frac{1}{t}\left(\mathbf{1} - \left(-\frac{1}{t}\overline{T_2^{-1}}\,\overline{T_1}T_2^{-1}\left(\mathbf{1} - \frac{1}{t}T_1\overline{T_2^{-1}}\,\overline{T_1}T_2^{-1}\right)^{-1}\right)T_1\right)\overline{T_2^{-1}}, \tag{8.9}$$

in $\mathcal{A}$. From the first formula of (8.9), one has that

$$\begin{aligned}
S_1 &= -\frac{1}{t}\overline{T_2^{-1}}\,\overline{T_1}T_2^{-1}\left(\mathbf{1} - \frac{1}{t}T_1\overline{T_2^{-1}}\,\overline{T_1}T_2^{-1}\right)^{-1} \\
&= -\frac{1}{t}\overline{T_2^{-1}}\,\overline{T_1}\left(T_2 - \frac{1}{t}T_1\overline{T_2^{-1}}\,\overline{T_1}T_2^{-1}T_2\right)^{-1} \\
&= -\frac{1}{t}\overline{T_2^{-1}}\,\overline{T_1}\left(T_2 - \frac{1}{t}T_1\overline{T_2^{-1}}\,\overline{T_1}\right)^{-1} \\
&= -\frac{1}{t}\overline{T_2^{-1}}\left(T_2\overline{T_1}^{-1} - \frac{1}{t}T_1\overline{T_2^{-1}}\,\overline{T_1}\,\overline{T_1}^{-1}\right)^{-1} \\
&= -\frac{1}{t}\overline{T_2^{-1}}\left(T_2\overline{T_1}^{-1} - \frac{1}{t}T_1\overline{T_2^{-1}}\right)^{-1} \\
&= -\frac{1}{t}\left(T_2\overline{T_1}^{-1}\overline{T_2} - \frac{1}{t}T_1\overline{T_2^{-1}}\,\overline{T_2}\right)^{-1} \\
&= \frac{1}{t}\left(-T_2\overline{T_1}^{-1}\overline{T_2} + \frac{1}{t}T_1\right)^{-1} = \left(-tT_2\overline{T_1}^{-1}\overline{T_2} + T_1\right)^{-1} = U,
\end{aligned} \tag{8.10}$$

where U is in the sense of (8.8), i.e., the S_1 of (8.9) is identical to U of (8.8) in $\mathcal{A}$, by (8.10). Now, let

$$\underset{\text{in } (8.9)}{S_1} = \underset{\text{in } (8.8)}{U} = \left(T_1 - tT_2\overline{T_1}^{-1}\overline{T_2}\right)^{-1} \in \mathcal{A},$$

by (8.10).

Also, in the second formula of (8.8), one has

$$V = -\left(T_1 - tT_2\overline{T_1}^{-1}\overline{T_2}\right)^{-1}(tT_2)\,\overline{T_1}^{-1},$$

and by (8.9),

$$S_2 = \frac{1}{t}\left(\mathbf{1} - \left(-\frac{1}{t}\overline{T_2^{-1}}\,\overline{T_1}T_2^{-1}\left(\mathbf{1} - \frac{1}{t}T_1\overline{T_2^{-1}}\,\overline{T_1}T_2^{-1}\right)^{-1}\right)T_1\right)\overline{T_2^{-1}},$$

in $\mathcal{A}$, under suitable invertibility assumptions.

Proposition 49 *Let $A, B \in \mathcal{A}$ be invertible elements of a fixed C^*-subalgebra $\mathcal{A}$ of the operator algebra $B(H)$ on a separable Hilbert space H. Then*

$$\left(A - \overline{BA}^{-1}B\right)^{-1} = A^{-1} + A^{-1}\overline{B}\left(\overline{A} - BA^{-1}\overline{B}\right)^{-1}BA^{-1},$$

and

$$\overline{A}^{-1}B\left(\overline{BA}^{-1}B - A\right)^{-1} = \left(BA^{-1}\overline{B} - \overline{A}\right)^{-1}BA^{-1}, \text{ in } \mathcal{A}. \tag{8.11}$$

Proof The above two formulas of (8.11) are shown by the formulas (3.9.25) and (3.9.26) of [1], respectively. □

By (8.11), one obtains the following corollary.

Corollary 50 *Let $A, B \in \mathcal{A}$ be invertible elements of a fixed C^*-algebra $\mathcal{A}$. Then*

$$A^{-1} + A^{-1}\overline{B}\left(\overline{A} - BA^{-1}\overline{B}\right)BA^{-1} = \left(A - \overline{BA}^{-1}B\right)^{-1}, \text{ in } \mathcal{A}. \tag{8.12}$$

Proof The formula (8.12) is shown by (8.11). Indeed,

$$\begin{aligned} &A^{-1} + A^{-1}\overline{B}\left(\overline{A} - BA^{-1}\overline{B}\right)BA^{-1} \\ &= \left(\mathbf{1} + A^{-1}\overline{B}\left(\overline{A} - BA^{-1}\overline{B}\right)^{-1}B\right)A^{-1} \\ &= \left(\mathbf{1} + A^{-1}\overline{B}\left(\mathbf{1} - \overline{A}^{-1}BA^{-1}\overline{B}\right)^{-1}\overline{A}^{-1}B\right)A^{-1} \\ &= \left(\mathbf{1} - A^{-1}\overline{BA}^{-1}B\right)^{-1}A^{-1} = \left(A - \overline{BA}^{-1}B\right)^{-1}, \end{aligned}$$

in $\mathcal{A}$, by applying the formulas of (8.11). □

If $S_1 \in \mathcal{A}$ is in the sense of (8.9), then one has

$$\overline{S_1} = \overline{U} = \overline{\left(T_1 - tT_2\overline{T_1}^{-1}\overline{T_2}\right)^{-1}} = \left(\overline{T_1} - t\overline{T_2}T_1^{-1}T_2\right)^{-1}, \tag{8.13}$$

in $\mathcal{A}$, by (8.10). Then, by (8.11) and (8.12),

$$\overline{S_1} = \overline{T_1}^{-1} + t\overline{T_1}^{-1}\overline{T_2}\left(T_1 - tT_2\overline{T_1}^{-1}\overline{T_2}\right)T_2\overline{T_1}^{-1}, \text{ in } \mathcal{A}. \tag{8.14}$$

Note and recall that, by the fourth formula of (8.8), we have

$$Z = \overline{T_1}^{-1} + \overline{T_2}\left(T_1 - tT_2\overline{T_1}^{-1}\overline{T_2}\right)^{-1}(tT_2)\overline{T_1}^{-1},$$

in $\mathcal{A}$. It shows that

$$\underset{\text{in (8.8)}}{Z} = \underset{\text{in (8.9)}}{\overline{S_1}}, \text{ in } \mathcal{A}, \tag{8.15}$$

by (8.13) and (8.14).

Now, consider the operator V in the second formula of (8.8),

$$V = -\left(T_1 - tT_2\overline{T_1}^{-1}\overline{T_2}\right)^{-1}(tT_2)\overline{T_1}^{-1},$$

and the operator S_2 of (8.9),

$$tS_2 = \left(\mathbf{1} - \left(-\frac{1}{t}\overline{T_2^{-1}}\overline{T_1}T_2^{-1}\left(\mathbf{1} - \frac{1}{t}T_1\overline{T_2^{-1}}\overline{T_1}T_2^{-1}\right)^{-1}\right)T_1\right)\overline{T_2^{-1}}$$

in $\mathcal{A}$. Then

$$\begin{aligned} tS_2 &= \left(\mathbf{1} - \left(-\frac{1}{t}\overline{T_2^{-1}}\overline{T_1}\left(T_2 - \frac{1}{t}T_1\overline{T_2^{-1}}\overline{T_1}T_2^{-1}T_2\right)^{-1}\right)T_1\right)\overline{T_2^{-1}} \\ &= \left(\mathbf{1} - \left(-\overline{T_2^{-1}}\left(tT_2\overline{T_1}^{-1} - T_1\overline{T_2^{-1}}\overline{T_1}\overline{T_1}^{-1}\right)^{-1}\right)T_1\right)\overline{T_2^{-1}} \\ &= \left(\mathbf{1} - \left(-tT_2\overline{T_1}^{-1}\overline{T_2} + T_1\overline{T_2^{-1}}\overline{T_2}\right)^{-1}T_1\right)\overline{T_2^{-1}} \\ &= \left(\mathbf{1} - \left(-tT_2\overline{T_1}^{-1}\overline{T_2} + T_1\right)^{-1}T_1\right)\overline{T_2^{-1}} \\ &= \left(\mathbf{1} - \left(T_1 - tT_2\overline{T_1}^{-1}\overline{T_2}\right)^{-1}T_1\right)\overline{T_2^{-1}} \\ &= \overline{T_2}^{-1} - \left(T_1 - tT_2\overline{T_1}^{-1}\overline{T_2}\right)^{-1}T_1\overline{T_2^{-1}}, \end{aligned} \tag{8.16}$$

in $\mathcal{A}$. Now, let's compare the operators V of (8.8) and the the operator tS_2 of (8.16) induced from the operator S_2 of (8.9).

$$tS_2 = \overline{T_2}^{-1} - \left(T_1 - tT_2\overline{T_1}^{-1}\overline{T_2}\right)^{-1} T_1\overline{T_2}^{-1}$$

by (8.16)

$$= \overline{T_2}^{-1} - \left(\left(\mathbf{1} - tT_2\overline{T_1}^{-1}\overline{T_2}T_1^{-1}\right)^{-1} T_1^{-1}T_1\overline{T_2}^{-1}\right)$$

$$= \overline{T_2}^{-1} - \left(\left(\mathbf{1} - tT_2\overline{T_1}^{-1}\overline{T_2}T_1^{-1}\right)^{-1} \overline{T_2}^{-1}\right)$$

$$= \left(\mathbf{1} - \left(\mathbf{1} - tT_2\overline{T_1}^{-1}\overline{T_2}T_1^{-1}\right)\right)\overline{T_2}^{-1} \tag{8.17}$$

$$= \left(\mathbf{1} - \left(\mathbf{1} + T_1^{-1}\left(\mathbf{1} - tT_2\overline{T_1}^{-1}\overline{T_2}T_1^{-1}\right)tT_2\overline{T_1}^{-1}\overline{T_2}\right)\right)\overline{T_2}^{-1} \tag{8.18}$$

since

$$(\mathbf{1} - AB)^{-1} = \mathbf{1} + A\,(\mathbf{1} - BA)^{-1}\,B,$$

in $\mathcal{A}$ under suitable invertibility assumptions for $A, B \in \mathcal{A}$; so, one can take

$$A = T_1^{-1}, \text{ and } B = tT_2\overline{T_1}^{-1}\overline{T_2},$$

in (8.17). Thus, by (8.18),

$$tS_2 = \left(-T_1^{-1}\left(\mathbf{1} - tT_2\overline{T_1}^{-1}\overline{T_2}T_1^{-1}\right)tT_2\overline{T_1}^{-1}\overline{T_2}\right)\overline{T_2}^{-1}$$

$$= -T_1^{-1}\left(\mathbf{1} - tT_2\overline{T_1}^{-1}\overline{T_2}T_1^{-1}\right)tT_2\overline{T_1}^{-1}$$

$$= -\left(T_1 - tT_2\overline{T_1}^{-1}\overline{T_2}\right)^{-1}(tT_2)\,\overline{T_1}^{-1} = V, \tag{8.19}$$

in $\mathcal{A}$. i.e., by (8.19), we have that

$$\underset{\text{where } S_2 \text{ is from (8.9)}}{tS_2} = \underset{\text{of (8.8)}}{V}, \text{ in } \mathcal{A}. \tag{8.20}$$

By (8.20), we obtain that

$$\underset{\text{of (8.9)}}{S_2} = \frac{1}{t}(tS_2) \underset{\text{by(8.18)}}{=} \frac{1}{t}V = \underset{\text{of (8.8)}}{W}, \text{ in } \mathcal{A},$$

where

$$W = -\overline{T_1}^{-1}\overline{T_2}\left(T_1 - tT_2\overline{T_1}^{-1}\overline{T_2}\right)^{-1}. \tag{8.21}$$

Theorem 51 *Let $t \neq 0$ in $\mathbb{R}$. Under suitable invertibility assumptions (in the sense of* **AN 8.1***), the invertibility on the t-conjugate $\mathcal{A}$-hypercomplexes $\mathfrak{H}_2^t(\mathcal{A})$ and the invertibility on the algebra $M_2(\mathcal{A})$ are equivalent, i.e.,*

$$\textit{the invertibility on } M_2(\mathcal{A}) \overset{\text{equi}}{=} \textit{the invertibility on } \mathfrak{H}_2^t(\mathcal{A}),\ \forall t \in \mathbb{R} \setminus \{0\}. \tag{8.22}$$

Proof For a nonzero scale $t \in \mathbb{R} \setminus \{0\}$, suppose the element $[(T_1, T_2)]_t$ is invertible "in the t-conjugate $\mathcal{A}$-hypercomplexes $\mathfrak{H}_2^t(\mathcal{A})$" with its inverse,

$$[(S_1, S_2)]_t = \begin{pmatrix} S_1 & tS_2 \\ \overline{S_2} & \overline{S_1} \end{pmatrix} \in \mathfrak{H}_2^t(\mathcal{A}),$$

where

$$S_1 = -\frac{1}{t}\overline{T_2}^{-1}\overline{T_1}T_2^{-1}\left(\mathbf{1} - \frac{1}{t}T_1\overline{T_2}^{-1}\overline{T_1}T_2^{-1}\right)^{-1},$$

and

$$S_2 = \frac{1}{t}\left(\mathbf{1} - \left(-\frac{1}{t}\overline{T_2^{-1}}\,\overline{T_1}T_2^{-1}\left(\mathbf{1} - \frac{1}{t}T_1\overline{T_2^{-1}}\,\overline{T_1}T_2^{-1}\right)^{-1}\right)T_1\right)\overline{T_2^{-1}},$$

in $\mathcal{A}$ by (6.2.8), or (8.3), under suitable invertibility assumptions. Then as a (2×2)-operator-block matrix,

$$[(T_1, T_2)]_t = \begin{pmatrix} T_1 & tT_2 \\ \overline{T_2} & \overline{T_1} \end{pmatrix} \in \mathfrak{H}_2^t(\mathcal{A}), \text{ in } M_2(\mathcal{A}),$$

it can have its inverse "in $M_2(\mathcal{A})$,"

$$\begin{pmatrix} U & V \\ W & Z \end{pmatrix} \in M_2(\mathcal{A}),$$

with

$$U = \left(T_1 - tT_2\overline{T_1}^{-1}\overline{T_2}\right)^{-1},$$
$$V = -\left(T_1 - tT_2\overline{T_1}^{-1}\overline{T_2}\right)^{-1}(tT_2)\overline{T_1}^{-1},$$
$$W = -\overline{T_1}^{-1}\overline{T_2}\left(T_1 - tT_2\overline{T_1}^{-1}\overline{T_2}\right)^{-1},$$

and

$$Z = \overline{T_1}^{-1} + \overline{T_2}\left(T_1 - tT_2\overline{T_1}^{-1}\overline{T_2}\right)^{-1}(tT_2)\overline{T_1}^{-1},$$

in $\mathcal{A}$, by (8.8). However, by (8.10), (8.13), (8.20) and (8.21), we have that

$$\begin{pmatrix} U & V \\ W & Z \end{pmatrix} = \begin{pmatrix} S_1 & tS_2 \\ \overline{S_2} & \overline{S_1} \end{pmatrix},$$

"in $\mathfrak{H}_2^t(\mathcal{A})$," inside $M_2(\mathcal{A})$. It shows that the invertibility on $M_2(\mathcal{A})$ implies that on $\mathfrak{H}_2^t(\mathcal{A})$.

Since $\mathfrak{H}_2^t(\mathcal{A})$ is a subalgebra of $M_2(\mathcal{A})$, the invertibility on $\mathfrak{H}_2^t(\mathcal{A})$ implies that on $M_2(\mathcal{A})$. Therefore, the equivalence (8.22) holds. □

So, we obtain the following main result of this section.

Corollary 52 *The invertibility on the t-conjugate $\mathcal{A}$-hypercomplexes $\mathfrak{H}_2^t(\mathcal{A})$ and the invertibility on $M_2(\mathcal{A})$ are equivalent, for all scales $t \in \mathbb{R}$.*

Proof It is shown by the equivalences (8.6) and (8.22). □

Funding
Open access funding provided by SCELC, Statewide California Electronic Library Consortium.

Data availability
The authors confirm that no data known is used in the manuscript.

Author details
[1]Department of Mathematics, Chapman University, 1 University Dr., Orange, CA 92866, USA, [2]Department of Mathematics and Statistics, St. Ambrose University, 421 Ambrose Hall, 518 W. Locust St., Davenport, IA 52803, USA.

Received: 15 September 2023 Accepted: 23 September 2023
Published online: 21 October 2023

References

1. Alpay, D.: Exercises in Applied Mathematics: With a View Toward Information Theory, Machine Learning, Wavelets and Statistical Physics. Springer, Berlin (to appear) (2023)
2. Alpay, D., Luna-Elizarraras, M.E., Shapiro, M., Struppa, D.: Gleason's problem, rational functions and spaces of left-regular functions: the split-quaternion settings. Isr. J. Math. **226**, 319–349 (2018)
3. Alpay, D., Cho, I.: Operators induced by certain hypercomplex systems. Opuscula Math. **43**(3), 275–333 (2023)
4. Alpay, D., Cho, I.: On scaled hyperbolic numbers induced by scaled hypercomplex numbers. Pure Appl. Funct. Anal. **33** (to appear) (2023)
5. Alpay, D., Cho, I.: Dynamical systems of operators induced by scaled hypercomplex rings. Adv. Appl. Clifford Algebras (2023). https://doi.org/10.1007/s00006-023-01272-0
6. Cho, I., Jorgensen, P.E.T.: Multi-variable quaternionic spectral analysis. Opuscula Math. **41**(3), 335–379 (2021)
7. Curtis, M.L.: Matrix Groups, ISBN: 0-387-90462-X. Published by Springer-Verlag, NY (1979)
8. Farid, F.O., Wang, Q., Zhang, F.: On the eigenvalues of quaternion matrices. Linear Multilinear Alg. **4**, 451–473 (2011)
9. Flaut, C.: Eigenvalues and eigenvectors for the quaternion matrices of degree two. An. St. Univ. Ovidius Constanta **10**(2), 39–44 (2002)
10. Girard, P.R.: Einstein's equations and Clifford algebra. Adv. Appl. Clifford Alg. **9**(2), 225–230 (1999)
11. Halmos, P.R.: Book, Linear Algebra Problem. Mathematical Association of America, Washington. ISBN: 978-0-88385-322-1. (1995)
12. Halmos, P. R.: Hilbert Space Problem Book. Springer, New York. ISBN: 978-038-790685-0 (1982)
13. Hamilton, W.R.: Lectures on Quaternions. Cambridge University Press, Cambridge (1853)
14. Kantor, I.L., Solodnikov, A.S.: Hypercomplex Numbers, an Elementary Introduction to Algebras. Springer, New York. ISBN: 0-387-96980-2 (1989)
15. Kravchenko, V.: Applied Quaternionic Analysis. Heldemann Verlag, Lemgo. ISBN: 3-88538-228-8 (2003)
16. Kobayashi, M.: Hyperbolic hopfield neural networks with directional multistate activation function. Neurocomputing **275**, 2217–2226 (2018)
17. Ratcliffe, J.: Foundations of Hyperbolic Manifolds, Grad. Text in Math., vol. 149. Springer, Berlin. ISBN:978-0-3873319-73 (2006)
18. Reynolds, W.F.: Hyperbolic geometry on a hyperboloids. Am. Math. Mon. **100**, 442–455 (1993)
19. Rodman, L.: Topics in Quaternion Linear Algebra. Princeton University Press, Princeton. ISBN:978-0-691-16185-3 (2014)
20. Rozenfeld, B.A.: The History of non-Eucledean Geometry: Evolution of the Concept of a Geometric Spaces. Springer, Berlin. ISBN: 978-038-796458-4 (1988)
21. Speicher, R.: Combinatorial Theory of the Free Product with Amalgamation and Operator-Valued Free Probability Theory. Amer. Math. Soc., Memoire, ISBN:978-0-8218-0693-7. Published by Amer. Math. Soc. (1998)
22. Sudbery, A.: Quaternionic analysis. Math. Proc. Camb. Philos. Soc. (1998). https://doi.org/10.1017/s0305004100053638
23. Taosheng, L.: Eigenvalues and eigenvectors of quaternion matrices. J. Central China Normal Univ. **29**(4), 407–411 (1995)
24. Vince, J.A.: Geometric Algebra for Computer Graphics. Springer, Berlin. ISBN: 978-1-84628-996-5 (2008)
25. Voiculescu, D.V., Dykema, K.J., Nica, A.: Variables, Free Random. American Mathematical Society. ISBN:978-0-8218-1140-5 (1992)
26. Voight, J.: Quaternion Algebra. Department of Mathematics, Dartmouth University. http://math.dartmouth.edu/~jvoight/quat-book.pdf (2019)

Publisher's Note