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A Dissertation Presented

By

William Emery Haver

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INTRODUCTION

Cellular mappings of a manifold onto itself possess many properties of homeomorphisms. In particular, for $n \neq 4$, a continuous function defined from a manifold onto itself is cellular if and only if it can be uniformly approximated by homeomorphisms. This thesis is a study of cellular mappings, spaces of cellular mappings and a class of mappings, called UV^{∞}-maps which are a natural generalization of cellular mappings for spaces which are not manifolds.

In Chapter one we prove that the space of cellular mappings from a manifold onto itself is a topological semi-group and that the space of all cellular mappings of B^n onto itself which are the identity on the boundary is locally contractible. The main theorem of Chapter two is that a mapping f of the n-sphere, $n \neq 4$, onto itself is cellular if and only if f has a continuous extension which maps the interior of the n+1 ball homeomorphically onto itself. This theorem is a higher dimensional analogue of a result of Floyd and Fort [11]. For higher dimensional manifolds with boundary, M^n, we show that if f maps the interior of M onto itself and the boundary of M onto itself and if f restricted to the interior is cellular, then f restricted to the boundary of M is also cellular.

Chapter three is concerned with showing that under certain conditions cellular mappings can be replaced in a canonical manner with bounded cellular mappings that agree with the original mappings on a given set. Similar techniques have proven valuable in studying homeomorphisms and spaces of homeomorphisms. In Chapter four we introduce a new type of covering property possessed by many metric spaces and show that possession of this property by the space of cellular mappings of B^n onto itself would show that the space of cellular mappings of a manifold onto itself is locally contractible.

In Chapter five we show that if $f:X \to Y$, Y a metric space, is a UV^∞-map, K a locally finite complex and $h:K \to Y$ is any continuous function, then for any $\varepsilon > 0$ there exists a mapping $g:K \to X$ such that fg is ε-homotopic to h.

CHAPTER I

SOME PROPERTIES OF CELLULAR MAPPINGS

In this chapter we prove some of the basic properties of cellular mappings and spaces of cellular mappings of a manifold onto itself. Proposition 1.1, which states that a mapping from a manifold onto itself which can be uniformly approximated by homeomorphisms is cellular, provides an important tool for dealing with cellular mappings, and while it is a consequence of a more general theorem announced by Finney [9] the simple proof is included here for completeness. The proof of this theorem implies that the space of all cellular mappings is a topological semigroup.

We next discuss the relationships between cellular mappings and certain classes of monotone mappings. The chapter is concluded with the proof that the space of all cellular mappings of B^n onto itself which are the identity on the boundary is locally contractible. In proving this theorem we show that for any manifold local contractibility at the identity map implies local contractibility of the space.

A compact mapping $f: M^n \rightarrow M^n$, from an n-manifold onto itself, is defined to be <u>cellular</u> if for each $y \in M^n$, $f^{-1}(y)$ is a cellular set; i.e., there is a sequence C_1, C_2, . . . of topological n-cells such that

$f^{-1}(y) = \bigcap_{i=1}^{\infty} C_i$ and $C_{i+1} \subset \text{Int } C_i$. Many properties of homeomorphisms are also possessed by cellular mappings. The first step for results in this direction is the following observation:

Proposition 1.1. Suppose $f: M^n \to M^n$ is an onto map which can be approximated uniformly by homeomorphisms. Then f is cellular.

Proof. First we show that f is a compact mapping. Let C be a compact subset of M^n. We choose $\varepsilon > 0$ small enough so that $\overline{N_{2\varepsilon}(C)} = \{x \in M^n | d(x,C) \leq 2\varepsilon\}$ is compact. Let h be a homeomorphism from M^n onto M^n such that $d(f,h) < \varepsilon$. Throughout this paper, if f and h are any functions, $d(f,h)$ is defined to be $\sup_{x \in M^n}(d(f(x), h(x)))$. For this h, $h(f^{-1}(C)) \subset \overline{N_{2\varepsilon}(C)}$. By the continuity of f, $f^{-1}(C)$ is closed, and hence $h(f^{-1}(C)))$ is a compact subset of the compact space $\overline{N_{2\varepsilon}(C)}$. By the continuity of h^{-1}, $h^{-1}(h(f^{-1}(C))) = f^{-1}(C)$ is compact.

Now it remains to be shown that if $b \in M^n$, then $f^{-1}(b)$ is a cellular set. It suffices to show that if U is an open subset of M^n, with $f^{-1}(b) \subset U$, then there exists an n-cell, C, containing $f^{-1}(b)$ and contained in U. (Since $N_1(f^{-1}(b)$ is open, this would show that there exists an n-cell, C_1, contained in $N_1(f^{-1}(b))$ and containing $f^{-1}(b)$. $N_{\frac{1}{2}}(f^{-1}(b)) \cap \text{Int } C_1$ is open and we therefore could obtain an n-cell C_2 contained in this open set and containing $f^{-1}(b)$. Continue inductively.)

Since f is a compact mapping, $f(\tilde{U})$ is a closed subset of M^n. We note that $\delta = d(f(\tilde{U}), f(f^{-1}(b)))= d(f(\tilde{U}), b)$ is a positive number. Since M^n is an n-manifold, there exists a positive number $\eta \leq \delta$ with the property that $N_{\eta/3}(b)$ is an n-cell. By our hypothesis, we can choose a homeomorphism h so that $d(h,f) < \eta/3$. Then $h^{-1}(N_{\eta/3}(b))$ is the desired n-cell, C.

First we show that $f^{-1}(b) \subset h^{-1}(N_{\eta/3}(b))$. Let $x \in f^{-1}(b)$, then $d(h(x), f(x)) < \eta/3$; i.e., $d(h(x),b) < \eta/3$, which implies that $h(f^{-1}(b)) \subset N_{\eta/3}(b)$. Therefore, $f^{-1}(b) \subset h^{-1}(N_{\eta/3}(b))$. Next, we show that $h^{-1}(N_{\eta/3}(b)) \subset U$. We suppose not. Then there exists $x \in h^{-1}(N_{\eta/3}(b)) \cap \tilde{U}$. $x \in h^{-1}(N_{\eta/3}(b))$ implies that $h(x) \in N_{\eta/3}(b)$ and $d(h(x),b) < \eta/3$. By choice of h, $d(f(x), h(x)) < \eta/3$. Therefore, $d(f(x),b) < 2\eta/3$. But, $x \in \tilde{U}$ implies that $d(f(x),b) > \eta$. This is the desired contradiction.

Combined with the following major result of Armentrout for $n \leq 3$ [2] and more recently of Siebenmann for $n \neq 4$ [23], the preceding proposition provides a characterization of those mappings which can be uniformly approximated by homeomorphisms.

<u>Theorem 1.2</u>. Suppose $f:M^n \to M^n$ is a cellular map of M^n onto M^n, $n \neq 4$, and suppose $g:M^n \to (0, \infty)$ is a

given continuous function. Then there exists a homeomorphism $h:M^n \to M^n$ such that $d(f(x), h(x)) < \sigma(x)$, for all $x \in M^n$.

This theorem is the analogue of a result of Hocking [12] which we will state after the necessary preliminaries. A mapping f is said to be cm^r if it is both compact and r-monotone (i.e., the inverse image of each point is k-acyclic in the sense of Vietoris, for all $k \leq r$). A space is ulc^r if it is uniformly locally connected in all dimensions $k \leq r$.

<u>Theorem 1.3</u>. Let M^2 be a ulc^1 2-manifold. A uniformly continuous mapping $f:M^2 \to M^2$ can be uniformly approximated by homeomorphisms if and only if f is cm^1.

As the following proposition demonstrates, a cellular map $f:M^n \to M^n$ is cm^n.

<u>Proposition 1.4</u>. Let $K \subset M^n$ be a cellular set. Then K is k-acylic for all $k \leq n$.

<u>Proof</u>: Let $\gamma = \{\gamma_1, \gamma_2, \ldots | \gamma_i$ is an ε_i-cycle and $\lim \varepsilon_i = 0\}$ be a k-dimensional infinite cycle in K. We wish to find a k-dimensional infinite chain κ in K such that $\partial\kappa = \gamma$. Let γ_i be an ε_i-cycle. By the cellularity of K, there exists a cell C_i such that $K \subset C_i \subset N_{\varepsilon_i}(K)$. Now, γ_i lies in C_i. Therefore, there exists an ε_i-chain κ_i^* such that $\partial\kappa_i^* = \gamma_i$ and each vertex of κ_i^* lies in C_i. We define κ_i by designating a

vertex v for each vertex v^* of κ_i^* in the following manner:

$$v = \begin{cases} v^*, \text{ if } v^* \varepsilon K \\ \text{any point of } K \text{ within } \varepsilon_i \text{ of } v^*, \text{ if } v^* \not\varepsilon K \end{cases}$$

Then $\partial\kappa_i = \gamma_i$ and κ_i is a k-dimensional $2\varepsilon_i$-chain. Since $\lim \varepsilon_i = 0$, $\lim 2\varepsilon_i = 0$.

This proposition leads to a natural question. Let $f:M^n \to M^n$ be cm^n. Is f cellular? Proposition 1.5 provides a partial answer for n = 2.

Proposition 1.5. Let M^2 be a ulc^1 2-manifold. A uniformly continuous mapping $f:M^2 \to M^2$ is cm^1 if and only if it is cellular.

Proof. If f is cellular it is cm^1 by proposition 1.4. If f is cm^1, theorem 3.1 implies it can be uniformly approximated by homeomorphisms and is therefore cellular according to proposition 1.1.

It will often be of interest to consider certain spaces of homeomorphisms and cellular mappings. Let M be a manifold. H(M) will designate the space of all homeomorphisms of M onto itself, Ce(M) the space of all cellular mappings of M onto itself. $H_\delta(M)$ will designate the space of all homeomorphisms of M onto itself which equal the identity when restricted to the boundary of M and $Ce_\delta(M)$ the space of all cellular mappings which equal the identity when restricted to the boundary. Each of these spaces is given the compact open topology, the topology generated by all sets of the form $N(C,U) = \{f \mid f(C) \subset U\}$, where C is compact and U is open in M. This topology agrees with the uniform topology for the special case where M is compact. In this instance each of the above mentioned spaces is a metric space with metric given by $d(f,g) = \sup_{x \in M} d(f(x), g(x))$. H(M) and $H_\delta(M)$ are known to be topological groups under composition of functions. Since cellular mappings which are not homeomorphisms have no inverses, Ce(M) and $Ce_\delta(M)$ are not topological groups. However, the following is true.

<u>Proposition 1.6</u>. Ce(M) and $Ce_\delta(M)$ are topological semigroups.

<u>Proof</u>. We must show that the composition of two cellular maps is cellular, or equivalently that if f is a cellular map and K is a cellular set, then $f^{-1}(K)$ is a cellular set. Since the composition of two compact mappings is compact, the proof of this statement is identical to thatof proposition 1.1 with "K" replacing "b".

A topological space, S, is <u>locally contractible</u> if given any point $s \in S$ and any neighborhood U of s, there exists a neighborhood V of s, a point $v_0 \in V$, and a homotopy $H: V \times I \to U$ such that $H(v,0) = v$ and $H(v,1) = v_0$ for all $v \in V$. Likewise, S is <u>locally contractible at s</u> if given any neighborhood U of s, there exists a neighborhood V of s, a point $v_0 \in V$ and a homotopy $H: V \times I \to U$

such that $H(v,0) = v$ and $H(v,1) = v_0$ for all $v \in V$.

Cernavskii [7] and Edwards and Kirby [8] have shown that if M is compact or equals E^n, then $H(M)$ is locally contractible. Alexander [1] demonstrated that if B^n is the n-dimensional ball, then $H_\delta(B^n)$ is locally contractible. We show in this section that $Ce_\delta(B^n)$ is also locally contractible for $n \neq 4$.

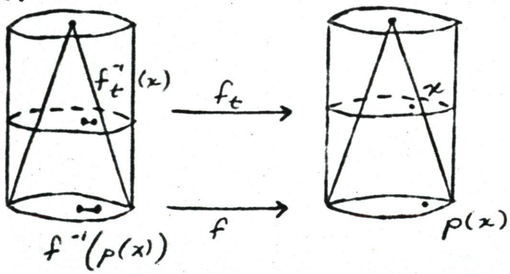
Theorem 1.7. $Ce_\delta(B^n)$ is locally contractible, $n \neq 4$.

Proof. We first show that if $\varepsilon > 0$, there exists a homotopy $H: N_\varepsilon(id) \times I \to N_\varepsilon(id)$ such that $H(f,0) = f$ and $H(f,1) = id$. We define H as follows:

$$H(f,t)(x) = f_t(x) = \begin{cases} \frac{1-t}{1+t} f\left(\frac{1+t}{1-t}x\right), & 0 \leq t < 1 \\ x, & t = 1, \end{cases}$$

where $f(x) = \begin{cases} f(x), & x \in B^n \\ x, & x \in R^n - B^n. \end{cases}$

We note that for each t, f_t is a cellular map, since it is the composition of a cellular map and two homeomorphisms. For each t, f_t restricted to the boundary of B^n is equal to the identity map. H is continuous by definition. The following diagram illustrates how this Alexander-type isotopy works in the case of cellular mappings.



Now, all that remains to be shown is local contractibility at an arbitrary point. (For H(M) or $H_\delta(M)$ this follows directly from local contractibility at the identity since these spaces are each topological groups.) However, in our case the implication must be demonstrated. We will show more generally that if Ce(M) or $Ce_\delta(M)$ is locally contractible at the identity map, it is locally contractible at every point. To this end we will assume that we have shown local contractibility at the identity. Let f be cellular and let $N_f(K,\eta)$ be given. Let $K' = \{m \in M \mid$ there exists $x \in K$ and $h \in N_f(K,\eta) \cap H(M)$ such that $m = h(x)\}$. Consider $N_{id}(K',\eta/2)$. We are assuming that there exists $N_{id}(D,\delta)$ and a homotopy $\phi: N_{id}(D,\delta) \times I \to N(K',\eta/2)$ such that:

1. $K' \subset D$
2. $\delta \leq \eta/2$
3. $\phi(g,t) \in N_{id}(K',\eta/2)$, for all $g \in N_{id}(D,\delta)$
4. $\phi(g,0) = g$, for all $g \in N_{id}(D,\delta)$
5. $\phi(g,1) = id$, for all $g \in N_{id}(D,\delta)$

Now, pick a homeomorphism $h: M \to M$ such that $d(h(x),f(x)) < \delta/2$, for all $x \in M$. In the case of $Ce_\delta(M)$, choose h so that in addition, h restricted to the boundary of M is the identity.

We will now define $H: N_f(h^{-1}(D),\delta/2) \times I \to N_f(K,\eta)$ as follows: $H(g,t) = (\phi(gh^{-1},t)) \circ h$, for all $g \in N_f(h^{-1}(D,\delta/2)\}$.

In order for this definition to make sense, we must show that $g \in N_f(h^{-1}(D), \delta/2)$ implies that $gh^{-1} \in N(D, \delta)$. But this follows, since if $x \in D$, then $d(gh^{-1}(x), x) < d(g(h^{-1}(x)), f(h^{-1}(x))) + d(f(h^{-1}(x)), h(h^{-1}(x))) < \delta/2 + \delta/2 = \delta$.

H is the proper map if we can show that:

1) $K \subset h^{-1}(D)$

2) $\delta/2 = \eta$

3) $H_t(g) \in N_f(K, \eta)$, for all $g \in N_f(h^{-1}(D), \delta/2)$

4) $H(g, 0) = g$

5) $H_1(g) = h$

It is clear that 1) follows since if $x \in K$, then $h(x) \in K'$ and $h^{-1}(h(x)) \in h^{-1}(K') \subset h^{-1}(D)$. 2) is obvious. To show 3), let $x \in K$, then $d(H(g,t)(x), f(x)) = d(\phi(gh^{-1}, t)(h(x)), f(x)) = d(\phi(gh^{-1}, t)(h(x)), h(x)) + d(h(x), f(x)) < \eta/2 + \eta/2 = \eta$, since $\phi(gh^{-1}, t) \in N(K', \eta/2)$. 4) and 5) are true by definition of H; i.e., $H(g, 0) = (\phi(gh^{-1}, 0)) \circ h = gh^{-1} \circ h = g$ and $H(g, 1) = (\phi(gh^{-1}, 1)) \circ h = id \circ h = h$.

We therefore have shown that for any M, local contractibility of Ce(M) or $Ce_\delta(M)$ at the identity implies local contractibility of the space at any point. This completes the proof of Theorem 1.7.

Mason [19] recently showed that $H_\delta(B^2)$ is an absolute retract. The proof depends upon the construction of a special basis for the topology and upon the fact that $H_\delta(B^2)$ is contractible and locally contractible. We have just shown that $Ce_\delta(B^n)$ is locally contractible and the Alexander Isotopy applied to cellular mappings demonstrates that $Ce_\delta(B^n)$ is contractible. This raises the question of whether $\overline{H_\delta(B^2)}$ is also an absolute retract.

Let $\overline{H(M)} = \{f:M \to M \mid f$ can be uniformly approximated by homeomorphisms$\}$.

<u>Corollary 1.8</u>. $\overline{H(B^n)}$ is locally contractible.

<u>Proof</u>. Same as theorem 1.7.

CHAPTER II

A CHARACTERIZATION THEOREM FOR CELLULAR MAPS

The main result of this chapter is that a mapping f of the n-sphere, ∂B^{n+1}, $n \neq 4$, onto itself is cellular if and only if f has a continuous extension which maps the interior of the n+1-ball B^{n+1} onto itself by a homeomorphism. This theorem is the higher dimensional analogue of a result of Floyd and Fort [11] which states that a mapping f of the 2-sphere ∂B^3 onto itself is monotone if and only if f has a continuous extension which maps the interior of the 3-ball, B^3, homeomorphically onto itself. Our theorem actually is an extension of Floyd and Fort's result since a mapping of a compact space onto itself is monotone if and only if it is cm^1 and, as we demonstrated in chapter 1, a mapping of a 2-sphere onto itself is cm^1 if and only if it is cellular.

<u>Lemma 2.1</u>. Suppose $f:\partial B^n \to \partial B^n$, for any n, can be approximated by homeomorphisms. Then f can be extended to a map which is a homeomorphism on the interior of B^n.

<u>Proof.</u> As we stated in chapter 1, $H(\partial B^n)$ is locally contractible. This implies, in particular, that $H(\partial B^n)$ is locally arcwise connected at the identity. In other words, given an $\varepsilon>0$, there exists a $\delta>0$ such that if

$f \in H(\partial B^n)$ and $d(f, id) < \delta$, then there exists $\Phi:[0,1] \to H(\partial B^n)$ such that $\Phi(0) = f$, $\Phi(1) = id$ and for all t, $d(\Phi(t), id) < \varepsilon$.

Since for all f, g, h in $H(\partial B^n)$, $d(f,g) = d(fh, gh)$, this implies that $H(\partial B^n)$ is uniformly locally arcwise connected, as the following argument indicates. Let $\varepsilon>0$ be given. There exists $\delta>0$ such that any element of $H(\partial B^n)$ within δ of the identity can be joined to the identity in $H(\partial B^n)$ by a curve of diameter less than ε. Suppose h and g are elements of $H(\partial B^n)$ and $d(h,g) < \delta$. Then $d(gh^{-1}, id) < \delta$ and there exists $\Phi:[0,1] \to H(\partial B^n)$ such that $\Phi(0) = hg^{-1}$, $\Phi(1) = id$ and $d(\Phi(t), id) < \varepsilon$, for all t. We define $\Psi:[0,1] \to H(\partial B^n)$ by $\Psi(t) = \Phi(t)g$ and note that $\Psi(0) = h$, $\Psi(1) = g$ and Ψ is a curve of diameter less than ε.

Now suppose f is as in the statement of Lemma 2.1. Using a standard technique, we now can construct a map $\phi:I \to H(\partial B^n)$ such that $0 \leq t < 1$ implies $\phi(t) \in H(\partial B^n)$ and $\phi(1) = f$. For each integer k, pick δ_k such that $d(g,h) < \delta_k$ implies g and h can be joined by a curve of diameter less than 1/k, and such that $\delta_{k+1} \leq \delta_k$. Also choose a sequence of homeomorphisms h such that $d(h_k,f) < \delta_k/2$. We now define ϕ by:

$$\phi(1 - 1/k) = h_k$$

$$\phi(1) = f.$$

Extend ϕ over all of I by the promised curves. ϕ is continuous since the diameters of the curves approach 0 as k increases and $\{h_k\}$ converges to f.

The following corollary is an immediate consequence of Lemma 2.1 and our characterization of cellular mappings when $n \neq 4$.

Corollary 2.2. Let $f:\partial B^{n+1} \to \partial B^{n+1}$, $n \neq 4$, be cellular. Then f can be extended to a map from B^{n+1} onto B^{n+1} in such a way that restricted to the interior of B^{n+1} it is a homeomorphism.

Various properties of mappings which are in general weaker than cellularity have been studied extensively [4, 15]. Here we make use of one such property. Let $f:X \to Y$ be continuous. Then f has property UV^{∞} if for each $y \in Y$ and each open set U containing $f^{-1}(y)$, there is an open set V containing $f^{-1}(y)$ and contained in U such that V is null-homotopic in U. This property was introduced by McMillan [18] in his characterization of cellular sets upon which our main theorem strongly depends.

Lemma 2.3. Let M be a manifold and $F:M \times (0,1] \to M \times (0,1]$ be a map such that $F^{-1}(M \times 1) = M \times 1$ and $F|M \times (0,1):M \times (0,1) \to M \times (0,1)$ is a homeomorphism, then $F|M \times \{1\}:M \times \{1\} \to M \times \{1\}$ is a UV^{∞}-map.

Proof. We identify M with $M \times 1$. We make use of the following auxiliary maps: for each ∂, define

$\pi_\partial : M \to M \times \{1-\partial\}$ by $\pi_\partial(x) = (x, 1-\partial)$ and $p : M \times (0,1] \to M$ by $p(x,t) = x$.

Let U' be open in M with $f^{-1}(b) \subset U'$. $U' \times (0,1]$ is open in $M \times (0,1]$. Therefore, there is a U such that:

a) U is open in $M \times (0,1]$.

b) $U \subset U' \times (0,1]$.

c) $f(U)$ is open in $M \times (0,1]$.

d) $F^{-1}(b) \subset U$.

Now choose $t_0 < 1$ and an open cylinder, C, about $b \times [t_0,1]$ such that $C \subset f(U)$. We note that:

$f^{-1}(C)$ is open in $M \times (0,1]$,

$f^{-1}(C) \subset U$,

$f^{-1}(b \times [t_0,1]) \subset f^{-1}(C)$.

Let $\eta = d(b,\widetilde{C})$, $\eta > 0$. Let δ be chosen so that

a) $N_{2\delta}(f^{-1}(b)) \subset f^{-1}(C)$

b) $d(x,y) < 2\delta \Rightarrow d(f(x), f(y)) < \eta$

Let $V = N_\delta(f^{-1}(b)) \cap M$. We note that if x is an element of $\pi_\delta(V)$, then $f(x)$ is an element of $N_\eta(b) \cap M \times (0,1) \subset C$.

Since C is a cell we can define a homotopy $G : C \times I \to C$ so that

1) $x \in \mathring{C} \Rightarrow G(x,t) \in C \cap (M \times (0,1))$

2) $G(x,0) = x$

3) there exists $z \in M \times (0,1)$ such that $G(x,1) = z$, for all $x \in C$.

We now can define the desired homotopy $H: V \times I \to U'$, by $H(x,t) = pf^{-1}(G(f\pi_\delta(x), t))$. Thus

$$H(x,0) = pf^{-1}[G(f\pi_\delta(x),0)] = pf^{-1}(f\pi_\delta(x)) = x$$

$$H(x,1) = pf^{-1}[G(f\pi_\delta(x),1)] = pf^{-1}(z) = \text{constant}.$$

The continuity of f follows from that of G, so all that remains to be shown is that $H(x,t) \in U'$, for all $x \in V$ and all $t \in I$. $x \in V \Longrightarrow \pi_\delta(x) \in \pi_\delta(V) \Longrightarrow f(\pi_\delta(x)) \in C \cap M \times (0,1) \Longrightarrow G(f\pi_\delta(x),t) \in C \cap \mathcal{B} \Longrightarrow f^{-1}$ is defined and $f^{-1}[G(f\pi_\delta(x), t)] \in f^{-1}(C) \subset J \subset U' \times (\frac{1}{2},1]$. Thus $p(f^{-1}[G(f\pi_\delta(x), t)] = H(x,t) \in U'$. This completes the proof of the lemma.

Let $M \subset X$. M is collared if there is a homeomorphism h taking $M \times (0,1]$ into a neighborhood of M with the property that $h(m,1) = m$, for all $m \in M$. M. Brown proved that the boundary of any manifold with boundary is collared [6]. Therefore, we have the following corollary.

Corollary 2.4. Let M be a manifold with boundary and let $f: M \to M$ mapping the interior of M onto the interior and the boundary onto the boundary be such that f restricted to the interior of M is a homeomorphism. Then $f|\partial M$ is a UV^∞- map.

As we stated before we will make use of an important theorem of McMillan [18] :

Theorem 2.5. Let X be a compact subset of the interior of a piecewise linear manifold M^n. Suppose

1) given any open set U of M^n which contains X, there exists an open V containing X and contained in U such that V is null-homotopic in U

2) for each open U containing X, there exists an open set V such that $X \subset V \subset U$ and each loop in $V - X$ is null-homotopic in $U - X$

3) $n \geq 5$ or $n = 3$ and some neighborhood of X can be embedded in E^3.

Using this theorem we will prove the following lemma:

<u>Lemma 2.5</u>. Suppose $f: \partial B^{n+1} \to \partial B^{n+1}$, $n = 3$ or $n \geq 5$, is a mapping of ∂B^{n+1} onto itself and can be extended to a map from B^{n+1} onto B^{n+1} in such a way that restricted to the interior of B^{n+1} it is a homeomorphism. Then f is cellular.

<u>Proof</u>. Let $x \in \partial B^{n+1}$. By corollary 2.4, condition 1 on McMillan's theorem is satisfied. If $n = 3$, for any $y \in \partial B^4 \cap \widetilde{f^{-1}(x)}$, $\partial B^4 - y$ is a neighborhood of $f^{-1}(x)$ which can be embedded in E^3; therefore, condition 3 is satisfied. So, all that remains is to show that condition 2 is satisfied.

Let $x \in M^n$ and let U be an open set containing $f^{-1}(x)$. We note that $f(f^{-1}(x)) = x$ is an element of the interior of $f(U)$. Since M^n is a manifold we can choose an open cell, C, containing x and contained in $f(U)$.

We then let $V = f^{-1}(C)$. Let $\gamma : S^1 \to V - f^{-1}(x)$. We wish to show that we can extend γ to a map $\Gamma : B^2 \to U - f^{-1}(x)$. By the definition of γ, $f\gamma$ is a mapping from S^1 into $C - x$. Since C is an n-cell with $n \geq 3$, $f\gamma$ can be extended to a map $G : B^2 \to C - x$. $G(B^2)$ is a compact subset of $C - x$. For each $y \in G(B^2)$, we note that $V - f^{-1}(x)$ is an open set containing $f^{-1}(y)$. By the previous lemma, we can choose an open set A_y such that:

$f^{-1}(y) \subset A_y \subset V - f^{-1}(x)$

A_y is null-homotopic in $V - f^{-1}(x)$

$f(A_y)$ is open, and

$f^{-1}(f(A_y)) = A_y$.

Therefore, $\{f(A_y) | y \in G(B^2)\}$ is an open cover of $G(B^2)$.

We now choose an open cover T_1 of $G(B^2)$ that star refines $\{f(A_y) | y \in G(B^2)\}$, i.e.,

a) for each $T_1 \in T_1$, $\{ T \in T_1 | T \cap T_1 \neq \phi\}$ is contained in some $f(A_y)$.

b) $T_1 \in T_1$ implies that T_1 is an open subset of M^n

c) $G(B^2) \subset \bigcup \{T | T \in T_1\}$

We repeat this process once more. For each $y \in G(B^2)$, choose $T_y \in T_1$ such that $y \in T_y$. We note that $f^{-1}(T_y)$ is an open set containing $f^{-1}(y)$. By the previous lemma, There exists an open set B_y such that $f^{-1}(y) \subset B_y \subset f^{-1}(T_y)$, B_y is null-homotopic in $f^{-1}(T_y)$ and $f(B_y)$ is open.

Therefore, $\{f(B_y) | y \in \Gamma(B^2)\}$ is an open cover of $G(B^2)$. We now choose a finite subcollection, T_2, of $\{f(B_y) | y \in G(B^2)\}$ that covers $G(B^2)$.

Triangulate B^2 such that if σ is a simplex of B^2, $G(\sigma)$ is contained in some element of T_2. We now are in a position to define the desired extension of the loop γ to all of B^2. First we will define the extension Γ on the 0-simplices of B^2. Let v be a vertex of B^2. If $v \in S^1$, let $\Gamma(v) = \gamma(v)$. If $v \notin S^1$, let $\Gamma(v)$ be some element of $f^{-1}(G(v))$. Next we will define Γ on the 1-simplices of B^2. Let σ be a 1-simplex of B^2. $G(\partial\sigma)$ is contained in some element of T_2, by our triangulation of B^2, so $\Gamma(\partial\sigma) \subset B_y$, for some y. B_y is null-homotopic in $f^{-1}(T)$ for some $T \in T_1$. We can therefore define Γ on all of σ so that $\Gamma(\sigma) \subset f^{-1}(T)$.

Finally, we wish to define Γ on the 2-skeleton of B^2. Let τ be a 2-simplex of B^2. $f(\Gamma(\partial\tau))$ is contained in $f(A_y)$ for some y, since T_1 star refines $\{f(A_y) | y \in G(B^2)\}$. We chose A_y so that $f^{-1}(f(A_y)) = A_y$; therefore, $\Gamma(\partial\tau) \subset A_y$. But A_y is null-homotopic in $V - f^{-1}(x)$ and we can extend Γ to all of τ and hence to all of $B2$ in such a way that $\Gamma(B^2) \subset V - f^{-1}(x)$. This completes the proof of lemma 2.5.

Note: In effect we have shown that property 2 is satisfied for $f^{-1}(x)$ if f has property UV^{∞}. Similar theorems have been proven by Lacher [16], Armentrout and Price [4] and Kozlowski [15]. We will provide an extension of Kozlowski's theorem later in this thesis.

Combining corollary 2.2 and lemma 2.5 we can state the main theorem of this chapter.

Theorem 2.6. A mapping f of the n-sphere ∂B^{n+1}, $n \neq 4$, onto itself is cellular if and only if f has a continuous extension which maps the interior of B^{n+1} homeomorphically onto itself.

We next consider the following general situation. Suppose $f:A \to B$ is an onto mapping such that f maps the interior of A onto the interior of B and the boundary of A onto the boundary of B. If the restriction of f to the interior of A has a certain property, does the restriction of f to the boundary of A have the same property? Whyburn proved that if A and B are locally connected compact continua and f restricted to the interior of A is monotone, then f restricted to the boundary is also monotone [24]. Another result of Whyburn is that if A and B are locally connected continua, then f quasi-open on the interior of A implies that f is quasi-open on the boundary of A. ($f:X \to Y$ is quasi-open if each $y \in Y$ is interior to the image of any open set in X which contains a compact component of $f^{-1}(y)$.) It would be interesting to determine whether certain conditions could be placed on A and B which would insure that if f restricted to the interior of A has property UV^∞ (f cm^r) then f restricted to the boundary of A has property UV^∞ (f cm^r). (See Chapter 1 for the definition of cm^r.) The following corollaries provide the corresponding result for cellular maps defined on an n-manifold, for $n \neq 4$ or 5; i.e., if $f:M^n \to M^n$ is as above and f restricted to the interior of M^n is cellular, then f restricted to the boundary of M^n is cellular.

<u>Corollary 2.7</u>. Let M be an n-manifold, $n \geq 5$, with boundary. Let f be a map of M onto M such that $f|\text{Int } M:\text{Int } M \to \text{Int } M$ is cellular and $f|\partial M:\partial M \to \partial M$. Then $f|\partial M$ is a UV^∞-map.

<u>Proof</u>. Define $g:\text{Int } M \to (0,\infty)$ by $g(m) = d(m,\partial M)$. Since $f|\text{Int } M$ is a cellular map, by Siebenmann's theorem (theorem 1.2), there is a homeomorphism h such that for all $x \in \text{Int } M$, $d(f(x), h(x)) < g(f(x))$. We define $F:M \to M$ by

$$F(x) = \begin{cases} h(x), & x \in \text{Int } M \\ f(x), & x \in \partial M \end{cases}$$

F is continuous, for suppose there is a sequence, $\{x_n\}$, of points in Int M which converge to $x \in M$. Let $\varepsilon > 0$ be given. By the continuity of f, there exists N such that $n > N \Longrightarrow d(f_n(x), f(x)) < \varepsilon/2$.

Then for such n:

$$d(F(x_n), F(x)) = d(h(x_n), f(x)) \leq$$
$$d(h(x_n), f(x_n)) + d(f(x_n), f(x)) < \varepsilon.$$

Thus, by corollary 2.4, $F|\partial M = f|\partial M$ is a UV^∞-map.

Corollary 2.8. Let M be an n-manifold with boundary, $n \geq 6$. Let $f:M \to M$ be such that $f|\text{Int } M$ is a cellular mapping of Int M onto Int M and $f|\partial M$ maps ∂M onto ∂M. Then $f|\partial M$ is a cellular mapping.

Proof. Since $n \geq 6$, the dimension of the boundary of M is ≥ 5 and the corollary follows immediately from corollary 2.7 and lemma 2.5.

CHAPTER III

OBTAINING BOUNDED CELLULAR MAPPINGS

An important procedure developed in recent years for dealing with certain homeomorphisms and spaces of homeomorphisms has been that of assigning to a given homeomorphism, f, a bounded homeomorphism which agrees with f on a given set. Kirby first developed such techniques to prove that a homeomorphism of R^n is stable if and only if it is isotopic to the identity.

The procedure has also proved to be a valuable tool in showing that: $H(M^n)$ is locally contractible if M^n is compact or equals E^n (Edwards and Kirby [8]); every cellular mapping of an n-manifold, $n \neq 4$, can be uniformly approximated by homeomorphisms (Siebenmann [23]); and in showing that the Hauptvermutung and the triangulation conjecture are false (Kirby and Siebenmann [14]).

In this chapter we develop the same techniques for a class of mappings on manifolds which can be uniformly approximated by homeomorphisms and therefore for a class of cellular mappings when $n \neq 4$. For instance, corollary 3.2 states (letting $r = 3/2$, $k = 0$, and $n \neq 4$) that there exists an $\varepsilon > 0$ and a mapping

$$\phi:\{f \in Ce(R^n) \mid d(f(x),x) < \varepsilon, \text{ for all } x \in 3/2B^n\} \longrightarrow$$

$$\{f \in Ce(R^n) \mid \text{for } x \in R^n - 3B^n,\ f(x) = x\} \text{ with}$$

the property that $\phi(f)|B^n = f|B^n$. Our proof will consist of considering a sequence of homeomorphisms, $\{h_i\}$, which converge to a mapping and then applying to each homeomorphism the techniques of Edwards and Kirby in a modified and slightly refined form.

Proposition 3.1. Let rB^n be a fixed n-ball, $r > 1$. Then there is an $\varepsilon > 0$ so that if $\overline{H}$ is the space of all functions, f, mapping $B^k \times 4B^n$ into $B^k \times R^n$ such that:

1. $f|\partial B^k \times 4B^n \cup [\frac{1}{2},1]B^k \times 3B^n = id$
2. if $x \in B^k \times rB^n$, then $d(f(x),x) < \varepsilon$
3. f can be uniformly approximated by homeomorphisms which equal the identity on $\partial B^k \times 4B^n \cup [\frac{1}{2},1]B^k \times 3B^n$.

And if $\overline{H}_1 = \{f \in \overline{H}: f|B^k \times (4B^n - 2rB^n) = id\}$, then there exists a continuous function $\phi:\overline{H} \to \overline{H}_1$ satisfying the condition that $\phi(f)|B^k \times B^n = f|B^k \times B^n$.

Proof. Let $H = \{f \in \overline{H} | f$ is a homeomorphism$\}$. The proof of this proposition will consist of assigning to each $(h,i) \in H \times N$ a homeomorphism, h^i, with the following properties:

a) $h|B^k \times B^n = h^i|B^k \times B^n$

b) $h^i|B^k \times (4B^n - 2rB^n) = id$

c) given $\eta > 0$, there exists $\delta_\eta > 0$ and an integer N such that if $i, j > N$ and

$$d(h(x), g(x)) < \delta_\eta \text{ for all } x \in B^k \times rB^n,$$

$$\text{then } d(h^i(x), g^i(x)) < \eta \text{ for all } x \in B^k \times 4B^n.$$

The construction of such an h^i for each pair (h, i) would complete the proof of the proposition, as the following argument indicates. Let $f \in \bar{H}$ and choose a sequence of homeomorphisms $\{h_i\}$ each of which is an element of H. We let $\phi(f) = \lim_{i \to \infty} h_i^i$.

By property a), $\phi(f)|B^k \times B^n = f|B^k \times B^n$ since for any $x \in B^k \times B^n$, $h_i^i(x) = h_i(x)$ and $h_i(x)$ converges to $f(x)$. Similarly property b) assures that $f|B^k \times (4B^n - 2rB^n)$ is the identity. Property c) assures that $\{h_i^i\}$ is a Cauchy sequence and therefore that $\phi(f)$ is a continuous function. For, let $\eta > 0$ be given. Pick an integer N_1 larger than the N of property c) such that $i, j > N_1$ implies that $d(h_i, h_j) < \delta_\eta$. Then $d(h_i^i, h_j^j) < \eta$.

$\phi(f)$ is independent of the choice of the sequence of homeomorphisms converging to f. For suppose $\{g_i\} \to f$, we shall show that $\lim_{i \to \infty} g_i^i = \phi(f)$. Let $\eta > 0$ be given. There is an N_1 such that $i > N_1$ implies $d(g_i, f) < \frac{1}{2}\delta_{\eta/2}$. There is an N_2 such that $i > N_2$ implies $d(h_i, f) < \frac{1}{2}\delta_{\eta/2}$. Choose N_3 so that $i > N_3$ implies $d(h_i^i, \phi(f)) < \eta/2$. Then if $i > \max(N_1, N_2, N_3)$, property c) implies that $d(g_i^i, h_i^i) < \eta/2$. In this case, $d(g_i^i, \phi(f)) \le d(g_i^i, h_i^i) + d(h_i^i, \phi(f)) < \eta/2 + \eta/2$. A similar argument yields that ϕ is continuous. To construct the homeomorphism h^i we shall make use of the diagram on the following page.

$$
\begin{array}{ccc}
B^k \times R^n & \xrightarrow{\quad h^1 \quad} & B^k \times R^n \\
\Big\uparrow{\scriptstyle \gamma^{-1}} & & \Big\uparrow{\scriptstyle \gamma^{-1}} \\
B^k \times R^n & \xrightarrow{\quad \tilde{h}^1 \quad} & B^k \times R^n \\
\Big\downarrow{\scriptstyle e} & & \Big\downarrow{\scriptstyle e} \\
B^k \times T^n & \xrightarrow{\quad \bar{h}^1 \quad} & B^k \times T^n \\
\Big\uparrow{\scriptstyle id} & & \Big\uparrow{\scriptstyle \omega_1} \\
B^k \times T^n & \xrightarrow{\quad \bar{h} \quad} & B^k \times T^n \\
\Big\uparrow{\scriptstyle i'} & & \Big\uparrow{\scriptstyle i'} \\
(B^k \times T^n) - (2D^k \times 2D^n) & \xrightarrow{\quad \tilde{h} \quad} & (B^k \times T^n) - (D^k \times D^n) \\
\Big\uparrow{\scriptstyle i} & & \Big\uparrow{\scriptstyle i} \\
B^k \times (T^n - 2D^n) & \xrightarrow{\quad \hat{h} \quad} & B^k \times (T^n - D^n) \\
\Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \alpha} \\
B^k \times 4B^n & \xrightarrow{\quad h \quad} & B^k \times R^n
\end{array}
$$

Let D^n, $2D^n$, $3D^n$, $4D^n$ be four concentric n-cells in $T^n - 2B^n$ such that $iD^n \subset \text{Int } (i+1)D^n$ for each i. In the same manner we let D^k, $2D^k$, $3D^k$, $4D^k$ be concentric k-cells in Int B^k such that $\frac{1}{2}B^k \subset D^k$ and $iD^k \subset \text{Int } (i+1)D^k$ for each i. Then let $\bar{\alpha}: T^n - D^n \to \text{Int } rB^n$ be a fixed immersion with the property that $\bar{\alpha}$ restricted to $\frac{r+1}{2} B^n$ is the identity. We choose our original $\varepsilon > 0$ small enough so that $h(B^k \times B^n) \subset B^k \times \frac{r+1}{2} B^n$. A theorem of Lees [17] assures that such an immersion exists. Let α denote the product immersion $id \times \bar{\alpha}: B^k \times (T^n - D^n) \to B^k \times \text{Int } rB^n$. For $h \in H$, we wish to canonically choose $\hat{h}: B^k \times (T^n - 2D^n) \to B^k \times (T^n - D^n)$ so that the lower square of the diagram commutes. To accomplish this let $\{U_i\}$ be a finite cover of $B^k \times (T^n - \text{Int } 2D^n)$ by open subsets of $B^k \times (T^n - D^n)$ with the property that if $U_i \cap U_j \neq \phi$, then $\alpha | U_i \cup U_j$ is a homeomorphism. For each U_i, consider a compact V_i such that $V_i \subset U_i$ and $\cup V_i$ covers $B^k \times (T^n - \text{Int } 2D^n)$. We choose our original ε small enough so that $h\alpha(W_i) \subset \alpha(U_i)$ for each of the finite number of i's. Given $h \in H$, let $\hat{h}|W_i = (\alpha|U_i)^{-1} h\alpha | W_i$. $\hat{h}$ is well defined since if $x \in W_i \cap W_j$, $h\alpha(x) \in \alpha(U_i \cap U_j)$ and we have required that α be a homeomorphism on $U_i \cup U_j$. $\hat{h}$ is thus the desired homeomorphism.

Finally, let $\gamma: \text{Int } 2rB^k \times 2rB^n \to R^k \times R^n$ be a homeomorphism which is a radial expansion and is the identity on $\frac{r+1}{2}B^k \times \frac{r+1}{2}B^n$. We extend $\tilde{h}^1$ by the identity to a homeomorphism $\tilde{h}^{1'}: R^k \times R^n \to R^k \times R^n$ and define $h^1: B^k \times R^n \to B^k \times R^n$ by

$$h^1(x) = \begin{cases} \gamma^{-1}\tilde{h}^1\gamma(x), & x \in B^k \times 2rB^n \\ x, & x \in B^k \times (R^n - \text{Int } 2rB^n). \end{cases}$$

Since $d(\bar{h}^1, id) < M$, h^1 is continuous and therefore is a homeomorphism. Property a) is satisfied since $\gamma e \omega_1^{-1} i'^{-1} i^{-1} \alpha(x) = x$ for all $x \in B^k \times \frac{r+1}{2} B^n$ and since $\varepsilon > 0$ was chosen small enough so that if $x \in B^k \times B^n$, then $h(x) \in B^k \times \frac{r+1}{2}B^n$. That property b) is satisfied was guaranteed by the choice of the homeomorphism γ.

To show that property c) is satisfied, suppose $\eta > 0$ is given. If $d(\tilde{h}^1(x), \tilde{\sigma}^1(x)) < \eta$, then $d(h^1(x), \sigma^1(x)) < \eta$ since γ is a radial expansion. $e: B^k \times R^n \to B^k \times T^n$ is uniformly continuous; therefore, there is a δ_1 such that if $d(\bar{h}^1(x), \bar{\sigma}^1(x)) < \delta_1$ for all $x \in B^k \times T^n$, then $d(\tilde{h}^1(x), \tilde{\sigma}^1(x)) < \eta$ for all $x \in B^k \times R^n$. We now wish to find an integer N and a positive number δ_2 such that if $d(\tilde{h}(x), \tilde{\sigma}(x)) < \delta_2$ for all $x \in (B^k \times T^n) - (2D^k \times 2D^n)$ and if $i, j > N$, $d(\bar{h}^1(x), \bar{\sigma}^1(x)) < \delta_2$ for all $x \in B^k \times T^n$:

let N be an integer such that $2/N < \delta_1$

pick δ_2 so that $\delta_2 < \delta_1/16$

We note that $\hat{h}|(B^k - 2D^k) \times (T^n - 2D^n)$ is the identity map, since h is the identity on $[\frac{1}{2},] B^k \times 3B^n$ and $\frac{1}{2}B^k \subset D^k$. Therefore, to obtain $\tilde{h}$, we extend $\hat{h}$ to be the identity on $(B^k - 2D^k) \times T^n$. Consider the restriction of $\tilde{h}$ to $(B^k \times T^n) - (3D^k \times 3D^n)$. By the Schoenflies theorem we can extend this restriction of $\tilde{h}$ to a homeomorphism $\bar{h}:B^k \times T^n \to B^k \times T^n$. This extension cannot be made to be canonical; i.e., if $\{\tilde{h}_i\}$ is a Cauchy sequence of homeomorphisms, it does not follow that $\{\bar{h}\}$ is a Cauchy sequence of homeomorphisms.

Until this point, the construction of the diagram is independent of i and varies only with the homeomorphism h. Consider $4D^k \times 4D^n$ to be $\{tx | x \in \partial(4D^k \times 4D^n)$, $0 \leq t \leq 4$. We then define the homeomorphism $\omega_i:B^k \times T^n \to B^k \times T^n$ which takes $3\frac{1}{2}D^k \times 3\frac{1}{2}D^n$ to $1/i\ D^k \times 1/i\ D^n$ by

a) $\omega_i | B^k \times T^n - (4D^k \times 4D^n) = \mathrm{id}$

b) $\omega_i(tx) = \begin{cases} (t - 3\frac{1}{2})(2)(4 - 1/i)x, & 3\frac{1}{2} \leq t \leq 4 \\ \frac{t}{3\frac{1}{2} i} x, & 0 \leq t \leq 3\frac{1}{2} \end{cases}$

Then $\bar{h}^i:B^k \times T^n \to B^k \times T^n$ is defined by $\bar{h}^i(x) = \omega_i \bar{h}(x)$. Let $\bar{e}:R^n \to T^n$ be a covering projection such that $\bar{e}| 2B^n$ is the identity and let $e:B^k \times R^n \to B^k \times T^n$ be equal to $\mathrm{id} \times \bar{e}$. Then $\bar{h}^i$ lifts to the homeomorphism $\tilde{h}^i:B^k \times R^n \to B^k \times R^n$. We note that $\bar{h}^i$ has the property that for some constant, M, $d(\bar{h}^i, \mathrm{id}) < M$.

We will consider the cases:

i) either $\bar{h}(x)$ or $\bar{g}(x)$ is not an element of $3\frac{1}{2}D^k \times 3\frac{1}{2}D^n$

ii) $\bar{h}(x) \in 3\frac{1}{2}D^k \times 3\frac{1}{2}D^n$ and $\bar{g}(x) \in 3\frac{1}{2}D^k \times 3\frac{1}{2}D^n$

In case i), one of $\bar{h}(x)$ and $\bar{g}(x)$ is an element of $(B^k \times T^n) - (3D^k \times 3D^n)$ by choice of the original ε and hence $\bar{h}(x) = \tilde{h}(x)$ and $\bar{g}(x) = \tilde{g}(x)$. Therefore, $d(\bar{h}(x), \bar{g}(x)) < \delta_1/16$. This implies that the radial distance between the two points and the distance in the sphere are each less than $\delta_1/16$. For any i and any j, ω_i and ω_j have the effect of increasing the radial distance between two points by a factor less than 8. Hence the radial distance between $\omega_i\bar{f}(x)$ and $\omega_j\bar{g}(x)$ is less than $8(\delta_1/16)$. The distance in the sphere remains the same or is decreased. Therefore, $d(\bar{h}^i(x), \bar{g}^j(x)) < \delta_1/16 + 8(\delta_1/16) < \delta_1$.

In case ii), $\bar{h}(x) \in 3\frac{1}{2}D^k \times 3\frac{1}{2}D^n$ and $\bar{g}(x) \in 3\frac{1}{2}D^k \times 3\frac{1}{2}D^n$ which implies that $\omega_i\bar{h}(x) \in 1/N\ D^k \times 1/N\ D^n$ and $\omega_j\bar{h}(x) \in 1/N\ D^k \times 1/N\ D^n$. Thus, $d(\bar{h}^i(x), \bar{g}^j(x)) < 2/N < \delta_1$.

If $d(\hat{h}(x), \hat{g}(x)) < \delta_2$ for all $x \in B^k \times (T^n - 2D^n)$, then $d(\tilde{h}(x), \tilde{g}(x)) < \delta_2$ for all $x \in (B^k \times T^n) - (2D^k \times 2D^n)$. Since α is uniformly continuous, there exists a $\delta > 0$ with the property that if $d(h(x), g(x)) < \delta$, for all $x \in B^k \times 4B^n$ then $d(\hat{h}(x), \hat{g}(x)) < \delta_2$ for all $x \in B^k \times (T^n - 2D^n)$. We have now shown that the construction of h^1 has the desired properties. This completes the proof of proposition 3.1.

Corollary 3.2. Let rB^n be a fixed n-ball, $r > 1$. Then there is an $\varepsilon > 0$ so that if $\overline{G}$ is the space of all functions, f, mapping $B^k \times 4B^n$ into $B^k \times R^n$ such that:

1) $f | \partial B^k \times 4B^n = id$
2) $x \in B^k \times rB^n$ implies $d(f(x),x) < \varepsilon$
3) f can be uniformly approximated by homeomorphisms which equal the identity on $\partial B^k \times 4B^n$

And if $\overline{G}_1 = \{f \in \overline{G}: f|B^k \times (4B^n - 2rB^n) = id\}$, then there exists a continuous function $\Psi:\overline{G} \to \overline{G}$ such that $\Psi(f)|B^k \times B^n = f|B^k \times B^n$.

Proof. Let $N_1 = \{(tx,v) \in B^k \times 4B^n | (1 - t) \leq 3/4(1 - 2d(v,3B^n))\}$

$$N_2 = \{(tx,v) \in B^k \times 4B^n | (1 - t) \leq \tfrac{1}{4}(1 - 2d(v,3B^n))\}$$

We define a homeomorphism

$$\gamma: \overline{(B^k \times 4B^n) - N_2} \to B^k \times 4B^n \text{ by}$$

$$\gamma(tx,v) = \begin{cases} (tx,v), & (tx,v) \notin N_1 \\ ([3t - \tfrac{1}{2} - 3d(v,3B^n)]x,v), & (tx,v) \in N_1 \end{cases}$$

Note that $\gamma | \overline{N_1 - N_2}$ maps $\overline{N_1 - N_2}$ onto N_1.

Next we define for each $f \in \overline{H}$ another function, $\tau(f)$ defined by

$$\tau(f)(x) = \begin{cases} \gamma^{-1} f \gamma(x), & x \in \overline{(B^k \times 4B^n) - N_2} \\ x, & x \in N_2 \end{cases}$$

$\tau(f)$ is an element of the set $\overline{H}$ of proposition 3.1. Thus if ϕ is the promised mapping of that proposition, $\phi(\tau(f)) \in \overline{H}_1$. Define $\sigma:\overline{H}_1 \to \overline{G}_1$ by $\sigma(h)(x) = \gamma h \gamma^{-1}(x)$. Finally we define $\Psi:\overline{G} \to \overline{G}_1$ by $\Psi(f) = \sigma\phi\tau(f)$.

Thus, if $x \in B^k \times B^n$,

$$\Psi(f)(x) = \gamma\phi(\gamma^{-1}f\gamma)\gamma^{-1}(x) =$$
$$\gamma(\phi(\gamma^{-1}f\gamma))(\gamma^{-1}(x)) = \gamma(\gamma^{-1}f\gamma)(\gamma^{-1}(x)) =$$
$$f(x), \text{ since } \gamma^{-1}(x) \in B^k \times B^n.$$

In the next chapter, it will be more convenient to work with this result in the following form.

Corollary 3.3. Let (a,b) be a pair of real numbers, $0 < a < b$. Then there is an $\varepsilon > 0$ so that if $\overline{G}$ is the space of all functions, f, mapping $B^k \times 4B^n$ into $B^k \times R^n$ such that:

1) $f|\partial B^k \times 4B^n = id$
2) $x \in B^k \times \frac{a+b}{2} B^n$ implies $d(f(x), x) < \varepsilon$
3) f can be uniformly approximated by homeomorphisms which equal the identity on $\partial B^k \times 4B^n$.

And if $\overline{G}_1 = \{f \in \overline{G}: f|B^k \times (4B^n - bB^n)\} = id$, then there exists a continuous function $\Psi:\overline{G} \to \overline{G}_1$ such that $\Psi(f)|B^k \times aB^n = f|B^k \times aB^n$.

CHAPTER IV

AN OPEN QUESTION AND A COVERING PROPERTY

This chapter is concerned with the problem of showing that the space of cellular mappings of an n-manifold onto itself is locally contractible. As was pointed out previously, it is known that the space of homeomorphisms of an n-manifold onto itself is locally contractible. It was demonstrated in chapter one that for $n \neq 4$, $Ce_\delta(B^n)$ is locally contractible and that for any n-manifold, M^n, $n \neq 4$, it suffices to show local contractibility of $Ce(M^n)$ at the identity. In this chapter we will show that for any $n \neq 4$ and any compact manifold M^n, $Ce(M^n)$ would be locally contractible if given $\varepsilon > 0$, there exists a continuous function $\phi_\varepsilon : Ce_\delta(B^n) \to H(B^n)$. It appears likely that Siebenmann's proof that any cellular mapping of M^n onto M^n can be uniformly approximated by homeomorphisms could be made to be canonical in the sense that we desire. However, I have not been able to demonstrate the truth of this conjecture. In the second part of this chapter we introduce a new type of covering property that is possessed by many metric spaces. If it could be shown that $Ce_\delta(B^n)$ has this property, then we could define directly the function ϕ_ε. Assuming the existence of such a ϕ_ε for all ε, we will prove:

<u>Proposition 4.1</u>. Suppose $n \neq 4$. There is a neighborhood Q of the inclusion $\eta: B^k \times 4B^n \to B^k \times R^n$ in $Ce(B^k \times 4B^n, \partial B^k \times 4B^n; B^k \times R^n)$ and a deformation of Q into $Ce(B^k \times 4B^n, \partial B^k \times 4B^n \cup B^k \times B^n; B^k \times R^n)$ modulo $\partial(B^k \times 4B^n)$.

<u>Proof</u>. Choose the neighborhood Q small enough so that the conditions of corollary 3.3 for $(1\frac{1}{2}, 2)$ will be met and associate with $f \in Q$ in a canonical manner the cellular map $\tilde{f}_1: B^k \times 4B^n \to B^k \times R^n$ with the properties:

a) $\tilde{f}_1 | B^k \times 1\frac{1}{2}B^n = f | B^k \times 1\frac{1}{2}B^n$

b) $\tilde{f}_1 | B^k \times (4B^n - 2\mathring{B}^n) \cup \partial B^k \times 4B^n = id$

Then associate with $\tilde{f}$ the homeomorphism $f_1 = \phi_{\varepsilon_1}(\tilde{f}_1)$, where ε_1 is small enough so that ff_1^{-1} is close enough to the identity to satisfy the hypothesis of corollary 3.3 for $(1\ 1/3, 1\frac{1}{2})$. Note that

a) $f_1: B^k \times 4B^n \to B^k \times R^n$ is a homeomorphism

b) $d(f_1(x), \tilde{f}_1(x)) < \varepsilon_1$, for all $x \in B^k \times 4B^n$

c) $d(f_1(x), f(x)) < \varepsilon_1$, for all $x \in B^k \times 1\frac{1}{2}B^n$

d) $f_1 | \partial(B^k \times 2B^n) = id$

e) $f_1 | B^k \times (4B^n - 2\mathring{B}^n) = id$

Now consider $ff_1^{-1}: B^k \times 4B^n \to B^k \times R^n$. Since $f_1^{-1} | \partial(B^k \times 2B^n)$ is the identity, there exists a canonical Alexander-type isotopy taking f to ff_1^{-1}, modulo the complement of $B^k \times 2B^n$. ff_1^{-1} is a cellular map which equals the identity on $\partial B^k \times 4B^n$ and is close enough to the identity to satisfy the conditions of corollary 3.3 for (1 1/3, 1½). Therefore, we can associate with ff_1^{-1} in a canonical manner a cellular map $\tilde{f}_2 : B^k \times 4B^n \to B^k \times R^n$ with the properties that

a) $\tilde{f}_2 | B^k \times 1\,1/3\, B^n = ff_1^{-1} | B^k \times 1\,1/3\, B^n$

b) $\tilde{f}_2 | B^k \times (4B^n - 1½\overset{o}{B}{}^n) \cup \partial B^k \times 4B^n = id$

Then associate with $\tilde{f}_2$ the homeomorphism $f_2 = \phi_{\varepsilon_2}(\tilde{f}_2)$, where ε_2 is small enough so that $ff_1^{-1}f_2^{-1}$ is close enough to the identity to satisfy the hypothesis of corollary 3.3 for (1½, 1 1/3). Note that

a) $f_2 : B^k \times 4B^n \to B^k \times R^n$ is a homeomorphism

b) $d(f_2(x), \tilde{f}_2(x)) < \varepsilon_2$, for all $x \in B^k \times 4B^n$

c) $d(f_2(x), ff_1^{-1}(x)) < \varepsilon_2$, for all $x \in B^k \times 1½B^n$

d) $f_2 | \partial(B^k \times 1½B^n) = id$

e) $f_2 | B^k \times (4B^n - 1½\overset{o}{B}{}^n) = id$

Now consider $f_1f_1^{-1}f_2^{-1} : B^k \times 4B^n \to B^k \times R^n$. Since $f_2^{-1} | \partial(B^k \times 1½B^n)$ is the identity, there exists a canonical Alexander-type isotopy taking ff_1^{-1} to $ff_1^{-1}f_2^{-1}$, modulo the complement of $B^k \times 1½B^n$.

Continuing inductively in this manner we define a sequence of cellular mappings $\tilde{f}_i:B^k \times 4B^n \to B^k \times R^n$ with the properties:

a) $\tilde{f}_i|B^k \times 1\ 1/i+1\ B^n = ff_1^{-1}. . .f_{i-1}^{-1}|$ $B^k \times 1\ 1/i+1\ B^n$

b) $\tilde{f}_i|B^k \times (4B^n - (1 + 1/i)\overset{o}{B}{}^n) \cup \partial B^k \times 4B^n =$ id

Also define a sequence of homeomorphisms $f_i:B^k \times 4B^n \to B^k \times R^n$ with the properties:

a) $d(f_i(x), \tilde{f}_i(x)) < \epsilon_i$, for all $x \in B^k \times 4B^n$

b) $d(f_i(x), ff_1^{-1}. . .f_{i-1}^{-1}(x)) < \epsilon_i$ for all $x \in B^k \times 1\ 1/i+1\ B^n$

c) $f_i|\partial B^k \times 4B^n \cup B^k \times (4B^n - (1 + 1/i)\overset{o}{B}{}^n) =$ id

where ϵ_i is chosen small enough so that $ff_1^{-1}. . .f_i^{-1}$ is close enough to the identity to satisfy corollary 3.3 for (1 1/i+2, 1 1/i+1). For each i we also define an isotopy taking $ff_1^{-1}. . .f_{i-1}^{-1}$ to $ff_1^{-1}. . .f_{i-1}^{-1} f_i^{-1}$.

Let $g:B^k \times 4B^n \to B^k \times R^n$ be the limit of the sequence $\{f, ff_1^{-1}, ff_1^{-1}f_2^{-1}, . . .\}$. Then g is continuous since if $x \in B^k \times B^n$, the sequence $\{f(x), ff_1^{-1}(x), . . .\}$ converges to x, if $x \notin B^k \times B^n$, the sequence becomes constant for sufficiently large i, and if $x \in B^k \times \partial B^n$ and $x_n \to x$, then $g(x_n)$ converges to $x = g(x)$. Since for any i, $ff_1^{-1}. . .f_i^{-1}$ is cellular, g is also cellular. The composition of the isotopies taking $ff_1^{-1} \ldots f_{i-1}^{-1}$ to $ff_1^{-1} \ldots f_{i-1}^{-1}f_i^{-1}$ provides the desired isotopy taking f to g.

Let M^n be a compact n-dimensional manifold. Using proposition 4.1 it is possible to consider a handlebody decomposition of M^n and construct the homotopy proving the following proposition. See Edwards and Kirby [8] for the details.

Proposition 4.2. Let M^n be a compact manifold. If given $\varepsilon > 0$ there is a continuous function $\phi_\varepsilon : \overline{H(B^n)} \to H(B^n)$, then $Ce(M^n)$ is locally contractible.

Proof. The construction of Edwards and Kirby shows that $Ce(M^n)$ is locally contractible at the identity and we have shown in chapter one that this implies local contractibility of the space.

Definition 4.3. A metric space (M,d) is said to have property (C) if given any $\varepsilon > 0$ there is a locally finite open cover $\mathcal{C}$ of M with diameter less than ε with the property that if $x \in M$ and n is any integer there exist at most n elements of $\mathcal{C}$ which contain x and have diameters greater than or equal to $\varepsilon/2^n$.

Proposition 4.4. Let X be a finite dimensional metric space. Then X has property (C).

Proof. Suppose X has covering dimension n. Let $\mathcal{U}$ be an open cover of X of diameter less than $\varepsilon/2^{n+1}$. Then there exists a locally finite open refinement $\mathcal{C}$ of $\mathcal{U}$ of order less than or equal to n+1 [20]. $\mathcal{C}$ meets the requirements of property (C) since if $i \leq n$ there are no sets of diameter greater than or equal to $\varepsilon/2^i$ and if $i \geq n+1$, there are at most n+1 sets containing any given point.

It can be shown that any locally finite polyhedron, any space that can be written as the union of countably many open finite dimensional spaces and Hilbert space under the usual metric all have property (C).

Proposition 4.5. If $\overline{H_\delta(B^n)}$ has property (C), then given any $\varepsilon > 0$ there is a map $\phi_\varepsilon : \overline{H_\delta(B^n)} \to H_\delta(B^n)$ such that if $f \in \overline{H_\delta(B^n)}$, then $d(f, \phi_\varepsilon(f)) < \varepsilon$.

Proof. Choose a locally finite cover $\mathcal{U}$ of $\overline{H_\delta(B^n)}$ of diameter less than $\varepsilon/2$ with the property that if $f \in \overline{H_\delta(B^n)}$ and j is any integer, there exist at most j members of $\mathcal{U}$ which contain f and have diameters greater than or equal to $\varepsilon/2^{j+1}$. Let $\eta : \overline{H_\delta(B^n)} \to N(\mathcal{U})$ be the standard map of $\overline{H_\delta(B^n)}$ into the nerve of the cover $\mathcal{U}$. The vertex of $N(\mathcal{U})$ corresponding to the set U_α is denoted by μ_α.

We next will define a mapping $\Psi:N(\mathcal{U}) \to H_\delta(B^n)$. Order the members of $\mathcal{U}$. We define Ψ_o from the 0-skeleton of $N(\mathcal{U})$ by letting $\Psi_o(\mu_\alpha) = h_\alpha$, where h_α is an element of U_α. If $<\mu_{\alpha_1}, \mu_{\alpha_2}>$ is a 1-simplex of $N(\mathcal{U})$ where $\alpha_1 < \alpha_2$, define Ψ_1 on this simplex by mapping $<\mu_{\alpha_1}, \mu_{\alpha_2}>$ barycentrically by the Alexander isotopy which takes h_{α_1} to h_{α_2}. Note that if $k \in <\mu_{\alpha_1}, \mu_{\alpha_2}>$ then $d(\Psi_1(k), h_{\alpha_2}) \leq d(h_{\alpha_1}, h_{\alpha_2})$. Next suppose $<\mu_{\alpha_1}, \mu_{\alpha_2}, \mu_{\alpha_3}>$ is a 2-simplex of $N(\mathcal{U})$ where $\alpha_1 < \alpha_2 < \alpha_3$. Define Ψ_2 on $<\mu_{\alpha_1}, \mu_{\alpha_2}, \mu_{\alpha_3}>$ by mapping the simplex barycentrically by the Alexander isotopy which takes $\Psi_1(<\mu_{\alpha_1},\mu_{\alpha_2}>)$ to h_{α_3}. The fact that the Alexander isotopy is canonical assures that Ψ_2 is continuous and the ordering of the members of $\mathcal{U}$ assures that Ψ_2 extends Ψ_1. If $k \in <\mu_{\alpha_1}, \mu_{\alpha_2}, \mu_{\alpha_3}>$ then there exists $k_1 \in <\mu_{\alpha_1}, \mu_{\alpha_2}>$ such that $\Psi_2(k)$ is on the isotopy taking $\Psi_1(k_1)$ to h_{α_3}. Hence $d(\Psi_2(k), h_{\alpha_3}) \leq d(\Psi_1(k_1), h_{\alpha_3}) \leq d(\Psi_1(k_1), h_{\alpha_2}) + d(h_{\alpha_2}, h_{\alpha_3}) \leq d(h_{\alpha_1}, h_{\alpha_2}) + d(h_{\alpha_2}, h_{\alpha_3})$. Continuing inductively, assume that Ψ_n has been defined on the n-skeleton of $N(\mathcal{U})$ using the Alexander isotopy in such a way that if $k \in <\mu_{\alpha_1},\ldots,\mu_{\alpha_n}>$, then $d(\Psi_n(k), h_{\alpha_n}) \leq d(h_{\alpha_1}, h_{\alpha_2}) + d(h_{\alpha_2}, h_{\alpha_3}) + \ldots d(h_{\alpha_{n-1}}, h_{\alpha_n})$. Let $<\mu_{\alpha_1},\ldots,\mu_{\alpha_{n+1}}>$ be an n+1 simplex of $N(\mathcal{U})$ with $\alpha_1 < \alpha_2 \ . \ . \ . < \alpha_n$. Define Ψ_{n+1} on $\langle \mu_{\alpha_1}, \ldots, \mu_{\alpha_{n+1}} \rangle$ by mapping this simplex barycentrically by the Alexander isotopy which takes $\Psi_n(\langle \mu_{\alpha_1}, \ldots, \mu_{\alpha_n} \rangle)$ to $h_{\alpha_{n+1}}$. Then if $k \in \langle \mu_{\alpha_1}, \ldots, \mu_{\alpha_{n+1}} \rangle$, $d(\Psi_{n+1}(k), h_{\alpha_{n+1}}) \leq d(h_{\alpha_1}, h_{\alpha_2}) + \ldots + d(h_{\alpha_n}, h_{\alpha_{n+1}})$. Let $\Psi = \lim \Psi_n$ and define $\phi_\varepsilon : \overline{H_\delta(B^n)} \rightarrow H_\sigma(B^n)$ by letting $\phi_\varepsilon(f) = \Psi\eta(f)$. If $f \in U_{\beta_1} \cap U_{\beta_2} \cap \ldots \cap U_{\beta_m}$, then $\eta(f) \in \langle \mu_{\beta_1}, \ldots, \mu_{\beta_m} \rangle$ and hence $d(\phi_\varepsilon(f), f) = d(\Psi\eta(f), f) \leq d(\Psi\eta(f), h_{\beta_m}) + d(h_{\beta_m}, f) \leq d(h_{\beta_1}, h_{\beta_2}) + \ldots + d(h_{\beta_{m-1}}, h_{\beta_m}) + d(h_{\beta_m}, f) < (\varepsilon/2^2 + \varepsilon/2^3 + \ldots + \varepsilon/2^{m+1}) + \varepsilon/2 < \varepsilon$.

CHAPTER V

FACTORIZATION OF UV^∞-MAPS

It was demonstrated in chapters one and two that every cellular map from an n-manifold onto itself was closed and had property UV^∞. In this chapter we will consider the more general situation of a closed UV^∞-map of a space X onto a metric space Y. First let us consider a property of any cellular mapping, f, of a manifold M^n onto itself, $n \neq 4$. Let $h:K \to Y$ be a map of any topological space into Y. Then given $\varepsilon > 0$, there is a map $\sigma:K \to X$ such that $d(f(\sigma(x)), h(x)) < \varepsilon$, for all $x \in K$. The proof of this statement is trivial. Let $w:M^n \to M^n$ be a homeomorphism with the property that $d(x(m), f(m)) < \varepsilon$, for all $m \in M^n$, and define σ by $\sigma(k) = w^{-1}(h(k))$. Then,

$$d(f(\sigma(k)), h(k)) = d(fw^{-1}(h(k)), h(k)) =$$
$$d(f(w^{-1}(h(k))), w(w^{-1}(h(k)))) < \varepsilon, \text{ since}$$
$$w^{-1}(h(k)) \in M^n.$$

It has been shown by Price [21], Kozlowski [15] and Lacher [16] that if $f:X \to Y$ is any closed UV^∞-map and K is a finite dimensional complex then $h:K \to Y$ can still be approximately factored through f. We will demonstrate the corresponding theorem for K a locally finite complex.

First we introduce some terminology that will be used throughout this chapter. A cover, T, of a space A is a collection of open sets whose union equals A. The diameter of a cover T is equal to the supremum of the diameters of the members of T. St(T, T) is defined to be the set of all points of A which are contained in a member of T intersecting T non-vacuously.

When there is no confusion we shall not distinguish between the complex K and the point set |K|. If B is a set and T a cover, h will be said to map B into T if there is some T ε T with h(B) ⊂ T. h will be said to map a complex K into T if h maps each simplex of K into T. $K^{(n)}$ will denote the n-th skeleton of the complex K. If α is a simplex of K, St(α, K) is defined to be the set of all points of K which are contained in a simplex of K which intersects α non-vacuously.

<u>Lemma 5.1.</u> Let $f:X \to Y$ be a closed UV^{∞}-map of X onto Y. Let $\mathcal{U}$ be a cover of Y. Then there exists a cover $\mathcal{V}$ of Y such that for each V ε $\mathcal{V}$ there is a U ε $\mathcal{U}$ so that

a) $St(V, \mathcal{V}) \subset U$

b) If $\gamma:S^{k-1} \to f^{-1}(St(V, \mathcal{V}))$ is given for any k, then there is an extension $\bar{\gamma}:B^k \to f^{-1}(U)$.

<u>Proof.</u> Choose a cover T of Y that star refines $\mathcal{U}$; i.e., if T ε T, there exists U ε $\mathcal{U}$ with St(T, T) ⊂ U. Such a cover exists since Y is paracompact. Then for

each point $y \in Y$, choose $T_y \in \mathcal{T}$ such that $y \in T_y$. Since f is a UV^∞-map, there is an open set, A, in X containing $f^{-1}(y)$ and contained in $f^{-1}(T_y)$ with the property that A is null-homotopic in $f^{-1}(T_y)$. Then choose V_y such that V_y is open in Y, $V_y \subset T_y$, $y \in V_y$ and $f^{-1}(V_y) \subset A$. Hence $f^{-1}(V_y)$ is null-homotopic in $f^{-1}(T_y)$. Let $\mathcal{V} = \{V_y | y \in Y\}$.

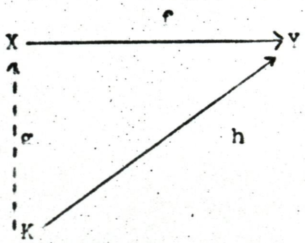
<u>Lemma 5.2</u>. Let K be a locally finite complex. Suppose $h:K \to Y$ is given and $\mathcal{U}$ is an open cover of Y. Let r be a positive integer. Then there exists a subdivision, K_1, of K so that

a) if σ is a r-simplex of K_1, then h maps $St(\sigma, K_1)$ into $\mathcal{U}$

b) if α is a simplex of K and $St(\alpha, K) \cap \overline{K - K^{(r-1)}} = \phi$, then α is a simplex of K_1.

<u>Proof</u>. Let $\tilde{K}$ be the first barycentric subdivision of K and let $L = \overline{\tilde{K} - \tilde{K}^{r-1}}$. Then define J to be the closed star of L in $\tilde{K}$. J is a locally finite complex (subcomplex of $\tilde{K}$) which can therefore be subdivided, forming $\tilde{J}$, in such a manner that if α is a simplex of $\tilde{J}$ then $h|\tilde{J}$ maps $St(\alpha, \tilde{J})$ into $\mathcal{U}$. If γ is any simplex of $\tilde{K}$ which is not contained in J but intersects J, subdivide γ by coning from the barycenter of γ over its boundary as subdivided by the formation of $\tilde{J}$. We have by this procedure constructed a subdivision, K_1, of K with the desired properties.

<u>Theorem 5.3</u>. Let $f:X \to Y$ be a closed UV^{∞}-map, K be a locally finite complex and $h:K \to Y$ be an arbitrary continuous function. Given any $\varepsilon > 0$ there is a map $\sigma:K \to X$ such that $d(f\sigma(k), h(k)) < \varepsilon$, for all $k \in K$; i.e., such that the following diagram commutes within ε.



<u>Proof</u>. Choose an open cover $\mathcal{U}$ of Y of diameter less than ε. Then choose a subcover $\mathcal{T}_1$ with the property that for each $T_1 \in \mathcal{T}_1$ there exists $U \in \mathcal{U}$ such that

a) $St(T_1, \mathcal{T}_1) \subset U$

b) given any n and any map $\gamma:S^{n-1} \to f^{-1}(St(T_1, \mathcal{T}_1))$, γ can be extended to a map $\bar{\gamma}:B^n \to f^{-1}(U)$.

Making use of lemma 5.2, we let K_1 be a subdivision of K so that h maps $St(K_1)$ into $\mathcal{T}_1$. Define $\sigma_0:K_1^0 \to X$ by letting $\sigma_0(v)$ be any element of $f^{-1}(h(v))$. Now let α be a 1-simplex of K_1^1. Let T_1 be an element of $\mathcal{T}_1$

with the property that $h(St(\alpha)) \subset T_1$. This implies that $\sigma_0(\partial\alpha) \subset f^{-1}(T_1)$. Therefore, we can extend σ_0 to a map σ_1 taking α into $f^{-1}(U)$ for some U containing T_1. We repeat this procedure for each 1-simplex of K_1 and thereby define $\sigma_1 : K_1^1 \to X$ with the property that if α is a simplex of K_1^1, then $f(\sigma(\alpha)) \cup h(St(\alpha, K_1))$ is contained in some member of $\mathcal{U}$.

Assume inductively that we have defined subdivisions $K_1, K_2, \ldots, K_n$ of K; covers $U = T_0, T_1, \ldots, T_n$ of Y and maps $\sigma_1, \ldots, \sigma_n$ with the following properties

a) $\sigma_i : K_i^i \to X$

b) if α is a simplex of K_i^i and $St(\alpha, K_i) \cap K_i - K_i^i = \phi$, then $j > i$ implies that $\sigma_i|\alpha = \sigma_j|\alpha$.

c) if α is a j-simplex of K_i^i and if k is the maximum of i and the dimension of $St(\alpha, K_i)$, then $h(St(\alpha,K_i)) \cup f\sigma_i(\alpha) \subset T_{k-j}$ for some $T_{k-j} \in T_{k-j}$

d) $K > K_1 > K_2 > \cdots > K_n$

e) if α is a j-simplex of K_j, h maps $St(\alpha,K_j)$ into T_j

f) for each $T_i \in T_i$ there is a $T_{i-1} \in T_{i-1}$ such that $St(T_i, T_i) \subset T_{i-1}$ and if $\gamma : S^{k-1} \to f^{-1}(St(T_i, T_i))$ is defined for any k, then there is an extension of γ, $\bar{\gamma} : B^k \to f^{-1}(T_{i-1})$

Note that we have completed such a procedure for $n = 1$. We now will define K_{n+1}, σ_{n+1} and $\mathcal{T}_{n+1}$ with the required properties.

Let $\mathcal{T}_{n+1}$ be a refinement of $\mathcal{T}_n$ such that for each $T_{n+1} \in \mathcal{T}_{n+1}$ there exists $T_n \in \mathcal{T}_n$ such that

a) $St(T_{n+1}, \mathcal{T}_{n+1}) \subset T_n$

b) for any k and any $\gamma: S^{k-1} \to f^{-1}(St(T_{n+1}, \mathcal{T}_{n+1}))$ there is an extension $\bar{\gamma}: B^k \to f^{-1}(T_n)$.

Let K_{n+1} be a subdivision of K_n such that

i) if α is a simplex of K_n^n and $St(\alpha, K_n^n) \cap \overline{K_n - K_n^n} = \phi$, then α is a simplex of K_{n+1}

ii) if α is an $(n+1)$-simplex of K_{n+1}, then h maps $St(\alpha, K_{n+1})$ into T_{n+1}.

We now define the map $\sigma_{n+1}: K_{n+1}^{n+1} \to X$. If v is a vertex of K_{n+1}, define σ_{n+1} at v, by

$$\sigma_{n+1}(v) = \begin{cases} \sigma_n(v), & v \text{ is a vertex of } K_n \\ \text{any element of } f^{-1}(h(v)), & v \text{ not a vertex of } K_n \end{cases}$$

We next define $\sigma_{n+1}|K_{n+1}^1$. If σ_1 is a 1-simplex of K_{n+1}, let j be the maximum of $n+1$ and the dimension of $St(\sigma_1, K_{n+1})$. Note that h maps $St(\sigma_1, K_{n+1})$ into $\mathcal{T}_j$ and choose $T_j \in \mathcal{T}_j$ with $h(St(\sigma_1, K_{n+1})) \subset T_j$. Choose T_{j-1} such that any mapping of S^k into $f^{-1}(St(T_j, \mathcal{T}_j))$ extends to a mapping of B^{k+1} into $f^{-1}(T_{j-1})$ and $St(T_j, \mathcal{T}_j) \subset T_{j-1}$. If $St(\sigma_1, K_{n+1}) \cap \overline{K_{n+1} - K_{n+1}^{n+1}} = \phi$, let $\sigma_{n+1}|\sigma_1 = \sigma_n|\sigma_1$.

Otherwise, note that $\sigma_{n+1}(\partial\sigma_1) \subset f^{-1}(St(T_1, T_1))$. Extend σ_{n+1} on σ_1 so that $\sigma_{n+1}(\sigma_1) \subset f^{-1}(T_{k-1})$. Note that

$$f\sigma_{n+1}(\sigma_1) \cup h(St(\sigma_1, K_{n+1})) \subset T_{k-1}.$$

Next assume (subinductive statement) that $\sigma_{n+1} | K^r_{n+1}$ has been defined in such a way that

i) if $\dim \sigma \leq r$ and $St(\sigma, K_{n+1}) \cap \overline{K_{n+1} - K^n_{n+1}} = \phi$ then $\sigma_{n+1}|\sigma = \sigma_n|\sigma$

ii) if σ is a simplex of K^r_{n+1} and if k is the maximum of n+1 and the dimension of $St(\sigma, K_{n+1})$, then there exists $T_{k-r} \in T_{k-r}$ such that $h(St(\sigma, K_{n+1})) \cup f\sigma_{n+1}(\sigma) \subset T_{k-r}$ and $\sigma_{n+1}(\sigma) \subset f^{-1}(T_{k-r})$.

We have demonstrated this subinductive statement for r = 1. Now let σ_{r+1} be an (r+1)-simplex of K^{r+1}_{n+1}. If $St(\sigma_{r+1}, K_{n+1}) \cap \overline{K_{n+1} - K^n_{n+1}} = \phi$, then define $\sigma_{n+1}|\sigma_{r+1}$ to be $\sigma_n|\sigma_{r+1}$. (Note that this extends $\sigma_{n+1}|\partial\sigma_{r+1}$, by assumption i.) σ_n satisfies the inductive statement. Therefore σ_{n+1} satisfies subinductive statement ii.

If $St(\sigma_{r+1}, K_{n+1}) \cap \overline{K_{n+1} - K^n_{n+1}} \neq \phi$, and γ is a simplex in the boundary of σ_{r+1}, we note that by ii) there exists $T_{k-r} \in T_{k-r}$ such that

$$h(St(\gamma, K_{n+1})) \cup f\sigma_{n+1}(\gamma) \subset T_{k-r}.$$

Pick one of these, call it T_{k-r}, and note that

$\sigma_{n+1}(\partial\sigma_{r+1}) \subset f^{-1}(St(T_{k-r}, T_{k-r}))$. We then extend g_{n+1} to σ_{r+1} in such a way that for some T_{k-r-1}, $g_{n+1}(\sigma_{r+1}) \subset f^{-1}(T_{k-r-1})$ and

$$f(g_{n+1}(\sigma_{r+1})) \cup h(St(\sigma_{r+1}, K_{n+1})) \subset T_{k-r-1},$$

since

$$h(St(\sigma_{r+1}, K_{n+1}))) \subset h(\bigcup_{\gamma \lneq \sigma_{r+1}} St(\gamma, K_{n+1})) \subset$$

$$h(\bigcup_{\gamma \lneq \sigma_{r+1}} St(\gamma, K_n)) \subset St(T_{k-r}, T_{k-r}) \subset T_{k-r-1}.$$

This completes the subinductive statement and with that the inductive statement. Properties a, d, e and f are trivially satisfied by definition. Property b is assured by property i) of the subinductive step and property c is satisfied by subinductive property ii).

We now define $g:K \to X$ by

$$g(x) = \lim_{n \to \infty} g_n(x).$$

For any $x \in K$ choose any simplex α containing x. The local finiteness of K assures that there is an integer N so that $St(\alpha, K) \cap \overline{K - K^N} = \phi$. Thus, for any $n \geq N$, $St(\alpha, K_n) \cap \overline{K - K_n^n} = St(\alpha, K_n) \cap \overline{K - K^n} \subset St(\alpha, K) \cap \overline{K - K^N} = \phi$. Hence for $n \geq N$, $g_n(x) = g_N(x)$. Therefore, g is well-defined and continuous. Let $x \in K$ and α be a simplex of maximal dimension containing x. Then there exists an integer N such that $St(\alpha,K) \cap \overline{K - K^N} = \phi$. Choose a simplex β in K_N^N containing x. Then $g(x) = g_N(x)$ and by inductive statement c, there is a T in some T_1 so that $h(St(\beta, K_N)) \cup fg_N(\beta) \subset T$. Since T_1 refines

$\mathcal{U}$, $h(x) \cup fg(x) \subset U$ for some U and therefore $d(h(x), fg(x)) < \varepsilon$.

Corollary 5.4. Let $f:X \to Y$ be a closed UV^∞-map, K a locally finite complex, L a subcomplex of K. Let $h:K \to Y$ and $g:L \to X$ be mappings such that for all $l \in L$, $f(g(l)) = h(l)$. Then given any $\varepsilon > 0$ there is a map $\tilde{g}:K \to L$ such that $\tilde{g}$ extends g and $d(f\tilde{g}(k), h(k)) < \varepsilon$, for all $k \in K$.

Proof. Proceed exactly as in theorem 5.3, except that if at any stage α is contained in L, define $\tilde{g}_n|\alpha$ to be $g|\alpha$. Since fg commutes with h, $fg(\alpha)$ will be contained in all the necessary T's.

Maps $u:A \to B$ and $v:A \to B$ are ε-homotopic if there exists a homotopy H taking u to v so that $d(H_t, H_{t'}) < \varepsilon$, for all t, t'.

Theorem 5.5. Let $f:X \to Y$ be a closed UV^∞-map and K a locally finite complex. Given a map $h:X \to Y$ and $\varepsilon > 0$, there exists a $g:K \to X$ such that fg is ε-homotopic to h.

Proof. For each nonnegative integer i, choose a cover $\mathcal{U}^i$ of Y of diameter less than $\varepsilon/4(i+1)$. For each $i \geq 1$ define sequences

$$K_{1,i}, K_{2,i}, \ldots$$
$$T_{-1,i} = \mathcal{U}^{i-1}, T_{0,i} = \mathcal{U}^i, T_{1,i}, T_{2,i} \ldots$$
$$g_{1,i}, g_{2,i}, \ldots$$

as in the proof of theorem 5.3, with the additional conditions that

for each $U_i \in \mathcal{U}^i$ there is a $U_{i-1} \in \mathcal{U}^{i-1}$ so that $St(U_i, \mathcal{U}^i) \subset U_{i-1}$ and if $\gamma: S^{k-1} \to f^{-1}(St(U_i, \mathcal{U}^i))$ is defined for any k, then there is an extension $\bar{\gamma}: B^k \to f^{-1}(U_{i-1})$.

$T_{p,i+1}$ refines $T_{p,i}$ for all p, i.

$K_{p,i+1}$ refines $K_{p,i}$ for all p, i.

Then let $g^i = \lim_{n\to\infty} g_{n,i}$.

For each positive integer i, we will define a homotopy $G^i: K \times I \to X$ such that

a) $G^i(k,0) = g^i(k)$ for all $k \in K$

b) $G^i(k,1) = g^{i+1}(k)$ for all $k \in K$

c) if $k \in K$, fG^i maps $k \times I$ into $\mathcal{U}^{i-1}$.

Fix i. Assume, inductively, that we have defined maps $G^i_1, G^i_2, \ldots, G^i_n$ in such a way that:

i) $G^i_j: K^j_{j,i} \times I \to X$ for $j = 1, 2, \ldots, n$

ii) $G^i_j(k,0) = g^i_j(k)$; $G^i_j(k,1) = g^{i+1}_j(k)$ for all $k \in K^j$

iii) if α is a simplex of $K_{j,i}$ and $St(\alpha, K_{j,i}) \cap \overline{K - K^j} = \phi$, then if $m > j$, $G^i_m|\alpha \times I = G^i_j|\alpha \times I$

iv) for every p-simplex $\beta \in K^j_{j,i}$, let k be the maximum of j and $St(\beta, K_{j,i})$.

Then there exists $T_{k-p-1,i} \in \mathcal{T}_{k-p-1,i}$ such that $f(G_j^i(\beta \times I)) \subset T_{k-p-1,i}$.

We now will define $G_{n+1}^i : K_{n+1,i}^{n+1} \times I \to X$. If v is a vertex of $K_{n+1,i}$, let j be the maximum of n+1 and the dimension of $St(v, K_{n+1,i})$. Then $fg_{n+1}(v) \cup h(v) \subset T_{j-1,i}$ for some $T_{j-1,i} \in \mathcal{T}_{j-1,i}$ and $fg_{n+2}(v) \cup h(v) \subset T_{j-1,i+1}$ for some $T_{j-1,i+1}$. But, $\mathcal{T}_{j-1,i+1}$ refines $\mathcal{T}_{j-1,i}$. Hence $fg_{n+1}(v) \cup fg_{n+2}(v) \subset f^{-1}(St(T_{j-1,i}))$ and we can define $G_{n+1}^i : K_{n+1,i}^0 \times I \to X$ in such a way that for any $v \in K_{n+1}^0$, $G_{n+1}^i(v \times I) \subset f^{-1}(T_{j-2,i})$ for some $T_{j-2,i} \in \mathcal{T}_{j-2,i}$, agreeing with G_n^i when appropriate.

Now, assume inductively that we have defined $G_{n+1}^i : K_{n+1,i}^p \times I \to X$ so that

a) if σ is a simplex of $K_{n,i}$ and $St(\sigma, K_{n,i}) \cap \overline{K_{n,i} - K_{n,i}^i} = \phi$ then $G_{n+1}^i | \sigma \times I = G_n^i | \sigma \times I$

b) if σ is a simplex of $K_{n+1,i}^p$ and if k is the maximum of n+1 and the dimension of $St(\sigma, K_{n+1,i})$, then there exists $T_{k-p-1,i} \in \mathcal{T}_{k-p-1,i}$ such that $f(G_{n+1}^i(\sigma \times I)) \subset T_{k-p-1,i}$

c) $G_{n+1}^i(k,0) = g_{n+1}^i(k)$, $G_{n+1}^i(k,1) = g_{n+1}^{i+1}(k)$ for all $k \in K_{n+1,i}^p$

Let σ be a (p+1)-simplex of $K_{n+1,i}$. If $St(\sigma, K_{n+1,i}) \cap \overline{K_{n+1,i} - K_{n+1,i}^{n+1}} = \phi$, let $G_{n+1} | \sigma \times I = G_n^i | \sigma \times I$.

Let k be the maximum of n+1 and the dimension of $St(\sigma, K_{n+1,i})$. For each simplex γ in the boundary of σ, there is a $T_{k-p-1,i}$ such that $f(G^i_{n+1}(\gamma \times I)) \subset T_{k-p-1,i}$. Also there is a member of $T_{k-p-1,i}$ that contains $f(g^i_{n+1}(\sigma))$ and another member of $T_{k-p-1,i}$ that contains $f(g^{i+1}_{n+1}(\sigma))$, since $T_{k-p-1,i+1}$ refines $T_{k-p-1,i}$. Define $G^i_{n+1}(k,0) = g^i_{n+1}(k)$, for all $k \in \sigma$ and $G^i_{n+1}(k,1) = g^{i+1}_{n+1}(k)$ for all $k \in \sigma$. Hence $G^i_{n+1}(\partial(\sigma \times I)) \subset f^{-1}(St(T_{k-p-1,i}, T_{k-p-1,i}))$ and we can extend G^i_{n+1} to $\sigma \times I$ such that $G^i_{n+1}(\sigma \times I) \subset f^{-1}(T_{k-p-2,i})$ for some $T_{k-p-2,i} \in T_{k-p-2,i}$. Let $G^i = \lim_{i\to\infty} G^i_n$. This completes the proof of the inductive statement. Note that for any point $k \in K$, there exists a simplex σ and an integer N such that $G^i|k \times I = G^i_N|k \times I$. and that for some $T_{-1,i} = U_{i-1} \in \mathcal{U}^{i-1}$ $fG^i(k \times I) \subset fG^i(\sigma \times I) \subset U_{i-1}$. So, the diameter of $fG^i(\sigma \times I)$ is less than $\epsilon/4i$.

For each i consider fG^i as mapping $K \times [1 - 1/i, 1 - 1/(i+1)]$ into Y and define $H:K \times I \to Y$ by

$$H(k,t) = \begin{cases} fG^i(k,t), & 1 - 1/i \le t \le 1 - 1/(i+1) \\ h(k), & t = 1 \end{cases}$$

H is continuous since for each i, $fG^i|K \times \{1 - 1/(i+1)\} = fG^{i+1}|K \times \{1 - 1/(i+1)\}$ and by choice of the covers $\mathcal{U}^i$, $\{fg^i\}$converges to h and the diameters of the defining homotopies approach 0 as $i \to \infty$. H is therefore an ε-homotopy taking fg^1 to h.

Corollary 5.6. Let $f:X \to Y$ be a closed UV^∞-map, K a locally finite complex and L a subcomplex of K. Let $h:X \to Y$ and $g:L \to X$ be mappings such that for all $l \in L$, $f(g(l)) = h(l)$. Then given any $\varepsilon > 0$ there exists a map $g:K \to X$ and an ε-homotopy taking fg to h.

Proof. The proof follows immediately from that of theorem 5.5.

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