Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

Simon Jubert

A Yau–Tian–Donaldson correspondence on a class of toric fibrations

Tome 73, nº 6 (2023), p. 2567-2604.

https://doi.org/10.5802/aif.3580

Les *Annales de l'Institut Fourier* sont membres du Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org e-ISSN : 1777-5310

Ann. Inst. Fourier, Grenoble
73, 6 (2023) 2567-2604

A YAU–TIAN–DONALDSON CORRESPONDENCE ON A CLASS OF TORIC FIBRATIONS

by **Simon JUBERT**

Abstract. — We establish a Yau–Tian–Donaldson type correspondence, expressed in terms of a single Delzant polytope, concerning the existence of extremal Kähler metrics on a large class of toric fibrations, introduced by Apostolov–Calderbank–Gauduchon–Tonnesen-Friedman and called *semi-simple principal toric fibrations*. We use that an extremal metric on the total space corresponds to a weighted constant scalar curvature Kähler metric (in the sense of Lahdili) on the corresponding toric fiber in order to obtain an equivalence between the existence of extremal Kähler metrics on the total space and a suitable notion of weighted uniform K-stability of the corresponding Delzant polytope. As an application, we show that the projective plane bundle $\mathbb{P}(\mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2)$, where $\mathcal{L}_i$ are holomorphic line bundles over an elliptic curve, admits an extremal metric in every Kähler class.

Résumé. — Nous établissons une correspondance du type Yau–Tian–Donaldson, exprimée en terme d'un polytope de Delzant, concernant l'existence de métriques Kähler extrémales sur une large classe de fibrations toriques définie par Apostolov–Calderbank–Gauduchon–Tonnesen-Friedman et appelée *semi-simple principal toric fibrations*. Nous utilisons qu'une extrémale sur l'espace total correspond à une métrique à courbure scalaire constante pondérée (dans le sens de Lahdili) sur la fibre torique correspondante pour obtenir une équivalence entre l'existence des métriques extrémales sur l'espace total et une notion appropriée de K-stabilité uniforme pondérée du polytope de Delzant correspondant. En tant qu'application, nous montrons que le fibré en plan projectif $\mathbb{P}(\mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2)$, où les $\mathcal{L}_i$ sont des fibrés holomorphes au dessus d'une courbe elliptique, admet une métrique extrémale dans chaque classe de Kähler.

1. Introduction

1.1. Motivation

A central problem in Kähler geometry, proposed by Calabi [15] in the 1980's, is to find a canonical Kähler metric in a given cohomology class of

Keywords: Differential geometry, complex geometry, complex analysis, toric geometry.
2020 *Mathematics Subject Classification:* 00X99.

a compact Kähler manifold. Calabi suggested looking for *extremal Kähler metrics*, characterized by the property that the flow of the gradient of the scalar curvature preserves the complex structure [16]. Constant scalar curvature Kähler (cscK for short) metrics and the much studied Kähler–Einstein metrics are particular examples of such metrics.

The existence of an extremal Kähler metric in a given Kähler class is conjecturally equivalent to a certain notion of *stability* through an extension of the Yau–Tian–Donaldson's (YTD) conjecture, introduced [62, 63] for the polarized case and [31] for a general Kähler class. This conjecture, its ramification [62, 63] and extension [8, 27, 46, 49, 62, 63, 61] have generated tremendous efforts in Kähler geometry and have led to many interesting developments during the last decades.

The question has been settled in some special cases, especially on smooth toric varieties [20, 21, 32, 43, 55, 66] where the relevant stability notion is expressed in terms of the convex affine geometry of the corresponding Delzant polytope, and is referred to as *uniform K-stability* of the polytope. Other special cases include Fano manifolds (see e.g. [13, 22, 23, 24, 56, 64]), total spaces of projective line bundles over a cscK base [6, 7] and certain varieties with a large symmetry group [28]. In general, though, the YTD conjecture is still open and it is expected that the relevant notion of stability would be the one of *relative uniform K-stability*, see e.g. [13, 30, 62, 63].

In [5], the authors introduced a class of fiber bundles, called *semi-simple rigid toric bundles*, which have toric Kähler fibers. They are obtained from *the generalized Calabi construction* [5, 7] involving a product of polarized Kähler manifolds, a certain principal torus bundle and a given toric Kähler manifold. In this paper, we are interested in the special case of *semi-simple principal toric fibration*. Namely, this is the case when the base is a global product of cscK Hodge manifolds, and there are no blow-downs, see [7] or Remark 3.2 below. Examples include the total space of the projectivisation of a direct sum of holomorphic line bundles over a compact complex curve, as well as the $\mathbb{P}^1$-bundle constructions over the product of cscK Hodge manifolds originally used by Calabi [15] and generalized in many subsequent works (see e.g. [6, 37, 44, 45, 47, 59, 65]). On any semi-simple rigid toric fibration, the authors of [5] introduced a class of Kähler metrics, called *compatible Kähler metrics*. A Kähler class containing a compatible Kähler metric is referred to as a *compatible Kähler class*. For any compatible Kähler metric on M, the momentum map of the toric Kähler fiber $(V, \omega_V, J_V, \mathbb{T})$ can be identified with the momentum map of the induced $\mathbb{T}$-action on (M, J, ω_M), so that the momentum image of M is the Delzant polytope P of $(V, \omega_V, J_V, \mathbb{T})$. In [7], the authors made the following conjecture:

Conjecture 1.1 ([7]). — *Let $(M, J, \omega, \mathbb{T})$ a semi-simple rigid toric fibration and P its associated Delzant polytope. Suppose $[\omega]$ is a compatible Kähler class. Then the following statements are equivalent:*

(1) *$(M, J, [\omega])$ admits an extremal Kähler metric;*
(2) *$(M, J, [\omega])$ admits a compatible extremal Kähler metric;*
(3) *P is weighted K-stable.*

In the third assertion, the notion of *weighted stability* is a weighted version of the notion of K-stability introduced in [32] (see Section 7.5) for appropriate values of the weight functions, and asks for the positivity of a linear functional defined on the space of convex piece-wise linear functions which are not affine-linear over P.

1.2. Main results

Our purpose in this paper is to solve Conjecture 1.1 for semi-simple principal toric fibrations.

Theorem 1. — *For M a semi-simple principal toric fibration, Conjecture 1.1 is true if we replace condition* (3) *with the notion of weighted uniform K-stability, see Definition 7.8.*

In the above statement, in order to define *uniform* (weighted) K-stability (see Definition 7.8), we use normalized continuous convex functions which are smooth in the interior of P and the usual L^1-norm of P. By C^0 density and continuity, this is equivalent to the uniform (weighted) K-stability of P, defined in terms of normalized convex piecewise linear functions and the L^1-norm.[(1)]

We split Theorem 1 in two statements: Theorem 2 and Theorem 3 below. Theorem 2 corresponds to the statement "(1) $\Leftrightarrow$ (2)" in Conjecture 1.1.

Theorem 2 (Theorem 6.1). — *Let $(M, J, \omega_M, \mathbb{T})$ be a semi-simple principal toric fibration with fiber $(V, J_V, \omega_V, \mathbb{T})$. Then, the following statements are equivalent:*

(1) After the submission of the first version of our article on the arXiv, we have been contacted by Yasufumi Nitta who kindly shared with us his manuscript with Shunsuke Saito in which the authors establish, in the case of a polarized toric variety, the equivalence between various notions of uniform K-stability of P. In particular, their result gives a strong evidence and establishes in a certain case the equivalence between the uniform weighted K-stability and and a suitable notion of weighted K-stability of P in Conjecture 1.1(3).

(1) *there exists an extremal Kähler metric in* $(M, J, [\omega_M], \mathbb{T})$*;*
(2) *there exists a compatible extremal Kähler metric in* $(M, J, [\omega_M], \mathbb{T})$*;*
(3) *there exists a weighted cscK metric in* $(V, J_V, [\omega_V], \mathbb{T})$ *for the weights defined in* (3.11) *below.*

In the third assertion, the notion of weighted cscK metric is in the sense of [49], see Section 2 for a precise definition. The equivalence (2) $\Leftrightarrow$ (3), established in [5], is recalled in Section 3. The main idea behind the proof of (1) $\Rightarrow$ (2) is to use that [20, 21, 42] the existence of an extremal Kähler metric implies a certain properness condition of the corresponding relative Mabuchi functional (see Theorem 5.2 below for a precise statement) and then shows that the continuity path of [19] can be made in the subspace of *compatible* Kähler metrics in $[\omega_M]$. The deep results [20, 21, 42] then yield the existence of an extremal Kähler metric in $[\omega_M]$ given by the generalized Calabi construction of [5].

Recently, building on the proof of Theorem 2, a similar statement was established in [9] for a larger class of fibrations associated to a certain class of principal $\mathbb{T}$-bundles over products of cscK Hodge manifolds, whose fiber is an arbitrary compact Kähler manifold containing $\mathbb{T}$ in its reduced automorphism group.

Theorem 3 below corresponds to the statement "(1) $\Leftrightarrow$ (3) " in Conjecture 1.1 and provides a *criterion* for verifying the equivalent conditions of Theorem 2, expressed in terms of the Delzant polytope of the fiber and data depending on the topology of M and the compatible Kähler class.

Theorem 3 (Theorem 7.12). — *Let* $(M, J, \omega_M, \mathbb{T})$ *be a semi-simple principal toric fibration with fiber* $(V, J_V, \omega_V, \mathbb{T})$ *and denote by* P *its associated Delzant polytope. Then there exists a weighted cscK metric in* $[\omega_V]$ *if and only if* P *is weighted uniformly* K*-stable, for the weights defined in* (3.11)*. In particular, the latter condition is necessary and sufficient for* $[\omega_M]$ *to admit an extremal Kähler metric.*

The strategy of proof of the above result consists in considering the extremal Kähler metrics on the total space $(M, J, \mathbb{T})$ as weighted (v, w)-cscK metrics on the corresponding toric fiber $(V, J_V, \omega_V, \mathbb{T})$ via Theorem 2. We then use the Abreu–Guillemin formalism and a weighted adaptation of the results in [17, 32, 66] to establish the equivalence on $(V, \omega_V, J_V, \mathbb{T})$: in one direction, namely showing that the existence implies that polytope is weighted uniformly K-stable, the argument follows from a straightforward modification of the result in [17], which appears in [53]. To show the other direction, we build on [32, 66] to obtain in Proposition 7.9 that the uniform weighted K-stability of the polytope implies a certain notion of coercivity of the weighted Mabuchi energy. We then show that the latter implies the properness of the Mabuchi energy of M, and we finally conclude by invoking again [20, 21, 42].

Finally, we will be interested in a certain class of almost Kähler metrics on a toric manifold $(V, \omega, \mathbb{T})$. They are, by definition, almost Kähler metrics such that the orthogonal distribution to the $\mathbb{T}$-orbits is involutive (see [51]) and we will refer to such metrics as *involutive almost Kähler metrics*. The idea of studying such metrics comes from [32] (see [5] for the weighted case), where it was conjectured that the existence of a weighted involutive csc almost Kähler metric is equivalent to the existence of a weighted cscK metric.

Proposition 1 (Proposition 8.2). — *Let $(V, \omega, \mathbb{T})$ be a toric manifold associated to a Delzant polytope P. Then, for the weights defined in* (3.11)*, the following statements are equivalent:*

1. *there exists a weighted cscK metric on $(V, \omega, \mathbb{T})$;*
2. *there exists an involutive weighted csc almost Kähler metric on $(V, \omega, \mathbb{T})$;*
3. *P is weighted uniformly K-stable.*

As an application of the above result, we study the existence of extremal Kähler metrics on the projectivisation $\mathbb{P}(\mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2)$ of a direct sum of line bundles $\mathcal{L}_i$ over a compact complex curve $S_{\mathbf{g}}$ of genus $\mathbf{g}$. In [7, Proposition 4], the authors established the existence of involutive weighted csc almost Kähler metrics on $\mathbb{P}(\mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2)$, depending on the degrees of the line bundles, the genus of the basis and the Kähler class. Combining with Proposition 1, we deduce the following:

Corollary 1. — *Let $M = \mathbb{P}(\mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2) \longrightarrow S_{\mathbf{g}}$ be a projective $\mathbb{P}^2$-bundle over a complex curve $S_{\mathbf{g}}$ of genus $\mathbf{g}$. If $\mathbf{g} = 0, 1$, then M is a Calabi dream manifold, i.e. M admits an extremal Kähler metric in each Kähler class. Furthermore, the extremal Kähler metrics are given by the generalized Calabi ansatz of* [5]*.*

When $\mathbf{g} = 0$, the existence part of Corollary 1 was already obtained in [50]. We prove in addition that these extremal metrics are given by the Calabi ansatz of [5].

1.3. Outline of the paper

Section 2 is a brief summary of the notion of weighted (v, w)-scalar curvature introduced by Lahdili [49]. In Section 3, we recall the construction and key results of semi-simple principal toric fibration established in [5, 7]. In Section 4, we introduce weighted distances, weighted functionals and weighted differential operators. Section 5 gives a brief exposition of the existence result of [20, 21, 42]. We explain why their argument works equally when the properness is relative to a maximal torus of the reduced group of automorphism (and not only a connected maximal compact subgroup). In Section 6, our main result, Theorem 2, is stated and proved. In Section 7 we review the basic facts of toric Kähler geometry and give the proof of Theorem 3. In Section 8, we show Corollary 1.

Acknowledgements

This paper is part of my PhD thesis. I am very grateful to my advisors Vestislav Apostolov and Eveline Legendre for their immeasurable help and invaluable advices. I would also like to thank Abdellah Lahdili for his careful reading and constructive criticism in the earlier versions, Yasufumi Nitta for his interest and to shared with me his manuscript with Shunsuke Saito, and the referee for their careful reading of our work and thoughtful suggestions that greatly improved the text. I am grateful to l'Université du Québec à Montréal and l'Université Toulouse III Paul Sabatier for their financial support.

2. The v-scalar curvature

In this section, we review briefly the notion of *weighted* v*-scalar curvature* introduced by Lahdili in [49]. Consider a smooth compact Kähler manifold (M, J, ω). We denote by $\mathrm{Aut}_{\mathrm{red}}(M)$ the reduced group of automorphisms whose Lie algebra $\mathfrak{h}_{\mathrm{red}}$ is given by the ideal of real holomorphic vector fields with zeros, see [36]. Let $\mathbb{T}$ be an ℓ-dimensional real torus in $\mathrm{Aut}_{\mathrm{red}}(M)$ with Lie algebra $\mathfrak{t}$. Suppose ω_0 is a $\mathbb{T}$-invariant Kähler form and consider the set of smooth $\mathbb{T}$-invariant Kähler potentials $\mathcal{K}(M,\omega_0)^{\mathbb{T}}$ relative to ω_0. For $\varphi \in \mathcal{K}(M,\omega_0)^{\mathbb{T}}$ we denote by $\omega_\varphi = \omega_0 + dd^c\varphi$ the corresponding Kähler metric. It is well known that the $\mathbb{T}$-action on M is ω_φ-Hamiltonian (see [36]) and we let $m_\varphi : M \longrightarrow \mathfrak{t}^*$ denote a ω_φ-momentum map of $\mathbb{T}$. It is also known [11, 39, 49] that $P_\varphi := m_\varphi(M)$ is a convex polytope in $\mathfrak{t}^*$ and we can normalize m_φ by

$$m_\varphi = m_0 + d^c\varphi, \tag{2.1}$$

in such a way that $P = P_\varphi$ is φ-independent, see [49, Lemma 1].

DEFINITION 2.1. — *For* $\mathrm{v} \in \mathcal{C}^\infty(P, \mathbb{R}_{>0})$ *we define the (weighted)* v-*scalar curvature of the Kähler metric* ω_φ, $\varphi \in \mathcal{K}(M, \omega_0)^{\mathbb{T}}$, *to be*

$$\mathrm{Scal}_{\mathrm{v}}(\omega_\varphi) := \mathrm{v}(m_\varphi)\,\mathrm{Scal}(\omega_\varphi) + 2\Delta_{\omega_\varphi}\big(\mathrm{v}(m_\varphi)\big) + \mathrm{Tr}\,\big(G_\varphi \circ (\mathrm{Hess}(\mathrm{v}) \circ m_\varphi)\big),$$

where Δ_{ω_φ} *is the Riemannian Laplacian associated to* $g_\varphi := \omega_\varphi(\cdot, J\cdot)$, $\mathrm{Hess}(\mathrm{v})$ *is the Hessian of* v *viewed as bilinear form on* $\mathfrak{t}^*$ *whereas* G_φ *is the bilinear form with smooth coefficients on* $\mathfrak{t}$, *given by the restriction of the Riemannian metric* g_φ *on fundamental vector fields and* $\mathrm{Scal}(\omega_\varphi)$ *is the scalar curvature of* (M, J, ω_φ).

In a basis $\boldsymbol{\xi} = (\xi_i)_{i=1\cdots\ell}$ of $\mathfrak{t}$ we have

$$\mathrm{Tr}\,\big(G_\varphi \circ (\mathrm{Hess}(\mathrm{v}) \circ m_\varphi)\big) = \sum_{1\leqslant i,j\leqslant \ell} \mathrm{v}_{,ij}(m_\varphi) g_\varphi(\xi_i, \xi_j)$$

where $\mathrm{v}_{,ij}$ stands for the partial derivatives of v in the dual basis of $\boldsymbol{\xi}$.

DEFINITION 2.2. — *Let* (M, J, ω_0) *be a compact Kähler manifold,* $\mathbb{T} \subset \mathrm{Aut}_{\mathrm{red}}(M)$ *a real torus with normalized momentum image* $P \subset \mathfrak{t}^*$ *associated to* $[\omega_0]$, *and* $\mathrm{v} \in \mathcal{C}^\infty(P, \mathbb{R}_{>0})$, $\mathrm{w} \in \mathcal{C}^\infty(P, \mathbb{R})$. *A* (v, w)*-cscK metric is a* $\mathbb{T}$*-invariant Kähler metric satisfying*

$$\mathrm{Scal}_{\mathrm{v}}(\omega_\varphi) = \mathrm{w}(m_\varphi). \tag{2.2}$$

The motivation for studying (2.2) is that many natural geometric problems in Kähler geometry correspond to (2.2) for suitable choices of v and w.For example, for $\mathbb{T}$ a maximal torus in $\mathrm{Aut}_{\mathrm{red}}(M)$, $\mathrm{v} \equiv 1$ and $\mathrm{w}_{\mathrm{ext}}$ a certain affine-linear function on $\mathfrak{t}^*$, the $(1, \mathrm{w}_{\mathrm{ext}})$-cscK metrics are the extremal metrics in the sense of Calabi. Another example, which will be the one of the main interest of this paper, is the existence theory of extremal Kähler metrics on a class of toric fibrations, which can be reduced to the study of (v, w)-cscK on the toric fiber for suitable choices of v and w. Weighted Kähler metrics have been extensively studied and related to a notion of (v, w)-weighted K-stability, see for example [8, 9, 10, 46, 49].

3. A class of toric fibrations

3.1. Semi-simple principal toric fibrations

Let $\mathbb{T}$ be an ℓ-dimensional torus. We denote by $\mathfrak{t}$ is Lie algebra and by $\Lambda \subset \mathfrak{t}$ the lattice of the generators of circle subgroups, so that $\mathbb{T} = \mathfrak{t}/2\pi\Lambda$. Consider $\pi_S : Q \longrightarrow (S, J_S)$ a principal $\mathbb{T}$-bundle over a $2d$-dimensional product of cscK Hodge manifold $(S, J_S, \omega_S) = \prod_{a=1}^{k}(S_a, J_a, \omega_a)$. Let $\theta \in \Omega^1(Q) \otimes \mathfrak{t}$ be a connection 1-form with curvature

$$\mathrm{d}\theta = \sum_{a=1}^{k} \pi_S^* \omega_a \otimes p_a \qquad p_a \in \Lambda \subset \mathfrak{t}. \tag{3.1}$$

The connection 1-form θ gives rise to a *horizontal* distribution $\mathcal{H} := \mathrm{ann}(\theta)$ and the tangent space splits as

$$TQ = \mathcal{H} \oplus \mathfrak{t},$$

where, by definition, $\mathcal{H}_s \overset{d_s\pi_S}{\cong} T_s S$ for all $s \in S$. The complex structure J_S acts on vector fields in $\mathcal{H}$ via the unique horizontal lift from TS defined via θ.

Now consider a 2ℓ-dimensional compact toric Kähler compact manifold $(V, J_V, \omega, \mathbb{T})$ with associated compact Delzant polytope P [29]. We will consider various actions of $\mathbb{T}$ in this paper. In order to avoid confusion, we specify on which $\mathbb{T}$ acts as a subscript, e.g. $\mathbb{T}_Q$ acts on Q. The interior P^0 is the set of regular value of the moment $m_\omega : V \longrightarrow P \subset \mathfrak{t}^*$ of $(V, \omega, \mathbb{T})$ and $V^0 := m_\omega^{-1}(P^0)$ is the open dense subset of points with regular $\mathbb{T}_V$-orbits. Introducing angular coordinates $t : V^0 \to \mathfrak{t}/2\pi\Lambda$ with respect to the Kähler structure (J_V, ω) (see e.g. [2]), we identify

$$V^0 \cong \mathbb{T} \times P^0 \quad \text{and} \quad T_x V^0 \cong \mathfrak{t} \oplus \mathfrak{t}^*. \tag{3.2}$$

for all $x \in V^0$. Notice that the first diffemorphism is $\mathbb{T}$-equivariant.

We consider the $2n = 2(\ell + d)$ dimensional smooth manifold

$$M^0 := Q \times_{\mathbb{T}} V^0,$$

where the $\mathbb{T}_{Q\times V^0}$-action is given by $\gamma \cdot (q, x) = (\gamma \cdot q, \gamma^{-1} \cdot x)$, $q \in Q$, $x \in V^0$, and $\gamma \in \mathbb{T}$. Using (3.2) we identify

$$M^0 \cong Q \times P^0. \tag{3.3}$$

We will still denote by $\pi_S : M^0 \longrightarrow S$ the projection. At the level of the tangent space we get

$$TM^0 = \mathcal{H} \oplus \mathcal{V}, \tag{3.4}$$

where, for all $s \in S$, $\mathcal{V}_s := \ker d_s\pi_S \cong \mathfrak{t} \oplus \mathfrak{t}^*$ is referred to as the *vertical space*. Since V^0 compactifies to V, the smooth manifold M^0 compactifies to a fiber bundle with fiber V

$$M := \overline{M^0} = Q \times_{\mathbb{T}} V.$$

By construction, M^0 is an open dense subset of M consisting of points with regular $\mathbb{T}_M$-orbits.

One can show that the almost complex structure $J_M \coloneqq J_S \oplus J_V$ on M^0 is integrable and extends on M as J_V extends to V. In other words, M is a compactification of a principal $(\mathbb{C}^*)^\ell$-bundle $\pi_S : (M^0, J) \longrightarrow (S, J_S)$.

3.2. Compatible Kähler metrics

Following [7], we introduce a family of Kähler metrics *compatible with the bundle structure*. In momentum-angular coordinates (m_ω, t), the Kähler form ω of $(V, J_V, \mathbb{T})$ is writen on V^0

$$\omega = \langle \mathrm{d}m_\omega \wedge \mathrm{d}t \rangle, \tag{3.5}$$

where the angle bracket denotes the contraction of $\mathfrak{t}^*$ and $\mathfrak{t}$. By (3.3), we can equivalently see θ on $M^0 = Q \times_{\mathbb{T}} V^0$ which satisfies $\theta(\xi^M) = \xi$ and $\theta(J\xi^M) = 0$, where ξ^M is the fundamental vector field defined by $\xi \in \mathfrak{t}$. Then, $\langle \mathrm{d}m_\omega \wedge \theta \rangle$ is well defined on M^0 and restricts to $\langle \mathrm{d}m_\omega \wedge \mathrm{d}t \rangle$ on the fibers. Thus, we define more generally

$$\omega \coloneqq \langle \mathrm{d}m_\omega \wedge \theta \rangle.$$

We choose the real constants c_a, $1 \leqslant a \leqslant k$, such that the affine-linear functions $\langle p_a, m_\omega \rangle + c_a$ are positive on P, where, we recall the elements $p_a \in \Lambda$ are defined by (3.1). We then define the 2-form on M^0

$$\widetilde{\omega} = \sum_{a=1}^{k} (\langle p_a, m_\omega \rangle + c_a)\, \omega_a + \langle \mathrm{d}m_\omega \wedge \theta \rangle, \tag{3.6}$$

which extends to a smooth Kähler form on (M, J) since ω does on (V, J_V). In the sequel, we fix the metrics ω_a, the 1-form θ and the constants c_a, noting that $p_a \in \mathfrak{t}$ are topological constants of the bundle construction. The Kähler manifold $(M, J, \widetilde{\omega}, \mathbb{T})$ is then a fiber bundle over S, with fiber the Kähler toric manifold $(V, J_V, \omega, \mathbb{T})$, obtained from the principal $\mathbb{T}$-bundle Q. Following [7], we define:

Definition 3.1. — *The Kähler manifold $(M, J, \widetilde{\omega}, \mathbb{T})$ constructed above is referred to as* a semi-simple principal toric fibration *and the Kähler metric given explicitly on M^0 by (3.6), is referred to as a* compatible Kähler metric. *The corresponding Kähler classes on (M, J) are called* compatible Kähler classes *and, in the above set up, are parametrized by the real constant c_a.*

Remark 3.2. — Let $(M, J, \widetilde{g}, \widetilde{\omega})$ be a compact Kähler $2n$-manifold endowed with an effective isometric hamiltonian action of an ℓ-torus $\mathbb{T} \subset \mathrm{Aut}_{\mathrm{red}}(M)$ and momentum map $m : M \to \mathfrak{t}^*$. Following [7], we say the action is rigid if for all x in M $R_x^* \widetilde{g}$ depends only on $m(x)$, where $R_x : \mathbb{T} \to \mathbb{T}\cdot x$

is the orbit map. This action is said semi-simple rigid if moreover, for any regular value x_0 of the momentum map, the derivative with respect to x of the family $\omega_S(x)$ of Kähler forms on the complex quotient (S, J_S) of (M, J) is parallel and diagonalizable with respect to $\omega_S(x_0)$.

By the result of [4, 5, 7], semi-simple principal toric fibrations $(M, J, \widetilde{\omega}, \mathbb{T})$ correspond to Kähler manifolds with a semi-simple rigid torus action, such that the Kähler quotient S is a global product of cscK manifolds, and there are no blow-downs.

The volume form of a compatible Kähler metric (3.6) satisfies

$$\widetilde{\omega}^{[n]} = \omega_S^{[d]} \wedge \mathrm{v}(m_\omega)\omega^{[\ell]} = \bigwedge_{a=1}^{k} \omega_a^{[d_a]} \wedge \mathrm{v}(m_\omega)\omega^{[\ell]} \tag{3.7}$$

where $\mathrm{v}(m_\omega) := \prod_{a=1}^{k} \big(\langle p_a, m_\omega\rangle + c_a\big)^{d_a}$, d_a is the complex dimension of S_a and $\omega^{[i]} := \frac{\omega^i}{i!}$ for $1 \leqslant i \leqslant n$. It follows from [5] and [49, §6] that the scalar curvature of a compatible metric is given by

$$\mathrm{Scal}(\widetilde{\omega}) = \sum_{a=1}^{k} \frac{\mathrm{Scal}_a}{\langle p_a, m_\omega\rangle + c_a} + \frac{1}{\mathrm{v}(m_\omega)} \mathrm{Scal}_{\mathrm{v}}(\omega), \tag{3.8}$$

where Scal_a is the constant scalar curvature of (S_a, J_a, ω_a) and $\mathrm{Scal}_{\mathrm{v}}(\omega)$ is the v-weighted scalar curvature of $(V, J_V, \omega, \mathbb{T})$, see Definition 2.1.

3.3. The extremal vector field

We now recall the definition of the extremal vector field on a general compact Kähler manifold M. To this end, we fix a maximal torus $T \subset \mathrm{Aut}_{\mathrm{red}}(M)$ and a Kähler class $[\widetilde{\omega}_0]$. Given any T-invariant Kähler metric $\widetilde{\omega} \in [\widetilde{\omega}_0]$, we consider the $L^2_{\widetilde{\omega}}$-orthogonal projection

$$\Pi_{\widetilde{\omega}} : L^2_{\widetilde{\omega}} \longrightarrow P^T_{\widetilde{\omega}}$$

where $P^T_{\widetilde{\omega}}$ is the space of $\widetilde{\omega}$-Killing potentials, which, by definition, is the space of function $f \in \mathcal{C}^\infty(M)^T$ such that the hamiltonian vector field $X := \widetilde{\omega}^{-1}(\mathrm{d}f)$ is holomorphic. Futaki and Mabuchi [35] showed that $\Pi_{\widetilde{\omega}}\big(\mathrm{Scal}(\widetilde{\omega})\big)$ does not depend on the chosen T-invariant Kähler metric $\widetilde{\omega}$ in $[\widetilde{\omega}_0]$. Therefore, with respect to the normalized moment map $m_{\widetilde{\omega}} : M \longrightarrow \mathrm{Lie}(T)^*$, see (2.1), one can write

$$\Pi_{\widetilde{\omega}}\big(\mathrm{Scal}(\widetilde{\omega})\big) = \langle \xi_{\mathrm{ext}}, m_{\widetilde{\omega}}\rangle + c_{\mathrm{ext}} =: \ell_{\mathrm{ext}}(m_{\widetilde{\omega}}) \tag{3.9}$$

where $\xi_{\mathrm{ext}} \in \mathrm{Lie}(T)$, $c_{\mathrm{ext}} \in \mathbb{R}$ and $\ell_{\mathrm{ext}} \in \mathrm{Aff}\big(\mathrm{Lie}(T)^*\big)$. See [49, Lemma 1] for more details.

Assume now $(M, J, \widetilde{\omega}, \mathbb{T})$ is a semi-simple principal toric fibration. Then by [7, Proposition 1], the extremal vector field is tangent to the fibers, i.e. $\xi_{\mathrm{ext}} \in \mathfrak{t}$.

Proposition 3.3. — *Let (M, J) be a semi-simple principal toric fibration and T_S be a maximal torus in the isometry group of $g_S := \sum_{a=1}^{k} g_a$, where g_a is the Riemannian metric of ω_a. Any compatible Kähler metric $\widetilde{\omega}$ on (M, J) is invariant by the action of a maximal torus $T \subset \mathrm{Aut}_{\mathrm{red}}(M)$ such that there exists an exact sequence*

$$\mathrm{Id} \longrightarrow \mathbb{T}_M \longrightarrow T \longrightarrow T_S \to \mathrm{Id}, \tag{3.10}$$

where T_S is a maximal torus in $\mathrm{Aut}_{\mathrm{red}}(S)$. Moreover, the extremal vector field ξ_{ext} belongs in the Lie algebra $\mathfrak{t}$ of $\mathbb{T}_M$.

As shown in [7], we get from (3.8) and (3.9):

Corollary 3.4. — *A compatible Kähler metric $\widetilde{\omega}$ on (M, J) is extremal if and only if its corresponding toric Kähler metric ω on (V, J_V) is (v, w)-cscK in the sense of Definition 2.2, where the weights are given by*

$$\begin{aligned} \mathrm{v}(x) &= \prod_{a=1}^{k} \left(\langle p_a, x\rangle + c_a\right)^{d_a} \\ \mathrm{w}(x) &= \mathrm{v}(x)\left(\ell_{\mathrm{ext}}(x) - \sum_{a=1}^{k} \frac{\mathrm{Scal}_a}{\langle p_a, x\rangle + c_a}\right), \end{aligned} \tag{3.11}$$

where $\ell_{\mathrm{ext}} \in \mathrm{Aff}(P)$ is defined in (3.9).

3.4. The space of functions

Any $\mathbb{T}_M$-invariant smooth function on M pulls back to a $\mathbb{T}_Q \times \mathbb{T}_V$-invariant function on $Q \times V$, and therefore descends to a $\mathbb{T}_V$-invariant smooth function on $S \times V$ (see Section 3.1). This gives rise to an isomorphism of Fréchet spaces

$$C^{\infty}(M)^{\mathbb{T}} \cong C^{\infty}(S \times V)^{\mathbb{T}_V}, \tag{3.12}$$

which we shall tacitly use throughout the paper. Moreover, by (3.3) we get

$$C^{\infty}(M^0)^{\mathbb{T}} \cong C^{\infty}(S \times P^0)$$

Given $f \in \mathcal{C}^{\infty}(M)^{\mathbb{T}}$, for any $s \in S$, we denote by $f_s \in \mathcal{C}^{\infty}(V)^{\mathbb{T}}$ the induced smooth function on V with respect to the identification (3.12). Similarly, for any $x \in V$, we denote by $f_x \in \mathcal{C}^{\infty}(S)$ the induced smooth function on S. It follows that on $\mathcal{C}^\infty(M)^{\mathbb{T}}$, the differential operator d splits as $\mathrm{d} = \mathrm{d}_S + \mathrm{d}_V$, where d_S and d_V, is the exterior derivative on S and V respectively. We get

$$\mathcal{C}^\infty(V)^{\mathbb{T}} \cong \{f \in \mathcal{C}^\infty(M)^{\mathbb{T}} \mid \mathrm{d}_S f_x = 0 \quad \forall x \in V\}, \tag{3.13}$$

showing that $\mathcal{C}^\infty(V)^{\mathbb{T}}$ is closed in $\mathcal{C}^\infty(M)^{\mathbb{T}}$ for the Fréchet topology.

3.5. The space of compatible potentials

We fix a reference Kähler metric $\widetilde{\omega}_0$ on (M, J), its corresponding Kähler metric ω_0 on (V, J_V) and the weights (v, w) given by (3.11). We denote by $\mathcal{K}(V, \omega_0)^{\mathbb{T}}$ the space of smooth Kähler potentials on V relative to ω_0 and denote by $\omega_\varphi = \omega_0 + \mathrm{d}_V \mathrm{d}^c_V \varphi$ the corresponding Kähler metric on (V, J_V). Similarly, we denote by $\mathcal{K}(M, \widetilde{\omega}_0)^{\mathbb{T}}$ the space of smooth $\mathbb{T}$-invariant Kähler potentials on (M, J) relative to $\widetilde{\omega}_0$ and we denote by $\widetilde{\omega}_\varphi = \widetilde{\omega}_0 + \mathrm{d}\mathrm{d}^c \varphi$ the corresponding Kähler metric. The following Lemma is established in [7, Lemma 7].

Lemma 3.5. — *Let $\omega_\varphi = \omega_0 + \mathrm{d}_V \mathrm{d}^c_V \varphi$ be a $\mathbb{T}$-invariant Kähler metric on (V, J_V) and denote by m_φ the moment map which satisfies normalization* (2.1)*. Then the compatible Kähler metric $\widetilde{\omega}_\varphi$ induced by ω_φ on M is given by $\widetilde{\omega}_\varphi = \widetilde{\omega}_0 + \mathrm{d}\mathrm{d}^c \varphi$, where φ is seen as a smooth function on M via* (3.12)*.*

It follows that $\mathcal{K}(V, \omega_0)^{\mathbb{T}}$ parametrizes the compatible Kähler metric on (M, J) given explicitly by (3.6) and will be referred to as *the space of compatible Kähler potentials*. It follows from Proposition 3.3 and Lemma 3.5:

Corollary 3.6. — *There is an embedding of Frechet spaces*

$$\mathcal{K}(V, \omega_0)^{\mathbb{T}} \hookrightarrow \mathcal{K}(M, \widetilde{\omega}_0)^{T}.$$

4. Weighted distance, functionals and operators

4.1. Weighted distance

Thanks to the work of Mabuchi [57, 58] it is well-known, that $\mathcal{K}(M, \widetilde{\omega}_0)^{\mathbb{T}}$ is an infinite dimensional Riemannian manifold when equipped with the Mabuchi metric:

$$\langle \dot{\varphi}_0, \dot{\varphi}_1 \rangle_\varphi = \int_M \dot{\varphi}_0 \dot{\varphi}_1 \widetilde{\omega}_\varphi^{[n]} \quad \forall \dot{\varphi}_0, \dot{\varphi}_1 \in T_\varphi \mathcal{K}(M, \widetilde{\omega}_0)^{\mathbb{T}}.$$

Furthermore, a path $(\varphi_t)_{t\in[0,1]} \in \mathcal{K}(M,\widetilde{\omega}_0)^{\mathbb{T}}$ connecting two points is a smooth geodesic if and only if

$$\ddot{\varphi}_t - |\mathrm{d}\dot{\varphi}_t|^2_{\varphi_t} = 0. \tag{4.1}$$

The following result is proven in [9, Lemma 5.6] in the more general context of semi-simple principal fiber bundles and follows easily from the expression (3.7) of the volume form of a compatible Kähler metric.

LEMMA 4.1. — *Let $\varphi \in \mathcal{K}(V,\omega_0)^{\mathbb{T}}$ and $f \in T_\varphi\mathcal{K}(V,\omega_0)^{\mathbb{T}}$, also viewed as an element of $T_\varphi\mathcal{K}(M,\widetilde{\omega}_0)^{\mathbb{T}}$. Then*

$$|\mathrm{d}f|^2_{\widetilde{\omega}_\varphi} = |\mathrm{d}f|^2_{\omega_\varphi}.$$

In particular, $\mathcal{K}(V,\omega_0)^{\mathbb{T}}$ is a totally geodesic submanifold of $\mathcal{K}(M,\widetilde{\omega}_0)^{\mathbb{T}}$ with respect to the Mabuchi metric.

In [38], Guan showed the existence of a smooth geodesic between two Kähler potentials on a toric manifold. The same argument shows the geodesic connectedness of two elements φ_0, $\varphi_1 \in \mathcal{K}(V,\omega_0)^{\mathbb{T}} \subset \mathcal{K}(M,\widetilde{\omega}_0)^{\mathbb{T}}$.

Remark 4.2. — On a general compact Kähler manifold (M,J,ω_0), Darvas [26] introduced the distance d_1 as

$$d_1(\varphi_0,\varphi_1) := \inf_{\varphi_t}\int_0^1\int_M |\dot{\varphi}_t|\omega_t^{[n]}, \tag{4.2}$$

where $\omega_t^{[n]}$ is the volume form associated to the metric $\omega_t = \omega + \mathrm{dd}^c\varphi_t$ and the infimum is taken over the space of smooth curves $\{\varphi_t\}_{t\in[0,1]} \subset \mathcal{K}(M,\omega_0)^{\mathbb{T}}$ joining φ_0 to φ_1. In the above formula, $\dot{\varphi}_t$ is the variation of φ_t with respect to t. It is showed in [29] that $d_1(\varphi_0,\varphi_1)$ equals to the length of the unique (weak) $C^{1,\bar{1}}$ geodesic [18] joining φ_0 and φ_1.

LEMMA 4.3. — *The distance d_1 restricts (up to a positive multiplicative constant) to the distance $d_{1,\mathrm{v}}$ on $\mathcal{K}(V,\omega_0)^{\mathbb{T}}$, defined by*

$$d^V_{1,\mathrm{v}}(\varphi_0,\varphi_1) := \inf_{\varphi_t}\int_0^1\int_V |\dot{\varphi}_t|\mathrm{v}(m_t)\omega_t^{[\ell]}. \tag{4.3}$$

We refer to [9, Corollary 5.5] for the proof. It follows also directly from (3.7) and the smooth geodesic connectedness of $\mathcal{K}(V,\omega_0)^{\mathbb{T}}$.

4.2. Weighted functionals

We consider the Mabuchi energy on $\mathcal{K}(M,\widetilde{\omega}_0)^{\mathbb{T}}$ relative to $\mathbb{T}$, characterized by its variation

$$\mathrm{d}_\varphi\mathcal{M}^{\mathbb{T}}(\dot\varphi) = -\int_M \dot\varphi\big(\mathrm{Scal}(\widetilde{\omega}_\varphi) - \Pi_{\widetilde{\omega}_\varphi}\big(\mathrm{Scal}(\widetilde{\omega}_\varphi)\big)\widetilde{\omega}_\varphi^{[n]}, \quad \mathcal{M}^{\mathbb{T}}(0)=0. \tag{4.4}$$

When restricted to $\mathcal{K}(V,\omega_0)^{\mathbb{T}} \subset \mathcal{K}(M,\widetilde{\omega}_0)^{\mathbb{T}}$, this functional is a special case of the weighted Mabuchi functional introduced in [49]. In our case, the following lemma, established in [7], follows directly from (3.8) and (3.11).

Lemma 4.4. — *The restriction of the Mabuchi relative energy $\mathcal{M}^{\mathbb{T}}$ to $\mathcal{K}(V,\omega_0)^{\mathbb{T}}$ is equal (up to a positive multiplicative constant) to the weighted Mabuchi energy, defined by*

$$\mathrm{d}_\varphi\mathcal{M}_{\mathrm{v},\mathrm{w}}(\dot\varphi) := -\int_V \big(\mathrm{Scal}_{\mathrm{v}}(\omega_\varphi) - \mathrm{w}(m_\varphi)\big)\dot\varphi\omega_\varphi^{[\ell]}, \quad \mathcal{M}_{\mathrm{v},\mathrm{w}}(0)=0, \tag{4.5}$$

with weights (v,w) *given by* (3.11), *and where $\varphi \in \mathcal{K}(V,\omega_0)^{\mathbb{T}}$ and $\dot\varphi \in T_\varphi\mathcal{K}(V,\omega_0)^{\mathbb{T}}$. In particular, compatible extremal Kähler metrics in $[\widetilde{\omega}_0]$ are critical points of $\mathcal{M}_{\mathrm{v},\mathrm{w}} : \mathcal{K}(V,\omega)^{\mathbb{T}} \longrightarrow \mathbb{R}$.*

The Aubin–Mabuchi functional $\mathcal{I} : \mathcal{K}(M,\widetilde{\omega}_0)^{\mathbb{T}} \longrightarrow \mathbb{R}$ is defined by

$$\mathrm{d}_\varphi\mathcal{I}(\dot\varphi) = \int_M \dot\varphi\widetilde{\omega}_\varphi^{[n]}, \quad \mathcal{I}(0)=0,$$

for any $\dot\varphi \in T_\varphi\mathcal{K}(M,\widetilde{\omega}_0)^{\mathbb{T}}$. By (3.7), its restriction to $\mathcal{K}(V,\omega_0)^{\mathbb{T}} \subset \mathcal{K}(M,\widetilde{\omega}_0)^{\mathbb{T}}$ is equal to (up to a positive multiplicative constant)

$$\mathrm{d}_\varphi\mathcal{I}_{\mathrm{v}}(\dot\varphi) := \int_V \dot\varphi\mathrm{v}(m_\varphi)\omega_\varphi^{[\ell]}, \quad \mathcal{I}_{\mathrm{v}}(0)=0 \tag{4.6}$$

for any $\dot\varphi \in T_\varphi\mathcal{K}(V,\widetilde{\omega}_0)^{\mathbb{T}}$. We define the space of $\mathcal{I}$-normalized relative Kähler potentials as

$$\mathring{\mathcal{K}}(M,\widetilde{\omega}_0)^{\mathbb{T}} := \mathcal{I}^{-1}(0) \subset \mathcal{K}(M,\widetilde{\omega}_0)^{\mathbb{T}}. \tag{4.7}$$

It is well known, see e.g. [36, chapter 4], that this space is totally geodesic in $\mathcal{K}(M,\widetilde{\omega}_0)^{\mathbb{T}}$. Similarly, we define

$$\mathring{\mathcal{K}}_{\mathrm{v}}(V,\omega_0)^{\mathbb{T}} := \mathcal{I}_{\mathrm{v}}^{-1}(0) \subset \mathcal{K}_{\mathrm{v}}(V,\omega_0)^{\mathbb{T}}. \tag{4.8}$$

It follows from (4.6) that we also have $\mathring{\mathcal{K}}_{\mathrm{v}}(V,\omega_0)^{\mathbb{T}} \subset \mathring{\mathcal{K}}(M,\widetilde{\omega}_0)^{\mathbb{T}}$.

4.3. Weighted differential operators

Following [7], we introduce the v-Laplacian of (V, J_V, ω) acting on smooth function

$$\Delta^V_{\omega,\mathrm{v}} f := \frac{1}{\mathrm{v}(m_\omega)}\delta\big(\mathrm{v}(m_\omega)\mathrm{d}_V f\big),$$

where δ is the formal adjoint of the differential d_V with respect to $\omega^{[\ell]}$. This definition immediately implies that $\Delta^V_{\omega,\mathrm{v}}$ is self-adjoint with respect to $\mathrm{v}(m_\omega)\omega^{[\ell]}$. Moreover, it follows from the computations in [7, Lemma 8] that $\Delta^V_{\omega,\mathrm{v}}$ can be alternatively expressed as

$$\Delta^V_{\omega,\mathrm{v}} f = \Delta_\omega f - \sum_{a=1}^{k} \frac{\mathrm{d}_a \mathrm{d}^c_V f(p^V_a)}{\langle p_a, m_\omega\rangle + c_a} \tag{4.9}$$

for any $f \in \mathcal{C}^\infty(V)$, where Δ_ω is the Laplacian with respect to ω and p^V_a is the fundamental vector field on V defined by $p_a \in \mathfrak{t}$. As in [49], we introduce the v-weighted Lichnerowicz operator of (V, J_V, ω) defined on the smooth functions $f \in \mathcal{C}^\infty(V)$ to be

$$\mathbb{L}^V_{\omega,\mathrm{v}} f := \frac{\delta\delta\big(\mathrm{v}(m_\omega)(D^-\mathrm{d}_V f)\big)}{\mathrm{v}(m_\omega)}, \tag{4.10}$$

where D is the Levi–Civita connection of ω, $D^-\mathrm{d}_V$ denotes the $(2,0)+(0,2)$ part of $D\mathrm{d}_V$ and $\delta : \otimes^p T^*V \longrightarrow \otimes^{p-1}T^*V$ is defined in any local orthogonal frame $\{e_1, \dots, e_{2n}\}$ by

$$\delta\psi := -\sum_{i=1}^{2n} e_i \lrcorner D_{e_i}\psi$$

where $\lrcorner$ denotes the interior product. The operator $\delta\delta$ is the formal adjoint of $D^-\mathrm{d}_V$ with respect to $\omega^{[\ell]}$. Hence, the v-weighted Lichnerowicz operator is self-adjoint with respect to the volume form $\mathrm{v}(m_\omega)\omega^{[\ell]}$. Let $\widetilde{\omega}$ be the compatible Kähler metric on (M, J) corresponding to ω. We denote by $\omega_S(x)$ the Kähler form on (S, J_S) induced by $\widetilde{\omega}$:

$$\omega_S(x) := \sum_{a=1}^{k} \big(\langle p_a, m_\omega(x)\rangle + c_a\big)\omega_a.$$

The following is etablished in the proof of [7, Lemma 8].

Proposition 4.5. — *Let f be a $\mathbb{T}$-invariant smooth function on M, seen as a $\mathbb{T}_V$-invariant function on $V \times S$ via (3.12). We denote by $\Delta_{\widetilde{\omega}}$ the Laplacian of $(M, J, \widetilde{\omega})$ and by Δ^S_x, respectively $\mathbb{L}^S_x$, the Laplacian, respectively the Lichnerowicz operator, of $(S, J_S, \omega_S(x))$. We then have*

$$\Delta_{\widetilde{\omega}} f = \Delta^V_{\omega,\mathrm{v}} f_s + \Delta^S_x f_x$$

Furthermore, the corresponding Licherowicz's operators $\mathbb{L}_{\tilde{\omega}}$, $\mathbb{L}^V_{\omega,\mathrm{v}}$ and $\mathbb{L}^S_x$ are related by

$$\mathbb{L}_{\tilde{\omega}} f = \mathbb{L}^V_{\omega,\mathrm{v}} f_s + \mathbb{L}^S_x f_x + \Delta^S_x \big(\Delta^V_{\omega,\mathrm{v}} f_s\big)_x + \Delta^V_\omega \big(\Delta^S_x f_x\big)_s + \sum_{a=1}^k Q_a(x)\Delta_a f_x \tag{4.11}$$

where Δ_a is the Laplacian with respect to (S_a, J_a, ω_a) and $Q_a(x)$ is a smooth function on V.

We fix a compatible Kähler metric

$$\widetilde{\chi} = \sum_{a=1}^k (\langle p_a, m_\chi\rangle + c_{a,\alpha})\omega_a + \chi$$

corresponding to a Kähler metric χ on (V, J_V), where m_χ is a moment map with respect to χ and $c_{a,\alpha}$ are constants depending on $\alpha := [\widetilde{\chi}]$ such that $\langle p_a, m_\chi\rangle + c_{a,\alpha} > 0$. Hashimoto introduced in [41] the operator $\mathbb{H}^{\tilde{\chi}}_{\tilde{\omega}} : \mathcal{C}^\infty(M)^{\mathbb{T}} \longrightarrow \mathcal{C}^\infty(M)^{\mathbb{T}}$ defined by

$$\mathbb{H}^{\tilde{\chi}}_{\tilde{\omega}} f := g_{\tilde{\omega}}\big(\widetilde{\chi}, \mathrm{dd}^c f\big) + g_{\tilde{\omega}}\big(\mathrm{d}\Lambda_{\tilde{\omega}}\widetilde{\chi}, \mathrm{d}f\big). \tag{4.12}$$

According to [41, Lemma 1], $\mathbb{H}^{\tilde{\chi}}_{\tilde{\omega}}$ is a second order elliptic self-adjoint differential operator with respect to $\widetilde{\omega}^{[n]}$. Furthermore, the kernel of $\mathbb{H}^{\tilde{\chi}}_{\tilde{\omega}}$ is the space of constant functions. We define the v-weighted *Hashimoto operator* $\mathbb{H}^\chi_{\omega,\mathrm{v}} : \mathcal{C}^\infty(V)^{\mathbb{T}} \to \mathcal{C}^\infty(V)^{\mathbb{T}}$ by

$$\mathbb{H}^\chi_{\omega,\mathrm{v}} f := g_\omega\big(\chi, \mathrm{d}_V\mathrm{d}^c_V f\big) + g_\omega\big(\mathrm{d}_V\Lambda_\omega\chi, \mathrm{d}_V f\big) + \frac{1}{\mathrm{v}(m_\omega)} g_\omega\big(\chi, \mathrm{d}_V \mathrm{v}(m_\omega)\wedge \mathrm{d}^c_V f\big).$$

Proposition 4.6. — *Let f be a $\mathbb{T}$-invariant smooth function on M, seen as a $\mathbb{T}_V$-invariant function on $V \times S$ via* (3.12). *The Hashimoto operator admits the following decomposition*

$$\mathbb{H}^{\tilde{\chi}}_{\tilde{\omega}} f = \mathbb{H}^\chi_{\omega,\mathrm{v}} f_s + \sum_{a=1}^k R_a(x)\Delta_a f_x,$$

where $R_a(x)$ is a smooth function on V depending on χ and α.

Proof. — For simplicity, we denote by m the moment map of ω and

$$q(m) := \sum_{a=1}^k \frac{\mathrm{d}_a\big(\langle p_a, m_\chi\rangle + c_{a,\alpha}\big)}{\langle p_a, m\rangle + c_a}.$$

Let $K \in \mathcal{C}^\infty(V)^{\mathbb{T}} \otimes \mathfrak{t}^*$ the generator of the $\mathbb{T}_V$-action. By definition $\mathrm{d}^c_V f(K)$ is a smooth $\mathbb{T}_V$-invariant $\mathfrak{t}^*$-valued function on V and induces a smooth $\mathbb{T}_M$-invariant $\mathfrak{t}^*$-valued function on M via (3.12). It is shown in the proof of [7, Lemma 8] that on M^0

$$\mathrm{dd}^c f = \langle \mathrm{d}_V(\mathrm{d}_V^c f_s(K))_s \wedge \theta\rangle + \langle d_S(\mathrm{d}_V^c f_s(K))_x \wedge \theta\rangle + \sum_{a=1}^k \mathrm{d}_V^c f_s(p_a^V)\omega_a + d_S \mathrm{d}_S^c f_x + \langle \mathrm{d}_S^c(\mathrm{d}_V^c f_s(K))_x, J\theta\rangle. \tag{4.13}$$

First, we recall the general identity

$$g_{\tilde\omega}(\mathrm{dd}^c f, \widetilde{\chi})\widetilde{\omega}^{[n]} = -\mathrm{dd}^c f \wedge \widetilde{\chi} \wedge \widetilde{\omega}^{[n-2]} - \Delta_{\tilde\omega} f \Lambda_{\tilde\omega}(\widetilde{\chi})\widetilde{\omega}^{[n]}. \tag{4.14}$$

From the expression of $\widetilde{\chi}$ and $\widetilde{\omega}$, we can see that

$$(\langle d_S(\mathrm{d}_V^c f_s(K))_x \wedge \theta\rangle + \langle \mathrm{d}_S^c(\mathrm{d}_V^c f_s(K))_x, J\theta\rangle) \wedge \widetilde{\chi} \wedge \widetilde{\omega}^{[n-2]} = 0.$$

A straightforward computation gives

$$\Lambda_{\tilde\omega}(\widetilde{\chi}) = \Lambda_\omega(\chi) + \sum_{a=1}^k \frac{\mathrm{d}_a\big(\langle p_a, m_\chi\rangle + c_{a,\alpha}\big)}{\langle p_a, m\rangle + c_a}. \tag{4.15}$$

From Proposition 4.5, (4.15) and (4.14) we have

$$g_{\tilde\omega}(\mathrm{dd}^c f_s, \widetilde{\chi}) = g_\omega(\mathrm{d}_V \mathrm{d}_V^c f_s, \chi) + \sum_{a=1}^k \frac{\mathrm{d}_a \mathrm{d}_V f_s(p_a^V)(\langle p_a, m_\chi\rangle + c_{a,\alpha})}{(\langle p_a, m\rangle + c_a)^2}. \tag{4.16}$$

Using (4.15) we get

$$g_{\tilde\omega}\big(d\Lambda_{\tilde\omega}(\widetilde{\chi}), df_s\big) = g_\omega\big(\mathrm{d}_V \Lambda_\omega(\chi), \mathrm{d}_V f_s\big) + g_\omega\big(\mathrm{d}_V q(m), \mathrm{d}_V^c f_s\big). \tag{4.17}$$

To summarize, we have shown

$$\mathbb{H}_{\tilde\omega}^{\tilde\chi} f_s = g_\omega(\mathrm{d}_V \mathrm{d}_V^c f_s, \chi) + g_\omega\big(\mathrm{d}_V \Lambda_\omega(\chi), \mathrm{d}_V f_s\big) + g_\omega\big(\mathrm{d}_V q(m), \mathrm{d}_V^c f_s\big) + \sum_{a=1}^k \frac{\mathrm{d}_a \mathrm{d}_V f_s(p_a^V)(\langle p_a, m_\chi\rangle + c_{a,\alpha})}{(\langle p_a, m\rangle + c_a)^2}. \tag{4.18}$$

Using (3.11) we have

$$\frac{1}{\mathrm{v}(m)} g_\omega\big(\chi, \mathrm{d}_V \mathrm{v}(m) \wedge \mathrm{d}_V^c f_s\big) = g_\omega\big(\mathrm{d}_V q(m), \mathrm{d}_V^c f_s\big) + \sum_{a=1}^k \frac{\mathrm{d}_a \mathrm{d}_V f_s(p_a^V)(\langle p_a, m_\chi\rangle + c_{a,\alpha})}{(\langle p_a, m\rangle + c_a)^2}. \tag{4.19}$$

From (4.18) and (4.19) we get

$$\mathbb{H}_{\tilde\omega}^{\tilde\chi} f_s = \mathbb{H}_{\omega,\mathrm{v}}^{\chi} f_s.$$

The term $\mathbb{H}_{\tilde\omega}^{\tilde\chi} f_x$ is obtained via similar computation. □

Remark 4.7. — Proposition 4.5 implies in particular that the restriction of $\mathbb{H}_{\tilde{\omega}}^{\tilde{\chi}}$ to the Frechet subspace $\mathcal{C}^\infty(V)^{\mathbb{T}} \subset \mathcal{C}^\infty(M)^{\mathbb{T}}$ coincides with $\mathbb{H}^{\chi}_{\omega,\mathrm{v}}$. It follows that $\mathbb{H}^{\chi}_{\omega,\mathrm{v}}$ is a self-adjoint (with respect to $\mathrm{v}(m_\omega)\omega^{[\ell]}$) second order elliptic operator.

5. An analytic criterion for the existence of extremal Kähler metrics

In this section we recall the existence results of extremal Kähler metrics in a given Kähler class, proved by Chen–Cheng [20, 21] in the constant scalar curvature case and extended by He [42] to the extremal case.

We fix a compact complex manifold (M, J), a maximal compact connected subgroup K of $\mathrm{Aut}_{\mathrm{red}}(M)$ and a K-invariant Kähler metric ω_0. Let ξ_{ext} denotes the corresponding extremal vector field, as explained in 3.3. Since the extremal vector field ξ_{ext} is central in the Lie algebra of K, it generates a group $\mathbb{T}_{\mathrm{ext}}$ in the center of the complexified group $G := K^{\mathbb{C}}$ of K. As in [42], we consider the space of $\mathbb{T}_{\mathrm{ext}}$-invariant Kähler potentials $\mathcal{K}(M, \omega_0)^{\mathbb{T}_{\mathrm{ext}}}$. The groupe G acts on $\mathring{\mathcal{K}}(M, \omega_0)^{\mathbb{T}_{\mathrm{ext}}}$ via the natural action on Kähler metrics in $[\omega_0]$ and the normalization (4.7). We introduce the distance $d_{1,G}$ relative to G

$$d_{1,G}(\varphi_1, \varphi_2) := \inf_{\gamma\in G} d_1(\varphi_1, \gamma\cdot\varphi_2), \tag{5.1}$$

where d_1 is defined in (4.2). Let $\mathcal{M}^{\mathbb{T}_{\mathrm{ext}}}$ be the Mabuchi energy relative to $\mathbb{T}_{\mathrm{ext}}$, see (4.4). We recall the following definition from [27]:

Definition 5.1. — *The relative Mabuchi energy $\mathcal{M}^{\mathbb{T}_{\mathrm{ext}}}$ is said proper with respect to $d_{1,G}$ if*

- *$\mathcal{M}^{\mathbb{T}_{\mathrm{ext}}}$ is bounded from below on $\mathcal{K}(M, \omega_0)^{\mathbb{T}_{\mathrm{ext}}}$;*
- *for any sequence $\varphi_i \in \mathring{\mathcal{K}}(M, \omega_0)^{\mathbb{T}_{\mathrm{ext}}}$, $d_{1,G}(0, \varphi_i) \to \infty$ implies that $\mathcal{M}^{\mathbb{T}_{\mathrm{ext}}}(\varphi_i) \to \infty$.*

Theorem 5.2. — *The $\mathbb{T}_{\mathrm{ext}}$-relative Mabuchi energy $\mathcal{M}^{\mathbb{T}_{\mathrm{ext}}}$ restricted to $\mathcal{K}(M, \omega_0)^K \subset \mathcal{K}(M, \omega_0)^{\mathbb{T}_{\mathrm{ext}}}$ is $d_{1,G}$-proper if and only if there exists an extremal Kähler metric in $(M, J, [\omega_0])$ with extremal vector fields ξ_{ext}. Moreover, the same assertion holds by replacing $\mathbb{T}_{\mathrm{ext}}$ by a maximal torus $T \subset \mathrm{Aut}_{\mathrm{red}}(M)$ in the maximal compact group K and $G = K^{\mathbb{C}}$ by the complexification $T^{\mathbb{C}}$ of T.*

The first assertion is established in [42, Theorem 3.1]. We can directly modify the argument to obtain the second. Indeed, in the one direction, suppose that $\mathcal{M}^T$ is $T^{\mathbb{C}}$-proper, in the sense that $\mathcal{M}^T$ is bounded from below on $\mathcal{K}(M,\omega_0)^T$ and for any sequence $\varphi_i \in \mathring{\mathcal{K}}(M,\omega_0)^T$, $d_{1,T^{\mathbb{C}}}(0,\varphi_i) \to \infty$ implies that $\mathcal{M}^T(\varphi_i) \to \infty$. Since $T \subset K$ is a maximal torus it must contain the center of K, i.e. $\mathbb{T}_{\mathrm{ext}} \subset T$. Hence, $\mathcal{M}^T|_{\mathcal{K}(M,\omega_0)^K} = \mathcal{M}^{\mathbb{T}_{\mathrm{ext}}}|_{\mathcal{K}(M,\omega_0)^K}$, where $\mathcal{K}(M,\omega_0)^K \subset \mathcal{K}(M,\omega_0)^T$ is the subspace of K-invariant ω_0-relative Kähler potentials. As any $T^{\mathbb{C}}$-orbit of an element of $\mathcal{K}(M,\omega_0)^K$ belongs to its G-orbit, the $d_{1,T^{\mathbb{C}}}$-properness of $\mathcal{M}^T$ implies that $\mathcal{M}^{\mathbb{T}_{\mathrm{ext}}}$ is $d_{1,G}$-proper when restricted to the subspace $\mathcal{K}(M,\omega_0)^K$. By [42, Theorem 3.1], this implies the existence of a K-invariant (and hence T-invariant) extremal Kähler metric in $[\omega_0]$.

Conversely, suppose $[\omega_0]$ admits a T-invariant extremal Kähler metric. Then the proof of [42, Theorem 3.7] yields the $T^{\mathbb{C}}$-properness of $\mathcal{M}^T$, should one have the uniqueness of the T-invariant extremal Kähler metrics modulo $T^{\mathbb{C}}$. Generalizing the result of Berman–Berndtsson [12] and Chen–Paun–Zeng [25], Lahdili showed, in the more general context of (v,w)-weighted metrics [48, Theorem 2, Remark 2], that the T-invariant extremal metrics are unique modulo the action of $T^{\mathbb{C}}$.

6. An analytic criterion in the case of semi-simple principal toric fibrations

This section is devoted to prove of the following result (where we use notation of Section 3).

Theorem 6.1. — *Let $(M,J,\widetilde{\omega}_0,\mathbb{T})$ be a semi-simple principal toric fibration with Kähler toric fiber $(V,J_V,\omega_0,\mathbb{T})$ and let (v,w) be the corresponding weight functions defined in (3.11). Then, the following statements are equivalent:*

(1) *there exists an extremal Kähler metric in $(M,J,[\widetilde{\omega}_0],\mathbb{T})$;*
(2) *there exists a compatible extremal Kähler metric in $(M,J,[\widetilde{\omega}_0],\mathbb{T})$;*
(3) *there exists a (v,w)-cscK metric in $(V,J_V,[\omega_0],\mathbb{T})$.*

The statement (2) $\Leftrightarrow$ (3) is established in Corollary 3.4 whereas the statement (2) $\Rightarrow$ (1) is clear. We focus on (1) $\Rightarrow$ (2).

We follow the argument of He [42] by restricting the continuity path of Chen [19] to compatible Kähler metrics. We consider the continuity path for $\varphi \in \mathcal{K}(V,\omega_0)^{\mathbb{T}} \subset \mathcal{K}(M,\widetilde{\omega}_0)^{\mathbb{T}}$ given by

$$t\big(\mathrm{Scal}_{\mathrm{v}}(\omega_\varphi) - \mathrm{w}(m_\varphi)\big) = (1-t)\big(\Lambda_{\omega_\varphi,\mathrm{v}}(\chi) - n\big), \quad t \in [0,1], \tag{6.1}$$

for some Kähler metric $\chi \in [\omega_0]$ that we will wisely choose in (6.5). In the above formula

$$\Lambda_{\omega_\varphi,\mathrm{v}}(\chi) := \Lambda_{\omega_\varphi}(\chi) + \sum_{a=1}^{k} \frac{\mathrm{d}_a\big(\langle p_a, m_\chi\rangle + c_a\big)}{\langle p_a, m_\varphi\rangle + c_a}$$

is a smooth function on V equal to $\Lambda_{\tilde{\omega}_\varphi}(\tilde{\chi})$. By definition, a solution φ_t at $t = 1$ corresponds to a compatible extremal metric on (M, J) or equivalently to a (v, w)-cscK on (V, J_V). For $t_1 \in (0, 1]$, we define

$$S_{t_1} := \{t \in (0, t_1] \mid (6.1) \text{ has a solution } \varphi_t \in \mathcal{K}(V, \omega_0)^{\mathbb{T}}\}. \tag{6.2}$$

We need to show that S_1 is open, closed and non empty.

6.1. Openness

Proposition 6.2. — *S_1 is open and non empty.*

For a compatible Kähler form $\tilde{\omega}$ on (M, J) corresponding to a Kähler metric ω on (V, J_V), we denote by $C^\infty(M, \tilde{\omega})^{\mathbb{T}}$ the space of $\mathbb{T}_M$-invariant smooth functions with zero mean value with respect to $\tilde{\omega}^{[n]}$ and by $\mathcal{C}^\infty_{\mathrm{v}}(V, \omega)^{\mathbb{T}} \subset C^\infty(M, \tilde{\omega})^{\mathbb{T}}$ the space of $\mathbb{T}_V$-invariant smooth functions with zero mean value with respect to $\mathrm{v}(m_\omega)\omega^{[\ell]}$. The following is an adaptation of [42, Lemma 3.2].

Lemma 6.3. — *S_1 is non empty.*

Proof. — Let ω a Kähler metric on (V, J_V) and $\tilde{\omega}$ its associate compatible Kähler metric on (M, J) via (3.6). Since $\Delta^V_{\omega,\mathrm{v}}$ is self-adjoint with respect to $\mathrm{v}(m_\omega)\omega^{[\ell]}$, it follows from the proof of Proposition 6.4 below that

$$\Delta^V_{\omega,\mathrm{v}} : \mathcal{C}^\infty_{\mathrm{v}}(V, \omega)^{\mathbb{T}} \longrightarrow \mathcal{C}^\infty_{\mathrm{v}}(V, \omega)^{\mathbb{T}} \tag{6.3}$$

is an isomorphism. Denote by $f \in C^\infty(M, \tilde{\omega})^{\mathbb{T}}$ the unique solution of

$$\Delta_{\tilde{\omega}} f = \mathrm{Scal}_{\mathrm{v}}(\omega) - \mathrm{w}(m_\omega). \tag{6.4}$$

By (6.3), $f \in \mathcal{C}^\infty_{\mathrm{v}}(V, \omega)^{\mathbb{T}}$. Now we choose

$$\tilde{\chi} := \tilde{\omega} - \mathrm{dd}^c \frac{f}{r}. \tag{6.5}$$

Since f is a $\mathbb{T}_V$-invariant smooth function on V, $\widetilde{\chi}$ is both Kähler and compatible for r sufficiently large by Lemma 3.5. We denote by χ its corresponding Kähler metric on (V, J_V). Then

$$\begin{aligned}\Delta_{\tilde{\omega}} f &= r\Delta_{\tilde{\omega}} f\frac{1}{r} = -r\Lambda_{\tilde{\omega}}\mathrm{dd}^c\frac{f}{r}\\ &= r\Lambda_{\tilde{\omega}}\left(\widetilde{\omega} - \mathrm{dd}^c\frac{f}{r} - \widetilde{\omega}\right)\\ &= r\big(\Lambda_{\omega,\mathrm{v}}(\chi) - n\big).\end{aligned}$$

Now let us write $r = t_0^{-1} - 1$, for $t_0 \in (0,1)$ sufficiently small. Then (ω, t_0) is solution of (6.1). $\square$

Now we show that S_1 is open. We fix (ω_{t_0}, t_0) the solution of (6.1) given by Lemma 6.3. Let $\widetilde{\omega}_{t_0} = \omega_0 + \mathrm{dd}^c\varphi_{t_0}$ be its associated compatible Kähler metric on (M, J), with $\varphi_{t_0} \in \mathcal{K}(V, \omega)^{\mathbb{T}}$. Let $\pi : \mathcal{C}^\infty(M)^{\mathbb{T}} \longrightarrow \mathcal{C}^\infty(M, \widetilde{\omega}_{t_0})^{\mathbb{T}}$ be the linear projection:

$$\pi(f) := f - \frac{1}{\int_M \widetilde{\omega}_{t_0}^{[n]}}\int_M f\widetilde{\omega}_{t_0}^{[n]}.$$

We consider

$$R : \mathring{\mathcal{K}}(M, \widetilde{\omega}_0)^{\mathbb{T}} \times [0,1] \longrightarrow \mathcal{C}^\infty(M)^{\mathbb{T}},$$

defined by

$$R(\varphi, t) := t\big(\mathrm{Scal}(\widetilde{\omega}_\varphi) - \Pi_{\tilde{\omega}_\varphi}\big(\mathrm{Scal}(\widetilde{\omega}_\varphi)\big) - (1-t)(\Lambda_{\tilde{\omega}_\varphi}(\widetilde{\chi}) - n).$$

The linearization of the composition $\pi \circ R$ at (φ_{t_0}, t_0) is given by

$$\begin{aligned}(6.6)\quad & D(\pi\circ R)(\varphi_{t_0}, t_0)[f,s]\\ &= \pi\left(\mathcal{L}_{\tilde{\omega}_{t_0}} f + s\big(\mathrm{Scal}(\widetilde{\omega}_{t_0}) - \Pi_{\tilde{\omega}_{t_0}}\big(\mathrm{Scal}(\widetilde{\omega}_{t_0})\big) + \Lambda_{\tilde{\omega}_{t_0}}(\widetilde{\chi}) - n\big)\right),\end{aligned}$$

where

$$\mathcal{L}_{\tilde{\omega}_{t_0}} = -2t_0\mathbb{L}_{\tilde{\omega}_{t_0}} + (1-t_0)\mathbb{H}^{\tilde{\chi}}_{\tilde{\omega}_{t_0}}.$$

Above we used the notation

$$\begin{aligned}\mathbb{L}_{\tilde{\omega}_{t_0}} f &:= \delta\delta D^-\mathrm{d}f\\ &= \frac{1}{2}\Delta^2_{\tilde{\omega}_{t_0}} f + g_{\tilde{\omega}_{t_0}}\big(\mathrm{dd}^c f, \mathrm{Ric}(\varphi_{t_0})\big) + \frac{1}{2}g_{\tilde{\omega}_{t_0}}\big(\mathrm{d}f, \mathrm{d}\,\mathrm{Scal}(\varphi_{t_0})\big),\end{aligned}$$

where D^-d and δ is introduced in (4.10) and and $\mathbb{H}^{\tilde{\chi}}_{\tilde{\omega}_{t_0}}$ is introduced in (4.12). Since $\mathcal{L}_{\tilde{\omega}_{t_0}}$ is a self-adjoint operator with respect to $\widetilde{\omega}_{t_0}^{[n]}$ we get

$$D(\pi\circ R)(\varphi_{t_0}, t_0)[f,s] = \mathcal{L}_{\tilde{\omega}_{t_0}} f.$$

By Proposition 4.5 and Proposition 4.6, the restriction of $\mathcal{L}_{\tilde{\omega}_{t_0}}$ to $\mathcal{C}^\infty(V)^{\mathbb{T}}$ is equal to $\mathcal{L}^V_{\omega_{t_0},\mathrm{v}}$, where

$$\mathcal{L}^V_{\omega_{t_0},\mathrm{v}} := -2t\mathbb{L}^V_{\omega_{t_0},\mathrm{v}} + (1-t)\mathbb{H}^\chi_{\omega_{t_0},\mathrm{v}}. \tag{6.7}$$

In the above equality, ω_{t_0} is the Kähler metric on (V, J_V) corresponding to $\widetilde{\omega}_{t_0}$. By Proposition 4.5 and Proposition 4.6 we obtain

$$\begin{aligned}\mathcal{L}_{\tilde{\omega}_{t_0}} f = \mathcal{L}^V_{\omega_{t_0},\mathrm{v}} f_s + t_0\mathbb{L}^S_x f_x + t_0\Delta^S_x\big(\Delta^V_{\omega_{t_0},\mathrm{v}} f_s\big)_x \\ + t_0\Delta^V_{\omega_{t_0},\mathrm{v}}\big(\Delta^S_x f_x\big)_s + \sum_{a=1}^k U_a(x)\Delta_a f_x\end{aligned} \tag{6.8}$$

for all $f \in \mathcal{C}^\infty(M)^{\mathbb{T}}$, where $U_a(x)$ is a smooth function on V. By [42, Lemma 3.1] the operator $\mathcal{L}_{\tilde{\omega}_{t_0}}$ extends to an isomorphism between Hölder spaces

$$\mathcal{L}_{\tilde{\omega}_{t_0}} : \mathcal{C}^{4,\alpha}(M, \widetilde{\omega}_{t_0})^{\mathbb{T}} \longrightarrow \mathcal{C}^{0,\alpha}(M, \widetilde{\omega}_{t_0})^{\mathbb{T}}, \tag{6.9}$$

where $\mathcal{C}^{4,\alpha}(M, \widetilde{\omega}_{t_0})^{\mathbb{T}}$ is the space of $\mathbb{T}_M$-invariant functions with regularity $(4,\alpha)$ with zero mean value with respect to $\widetilde{\omega}^{[n]}_{t_0}$ and similarly for $\mathcal{C}^{0,\alpha}(M, \widetilde{\omega}_{t_0})^{\mathbb{T}}$. By (6.7), the restriction of the operator $\mathcal{L}_{\tilde{\omega}_{t_0}}$ to the space $\mathcal{C}^{4,\alpha}_{\mathrm{v}}(V, \omega_{t_0})^{\mathbb{T}}$ is equal to $\mathcal{L}^V_{\omega_{t_0},\mathrm{v}}$, where $\mathcal{C}^{4,\alpha}_{\mathrm{v}}(V, \omega_{t_0})^{\mathbb{T}}$ is the space of $\mathbb{T}_V$-invariant functions of regularity $(4,\alpha)$ with zero mean value with respect to $\mathrm{v}(m_{t_0})\omega^{[\ell]}_{t_0}$.

Proposition 6.4. — *The differential operator $\mathcal{L}^V_{\omega_{t_0},\mathrm{v}} : \mathcal{C}^{4,\alpha}_{\mathrm{v}}(V, \omega_{t_0})^{\mathbb{T}} \longrightarrow \mathcal{C}^{0,\alpha}_{\mathrm{v}}(V, \omega_{t_0})^{\mathbb{T}}$ is an isomorphism.*

Proof. — Since $\mathcal{L}^V_{\omega_{t_0},\mathrm{v}}$ is the restriction of an injective operator, it is enough to prove the surjectivity. We proceed analogously to the proof of [7, Lemma 8].

We denote by $L^2_{0,\mathrm{v}}(V)^{\mathbb{T}}$ the completion for the L^2-norm of $\mathcal{C}^{0,\alpha}_{\mathrm{v}}(V, \omega_{t_0})^{\mathbb{T}}$. We argue by contradiction. Assume $\mathcal{L}^V_{\omega_{t_0},\mathrm{v}} : \mathcal{C}^{4,\alpha}_{\mathrm{v}}(V, \omega_{t_0})^{\mathbb{T}} \to \mathcal{C}^{0,\alpha}_{\mathrm{v}}(V, \omega_{t_0})^{\mathbb{T}}$ is not surjective. Then, there exists $\phi \in L^2_{0,\mathrm{v}}(V)^{\mathbb{T}}$ satisfying

$$\int_V \mathcal{L}^V_{\omega_{t_0},\mathrm{v}}(f)\phi\mathrm{v}(m_{t_0})\omega^{[\ell]}_{t_0} = 0 \tag{6.10}$$

for all $f \in \mathcal{C}^{4,\alpha}_{\mathrm{v}}(V, \omega_{t_0})^{\mathbb{T}}$. We claim it implies

$$\int_M \mathcal{L}_{\tilde{\omega}_{t_0}}(f)\phi\widetilde{\omega}^{[n]}_{t_0} = 0 \tag{6.11}$$

for all $f \in \mathcal{C}^{4,\alpha}(M, \widetilde{\omega}_{t_0})^{\mathbb{T}}$, which contradicts the surjectivity of $\mathcal{L}_{\widetilde{\omega}_{t_0}}$: $\mathcal{C}^{4,\alpha}(M, \widetilde{\omega}_{t_0})^{\mathbb{T}} \longrightarrow \mathcal{C}^{0,\alpha}(M, \widetilde{\omega}_{t_0})^{\mathbb{T}}$ established in [42, Lemma 3.1]. Therefore, it is sufficient to show "(6.10) ⇒ (6.11)". For this we argue similar to the proof of [7, Lemma 8] using that the image of $\mathcal{L}^V_{\omega_{t_0},\mathrm{v}}$ is L^2-orthogonal to the subspace of constant functions with respect to $\mathrm{v}(m_{t_0})\omega_{t_0}^{[\ell]}$. □

By the Implicit Function Theorem applied to $\mathcal{L}^V_{\omega_{t_0},\mathrm{v}} : \mathcal{C}^{4,\alpha}_{\mathrm{v}}(V, \omega_{t_0})^{\mathbb{T}} \longrightarrow \mathcal{C}^{0,\alpha}_{\mathrm{v}}(V, \omega_{t_0})^{\mathbb{T}}$, we get a sequence of solutions $\{\varphi_{t_i}\}_{i\in\mathbb{N}}$ of (6.1) of regularity $\mathcal{C}^{4,\alpha}$. By a well-known bootstrapping argument, any solution of (6.1), of regularity $\mathcal{C}^{4,\alpha}$ is in fact smooth. This concludes the proof of Proposition 6.2.

6.2. Closedness

Proposition 6.5. — *S_1 is closed*

Proof. — By hypothesis, there exists an extremal Kähler metric in $[\widetilde{\omega}_0]$. By Theorem 5.2, the relative Mabuchi energy $\mathcal{M}^T$ is $d_{1,T^{\mathbb{C}}}$-proper. Let $\{\varphi_i\}_{i\in\mathbb{N}} \subset \mathcal{K}(V, \omega_0)^{\mathbb{T}}$ be a sequence of solutions of (6.1) given by Proposition 6.2 with $t_i \to t_1 < 1$. By Corollary 3.6, the sequence $\{\varphi_i\}_{i\in\mathbb{N}}$ lies in $\mathcal{K}(M, \widetilde{\omega}_0)^T$. Consequently, the same argument as in [42, Lemma 3.3] shows the existence of a smooth limit $\varphi_{t_1} \in \mathcal{K}(M, \widetilde{\omega}_0)^T$. Moreover, it follows from (3.10) and (3.12), that $\mathcal{K}(V, \omega_0)^{\mathbb{T}}$ is closed in $\mathcal{K}(M, \widetilde{\omega}_0)^T$ for the Frechet topology. Then, the Kähler potential limit φ_{t_1} belongs to $\mathcal{K}(V, \omega)^{\mathbb{T}}$. In particular $\widetilde{\omega}_{\varphi_{t_1}}$ is a compatible Kähler metric.

Let $\widetilde{\varphi}_{t_i} \in \mathring{\mathcal{K}}(V, \omega_0)^{\mathbb{T}}$ (see (4.8)) be the solution of (6.1) at t_i for t_i increasing to 1. By Theorem 5.2, $\mathcal{M}^T$ is $d_{1,T^{\mathbb{C}}}$-proper. Then, by Corollary 3.6, we get a bound with respect to $d_{1,T^{\mathbb{C}}}$, that is

$$\sup_{i\in\mathbb{N}} d_{1,T^{\mathbb{C}}}(0, \widetilde{\varphi}_{t_i}) < \infty.$$

By definition of $d_{1,T^{\mathbb{C}}}$, there exists $\gamma_i \in T^{\mathbb{C}}$ and $\varphi_{t_i} \in \mathring{\mathcal{K}}(M, \widetilde{\omega}_0)^T$ such that $\omega_{\varphi_{t_i}} = \gamma_i^* \omega_{\tilde{\varphi}_{t_i}}$, and

$$\sup_{i\in\mathbb{N}} d_1(0, \varphi_{t_i}) < \infty.$$

By definition γ_i preserves J. Moreover, the form $\widetilde{\omega}_{\varphi_{t_i}}$ is not compatible in general since the connection form θ and the base Kähler metrics ω_a may change by the action of γ_i. However, by Proposition 3.3, the $T^{\mathbb{C}}$-action commutes with the $\mathbb{T}_M$-action. Then, for each t_i, the $\mathbb{T}_M$-action is still rigid and semi-simple (see Remark 3.2). According to [5], $\widetilde{\omega}_{\varphi_{t_i}}$ is given by the generalized Calabi ansatz, with a fixed stable quotient $S = \prod_{a=1}^k S_a$

with respect to the complexified action $\mathbb{T}_M^{\mathbb{C}}$. Thus, there exists a connection 1-form θ_{Q,t_i} with curvature

$$\mathrm{d}\theta_{t_i} = \sum_{a=1}^{k} \pi_S^*(\omega_{a,t_i}) \otimes p_{a,t_i} \quad p_{a,t_i} \in \Lambda$$

such that $\widetilde{\omega}_{\varphi_{t_i}}$ is given by

$$\widetilde{\omega}_{\varphi_{t_i}} = \sum_{a=1}^{k} (\langle p_{a,t_i}, m_{\varphi_{t_i}} \rangle + c_{a,t_i}) \pi_S^*(\omega_{a,t_i}) + \langle \mathrm{d}m_{\varphi_{t_i}}, \theta_{t_i} \rangle.$$

Since $\widetilde{\omega}_{\varphi_{t_i}} \in [\widetilde{\omega}_0]$, $c_{a,t_i} = c_a$ and $p_{a,t_i} = p_a$. By [42, Theorem 3.5], $\widetilde{\omega}_{t_i}$ converge smoothly to an extremal metric ω_{φ_1}. Furthermore, by Propositon 3.3, the extremal vector field ξ_{ext} of $[\widetilde{\omega}_0]$ relatif to T is in the Lie algebra $\mathfrak{t}$ of $\mathbb{T}_M$. Then, by Corollary 3.4 and the smooth convergence of $\widetilde{\omega}_{\varphi_{t_i}}$ to $\widetilde{\omega}_{\varphi_1}$, we get

$$\langle m_{\varphi_1}, \xi_{ext} \rangle + c_{ext} = \sum_{a=1}^{k} \frac{\mathrm{Scal}(\omega_{a,1})}{\langle p_a, m_{\varphi_1} \rangle + c_a} + \frac{1}{\mathrm{v}(m_{\varphi_1})} \mathrm{Scal}_{\mathrm{v}}(\omega_{\varphi_1}), \tag{6.12}$$

where ω_{φ_1} is the Kähler metric on (V, J_V) corresponding to $\widetilde{\omega}_{\varphi_1}$. Taking the exterior differential d_{S_a} on S_a in (6.12) we get $d_{S_a} \mathrm{Scal}(\omega_{a,1}) = 0$ for all $1 \leqslant a \leqslant k$, i.e. $\omega_{a,1}$ has constant scalar curvature. Yet, $[\omega_{a,1}] = [\omega_a]$, showing that $\mathrm{Scal}(\omega_{a,1}) = \mathrm{Scal}_a$. By definition of $\mathrm{w} \in \mathcal{C}^\infty(P, \mathbb{R})$, we get

$$\mathrm{Scal}_{\mathrm{v}}(\omega_{\varphi_1}) = \mathrm{w}(m_{\varphi_1}). \qquad \square$$

Corollary 6.6. — *In a compatible Kähler class, the extremal metrics are given by the Calabi ansatz of* [5]. *Equivalently, in a compatible Kähler class, the extremal metrics are induced by* (v, w)*-cscK metrics on* (V, J_V) *via* (3.6) *for a suitable connection* θ *and suitable Kähler metric* ω_a.

Proof. — Suppose there exists an extremal metric ω_1 in $[\widetilde{\omega}]$. By a result of Calabi [16], ω_1 is invariant by some maximal torus $T \subset \mathrm{Aut}_{\mathrm{red}}(M)$. Conjugating if necessary, we can assume that $\mathbb{T}_M \subset T$. By Theorem 6.1, there exists a compatible extremal metric ω_2 in $[\widetilde{\omega}]$. By Lemma 3.6, ω_2 is T-invariant. Then, by unicity of extremal Kähler metrics invariant by a maximal torus of the reduced automorphism group [12, 25, 49], there exists $\gamma \in T^{\mathbb{C}}$ such that $\omega_1 = \gamma^* \omega_2$. Since $\mathbb{T}_M \subset T$, the action of $\mathbb{T}$ on (M, J, ω_1) is still rigid and semi-simple, see Remark 3.2. Thus, according to [5], ω_1 is given by the Calabi ansatz. $\square$

7. Weighted toric K-stability

7.1. Complex and symplectic points of view

In view of Section 8, where we will consider almost Kähler structures on toric varieties, we briefly recall the well-known correspondence between symplectic and Kähler potentials of toric Kähler manifolds, established and widely used over the years, notably in [4, 5, 32, 40]. We use the notation and conventions of [3], which differ in places from those used in [1, 32].

Let $(V, \omega_0, \mathbb{T})$ be a toric symplectic manifold classified by its *labelled integral Delzant polytope* $(\mathrm{P}, \mathbf{L})$ [4, 29], where $\mathbf{L} = (L_j)_{j=1\dots k}$ is the collection of non-negative defining affine-linear functions for P, with dL_j being primitive elements of the lattice Λ of circle subgroups of $\mathbb{T}$. Choose a Kähler structure (g, J) on $(V, \omega_0, \mathbb{T})$ and denote by (m_0, t_J) the associated moment map, i.e. $m_0 : V \longrightarrow \mathfrak{t}^*$ is the moment map of $(V, \omega_0, \mathbb{T})$ and $t_J : V^0 \longrightarrow \mathfrak{t}/2\pi\Lambda$ is the angular coordinates (unique modulo an additive constant) depending on the complex structure J (see [5, Remark 3]). These coordinates are symplectic, i.e. ω_0 is given by (3.5) for its respective moment-angular coordinates. The Kähler structure (g, J) is defined on V^0 by a smooth strictly convex function u on P^0 via

$$g = \langle \mathrm{d}m_0, \mathbf{G}, \mathrm{d}m_0 \rangle + \langle \mathrm{d}t_J, \mathbf{H}, \mathrm{d}t_J \rangle \quad \text{and} \quad J\,\mathrm{d}m_0 = \langle \mathbf{H}, \mathrm{d}t_J \rangle, \tag{7.1}$$

where $\mathbf{G} := \mathrm{Hess}(u)$ is a positive definite $S^2\mathfrak{t}$-valued function and $\mathbf{H}$ is $S^2\mathfrak{t}^*$-valued function on P^0 and inverse of $\mathbf{H}$ (when seen $\mathbf{H} : \mathfrak{t} \longrightarrow \mathfrak{t}^*$ and $\mathbf{G} : \mathfrak{t}^* \longrightarrow \mathfrak{t}$ in each point in P^0) and $\langle \cdot, \cdot, \cdot \rangle$ denote the point wise contraction $\mathfrak{t}^* \times S^2\mathfrak{t} \times \mathfrak{t}^*$ or its dual. It is shown in [5, Lemma 3], that for two $\mathbb{T}$-invariant Kähler structures on $(V, \omega_0, \mathbb{T})$, given explicitly on V^0 by (7.1) with the same matrix $\mathbf{H}$, there exists a $\mathbb{T}$-equivariant Kähler isomorphism between them.

Conversely, smooth strictly convex functions u on P^0 define $\mathbb{T}$-invariant ω_0-compatible Kähler structures on V^0 via (7.1). The following Proposition established in [5] gives a criterion for the metric to compactify.

PROPOSITION 7.1. — *Let $(V, \omega, \mathbb{T})$ be a compact toric symplectic 2ℓ-manifold with momentum map $m_\omega : V \to P$ and u be a smooth strictly convex function on P^0. Then the positive definite $S^2\mathfrak{t}^*$-valued function $\mathbf{H} := \mathrm{Hess}(u)^{-1}$ on P^0 comes from a $\mathbb{T}$-invariant, ω-compatible Kähler metric g via* (7.1) *if and only if it satisfies the following conditions:*

- [smoothness] $\mathbf{H}$ *is the restriction to P^0 of a smooth $S^2\mathfrak{t}^*$-valued function on P;*

- [boundary values] *for any point y on the codimension one face $F_j \subset P$ with inward normal u_j, we have*

$$\mathbf{H}_y(u_j, \cdot) = 0 \text{ and } (\mathrm{d}\mathbf{H})_y(u_j, u_j) = 2u_j, \tag{7.2}$$

  where the differential $\mathrm{d}\mathbf{H}$ *is viewed as a smooth* $S^2\mathfrak{t}^* \otimes \mathfrak{t}$*-valued function on* P*;*
- [positivity] *for any point y in the interior of a face $F \subseteq P$, $\mathbf{H}_y(\cdot,\cdot)$ is positive definite when viewed as a smooth function with values in $S^2(\mathfrak{t}/\mathfrak{t}_F)^*$, where $\mathfrak{t}_F \subset \mathfrak{t}$ the vector subspace spanned by the inward normals u_j in $\mathfrak{t}$ to the codimension one faces F.*

Definition 7.2. — *Let $\mathcal{S}(\mathrm{P}, \mathbf{L})$ be the space of smooth strictly convex functions on the interior of P^0 such that $\mathbf{H} = \mathrm{Hess}(u)^{-1}$ satisfies the conditions of Proposition 7.1.*

Remark 7.3. — In the above Proposition and Definition, we use as a model metric the Guillemin Kähler metric (g_0, J_0) [40], given by (7.1) for the symplectic potential

$$u_0 \coloneqq \frac{1}{2}\sum_{j=1}^{d} L_j \log L_j,$$

where $L_j, j = 1, \ldots, d$ are the affine-linear functions defining the polytope. This introduces a discrepancy of a factor $1/2$ with respect to the normalization used in [32], which in turn will result in some obvious modification of the formula for the (weighted) Futaki invariant in Section 7.3.

Thus, there exists a bijection between $\mathbb{T}$-equivariant isometry classes of $\mathbb{T}$-invariant ω_0-compatible Kähler structures and $\mathrm{S}^2(\mathfrak{t}^*)$-valued smooth functions $\mathbf{H} = \mathrm{Hess}(u)^{-1}$, where $u \in \mathcal{S}(\mathrm{P}, \mathbf{L})$.

We fix the Guillemin Kähler structure J_0 on $(V, \omega_0, \mathbb{T})$. Consider another ω_0-compatible Kähler structure J_u defined by a symplectic potential $u \in S(\mathrm{P}, \mathbf{L})$ via (7.1). Donaldson shows that [33] there is a biholomorphism $\Phi_u : (V, J_u) \cong (V, J_0)$. Let $\omega_u \coloneqq \Phi_u^*(\omega_0)$ and ϕ_u and ϕ_{u_0} be the Legendre transforms of u and u_0, respectively. By [40], we have that

$$\omega_u = \omega_0 + \mathrm{dd}^c_{J_0}\varphi_u, \qquad \varphi_u(y_0) \coloneqq \phi_u(y_0) - \phi_{u_0}(y_0), \tag{7.3}$$

where $y_0 = \nabla u_0$ are the pluriharmonic coordinates with respect to J_0.

Conversely, using the dual Legendre transform, any $\mathbb{T}$-invariant Kähler potential $\varphi \in \mathcal{K}(V, \omega_0)^{\mathbb{T}}$ gives rise to a symplectic potential $u \in \mathcal{S}(\mathrm{P}, \mathbf{L})$ through (7.3). The key point of this correspondence is that, as show in [38], the path $\varphi_{u_t} \in \mathcal{K}(V,\omega_0)^{\mathbb{T}}$ corresponding to a path $u_t \in \mathcal{S}(\mathrm{P},\mathbf{L})$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}u_t = -\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t. \tag{7.4}$$

7.2. Generalized Abreu's equation

Thanks to Abreu [1], the scalar curvature $\mathrm{Scal}(u)$ associated to a symplectic potential $u \in \mathcal{S}(\mathrm{P},\mathbf{L})$ is expressed by

$$\mathrm{Scal}(u) = \sum_{i,j=1}^{\ell} -(H^u_{ij})_{,ij}, \tag{7.5}$$

where the partial derivatives and the inverse Hessian $(H^u_{ij}) = \mathrm{Hess}(u)^{-1}$ of u is taken in a fixed basis $\boldsymbol{\xi}^*$ of $\mathfrak{t}^*$.

From [49, §6] and the computation of [10, §3], the v-scalar curvature associated to a symplectic potential $u \in \mathcal{S}(\mathrm{P},\mathbf{L})$ and a positive weight function v is given by

$$\mathrm{Scal}_{\mathrm{v}}(u) = -\sum_{i,j=1}^{\ell} (\mathrm{v}H^u_{ij})_{,ij}. \tag{7.6}$$

Let $\mathrm{v} \in \mathcal{C}^\infty(P,\mathbb{R}_{>0})$ and $\mathrm{w} \in \mathcal{C}^\infty(P,\mathbb{R})$. According to Definition 2.2, a Kähler structure (J_u, g_u) on $(V,\omega_0,\mathbb{T})$, associated with a symplectic potential $u \in \mathcal{S}(\mathrm{P},\mathbf{L})$, is (v,w)-cscK if and only if it satisfies

$$-\sum_{i,j=1}^{\ell} (\mathrm{v}H^u_{ij})_{,ij} = \mathrm{w}. \tag{7.7}$$

This formula generalizes the expression (7.5) and is referred to as *the generalized Abreu equation*. This equation has been studied for example in [7, 52, 53, 54].

7.3. Weighted Donaldson–Futaki invariant

Following [32, 49, 52], for $\mathrm{v} \in \mathcal{C}^\infty(P,\mathbb{R}_{>0})$ and $\mathrm{w} \in \mathcal{C}^\infty(P,\mathbb{R})$, we introduce the (v,w)-Donaldson–Futaki invariant

$$\mathcal{F}_{\mathrm{v},\mathrm{w}}(f) := 2\int_{\partial P} f\mathrm{v}\,\mathrm{d}\sigma - \int_P f\mathrm{w}\,\mathrm{d}x, \tag{7.8}$$

for all continuous functions f on P, where $\mathrm{d}\sigma$ is the induced measure on each face $F_i \subset \partial P$ by letting $\mathrm{d}L_i \wedge \mathrm{d}\sigma = -\mathrm{d}x$, where $\mathrm{d}x$ is the Lesbegue measure on P.

Convention 7.4. — *The weights* $\mathrm{v} > 0$ *and* $\mathrm{w} \in C^\infty(P, \mathbb{R})$ *satisfy*

$$\mathcal{F}_{\mathrm{v},\mathrm{w}}(f) = 0 \tag{7.9}$$

for all f *affine-linear on* P*.*

Integration by parts (see e.g. [32]) reveals that (7.9) is a necessary condition for the existence of (v, w)-cscK metric on $(V, \omega_0, \mathbb{T})$.

Remark 7.5. — Notice that in the case of semi-simple principal toric fibrations, the weights given by (3.11) satisfy (7.9) above.

7.4. The weighted Mabuchi energy

The volume form $\omega_0^{[\ell]}$ on V is pushed forward to the measure $\mathrm{d}x$ via the moment map m_0. Seen as functional on $\mathcal{S}(\mathrm{P}, \mathbf{L})$ via (7.4), the weighted Mabuchi energy $\mathcal{M}_{\mathrm{v},\mathrm{w}}$ satisfies

$$d_u\mathcal{M}_{\mathrm{v},\mathrm{w}}(\dot{u}) = \int_P \left(- \sum_{i,j=1}^{\ell} \left(\mathrm{v}H^u_{ij}\right)_{,ij} - \mathrm{w} \right) \dot{u} \, \mathrm{d}x.$$

From [10, Lemma 6] (see also Lemma 7.6 below) we get

$$d_u\mathcal{M}_{\mathrm{v},\mathrm{w}}(\dot{u}) = \mathcal{F}_{\mathrm{v},\mathrm{w}}(\dot{u}) - \int_P \sum_{i,j=1}^{\ell} \mathrm{v}H^u_{ij}\dot{u}_{,ij} \, \mathrm{d}x,$$

where $\mathcal{F}_{\mathrm{v},\mathrm{w}}$ is the Donaldson–Futaki invariant defined in (7.8). Using that the derivative of $\operatorname{tr}\mathbf{H}^{-1}\mathrm{d}\mathbf{H}$ is $\mathrm{d}\log\det\mathbf{H}$, we get

$$\mathcal{M}_{\mathrm{v},\mathrm{w}}(u) = \mathcal{F}_{\mathrm{v},\mathrm{w}}(u) - \int_P \log\det \mathrm{Hess}(u)\,\mathrm{Hess}(u_0)^{-1} \mathrm{v} \, \mathrm{d}x.$$

We denote by $\mathcal{CV}^\infty(P)$ the set of continuous convex functions on P which are smooth in the interior P^0. Using the same argument than [32, Lemma 3.3.5], since v is smooth and P compact, we get:

Lemma 7.6. — *Let* $\mathbf{H}$ *be any smooth* $S^2\mathfrak{t}^*$*-valued function on* P *which satisfies the boundary condition* (7.2) *of Proposition 7.1, but not necessarily the positivity condition. For any* $\mathrm{v} \in \mathcal{C}^\infty(P, \mathbb{R}_{>0})$ *and* $f \in \mathcal{CV}^\infty(P)$*:*

$$\int_P \sum_{i,j=1}^{\ell} \left(\mathrm{v}H_{ij}\right) f_{,ij} \, \mathrm{d}x = \int_P \left(\sum_{i,j=1}^{\ell} \left(\mathrm{v}H_{ij}\right)_{,ij} \right) f \, \mathrm{d}x + 2 \int_{\partial P} f\mathrm{v} \, \mathrm{d}\sigma. \tag{7.10}$$

In particular, $\int_P \sum_{i,j=1}^{\ell} \left(\mathrm{v}H_{ij}\right) f_{,ij} \, \mathrm{d}x < \infty$*.*

The following result and proof are generalizations of [32, Proposition 3.3.4].

Proposition 7.7. — *Let* $\mathrm{v} \in \mathcal{C}^\infty(P, \mathbb{R}_{>0})$ *and* $\mathrm{w} \in \mathcal{C}^\infty(P, \mathbb{R})$. *The Mabuchi energy* $\mathcal{M}_{\mathrm{v},\mathrm{w}}$ *extends to the set* $\mathcal{CV}^\infty(P)$ *as functional with values in* $(-\infty, +\infty]$. *Moreover, if there exists* $u \in \mathcal{S}(\mathrm{P}, \mathbf{L})$ *corresponding to a* (v, w)*-cscK metric, i.e. which satisfies* (7.7)*, then* u *realizes the minimum of* $\mathcal{M}_{\mathrm{v},\mathrm{w}}$ *on* $\mathcal{CV}^\infty(P)$.

Proof. — The linear term $\mathcal{F}_{\mathrm{v},\mathrm{w}}$ is well-defined on $\mathcal{CV}^\infty(P)$. We then focus on the nonlinear term of $\mathcal{M}_{\mathrm{v},\mathrm{w}}$. Let $u \in \mathcal{S}(\mathrm{P}, \mathbf{L})$ and $h \in \mathcal{CV}^\infty(P)$. Suppose $\det \mathrm{Hess}(h) \neq 0$. By convexity of the functional $-\log\det$ on the space of positive definite matrices, we get:

$$-\log\det \mathrm{Hess}(h) + \log\det \mathrm{Hess}(u) \geqslant -\mathrm{Tr}\left(\mathrm{Hess}(u)^{-1}\,\mathrm{Hess}(f)\right),$$

where $f = h - u$. Turning this around and multiplying by v, we obtain:

$$\mathrm{v}\log\det \mathrm{Hess}(h) \leqslant \mathrm{v}\log\det \mathrm{Hess}(u) + \mathrm{v}\,\mathrm{Tr}\left(\mathrm{Hess}(u)^{-1}\mathrm{Hess}(f)\right).$$

By linearity of (7.10), the equality still holds when we replace f by a difference of two functions in $\mathcal{CV}^\infty(P)$. In particular, this shows that the function $\mathrm{v}\,\mathrm{Tr}\left(\mathrm{Hess}^{-1}(u)\,\mathrm{Hess}(f)\right)$ is integrable on P and hence, by the previous inequality, $\mathrm{v}\log\det \mathrm{Hess}(h)$ is integrable too. Now if the determinant of h is equal to 0, we define the value of $\mathcal{M}_{\mathrm{v},\mathrm{w}}(h)$ to be $+\infty$. Then $\mathcal{M}_{\mathrm{v},\mathrm{w}}$ is well-defined on $\mathcal{CV}^\infty(P)$. Suppose u satisfies (7.7). If $\det \mathrm{Hess}(f) = 0$, then we trivially get $\mathcal{M}_{\mathrm{v},\mathrm{w}}(u) \leqslant \mathcal{M}_{\mathrm{v},\mathrm{w}}(f)$. Now, admit $\det \mathrm{Hess}(f) \neq 0$ and consider the function $g(t) = \mathcal{M}_{\mathrm{v},\mathrm{w}}(u + tf)$. The function g is therefore a convex function. Moreover, g is differentiable at $t = 0$ with

$$g'(0) = -\int_P \left(\sum_{i,j=1}^{\ell} (\mathrm{v}H^u_{ij})_{,ij} - \mathrm{w}\right) f\,\mathrm{d}x,$$

which is equal to 0 by hypothesis on u. Then $\mathcal{M}_{\mathrm{v},\mathrm{w}}(u) \leqslant \mathcal{M}_{\mathrm{v},\mathrm{w}}(f)$ by the convexity of g. □

7.5. Properness and (v, w)-uniform K-Stability

Following [32, 60] (see also [3, Chapter 3.6]), we fix $x_0 \in P^0$ and consider the following normalization

$$\mathcal{CV}^\infty_*(P) \coloneqq \{f \in \mathcal{CV}^\infty(P) \mid f(x) \geqslant f(x_0) = 0\}. \tag{7.11}$$

Then, any $f \in \mathcal{CV}^\infty(P)$ can be written uniquely as $f = f^* + f_0$, where f_0 is affine-linear and $f^* = \pi(f) \in \mathcal{CV}^\infty_*(P)$, where $\pi : \mathcal{CV}^\infty(P) \longrightarrow \mathcal{CV}^\infty_*(P)$ is the linear projection.

Definition 7.8. — *A Delzant polytope* $(\mathrm{P}, \mathbf{L})$ *is* (v, w)*-uniformly K-stable if there exists* $\lambda > 0$ *such that*

$$\mathcal{F}_{\mathrm{v},\mathrm{w}}(f) \geqslant \lambda \|f^*\|_1 \tag{7.12}$$

for all $f \in \mathcal{CV}^\infty(P)$*, where* $\|\cdot\|_1$ *denotes the* L^1*-norm on* P*.*

Proposition 7.9. — *Suppose* $(\mathrm{P}, \mathbf{L})$ *is* (v, w)*-uniformly K-stable. Then there exists* $C > 0$ *and* $D \in \mathbb{R}$ *such that*

$$\mathcal{M}_{\mathrm{v},\mathrm{w}}(u) \geqslant C\|u^*\|_1 + D$$

for all $u \in \mathcal{S}(\mathrm{P}, \mathbf{L})$*.*

Proof. — This result when $\mathrm{v} = 1$ is due to [32, 66]. The proof is an adaptation of the exposition in [3].

Let $u_0 \in \mathcal{S}(\mathrm{P}, \mathbf{L})$ be the Guillemin Kähler potential. We consider $\mathcal{F}_{\mathrm{v},\mathrm{w}_0}$ where $\mathrm{w}_0 := \mathrm{Scal}_\mathrm{v}(u_0)$. For any $f \in \mathcal{CV}^\infty_*(P)$ there exists $C > 0$ such that

$$|\mathcal{F}_{\mathrm{v},\mathrm{w}}(f) - \mathcal{F}_{\mathrm{v},\mathrm{w}_0}(f)| \leqslant 2C\|f\|_1.$$

Since $(\mathrm{P}, \mathbf{L})$ is (v, w)-uniformly stable we get

$$|\mathcal{F}_{\mathrm{v},\mathrm{w}_0}(f) - \mathcal{F}_{\mathrm{v},\mathrm{w}}(f)| \leqslant C_1\mathcal{F}_{\mathrm{v},\mathrm{w}}(f) - C\|f\|_1,$$

where C_1 is a positive constant depending (7.12). We deduce that

$$\mathcal{F}_{\mathrm{v},\mathrm{w}_0}(f) \leqslant \widetilde{C}\mathcal{F}_{\mathrm{v},\mathrm{w}}(f) - C\|f\|_1, \tag{7.13}$$

where $\widetilde{C} := C_1 + 1$. By Proposition 7.7, the Mabuchi energy extends to $\mathcal{CV}^\infty_*(P)$. Then, by (7.13) and (7.9), for any $u \in \mathcal{S}(\mathrm{P}, \mathbf{L})$,

$$\begin{aligned}
\mathcal{M}_{\mathrm{v},\mathrm{w}}(u) &= \mathcal{F}_{\mathrm{v},\mathrm{w}}(u^*) - \int_P \mathrm{v} \log\det \mathrm{Hess}(u^*)\,\mathrm{Hess}(u_0)^{-1}\,\mathrm{d}x \\
&\geqslant \widetilde{C}\mathcal{F}_{\mathrm{v},\mathrm{w}_0}(u^*) + C\|u^*\|_1 - \int_P \mathrm{v} \log\det \mathrm{Hess}(u^*)\,\mathrm{Hess}(u_0)^{-1}\,\mathrm{d}x \\
&= \mathcal{M}_{\mathrm{v},\mathrm{w}_0}(\widetilde{C}u^*) + \int_P \mathrm{v} \log\det \mathrm{Hess}(\widetilde{C}u^*)\,\mathrm{Hess}(u^*)^{-1}\,\mathrm{d}x + C\|u^*\|_1 \\
&= \mathcal{M}_{\mathrm{v},\mathrm{w}_0}(\widetilde{C}u^*) + n\log\widetilde{C}\int_P \mathrm{v}\,\mathrm{d}x + C\|u^*\|_1.
\end{aligned}$$

The Mabuchi energy $\mathcal{M}_{\mathrm{v},\mathrm{w}_0}$ reaches its minimum at the potential $u_0 \in \mathcal{S}(\mathrm{P}, \mathbf{L})$, which is solution of

$$\mathrm{Scal}_\mathrm{v}(u_0) = \mathrm{w}_0.$$

In particular, $\mathcal{M}_{\mathrm{v},\mathrm{w}_0}$ is bounded from below on $\mathcal{CV}^{\infty}(P)$ by Proposition 7.7. Letting $D := \inf_{\mathcal{CV}^\infty} \mathcal{M}_{\mathrm{v},\mathrm{w}_0} + n \log \widetilde{C} \int_P \mathrm{v}\, \mathrm{d}x$ we get the result. □

7.6. Existence of (v,w)-cscK is equivalent to (v,w)-uniform K-stability

The following is established in [53, Theorem 2.1] and is due to [17] when $\mathrm{v} = 1$.

Proposition 7.10. — *Suppose there exists an (v,w)-cscK metric in $[\omega_0]$, i.e. (7.7) admits a solution $u \in \mathcal{S}(\mathrm{P},\mathbf{L})$. Then P is (v,w)-uniformly K-stable.*

We now focus on the converse. We consider the space of normalized Kähler potentials (4.8) and normalized symplectic potentials

$$\mathring{\mathcal{S}}_{\mathrm{v}}(\mathrm{P},\mathbf{L}) := \left\{ u \in \mathcal{S}(\mathrm{P},\mathbf{L}) \,\middle|\, \int_P u\mathrm{v}\, \mathrm{d}x = \int_P u_0 \mathrm{v}\, \mathrm{d}x \right\}. \tag{7.14}$$

Lemma 7.11. — *For any $\mathring{u}_t \in \mathring{\mathcal{S}}_{\mathrm{v}}(\mathrm{P},\mathbf{L})$, the corresponding Kähler potential $\varphi_t = \varphi_{\mathring{u}_t}$ obtained via (7.3) belongs to the space of normalized Kähler potential $\mathring{\mathcal{K}}_{\mathrm{v}}(V,\omega_0)^{\mathbb{T}}$ defined in (4.8). Conversely, any path in $\mathring{\mathcal{K}}_{\mathrm{v}}(V,\omega_0)^{\mathbb{T}}$ comes from a path $\mathring{u}_t$ in $\mathring{\mathcal{S}}_{\mathrm{v}}(\mathrm{P},\mathbf{L})$.*

Proof. — By [14, Lemma 2.4], the functional $\mathcal{I}_{\mathrm{v}}$ defined in (4.6), is also characterized by its variation for general weights $\mathrm{v} \in \mathcal{C}^\infty(P,\mathbb{R}_{>0})$. Then, a path $\varphi_t \in \mathcal{K}_{\mathrm{v}}(V,\omega_0)^{\mathbb{T}}$ starting from 0 belongs to $\mathring{\mathcal{K}}_{\mathrm{v}}(V,\omega_0)^{\mathbb{T}}$ if and only if

$$\int_V \dot{\varphi}_t \mathrm{v}(m_{\varphi_t})\omega_t^{[\ell]} = 0 \tag{7.15}$$

for all $\dot{\varphi}_t \in T_{\varphi_t}\mathcal{K}_{\mathrm{v}}(V,\omega_0)^{\mathbb{T}}$. By pushing-forward the measure $\omega_t^{[\ell]}$ via $m_{\omega_{\varphi_t}}$ and using (7.4) we get that (7.15) is equivalent to

$$\int_P \dot{u}_t \mathrm{v}\, \mathrm{d}x = 0,$$

where u_t is the path corresponding to φ_t via (7.3). The conclusion follows from the convexity of $\mathcal{S}(\mathrm{P},\mathbf{L})$. □

Theorem 7.12. — *Let $(M, J, \widetilde{\omega}_0, \mathbb{T})$ be a semi-simple principal toric fibration with Kähler toric fiber $(V, J_V, \omega_0, \mathbb{T})$. Let (v,w) be the weights defined in (3.11) and denote by P the Delzant polytope associated to $(V,\omega_0,\mathbb{T})$. Then there exists a (v,w)-weighted cscK metric in $[\omega_0]$ if and only if P is (v,w)-uniformly K-stable. In particular, the latter condition is necessary and sufficient for $[\widetilde{\omega}_0]$ to admit an extremal Kähler metric.*

Proof. — Suppose there exists a (v, w)-cscK metric in $[\omega_0]$. By Proposition 7.10, P is (v, w)-uniformly K-stable.

Conversely, suppose P is (v, w)-uniformly K-stable. We are going to show that there are uniform positive constants $\widetilde{A}$ and $\widetilde{B}$ such that

$$\mathcal{M}_{\mathrm{v},\mathrm{w}}(\varphi) \geqslant \widetilde{A} \inf_{\gamma\in\mathbb{T}^{\mathbb{C}}} d^V_{1,\mathrm{v}}(0,\gamma\cdot\varphi) - \widetilde{B}, \tag{7.16}$$

where $d^V_{1,\mathrm{v}}$ is defined in Lemma 4.3 and $\mathcal{M}_{\mathrm{v},\mathrm{w}}$ is the weighted Mabuchi energy of the Kähler toric fiber $(V, J_V, [\omega_0], \mathbb{T})$, see (4.5). For all $\varphi \in \mathring{\mathcal{K}}(V,\omega_0)^{\mathbb{T}}$, there exists $\gamma \in \mathbb{T}^{\mathbb{C}}$ such that the symplectic potential $u_{\gamma\cdot\varphi}$ corresponding to $\gamma\cdot\varphi$ satisfies $d_{x_0} u_{\gamma\cdot\varphi} = 0$. By Lemma 7.11 and the inequality in [3, (66)], we have

$$d^V_{1,\mathbb{T}^{\mathbb{C}}}(0,\gamma\cdot\varphi) \leqslant A\int_P |u^*_\varphi - u^*_0|\,\mathrm{d}x \leqslant A\|u^*_\varphi\|_1 + B, \tag{7.17}$$

for some uniform constants $A > 0$ and $B > 0$, where $d^V_{1,\mathbb{T}}$ is the d_1 distance relative to $\mathbb{T}^{\mathbb{C}}$ (see (5.1)) on $\mathcal{K}(V,\omega_0)^{\mathbb{T}}$, $u_\varphi \in S(\mathrm{P},\mathbf{L})$ is the symplectic potential corresponding to φ and u^*_φ is its normalization in $S(\mathrm{P},\mathbf{L})\cap\mathcal{CV}^\infty_*(P)$, see (7.11). Since $\mathrm{v} > 0$ on P, we have for the weighted distance $d^V_{1,\mathrm{v}} \leqslant C d^V_1$. Then (7.16) follows from (7.17) and Proposition 7.9.

Let T be the maximal torus in $\mathrm{Aut}_{\mathrm{red}}(M)$ containing $\mathbb{T}_M$ and satisfying (3.10). By Lemma 4.4, and our choice of normalization (4.8), the Mabuchi energy $\mathcal{M}^T$ restricts to $\mathcal{M}_{\mathrm{v},\mathrm{w}}$ on $\mathring{\mathcal{K}}_{\mathrm{v}}(V,\omega_0)^{\mathbb{T}}$. We denote by $d_{1,T^{\mathbb{C}}}$ the d_1 distance relative to $T^{\mathbb{C}}$ on $\mathcal{K}(M,\widetilde{\omega}_0)^T$. Since any $\mathbb{T}^{\mathbb{C}}_M$-orbit lies in a $T^{\mathbb{C}}$-orbit, by (7.16) and Lemma 4.3, $\mathcal{M}^T$ is $d_{1,T^{\mathbb{C}}}$-proper on $\mathring{\mathcal{K}}_{\mathrm{v}}(V,\omega_0)^{\mathbb{T}}$ in the sense of Definition 5.1.

In the proof of Theorem 6.1 "(1) $\Rightarrow$ (2)", we have used the $d_{1,T^{\mathbb{C}}}$-properness only on sequences included in $\mathcal{K}(V,\omega_0)^{\mathbb{T}}$. It allows us to obtain the existence of a (v, w)-cscK metric by the same argument.

The last assertion follows from Theorem 6.1. □

8. Applications

8.1. Almost Kähler metrics

As observed in [32], for fixed angular coordinates $\mathrm{d}t_0$ with respect to a reference Kähler structure J_0, one can use (7.1) to define a $\mathbb{T}$-invariant *almost-Kähler* metric on V, as soon as $\mathbf{H}$ satisfies the smoothness, boundary value and positivity conditions of Proposition 7.1, even if the inverse matrix $\mathbf{G} := \mathbf{H}^{-1}$ is not necessarily the Hessian of a smooth function. We shall refer to such almost Kähler metric as *involutive*. One can further use such involutive AK metrics on V to build a compatible metric $\widetilde{g}_{\mathbf{H}}$ on M, by the formula

$$\widetilde{g}_{\mathbf{H}} = \sum_{a=1}^{k} \big(\langle p_a, m\rangle + c_a\big) g_a + \langle dm, \mathbf{G}, dm\rangle + \langle \theta, \mathbf{H}, \theta\rangle.$$

It is shown in [7] that $\widetilde{g}_{\mathbf{H}}$ is extremal AK on M in the sense of [51] (i.e. the hermitian scalar curvature of $\widetilde{g}_{\mathbf{H}}$ is a Killing potential) if and only if $\mathbf{H}$ satisfies the equation

$$-\sum_{i,j} \big(\mathrm{v}H_{ij}\big)_{,ij} = \mathrm{w}, \tag{8.1}$$

for (v, w) defined in (3.11). We shall more generally consider involutive AK metrics satisfying the equation (8.1) weight functions $\mathrm{v} > 0$ and w. For such AK metrics we say that $(V, J_{\mathbf{H}}, , g_{\mathbf{H}}, \omega, \mathbb{T})$ is an *involutive* (v, w)*-csc almost Kähler metric*.

The point of considering involutive (v, w)-csc almost Kähler metrics is that (8.1) is a linear indeterminate PDE for the smooth coefficients of $\mathbf{H}$ (which would therefore admit infinitely many solutions if we drop the positivity assumption of $\mathbf{H}$), which in some special cases is easier to solve explicitly, as demonstrated in [7]. On the other hand, it was observed in [32] (see [7] for the weighted case) that the existence of a (v, w)-csc almost Kähler metric implies that $\mathcal{F}_{\mathrm{v},\mathrm{w}}(f) \geqslant 0$ with equality iff $f = 0$, and it was conjectured that the existence of a (v, w)-csc almost Kähler metric is equivalent to the existence of a (v, w)-cscK metric on $(V, \omega, \mathbb{T})$. E. Legendre [50] observed that the existence of an involutive extremal almost Kähler metric implies the stronger yet uniform stability of P, and thus confirmed the conjecture in the case where $\mathrm{v} = 1$ and $\mathrm{w} = \ell_{\mathrm{ext}}$, see Section 3.3. Our additional observation is that the same arguments as in the proof of Proposition 7.10 show the following.

PROPOSITION 8.1. — *Let* $(V, \omega, \mathbb{T})$ *be a toric manifold associated to Delzant polytope* P*. Let* $\mathrm{v} \in \mathcal{C}^\infty(P, \mathbb{R}_{>0})$ *and* $\mathrm{w} \in \mathcal{C}^\infty(P, \mathbb{R})$ *such that* w *satisfies* (7.9)*. Suppose there exists an involutive* (v, w)*-csc almost Kähler metric on* $(V, \omega, \mathbb{T})$*, i.e. there exists* $\mathbf{H}$ *satisfying the smoothness, boundary value and positivity conditions of Proposition 7.1 and equation* (8.1)*. Then* P *is* (v, w)*-uniformly K-stable.*

Combining this result (for the special weights (v, w) associated to a semi-simple principal torus bundle via (3.11)) with Theorem 7.12, we deduce:

Proposition 8.2. — *Let* $(V,\omega,\mathbb{T})$ *be a toric manifold associated to Delzant polytope* P*. Let* (v,w) *be weights defined in* (3.11)*. Then the following statements are equivalent:*

(1) $\exists$ *a* (v,w)*-cscK metric on* $(V,\omega,\mathbb{T})$*;*
(2) $\exists$ *an involutive* (v,w)*-csc almost Kähler metric on* $(V,\omega,\mathbb{T})$*;*
(3) P *is* (v,w)*-uniformly K-stable in the sense of Definition 7.8.*

8.2. Proof of Corollary 1

Let (S,J_S) be a compact complex curve of genus $\mathbf{g}$ and $\mathcal{L}_i \longrightarrow S$ an holomorphic line bundle, $i=0,1,2$. We consider $(M,J) \coloneqq \mathbb{P}(\mathcal{L}_0\oplus\mathcal{L}_1\oplus\mathcal{L}_2)$. Since the biholomorphism class of M is invariant by tensoring $\mathcal{L}_0\oplus\mathcal{L}_1\oplus\mathcal{L}_2$ with a line bundle, we can suppose without loss of generality that $(M,J)=P(\mathcal{O}\oplus\mathcal{L}_1\oplus\mathcal{L}_2)$, where $\mathcal{O}\longrightarrow S$ is the trivial line bundle and $\boldsymbol{p}_i \coloneqq \deg\left(\mathcal{L}_i\right)\geqslant 0$, $i=1,2$. Suppose $\boldsymbol{p}_1=\boldsymbol{p}_2=0$, then by a result of Fujiki [34] (M,J) admits an extremal metric (see also [7, Remark 2]) in every Kähler class. If $\boldsymbol{p}_2=\boldsymbol{p}_1>0$ or $\boldsymbol{p}_2>\boldsymbol{p}_1=0$ and $\mathbf{g}=0,1$, there exists an extremal metric in every Kähler class by [6, Theorem 6]. We then suppose $\boldsymbol{p}_2>\boldsymbol{p}_1>0$.

Suppose S is of genus $\mathbf{g}=0$, i.e. $(S,J_S)=\mathbb{CP}^1$. Then M is a toric variety. Using the existence of an extremal almost Kähler metric of involutive type compatible with any Kähler metric on M (see [7, Proposition 4]) and the Yau–Tian–Donaldson correspondence on toric manifold, it is shown in [50] that there exists an extremal Kähler metric in every Kähler class of $\mathbb{P}(\mathcal{O}\oplus\mathcal{L}_1\oplus\mathcal{L}_2)\longrightarrow\mathbb{CP}^1$. Observe that by applying Proposition 8.2 and Theorem 6.1 we obtain that these extremal metrics are given by the Calabi ansatz of [5].

Now suppose that (S,J_S) in an elliptic curve, i.e. $\mathbf{g}=1$. The complex manifold (M,J) is not toric. However, it is shown in [7] that (M,J) is a semi-simple principal toric fibration. By the Leray–Hirch Theorem $H^2(M,\mathbb{R})$ is of dimension 2. In particular, up to scaling any Kähler class on (M,J) is compatible. It is shown in [6, Proposition 4], that M admits an extremal almost Kähler metric in any compatible Kähler class. Then, using Proposition 8.2 and Theorem 7.12, we conclude that there exists an extremal Kähler metric in every Kähler class. Furthermore, by Theorem 6.1, the extremal Kähler metrics are of the form of (3.6), i.e. are given by the generalized Calabi ansatz.

Remark 8.3. — We conclude by pointing out that if $\mathbf{g}\geqslant 2$, it is shown in [7, Theorem 2] that there exists an extremal Kähler metric in sufficiently *small* compatible Kähler classes. On the other hand, by [7, Proposition 2], if $\mathbf{g} > 2$ and $\boldsymbol{p}_1, \boldsymbol{p}_2$ satisfying $2(\mathbf{g}-1) > \boldsymbol{p}_1 + \boldsymbol{p}_2$, there is no extremal Kähler metric in sufficiently *big* Kähler classes.

BIBLIOGRAPHY

[1] M. Abreu, "Kähler geometry of toric varieties and extremal metrics", *Internat. J. Math.* **9** (1998), no. 6, p. 641-651.

[2] ———, "Kähler geometry of toric manifolds in symplectic coordinates", in *Symplectic and contact topology: interactions and perspectives (Toronto, ON/Montreal, QC, 2001)*, Fields Inst. Commun., vol. 35, Amer. Math. Soc., Providence, RI, 2003, p. 1-24.

[3] V. Apostolov, "The Kähler geometry of toric manifolds", Lecture notes availabe at `http://profmath.uqam.ca//~apostolo/notes.html`.

[4] V. Apostolov, D. M. J. Calderbank & P. Gauduchon, "Hamiltonian 2-forms in Kähler geometry. I. General theory", *J. Differential Geom.* **73** (2006), no. 3, p. 359-412.

[5] V. Apostolov, D. M. J. Calderbank, P. Gauduchon & C. W. Tønnesen-Friedman, "Hamiltonian 2-forms in Kähler geometry. II. Global classification", *J. Differential Geom.* **68** (2004), no. 2, p. 277-345.

[6] ———, "Hamiltonian 2-forms in Kähler geometry. III. Extremal metrics and stability", *Invent. Math.* **173** (2008), no. 3, p. 547-601.

[7] ———, "Extremal Kähler metrics on projective bundles over a curve", *Adv. Math.* **227** (2011), no. 6, p. 2385-2424.

[8] V. Apostolov, D. M. J. Calderbank & E. Legendre, "Weighted K-stability of polarized varieties and extremality of Sasaki manifolds", to appear in *Adv. Math.*, `https://arxiv.org/abs/2012.08628`.

[9] V. Apostolov, S. Jubert & A. Lahdili, "Weighted K-stability and coercivity with applications to extremal Kähler and Sasaki metrics", to appear in *Geometry and Topology*, `https://arxiv.org/abs/2104.09709`.

[10] V. Apostolov & G. Maschler, "Conformally Kähler, Einstein–Maxwell geometry", *J. Eur. Math. Soc. (JEMS)* **21** (2019), no. 5, p. 1319-1360.

[11] M. F. Atiyah, "Convexity and commuting Hamiltonians", *Bull. London Math. Soc.* **14** (1982), no. 1, p. 1-15.

[12] R. J. Berman & B. Berndtsson, "Convexity of the K-energy on the space of Kähler metrics and uniqueness of extremal metrics", *J. Amer. Math. Soc.* **30** (2017), no. 4, p. 1165-1196.

[13] R. J. Berman, S. Boucksom & M. Jonsson, "A variational approach to the Yau–Tian–Donaldson conjecture", *J. Amer. Math. Soc.* **34** (2021), no. 3, p. 605-652.

[14] R. J. Berman & D. Witt-Nyström, "Complex optimal transport and the pluripotential theory of Kähler-Ricci solitons", `https://arxiv.org/abs/1401.8264`.

[15] E. Calabi, "Extremal Kähler metrics", in *Seminar on Differential Geometry*, Ann. of Math. Stud., vol. 102, Princeton Univ. Press, Princeton, N.J., 1982, p. 259-290.

[16] ———, "Extremal Kähler metrics. II", in *Differential geometry and complex analysis*, Springer, Berlin, 1985, p. 95-114.

[17] B. Chen, A.-M. Li & L. Sheng, "Uniform K-stability for extremal metrics on toric varieties", *J. Differential Equations* **257** (2014), no. 5, p. 1487-1500.

[18] X. Chen, "The space of Kähler metrics", *J. Differential Geom.* **56** (2000), no. 2, p. 189-234.

[19] ———, "On the existence of constant scalar curvature Kähler metric: a new perspective", *Ann. Math. Qué.* **42** (2018), no. 2, p. 169-189.

[20] X. Chen & J. Cheng, "On the constant scalar curvature Kähler metrics (I) – A priori estimates", *J. Amer. Math. Soc.* **34** (2021), no. 4, p. 909-936.

[21] ———, "On the constant scalar curvature Kähler metrics (II) – Existence results", *J. Amer. Math. Soc.* **34** (2021), no. 4, p. 937-1009.

[22] X. Chen, S. Donaldson & S. Sun, "Kähler–Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities", *J. Amer. Math. Soc.* **28** (2015), no. 1, p. 183-197.

[23] ———, "Kähler–Einstein metrics on Fano manifolds. II: Limits with cone angle less than 2π", *J. Amer. Math. Soc.* **28** (2015), no. 1, p. 199-234.

[24] ———, "Kähler–Einstein metrics on Fano manifolds. III: Limits as cone angle approaches 2π and completion of the main proof", *J. Amer. Math. Soc.* **28** (2015), no. 1, p. 235-278.

[25] X. Chen, M. Paun & Y. Zeng, "On deformation of extremal metrics", `https://arxiv.org/abs/1506.01290v2`.

[26] T. Darvas, "The Mabuchi completion of the space of Kähler potentials", *Amer. J. Math.* **139** (2017), no. 5, p. 1275-1313.

[27] T. Darvas & Y. A. Rubinstein, "Tian's properness conjectures and Finsler geometry of the space of Kähler metrics", *J. Amer. Math. Soc.* **30** (2017), no. 2, p. 347-387.

[28] T. Delcroix, "The Yau–Tian–Donaldson conjecture for cohomogeneity one manifolds", `https://arxiv.org/abs/2011.07135`.

[29] T. Delzant, "Hamiltoniens périodiques et images convexes de l'application moment", *Bull. Soc. Math. France* **116** (1988), no. 3, p. 315-339.

[30] R. Dervan, "Uniform stability of twisted constant scalar curvature Kähler metrics", *Int. Math. Res. Not. IMRN* (2016), no. 15, p. 4728-4783.

[31] R. Dervan & J. Ross, "K-stability for Kähler manifolds", *Math. Res. Lett.* **24** (2017), no. 3, p. 689-739.

[32] S. K. Donaldson, "Scalar curvature and stability of toric varieties", *J. Differential Geom.* **62** (2002), no. 2, p. 289-349.

[33] ———, "Extremal metrics on toric surfaces: a continuity method", *J. Differential Geom.* **79** (2008), no. 3, p. 389-432.

[34] A. Fujiki, "Remarks on extremal Kähler metrics on ruled manifolds", *Nagoya Math. J.* **126** (1992), p. 89-101.

[35] A. Futaki & T. Mabuchi, "Bilinear forms and extremal Kähler vector fields associated with Kähler classes", *Math. Ann.* **301** (1995), no. 2, p. 199-210.

[36] P. Gauduchon, "Calabi's extremal Kähler metrics: An elementary introduction", Lecture Notes.

[37] D. Guan, "Existence of extremal metrics on compact almost homogeneous Kähler manifolds with two ends", *Trans. Amer. Math. Soc.* **347** (1995), no. 6, p. 2255-2262.

[38] ———, "On modified Mabuchi functional and Mabuchi moduli space of Kähler metrics on toric bundles", *Math. Res. Lett.* **6** (1999), no. 5-6, p. 547-555.

[39] V. Guillemin & S. Sternberg, "Convexity properties of the moment mapping", *Invent. Math.* **67** (1982), no. 3, p. 491-513.

[40] V. Guillemin, "Kaehler structures on toric varieties", *J. Differential Geom.* **40** (1994), no. 2, p. 285-309.

[41] Y. Hashimoto, "Existence of twisted constant scalar curvature Kähler metrics with a large twist", *Math. Z.* **292** (2019), no. 3-4, p. 791-803.

[42] W. He, "On Calabi's extremal metric and properness", *Trans. Amer. Math. Soc.* **372** (2019), no. 8, p. 5595-5619.

[43] T. Hisamoto, "Stability and coercivity for toric polarizations", https://arxiv.org/abs/1610.07998.

[44] A. D. Hwang, "On existence of Kähler metrics with constant scalar curvature", *Osaka J. Math.* **31** (1994), no. 3, p. 561-595.

[45] A. D. Hwang & M. A. Singer, "A momentum construction for circle-invariant Kähler metrics", *Trans. Amer. Math. Soc.* **354** (2002), no. 6, p. 2285-2325.

[46] E. Inoue, "Constant μ-scalar curvature Kähler metric – formulation and foundational results", to appear in *J. Geom. Anal.*, https://arxiv.org/abs/1902.00664.

[47] N. Koiso & Y. Sakane, "Nonhomogeneous Kähler–Einstein metrics on compact complex manifolds", in *Curvature and topology of Riemannian manifolds (Katata, 1985)*, Lecture Notes in Math., vol. 1201, Springer, Berlin, 1986, p. 165-179.

[48] A. Lahdili, "Convexity of the weighted Mabuchi functional and the uniqueness of weighted extremal metrics", https://arxiv.org/abs/2007.01345.

[49] ———, "Kähler metrics with constant weighted scalar curvature and weighted K-stability", *Proc. Lond. Math. Soc. (3)* **119** (2019), no. 4, p. 1065-1114.

[50] E. Legendre, "A note on extremal toric almost Kähler metrics", https://arxiv.org/abs/1811.05693.

[51] M. Lejmi, "Extremal almost-Kähler metrics", *Internat. J. Math.* **21** (2010), no. 12, p. 1639-1662.

[52] A.-M. Li, Z. Lian & L. Sheng, "Some estimates for the generalized Abreu equation", *Differential Geom. Appl.* **48** (2016), p. 87-103.

[53] ———, "Interior regularity for the generalized Abreu equation", *Internat. J. Math.* **28** (2017), no. 7, article no. 1750053 (18 pages).

[54] A.-M. Li, L. Sheng & G. Zhao, "Differential inequalities on homogeneous toric bundles", *J. Geom.* **108** (2017), no. 2, p. 775-790.

[55] C. Li, "Geodesic rays and stability in the cscK problem", accepted by *Ann. Sci. Éc. Norm. Supér.*, https://arxiv.org/abs/2001.01366v3.

[56] C. Li, G. Tian & F. Wang, "The uniform version of Yau-Tian-Donaldson conjecture for singular Fano varieties", https://arxiv.org/abs/1903.01215.

[57] T. Mabuchi, "K-energy maps integrating Futaki invariants", *Tohoku Math. J. (2)* **38** (1986), no. 4, p. 575-593.

[58] ———, "Some symplectic geometry on compact Kähler manifolds. I", *Osaka J. Math.* **24** (1987), no. 2, p. 227-252.

[59] Y. Sakane, "Examples of compact Einstein Kähler manifolds with positive Ricci tensor", *Osaka J. Math.* **23** (1986), no. 3, p. 585-616.

[60] G. W. Schwarz, "Smooth functions invariant under the action of a compact Lie group", *Topology* **14** (1975), p. 63-68.

[61] Z. Sjöström Dyrefelt, "K-semistability of cscK manifolds with transcendental cohomology class", *J. Geom. Anal.* **28** (2018), no. 4, p. 2927-2960.

[62] G. Székelyhidi, "Extremal metric and K-stability (Phd Thesis)", https://arxiv.org/abs/math/0611002.

[63] ———, "Filtrations and test-configurations", *Math. Ann.* **362** (2015), no. 1-2, p. 451-484, With an appendix by Sebastien Boucksom.

[64] G. Tian, "K-stability and Kähler–Einstein metrics", *Comm. Pure Appl. Math.* **68** (2015), no. 7, p. 1085-1156.

[65] C. W. Tønnesen-Friedman, "Kähler Yamabe minimizers on minimal ruled surfaces", *Math. Scand.* **90** (2002), no. 2, p. 180-190.

[66] B. Zhou & X. Zhu, "Relative K-stability and modified K-energy on toric manifolds", *Adv. Math.* **219** (2008), no. 4, p. 1327-1362.

Manuscrit reçu le 27 août 2021,
révisé le 15 février 2022,
accepté le 12 mai 2022.

Simon JUBERT
Departement de Mathématiques
UQAM, C.P. 8888
Succursale Centre-ville, Montréal (Quebec)
H3C 3P8 (Canada)
Institut de Mathématiques de Toulouse
Université Paul Sabatier
118 route de Narbonne
31062, Toulouse (France)
simonjubert@gmail.com