ALGEBRAIC COMBINATORICS

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Volume 6, issue 5 (2023), p. 1177-1188.
https://doi.org/10.5802/alco.310

Algebraic Combinatorics is published by The Combinatorics Consortium
and is a member of the Centre Mersenne for Open Scientific Publishing
www.tccpublishing.org www.centre-mersenne.org
e-ISSN: 2589-5486

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ABSTRACT We construct a natural idempotent in the descent algebra of a finite Coxeter group. The proof is uniform (independent of the classification). This leads to a simple determination of the spectrum of a natural matrix related to descents. Other applications are discussed.

1. INTRODUCTION

The study of permutation statistics is a venerable branch of combinatorics, stretching back at least to the time of Euler, and continuing to the present day.(1) Let $\pi \in \mathfrak{S}_n$ be an element of the symmetric group on $[n] := \{1, 2, \dots, n\}$. Some classical and much studied permutation statistics on $\mathfrak{S}_n$ include the *inversion number* $\operatorname{inv}(\pi)$, *descent number* $\operatorname{des}(\pi)$, and *major index* $\operatorname{maj}(\pi)$. They are defined in terms of the *inversion set* $\operatorname{Inv}(\pi)$ and *descent set* $\operatorname{Des}(\pi)$ as follows:

$$\begin{aligned} \operatorname{Inv}(\pi) &:= \{(i,j) : 1 \leqslant i < j \leqslant n, \pi(i) > \pi(j)\} & \operatorname{inv}(\pi) &:= |\operatorname{Inv}(\pi)| \\ \operatorname{Des}(\pi) &:= \{i : 1 \leqslant i < n, \pi(i) > \pi(i+1)\} & \operatorname{des}(\pi) &:= |\operatorname{Des}(\pi)|, \end{aligned}$$

and

$$\operatorname{maj}(\pi) = \sum_{i \in \operatorname{Des}(\pi)} i.$$

We will also have need of the *ascent number* $\operatorname{asc}(\pi) = |\{i : 1 \leqslant i < n, \pi(i) < \pi(i+1)\}|$.

As is well-known, the symmetric group $\mathfrak{S}_n$ is the Coxeter group of type A_{n-1}. (For background on Coxeter groups, see, e.g. [7, 9, 20, 22].) Many permutation statistics make sense in the more general setting of Coxeter groups. Let (W, S) be a finite irreducible Coxeter group. Any element $w \in W$ can be written as a word in the simple reflections S, and the minimum number of reflections required is $\ell(w)$, the *length* of w. It is not difficult to see [7, Prop. 1.5.2] that for a permutation π, $\operatorname{inv}(\pi) = \ell(\pi)$. For $w \in W$ the *(right) descent set* $D_R(w)$ is

$$D_R(w) = \{s \in S : \ell(ws) < \ell(w)\}.$$

For a permutation π, $\operatorname{des}(\pi) = |D_R(\pi)|$ [7, Prop. 1.5.3]. We also observe that, by [7, Prop. 1.4.2], $\ell(ws) = \ell(w) \pm 1$, so that we can just as well write

$$D_R(w) = \{s \in S : \ell(ws) = \ell(w) - 1\}.$$

Manuscript received 15th October 2022, revised 6th April 2023, accepted 8th April 2023.

KEYWORDS. Coxeter group, reflection representation, permutation representation, descents, descent algebra, idempotents, central limit theorems.

(1)For a sampling of some of these ideas, see, e.g. [8, 12, 14, 16, 17, 18, 23, 34].

In [28], the author, motivated by a consideration of the distance spectra of Cayley graphs on Coxeter groups, observed that, for many types of Coxeter groups, the matrix $\ell(uv^{-1})|_{u,v\in W}$ has integral eigenvalues. These eigenvalues were computed explicitly in types A, D, and E. In [27], Reiner, Saliola, and Welker independently observed the integrality of the eigenvalues, and explained this result in terms of the presence of a twisted Gel'fand pair. In [24], Randriamaro generalized these results in a different direction, by introducing the polynomial-valued statistics $\mathrm{inv}_X(\pi) := \sum_{(i,j)\in \mathrm{Inv}(\pi)} X_{ij}$ and $\mathrm{des}_X(\pi) := \sum_{i\in \mathrm{Des}(\pi)} X_i$, where the X_i and X_{ij}'s are indeterminates. He then computed the spectrum of the matrices $\mathrm{inv}_X(\sigma\tau^{-1})$ and $\mathrm{des}_X(\sigma\tau^{-1})$, where σ and τ range over the symmetric group.(2)

In a recent paper [38], Vershik and Tsilevich independently rediscovered some of these results, and introduced a very elegant representation theoretic approach affording a uniform treatment of cases, allowing them to determine simultaneously the spectra of the matrices $\mathrm{inv}(\sigma\tau^{-1})$, $\mathrm{des}(\sigma\tau^{-1})$, and $\mathrm{maj}(\sigma\tau^{-1})$. In the course of their investigations they discovered that certain naturally defined elements of $\mathbb{C}\mathfrak{S}_n$, the group algebra of $\mathfrak{S}_n$, possess some very nice properties. One of these, which they call $u_{\widetilde{\mathrm{des}}}$, is our main interest here.

To motivate its definition, we first observe that the mean value of des is $(n-1)/2$.(3) Vershik and Tsilevich define the new *centered* statistic $\widetilde{\mathrm{des}}$, which is just des shifted so as to have zero mean:

$$\widetilde{\mathrm{des}}(\pi) = -\frac{1}{2}(\mathrm{asc}(\pi) - \mathrm{des}(\pi)) = \mathrm{des}(\pi) - \frac{1}{2}(n-1).$$

They then defined the group algebra element

$$u_{\widetilde{\mathrm{des}}} := \sum_{\pi\in\mathfrak{S}_n} \widetilde{\mathrm{des}}(\pi)\pi,$$

and proved [38, Cor. 2] that it is a quasi-idempotent of $\mathbb{C}\mathfrak{S}_n$:

$$u_{\widetilde{\mathrm{des}}}^2 = -(n-1)!\,u_{\widetilde{\mathrm{des}}}. \tag{1}$$

They obtained this result by first obtaining the spectrum of $\widetilde{\mathrm{des}}(\sigma\tau^{-1})|_{u,v\in\mathfrak{S}_n}$. The connection between the two results is that $\widetilde{\mathrm{des}}(\sigma\tau^{-1})|_{u,v\in\mathfrak{S}_n}$ is the representation matrix of $u_{\widetilde{\mathrm{des}}}$ in $\mathbb{C}\mathfrak{S}_n$. In general, for any finite group W, if $b = \sum_{w\in W} b(w)w$ is any element of the group algebra $\mathbb{C}W$, then for any $u\in W$,

$$bu = \sum_{w\in W} b(w)wu = \sum_{v\in W} b(vu^{-1})v,$$

so the (u,v) entry of the representation matrix of b on $\mathbb{C}W$ is just $b(uv^{-1})$.(4)

In this work we propose to reverse this development, and in the process, extend the result to all (finite) Coxeter groups. Specifically, we will prove the following. Let (W,S)

(2)For recent work in the same vein, see [26].

(3)As noted by Vershik and Tsilevich, this formula can be proven starting from the Eulerian polynomial. A more direct proof proceeds by random sampling of the set of all permutations on $[n]$. Define the random variable $X_i(\pi)$ to be 1 if there is a descent at position i in π, and zero otherwise. Then $\mathrm{des}(\pi) = \sum_{i=1}^{n-1} X_i$. As there is an equal chance for the position i of a random permutation to be a descent or an ascent, we have $\mathbb{E}(X_i) = 1/2$ for all i. The result now follows by linearity of expectation.

(4)In what follows we will actually be concerned with group matrices of the form $b(u^{-1}v)$. But the two matrices $b(uv^{-1})$ and $b(u^{-1}v)$ just differ by a relabeling, and are therefore similar. (Actually, in [28] there is an inadvertent sleight of hand at one point, switching $\ell(uv^{-1})$ to $\ell(u^{-1}v)$, but with no ill effects.)

be as above, and let $A_R(w) = \{s \in S : \ell(ws) > \ell(w)\}$ be the *(right) ascent set* of w. Define

$$\widetilde{d} := \sum_{w \in W} \widetilde{d}(w) w \in \mathbb{C}W,$$

where

$$\widetilde{d}(w) = |A_R(w)| - |D_R(w)|$$

is the natural generalization (up to a constant) of $\widetilde{\mathrm{des}}$ to all Coxeter groups.(5)

THEOREM 1.1. *The element $\widetilde{d}$ is a quasi-idempotent of $\mathbb{C}W$. In particular,*

$$\widetilde{d}^2 = \frac{2|W|}{h}\widetilde{d},$$

where h is the Coxeter number of W.

REMARK 1.2. It is well-known that $|W| = d_1 d_2 \cdots d_{|S|}$ [20, Theorem 3.9], where the d_i are the invariant degrees of W, and that $h = d_{|S|}$ [20, Proposition 3.17 and Theorem 3.19], so $|W|/h$ is always integral. This can also be seen by noting that the centralizer of a Coxeter element consists only of its powers [22, Theorem 29-5], so $|W|/h$ is the size of the conjugacy class of a Coxeter element.

COROLLARY 1.3. *The element $\frac{h}{2|W|}\widetilde{d}$ is an idempotent in $\mathbb{C}W$.*

We recover (1) from Theorem 1.1 by observing that, in type A_{n-1}, $\widetilde{\mathrm{des}}(\pi) = -\widetilde{d}(\pi)/2$ and $h = n$. Additionally, we get the following.

COROLLARY 1.4. *For $u, v \in W$ let $M_{u,v} = \widetilde{d}(u^{-1}v)$. Then the (eigenvalue, multiplicity) pairs of M are $(0, |W| - N)$ and $(2|W|/h, N)$, where N is the number of positive roots.*

From this, we can retrieve the spectrum of $\mathrm{des}(u^{-1}v)$ in type A_{n-1} obtained in [24] and [38]. (See below.)

To explain the connection of all this with the descent algebra requires a little background. Let (W, S) be a finite Coxeter system. Let W_J be the parabolic subgroup of W generated by $J \subseteq S$, and let W^J be the distinguished transversal of left coset representatives of W/W_J consisting of minimum length elements [20, Proposition 1.10]. Define

$$x_J := \sum_{w \in W^J} w$$

Let X_{JK} be the distinguished transversal of double coset representatives of $W_J \backslash W / W_K$ consisting of minimum length elements. Solomon showed [33] that the $\{x_J\}_{J \subseteq S}$ form a subalgebra Σ of $\mathbb{C}W$ called the *descent algebra*:

$$x_J x_K = \sum_L a_{JKL} x_L,$$

with structure constants given by

$$a_{JKL} = |\{x \in X_{JK} : x^{-1} W_J x \cap W_K = W_L\}|.$$

The reason for the name 'descent algebra' becomes more readily apparent from the second basis of Σ discovered by Solomon, consisting of elements of the form

$$y_K := \sum_{J \supseteq K} (-1)^{|J-K|} x_J = \sum_{y \in W : D_R(y) = \overline{K}} y.$$

(5)For a proof that $\widetilde{d}$ is centered, see Remark 2.3 below.

where $\overline{K} := S - K$. After a little manipulation we find that

$$\widetilde{d} = |S||W|\mathcal{I} - 2 \sum_{K \subseteq S} |\overline{K}| y_K,$$

where $\mathcal{I} := \frac{1}{|W|} \sum_w w$ is the trivial idempotent of $\mathbb{C}W$ (which is, of course, also in the descent algebra).

Vershik and Tsilevich observe [38] that one could, in theory, obtain the spectrum of $\widetilde{d}$ (or in their case, $\widetilde{\mathrm{des}}$) by using properties of the descent algebra. Indeed, as they remark, that was one of their original motivations for considering the problem.(6) But, as they point out, the complicated nature of the structure constants makes this an unpleasant task.(7) Moreover, the main point of their article was to use a simpler method to obtain a wider variety of results in the case of the symmetric group, by computing the spectra of the matrices $\mathrm{des}(\sigma\tau^{-1})$, $\mathrm{maj}(\sigma\tau^{-1})$, and $\mathrm{inv}(\sigma\tau^{-1})$, the last of which is not related (at least not directly) to the descent algebra.

In fact, a more general result of this nature, covering all Coxeter groups, was obtained earlier by Randriamaro [25]. He showed that, somewhat remarkably, the spectrum of a general descent algebra element of the form $b = \sum_{J \subseteq S} \lambda_J x_J$ is *integral* whenever the λ_J are integral. He did this by providing an explicit formula. Let c_J be the Coxeter element of W_J, namely the product of the elements of J taken in some fixed order, and write $\overline{c_J}$ for the conjugacy class of c_J. Write $[J]$ for the class of parabolic subgroups of the form W_K conjugate to W_J. Let $\{J_i\}_{i=1}^p$ be representatives of the equivalence classes of parabolic subgroups. Then the eigenvalues of b can be written

$$\sum_{i=1}^{p} a_{J_i J_j J_j} \sum_{K_i \in [J_i]} \lambda_{K_i},$$

one for each j. The corresponding multiplicities are $|\overline{c_{J_j}}|$. In [25, Theorem 2.5], Randriamaro gives an explicit formula for the structure constants a_{JKK} appearing above, involving normalizers of parabolic subgroups. In theory, therefore, it ought to be possible to obtain Theorem 1.1 from Randriamaro's results. In practice, however, the computations rapidly become unwieldy, and do not lend themselves to simple analysis. Instead, we will obtain Theorem 1.1 by employing some results found in [28].

One more remark is necessary. In [4], Bergeron, Bergeron, Howlett, and Taylor construct a complete set of primitive, pairwise orthogonal idempotents of Σ. Bidigare showed in his thesis [6] that the descent algebra of a Coxeter group is anti-isomorphic to a subalgebra of the face algebra of the corresponding hyperplane arrangement. (See also [5, 10, 30].) Building on this work, Saliola constructed [32, Theorem 5.2] inductively a complete set of primitive orthogonal idempotents in the face algebra of a hyperplane arrangement. In [31] he describes how this translates to a set of idempotents in the descent algebra. It is possible that these results could also be used to obtain Theorem 1.1.

2. The permutation representation

To begin our proof of Theorem 1.1 we recall some facts from [28]. Let V be the reflection representation of W equipped with the usual inner product $(\cdot,\cdot)$. Let $\Phi \subset V$ be the root system associated to W and choose a positive root system Π and a

(6)This line of reasoning is carried out in [36] and [38].

(7)For an idea of how unpleasant the structure constants can be in the simplest case of the symmetric group, the reader is invited to examine [19]. The results in [19] are rendered a little more comprehensible in [36] and [38]. Arguably, Atkinson [2, 3] or Willigenburg [37] provide the simplest approach to the descent algebra of the symmetric group.

corresponding simple root system Δ, so that $S = \{s_\alpha : \alpha \in \Delta\}$. We write $\alpha > 0$ for $\alpha \in \Pi$ and $\alpha < 0$ for $\alpha \in -\Pi$.

Let Ψ be the vector space direct sum of the one dimensional subspaces spanned by the root vectors in Φ. To distinguish root vectors as elements of Ψ as opposed to elements of V we use Dirac's bra-ket notation. Thus, vectors in Ψ are denoted by $|\psi\rangle$ and dual vectors by $\langle\psi|$. The standard inner product on Ψ is given by $\langle\alpha|\,\beta\rangle = \delta_{\alpha\beta}$, where $\alpha, \beta \in \Phi$ and $\delta_{\alpha\beta}$ is the Kronecker delta. Then, for all $w \in W$ and $\alpha \in \Phi$, the permutation action of W on Φ is given by

$$w\,|\alpha\rangle := |w\alpha\rangle\,.$$

As $w\alpha = w\beta$ if and only if $\alpha = \beta$, we have

$$\langle w\alpha|\, w\beta\rangle = \delta_{w\alpha,w\beta} = \delta_{\alpha,\beta} = \langle\alpha|\,\beta\rangle\,,$$

which shows that the permutation representation is orthogonal. In particular, if T denotes 'transpose',

$$w^T\,|\alpha\rangle = w^{-1}\,|\alpha\rangle = \left|w^{-1}\alpha\right\rangle,$$

so

$$\langle\alpha|\,w = (w^T\,|\alpha\rangle)^T = (\left|w^{-1}\alpha\right\rangle)^T = \left\langle w^{-1}\alpha\right|.$$

Lastly, we define

$$|\psi_w\rangle := \sum_{\alpha>0} w\,|\alpha\rangle\,.$$

Next, we prove a series of lemmas leading to the main result.

Lemma 2.1. *Let $\ell(w)$ be the length of $w \in W$. Then*

$$\ell(u^{-1}v) = N - \langle\psi_u|\,\psi_v\rangle\,,$$

where $N = |\Pi|$.

Proof. We have

$$\begin{aligned}\langle\psi_u|\,\psi_v\rangle &= \sum_{\alpha>0,\beta>0} \langle\alpha|\,u^{-1}v\,|\beta\rangle = \sum_{\alpha>0,\beta>0} \langle u\alpha|\,v\beta\rangle = \sum_{\alpha>0,\beta>0} \delta_{u\alpha,v\beta} \\ &= \sum_{\alpha>0,\beta>0} \delta_{\alpha,u^{-1}v\beta} = |\{\beta > 0 : u^{-1}v\beta > 0\}|.\end{aligned}$$

Hence, $\langle\psi_u|\,\psi_v\rangle$ counts the number of positive roots sent to positive roots by $u^{-1}v$. So $N - \langle\psi_u|\,\psi_v\rangle$ is the number of positive roots sent to negative roots by $u^{-1}v$. But it is well-known (e.g. [20, Corollary 1.7]) that this number is the same as $\ell(u^{-1}v)$. ☐

Lemma 2.2. *We have*

$$\widetilde{d}(u^{-1}v) = \langle\psi_u|\,\varphi_v\rangle\,,$$

where

$$|\varphi_v\rangle := \sum_{s\in S}(|\psi_v\rangle - |\psi_{vs}\rangle).$$

Proof. By definition,

$$\begin{aligned}\widetilde{d}(w) &= |A_R(w)| - |D_R(w)| \\ &= |\{s \in S : \ell(ws) = \ell(w) + 1\}| - |\{s \in S : \ell(ws) = \ell(w) - 1\}| \\ &= \sum_{s\in S}(\ell(ws) - \ell(w)).\end{aligned}$$

Thus, by Lemma 2.1,

$$\begin{aligned}\widetilde{d}(u^{-1}v) &= \sum_{s\in S}(\ell(u^{-1}vs) - \ell(u^{-1}v)) \\ &= \sum_{s\in S}(\langle\psi_u|\ \psi_v\rangle - \langle\psi_u|\ \psi_{vs}\rangle) \\ &= \langle\psi_u|\ \varphi_v\rangle\,. \end{aligned}$$ □

Remark 2.3. Using the representation of $\widetilde{d}$ given in the proof of Lemma 2.2 it is easy to see that $\widetilde{d}$ is centered. Explicitly, we have

$$\sum_{w\in W}\widetilde{d}(w) = \sum_{s\in S}\sum_{w\in W}(\ell(ws) - \ell(w)) = 0,$$

because the map $w \mapsto ws$ is a bijection of W.

Lemma 2.4. *We have*

$$|\varphi_u\rangle = u\sum_{\gamma\in\Delta}\left|\gamma^-\right\rangle,$$

where

$$\left|\gamma^-\right\rangle := |\gamma\rangle - |-\gamma\rangle\,.$$

Proof. Let $\gamma\in\Delta$. According to [20, Prop. 1.4], the simple reflection s_γ permutes all the positive roots amongst themselves, except for γ, which satisfies $s_\gamma\gamma = -\gamma$. Hence,

$$\begin{aligned}\left|\psi_{us_\gamma}\right\rangle &= \sum_{\delta>0} u\,|s_\gamma\delta\rangle \\ &= \sum_{\delta>0,\delta\neq\gamma} u\,|s_\gamma\delta\rangle + u\,|-\gamma\rangle \\ &= \sum_{\varepsilon>0} u\,|\varepsilon\rangle - u(|\gamma\rangle - |-\gamma\rangle) \\ &= |\psi_u\rangle - u\left|\gamma^-\right\rangle. \end{aligned}$$

Thus,

$$|\varphi_u\rangle = \sum_{\gamma\in\Delta}(|\psi_u\rangle - \left|\psi_{us_\gamma}\right\rangle) = u\sum_{\gamma\in\Delta}\left|\gamma^-\right\rangle.$$ □

The following proposition is critical.

Proposition 2.5. *For $\alpha,\beta\in\Phi$ we have*

$$\sum_{u\in W}\langle\alpha|\ \varphi_u\rangle\langle\psi_u|\ \beta\rangle = \frac{|W|}{h}(\delta_{\alpha,\beta} - \delta_{\alpha,-\beta}),$$

where $\delta_{\alpha,\beta}$ is the Kronecker delta.

Proof. We have

$$\begin{aligned}\sum_{u\in W}\langle\alpha|\ \varphi_u\rangle\langle\psi_u|\ \beta\rangle &= \sum_{u\in W,\gamma\in\Delta}\langle\alpha|\,u\left|\gamma^-\right\rangle\langle\psi_u|\ \beta\rangle \\ &= \sum_{u\in W,\gamma\in\Delta}\left\langle u^{-1}\alpha\right|\left.\gamma^-\right\rangle\langle\psi_e|\ u^{-1}\beta\rangle \\ &= \sum_{u\in W,\gamma\in\Delta}(\delta_{u^{-1}\alpha,\gamma} - \delta_{u^{-1}\alpha,-\gamma})\chi(u^{-1}\beta>0) \\ &= \sum_{u\in W,\gamma\in\Delta}(\delta_{u\alpha,\gamma} - \delta_{u\alpha,-\gamma})\chi(u\beta>0), \qquad (*)\end{aligned}$$

where $\chi(P) = 1$ or $\chi(P) = 0$ according as the proposition P be true or false.

Consider first the terms for which $\beta = \alpha$. Then the second term in the sum $(*)$ becomes

$$\sum_{u \in W, \gamma \in \Delta} \delta_{u\alpha,-\gamma} \chi(u\alpha > 0),$$

which vanishes because all the simple roots are positive. (We cannot have both $u\alpha > 0$ and $u\alpha < 0$.) So, we must count all the group elements u such that $u\alpha = \gamma$, or, equivalently, the set of group elements u such that $u\gamma = \alpha$, for some fixed α. Call this number g.

Let $\operatorname{stab}(\alpha)$ be the stabilizer of α. The claim is that $g = |\operatorname{stab}(\alpha)|$. By [20, Corollary 1.5], there exists a $w \in W$ such that $w\gamma = \alpha$. Certainly $\operatorname{stab}(\alpha) w\gamma = \alpha$. Suppose $w'\gamma = \alpha$. Then $w'w^{-1} \in \operatorname{stab}(\alpha)$, so $w' \in \operatorname{stab}(\alpha)w$, and the claim follows.

We want to show that

$$\sum_{\gamma \in \Delta, \gamma \in \operatorname{orb}(\alpha)} |\operatorname{stab}(\alpha)| = \frac{|W|}{h},$$

where $\operatorname{orb}(\alpha)$ is the orbit of α. By the orbit-stabilizer theorem, if W acts transitively on X, then for all $x \in X$ we have $|W| = |\operatorname{orb}(x)||\operatorname{stab}(x)|$. It is known that there are at most two orbits of positive roots for an irreducible Coxeter group, but we need not use this fact here, which relies on the classification. Instead, we appeal to a result of Steinberg [35, Corollary 6.5], which does not rely on the classification, and which states that, if the simple roots S are partitioned into transitive sets of $n_1, n_2, \ldots, n_r$ elements, then the set of all roots is partitioned into transitive sets of $n_1h, n_2h, \ldots, n_rh$ elements, and these sets correspond. Using Steinberg's result, we have

$$\sum_{\gamma \in \Delta, \gamma \in \operatorname{orb}(\alpha)} |\operatorname{stab}(\alpha)| = \sum_{\gamma \in \Delta, \gamma \in \operatorname{orb}(\alpha)} \frac{|W|}{|\operatorname{orb}(\alpha)|} = |W| \frac{n_i}{n_i h} = \frac{|W|}{h},$$

where i specifies the orbit of α containing γ.

Now suppose that $\beta = -\alpha$ in $(*)$. Then the exact same reasoning as above shows that the right hand side of $(*)$ equals $-|W|/h$.

Finally, suppose that $\beta \neq \alpha, -\alpha$. Define $W_1 := \{u \in W : u\beta > 0\}$. Suppose $u \in W_1$ satisfies $u\alpha = \gamma$. Observe that $u\beta = \pm\gamma$ is impossible. Now $s_\gamma u\alpha = -\gamma$. Moreover, $s_\gamma u\beta > 0$, because, as mentioned previously, s_γ permutes the positive roots not equal to γ. But then

$$\sum_{u \in W_1} (\delta_{u\alpha,\gamma} - \delta_{u\alpha,-\gamma}) = \sum_{u \in W_1} (\delta_{u\alpha,\gamma} - \delta_{s_\gamma u\alpha,\gamma}) = 0,$$

because $u \mapsto s_\gamma u$ is a bijection of W_1. ☐

Corollary 2.6. *Let* $|\xi\rangle = \sum_{\alpha \in \Phi} \xi_\alpha |\alpha\rangle$ *and* $|\eta\rangle = \sum_{\beta \in \Phi} \eta_\beta |\beta\rangle$. *Assume that* $\eta_{-\beta} = -\eta_\beta$. *Then*

$$\sum_{u \in W} \langle \xi | \varphi_u \rangle \langle \psi_u | \eta \rangle = \frac{2|W|}{h} \langle \xi | \eta \rangle .$$

Proof. We have

$$\begin{aligned}\sum_{u\in W} \langle\xi|\,\varphi_u\rangle\,\langle\psi_u|\,\eta\rangle &= \frac{|W|}{h}\sum_{\alpha,\beta\in\Phi}\xi_\alpha\eta_\beta(\delta_{\alpha,\beta}-\delta_{\alpha,-\beta})\\ &= \frac{|W|}{h}\sum_{\alpha\in\Phi}\xi_\alpha(\eta_\alpha-\eta_{-\alpha})\\ &= \frac{2|W|}{h}\sum_{\alpha\in\Phi}\xi_\alpha\eta_\alpha\\ &= \frac{2|W|}{h}\langle\xi|\,\eta\rangle\,. \end{aligned}$$ □

Proof (of Theorem 1.1). We have

$$\widetilde{d}^2 = \sum_{u,w\in W}\widetilde{d}(u)\widetilde{d}(w)uw = \sum_{v\in W}\left(\sum_{u\in W}\widetilde{d}(u)\widetilde{d}(u^{-1}v)\right)v,$$

so we must show that, for every $v\in W$,

$$(\dagger)\qquad \sum_{u\in W}\widetilde{d}(u)\widetilde{d}(u^{-1}v) = \frac{2|W|}{h}\widetilde{d}(v).$$

From Lemma 2.2 we have

$$\sum_{u\in W}\widetilde{d}(u)\widetilde{d}(u^{-1}v) = \sum_{u\in W}\langle\psi_e|\,\varphi_u\rangle\,\langle\psi_u|\,\varphi_v\rangle\,.$$

But by Lemma 2.4, $|\varphi_v\rangle = \sum_{\gamma\in\Delta}(|v\gamma\rangle - |-v\gamma\rangle)$, so by Corollary 2.6 we get

$$\sum_{u\in W}\langle\psi_e|\,\varphi_u\rangle\,\langle\psi_u|\,\varphi_v\rangle = \frac{2|W|}{h}\langle\psi_e|\,\varphi_v\rangle\,.$$

Applying Lemma 2.2 again yields Equation (†). □

Closer examination of the proof of Theorem 1.1 given above reveals that we could have replaced $\langle\psi_e|$ by any other element of the form $\langle\psi_t|$ for some $t\in W$. This yields the following, which is in fact equivalent to (†).

Proposition 2.7. *For any $x,z\in W$,*

$$\sum_{y\in W}\widetilde{d}(x^{-1}y)\widetilde{d}(y^{-1}z) = \frac{2|W|}{h}\widetilde{d}(x^{-1}z).$$

Proof. Substitute $u=x^{-1}y$ and $v=x^{-1}z$ into (†). □

3. The spectrum of $\widetilde{d}$

If we define a matrix M such that $M_{x,y}=\widetilde{d}(x^{-1}y)$, then by Proposition 2.7 the minimal polynomial of M is

$$M\left(M-\frac{2|W|}{h}\right)=0.$$

In particular, M has only two eigenvalues, namely 0 and $2|W|/h$. Evidently, the multiplicity of the nonzero eigenvalue is just the rank of M.

Proposition 3.1. *The rank of M is N, the number of positive roots.*

Proof. We compute the trace of M in two ways. On the one hand, we can sum the diagonal elements to get

$$\operatorname{Tr} M = \sum_{x\in W} \widetilde{d}(x^{-1}x) = \sum_{x\in W} \widetilde{d}(1) = |W||A_R(1)| = |W|n,$$

where n is the number of simple roots. On the other hand, if m is the multiplicity of the nonzero eigenvalue, then summing the eigenvalues gives $2|W|m/h$. Equating the two expressions and using $nh = 2N$ [20, Prop. 3.18] shows that $m = N$. $\square$

Corollary 1.4 is now immediate.

To show that this reproduces the spectrum of $\operatorname{des}(\sigma^{-1}\tau)$ in type A_{n-1} discovered in [24] and [38], we proceed as follows. From the discussion given in the introduction

$$\operatorname{des}(\pi) = \widetilde{\operatorname{des}}(\pi) + (n-1)/2 = -\frac{1}{2}\widetilde{d}(\pi) + \frac{1}{2}(n-1)$$

Writing $Q_{u,v} = \operatorname{des}(u^{-1}v)$, we get

$$Q = -\frac{1}{2}M + \frac{n-1}{2}J,$$

where J is the all-ones matrix of size $|W|$. But M and Q have constant (and equal) row and column sums,[(8)] hence they both commute with J. In particular, the eigenvectors of Q and M divide into two classes, the all-ones vector $\boldsymbol{\jmath}$, and everything orthogonal to $\boldsymbol{\jmath}$. By construction, $M\boldsymbol{\jmath} = 0$, because $\widetilde{d}$ is centered, so $Q\boldsymbol{\jmath} = n!(n-1)/2\boldsymbol{\jmath}$. For the remaining eigenvectors $\boldsymbol{s}$ we have $Q\boldsymbol{s} = -\frac{1}{2}M\boldsymbol{s}$. By Corollary 1.4, we have $|W| - N = n! - \binom{n}{2}$ zero eigenvalues of M, one of which corresponds to $\boldsymbol{\jmath}$, so the multiplicity of zero as an eigenvalue of Q is $n! - \binom{n}{2} - 1$. The remaining $N = \binom{n}{2}$ eigenvalues of Q are just $-(2|W|/h)/2 = -(n-1)!$. Collecting results yields the following.

Theorem 3.2 ([24, Corollary 1.2]; [38, Theorem 4]). *The (eigenvalue, multiplicity) pairs of the matrix* $\operatorname{des}(\sigma^{-1}\tau)$ *in type* A_{n-1} *are*

$$\left[\frac{1}{2}n!(n-1), 1\right], \quad \left[0, n! - \binom{n}{2} - 1\right], \quad \left[-(n-1)!, \binom{n}{2}\right].$$

4. More group algebra computations

In [38], Vershik and Tsilevich also define a group algebra element $u_{\widetilde{\text{inv}}}$ related to inversions, and show [38, Corollary 2] that $u_{\widetilde{\text{inv}}}u_{\widetilde{\text{des}}}$ is proportional to $u_{\widetilde{\text{inv}}}$. Using the techniques introduced above, we can show that this result holds in the more general Coxeter setting. First, we obtain a centered length statistic.

Lemma 4.1. *We have*

$$\mathbb{E}(\ell) = \frac{1}{|W|}\sum_{w\in W} \ell(w) = \frac{N}{2}.$$

Proof. Let w_0 be the longest element in W. Then (e.g. [7], Proposition 2.3.2) we have

$$\ell(ww_0) = \ell(w_0) - \ell(w) = N - \ell(w).$$

But the map $w \to ww_0$ is a bijection of W, so

$$\sum_{w\in W} \ell(w) = \sum_{w\in W} \ell(ww_0) = N|W| - \sum_{w\in W} \ell(w). \qquad \square$$

(8) This is immediate from the fact that they are group matrices of the form $f(u^{-1}v)$, because $\sum_u f(u^{-1}v) = \sum_x f(x) = \sum_v f(u^{-1}v)$.

Define

$$\widetilde{\ell}(w) := \ell(w) - \frac{N}{2},$$

and set

$$\widetilde{\ell} = \sum_{w\in W} \widetilde{\ell}(w) w.$$

Theorem 4.2. *We have*

$$\widetilde{d}\widetilde{\ell} = \frac{2|W|}{h}\widetilde{\ell}.$$

Remark 4.3. Our result holds for $\widetilde{d}\widetilde{\ell}$ rather than $\widetilde{\ell}\widetilde{d}$ due to a slight difference in our conventions from those of [38].

Proof. We have

$$\widetilde{d}\widetilde{\ell} = \sum_{u,w} \widetilde{d}(u)\widetilde{\ell}(w)uw = \sum_{v\in W}\left(\sum_{u\in W}\widetilde{d}(u)\widetilde{\ell}(u^{-1}v)\right)v,$$

so it suffices to show that

$$\sum_{u\in W}\widetilde{d}(u)\widetilde{\ell}(u^{-1}v) = \frac{2|W|}{h}\widetilde{\ell}(v).$$

From Lemmas 2.1 and 2.2 we obtain

$$\sum_{u\in W}\widetilde{d}(u)\widetilde{\ell}(u^{-1}v) = \sum_{u\in W}\langle\psi_e|\ \varphi_u\rangle\left(\frac{1}{2}N - \langle\psi_u|\ \psi_v\rangle\right).$$

We claim the first part of the sum vanishes. To see this, use Lemma 2.4 to write the sum as

$$\sum_{u}\langle\psi_e|\ \varphi_u\rangle = \sum_{u,\gamma\in\Delta}\left\langle\psi_e\,|u|\ \gamma^-\right\rangle = \sum_{u,\gamma\in\Delta}\left\langle\psi_{u^{-1}}|\ \gamma^-\right\rangle = \sum_{u,\gamma\in\Delta}\left\langle\psi_u|\ \gamma^-\right\rangle.$$

But for any $\alpha\in\Phi$,

$$\sum_{u\in W} u\,|\alpha\rangle = \sum_{\beta\in\Phi} f_\beta(|\beta\rangle + |-\beta\rangle),$$

for some constants f_β, because if $u\alpha=\beta$ then $us_\alpha\alpha = -\beta$, and $u\mapsto us_\alpha$ is a bijection of W. As $(\langle\beta| + \langle-\beta|)(|\gamma\rangle - |-\gamma\rangle) = 0$, the claim follows.

As for the second part of the sum, by Proposition 2.5 we have

$$\begin{aligned}
-\sum_{u\in W}\langle\psi_e|\ \varphi_u\rangle\langle\psi_u|\ \psi_v\rangle &= -\sum_{u\in W,\alpha>0,\beta>0}\langle\alpha|\ \varphi_u\rangle\langle\psi_u|\ v\beta\rangle\\
&= -\frac{|W|}{h}\sum_{\alpha>0,\beta>0}(\delta_{\alpha,v\beta} - \delta_{\alpha,-v\beta})\\
&= -\frac{|W|}{h}\left(|\{\beta>0 : v\beta>0\}| - |\{\beta>0 : v\beta<0\}|\right)\\
&= -\frac{|W|}{h}(N - 2\ell(v))\\
&= \frac{2|W|}{h}\widetilde{\ell}(v).
\end{aligned}$$

□

5. Additional remarks

5.1. Other statistics. In this work we have discussed a simple, centered, idempotent related to descents in Coxeter groups. But of course there are many other statistics. Randriamaro, Tsilevich, and Vershik ([24, 26, 36, 38]) have obtained the spectra for group algebra elements on the symmetric group related to major indices, excedances, peaks, valleys, double descents, and double ascents, just to name a few. It turns out that these all have simple forms. This raises the question of whether or not any of these statistics have natural extensions and simple descriptions in the more general setting of Coxeter groups. Although some of these statistics have been extended to other Coxeter types (for instance, the flag major index of type B defined in [1]), it is not clear if there are any natural candidates valid for all Coxeter groups. Nevertheless, this may be a line worth investigating.

5.2. Central limit theorems. In addition to their combinatorial significance, permutation statistics have been examined by statisticians interested in their limiting distributions (see, e.g. [11, 13, 21, 29]). For instance, it is known that the descent statistic (normalized by mean and variance) has a limiting normal distribution. Recently, there has been much interest in the two-sided descent statistic $T(\pi) = \mathrm{des}(\pi) + \mathrm{des}(\pi^{-1})$. Chatterjee and Diaconis [13] showed that this statistic obeys a central limit theorem. This was extended to all Coxeter groups by Brück and Röttger [11] and Feray [15]. In the course of the proofs of these results, one needs the moments of the distribution $T(\pi)$. For instance, one needs $\mathbb{E}(\mathrm{des}(\pi)\,\mathrm{des}(\pi^{-1}))$. This can be obtained easily, and for all Coxeter groups, from Proposition 2.7. We have (with $n = |S|$)

$$\mathbb{E}(\widetilde{d}(u)\widetilde{d}(u^{-1})) = \frac{1}{|W|}\sum_{u\in W}\widetilde{d}(u)\widetilde{d}(u^{-1}) = \frac{2n}{h} = \frac{n^2}{N}.$$

By construction, $\mathbb{E}(\widetilde{d}) = 0$, so using the fact that $\widetilde{d}(u) = n - 2\,\mathrm{des}(u)$ we get

$$\mathbb{E}(\mathrm{des}(u)\,\mathrm{des}(u^{-1})) = \frac{n^2}{4}\left(1 + \frac{1}{N}\right).$$

For type A_{n-1} this agrees with the computation in [13].

Acknowledgements. I would like to thank the referees for their careful reading of the manuscript and for their corrections and suggestions, which led to a much improved exposition.

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