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"Impact of step size on convergence in swarmalator systems"

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Abstract

Group robotics is one of the key areas of the development of robotic systems. This is due to the fact that for a wide class of practical tasks, the use of a group of relatively simple robots is much more efficient than using a single large multi-purpose device. The modern development of computer technology and communication systems opens up wide opportunities for the construction of such systems.

The most progressive and effective approach is the implementation of the collective behavior of robots according to the swarm principle, when each of them interacts only with neighboring individuals, synchronously exchanging the collected information about the environment and their condition. Such a group compensates for the weakness of its detection and communication devices by joining a team.

The problems of introducing group robotics into the modern world are studied in this thesis. If they combine two concepts, synchronization and swarming, they are called a swarmalator. In swarmalator systems, the movement of the robots is governed by differential equations. These equations are solved with the Euler method, where the location and phase are determined. The Euler method is time-discrete and allows the integration of first-order differential equations. Therefore, there is a step size to be chosen.

The main task is to study group movement, which is based on transmitting information with a definite step size. The step value affects how often the swarmalators share their location and phase. Three main conclusions are made. The first research is what happens when varying the step size - is it most optimal to use with small step sizes? The second conclusion is that when increasing the step size with a small increment or using randomization of the step size. Such methods are typically, more optimal to use with a gradual increase in the step size because the convergence time is lower. The third is when decreasing the step size using a small increment. The results showed that this method is optimal to use when the step size exceeds 1. The states converge at a rather large interval, compared with previous results, but at the same time with a large value of the convergence time.

The values of optimal step sizes are presented and analyzed. As performance criteria, we consider the computational power that is required, the average convergence time, the coupling probability and the step size. The behavior of all parameters is graphically represented in plots. The conclusions are based on the simulations done for the results.

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Contents

CHAPTER 1

Introduction

The idea of solving comparatively simple technical problems with a group of synchronized systems, robots or machines, has long been at the centre of discussion by both artificial intelligence specialists and robotics.

Probably, the first working results in the form of real projects in the area of group robotics, which is the creation of systems that interact with each other, appeared in the last century. As well the concept of synchronicity has appeared and is successfully developing.

The concept of synchronicity can be seen in nature, in living beings - we encounter it every single day. For example, flashing fireflies, heartbeat and neurons inside a person. Hence, synchronicity can be observed not only outside, but also inside, it can be both invisible and visible. Over time, many scientists began to deal with the topic of synchronization, study it and formulate mathematical models of synchronization. Subsequently, synchronicity and asynchronicity began to be applied to various technical systems, such as different network typologies, communication latency and communication collision avoidance.

Another important concept is swarming. Certain animals of the same type in nature are grouped. This association takes place based on an instinct laid down by nature. It helps groups of animals survive in nature, look for food and protect themselves from dangers. Examples are birds flock and fish school. This concept interested [1], [2], [3] and [4] scientists and they start to create mathematical models trying to bring swarming to life. When trying to implement, there are many tasks, such as attraction for grouping together, but at the same time avoiding collisions exchange. Thus, theoretical models are gradually being introduced into robotic systems, so new emerging problems such as mobility, delays and connectivity have to be taken into account.

These two concepts have quite a lot in common. So the synchronicity and swarming are combined into one general model called the swarmalator [5]. The swarmalator creates a model, which synchronizes an internal phase and swarm. It is important to note that when the distance between adjacent swarmalators and their phases changes, this is reflected in the swarmalarors themselves, their phases and their location. This interaction leads to new system dynamics. Thus, appear new states of swarming. Further, swarmalators start to be used in technical systems. Accordingly, it is necessary to take into consideration several limitations of movements, communications and locations. These limitations are related to physical capabilities, delays, loss of information and accuracy.

However, it should be noted that so far a lot of research in this area remains at the theoretical, model level [6], [7], [8] and [9]. This is seen in the numerous tasks that are performed by swarmalators, among which it should be highlighted. Currently, some of the main challenges are: the task of distributing robots and drones so that communication is maintained between them and at the same time the desired area is covered; the task of coordinating the movement of swarmalators and their group movement in the selected zone; the task of creating different geometric shapes by changing the location of individual swarms; the task of exchanging messages between robots and drones with each position and phase change.

Differential equations are given for each swarmalator. These equations determine the location and phase. There are various mathematical methods for solving differential equations. The simplest method is the Euler method because it allows the integration of first-order differential equations but with low accuracy. The concept of the Euler method is that with each step we find the derivative of the previous location (phase), which is multiplied by time, which we will later call a step size, and append it to the current calculation of the location (phase). This calculation is used to find the location of swarmalators and their phase. However, it is important to mention that each swarmalator has a coupling probability. This is the probability that the swaramalator has sent its location and phase [10].

The main task for this project is to choose the optimal value of the step size. The step size can be either small or large. This can affect the swarmalator in various ways.

We can say that small step size is a fairly sound solution, which in most cases will ensure maximum convergence for solving the problem. Similarly, updating the system parameters will occur more often, but will lead to high costs for calculations and the need for huge computing power because calculations will occur frequently. Although a large step size can be a good solution, which saves computing power, the results obtained may be too far from the real value. Additionally, the result depends on the state and several system parameters of swarmulator as the number of swarmalators, maximum number of step sizes, speed, and area size.

This thesis simulates various values of step sizes using different concepts such as varying step sizes, adding a small value to the step size or randomizing the step size.

These concepts are suggested for swarmalators for two reasons:

- minimization of the computational power that is required.

- decrease in the average convergence time.

For this reason, we have started selecting the optimal step size, which will subsequently reduce the strain on the wireless network and speed up the work of swarmalators.

The thesis consists of the following chapters:

Chapter 2 represents the state of the art. We start with the description of the oscillators. We present the general concept as published in nature of synchronization, the beginning of knowledge and the practical implementation in modern times. Next, we describe the general concept of swarming, give an overview of the origin of knowledge and a short description of how it is currently applied. The chapter ends with the characteristics of the swarmalators and how they began to develop today, based on scientific and methodological literature.

Chapter 3 gives a brief overview of the swarmalator model. This chapter of the thesis is based on the theory set out in the previous chapter. So there is a description of the chosen methodology, and the essence of practical work is described here. All necessary parameters and states are discussed and presented by formulas and tables.

Chapters 4, 5 and 6 represent the experimental part with the help of simulation tools. The description begins with a review of the methods that were used during the study. Additionally, the practical part includes all plots of the states and their behaviours are shown. We describe three different methods for changing the step size. Finally, the analysis of the collected information was done for each part of the simulations.

Chapter 7 presents the results and conclusions. In the results, we combine all the results obtained for all types of simulations, and analyzed them for each state. In conclusions, we have provided the most optimal step sizes that can be used for various tasks. Each value will be optimal for a specific type of task.

CHAPTER 2

State of the art

A brief overview of the state of the art related to this work is given in this chapter. Firstly, the basic description of the meaning of oscillators. Secondly, we describe the meaning of swarming. The next section helps us to bring both meanings into a general concept, which we further call swarmalators [5].

2.1 Oscillators

A general concept in nature

Nature has been creating various biological systems for millions of years, using synchronization in everyday life. One of the most common mentions was about flashing fireflies and crickets chirping in unison. This process unites and simultaneously protects them in nature. Scientists also take attention to the heart cells because it has synchronization and they also mention the synchronized menstrual cycle of every woman [11]. In the future, this will help humanity understand where life came from and up to the analysis of human behavior [12]. This suggests that synchronization can be both outside, as example of nature, and inside, as the menstrual cycle of women. More examples are presented in the article by R. E. Mirollo and S. H. Strogatz [13].

The beginning of knowledge

One of the initial books on the study of synchronization was the book by the American scientist N. Wiener. He is the first one who was able to give a way for future scientists to begin studying the connection between a machine and a living entity using a mathematical description. [14]

One of the next persons who could describe the synchronization of all biological oscillators was Winfree [15]. Using two-phase functions, he described the relationship between the oscillators. The first one is the function of influence, how it affected others, and the second is the sensitivity function, which determined how the generator itself would receive signals and its reaction. His description is applicable only in the case when the connection between the oscillators is weak and it is possible to neglect the changes in the amplitude of the oscillations. This model allows considering only the dynamics of their phases.

Subsequently, the concept of Winfree was further developed in the book of Kuramoto, in which he presented a more structured model from the point of view of mathematical analysis [16]. It became the first book for studying synchronization processes, which introduced order parameters to quantify a level of synchronization. Kuramoto was also able to clarify more precisely the populations of interconnected oscillators.

Later, different scientists began to do different studies on all kinds of variations and generalizations of the model. They began to take into account various factors, such as frequency distributions, the presence of various noises, the complex configuration of neighboring connections, and the study of the influence of inertia. Many of them are presented in books [17], [18], [19] and [20].

Nowadays, scientists focused on learning the issue of networks on splay state stabilization, which makes it possible to obtain formal guarantees of reliability. In the papers [21], [22] and [23] authors study the influence of weak and strict desynchronization using the Lyapunov function. Additionally, various cluster states were received when the coupling form was changed [24] and [25].

Further, scientists presented results that were robust and reliable for slightly unidentified network elements, which combined two parameters reliability even [26].

Thus, the states discussed earlier can be effective not only for networks but also for robotic systems.

Practical implementation

Nowadays, researchers often face synchronization in practice, which sometimes needs to be achieved or, conversely, desynchronization is required. So, for example, many different technologies need to process and analyze a huge amount of information. The article describes big data strategies used in synchrotrons [27].

Also, it describes time synchronization, which is very important to us in everyday life, because when we use networks to send and receive emails, it is necessary to synchronize the action so that the new file can be overwritten by an older file. For instance, various files and databases, all with timestamps. In papers [28] and [29] the synchronization problem and requirements for cellular technologies are covered.

A good example of the use of synchronization is time synchronization in the underwater world, for which a new scheme consisting of two stages like TSHL was proposed, where the first phase is the stage of evaluation of the skew of clock pulses. Further, the first phase has the estimated skew and based on this the second phase is the stage of estimating the displacement of clock pulses [30].

The coupled oscillators can be used in technical systems, robots and nature by using different communications, like the robot simulation, which is inspired by firefly [31], [32] and [33].

2.2 Swarming

A general concept in nature

Collective association into a herd, a swarm, or a crowd is inherent in living beings. Regardless of the size of living beings, they unite with each other. For example, bacteria [1], insects [2], birds [3] and [4] and larger living creatures.

Swarming usually coordinates a group of objects. The idea itself originated when attention was drawn to a swarm of birds that are grouped into flocks and exist as a whole. It is also a good realistic example that helps to study group behavior in practice, to study the basic facts that can influence swarms [3].

Thus, they can easily adapt to environmental conditions. It is interesting to note that nature itself is based only on surprisingly simple laws and rules. Based on this, many scientists are inspired by these characteristics of living beings. Researchers are engaged in the development of innovative designs [34].

The beginning of knowledge

The mathematical analysis of swarming is a difficult task for scientists. Many swarming methods have already been presented, one of which can be seen in Reynolds work [35]. He presented a theoretical model of the birds who flock. The boid model, which makes a simulation, needs to have a matching speed with neighbors, avoid collision and be close to each of the groups. However, many aspects were not taken into account, such as turning the head and flapping the wings.

Based on Reynolds work, the scientists proposed a more realistic model that took into account additional conditions. These included that birds are looking for food and more thoroughly considered the distance and grouping between the creatures. It was measured by vectors and a flow diagram of the system was presented, which described the system cycle for each new position [36].

In the future, studies were conducted at the sheep stage, a multi-stage genetic algorithm was presented. The algorithm simulates two conditions: the first one - is inheritance among individual sheep and the second one - is inheritance among sheep herds [37]. Further, the method was tested and presented in the new paper with the calculation results for simple and multi-stage [38].

The centuries-old problem is in the proper tracking and coordination of objects. The main idea is that the swarm should actuate and feel in the physical world, coordinate a large number of swarms and thereby increase productivity with an increase in swarms. The paper describes a few methods, which simultaneously find several goals at the same time, track them and balance the new optimums [39].

Practical implementation

One of the very first implementations of welding turned out to be the Robot Sheepdog Project. The work was to control the behavior of the nature of the ducks' flocking in practice [40].

Subsequently, the behavior of mobile robots was analyzed by experiments. The paper simulates a sequential change in the characteristics of sensor systems. When the number of neighbors increases, the stability of noise and the speed also increase [41].

For a long time, there are such tasks as modeling, functionalities, mechanics, organizations, apps and others for swarms. Thus, the characteristics and nowadays technologies of drones and the swarm of drones were discussed [42]. As an example, the speed and sensors of the quadrocopter were considered.

2.3 Swarmalators - Sync and Swarm

A general concept

Systems that combine two concepts swarm and oscillators are called a swarmalator system or swarmalator. For the first time, a simple model of the swarmalator system was proposed by K. O'Keeffe, H. Hong, and S. Strogatz. This system made it possible to analytically study various states depending on the location and phase of the model [5].

By changing the location and phase, one by one, new stable states were found. The stable states are possible just because of the stabilizing influence of the phases. Thus, this interaction between the swarmalators, their locations and places lead to new results. We can observe new stable swarming states, with stable phases [7].

A. Barcis and C. Bettstetter in their article describe the influence of discrete-time interactions and temporary coordination. Additionally, they presented the combined model into a single whole temporal and spatial coordination [43].

In addition, the influence of different probability values, the influence of the distance between robots, speed and acceleration were analyzed for five different states, the static and phase wave states: sync, async, phase wave; splintered and active [32].

Further, the swarmalator model is implemented in practice. For instance, such mechanisms and living beings as robots, bacteria, microswimmers, vinegar eels and colloids, are described in detail in the review of swarmalators [9] and [44].

The beginning of knowledge

O'Keeffe presents the swarmalator models in the real life [5]. One of the most famous studies began with frog choruses. The subject of the study was male Japanese tree frogs. The antisynchronization and wavy antisynchronization were presented and realized in the mathematical model. The main idea is to study the stability of two-cluster and wavy antisynchronization [45].

Another example of swarmalaror is myxobacteria. Here presented the mathematical model that describes the waves, which are formed when passing through "ripple phase". The article describes the phenomena when the myxobacterial waves pass through each other without collision. Subsequently, based on experiments, myxobacteria, which is an oscillator, have a phase of the waves and they depend on the density [46].

Afterwards, an important discovery was made by scientists, they found a connection between the dynamics of movement and the internal state [47].

Theoretical and practical implementation

The very first step in the development swarmalators was made by Tanaka. The scientist makes a description of a model with interacting elements. These elements have internal dynamics, which are related to the internal state [6].

Afterwards, Tanaka and other scientists describe the states and their configurations. Eventually, it was obtained that the received static states have a hierarchical composition of clusters [48].

At present, two-way interaction or swarmalator is considered in more detail. The agents with common opinions moving closer to each other and with various opinions become further apart [49].

For today in one of the last articles, O'Keeffe and colleagues describe the swarmalators which are on the ring [7] and [8], additionally, he makes a description of the states and presents the conditions for the appearance of these states [50]. Further, the scientist explore the effects of delayed intercommunications on swarming and synchronised models and identified stable collective states for them [51].

Based on the theoretical knowledge gained over all previous years, many studies of swarmalators in practice have been conducted. Scientists at first implemented the model of the swarmalator in the lab. The scientists make a model from the robots [9]. In addition, the scientists make simulations with the multi-robot systems [43], in which they took into account the low refresh rate and delays.

For further study, the following parameters were selected: position, limits of the speed, limits of the acceleration and physical size of robots. These parameters are the most important because it is necessary to avoid robot collisions and try to minimize the distance between robots in some cases [10].

CHAPTER 3

Swarmalators

3.1 Introduction to the model

For this thesis, we need to understand the principle of building the swarmalator model because in the further parts we will use these concepts for calculations.

Each swarmalator model has a location and an internal phase. We call the location x_i and internal phase θ_i. Two equations describe how the location and internal phase of swarmalators change. Each swarmalator i from 1 to N has its own x_i and θ_i.

Main equations:

$$\dot{x_i} = \frac{1}{N}\sum_{j=1}^{N}\frac{x_j - x_i}{||x_j - x_i||}(1 + J\cos(\theta_j - \theta_i)) - \frac{x_j - x_i}{||x_j - x_i||^2} \tag{3.1}$$

$$\dot{\theta}_i = \frac{K}{N}\sum_{j=1}^{N}\frac{\sin(\theta_j - \theta_i)}{\|x_j - x_i\|} \tag{3.2}$$

where $i = 1, ..., N$ with two parameters J and K.

In the first case, looking at Equation 3.1: the swarmalator changes its position based on the position of other swarmalators and their phases.

The coefficient J can be:

- greater than zero: then the swarmalators with similar phases attract each other and with different phases repel from each other.
- less than zero: then the opposite process occurs, with the same phases repelled, and with different attracted.

Table 3.1: Values of parameters J and K for five states: *static sync, static async, static phase wave, splintered phase wave and active phase wave* [10].

	Static sync	Static async	Static phase wave	Splintered phase wave	Active phase wave
J	0.1	0.1	1	1	1
K	1	-1	0	-0.1	-0.75

In the second case, looking at Equation 3.2: the swarmalator changes its phase based on the position of other swarmalators and from their phase.

The coefficient K can be:

- greater than zero: then the swarmalators synchronize the phases.
- less than zero: then the neighboring swarmalators maximize their phase difference and make the desynchronization.
- equal to 0, then the phase is not changing.

We get different states with various values of coefficients J and K (see Table 3.1).

The swarmalators are used in various fields of engineering. In order to be able to control them, they must exchange their locations and phases using messages. Thus, they must have a continuous connection all the time.

The discrete time connection is based on the Euler method [52]. The idea of Euler's methods is to replace a fragment of the graph with a polyline, but so that the resulting polyline is as close as possible to the original one.

This method allows integrating first-order differential equations. Its accuracy is low, but in some cases, for example, in electric drive control systems, it is used quite often. Based on the Euler's method, it is easier to understand the algorithms of other, more efficient methods. The geometric meaning of such method is the approximation of the solution on the interval by the segment of the tangent drawn to the graph of the solution at the point. As can be seen on the Fig. 3.1, at each new step, the approximate solution moves to another member of the solution family. As a result, the sampling error accumulates, which linearly depends on the step size, see Equation 3.3 for location and Equation 3.4 for internal phase.

$$x(t+1) = x(t) + \dot{x}(t)\Delta t \tag{3.3}$$

$$\theta(t+1) = \theta(t) + \dot{\theta}(t)\Delta t \tag{3.4}$$

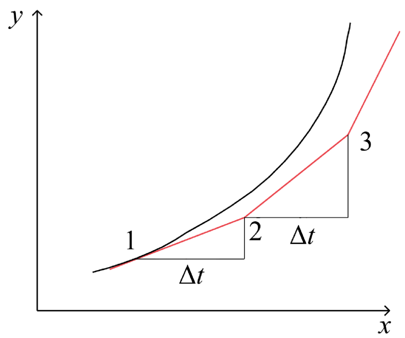


Figure 3.1: Geometric interpretation of the modified Euler method, where 1 is the starting point and the slope of the tangent line (x_0, y_0), 2 is the slope of the tangent line from recursively continue point 1 (x_1, y_1) and 3 is the slope of the tangent line from recursively continue point 2 (x_2, y_2). Step size is Δt.

Thus, a first order accuracy has the Euler method. A practical consequence of this fact is the expectation that as the step size decreases, the approximate solution will become more and more accurate and, as the step size tends to zero, it will converge to an exact solution with a linear velocity in step size. We expect that when the step size is halved, the error will decrease by more times. Very slow convergence with decreasing the step size is characteristic of first-order methods and serves as an obstacle to their widespread use. Therefore, it is important to choose the value that is most optimal for the swarmalators under different conditions.

It follows from the geometric illustration (Fig. 3.2) that:

- There is an error at each step.
- The error is greater with larger steps.
- The error accumulates during subsequent calculations.

Thus, it is very important to choose the optimal value of the step size because smaller value of Δt implies that the function is closer to the real function. This is shown at the Figure 3.2 where it is illustrates what happens as the step size gets smaller and smaller.

Obviously, the faster the function changes, the smaller the step size should be such that a sufficiently smooth graph of the function in question can be drawn through the found set of points.

3.2 Description of the states

We pay attention to Table 3.1, where the values of J and K are given, leading to one of the five states presented in Fig. 3.3 and 3.4 [11]:

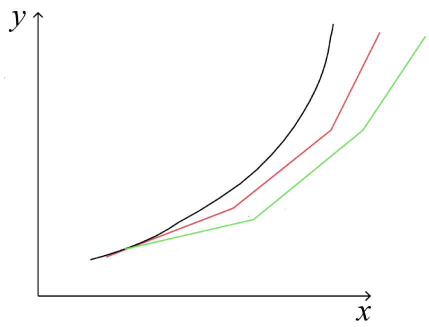


Figure 3.2: Geometric interpretation of the modified Euler method, with different step sizes Δt. The black function is original. The red line is the one with the smallest Δt. The green line is the one with the largest Δt.

1. Static sync: in Fig 3.3a the state is shown. This is the first stationary state. All swarmalators have the same color, which indicates a synchronized phase and they are symmetrically distributed in a circled disk. This state occurs for K larger a 0 and various J.

2. Static async: in Fig 3.3b the state is shown. This is the second stationary state. All swarmalators are symmetrically distributed in a circled disk, as in the static sync case. With different locations, we can notice various phases. They have complete phase desynchronization. This state occures for K and J less then 0 and for J more then 0.

3. Static phase wave: in Fig 3.3c the state is shown. This is the last one stationary state. For this state the coefficients are K equal to 0 and J more then 0. We may notice that the same colors are grouped together, which means that the same phases are located next to each other. They are located around the ring, and are grouped closer to the same phases, so we call it the static phase wave state.

4. Splintered phase wave: in Fig 3.4a the state is shown. We take a look at the first non-stationary state. For this state the coefficient K should be equal to 0 or less then 0. It is splintered into disconnected clusters, which are grouped into different phase groups. Between each pair of phase group, we can see the gaps, which separate different phases. In this state, small amplitude oscillations are performed both in position and in phase relative to their average values.

5. Active phase wave: in Fig 3.4b the state is shown. We see the second non-stationary state. The amplitude of oscillations is increase to compare with the splintered phase wave. In the future, it will begin to perform regular cycles both in spatial angle and phase. Afterwords, they split into the counter-rotating subgroups without gaps. The swarmalators are non-stationary to compare with three previous static states. For this state the coefficients: K is less then 0 and J is greater then 0.

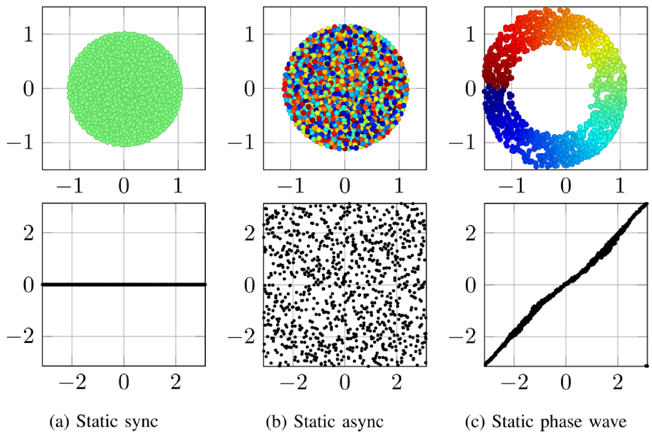


Figure 3.3: Converged three static states with stochastic coupling and memory initialized with 0 for coupling probability $p = 0.01$, parameters J and K, Table 3.1. Upper row are locations and phases. Lower row are phases over polar angle of their locations. Figure taken from [10].

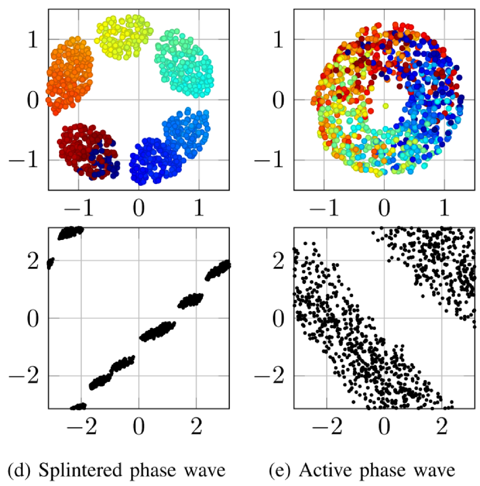


Figure 3.4: Converged two phase wave states with stochastic coupling and memory initialized with 0 for coupling probability $p = 0.01$, parameters J and K, Table 3.1. Upper row are locations and phases. Lower row are phases over polar angle of their locations. Figure taken from [10].

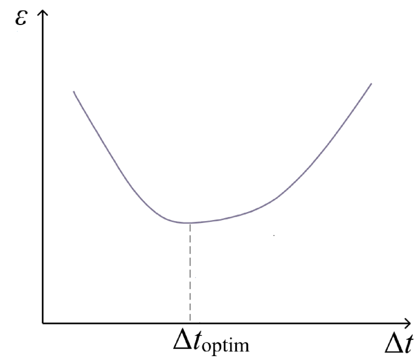


Figure 3.5: Choosing the optimal integration step size.

We can also divide the five states into two groups, stationary and dynamic states.

The first static group is the stationary state. They are presented in Fig. 3.3. In this case, the swarmalators relocate around and quickly change the phases. We notice the high values of speed and angular displacement for each swarmalator. Then, these values are decreasing and get closer to zero. Subsequently, it is possible to determine which of the three static states is formed: the static sync, static async or static phase wave. Finally, the swarmalarors slow down and stop and we can say that convergence has been achieved.

The second dynamic group is non-stationary states. They are presented in Fig. 3.4. In this case, the swarmalarors move around continuously without stopping. Thus, it is harder to determine convergence, because we are faced with moving states. According to the gaps, because clusters are divided by phase, we can determine which state it is.

3.3 Concept of the optimal step size

One of the most important tasks for swarmalators that an engineer who composes programs for solving differential equations faces is the choice of an appropriate step size.

If the step size is too small, then the calculations require an unjustifiably long machine time, and the number of errors in individual steps that add up to a total error are very large, Fig. 3.5.

If, on the contrary, a sufficiently large step size is selected, then the error ϵ caused by the truncation of the series is significant, and there will also be an unacceptably large accumulated total error as well, Fig. 3.5. Also, the convergence time decreases greatly with increasing the step size.

In the research project, we focus on the amount of calculation power that is required for a convergence of the system. The result will depend on the state and on several system parameters, for example, on the speed, acceleration and distance of the simulator.

In particular, the amount of computational power is determined by the step size. If larger a step size is taken, it implies that an update of the movement speed and phase update is calculated less often, and vice versa. A too-large step size, albeit saving computational effort, would prevent the system from converging.

The optimal step size is unknown and to find the optimal value we consider the following methods in the work:

- Impact of the varied step size to find the critical values, for which the system stops converging in a reliable manner.
- Impact of the increment to improve convergence while still partly using large step sizes.
- Impact of the standard deviation when using randomization of the step sizes.

3.4 Simulation tool

In the previous section, we considered different methods for changing the step size for five states.

The essence of the computational experiment is that with the help of some computer program, we conduct a numerical study of the step size for the models under consideration, varying various incoming parameters.

The method of choice for this tasks is Monte-Carlo simulations. They will be extend using the statistics software package R [53], for which an implementation of the swarmalator model is already available from the supervisor. The goal is to work with this implementation to perform extensive simulations, and extend it to cover the simulations with increment and standard deviation.

The simulation tool used in the work was developed for previous papers. The main idea of this simulation is to find the values of convergence time. We are taking into account the fact that the values of the coupling probability and step size are set personally, by the selection method. The key thing is to pay attention to reducing computing power and minimizing convergence time. The simulation performed in the R Studio, the method of choice for this task is Monte-Carlo simulations.

We use several files:

- "std_param.r", where the basic parameters of variables and constants are set.
- "utilities.r", where we calculate the distance between swarmalators.
- "sim_v2.r", where we define the main parameters for the calculation model and the main functions for the simulations.

```
# system parameters
NODE_COUNT <- 100

NUM_STEPS <- 10000
DT <- 0.01

# plot every .. steps
PLOT_STEPS <- 10;

TX_P <- 0.01

# parameters of the model
# following states are supported:
STATIC_SYNC <- 1
STATIC_ASYNC <- 2
STATIC_PHASE_WAVE <- 3
SPLINTERED_PHASE_WAVE <- 4
ACTIVE_PHASE_WAVE <- 5
```

Listing 1: Struct interface of "std_param.r".

- "convTime_stepsize.r", where the main parameters of the step sizes, coupling probabilities and creation of a separate file are set. All calculations of the step sizes and convergence time for all five states, previously discussed, are written into a configuration file. This file is generated with a tool and is independent of the simulations. The same simulation is run many times to find the optimal parameters and replayed infinitely times.

There are a few global variables, which are used in each simulation. They are specified in the file "std_param.r". The file contains the basic parameters. Furthermore, we use "utilities.r" as a source file for "sim_v2.r". Afterwards, the files "std_param.r" and "sim_v2.r" are used as a source files for "convTime_stepsize.r". Listing 1 shows the main parameters, which we are more focused on.

Listing 2 is an example of the section of code from "convTime_stepsize.r". The results obtained are recorded to a file during the calculation named "temp_stepsize.out" and at the end of the simulation they are written to a file "sim_stepsize.out". Listing 2 shows the main function "startAnalysis", which according to the specified parameters, will create a table of calculated values with a given step and convergence time for all five states separately. To do this, we set the number of steps, the value of the step size, the interval with which the step will be counted using the loop and the coupling probability value.

```
source( 'sim_v2.r' );
source( 'std_param.r' );

REPETITIONS <- 10;
startAnalysis <- function( filename = "sim_stepsize.out" )
{
        file.remove( "temp_stepsize.out" );
        # delete temp file and save data into it
        RND_SND <<- 1;
        ABORT_WHEN_CONVERGED <<- 1;
        PLOT_STEPS <<- 0;
        # no plotting
        USE_MEM <<- MEM_INIT_ZERO;
        # use memory with random values
        NUM_STEPS <<- 50000 # number of steps
        for( stepsize_exp in 1:-1 )
        # iterate over sending probabilities
        {
          for( stepsize_digit in 1:9 )
          {
                DT <<- stepsize_digit * 10^-stepsize_exp;
                TX_P <<- 1; # coupling probability

                # simulation: varying the step size

                # p=1 and step size from 0.1 to 90
                ...
                write( c( DT, avg_ct, stddev_ct, min_ct, max_ct ),
                file="temp_stepsize.out", ncolumns=1+4*5, append=TRUE );
                # store results in file
                }
        }
        file.remove( filename );
        file.rename( "temp_stepsize.out", filename );
        # move simulation results to output file
}
```

Listing 2: Struct interface of "convTime_stepsize.r".

CHAPTER 4

Impact of step size

In this part of the thesis, we give an overview of the concept and then find the critical values of the step size, for which the system stops converging in a reliable manner. To find the values we use a simulation tool.

4.1 Varying the step size

In the section, we consider simulations for which we change the step size. The main idea is to find the critical values, for which all five states stop to converge: *static sync*, *static async*, *static phase wave*, *splintered phase wave* and *active phase wave*. For this analysis, simulations are carried out in the interval of the step size from 0.1 to 6 and the coupling probability varied from a minimum value of 0.1 to a maximum of 1.

We take a look at the convergence time (CT) and the step size $\Delta t \in [0.1; 1]$, Fig. 4.1a.

Firstly, consider the case when the coupling probability is equal to 1. We can see that all the graphs are flat on the whole interval. The convergence time is around 30 for the *static sync*, *static async*, *active phase wave* and around 120 for the *static* and *splintered phase waves*. Important to notice is that the *static sync* state has an increase to 350 when $\Delta t = 1$. At the same time, the *static phase wave* does not converge at the step size equal to 1, Fig. 4.1a.

Then, we gradually decrease our probability value by 0.1 each time at the same step size interval and see how the CT of states changes.

When the coupling probability is equal to 0.6 (Fig. 4.1b) and step size $\Delta t = 0.4$ the *active phase wave* start to grow significantly and with a step size value of 0.5, the state assumes a greater convergence time than *splintered phase wave*. Moreover, when Δt is equal to 0.7, 0.8 and 0.9, this state does not converge, and when $\Delta t = 1$ it again converges with a convergence time close to 1400.

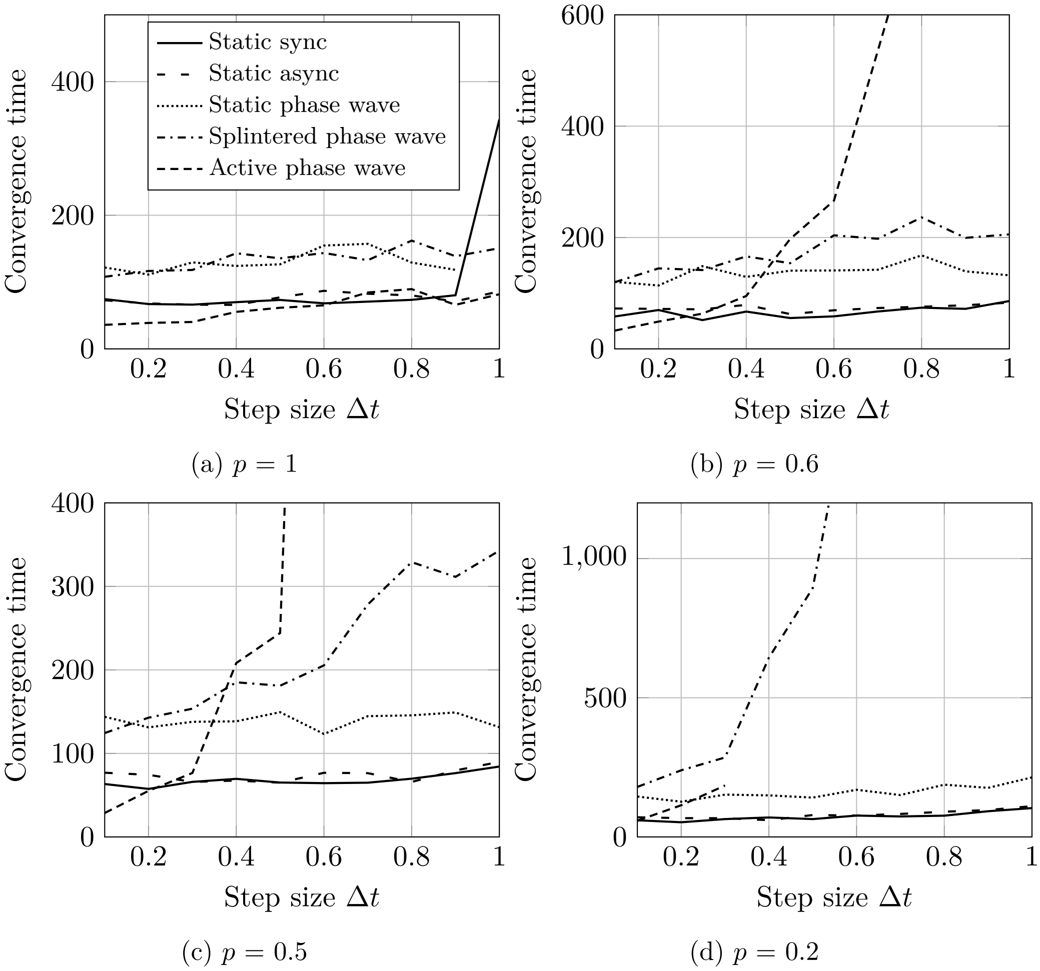


Figure 4.1: Convergence time with the step size $\triangle t$ for *static sync, static async, static phase wave, splintered phase wave and active phase wave.* $\triangle t$ from 0.1 to 1 by 0.1. One simulation is made for each coupling probability. 100 swarmalators.

Next, the coupling probability is decreased to 0.5. We can notice that when the step size is equal to 0.6 the *splintered phase wave* starts to grow with increasing Δt and reaches a convergence value equal to 380, Fig. 4.1c.

When the probability value reaches a very small value of 0.2, it can be seen that the convergence time of the *static sync*, *static async*, *static phase wave* have not changed, compared with the previous simulations despite the small probability value. What is important here is that the *splintered phase wave* increases even at a greater pace after $\Delta t = 0.3$ and stops to converge at step size from 0.7 to 1. Also, the *active phase wave* stops to converge completely at a step size greater than 0.3, see Fig. 4.1d.

Cases were considered when we took the step size value from 0.1 to 1. What will happen

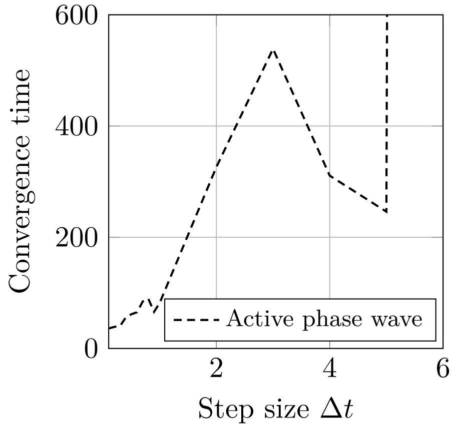


Figure 4.2: Convergence time with step size Δt for the *active phase wave*. Coupling probability $p = 1$.

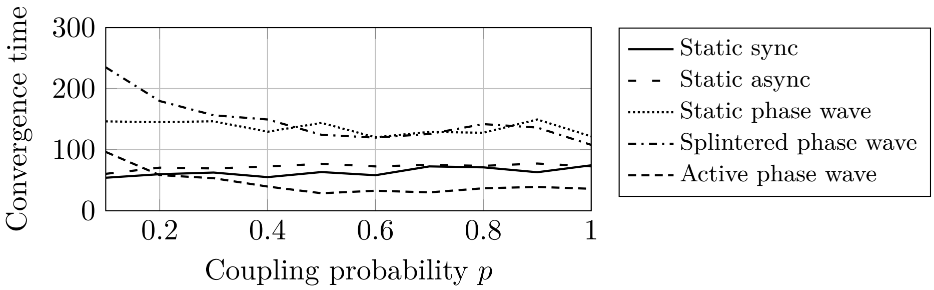
to the states if we increase the step size?

One of the simulations turned out to be the most interesting, which is because, with a probability value of 1, the following results were obtained, presented in Fig. 4.2. As you can notice previously, all states stop converging at step size value greater than 1, at any probability. However, at $p = 1$, the active phase continues to converge. Moreover, the active phase wave converges until the step size exceeds the value equal to 6 in which the CT is around 31000.

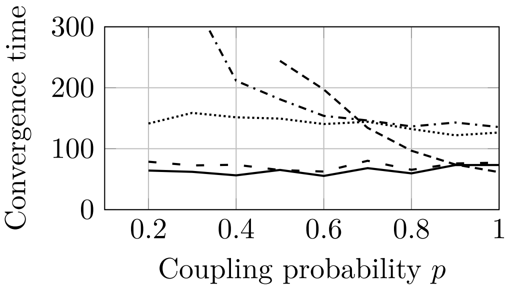
4.2 Varying the coupling probability

Earlier we looked at how the value of the convergence depends on the step size Δt. We use the previous results and build the dependence of convergence time on the coupling probability when Δt has a constant value. Referring to the previous calculations, we study the value of coupling probability p from 0.1 to 1 and take the constant value of Δt from 0.1 to 1.

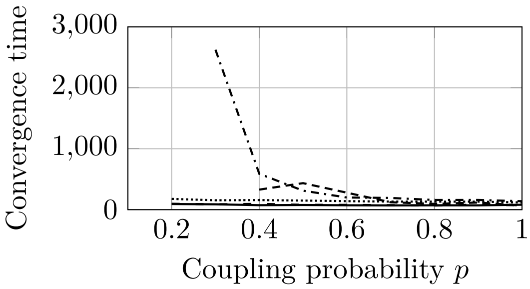
Thus, based on the values obtained, we build a plot for a visual understanding of the results obtained. When $\Delta t = 0.1$, Fig. 4.3a, all states from 0.1 to 1 converge. It is important to notice that the value of CTs of static states are not changing much on the whole interval. The CT is around 150 for the *static phase wave* and around 80 for the other two static states. If we take our attention on the *splintered* and *active phase waves*, they decrease from 0.1 to 0.6 and after that, the states become flat on the remaining interval of probabilities. This decline in the convergence time can be connected with unstable states, which means that the state needs more time to break up into clusters and adjust the phases. We can assume that *splintered phase wave* needs more time because of making gaps, which is particularly slower when the probability is low.



(a) $\Delta t = 0.1$



(b) $\Delta t = 0.5$



(c) $\Delta t = 0.9$

Figure 4.3: Convergence time of the five states when varying the coupling probability p. Parameters: 100 swarmalators.

Subsequently, we graphically present the results obtained for all five states when the Δt will be increased to the value of 0.5, Fig. 4.3b. Analyzing the result obtained, we can notice that with the given step size value, all static states stop converging with a coupling probability from 0.1 to 0.2. In addition, the *splintered phase wave* does not converge at a single point for a coupling probability equal to 0.1. However, when the probability value is 0.2 and CT = 895, the *splintered phase wave* begins to decrease exponentially to the value of convergence time 135 when p=1. It is also worth paying attention to the *active phase wave*, which converges only on the interval of coupling probability between 0.5 and 1. Thus, we can assume that the step size $\Delta t = 0.5$ is the threshold value for all static states and for the splintered phase wave when the coupling probability is small.

Having done several simulations, increasing the value of the step size, it is worth focusing on the case when $\Delta t = 0.9$, Fig. 4.3c. The *splintered phase wave* already stops to converge at the $\Delta t = 0.3$. If we compare with the previous simulation, it is important to note that the *active phase wave* again begins to converge at the step size equal to 0.4.

4.3 Analysis and discussions

In this part, we begin with careful consideration of one simulation in more detail and repeat it 10 times. This number of simulations is optimal in time and they help us

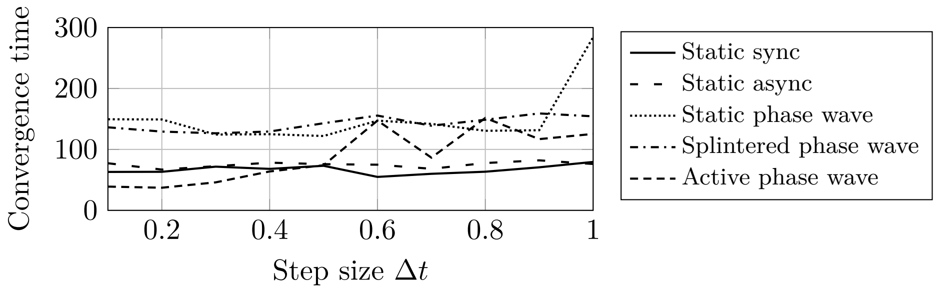


Figure 4.4: Convergence time with step size Δt from 0.1 to 1 for *static sync, static async, static phase wave, splintered phase wave, active phase wave.* Coupling probability $p = 0.9$.

to process and analyze the results for each state. Therefore, our responsibility is to understand how, with the same selected parameters, the convergence time changes. For example, we take a few simulations with step size Δt from 0.1 to 4 and constant coupling probability $p = 0.9$.

Fig. 4.4 shows that all states stop converging when the step size becomes more than one, so further we will consider the step size interval from 0.1 to 1 and see at what values the states stop converging.

We make 10 simulations and summarize everything in tables. For each Δt, starting from 0.1 to 1. In addition, for each state and Δt, we will calculate the mean value from the simulations.

Firstly, we take a look at the convergence time with a coupling probability p = 0.9 and an Euler step size $\Delta t = 0.1$ (see Table 4.1). We can see that for all five states the system converges into stable patterns. The convergence times are between 70 and 80 for static sync and async, around 30 for active phase wave, and about 135 for the phase wave states.

The differences could be due to the various complexity of the patterns that require different durations to converge. For example, in the splintered phase wave state it takes a while until the clusters separate from each other and form stable distinct shapes.

In contrast, the patterns for the static sync and async cases are rather regular and simple and converge quicker. The reason that the active phase wave converges that fast is not clear, but could be related to the particular metric that determines whether or not a pattern is already emerged / convergence has already occurred. This sheds light on a particular weakness of comparing convergence times of different states: Due to their different nature, no common metric of convergence can be defined, but different states have their own, individual metric (except that the three static cases use a common metric). Thus, convergence times also depend on the particular metrics involved which can affect the validity of the comparison.

Further simulations have to verify the following aspects:

Table 4.1: Convergence time at $\Delta t = 0.1$ and at coupling probability $p = 0.9$ for *static sync, static async, static phase wave, splintered phase wave and active phase wave.*

Run number	Static sync	Static async	Static phase wave	Splintered phase wave	Active phase wave
1	63.07	77.31	149.47	136.26	38.99
2	66.61	84.32	147.53	122.95	33.35
3	69.03	75.59	134.31	166.37	42.03
4	70.36	82.1	118.88	128.28	36.56
5	66.38	3.73	118.32	146.34	32.09
6	77.97	67.21	136.93	117.8	30.38
7	67.11	76.07	133.88	115.42	30.28
8	76.61	71.46	143.54	120.22	37.46
9	78.74	84.85	124.61	166.93	36.72
10	67.23	76.73	143.05	124.87	29.78
Mean	70.31	76.94	135.05	134.54	31.11

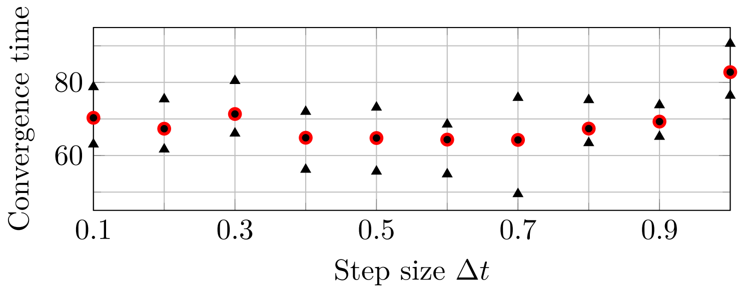
- For which values of Δt do we see convergence?
- Is the convergence time different for these other values?
- If yes, what are the largest Δt for which the system converges?
- Which values for Δt are optimal in terms of convergence time?

In the following, we will explain the results of the subsequent simulations: *static sync, static async, static phase wave, splintered phase wave, active phase wave.*

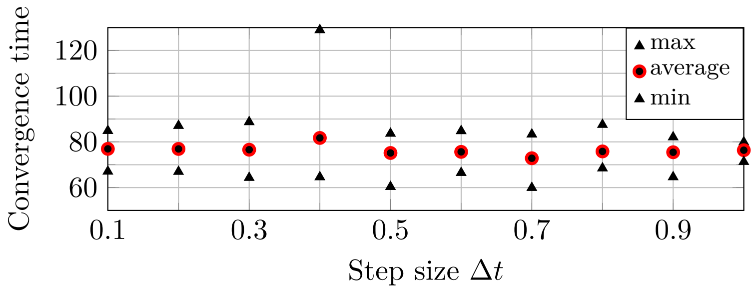
Based on the results obtained, we have constructed graphs of the dependence of the convergence time on the step size. We denoted the maximum, minimum, and average convergence time for all simulations, Fig. 4.5 for static sync and async, Fig. 4.6 for static, splintered phase wave and active phase wave.

The convergence time of the static sync state is approximately 70, Fig. 4.5a. It experiences a small change in convergence only at step size equal to 1, where it exceeds the convergence time equal to 82.

As for the static async state, see Fig. 4.5b, it is almost constant throughout the interval and also fluctuates around 75, which is similar to the static sync. The exception is when the step size value is equal to 0.4. Interestingly, out of all 10 experiments, one value was



(a) *Static sync*



(b) *Static async*

Figure 4.5: Average, max and min value of the convergence time for two static states with the step size Δt from 0.1 to 1 and coupling probability $p = 0.9$. 10 runs for each Δt.

formed that differs from all the others. In this Δt, the average value of CT is 80, and in the third experiment we got a CT greater than 120.

Thus, the largest value of step size for the static sync and async states is 1 at which the states converge. It seems that the values of CT do not differ much from each other over the entire interval from 0.1 to 1. Hence, any Δt value with a given probability $p = 0.9$ will be an optimal value.

Next, it is worth paying attention to the static phase wave state, since we see in Fig. 4.6a that the average value for the entire interval is approximately 130. However, the state ceases to converge in the third experiment at a step size equal to 0.9, that is, we received only 9 experiments for analysis, see Tab. 4.2. Further, when the step size value is 1, then in the second experiment and the tenth, the state does not converge, Tab. 4.3. In addition, we have obtained the convergence time values around 4800 and 2100.

That is, already at the value of the step size $\Delta t = 0.9$ and $\Delta t = 1$, in some experiments the static phase wave already stops to converge. The largest value for the step size is 0.8 in which the state is converge with all previously performed experiments. Speaking about the optimal values of the step size for a given probability, the best values will be from 0.1 to 0.8.

We also pay attention to the splintered phase wave, in this case, over the step size interval

Table 4.2: Convergence time at $\Delta t = 0.9$ and at coupling probability $p = 0.9$ for *static sync, static async, static phase wave, splintered phase wave and active phase wave.*

Run number	Static sync	Static async	Static phase wave	Splintered phase wave	Active phase wave
1	70.65	82.08	131.49	159.03	116.91
2	70.11	70.65	115.65	181.35	115.92
3	66.33	86.94	NA	155.43	121.32
4	65.52	76.5	168.21	175.95	161.46
5	73.8	64.62	137.97	154.44	126.27
6	73.44	76.23	124.29	151.47	237.78
7	70.29	77.22	134.01	164.97	134.64
8	69.48	69.84	150.3	179.19	212.22
9	68.04	78.12	150.75	172.71	113.04
10	65.16	72.36	115.2	157.23	146.52
Mean	69.28	75.46	136.43	165.18	148.61

from 0.1 to 1, the state approximately grows from 130 to 170 and increases the CT value by about 5-10 each step, Fig. 4.6b. The state converges on the interval of step size from 0.1 to 1 in each experiment. When the value of Δt becomes larger than 1, the splintered phase wave stops to converge as in previous states.

The behavior of the active phase wave is the most interesting as we see from the graph in Fig. 4.6c, the smallest convergence value is about 30 at step 0.1, from higher step sizes the graph begins to grow exponentially, and a stronger growth can be noticed at step size 0.5, which increases to about 75. A very curious fact is that with a step size value of 0.7, in experiment 7, see Tab. 4.4, the state does not converge.

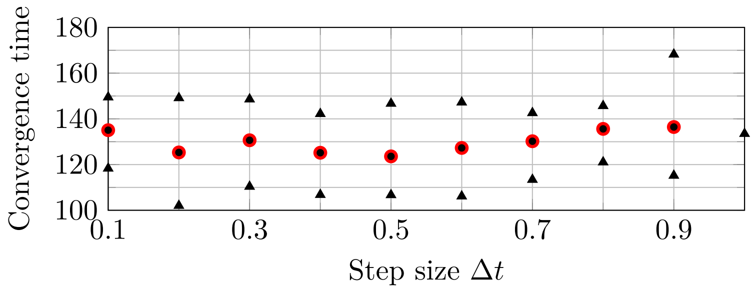
After that, with an increase in the step size, the active phase wave begins to converge again and the gap between the values of the CT for each experiment also increases. It seems quite interesting that the active phase wave does not converge in the intermediate step value between 0.1 and 1. Thus, the optimal values of the step size are from 0.1 to 1, except for 0.7. But the best case is if we take values where the spread of the CT is smaller, from 0.1 to 0.5, so we approximately know the value of convergence.

To sum up, from the experiments done, the optimal value of the step size for all states is the values from 0.1 to 0.6, because with larger step size values, as we have already noticed, some states begin to diverge, such as the static phase wave, with large step values and active phase wave at the intermediate point. It also makes no sense to take a

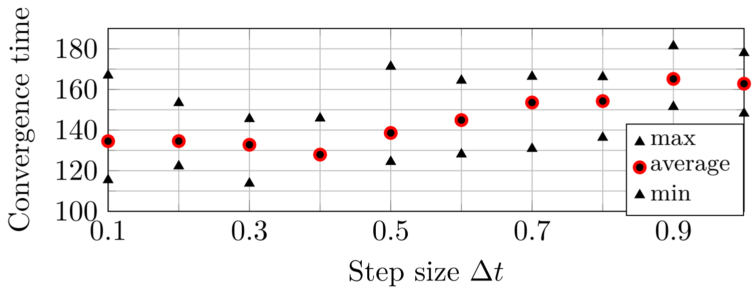
Table 4.3: Convergence time at $\Delta t = 1$ and at coupling probability $p = 0.9$ for *static sync, static async, static phase wave, splintered phase wave and active phase wave.*

Run number	Static sync	Static async	Static phase wave	Splintered phase wave	Active phase wave
1	79.5	76.1	284.2	154.3	125.4
2	80.1	73.5	NA	169.6	98.9
3	76.4	71.3	4845.3	161.8	121.7
4	85.2	74	534	178	87.9
5	89.5	83.4	133.5	156.7	227.5
6	81	77.8	865.6	164.8	161.8
7	90.6	73.6	2108.2	161.3	146
8	81.4	77.1	182.9	148.2	133.6
9	81.3	79.9	155.3	156.2	135.4
10	83	76.9	NA	177.3	121.3
Mean	82.8	76.36	1138.63	162.82	135.95

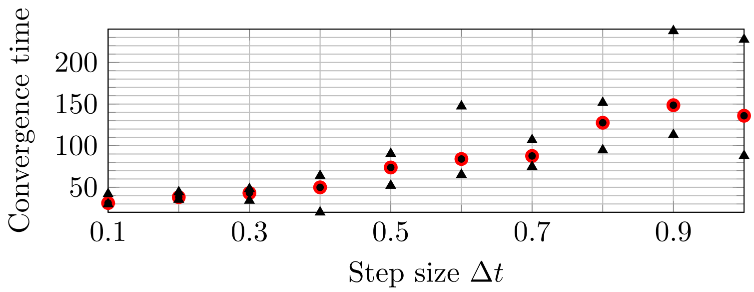
step size greater than 1, since none of the states converge at the given probability.



(a) *Static phase wave*



(b) *Splintered phase wave*



(c) *Active phase wave*

Figure 4.6: Average, max and min value of the convergence time for three phase wave states with the step size Δt from 0.1 to 1 and coupling probability $p = 0.9$. 10 runs for each Δt.

Table 4.4: Convergence time at $\Delta t = 0.7$ and at coupling probability $p = 0.9$ for *static sync, static async, static phase wave, splintered phase wave and active phase wave.*

Run number	Static sync	Static async	Static phase wave	Splintered phase wave	Active phase wave
1	59.99	68.25	141.89	138.88	86.52
2	59.64	73.43	139.72	164.29	71.68
3	67.2	75.53	130.69	154	102.97
4	66.64	72.8	125.51	130.83	82.95
5	61.67	73.99	113.47	159.6	106.82
6	75.81	59.92	120.12	150.99	85.19
7	69.44	83.3	127.05	146.72	NA
8	49.49	76.09	134.12	176.12	93.38
9	63.21	73.57	126.98	147.98	74.69
10	69.58	71.54	142.59	166.32	84.7
Mean	64.27	72.84	130.21	153.57	87.66

CHAPTER 5

Impact of a time-variant step size

In the section, we consider simulations for which we increase and decrease the step size by a small amount in each step, which we call τ. For instance, τ is equal to 0.00002, 0.0002 and 0.002 for all five states: *static sync, static async, static phase wave, splintered phase wave and active phase wave.*

For these simulations, we need to add to line 15 of the code "sim_v2.r" that we subtract or add an increment from the given step size. The example code specifies the increment subtraction, see Listing 3. It is mandatory that this value be added to the cycle because, in this case, we change the step size by a small value, increment τ. Thus, the step size iterates over steps.

Thus, further simulations have to verify the following aspects:

- How does increasing the step size affect convergence time?
- How does a decrease for the same value of τ affect convergence time?
- For which values of $\Delta t + \tau$ and Δt - τ do we see convergence?
- What is the largest step size for which the states converge?
- What is the optimal τ for each state?

5.1 Small increment

On the one hand, we consider the simulation when the step size is decreased by τ=0.00002 and the coupling probability is equal to 1, Fig. 5.1a. As we can notice for all five states the system converges to stable patterns from 0.1 to 0.9. The convergence times are between 70 and 80 for static sync and async. These states stop converging when the step

```
sim <- function()
# main simulation function
{
        # setup model parameters
        # A = B = 1 is Strogatz model
        A <- 1;
        B <- 1;

        J <- J_LIST[SCENARIO];
        K <- K_LIST[SCENARIO];
        ...
        # iterate over steps
        for( ti in 1:NUM_STEPS )
        {
          DT <- DT - 0.0002; # add increment
        ...
        }
}
```

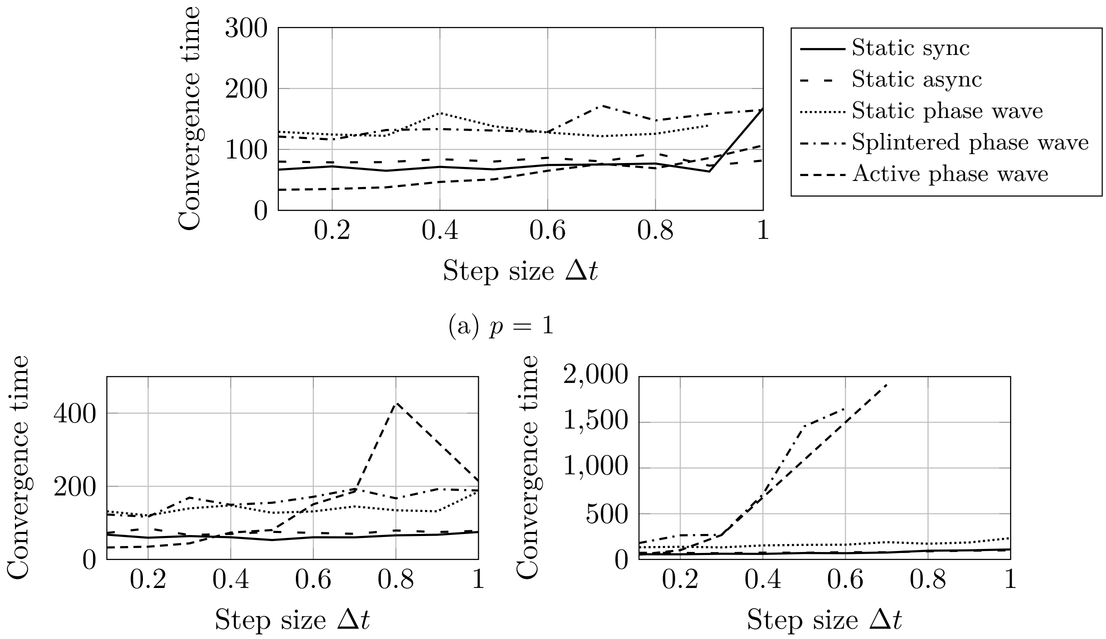
Listing 3: Struct interface of "sim_v2.r" with the step size increase using the increment τ.

size is about 1. For static and splintered phase waves the CT is from 120 to 160. It is important to see that the static phase wave stops converging when the step size is larger than 0.9. The most curious result is the active phase wave, see Fig. 5.3b, the CT of the state is around 25 on the interval from 0.1 to 1. Compared with other states, it does not stop at the step size equal to 1, that is, it continues to converge till the step size is about 6. The convergence time reaches a maximum value of 30000 in this step size.

Additionally, it is important to note that we begin to decrease the probability value each time and with a value of p = 0.7, all states stop converging with a step size greater than 1, Fig. 5.1b. As for the active phase, we can notice a peak at a step size equal to 0.8. We can assume that this is critical for this state, because each time the probability decreases, the active phase wave takes fewer and fewer values at which it can converge. Reducing the probability strongly affects the convergence of the active phase wave.

As soon as the probability decreases to 0.2, Fig 5.1c, the coupling probability of signal transmission becomes very small, then the active phase wave converges on the interval from 0.1 to 0.7 and after the state does not converge at all. Moreover, with this probability, the splintered phase wave is already beginning to diverge at a step size from 0.6.

Accordingly, step reduction by increment effects the splintered and active phase waves, while the convergence time of static states varies slightly. Regardless of the coupling probability for all five states, a small step value will be the most optimal option, up to 0.5.



(a) $p = 1$

(b) $p = 0.7$

(c) $p = 0.2$

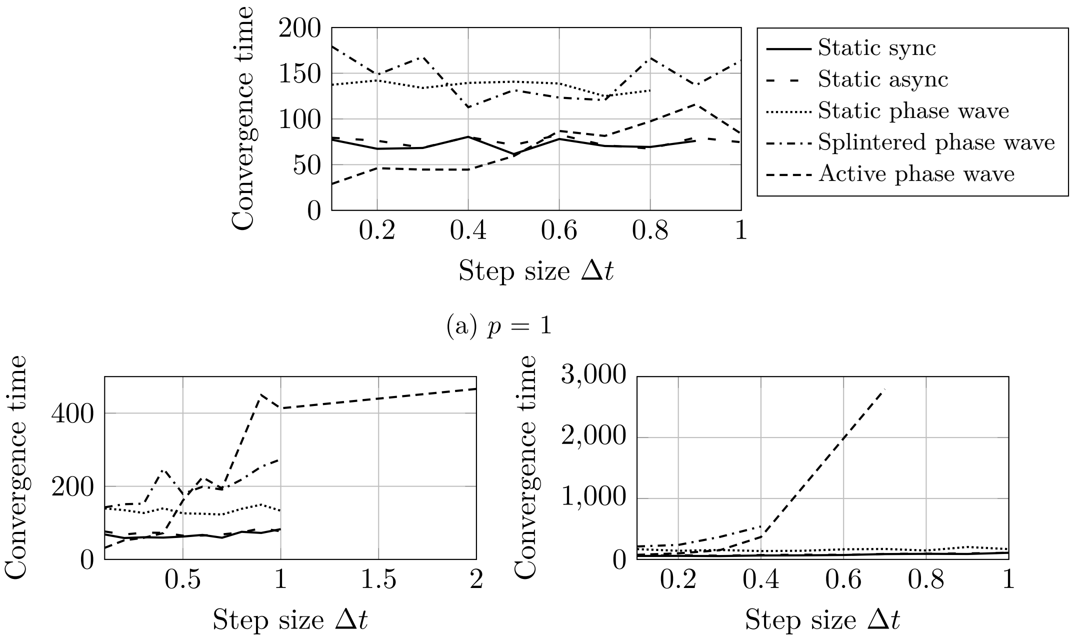
Figure 5.1: Convergence time of five states when the time-varying step size. Decrease the step size for τ= 0.00002. Parameters: 100 swarmalators.

On the other hand, we consider the simulation when the step size is increased by τ=0.00002. With the coupling probability equal to 1, see Fig. 5.2a, the convergence times are similar for the simulation when we decrease the step size for the same value, Fig. 5.1a.

However, in this case, the static phase wave stops to converge at step size around 0.8. The state ends up converging at a smaller step value. Also, the static sync stops converging earlier, when the step size is around 0.9. This also suggests that the static state converges at a smaller step size. Looking at the active phase wave in Fig. 5.3a, it again shows the longest convergence result, that is, it converges to about 5 over the entire range of values. As we can see from the figure, after unity there is a sharp increase in convergence time, but not exceeding 500.

Also consider Fig. 5.2b, the case when the coupling probability is equal to 0.6. We see that the active phase wave already ceases to converge at the step size around 2. Compared with the previous result, in this simulation, even with a smaller probability, the state converges with a larger step value. However, such a value of probability is better for static sync and phase wave because as the coupling probability decreases, the states increase the values of the step size at which convergence is maintained.

As soon as the coupling probability has decreased to 0.2, Fig. 5.2c, the splintered phase wave begins to diverge at a step approximately equal to 0.4. In addition, the active



(a) $p = 1$

(b) $p = 0.6$

(c) $p = 0.2$

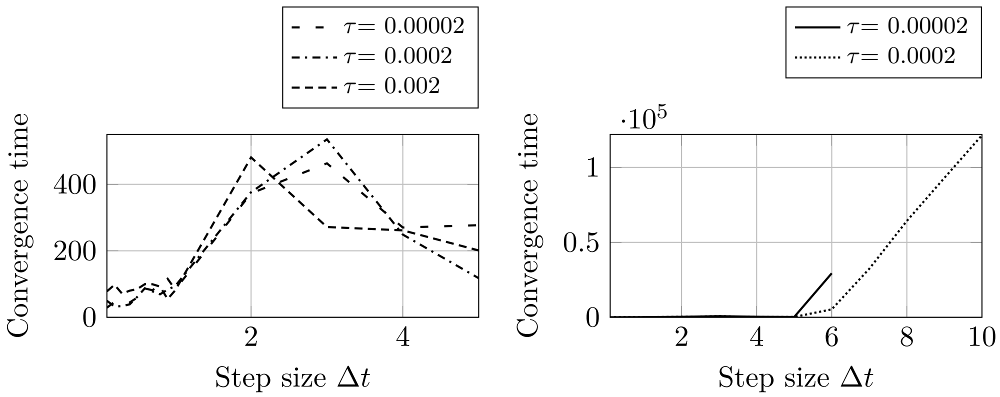
Figure 5.2: Convergence time of five states when the time-varying step size. Increase the step size for τ= 0.00002. Parameters: 100 swarmalators.

phase wave stops to converge at the step size of around 0.7 and the value of the CT is about 2800 at the last point.

Thus, the convergence times of two different simulations are approximately the same for both cases, when we increase and decrease the step size. On the contrary, the best convergence results were obtained by decreasing the step size by τ= 0.00002. Firstly, static sync and async states converge with larger step size with the coupling probability equal to 1. Secondly, the splintered phase wave keeps the convergence longer with the same coupling probability.

5.2 Medium increment

Firstly, we consider the simulation when the step size decreased by τ= 0.0002. Fig. 5.4a shows the step size interval from 0.1 to 10 when the coupling probability is equal to 1. It can be noted that in the entire interval, except for step 0.1, the static sync and async converge and the convergence time lies about 60, for the static and splintered phase waves the CTs are around 130 and for the active phase wave the Cts are about 65. But with a step greater than 1, the CTs of all states grow rapidly. It reaches about 45000 for the static sync and 55000 for the async. Two phase waves get the highest value of the CT around 50000 in step 10. Looking at the active phase wave, we can notice that the



(a) Increase step size

(b) Decrease step size

Figure 5.3: Convergence time of the active phase wave when the time-varying step size. Decrease and increase the step size for τ. Parameters: 100 swarmalators. Coupling probability $p = 1$.

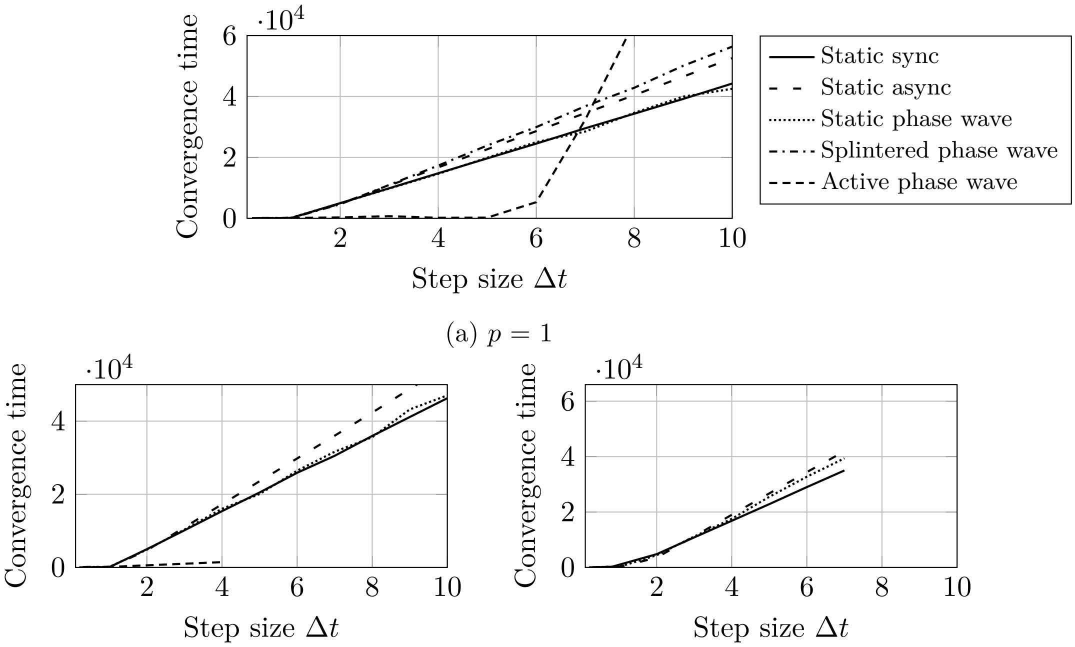
largest value of the CT gets at step size around 10 and is equal to 120000. As the step increases more than 10, none of the models converge.

As we start to decrease the value of the coupling probability, the splintered and active phase waves stop to converge at a step size around 1 and another one around 4, Fig. 5.4b.

At the end, the coupling probability is equal to 0.1, Fig. 5.4c. We can notice that the decrease in probability did not affect the static phases in any way. But the last two phase waves stopped converging on the entire interval, even with a small step size.

If we compare with the previous simulation with a high coupling probability equal to 1, Fig. 5.1a, then obviously huge changes can be seen ib this case because the value of the step size at which all states converge has significantly increased. Additionally, as soon as the step value was reduced, the convergence of the three static states increased. In the previous simulation, the convergence of the states stopped at about 1. And in this case, it already reaches a step value of 10. Talking about the splintered phase wave, it can be seen that with the same significant growth, at step size about 2, the state finished converging. In this case, it again reaches the value of step 10. Paying attention to the active phase wave, it converged at step size around 6, and again the value doubled. However, with a small step size of about 0.1, this step change is poorly reflected.

Secondly, we consider the simulation when the step size increased by τ= 0.0002 and the case when the coupling probability is equal to 1, Fig. 5.5a. The convergence times for static sync and async are between 60 and 120, for static and splintered phase waves are from 150 and 200. For the active phase wave, the CT is between 30 and 80. As we can see, the static sync stops to converge at step around 0.9, and the static async and the splintered phase wave stop to converge with Δt around 1. The static phase wave diverges at step 0.7. The active phase wave converges when the value of step size from 0.1 and



(a) $p = 1$

(b) $p = 0.9$

(c) $p = 0.1$

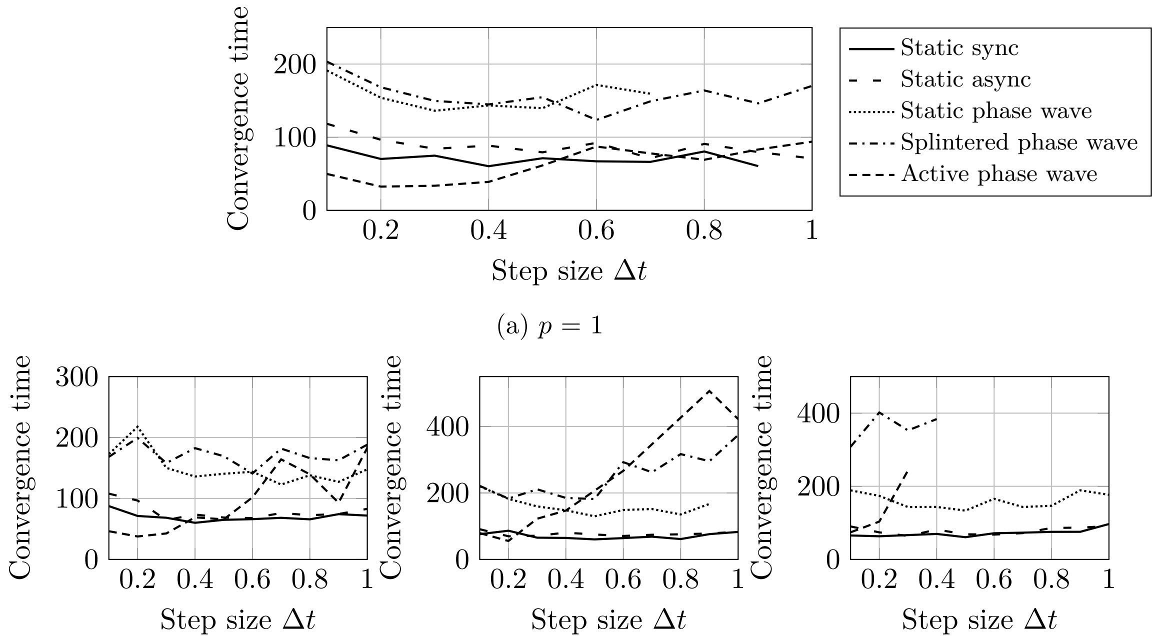
Figure 5.4: Convergence time of five states when the time-varying step size. Increase the step size for τ= 0.0002. Parameters: 100 swarmalators.

exceeds the value of 1. The convergence continues up to 5 and the convergence time is about 120, we see the behavior in Fig. 5.3a.

However, when we decrease the coupling probability to 0.8, we can notice in Fig. 5.5b that the static sync and phase wave continue to converge with a larger step size, around 1 each. But the active phase wave with a lower coupling probability already ceases to converge at a step size around 3. Compared to the previous result, the step size at which the state converges has significantly decreased from 5 to 2.

As the probability decreases with each step, there is also a decrease in the Δt at which there will be convergence, it is important to note the probability equal to 0.5, we can see that the active phase wave maximally reduces the size of the step at which the state converge, the state stops to converge around 1. That is, already at a step size of more than 0.9 the static phase wave no longer converges, Fig. 5.5c.

Next, we pay attention to a smaller coupling probability equal to 0.3, Fig. 5.5d. In this case, all states cease to converge at step size from 0.1 to 0.3, important to see because, in previous experiments, the decrease in probability did not affect the small values of step. As for the splintered and active phase waves, they cease to converge over the interval from 0.1 to 0.3 and from 0.1 to 0.4, approximately, because of the probability, which is too small for counter-rotating subgroups.



(a) $p = 1$

(b) $p = 0.8$ (c) $p = 0.5$ (d) $p = 0.3$

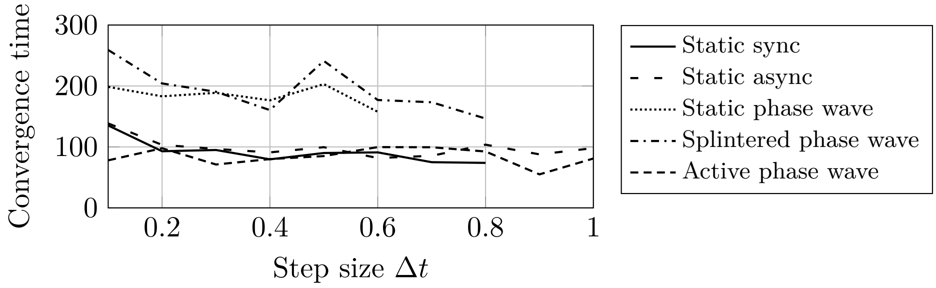
Figure 5.5: Convergence time of five states when the time-varying step size. Increase the step size for τ= 0.0002. Parameters: 100 swarmalators.

Therefore, with increasing the increment, the optimal probability value will be an intermediate value that will be less than 0.9, since we noticed the effect of convergence on static states at the maximum probability value. So as the probability decreases, static models change their behavior in the best way.

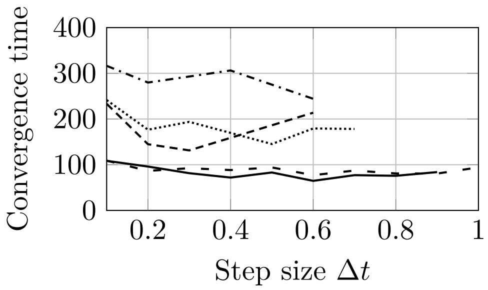
5.3 Large increment

This time we decrease and increased the step size by a slightly larger amount equal to 0.002.

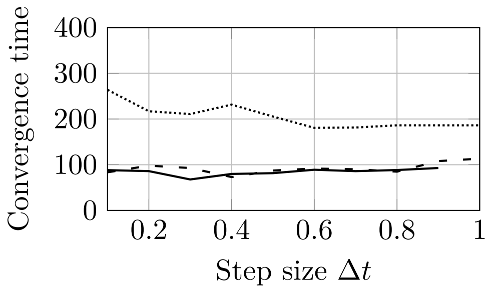
When we start reducing the step size, none of the states converge at any value. However, when decreasing the step size by τ= 0.002 and with coupling probability equal to 1, see Fig. 5.6a, we got that the CT of static sync, static async and active phase wave are between 75 and 140. The CTs of the static and splintered phase waves are between 150 and 200. Take a look at the static sync and static phase wave which stop to converge at the step size of about 0.8 with a high probability value. As for the static phase wave and async, one ceases to converge with the step size around 0.6 and another one around 1. In Fig. 5.3a we see the active phase wave, which converges till the step size is equal to 5. The convergence time of the state increase and has a peak of convergence around 500 when $\Delta t = 2$.



(a) $p = 1$



(b) $p = 0.7$



(c) $p = 0.3$

Figure 5.6: Convergence time of five states when the time-varying step size. Decrease the step size for τ= 0.002. Parameters: 100 swarmalators.

With each time decreasing the coupling probability, it should be noted the case when p=0.7, Fig. 5.6b. To compare with previous simulations with higher probability, each of the states reduces the step at which it will converge. Eventually, the splintered and active phase waves stop to converge at 0.6 and the static async at 0.9. On the contrary, the static sync and phase waves increase the value of the step at which convergence occurs. The increase is approximately 0.1 for both of them.

As soon as the coupling probability has decreased to 0.3, Fig. 5.6c, the static phase wave converges with a larger step size and a lower probability. As we can notice the state still converge when the step size is equal to 1. In contrast, with decreasing the coupling probability to 0.3 the splintered and active phase waves do not converge at all over the entire interval.

Unambiguously, the most optimal coupling probability is the maximum value equal to 1 for all states, because maximum convergence is noted at the maximum step size. However, if we consider static sync and async separately, then convergence for them is optimal with a decrease in the probability value, convergence is observed with a larger step size.

5.4 Analysis of the coupling probability

5.4.1 Increased step size

Based on previous simulations, by changing the step size with a varying small τ, it is worth analyzing how the states behave when the coupling probability changes from 0.1 to 1. In this case, we will take a constant step size Δt equal to 0.1 and will increase to the value τ.

At first, τ is equal to 0.00002. Then we gradually increase and the next simulation considers τ= 0.0002. In the end, we look at the τ equal to 0.002.

The graphs in Fig. 5.7 show the dependence of the convergence time on the coupling probability for all five states with step size $\Delta t + \tau$ with $\Delta t = 0.1$. The τ= 0.00002, Fig. 5.7a. The convergence time of static sync and async are around 60 in the whole interval of probability, the change in probability, as we can see, does not affect these two states in any way. As for the active phase wave when the coupling probability is from 0.1 to 0.3 the value of the CT is larger than the CT of the two previous states and it is decreasing from 100 to 60. As soon as the step size is greater than 0.3, then the CT of the state is reduced to 40 for the entire remaining interval. The static phase wave has a constant value of CT and which is around 190. The fifth state, the splintered phase wave, is similar in its behavior to the static phase wave on the interval when the coupling probability is larger than 0.2. While with a low probability, we can notice a decrease in the function from 410 to 200.

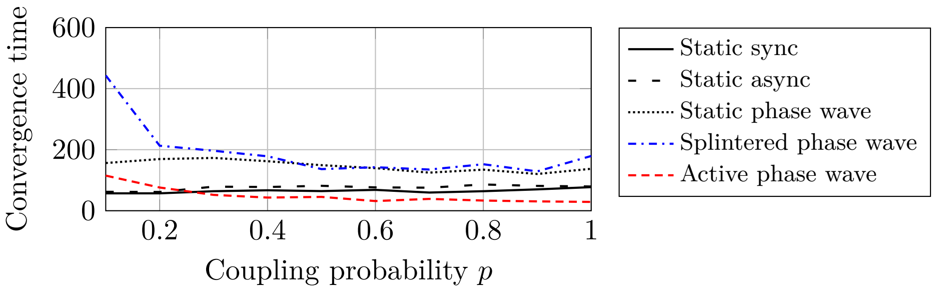
Next, we take a look at Fig. 5.7b where τ equal to 0.0002. When we increase the step size and we can see that the active phase wave has a larger increase of the convergence time equal to 200 with the same probability as previously. A rather strong change can be seen in the behavior of the splintered phase wave, which at a coupling probability of 0.4 begins to increase to the CT around 500. The state ceases to converge at a value of p less than 0.2.

Finally, when we increase the step size to 0.002 we can see that the active phase wave at p = 0.9 starts to be greater than static sync and async states, at p = 0.6 exceeds the CT of the static phase wave and stops to converge at p = 0.4 with CT around 1000, Fig. 5.7c. As for the splintered phase wave, it greatly reduces the convergence interval and ceases to converge at 0.5.

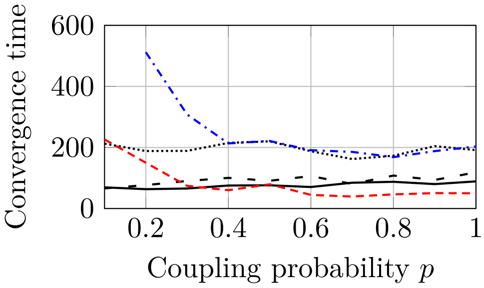
Thus, as soon as we increase the step size value by an insignificant random value τ, this does not significantly affect the static states, but at the phase waves we can notice that they sharply increase their values and on the next point of the coupling probability stops converging.

5.4.2 Decreased step size

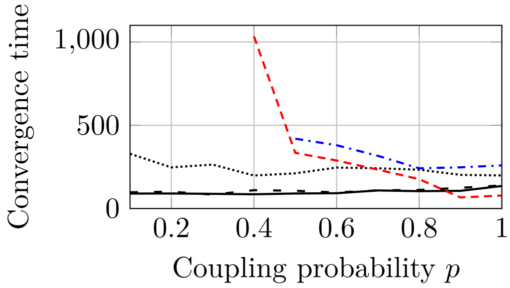
In this part, we consider the most interesting case when, with a step size Δt decrease of τ= 0.0002, Fig. 5.8. our models continued to converge at a step size equal to 10. Thus,



(a) τ= 0.00002



(b) τ= 0.0002



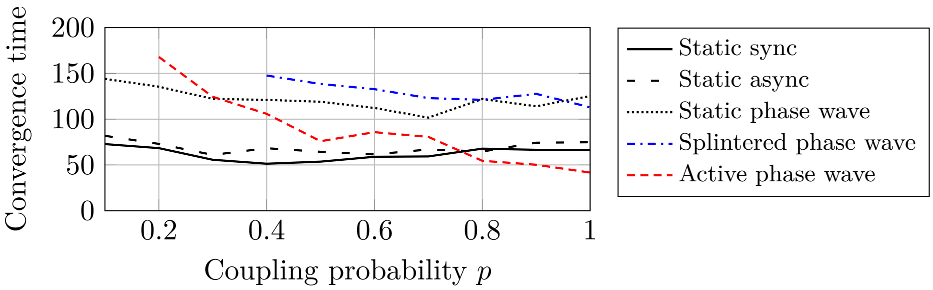
(c) τ= 0.002

Figure 5.7: Convergence time of the five states when varying the coupling probability p from 0.1 to 1. Increasing the step size by τ. Parameters: 100 swarmalators, $\Delta t = 0.1$.

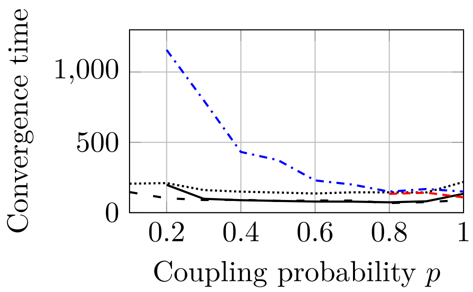
we analyze how the probability will change in the interval from 0.1 to 1 if we change the value of the step size Δt. Firstly, consider the case when $\Delta t = 0.4$. We take such a value because when the step size is smaller, no state converges. Secondly, we run the simulation when $\Delta t = 1$ and ultimately we consider a higher value of step size which is equal to 4.

We take a look at Fig. 5.8a that shows the dependence of the convergence time on the coupling probability with a constant step size $\Delta t = 0.4$ and τ= 0.0002. The convergence times are between 60 and 80 for static sync and async, and between 100 and 150 for the static phase wave. It is important to notice the behavior of the splintered and active phase waves. The first CT is almost constant over the entire interval of about 140. However, it ceases to converge when the coupling probability is less than 0.4. On the second case CT slowly decreases when we increase the coupling probability, where the CT is from 50 to 160, and the CT = 160 is the critical value with p = 0.2 because the state stops to converge.

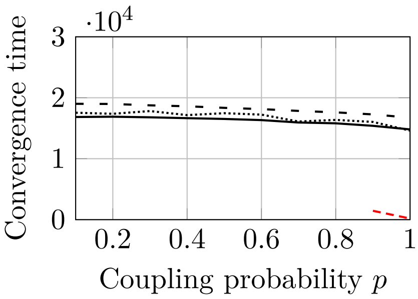
The next step is very interesting because here we increase the step size Δt which is equal to 1, and the τ is not changing, Fig. 5.8b. Note that, firstly, the state is already experiencing difficulties in convergence with a small probability. Also, the splintered phase wave, on the contrary, increased its convergence, with a lower probability, and continues to converge, up to p = 0.2. However, the convergence time has increased and



(a) $\Delta t = 0.4$



(b) $\Delta t = 1$



(c) $\Delta t = 4$

Figure 5.8: Convergence time of the five states when varying the coupling probability p from 0.1 to 1. Decreasing the step size Δt each simulation by the same τ= 0.0002. Parameters: 100 swarmalators.

reached 1000. As for the active phase wave, it converges only at high probability values of 0.8, 0.9 and 1.

When the step size Δt increased to 4, we can notice that static states converge on the whole interval with any value of coupling probability, Fig. 5.8c. The convergence time is between 140000 and 160000. As for phase waves, the splintered state is not converged for any probability values and the active state still converges at maximum probability values equal to 0.9 and 1.

To sum up, static states converge at any step size increase over the entire probability interval from 0.1 to 1. Only when the Δt increases more than 1, the convergence time values increase significantly. As for the splintered state, as soon as the Δt becomes greater than 1, the state ceases to converge at any step value. A fairly good result is obtained in the intermediate value for this state. The active phase continues to converge, but each time the probability interval at which it can converge decreases. For this state, a good result can be reached when the step size is small.

5.5 Analysis of increment

Based on the simulations, we construct the dependence of the convergence time on the increment. In this case, we take for comparison a different step size (Δt equal to 0.1, 0.3, 0.7 and 1) from which the increment is subtracted or added.

Fig. 5.9a shows the behavior of the static sync state. We can see that the state converges on the entire interval with a step size equal to 0.3 and 0.7. The convergence time lies from 60 to 90 for both steps. As we can notice the static sync with the step size $\Delta t = 0.1$ does not converge on the τ= -0.0002. With the remaining values of the step sizes, the convergence times are between 60 and 140. However, when increasing the step size to 1, we can say that the state converges, but only at the initial points. The state stopped to converge when τ is greater than 0.00002. We get that for this static state, with these τ values, it is most optimal to choose intermediate values of steps.

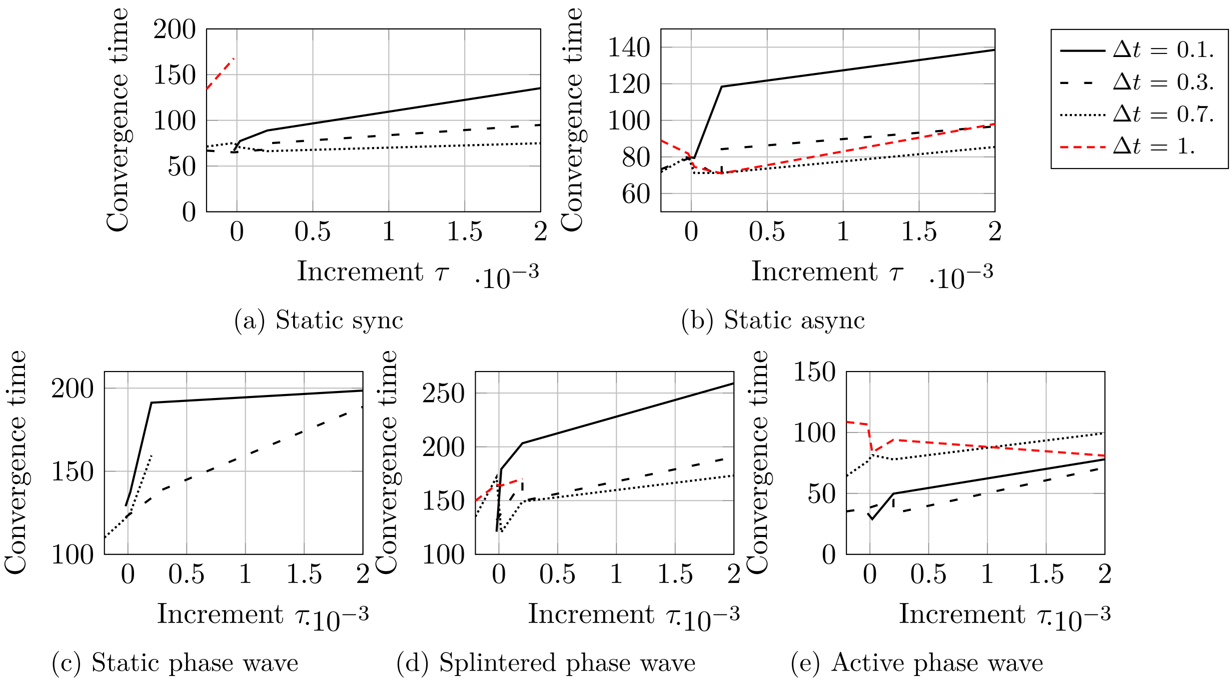
Next, we take a look at the static async state, Fig. 5.9b. The behavior of the state is similar with Δt equal to 0.3, 0.7 and 1, where the CTs are between 65 and 100. But as for the step size being equal to 0.1, the state does not converge with τ= -0.0002. The convergence time gradually increases over the entire interval from 80 to 140. In this case, the optimal values of the step size are larger than 0.1.

Additionally, we pay attention to the static phase wave, which is shown in Fig. 5.9c. With the τ values selected, the state does not converge at $\Delta t = 1$ at any point. When the step size is reduced to 0.7, the state converges when the τ is less than 0.002 with the CT from 110 to 160. As soon as we reduced the step size to 0.3, we notice a smoother behavior of the graph, compared to the step size equal to 0.1. For these two step sizes, the growing convergence times are from 120 to 190. In this case, it is worth paying attention to the choice of τ, it affects which step is better to choose for this state. For small τ, 0.7 is optimal, and for larger τ, smaller step size.

The splintered phase wave converges with τ greater than -0.00002 and the step size equal to 0.1 and 0.3, Fig. 5.9d. The CT is from 120 to 250 for $\Delta t = 0.1$ and from 120 to 190 for $\Delta t = 0.3$. It is interesting to mention that with $\Delta t = 0.7$ the state converges on the whole interval of $\tau$with the CT around 150. But with the increasing of the step size to 1, the state converges on the interval of τtill 0.002 with the CT around 160. We can see that the optimal value of the step is around 0.7.

Finally, the active phase wave has the best convergence with such τ values. At any chosen step size, the state converges over the entire interval. For the small values of steps the CTs are around 60 and for larger values around 90.

Thus, when choosing the best step size value for τ, it is worth paying attention to the step equal to 0.7, because with this value the optimal convergence over the entire interval for all five states and in addition to this, with a small value of the convergence times.



(a) Static sync (b) Static async

(c) Static phase wave (d) Splintered phase wave (e) Active phase wave

Figure 5.9: Convergence time of five states when the step size equal to 1, 0.7, 0.3 and 0.1. Parameters: 100 swarmalators, the coupling probability $p = 1$, τ is equal to -0.0002, -0.00002, 0.00002, 0.0002 and 0.002.

CHAPTER 6

Impact of a random step size

6.1 Randomization of the step sizes

In this section, we explore whether randomization of the step sizes can help the system to converge. For this reason, we use a random step size following a normal distribution. It is simple to use, manipulate algebraically and derive formulas. Thus, with the help of a normal distribution, it is possible to obtain results that can be easily applied.

Three arguments are used in the function $rnorm(n, mean = 0, SD = 1)$, which generates a vector of random values:

- n: "the number of random variables to generate";
- mean: "mean value, if not specified defaults to 0";
- SD: "the standard deviation, σ", if not specified, standard normal distribution equal to 1.

The step size depends on our initially selected value, the number of swarmalators, the number of steps and the standard deviation σ. We concentrate on the most important factor,which is σ. Thus, we vary the value of the standard deviation and see how it affect the convergence time of all five states.

We add the variable "DTv" to our code, which consists of the function "rnorm", see Listing 4. Further, we change the parameters depending on the input values.

A piece of code is presented for a simulation where the standard deviation is equal to 0.01, Listing 4. The mean value is equal to 0. It is also taken into account that the step size values can only be strictly positive, which is ensured in line 10. For further simulations with different values of standard deviation, the same structure of code is used.

```
...
DTv <- rep(DT,NODE_COUNT)
SD <- 0.01
...
# simulation main
sim <- function()
{
        ...
        DTv <- DT + rnorm( NODE_COUNT, 0, SD );
        DTv[DTv <= 0] <- 0.000001;  # values greater than 0
        ...
        # actually change position and phase
                coord <- coord + coord_dot*DTv;
                phase <- phase + phase_dot*DTv;
        ...
}
```

Listing 4: Struct interface of "sim_v2.r" with the step size randomized using the standard deviation σ.

The standard deviation shows us how distributed the data is. This is an indicator of how far each observed value is from the average. We will increase the value of the standard deviation in each simulation and choose the value at which the most favorable convergence time occurs. We use values of σ equal to 0, 0.001, 0.005, 0.01, 0.05 and 0.1.

In the following, we consider simulations, when we increase the standard deviation from 0 to 0.1. It is important to notice that the larger our standard deviation, the larger the spread. Thus, the values of step sizes are completely randomly chosen for all five states: *static sync, static async, static phase wave, splintered phase wave and active phase wave.*

Therefore, further simulations have to verify the following aspects:

- How does the increase of the standard deviation affect the convergence time?
- For which values of the step size do we see convergence?
- What is the largest step size for which the states converge?
- What is the optimal σ for each state?

6.1.1 Baseline scenario

Firstly, we consider the "baseline scenario". We have a standard deviation equal to zero. This shows that there is no variability in our data set at all. Since our data can only have one value, this value represents the mean of the sample. In this case, if all our data values are the same, there will be no difference.

We take a look at the convergence time with a coupling probability p = 0.9 and an Euler step size is equal to Δt + rnorm(100,0,σ), Fig. 6.1a. The convergence times of the static sync and async are around 80, from 120 to 140 for the static phase wave, between 120 and 170 for the splintered phase wave and from 40 to 160 for the active phase wave. We can notice that for all five states the system converges to stable patterns up the step size equal to 1. The exception is the phase waves, the static converges till 0.9 and the active continues to converge around 2.

We take attention to the next simulation when we decrease the coupling probability to 0.7, Fig. 6.1b Interesting to see that the splintered phase wave converges till the step size is equal to 1 with a lower probability. At the same time, the active phase wave ceases to converge at step size 1, when the probability decreases.

At the moment when the coupling probability decreases to 0.2, we can notice that the splintered and active phase waves reduce the value of the step size at which convergence is maintained. The first one stops to converge at a step size equal to 0.6 and another one at 0.7, Fig. 6.1c.

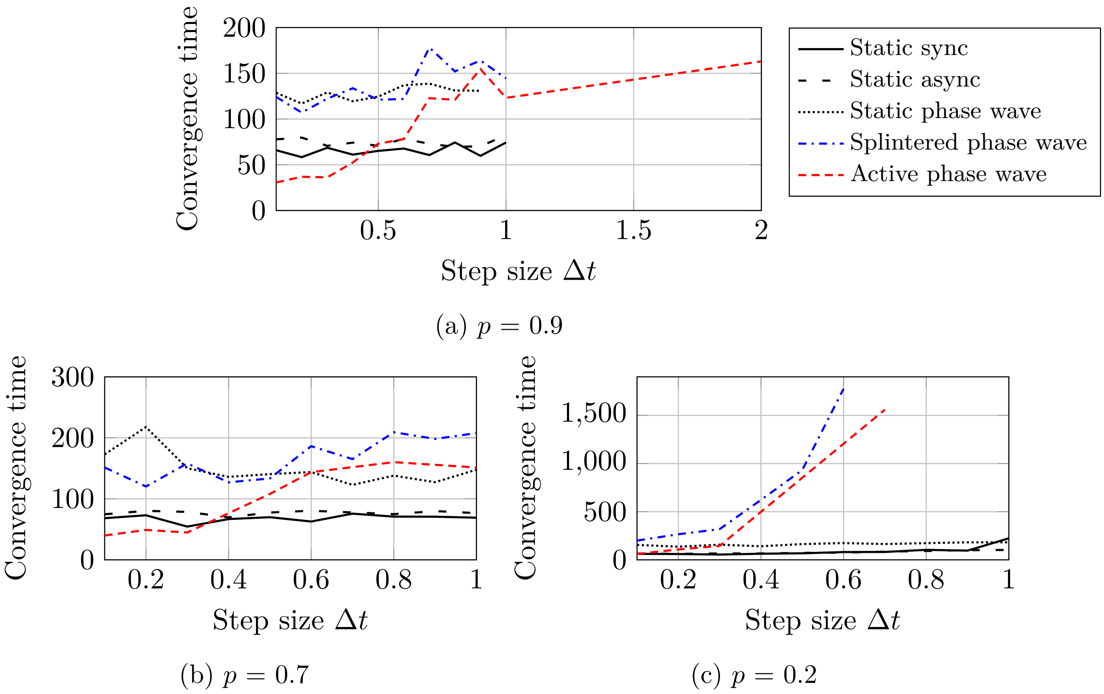
Hence, we can note that for the value when σ= 0, then the most favorable simulation is with a coupling probability of 0.9, since all states converge at the maximum step size at this time.

6.1.2 Increasing the spread of the step sizes

Next, we consider a standard deviation equal to 0.005. We increase the observed step size from the average value. The increase in standard deviation means that the step sizes are clustered around the mean. We take a look at the convergence time with a maximum coupling probability p = 1 and an Euler step size is equal to Δt + rnorm(100,0,σ), Fig. 6.2a. The CTs of the static sync and async are around 90 and between 120 to 160 for the static and splintered phase waves. As for the active phase wave, the CT of the state increases from 20 to 90 when the step size is from 0.1 to 0.5. However, as the step grows the CT repeats the behavior of the static sync and async. The longest convergence has the active state which is till 6, Fig. 6.2b. The static phase wave stops to converge earlier than others at step size 0.8, the static sync at 0.9, and the static async and splintered at 1.

As the probability decreases, we can notice that the static sync and the splintered phase wave converge with a lower probability and a larger step size, both stop to converge at a step equal to 1. On the contrary, the splintered and active phase waves converge only with a smaller step size and stop to converge at 0.9 and another state at 1, Fig. 6.2c.

To compare with the coupling probability equal to 0.5, the CT of the splintered phase wave again converge with a higher step size than previously. The state stops to converge at step size around 1. It is important to look at the active phase wave which is significantly



(a) $p = 0.9$

(b) $p = 0.7$

(c) $p = 0.2$

Figure 6.1: Convergence time of five states when we consider a "base line scenario" with no randomization of the step size. Parameters: 100 swarmalators, the standard deviation equal to zero.

changing. The active state does not converge with the step size larger than 0.4 as we can see in the Fig. 6.2d but at the step size equal to 1 again appears the convergence.

When the coupling probability decreases to 0.2, see Fig. 6.2e, the convergence ends for the active phase wave at a smaller step size, which is 0.3. At the same time, the CT of the splintered phase wave increases and already finishes converging the same way earlier at the step size equal to 0.7, CT around 3800.

Thus, one of the most optimal solutions for the states is when the probability is equal to 1 or 0.9, because the states converge at the largest step size, with less convergence time. As for the minimum probability value, the critical one is when p = 0.6, after which the step size is already significantly reduced by which the active phase wave converges.

6.1.3 Wider range of the step sizes

In addition, we consider that we have a standard deviation equal to 0.05. We increase the observed step size from the average value. In these simulations, the step sizes are clustered around the mean but a little further than in the previous case.

We take a look at the convergence time with a maximum coupling probability p = 1 and an Euler step size equal to Δt + rnorm(100,0,σ), Fig. 6.3a. The CTs of the static

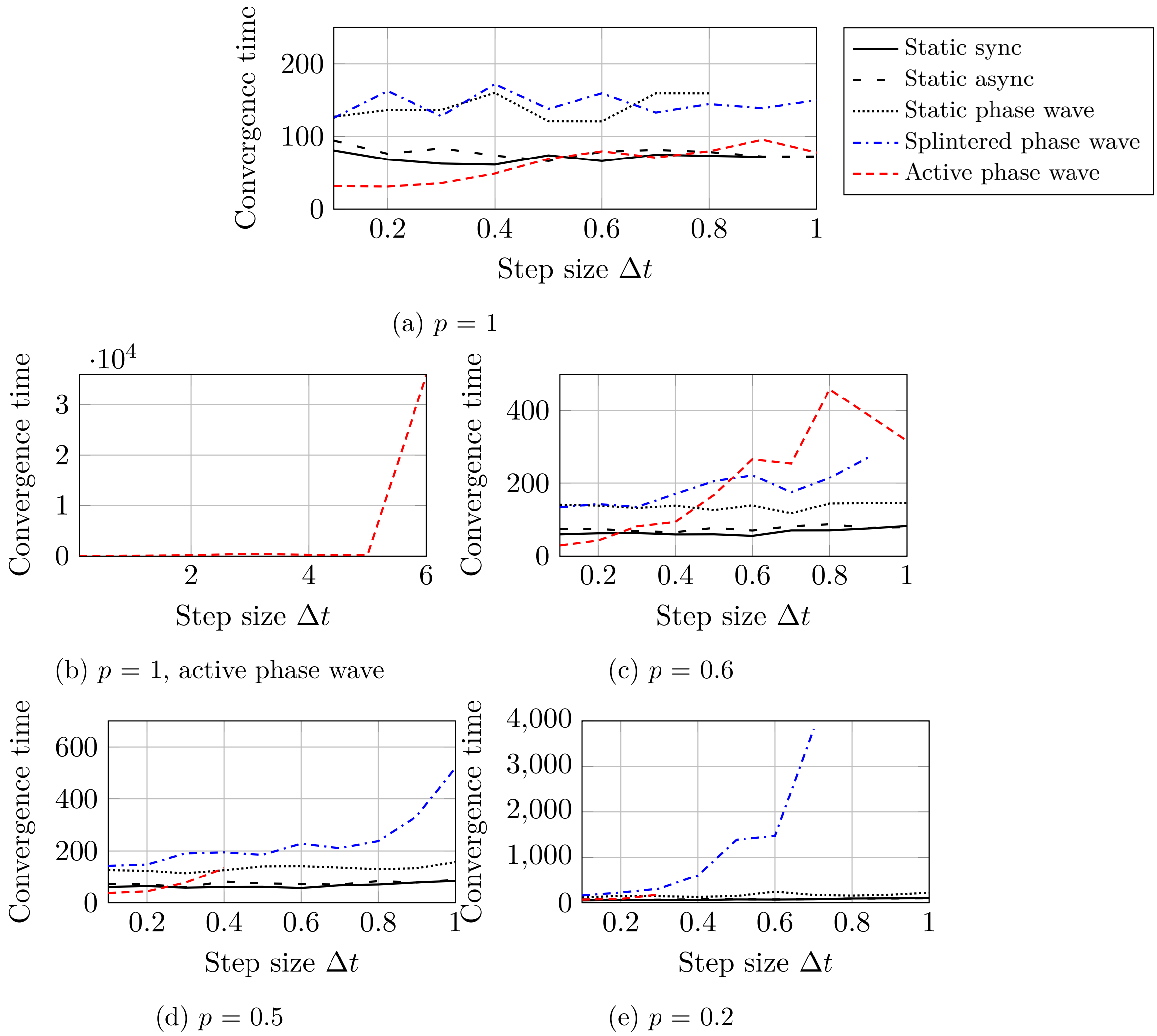


Figure 6.2: Convergence time of five states when the randomization of the step sizes. Parameters: 100 swarmalators, the standard deviation equal to 0.005.

sync and async are around 90 and between 120 to 160 for the static and splintered phase waves. As for the active phase wave, the CT of the state increases from 20 to 90 when the step size is from 0.1 to 0.5. However, as the step grows the CT repeats the behavior of the static sync and async. The longest convergence time, which is about 26300, occurs at the active state with Δt around 6, Fig. 6.3b. Also the static phase wave stops to converge earlier than others at a step size around 0.8, the static sync at 0.9, the static async and splintered stops around 1. It is important to mention that none of the states converge at σ= 0.05 on step sizes of approximately 0.1 to 0.2.

When the coupling probability increases to 0.7, we see the effect already in the active phase wave, which ends up converging at a step size of about 2 at the CT of 5465.5,

Fig. 6.3c. However, the static phase wave on the contrary increases the step size to 2 with decreasing probability.

As soon as we have reduced the coupling probability to 0.4, the active phase wave stops converging at a step size of about 1 and the convergence time itself has increased compared to the previous simulation. We also notice the growth of the CT of the splintered phase wave. The convergence time has increased approximately twice but the state still converges on the entire interval, Fig. 6.3d.

Next, when the coupling probability is reduced to 0.2, then the active and splintered phase waves stop to converge at step sizes of about 0.4 and 0.6. The splintered state reaches a peak of the CT around 2000 at the last point of convergence, Fig. 6.3e.

Thus, by decreasing the coupling probability, we can notice a significant impact on the splintered and active phase waves. The optimal p is up to 0.4 because then a strong influence on the states is noticed and they begin to diverge. As for static states, any probability value will be suitable for them, with the convergence to about 1 for all static states.

6.2 Changing the coupling probability

In this part, we analyze the dependence of the convergence time on the coupling probability for a certain step size equal to 0.2, but using different values of standard deviation which are equal to 0, 0.005 and 0.05.

Firstly, we take a look at Fig. 6.4a, when we have $\Delta t = 0.2$, $\sigma = 0$ and the interval of p from 0.1 to 1. The static states converge over the entire interval with a convergence time of about 60. For phase wave the CT is about 180. As for the splintered and active phase waves, the first one has an increase with the coupling probability smaller than 0.2 and another one does not converge to Δt around 0.2. The convergence times are from 180 to 700 and from 50 to 70.

As the standard deviation increases to 0.005, the active phase wave begins to converge over the entire interval, especially with a probability equal to 0.1. But the splintered phase wave increases even more with a small coupling probability, between 0.1 and 0.2 and reaches the CT = 790, Fig. 6.4b.

If we increase the standard deviation to 0.05, the splintered phase wave experiences an even greater peak at probability 0.1 the convergence time value already reaches 800. It is important to notice that the static phase wave experiences an increase in the CT = 300 when $p = 1$, Fig. 6.4c.

Based on the results obtained, at a step size of about 0.2, the change in standard deviation affects the phase waves to a greater extent. For all states to converge and the time values to be the most optimal, it is better to choose σ around 0.005.

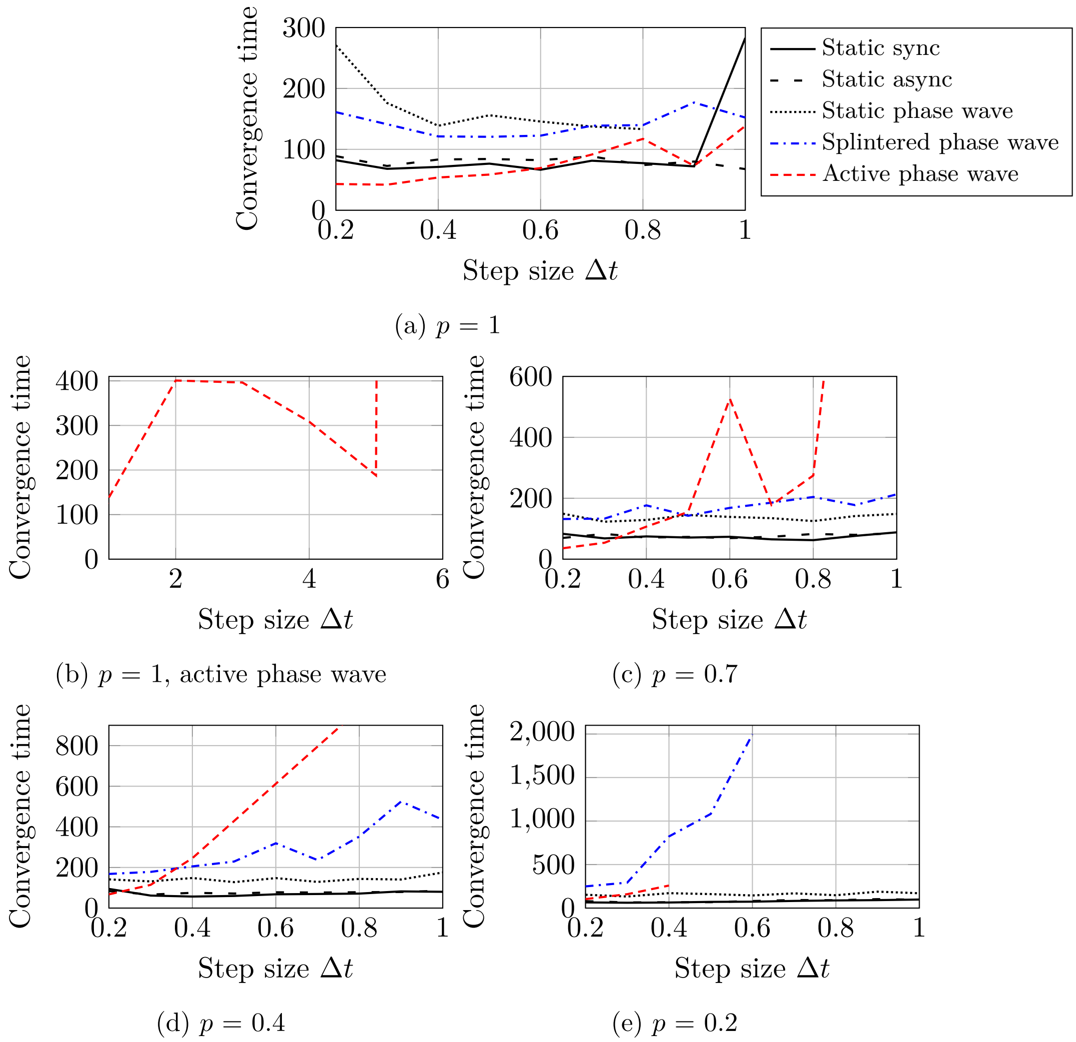
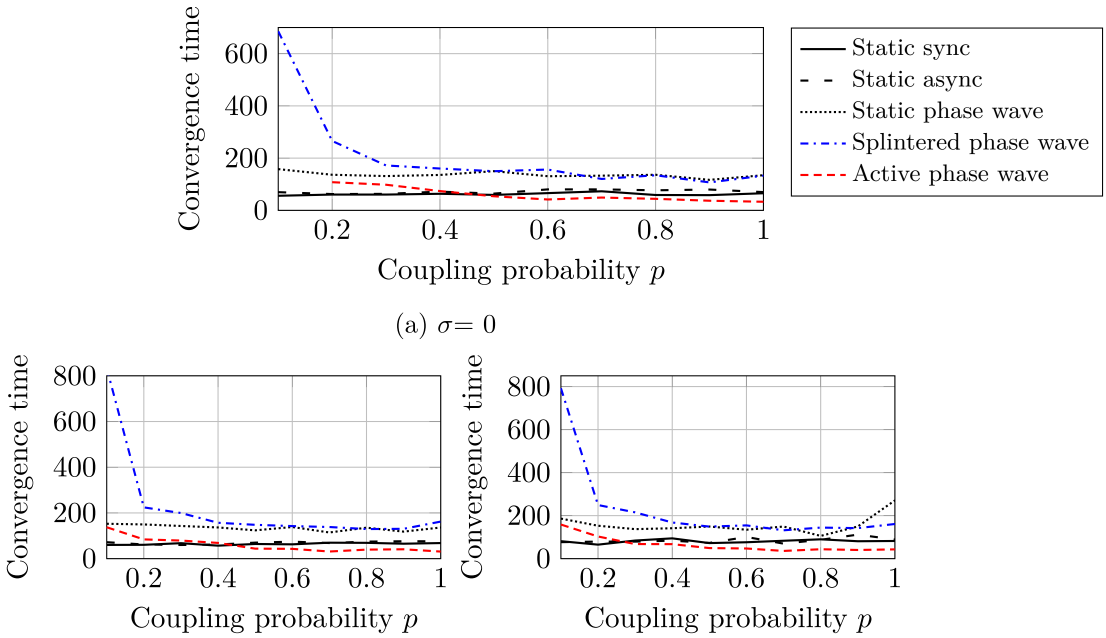


Figure 6.3: Convergence time of five states when the randomization of the step sizes. Parameters: 100 swarmalators, the standard deviation equal to 0.05.

6.3 Probabilistic dependence at the standard deviation and various step sizes

From the simulations performed to select the most optimal σ, it turned out that σ=0.005 is the most optimal for a step size of about 0.2. We will conduct further analysis of this σ and see how it affects the probability when the step size changes, Fig. 6.5.

For the beginning we take a look at the results for $\Delta t = 0.1$, Fig. 6.5a. All states converge on the whole interval of probability. The CTs of static sync and async are from 60 to 90. The convergence time of the active phase wave is from 60 to 100 when p is smaller than 0.3 and around 60 with larger p. As we can see the behavior of the static



(a) $\sigma = 0$

(b) $\sigma = 0.005$

(c) $\sigma = 0.05$

Figure 6.4: Convergence time of five states when the coupling probability p from 0.1 to 1. Parameters: 100 swarmalators, $\Delta t = 0.2$.

and splintered phase waves is similar with the coupling probability greater than 0.2. In lower values of probability the static phase wave does not change, but another one state increases to 280 at $p = 0.1$.

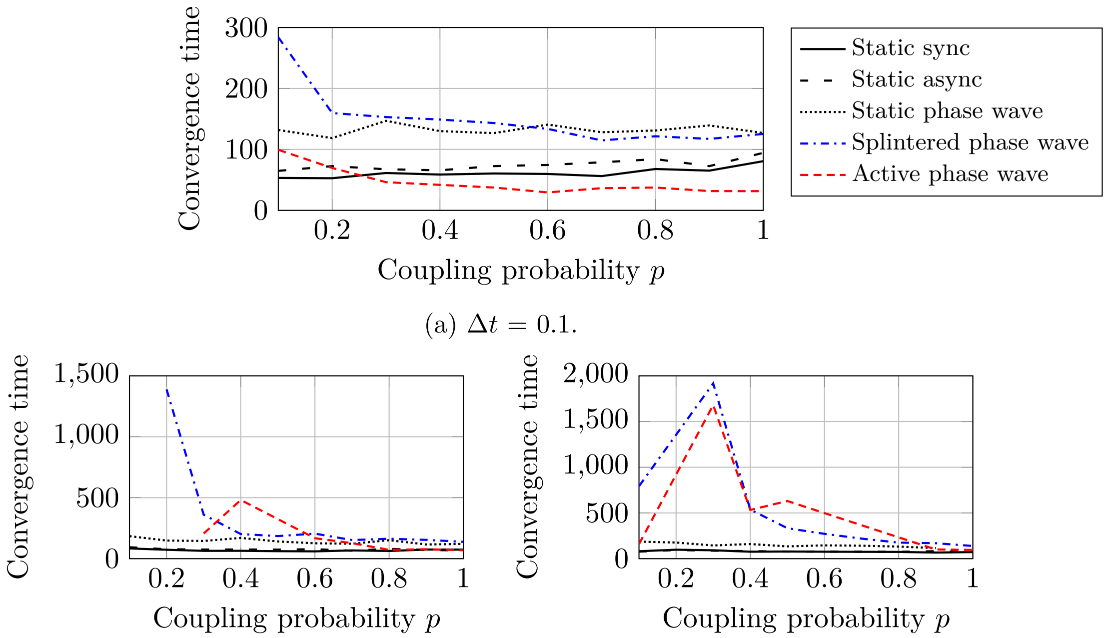
Next, we increase the step size and $\Delta t = 0.5$, Fig. 6.5b. It can be noted a strong influence on the splintered and active phase waves. One of them stops converging when the coupling probability is less than 0.2 with the CT around 1400 and 0.3 for another one with the CT around 200.

Nonetheless, with an increase in the step size to 0.9 the splintered and active phase waves again converge on the whole interval of p. Some peaks of time values remain at $p = 0.3$ with CT around 1700 for both states, Fig. 6.5c.

Thus, the optimum is to take attention to the splintered and active phase waves. For them it is better to choose small values of Δt or closer to one, if we take $\sigma = 0.005$. The standard deviation has no visible impact on other static states.

6.4 Analysis of the standard deviation

Based on the simulations, we construct the dependence of the convergence time on the standard deviation. In this case, we take previous values of σ equal to 0, 0.005, 0.05 and make additional simulations with σ equal to 0.0001, 0.001 and 0.1. This way we get more



(a) $\Delta t = 0.1$.

(b) $\Delta t = 0.5$.

(c) $\Delta t = 0.9$.

Figure 6.5: Convergence time of the five states when the randomization of the step size. Parameters: 100 swarmalators, rnorm(100, 0, 0.005), the standard deviation equal to 0.005.

points to build a graphical dependency. We present each state separately at different step sizes, which are equal to 0.1, 0.3. 0.7 and 1, Fig. 6.6.

We start with the static sync in Fig. 6.6a. With a step size equal to 0.3 and 0.7, we can notice convergence over the entire interval and the convergence times are from 60 to 120 and around 70, respectively. At $\Delta t = 0.1$ and $\Delta t = 1$ the state stops converging when σ greater than 0,01 and 0.05. The convergence times are around 60 and 300 for small and large step sizes.

We take a look at the static async state, which converges at the steps under consideration except for $\Delta t = 0.1$, Fig. 6.6b. At step sizes equal to 0.7 and 1, the convergence times are about 90 over the entire interval of the standard deviation. When reducing the step size to 0.3, we note that at σ= 0.1 there is a significant increase in the convergence time. However, if the step size is further reduced to 0.1, the state converges only to σ equal to 0.01 with the CT around 80.

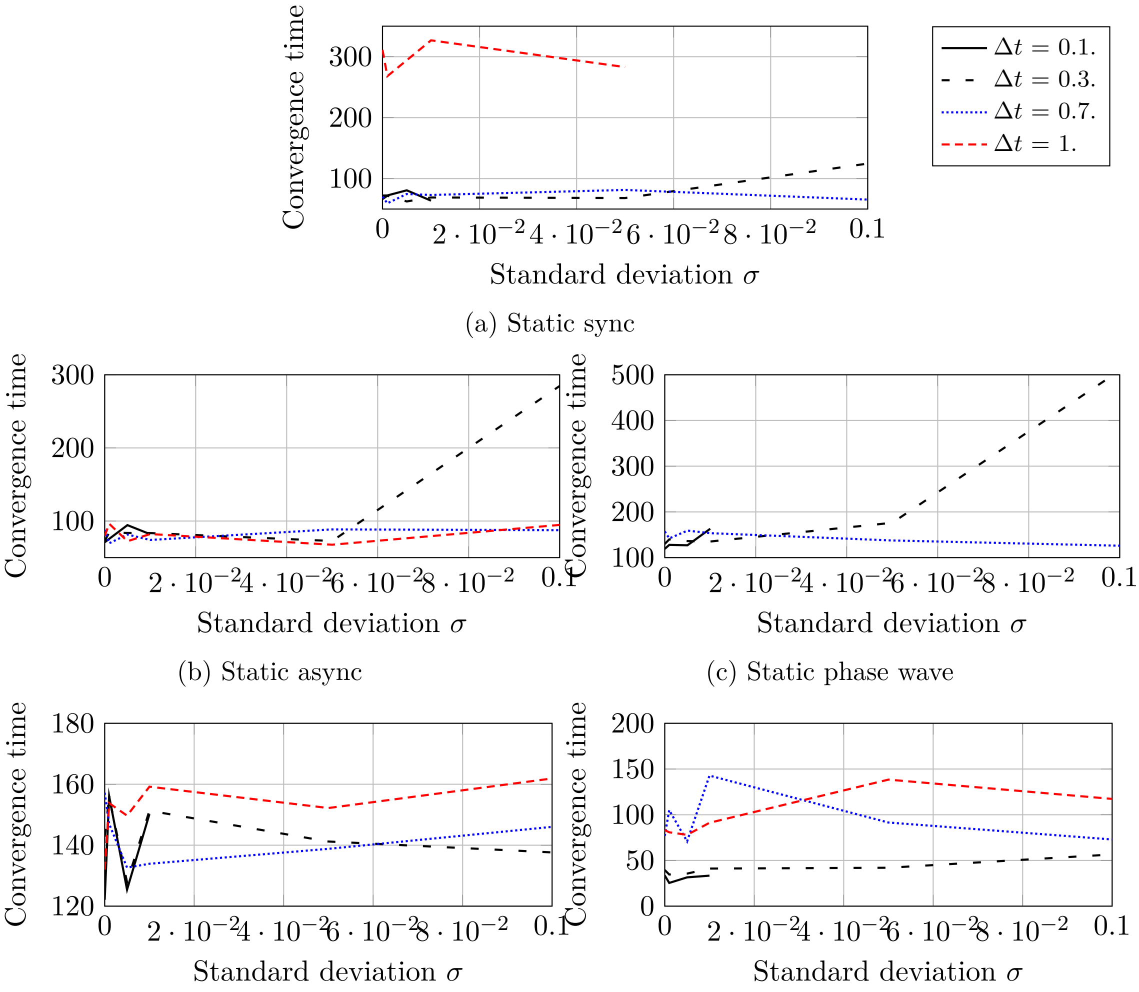
Additionally, paying attention to the static phase wave, Fig. 6.6c, it can be noted that it is similar to the static async, Fig. 6.6b, but with slightly larger values of convergence time. Furthermore, at a step size $\Delta t = 1$, the state does not converge at any given σ.

Next, we take a look at the convergence of the splintered phase wave, Fig. 6.6d. When the

$\Delta t = 0.1$, the state stops to converge at σ= 0.01 and the convergence time is between 125 and 160. When we increase the step size to 0.3, 0.7 and 1 with any chosen value of the standard deviation the state converges. The CTs are from 130 to 160 for these Δt.

Finally, analyzing the active phase wave, we can say that with $\Delta t = 0.1$ the behavior of the state is similar for all static states, Fig 6.6a, 6.6b and 6.6c. As the step increases, the state converges over the entire σ interval from 0 to 0.1. When $\Delta t = 0.3$ the state converges with the CT around 50, when $\Delta t = 0.7$ and $\Delta t = 1$ the CTs are between 70 and 145. At the step size equal to 1, the active state increases at the interval, and at the $\Delta t = 0.7$ on the contrary decreases and both intersect on the standard deviation of about 0.03 with CTs around 120.

Thus, analyzing each state separately, for different values of the step size and the selected σ, we can say that with a small step size of 0.1, none of the models converges on the entire σ interval from 0.1 to 1. When increasing the step size to 1, the splintered phase wave does not converge at any point of the σ. As soon as we start to increase the step size, the static async and phase wave, the splintered and active phase waves are quite similar in their behavior. Then for all states, it is optimal to choose a step size around 0.7.



(a) Static sync

(b) Static async

(c) Static phase wave

(d) Splintered phase wave

(e) Active phase wave

Figure 6.6: Convergence time of five states when the step size is equal to 1, 0.7, 0.3 and 0.1. Parameters: 100 swarmalators, the coupling probability $p = 1$, σ is equal to 0, 0.001, 0.0005, 0.01, 0.05 and 0.1.

CHAPTER 7

Results and conclusions

7.1 Summary of results

In the following, based on the results obtained, which are presented in the previous chapters of the thesis, we summarize all three different simulations with the impact of the varied step size, increased or decreased increment and standard deviation.

Fig. 7.1 and 7.2 show different scenarios for choosing a step size for all five states separately.

Paying attention to the static sync, Fig. 7.1a, with a small step size from 0.1 to 1, when we vary the step size in different ways, this does not particularly affect the values of the convergence time and convergence. When we increase the step size with increment τ, which is τ= 0.0002, or apply randomization with the standard deviation σ, which is σ=0.005, the state stops converging at a step size of about 1. But as soon as we decrease the step size with increment τ, the step size immediately increases significantly, and the state converges to about 10, Fig. 7.2. The important thing to note is that with a small Δt, this static state does not converge and also with a large step size equal to 10, we see a rather long convergence time compared to the interval of the step size from 0.1 to 1.

Next, we take a look at the static async in Fig. 7.1b. Here, when we increase the step size with increment τ, or apply randomization of the step size with the standard deviation σ, with small step sizes, the state converges a little longer than usual. On the contrary, with large values of CTs, the static async converges faster than with a constant step size. Also, when we decrease the step size with increment τ, Fig. 7.2, the state converges with large step size, up to 10, with the same parameters as other simulations with this state, but it does not converge with small values of step sizes.

Analyzing the static phase wave in Fig. 7.1c, we can note that when we increase the step size with increment or apply standard deviation, the state converges only with smaller step sizes and larger convergence times. Additionally, if the randomization is applied then the static phase wave converges with larger step sizes, up to 10 as we see in Fig. 7.2. However, this method of changing the step size is not applicable for small Δt because the state diverges to about 0.4.

Of course, it is important to consider the splintered phase wave, Fig. 7.1d, which, when applied an increment for decreasing or increasing the step size, takes good values starting from the middle of the interval of Δt from 0.1 to 1, around 0.6. If we look at the influence of the standard deviation σ, we can say that very similar behavior as the above states. Namely, it shows a perfect convergence with large step sizes, but also with large convergence times, Fig. 7.2.

It seems that one of the most interesting cases that should be paid attention is to the active phase wave, Fig. 7.1e. With step size from 0.1 to 1, we do not particularly notice any changes when adding an increment or using the standard deviation for randomization. When we increase the step size, we can notice that after a step size of about 3, adding an increment has a positive effect, and the value of the CT decreases. Also, the increment subtraction reduces the convergence time and converges at a step size of about 10, Fig. 7.2. As for the use of σ, after a step size of about 1, the time decreases, but this does not affect the increase in the step size at which the model continues to converge.

To sum up, using the simple Δt has a positive effect at small values, on the initial interval of the thesis for all five states. As soon as we start increasing the step size, when adding or subtracting the increment τ, it has a slight effect on all states, that is, somewhere the convergence time becomes a little longer, and somewhere a little less, but to a greater extent, everything depends on the experiment. When the step size increases to about 1, then definitely all static states and the splintered phase wave cease to converge at a given step size, and just, in this case, the increment step change has a positive effect on the convergence of all states. But it is important to note here that it is the increment subtraction that increases the step size at which the states continue to converge.

7.2 Conclusions

The thesis shows methods such as increment addition or subtraction in each step, as well as the use of randomization for different step sizes. The results of simulations on the selection of optimal values for different step sizes are presented in Table 7.1.

Analyzing the results obtained, summarized in Table 7.1, we can say that for small step sizes, up to 0.2, it is optimal to use all three cases, except for increment subtraction. The case when a swarmalator shares its location and phase vary often.

As the step size increases, when the step takes an intermediate value, then this is the optimal situation, in which any case of changing the step size will be optimal (Table 7.1). In this instance, information is exchanged in the same way, but only the simulators exchange messages less often. It is used in mechanisms where the exchange of information every second is not a critical option.

Table 7.1: The step size Δt for converge states *static sync, static async, static phase wave, splintered phase wave and active phase wave*, where ① - for impact of varied step size, ② - for impact of the increment subtraction, ③ - for the adding increment, ④ - for the impact of the standard deviation and ✗ - no one state converge.

Step size Δt	Static sync	Static async	Static phase wave	Splintered phase wave	Active phase wave
0.1 - 0.2	① ③ ④	① ③ ④	① ③ ④	① ③ ④	① ③ ④
0.2 - 0.3	① ② ③ ④	① ③ ④	① ③ ④	① ③ ④	① ② ③ ④
0.3 - 0.4	① ② ③ ④	① ② ③ ④	① ③ ④	① ③ ④	① ② ③ ④
0.4 - 0.7	① ② ③ ④	① ② ③ ④	① ② ③ ④	① ② ③ ④	① ② ③ ④
0.7 - 0.8	① ② ③ ④	① ② ③ ④	① ② ④	① ② ③ ④	① ② ③ ④
0.8 - 0.9	① ② ③ ④	① ② ③ ④	① ②	① ② ③ ④	① ② ③ ④
0.9 - 1	① ②	① ② ③ ④	②	① ② ③ ④	① ② ③ ④
1 - 6	②	②	②	②	① ② ③ ④
6 - 10	✗	✗	✗	✗	②
10 - ...	✗	✗	✗	✗	✗

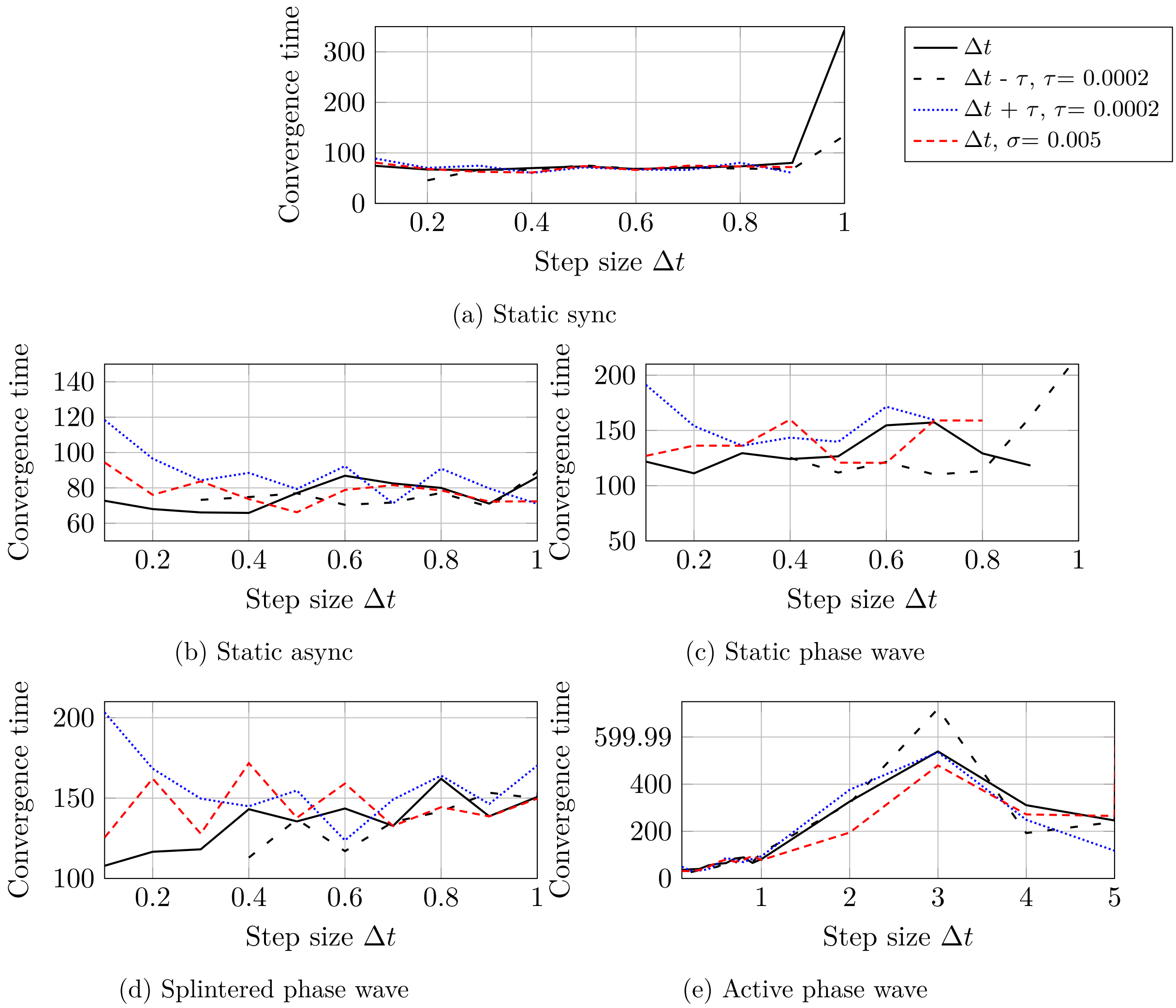


Figure 7.1: Convergence time of five states with the step size. The coupling probability $p = 1$.

When the step is increased to values greater than 1 (Table 7.1), which means that sending messages is rare. We get the following: The states converge only when the increment is subtracted and also during the active phase wave of long convergence. This means that this case, the transfer and exchange of information is quite rare. This case can be suitable for a scenario where the movement of mechanisms does not change much over time and the exchange of messages may occur less frequently.

Finally, concluding from all the simulations obtained, we can say that as soon as we take a small step size value, the convergence time will be as fast as possible. But when we gradually begin to increase the value of the step size, then, accordingly, this causes on increase in the convergence time. Thus, the convergence time is the time that passes

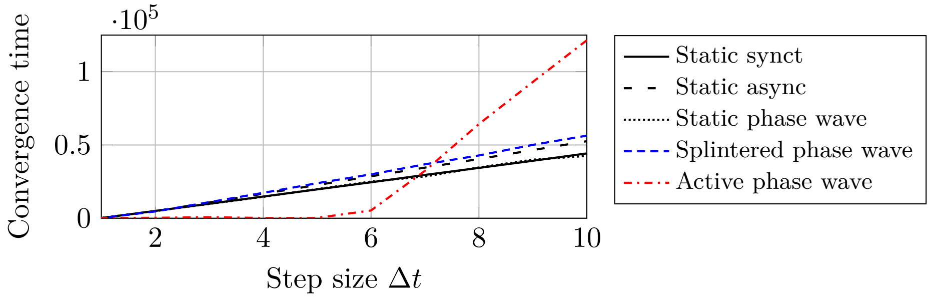


Figure 7.2: Convergence time with step size Δt for the *static sync, static asyn, static phase wave, splintered phase wave and active phase wave.* Coupling probability $p = 1$. Δt - τ, τ= 0.0002.

from the moment of perception of information to the reaction to it. In other words, it is the ability to detect, process and respond to the information received. Convergence time includes perception, information processing, and response. Note that if any of these processes is disrupted, this will negatively affect the convergence time or the signal will not be sent, transmitted or received.

To sum up, a wide range of simulations is discussed with the selection of the most optimal values. Plots for three different parts of simulations are presented. The general idea is obtained by the most optimal value of the step size, coupling probability, increment and standard deviation. Due to the results obtained, it is definitely worth paying attention to the system parameters because the application can be different and for each task, it is a specific step value will be optimal.

7.3 Further work

The results of the work done have been obtained by simulation. In the further, it would be interesting to check the optimal step size values in practice. Most of the results can be checked first indoors without interference and in an open space under different weather conditions. Hence, we could choose the values that will be optimal under different conditions.

In addition, by conducting the experiments, it would be interesting to see the effect of the step size and coupling probability. It will be important to consider how they change with the same input values because we can only track the coupling probability itself and see what values are obtained on average during simulations.

In the future, it would be interesting to consider and graphically depict the dependence of the fixed step size on the coupling probability. The plot will show the values at which convergence is stable and unstable for five different states.

Similarly, it would be useful to show how the value of the smallest distance between swarmalators depends on the coupling probability. We could plot the graphical dependence of the distance on the probability of signal transmission.

Hence, the goal would be to apply the same methods to the distance as we applied to the step size in this paper:

1. For the beginning, check the impact of the varied distance.
2. Secondly, check the impact of the increment to increase the distance.
3. Finally, apply the standard deviation to the distance.
4. Subsequently, check these results in practice.

List of Figures

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