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Harmonic analysis / *Analyse harmonique*

On the boundedness of a family of oscillatory singular integrals

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Abstract. Let $\Omega \in H^1(\mathbb{S}^{n-1})$ with mean value zero, P and Q be polynomials in n variables with real coefficients and $Q(0) = 0$. We prove that

$$\left|\text{p.v.}\int_{\mathbb{R}^n} e^{i(P(x)+1/Q(x))}\frac{\Omega(x/|x|)}{|x|^n}\,\mathrm{d}x\right| \le A\|\Omega\|_{H^1(\mathbb{S}^{n-1})}$$

where A may depend on n, $\deg(P)$ and $\deg(Q)$, but not otherwise on the coefficients of P and Q.

The above result answers an open question posed in [13]. Additional boundedness results of similar nature are also obtained.

Keywords. oscillatory integrals, singular integrals, Calderón–Zygmund kernels, Hardy spaces.

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1. Introduction

The study of oscillatory singular integrals has a long-standing history ([1, 4, 7–9, 11, 12]). For the specific topic considered in this paper, we shall begin with a well-known result of Stein and Wainger in [12] and its extension by Stein in [10].

Let $n \ge 2$, $K(x)$ be a Calderón–Zygmund kernel given by

$$K(x) = \frac{\Omega(x/|x|)}{|x|^n} \tag{1}$$

where $\Omega : \mathbb{S}^{n-1} \to \mathbb{C}$ is integrable over the unit sphere $\mathbb{S}^{n-1}$ with respect to the induced Lebesgue measure σ and satisfies

$$\int_{\mathbb{S}^{n-1}} \Omega(x)\mathrm{d}\sigma(x) = 0. \tag{2}$$

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For $d \in \mathbb{N}$, let $\mathscr{P}_{n,d}$ denote the space of real-valued polynomials in n variables whose degrees do not exceed d. It was proves in [10] that, if $\Omega \in L^\infty(\mathbb{S}^{n-1})$ and $P \in \mathscr{P}_{n,d}$, then

$$\left|\text{p.v.} \int_{\mathbb{R}^n} e^{iP(x)} K(x) \mathrm{d}x\right| \le C_{n,d} \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}$$

where $C_{n,d}$ is independent of the coefficients of P.

In the recent paper [13] the authors obtained an extension of the above result in which the phase functions belong to a certain class of rational functions while Ω is allowed to be in a block space $B_q^{0,0}(\mathbb{S}^{n-1})$. Their result can be described as follows.

Theorem 1 ([13]). *Let $q > 1$ and $K(x)$ be a Calderón–Zygmund kernel given by* (1)–(2). *Let $P(x), Q(x) \in \mathscr{P}_{n,d}$ such that $Q(0) = 0$ and $\Omega \in B_q^{0,0}(\mathbb{S}^{n-1})$. Then*

$$\left|\text{p.v.} \int_{\mathbb{R}^n} e^{i(P(x)+1/Q(x))} K(x) \mathrm{d}x\right| \le A \tag{3}$$

where A may depend on $\|\Omega\|_{B_q^{0,0}(\mathbb{S}^{n-1})}$, n and d but not otherwise on the coefficients of P and Q.

The definition of $B_q^{0,v}(\mathbb{S}^{n-1})$ for $v > -1$ and $q > 1$ can be found in [13]. It had been known that the bound (3) also holds for all $\Omega \in L\log L(\mathbb{S}^{n-1})$, which was proved by Folch-Gabayet and Wright in [5].

Let $H^1(\mathbb{S}^{n-1})$ denote the Hardy space over the unit sphere. An important question, posed by the authors of [13], is whether the bound (3) continues to hold under the condition $\Omega \in H^1(\mathbb{S}^{n-1})$ (with the same phase functions $P(x) + 1/Q(x)$). This is a very natural question because both $B_q^{0,0}(\mathbb{S}^{n-1})$ and $L\log L(\mathbb{S}^{n-1})$ are proper subspaces of $H^1(\mathbb{S}^{n-1})$.

Our first result answers the above question in the affirmative.

Theorem 2. *Let $K(x)$ be a Calderón–Zygmund kernel given by* (1)–(2). *Let $P(x), Q(x) \in \mathscr{P}_{n,d}$ such that $Q(0) = 0$. Suppose that $\Omega \in H^1(\mathbb{S}^{n-1})$. Then*

$$\left|\text{p.v.} \int_{\mathbb{R}^n} e^{i(P(x)+1/Q(x))} K(x) \mathrm{d}x\right| \le A \|\Omega\|_{H^1(\mathbb{S}^{n-1})} \tag{4}$$

where A may depend on n and d but not otherwise on the coefficients of P and Q.

As usual, Theorem 2 implies the uniform boundedness of oscillatory singular integral operators of the following type on $L^2(\mathbb{R}^m)$:

$$f \to \text{p.v.} \int_{\mathbb{R}^n} f(u_1 - P_1(y), \ldots, u_m - P_m(y)) e^{i/Q(y)} |y|^{-n} \Omega(y/|y|) \mathrm{d}y,$$

where $P_1, \ldots, P_m, Q$ are polynomials and Ω is a function in $H^1(\mathbb{S}^{n-1})$ with a zero mean-value. The proof of Theorem 2 will be given in Section 2.

The general question about whether the condition $Q(0) = 0$ can be removed is open. But for $\deg(Q) \le 1$, this is known to be the case.

Theorem 3 ([5, 13]). *Let $K(x)$ be a Calderón–Zygmund kernel given by* (1)–(2). *Let $P(x) \in \mathscr{P}_{n,d}$ and $Q(x) = a + v \cdot x$ where $a \in \mathbb{R}$ and $v \in \mathbb{R}^n$. Suppose that $\Omega \in L\log L(\mathbb{S}^{n-1})$ or $\Omega \in B_q^{0,0}(\mathbb{S}^{n-1})$ for some $q > 1$. Then*

$$\left|\text{p.v.} \int_{\mathbb{R}^n} e^{i(P(x)+1/Q(x))} K(x) \mathrm{d}x\right| \le A \tag{5}$$

where A may depend on n, d and the respective norm of Ω, but not otherwise on a, v and the coefficients of P.

We have the following extension of Theorem 3:

Theorem 4. *Let $P(x) \in \mathscr{P}_{n,d}$. Let $l \in \mathbb{N}$, $h(x)$ be a nonzero real-valued homogeneous polynomial of degree l and $Q(x) = a + h(x)$. Then for every Calderón–Zygmund kernel $K(x)$ given by* (1)–(2) *with an $\Omega(\cdot)$ in $H^1(\mathbb{S}^{n-1})$,*

$$\left|\text{p.v.}\int_{\mathbb{R}^n} e^{i(P(x)+1/Q(x))}K(x)\mathrm{d}x\right| \le A \tag{6}$$

where A may depend on $\|\Omega\|_{H^1(\mathbb{S}^{n-1})}$, n, d and l but not otherwise on the coefficients of $P(x)$ and $Q(x)$.

The proof of Theorem 4 will be given in Section 3.

The following is an important estimate due to E. M. Stein:

Theorem 5. *Let $\Omega \in L\log L(\mathbb{S}^{n-1})$ and $d \in \mathbb{N}$. For every homogeneous polynomial of degree d on $\mathbb{R}^n$ $P(x) = \sum_{|\alpha|=d} a_\alpha x^\alpha$, let $m_P = \sum_{|\alpha|=d}|a_\alpha|$. Then there exists a constant $C_{n,d,\Omega} > 0$ which is independent of $\{a_\alpha\}$ such that*

$$\int_{\mathbb{S}^{n-1}} |\Omega(x)|\left|\log\left(\frac{|P(x)|}{m_P}\right)\right|\mathrm{d}\sigma(x) \le C_{n,d,\Omega} \tag{7}$$

holds whenever $m_P \neq 0$.

What happens if $P(x)$ is a general polynomial instead of a homogeneous polynomial? For $P(x) = \sum_{|\alpha|\le d} a_\alpha x^\alpha$, the direct analogue of (7), where m_P is replaced by $\sum_{|\alpha|\le d}|a_\alpha|$, is clearly false. This is due to the fact that, unlike $P \to m_P$ for homogeneous polynomials of a fixed degree,

$$\sum_{|\alpha|\le d} a_\alpha x^\alpha \to \sum_{|\alpha|\le d}|a_\alpha|$$

is not a norm on $\mathscr{P}_{n,d}\big|_{\mathbb{S}^{n-1}}$. To remedy this situation, we can simply replace the above with any norm on $\mathscr{P}_{n,d}\big|_{\mathbb{S}^{n-1}}$ (e.g. $\|\cdot\|_\infty$) to arrive at the following extension of Theorem 5:

Theorem 6. *Let $\|\cdot\|$ be a norm on $\mathscr{P}_{n,d}\big|_{\mathbb{S}^{n-1}}$. Then there exists a positive constant C which depends on n, d and $\|\cdot\|$ only such that*

$$\int_{\mathbb{S}^{n-1}} |\Omega(y)|\left|\log\left(\frac{|P(x)|}{\|P|_{\mathbb{S}^{n-1}}\|}\right)\right|\mathrm{d}\sigma(x) \le C(1+\|\Omega\|_{L\log L(\mathbb{S}^{n-1})}) \tag{8}$$

holds for all $\Omega \in L\log L(\mathbb{S}^{n-1})$ and all $P \in \mathscr{P}_{n,d}$ not vanishing identically over $\mathbb{S}^{n-1}$.

Since any two norms on a finite dimensional space are equivalent, one recovers (7) when applying (8) to homogeneous polynomials.

More broadly, results such as Theorem 6 can be framed in terms of functions of finite type and compactness, as is done in the theorem below.

Theorem 7. *Let M be a compact smooth submanifold of $\mathbb{R}^n$, $\sigma = \sigma_M$ be the induced Lebesgue measure on M and U be an open subset of $\mathbb{R}^m$. Let $f \in C^\infty(M \times U)$ such that, for every $u \in U$, $f(\cdot, u)$ does not vanish to infinite order at any point in M. Then, for every compact subset W of U, there exists a positive constant $C = C(M, n, m, f, W)$ such that*

$$\sup_{u\in W}\int_M |\Omega(y)|\log(|f(y,u)|)|\mathrm{d}\sigma(y) \le C(1+\|\Omega\|_{L\log L(M)}) \tag{9}$$

holds for all $\Omega \in L\log L(M)$.

The proof of Theorem 7 will be based on Malgrange preparation theorem. It will be given in Section 4 where one will also see how Theorem 6 follows as a simple consequence. As an application of Theorem 7, we have the following:

Theorem 8. *Let $b \in \mathbb{R}\backslash\{0\}$ and $P(x) \in \mathscr{P}_{n,d}$. Let $\rho \in \mathbb{R}^+$, U be an open subset of $\mathbb{R}^m$ and W be a compact subset of U. Let $\psi \in C^\infty(\mathbb{S}^{n-1} \times U)$ such that, for every $u \in U$, $\psi(\cdot, u)$ does not vanish to infinite order at any point in $\mathbb{S}^{n-1}$. For $(x,u) \in (\mathbb{R}^n\backslash\{0\}) \times U$, let*

$$\Phi(x,u) = b|x|^\rho \psi(x/|x|, u). \tag{10}$$

Then for every Calderón–Zygmund kernel $K(x)$ given by (1)–(2) *with an $\Omega(\cdot)$ in $L\log L(\mathbb{S}^{n-1})$,*

$$\sup_{u\in W}\left|\text{p.v.}\int_{\mathbb{R}^n} e^{i(P(x)+1/\Phi(x,u))}K(x)\mathrm{d}x\right| \le A \tag{11}$$

where A may depend on $\|\Omega\|_{L\log L(\mathbb{S}^{n-1})}$, ψ, W, n, d and ρ but not otherwise on b and the coefficients of $P(x)$.

In the rest of the paper we shall use $A \lesssim B$ to mean that $A \le cB$ for a certain constant c which depends on some essential parameters only.

2. Proof of Theorem 2

We shall now prove (4) under the assumptions of Theorem 2. By (2) and the atomic decomposition of $H^1(\mathbb{S}^{n-1})$ (see [2] and [3]), it suffices to prove that

$$\left|\text{p.v.}\int_{\mathbb{R}^n} e^{i(P(x)+1/Q(x))}\frac{\Omega(x/|x|)}{|x|^n}\mathrm{d}x\right| \le A \tag{12}$$

under the assumption that $\Omega(\cdot)$ is a regular atom on $\mathbb{S}^{n-1}$, i.e. $\Omega(\cdot)$ enjoys the following additional properties:

(a) $\operatorname{supp}(\Omega) \subseteq \mathbb{S}^{n-1} \cap B(\zeta_0, \delta)$ for some $\zeta_0 \in \mathbb{S}^{n-1}$ and $\delta > 0$ where $B(\zeta_0,\delta) = \{y \in \mathbb{R}^n : |y-\zeta_0| < \delta\}$; and

(b) $\|\Omega\|_\infty \le \delta^{-n+1}$.

If $\delta \ge 1/4$, (12) follows from (b) and Theorem 1. Thus, we may assume that $0 < \delta < 1/4$. By using a rotation if necessary, we may also assume that $\zeta_0 = (0,\dots,0,1)$. For any $x = (x_1,\dots,x_{n-1},x_n) \in \mathbb{R}^n$, we write $x = (\widetilde{x}, x_n)$ where $\widetilde{x} = (x_1,\dots,x_{n-1})$. We also extend the definition of $\Omega(\cdot)$ from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n\backslash\{0\}$ by using $\Omega(x) = \Omega(x/|x|)$. We define $\Omega_\delta : \mathbb{R}^n\backslash\{0\} \to \mathbb{C}$ by

$$\Omega_\delta(x) = (\delta^{n-1}|x|^n)\frac{\Omega(\delta\widetilde{x}, x_n)}{|(\delta\widetilde{x}, x_n)|^n}.$$

Then $\Omega_\delta(\cdot)$ is homogeneous of degree 0. It is well-known that, by the theory of Calderón–Zygmund operators, (2) implies that

$$\int_{\mathbb{S}^{n-1}} \Omega_\delta(y)\mathrm{d}\sigma(y) = 0. \tag{13}$$

Next we will show that $\|\Omega_\delta\|_\infty \lesssim 1$. To see this, we assume that $\Omega_\delta(x) \ne 0$ for some $x \in \mathbb{R}^n\backslash\{0\}$. Then

$$\eta := \left|\frac{(\delta\widetilde{x}, x_n)}{|(\delta\widetilde{x}, x_n)|} - \zeta_0\right| < \delta.$$

By (b) and $x_n = |(\delta\widetilde{x}, x_n)|(1-\eta^2/2)$,

$$\begin{aligned}
|\Omega_\delta(x)| &\le \left(\frac{|x|}{|(\delta\widetilde{x}, x_n)|}\right)^n \\
&= (\delta|(\delta\widetilde{x}, x_n)|)^{-n}\big(|(\delta\widetilde{x}, x_n)|^2 + (\delta^2-1)x_n^2\big)^{n/2} \\
&= \delta^{-n}\big(1 + (\delta^2-1)(1-\eta^2/2)^2\big)^{n/2} \\
&= \delta^{-n}\big(\delta^2(1-\eta^2/2)^2 + \eta^2(1-\eta^2/4)\big)^{n/2} \lesssim 1.
\end{aligned}$$

Let $P_\delta(x) = P(\delta\tilde{x}, x_n)$ and $Q_\delta(x) = Q(\delta\tilde{x}, x_n)$. Then $P_\delta(\cdot), Q_\delta(\cdot) \in \mathscr{P}_{n,d}$, $\deg(P_\delta) = \deg(P)$, $\deg(Q_\delta) = \deg(Q)$ and $Q_\delta(0) = Q(0) = 0$. It follows from Theorem 1 that

$$\left|\text{p.v.}\int_{\mathbb{R}^n} e^{i(P(x)+1/Q(x))}\frac{\Omega(x)}{|x|^n}\,\mathrm{d}x\right| = \left|\text{p.v.}\int_{\mathbb{R}^n} e^{i(P_\delta(x)+1/Q_\delta(x))}\frac{\Omega(\delta\tilde{x}, x_n)}{|(\delta\tilde{x}, x_n)|^n}\delta^{n-1}\,\mathrm{d}x\right|$$
$$= \left|\text{p.v.}\int_{\mathbb{R}^n} e^{i(P_\delta(x)+1/Q_\delta(x))}\frac{\Omega_\delta(x)}{|x|^n}\,\mathrm{d}x\right| \le A$$

where A depends on n and d only. This proves Theorem 2.

3. Proof of Theorem 4

First let us recall the following version of van der Corput's lemma.

Lemma 9.

(i) *Let ϕ be a real-valued C^k function on $[a,b]$ satisfying $|\phi^{(k)}(x)| \ge 1$ for every $x \in [a,b]$. Suppose that $k \ge 2$, or that $k = 1$ and ϕ' is monotone on $[a,b]$. Then there exists a positive constant c_k such that*

$$\left|\int_a^b e^{i\lambda\phi(x)}\,\mathrm{d}x\right| \le c_k|\lambda|^{-1/k}$$

for all $\lambda \in \mathbb{R}$. The constant c_k is independent of λ, a, b and ϕ.

(ii) *Let ϕ and c_k be the same as in* (i). *If $\psi \in C^1([a,b])$, then*

$$\left|\int_a^b e^{i\lambda\phi(x)}\psi(x)\,\mathrm{d}x\right| \le c_k|\lambda|^{-1/k}(\|\psi\|_{L^\infty([a,b])} + \|\psi'\|_{L^1([a,b])})$$

holds for all $\lambda \in \mathbb{R}$.

We will now give the proof of Theorem 4. Since the case $a = 0$ is already covered by Theorem 5, we shall assume that $a \ne 0$. Initially we will assume that $\Omega \in L^\infty(\mathbb{S}^{n-1})$.

For $\omega \in \mathbb{S}^{n-1}$, let

$$\theta = \theta(\omega) = \left|\frac{a}{h(\omega)}\right|^{1/l}.$$

Then

$$\text{p.v.}\int_{\mathbb{R}^n} e^{i(P(x)+1/Q(x))}K(x)\,\mathrm{d}x = \int_{\mathbb{S}^{n-1}}\Omega(\omega)I(\omega)\,\mathrm{d}\sigma(\omega)$$

where

$$I(\omega) = \int_0^\infty e^{i(P(r\omega)+1/(a+h(\omega)r^l))}\frac{\mathrm{d}r}{r} = I_1(\omega) + I_2(\omega) + I_3(\omega)$$

where

$$I_1(\omega) = \int_0^{\alpha\theta} e^{i(P(r\omega)+1/(a+h(\omega)r^l))}\frac{\mathrm{d}r}{r},$$
$$I_2(\omega) = \int_{\alpha\theta}^{\beta\theta} e^{i(P(r\omega)+1/(a+h(\omega)r^l))}\frac{\mathrm{d}r}{r}$$

and

$$I_3(\omega) = \int_{\beta\theta}^\infty e^{i(P(r\omega)+1/(a+h(\omega)r^l))}\frac{\mathrm{d}r}{r}$$

for some suitable constants α and β. Since $|I_2(\omega)| \le \ln(\beta/\alpha)$, it suffices to show that there exist $0 < \alpha < \beta$ such that

$$\int_{\mathbb{S}^{n-1}}\Omega(\omega)I_j(\omega)\,\mathrm{d}\sigma(\omega) = O(1) \tag{14}$$

for $j = 1$ and $j = 3$.

The estimate (14) for $j = 3$ follows from a slight modification of the proof of Theorem 1 in [5]. Details will be omitted. Below we shall show how to obtain (14) for $j = 1$ with an appropriate selection of α.

Let

$$\phi_\omega(r) = P(r\omega) + \frac{1}{a + h(\omega)r^l}.$$

In order to apply van der Corput's lemma, we shall need to obtain appropriate lower bounds for at least one of the derivatives of $\phi_\omega(\cdot)$ near 0. When $l = 1$, this can be done with any derivative of $\phi_\omega(\cdot)$ whose order exceeds the degree of $P(\cdot)$. When $l > 1$, one needs to be more selective as demonstrated below.

Let $g(t) = (1 \pm t^l)^{-1}$. Then, for $k = 0, 1, 2, \dots,$

$$\left|\frac{\mathrm{d}^s g(0)}{\mathrm{d}t^s}\right| = \begin{cases} s! & \text{if } l \mid s \\ 0 & \text{if } l \nmid s. \end{cases}$$

Let $k_0 \in \mathbb{N}$ such that $k_0 l > \max\{\deg(P), 1\}$. By $|g^{(k_0 l)}(0)| = (k_0 l)! \geq 1$, there exists an $\alpha \in (0,1)$ such that $|g^{(k_0 l)}(0)| \geq 1/2$ for $|t| \leq \alpha$. By

$$\phi_\omega(r) = P(r\omega) + a^{-1} g(r/\theta),$$

we have

$$\begin{aligned} |\phi_\omega^{(k_0 l)}(r)| &= (|a|\theta^{k_0 l})^{-1} |g^{(k_0 l)}(r/\theta)| \\ &\geq (2|a|\theta^{k_0 l})^{-1} \end{aligned}$$

for $r \in (0, \alpha\theta]$.

Let $b = \min\{|a|, 1\}$. If $|a| \geq 1$, then

$$\int_{(b^{1/(k_0 l)}\alpha\theta,\, \alpha\theta]} e^{i\phi_\omega(r)} \frac{\mathrm{d}r}{r} = 0.$$

If $|a| < 1$, then $b = |a|$ and by Lemma 9,

$$\left|\int_{(b^{1/(k_0 l)}\alpha\theta,\, \alpha\theta]} e^{i\phi_\omega(r)} \frac{\mathrm{d}r}{r}\right| \lesssim \frac{1}{((2|a|\theta^{k_0 l})^{-1})^{1/(k_0 l)}} \cdot \frac{1}{|a|^{1/(k_0 l)}\alpha\theta} \lesssim 1.$$

Thus, we always have

$$\left|\int_{(b^{1/(k_0 l)}\alpha\theta,\, \alpha\theta]} e^{i\phi_\omega(r)} \frac{\mathrm{d}r}{r}\right| \lesssim 1. \tag{15}$$

Therefore, it suffices to show that

$$\int_{\mathbb{S}^{n-1}} \Omega(\omega)\left(\int_0^{\alpha\theta b^{1/(k_0 l)}} e^{i\phi_\omega(r)} \frac{\mathrm{d}r}{r}\right) \mathrm{d}\sigma(\omega) = O(1). \tag{16}$$

Let

$$q(x) = \left(\frac{1}{a}\right) \sum_{j=0}^{k_0 - 1} \left(-\frac{h(x)}{a}\right)^j.$$

For any $\omega \in \mathbb{S}^{n-1}$ and $0 < r \leq \alpha\theta b^{1/(k_0 l)}$, by $0 \leq b \leq 1$,

$$\left|\frac{h(\omega)r^l}{a}\right| \leq b^{1/k_0}\alpha^l \left(\left|\frac{h(w)}{a}\right|\theta^l\right) \leq \alpha^l < 1,$$

which implies that

$$\begin{aligned} |\phi_\omega(r) - (P(r\omega) + q(rw))| &= |a|^{-1} \left|\left(1 + \frac{h(w)r^l}{a}\right)^{-1} - \sum_{j=0}^{k_0 - 1}\left(-\frac{h(w)r^l}{a}\right)^j\right| \\ &\lesssim |a|^{-1}\left|\frac{h(w)r^l}{a}\right|^{k_0} = |a|^{-1}\theta^{-k_0 l} r^{k_0 l}. \end{aligned}$$

Thus,

$$\left|\int_0^{\alpha\theta b^{1/(k_0 l)}}\left(e^{i\phi_\omega(r)}-e^{i(P(r\omega)+q(r\omega))}\right)\frac{\mathrm{d}r}{r}\right|\lesssim|a|^{-1}\theta^{-k_0 l}\int_0^{\alpha\theta b^{1/(k_0 l)}} r^{k_0 l-1}\mathrm{d}r\lesssim\alpha^{k_0 l}|a|^{-1}b\lesssim 1,$$

which immediately gives

$$\int_{\mathbb{S}^{n-1}}\Omega(\omega)\int_0^{\alpha\theta b^{1/(k_0 l)}}\left(e^{i\phi_\omega(r)}-e^{i(P(r\omega)+q(r\omega))}\right)\frac{\mathrm{d}r}{r}\mathrm{d}\sigma(\omega)=O(1). \tag{17}$$

By an inequality on page 334 of [10],

$$\left|\text{p.v.}\int_{|x|\le\alpha|a|^{1/l}b^{1/(k_0 l)}m_h^{-1/l}} e^{i(P(x)+q(x))}K(x)\mathrm{d}x\right|\le A,$$

i.e.

$$\int_{\mathbb{S}^{n-1}}\Omega(\omega)\int_0^{\alpha|a|^{1/l}b^{1/(k_0 l)}m_h^{-1/l}} e^{i(P(r\omega)+q(r\omega))}\frac{\mathrm{d}r}{r}\mathrm{d}\sigma(\omega)=O(1). \tag{18}$$

Trivially, we have

$$\left|\int_{\alpha|a|^{1/l}b^{1/(k_0 l)}m_h^{-1/l}}^{\alpha\theta b^{1/(k_0 l)}} e^{i(P(r\omega)+q(r\omega))}\frac{\mathrm{d}r}{r}\right|\lesssim\left|\ln\left(\frac{|h(\omega)|}{m_h}\right)\right|.$$

It follows from Theorem 5 that

$$\int_{\mathbb{S}^{n-1}}\Omega(\omega)\int_{\alpha|a|^{1/l}b^{1/(k_0 l)}m_h^{-1/l}}^{\alpha\theta b^{1/(k_0 l)}} e^{i(P(r\omega)+q(r\omega))}\frac{\mathrm{d}r}{r}\mathrm{d}\sigma(\omega)=O(1). \tag{19}$$

By (17)–(19), we obtain (16). This proves (6) for $\Omega\in L^\infty(\mathbb{S}^{n-1})$. By applying the argument used in the proof of Theorem 2, one then obtains (6) for $\Omega\in H^1(\mathbb{S}^{n-1})$. Details are omitted.

4. Nonvanishing of infinite order

Let M be a smooth k-dimensional submanifold of $\mathbb{R}^n$, $f: M\to\mathbb{R}$ be a C^∞ function and $p\in M$. We say that f does not vanish to infinite order at p if there is a chart (U_p,φ) around p such that $\varphi(p)=0$ and $D^\alpha(f\circ\varphi^{-1})(0)\neq 0$ for some $\alpha\in(\mathbb{N}\cup\{0\})^k$. For $r>0$, let $B_k(r)$ denote the open ball in $\mathbb{R}^k$ which is centered at the origin and has radius r. We begin with the following:

Lemma 10. *Let $k,m\in\mathbb{N}$, $x\in\mathbb{R}^k$, $y\in\mathbb{R}^m$ and $R>0$. Let $g(x,y)\in C^\infty(B_k(R)\times B_m(R))$ such that*

$$\frac{\partial^\alpha g(0,0)}{\partial x^\alpha}\neq 0 \tag{20}$$

holds for some $\alpha\in(\mathbb{N}\cup\{0\})^k$. Then there exists an $r\in(0,R/3)$ such that, for every $\delta\in\left(0,(\max\{|\alpha|,1\})^{-1}\right)$ and every C^∞ function $h:\mathbb{R}^k\to\mathbb{R}$ which is supported in $B_k(r)$,

$$\sup_{y\in B_m(r)}\int_{B_k(r)}|g(x,y)|^{-\delta}|h(x)|\mathrm{d}x<\infty. \tag{21}$$

Proof. If (20) holds with $|\alpha|=0$, i.e. $g(0,0)\neq 0$, then (21) follows trivially by continuity.

Now suppose that $g(0,0)=0$ and let

$$l=\min\left\{|\alpha|:\alpha\in(\mathbb{N}\cup\{0\})^k \text{ and } \frac{\partial^\alpha g(0,0)}{\partial x^\alpha}\neq 0\right\}. \tag{22}$$

By an argument on p. 317 of [10], we may assume that

$$\frac{\partial^l g(0,0)}{\partial x_k^l}\neq 0.$$

By (22) we also have

$$\frac{\partial^j g(0,0)}{\partial x_k^j}=0$$

for $j = 0, 1, \dots, l-1$. Let $\tilde{x} = (x_1, \dots, x_{k-1})$. By Malgrange preparation theorem ([6]), there exist $r \in (0, R/3)$, $\eta_0 > 0$, $a_0(\tilde{x}, y), \dots, a_{l-1}(\tilde{x}, y) \in C^\infty(B_{k-1}(r) \times B_m(r))$ and $c(x, y) \in C^\infty(B_k(r) \times B_m(r))$ such that, for all $(x, y) \in B_k(r) \times B_m(r)$,

$$g(x, y) = c(x, y)(x_k^l + a_{l-1}(\tilde{x}, y)x_k^{l-1} + \cdots + a_0(\tilde{x}, y))$$

and $|c(x, y)| \geq \eta_0$. Thus, for any $\delta \in (0, 1/l)$ and any C^∞ function $h(x)$ supported on $B_k(r)$,

$$\begin{aligned}&\sup_{y \in B_m(r)} \int_{B_k(r)} |g(x, y)|^{-\delta} |h(x)| \mathrm{d}x \\ &\qquad \lesssim \sup_{y \in B_m(r)} \int_{B_{k-1}(r)} \int_{|x_k| < r} (x_k^l + a_{l-1}(\tilde{x}, y)x_k^{l-1} + \cdots + a_0(\tilde{x}, y))^{-\delta} \mathrm{d}x_k \mathrm{d}\tilde{x} < \infty. \qquad \square\end{aligned}$$

Proof of Theorem 7. Let M be a compact smooth submanifold of $\mathbb{R}^n$ and U be an open subset of $\mathbb{R}^m$. Let $f \in C^\infty(M \times U)$ such that, for every $u \in U$, $f(\cdot, u)$ does not vanish to infinite order at any point in M. Suppose that W is a compact subset of U. By Lemma 10 and the compactness of M and W, there exist $\delta = \delta_{f,W} > 0$ and $C = C(M, n, m, f, W)$ such that

$$\sup_{u \in W} \int_M |f(y, u)|^{-\delta} \mathrm{d}\sigma(y) \leq C. \tag{23}$$

For any $\Omega \in L \log L(M)$ and $u \in W$, it follows from (23) that

$$\int_{\{y \in M : |\Omega(y)| < |f(y,u)|^{-\delta/2}\}} |\Omega(y)| |\log(|f(y, u)|)| \mathrm{d}\sigma(y) \lesssim \int_M |f(y, u)|^{-\delta} \mathrm{d}\sigma(y) \lesssim 1.$$

On the other hand, we have trivially that

$$\int_{\{y \in M : |\Omega(y)| \geq |f(y,u)|^{-\delta/2}\}} |\Omega(y)| |\log(|f(y, u)|)| \mathrm{d}\sigma(y) \lesssim \|\Omega\|_{L \log L(M)}.$$

Thus (9) holds and the proof of Theorem 7 is now complete. $\square$

Proof of Theorem 6. Let $m = \dim(\mathscr{P}_{n,d}|_{\mathbb{S}^{n-1}})$ and $p_1(x), \dots, p_m(x) \in \mathscr{P}_{n,d}$ such that $\{p_j|_{\mathbb{S}^{n-1}} : 1 \leq j \leq m\}$ forms a basis for $\mathscr{P}_{n,d}|_{\mathbb{S}^{n-1}}$. Define $f : \mathbb{S}^{n-1} \times (\mathbb{R}^m \backslash \{0\}) \to \mathbb{R}$ by

$$f(x, u) = \sum_{j=1}^m u_j p_j(x)$$

for $x \in \mathbb{S}^{n-1}$ and $u = (u_1, \dots, u_m) \in \mathbb{R}^m \backslash \{0\}$. For each $u \in \mathbb{R}^m \backslash \{0\}$, $f(\cdot, u)$ is not identically zero on $\mathbb{S}^{n-1}$ which, by real-analyticity, implies that it does not vanish to infinite order at any point in $\mathbb{S}^{n-1}$. By Theorem 7,

$$\sup_{u \in \mathbb{S}^{m-1}} \int_{\mathbb{S}^{n-1}} |\Omega(x)| |\log(|f(x, u)|)| \mathrm{d}\sigma(x) \leq C(1 + \|\Omega\|_{L \log L(\mathbb{S}^{n-1})}), \tag{24}$$

which implies that (8) holds when the norm is given by

$$\sum_{j=1}^m u_j p_j \big|_{\mathbb{S}^{n-1}} \to \left(\sum_{j=1}^m u_j^2 \right)^{1/2}.$$

Since any two norms on $\mathscr{P}_{n,d}\big|_{\mathbb{S}^{n-1}}$ are equivalent, Theorem 6 is proved. $\square$

Proof of Theorem 8. By assumption and Theorem 7,

$$\sup_{u \in W} \int_{\mathbb{S}^{n-1}} |\Omega(\omega)| |\log(|\psi(\omega, u)|)| \mathrm{d}\sigma(\omega) \leq C(1 + \|\Omega\|_{L \log L(\mathbb{S}^{n-1})}). \tag{25}$$

One can then adopt the arguments in the proof of Theorem 1 in [5], at times using (25) instead of (7), to finish the proof. Details are omitted. $\square$

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