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Harmonic analysis / *Analyse harmonique*

Equivalence of K-functionals and modulus of smoothness generated by a Dunkl type operator on the interval $(-1, 1)$

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Abstract. Our aim in this paper is to show that the modulus of smoothness and the K-functionals constructed from the Sobolev-type space corresponding to the Dunkl operator are equivalent on the interval $(-1, 1)$.

Keywords. Fourier–Dunkl series, Dunkl transform, generalized translation operator, K-functionals, modulus of smoothness.

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1. Introduction

As it is well-known, the modulus of smoothness generated by the standard translation is equivalent with the Peetre's K-functional, see e.g. [6, p. 171]. This property is extended to Dunkl translation by E. S. Belkina and S. S. Platonov (see [2]) and Bessel translation in [12]. In this paper, we prove the counterparts of results obtained in [2], i.e., we establish the equivalence between K-functionals and modulus of smoothness in the Dunkl context (on $(-1, 1)$) by using Fourier–Dunkl expansions introduced in [4], instead of Dunkl transform. The orthonormal system associated with this kind of series is a generalization of the trigonometric one (in particular, the periodicity is lost).

Hereinafter the symbol α stands for a real value such that $\alpha > -1$. We consider the Dunkl operator Λ_α associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$ given by

$$\Lambda_\alpha f(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x) + \frac{2\alpha+1}{x}\left(\frac{f(x) - f(-x)}{2}\right).$$

The initial value problem

$$\begin{cases} \Lambda_\alpha f(x) = i\lambda f(x), & \lambda \in \mathbb{R} \\ f(0) = 1 \end{cases}$$

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has a unique solution $E_\alpha(i\lambda.)$ (called the *Dunkl kernel*) given by:

$$E_\alpha(i\lambda x) = j_\alpha(\lambda x) + \frac{i\lambda x}{2(\alpha+1)} j_{\alpha+1}(\lambda x), \ \ x \in \mathbb{R}, \tag{1}$$

where j_α is the normalized Bessel function of the first kind defined by

$$j_\alpha(x) = 2^\alpha \Gamma(\alpha+1) \frac{J_\alpha(x)}{x^\alpha} \tag{2}$$

and J_α is the Bessel functions of the first kind of order α

$$J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{n=0}^{\infty} \frac{\left(\frac{ix}{2}\right)^{2n}}{n!\Gamma(n+\alpha+1)}.$$

From [15], for all $x \in \mathbb{R}$, we have

$$|E_\alpha(ix)| \leq 1 \quad \text{and} \quad |E'_\alpha(ix)| \leq 1. \tag{3}$$

Let $L^p\left((-1,1), d\mu_\alpha\right), p \geq 1$, denote the Lebesgue spaces on the interval $(-1,1)$ endowed with the norm

$$\|f\|_{\alpha,p} = \left(\int_{-1}^{1} |f(t)|^p d\mu_\alpha(x)\right)^{\frac{1}{p}}.$$

where $d\mu_\alpha(x) = \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^{-1} |x|^{2\alpha+1} dx$. The Dunkl transform is a generalization of the Fourier transform. It is defined for $f \in L^1\left((-1,1), d\mu_\alpha\right)$ by the identity (see [7,9])

$$\mathscr{F}_\alpha f(y) = \int_{\mathbb{R}} f(x) E_\alpha(-iyx) d\mu_\alpha(x), \quad y \in \mathbb{R}.$$

The Fourier transform corresponds with the case $\alpha = -1/2$ because $E_{-1/2}(ix) = e^{ix}$ and $d\mu_{-1/2}$ is, up to a multiplicative factor, the Lebesgue measure on $\mathbb{R}$.

2. Equivalence of K-functionals and modulus of smoothness generated by a Dunkl type operator on the interval $(-1,1)$.

Let $\{\lambda_n := \lambda_{\alpha+1,n}, n \in \mathbb{N}\}$ be the increasing sequence of positive zeros of $J_{\alpha+1}$. It is proved in [10] that

$$\lambda_n \leqslant n\pi + \alpha\pi/2 + \pi/4 \quad \text{for } \alpha > -1/2. \tag{4}$$

In [8] we find the following inequality

$$\lambda_n > \alpha + n\pi - \frac{\pi}{2} + \frac{3}{2}, \quad \alpha > -1, \quad n = 1,2,\ldots$$

then

$$\lambda_n > n, \quad \alpha > -1, \quad n = 1,2,\ldots. \tag{5}$$

When the range of α is fixed, like $-1 < \alpha \leqslant -\frac{1}{2}$ (see Schafheitlin in [18, p. 490]) and no (essential) restriction on n :

$$n\pi + \alpha\pi/2 + \pi/4 < \lambda_n < n\pi \quad \left(-1 < \alpha \leqslant -\frac{1}{2}\right). \tag{6}$$

Let $c_\alpha = \pi + \max\{0, \alpha\pi/2 + \pi/4\}$. Combining (4), (5), and (6) gives

$$n < \lambda_n < c_\alpha n \quad \text{for all } \alpha > -1. \tag{7}$$

The real-valued function $\Im E_\alpha(ix) = \frac{x}{2(\alpha+1)} j_{\alpha+1}(x)$ is odd and its zeros are $\{\lambda_n, n \in \mathbb{Z}\}$ where $\lambda_{-n} = -\lambda_n$ and $\lambda_0 = 0$.

Theorem 1 in [4] establishes that $\{E_\alpha(i\lambda_n x)\}_{n\in\mathbb{Z}}$ is a complete orthogonal system in $L^2\left((-1,1), d\mu_\alpha\right)$. That is to say

$$\int_{-1}^{1} E_\alpha(i\lambda_n x) \overline{E_\alpha(i\lambda_m x)} d\mu_\alpha(x) = \|E_\alpha(i\lambda_n \cdot)\|^2_{2,\alpha} \delta_{nm}.$$

For each appropriate function f on $(-1,1)$, we define its Fourier series related to the system $\{E_\alpha(i\lambda_n x)\}_{n\in\mathbb{Z}}$, which are called Fourier–Dunkl series, as

$$f \sim \sum_{n\in\mathbb{Z}} c_n(f) E_\alpha(i\lambda_n x)\theta_n, \quad c_n(f) = \int_{-1}^{1} f(y)\overline{E_\alpha(i\lambda_n y)}\mathrm{d}\mu_\alpha(y).$$

and

$$\theta_n = \|E_\alpha(i\lambda_n\cdot)\|_{2,\alpha}^{-2}.$$

We notice that Ó. Ciaurri and his collaborators have studied in [3] the weighted norm convergence of the Fourier–Dunkl series and proved in [5] an uncertainty inequality associated to this system. From [4, Lemma 1] we have

$$\theta_n = \frac{2^\alpha \Gamma(\alpha+1)}{|j_\alpha(\lambda_n)|^2}, \quad n\in\mathbb{Z}\setminus\{0\} \quad (\text{we recall that } \lambda_n := \lambda_{\alpha+1,n}) \tag{8}$$

and $\theta_0 = 2^{\alpha+1}\Gamma(\alpha+2)$. The following asymptotic formulas hold for the Bessel function $J_\alpha(u)$ ([18, p. 490]):

$$J_\alpha(u) = \sqrt{\frac{2}{\pi u}}\left[\cos\left(u - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{u}\right)\right], u \to \infty. \tag{9}$$

Combining (8) and (9) gives

$$\theta_n \sim \pi|\lambda_n|^{2\alpha+1}, |n| \to \infty. \tag{10}$$

The sequence $\{c_n(f), n\in\mathbb{Z}\}$ is called the discrete Fourier–Dunkl transform of f. We define the weighted spaces $l^p(\mathbb{Z},(\theta_n)_{n\in\mathbb{Z}})$ by

$$l^p(\mathbb{Z},(\theta_n)_{n\in\mathbb{Z}}) = \left\{(x_n)_{n\in\mathbb{Z}} : \mathbb{Z} \longrightarrow \mathbb{C} : \left(\sum_{n\in\mathbb{Z}} |x_n|^p \theta_n\right)^{1/p} < +\infty\right\}.$$

If $f \in L^2((-1,1),\mathrm{d}\mu_\alpha)$, then the sequence $\{c_n(f), n\in\mathbb{Z}\}$ belongs to $l^2(\mathbb{Z},(\theta_n)_{n\in\mathbb{Z}})$ and we have

$$\|f\|_{2,\alpha} = \sqrt{\sum_{n=-\infty}^{\infty} \left|c_n(f)\right|^2 \theta_n}. \tag{11}$$

The Dunkl translation operator of a function f is defined for all $h\in\mathbb{R}$ by (see [1, 13])

$$\tau_y^\alpha f(x) = \sum_{n=0}^{\infty} \Lambda_\alpha^n f(x)\frac{y^n}{\gamma_{n,\alpha}}, \quad \alpha > -1 \tag{12}$$

where Λ_α^0 is the identity operator, $\Lambda_\alpha^{n+1} = \Lambda_\alpha\left(\Lambda_\alpha^n\right)$, and

$$\gamma_{n,\alpha} = \begin{cases} 2^{2k} k!(\alpha+1)_k & \text{if } n = 2k, \\ 2^{2k+1} k!(\alpha+1)_{k+1} & \text{if } n = 2k+1. \end{cases}$$

The definition (12) is valid only for C^∞-functions, and assuming also that the series on the right is convergent. In particular, this can be guaranteed when f is a polynomial, because the operator Λ_α applied to a polynomial of degree k generates a polynomial of degree $k-1$, so the series (12) has only a finite number of nonzero summands. In the case $\alpha = -1/2$, the translation $\tau_y^\alpha f$ is just the Taylor expansion of a function f around a fixed point x, that is,

$$f(x+y) = \sum_{n=0}^{\infty} f^{(n)}(x)\frac{y^n}{n!}.$$

Some properties of the translation operator, including an integral expression, can be found in [11, 14, 16, 17]. For our purposes, we only need the identity [13, formula (4.2.2)]

$$\tau_h^\alpha(E_\alpha(i\lambda x)) = E_\alpha(i\lambda h)\,E_\alpha(i\lambda x) \text{ for all } x, h\in\mathbb{R}. \tag{13}$$

That resembles the classical

$$e^{\lambda(h+x)} = e^{\lambda h}e^{\lambda x}.$$

The scalar product in the Hilbert space $L^2\left((-1,1),\mathrm{d}\mu_\alpha\right)$ obeys the formula

$$(f,g)=\int_{-1}^{1} f(x)\overline{g(x)}\mathrm{d}\mu_\alpha(x).$$

We denote by $\mathscr{E}=\mathscr{E}((-1,1))$, the set of all infinitely differentiable with compact support included in the interval $(-1,1)$. By the partial integration one can verify the relation

$$\int_{-1}^{1}\Lambda_\alpha f(x)\overline{g(x)}\mathrm{d}\mu_\alpha(x)=\frac{f(1)\overline{g(1)}-f(-1)\overline{g(-1)}}{2^{\alpha+1}\Gamma(\alpha+1)}-\int_{-1}^{1} f(x)\overline{\Lambda_\alpha g(x)}\mathrm{d}\mu_\alpha(x). \tag{14}$$

Then for any functions $f,g\in\mathscr{E}$, we have

$$(\Lambda_\alpha f,g)=-(f,\Lambda_\alpha g). \tag{15}$$

As usual, we endow the space $\mathscr{E}$ with a topology; this turns it into a topological vector space. Let $\mathscr{E}'$ stand for the set of generalized functions, i.e., linear continuous functionals on the space $\mathscr{E}$. We denote the value of a functional $f\in\mathscr{E}'$ on a function $\varphi\in\mathscr{E}$ by $\langle f,\varphi\rangle$.The space $L^2\left((-1,1),\mathrm{d}\mu_\alpha\right)$ is embedded into $\mathscr{E}'$, provided that for $f\in L^2\left((-1,1),\mathrm{d}\mu_\alpha\right)$ and $\varphi\in\mathscr{E}'$ we put

$$\langle f,\varphi\rangle=\int_{-1}^{1} f(x)\varphi(x)\mathrm{d}\mu_\alpha(x)$$

We can extend the action of Dunkl operator Λ_α onto the space of generalized functions $\mathscr{E}'$, putting

$$\langle\Lambda_\alpha f,\varphi\rangle=-\langle f,\Lambda_\alpha\varphi\rangle, f\in\mathscr{E}',\varphi\in\mathscr{E}.$$

In particular, the action of the operator $\Lambda_\alpha f$ is defined for any function $f\in L^2\left((-1,1),\mathrm{d}\mu_\alpha\right)$, but, generally speaking, $\Lambda_\alpha f$ is a generalized function.

Analogously to Λ_α, we can extend the operator τ_h^α by continuity on the whole space $L^2\left((-1,1),\mathrm{d}\mu_\alpha\right)$. Indeed, Let P be a vector space generated by the system $\{E_\alpha(i\lambda_n x)\}_{n\in\mathbb{Z}}$ and $f\in P$, then f can be written as $f(x)=\sum_{n=-m}^{m}c_nE_\alpha(i\lambda_n x)\theta_n$. Using (3) and (13) we check easily that

$$\|\tau_h^\alpha f\|_{2,\alpha}\le\|f\|_{2,\alpha}. \tag{16}$$

As P is a dense subspace of $L^2\left((-1,1),\mathrm{d}\mu_\alpha\right)$, it follows from (16) that τ_h^α can be extended by continuity to a bounded operator in $L^2\left((-1,1),\mathrm{d}\mu_\alpha\right)$. The extended operator is also denoted by τ_h^α; inequality (16) remains valid for it.

For every function $f\in L^2\left((-1,1),\mathrm{d}\mu_\alpha\right)$ we define the differences $\Delta_h^m f$ of order, $m\in\mathbb{N}=\{1,2,3,\ldots\}$, with step $h\in\mathbb{R}$ by the formula $\Delta_h^1 f(t)=\Delta_h f(t)=(\tau_h^\alpha-I)f(t)$, where I is the identity operator in $L^2\left((-1,1),\mathrm{d}\mu_\alpha\right)$ and for $m>1$

$$\Delta_h^m f(t)=\Delta_h(\Delta_h^{m-1}f(t))=(\tau_h^\alpha-I)^m f(t)=\sum_{i=0}^{m}(-1)^{m-1}\binom{m}{i}(\tau_h^\alpha)^i f(t),$$

where

$$(\tau_h^\alpha)^0 f(t)=f(t),\quad (\tau_h^\alpha)^i f(t)=\tau_h^\alpha((\tau_h^\alpha)^{i-1}f(t)),\quad i=1,2,\ldots,m.$$

The moduli of smoothness generated by general translations are defined as follows.

$$\omega_m(f,\delta)_{2,\alpha}:=\sup_{0<h\le\delta}\left\|\Delta_h^m f\right\|_{2,\alpha},\quad \delta>0,\quad f\in L^2\left((-1,1),\mathrm{d}\mu_\alpha\right).$$

Let $W_{2,\alpha}^m$ be the Sobolev space constructed by the operator Λ_α, i.e.,

$$W_{2,\alpha}^m:=\left\{f\in L^2\left((-1,1),\mathrm{d}\mu_\alpha\right):\Lambda_\alpha^j f\in L^2\left((-1,1),\mathrm{d}\mu_\alpha\right), j=1,2,\ldots,m\right\}.$$

Then the corresponding K-functional is

$$K(f,t,W_{2,\alpha}^m):=\inf\left\{\left\|f-g\right\|_{2,\alpha}+t\left\|\Lambda_\alpha^m g\right\|_{2,\alpha}:g\in W_{2,\alpha}^m\right\}$$

where $f\in L^2\left((-1,1),\mathrm{d}\mu_\alpha\right)$ and $t>0$. For brevity, we denote

$$K_m(f,t)_{2,\alpha}:=K(f,t,W_{2,\alpha}^m).$$

The following theorem establishes an equivalence between the modulus of smoothness and the K-functional. It is analogous to the theorem on the equivalence between the modulus of smoothness and the K-functional in classical approximation theory.

Theorem 1. *One can find positive numbers $C_1 = C_1(m,\alpha)$ and $C_2 = C_2(m,\alpha)$ which satisfy the inequality*

$$C_1\omega_m(f,\delta)_{2,\alpha} \leq K_m(f,\delta^m)_{2,\alpha} \leq C_2\omega_m(f,\delta)_{2,\alpha}$$

where $f \in L^2\left((-1,1), \mathrm{d}\mu_\alpha\right)$ and $\delta > 0$.

Using that $E_{-1/2}(ix) = \mathrm{e}^{ix}$, it is easy to check that Theorem 1 reduces to an equivalence between the modulus of smoothness and the K-functional on the basis of Fourier series and usual translation.

When we consider real even and odd functions the Fourier–Dunkl series can be seen as Fourier–Dini and Fourier–Bessel series respectively. From this fact, applying Theorem 1 to even or odd functions, we can deduce analogs of Theorem 1 for these kinds of series.

3. Proof

Some technical facts are needed to prove our result. They are included in the next lemmas.

Lemma 2.

(i) *For $x \in \mathbb{R}$ the following inequality is fulfilled*

$$|1 - E_\alpha(ix)| \leq |x|.$$

(ii) *For $|x| \geq 1$, there exists a certain constant $c > 0$ which depends only on α such that*

$$|1 - E_\alpha(ix)| \geq c.$$

Proof. (i). It follows by the estimates provided in (3) together with standard application of Lagrange's mean value theorem.

(ii). The asymptotic formulas (9) imply that $j_\alpha(x) \to 0$ as $|x| \to \infty$. Consequently, a number $\eta > 0$ exists such that with $|x| \geq \eta$ the inequality $\left|j_\alpha(x)\right| \leq 1/2$ is true. Let

$$m = \min_{1\leq|x|\leq\eta} \left|1 - j_\alpha(x)\right|.$$

With $|x| \geq 1$ we get the inequality $\left|1 - j_\alpha(x)\right| \geq c$, where $c = \min\{m, 1/2\}$. Taking only the reel part of $1 - E_\alpha(i\lambda_n x)$ gives

$$c \leq \left|1 - j_\alpha(x)\right| \leq |1 - E_\alpha(i\lambda_n x)|.$$

□

Proposition 3. *Let $x \in (-1,1)$ and $h \in \mathbb{R}$. If $f \in L^2\left((-1,1), \mathrm{d}\mu_\alpha\right)$ with*

$$f(x) = \sum_{n\in\mathbb{Z}} c_n(f) E_\alpha(i\lambda_n x)\theta_n,$$

then

$$\tau_h^\alpha f(x) = \sum_{n\in\mathbb{Z}} c_n(f) E_\alpha(i\lambda_n h) E_\alpha(i\lambda_n x)\theta_n. \tag{17}$$

Proof. By product formula (13) of τ_h^α, we have

$$\tau_h^\alpha E_\alpha(i\lambda_n x) = E_\alpha(i\lambda_n h) E_\alpha(i\lambda_n x).$$

So, for any Dunkl-polynomial function

$$\mathcal{Q}_N(x) = \sum_{n=-N}^{N} c_n(f) E_\alpha(i\lambda_n x)\theta_n,$$

(this can be considered as a generalization of the trigonometric polynomial in the classical case $\alpha = -1/2$) since τ_h^α is linear, we have

$$\tau_h^\alpha \mathscr{Q}_N(x) = \sum_{n=-N}^{N} c_n(f) E_\alpha(i\lambda_n h) E_\alpha(i\lambda_n x)\theta_n. \tag{18}$$

By using the fact that τ_h^α is extended to a continuous linear operator in $L^2\left((-1,1), \mathrm{d}\mu_\alpha\right)$ and the set of all polynomials $\mathscr{Q}_N(x)$ is everywhere dense in $L^2\left((-1,1), \mathrm{d}\mu_\alpha\right)$, passage to the limit in (18) gives the desired equality. □

Corollary 4. *Let $f \in L^2\left((-1,1), \mathrm{d}\mu_\alpha\right)$ and $h \in \mathbb{R}$, then*

$$\left\|\Delta_h^m f\right\|_{2,\alpha} \leq 2^m \|f\|_{2,\alpha}. \tag{19}$$

Proof. Let $h \in \mathbb{R}$. According to the formula (17), we obtain

$$\begin{aligned} c_n\left(\Delta_h^1 f\right) &= c_n\left(\tau_h^\alpha f\right) - c_n(f) \\ &= \left(E_\alpha(i\lambda_n h) - 1\right) c_n(f). \end{aligned}$$

Using induction with respect to m, we have

$$c_n\left(\Delta_h^m f\right) = \left(E_\alpha(i\lambda_n h) - 1\right)^m c_n(f). \tag{20}$$

Then

$$c_n\left(\Delta_h^m f\right) \leq 2^m c_n(f).$$

□

Lemma 5. *If $f \in \mathscr{E}$, we get*

$$c_n(\Lambda_\alpha f) = i\lambda_n c_n(f) \tag{21}$$

for all $n \in \mathbb{Z}$.

Proof. Let $f \in \mathscr{E}$, we put $c_n(f) = \int_{-1}^{1} f(y)\overline{E_\alpha\left(i\lambda_n y\right)}\mathrm{d}\mu_\alpha(y)$. It follows from (15) that

$$\begin{aligned} c_n(\Lambda_\alpha f) &= \int_{-1}^{1} \Lambda_\alpha f(y) E_\alpha\left(-i\lambda_n y\right) \mathrm{d}\mu_\alpha(y), \\ &= -\int_{-1}^{1} f(y) \Lambda_\alpha E_\alpha\left(-i\lambda_n y\right) \mathrm{d}\mu_\alpha(y), \\ &= i\lambda_n \int_{-1}^{1} f(y) E_\alpha\left(-i\lambda_n y\right) \mathrm{d}\mu_\alpha(y), \\ &= i\lambda_n c_n(f). \end{aligned}$$

Then the equality (21) is valid in $\mathscr{E}$. □

Remark 6. Using induction with respect to m and Lemma 5, we can see that for all $f \in W_{2,\alpha}^m$

$$c_n(\Lambda_\alpha^m f) = (i\lambda_n)^m c_n(f) \tag{22}$$

for all $n \in \mathbb{Z}$ and $m = 0, 1, 2, \ldots$.

Lemma 7. *Assume that $\delta > 0$ and $f \in W_{2,\alpha}^m$. The following inequality is true:*

$$\omega_m(f, \delta)_{2,\alpha} \leq \delta^m \left\|\Lambda^m f\right\|_{2,\alpha}.$$

Proof. Let $h \in (0, \delta]$. According to the formula (20), we have

$$c_n\left(\Delta_h^m f\right) = \left(E_\alpha(i\lambda_n h) - 1\right)^m c_n(f).$$

It follows from the Parseval identity (11) and Lemma 2 that

$$\begin{aligned} \left\|\Delta_h^m f\right\|_{2,\alpha}^2 = \sum_{n\in\mathbb{Z}} \left(1 - E_\alpha(i\lambda_n h)\right)^{2m} \left|c_n(f)\right|^2 \theta_n &\leq h^{2m} \sum_{n\in\mathbb{Z}} \left(\frac{1 - E_\alpha(i\lambda_n h)}{\lambda_n h}\right)^{2m} \left|\lambda_n^m c_n(f)\right|^2 \theta_n \\ &\leq h^{2m} \left\|\Lambda_\alpha^m f\right\|_{2,\alpha}^2 \leq \delta^{2m} \left\|\Lambda_\alpha^m f\right\|_{2,\alpha}^2. \end{aligned}$$

Calculating the supremum with respect to all $h \in (0,\delta]$, we obtain $\omega_m(f,\delta)_{2,\alpha} \le \delta^m \left\|\Lambda_\alpha^m f\right\|_{2,\alpha}$. □

Definition 8. *For any function $f \in L^2\left((-1,1), \mathrm{d}\mu_\alpha\right)$ and any number $\sigma > 0$, we define the function*

$$\mathscr{P}_\sigma(f)(t) := \sum_{n\in\mathbb{Z}} \mathscr{X}_\sigma(\lambda_n) c_n(f) E_\alpha\left(i\lambda_n t\right)\theta_n$$

where $\mathscr{X}_\sigma(n)$ is the characteristic function defined by

$$\mathscr{X}_\sigma(\lambda_n) := \begin{cases} 1 & \text{if } |\lambda_n| \le \sigma, \\ 0 & \text{if } |\lambda_n| > \sigma. \end{cases}$$

Proposition 9. *Let $\sigma > 0$. For any function $f \in L^2\left((-1,1), \mathrm{d}\mu_\alpha\right)$ the following inequality is true:*

$$\|f - \mathscr{P}_\sigma(f)\|_{2,\alpha} \le C\|\Delta^m_{\frac{1}{\sigma}} f\|_{2,\alpha}.$$

Proof. Using the Parseval equality, we obtain

$$\|f - \mathscr{P}_\sigma(f)\|^2_{2,\alpha} = \sum_{n\in\mathbb{Z}} (1 - \mathscr{X}_\sigma(\lambda_n))|c_n(f)|^2\theta_n, \tag{23}$$

$$= \sum_{n\in\mathbb{Z}} \frac{(1-\mathscr{X}_\sigma(\lambda_n))}{\left(1 - E_\alpha\left(i\lambda_n\sigma^{-1}\right)\right)^{2m}} \left(1 - E_\alpha\left(i\lambda_n\sigma^{-1}\right)\right)^{2m} |c_n(f)|^2\theta_n. \tag{24}$$

Note that $C_1 \le |1 - E_\alpha\left(ix\right)|$ with $|x| \ge 1$ (see Lemma 2). Hence

$$\sup_{n\in\mathbb{Z}} \frac{1-\mathscr{X}_\sigma(\lambda_n)}{1 - E_\alpha\left(i\lambda_n\sigma^{-1}\right)} \le \sup_{|x|\ge1} \frac{1}{1 - E_\alpha\left(ix\right)} \le \frac{1}{C_1}. \tag{25}$$

Relations (24) and (25) give

$$\|f - \mathscr{P}_\sigma(f)\|_{2,\alpha} \le C\|\Delta^m_{\frac{1}{\sigma}} f\|_{2,\alpha}$$

where $C = \dfrac{1}{C_1^m}$. □

Corollary 10. *For any function $f \in L^2\left((-1,1), \mathrm{d}\mu_\alpha\right)$ the following inequality is true:*

$$\|f - \mathscr{P}_\sigma(f)\|_{2,\alpha} \le C\omega_m\left(f, \frac{1}{\sigma}\right)_{2,\alpha}.$$

Proposition 11. *Suppose that $f \in L^2\left((-1,1), \mathrm{d}\mu_\alpha\right)$, $m \in \mathbb{N}$, and $\sigma > 0$. Then we have*

$$\|\Lambda_\alpha^m \mathscr{P}_\sigma(f)\|_{2,\alpha} \le C_3\sigma^m\|\Delta^m_{\frac{1}{\sigma}} f\|_{2,\alpha}.$$

Proof. Using the Parseval equality, we obtain

$$\|\Lambda_\alpha^m \mathscr{P}_\sigma(f)\|^2_{2,\alpha} = \sum_{n\in\mathbb{Z}} \lambda_n^{2m}\mathscr{X}_\sigma(\lambda_n)|c_n(f)|^2\theta_n, \tag{26}$$

$$= \sigma^{2m} \sum_{n\in\mathbb{Z}} \frac{\mathscr{X}_\sigma(\lambda_n)(\lambda_n\sigma^{-1})^{2m}}{\left(1 - E_\alpha\left(i\lambda_n\sigma^{-1}\right)\right)^{2m}} \left(1 - E_\alpha\left(i\lambda_n\sigma^{-1}\right)\right)^{2m} |c_n(f)|^2\theta_n. \tag{27}$$

Note that

$$\sup_{n\in\mathbb{Z}} \left|\frac{\mathscr{X}_\sigma(\lambda_n)\lambda_n\sigma^{-1}}{1 - E_\alpha\left(i\lambda_n\sigma^{-1}\right)}\right| \le \sup_{|x|\le1} \left|\frac{x}{1 - E_\alpha\left(ix\right)}\right| = C_2. \tag{28}$$

Then formula (27) yields

$$\|\Lambda_\alpha^m \mathscr{P}_\sigma(f)\|_{2,\alpha} \le C_3\sigma^m\|\Delta^m_{\frac{1}{\sigma}} f\|_{2,\alpha}$$

where $C_3 = C_2^m$. □

Corollary 12. *For any function $f \in L^2\left((-1,1), \mathrm{d}\mu_\alpha\right)$ the following inequality is true:*

$$\|\Lambda_\alpha^m \mathscr{P}_\sigma(f)\|_{2,\alpha} \le C_3\sigma^m\omega_m\left(f, \frac{1}{\sigma}\right)_{2,\alpha}.$$

Proof of Theorem 1.

Proof of the inequality $2^{-m}\omega_m(f,\delta)_{2,\alpha} \le K_m(f,\delta^m)_{2,\alpha}$**.**

Let $h \in (0,\delta]$, $g \in W^m_{2,\alpha}$. Using Lemma 7 and inequality (19), we obtain

$$\begin{aligned}\left\|\Delta^m_h f\right\|_{2,\alpha} \le \left\|\Delta^m_h f - g\right\|_{2,\alpha} + \left\|\Delta^m_h g\right\|_{2,\alpha} &\le 2^m \left\|f-g\right\|_{2,\alpha} + h^m \left\|\Lambda^m_\alpha f\right\|_{2,\alpha}\\ &\le 2^m \left(\left\|f-g\right\|_{2,\alpha} + h^m \left\|\Lambda^m_\alpha f\right\|_{2,\alpha}\right).\end{aligned}$$

Calculating the supremum with respect to $h \in (0,\delta]$ and the infimum with respect to all possible functions $g \in W^m_{2,\alpha}$ we obtain

$$2^{-m}\omega_m(f,\delta)_{2,\alpha} \le K_m(f,\delta^m)_{2,\alpha}.$$

Proof of the inequality $K_m(f,\delta^m)_{2,\alpha} \le C_1\omega_m(f,\delta)_{2,\alpha}$**.**

Since $\mathscr{P}_\sigma(f) \in W^m_{2,\alpha}$ by the definition of a K-functional we have

$$K_m(f,\delta^m)_{2,\alpha} \le \left\|f - \mathscr{P}_\sigma(f)\right\|_{2,\alpha} + \delta^m \left\|\Lambda^m_\alpha \mathscr{P}_\sigma(f)\right\|_{2,\alpha}.$$

Using Corollaries 10 and 12, this gives

$$K_m(f,\delta^m)_{2,\alpha} \le \omega_m\left(f,\frac{1}{\sigma}\right)_{2,\alpha} + C_3(\delta\sigma)^n\omega_m\left(f,\frac{1}{\sigma}\right)_{2,\alpha}.$$

Since σ is an arbitrary positive value, choosing $\sigma = 1/\delta$, we obtain the inequality. □

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