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Integral representation of vertical operators on the Bergman space over the upper half-plane

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Abstract. Let Π denote the upper half-plane. In this article, we prove that every vertical operator on the Bergman space $\mathscr{A}^2(\Pi)$ over the upper half-plane can be uniquely represented as an integral operator of the form

$$\left(S_\varphi f\right)(z) = \int_\Pi f(w)\varphi(z-\overline{w})d\mu(w),\ \forall\ f \in \mathscr{A}^2(\Pi),\ z \in \Pi,$$

where φ is an analytic function on Π given by

$$\varphi(z) = \int_{\mathbb{R}_+} \xi\sigma(\xi)e^{iz\xi}d\xi,\ \forall\ z \in \Pi$$

for some $\sigma \in L^\infty(\mathbb{R}_+)$. Here $d\mu(w)$ is the Lebesgue measure on Π. Later on, with the help of above integral representation, we obtain various operator theoretic properties of the vertical operators.

Also, we give integral representation of the form S_φ for all the operators in the C^*-algebra generated by Toeplitz operators $T_{\mathbf{a}}$ with vertical symbols $\mathbf{a} \in L^\infty(\Pi)$.

Keywords. Bergman space, multiplication operator, reducing subspace, Toeplitz operator.

Mathematical subject classification (2010). 30H20, 47A15, 47B35, 47G10.

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1. Introduction

This paper is devoted to the integral representation of vertical operators on the Bergman space over the upper half-plane.

Let $\Pi = \{z = x + iy \in \mathbb{C} : y > 0\}$ be the upper half-plane, and let $d\mu(z) = dxdy$ be the standard Lebesgue plane measure on Π. The Bergman space $\mathscr{A}^2(\Pi)$ is the closed subspace of $L^2(\Pi, d\mu)$ which consists of all funtions analytic in Π. It is well known that $\mathscr{A}^2(\Pi)$ is a reproducing kernel Hilbert space with the reproducing kernel given by

$$K_{\Pi,w}(z) = -\frac{1}{\pi(z-\overline{w})^2}, \forall\, z, w \in \Pi.$$

Let $\mathscr{B}(\mathscr{A}^2(\Pi))$ denote the collection of all bounded linear operators on $\mathscr{A}^2(\Pi)$. For every $h \in \mathbb{R}$, let $H_h : \mathscr{A}^2(\Pi) \to \mathscr{A}^2(\Pi)$ be the horizontal translation operator defined by

$$\left(H_h f\right)(z) = f(z-h), \quad \forall\, f \in \mathscr{A}^2(\Pi),\ z \in \Pi.$$

The operator H_h is unitary on $\mathscr{A}^2(\Pi)$ for all $h \in \mathbb{R}$. An operator $T \in \mathscr{B}(\mathscr{A}^2(\Pi))$ is said to be vertical (or horizontal translation invariant) if

$$TH_h = H_h T, \quad \forall\, h \in \mathbb{R}.$$

As $\mathscr{A}^2(\Pi)$ is a reproducing kernel Hilbert space, every operator $T \in \mathscr{B}(\mathscr{A}^2(\Pi))$ can be uniquely written as an integral operator of the form

$$(Tf)(z) = \int_\Pi f(w) A_T(z,\overline{w})\, d\mu(w), \quad z \in \Pi, \tag{1}$$

where $A_T(z,\overline{w}) := \overline{(T^* K_{\Pi,z})(w)} = \overline{\langle T^* K_{\Pi,z}, K_{\Pi,w}\rangle_{\mathscr{A}^2}} = \overline{\langle K_{\Pi,z}, TK_{\Pi,w}\rangle_{\mathscr{A}^2}} =: \overline{A_{T^*}(w,\overline{z})}$. It can be easily seen that $A_T(\cdot,\overline{(\cdot)})$ is defined on $\Pi \times \Pi$ and $A_T(\cdot,\overline{w}),\ \overline{A_T(z,\overline{(\cdot)})} \in \mathscr{A}^2(\Pi)$. It is now natural to ask the following question:

Question. *Characterize all the functions* $A(\cdot,\overline{(\cdot)})$ *on* $\Pi \times \Pi$ *with* $A(\cdot,\overline{w}),\ \overline{A(z,\overline{(\cdot)})} \in \mathscr{A}^2(\Pi)$ *for all* $z, w \in \Pi$ *such that the integral operator*

$$\left(T_A f\right)(z) = \int_\Pi f(w) A(z,\overline{w})\, d\mu(w), \quad z \in \Pi,$$

is bounded on $\mathscr{A}^2(\Pi)$.

In the present article, we consider the following class of integral operators:

For a function φ on the upper half-plane such that $\varphi((\cdot)-\overline{w}),\ \overline{\varphi(z-\overline{(\cdot)})} \in \mathscr{A}^2(\Pi)$ for each $z, w \in \Pi$, we formally define an integral operator $S_\varphi : \mathscr{A}^2(\Pi) \to \mathscr{A}^2(\Pi)$ by

$$\left(S_\varphi f\right)(z) = \frac{1}{\pi}\int_\Pi f(w)\varphi(z-\overline{w})\, d\mu(w), \quad z \in \Pi,\ f \in \mathscr{A}^2(\Pi). \tag{2}$$

By Cauchy–Schwarz inequality, we have $|(S_\varphi f)(z)| \le \|f\|_{\mathscr{A}^2}\ \|\overline{\varphi(z-\overline{(\cdot)})}\|_{\mathscr{A}^2}$ for all $f \in \mathscr{A}^2(\Pi)$ and $z \in \mathbb{C}$. Also, $(S_\varphi K_{\Pi,p})(\cdot) = \varphi((\cdot)-\overline{p}) \in \mathscr{A}^2(\Pi)$ for all $p \in \Pi$. As $\operatorname{span}\{K_{\Pi,p} : p \in \Pi\}$ is dense in $\mathscr{A}^2(\Pi)$, S_φ is densely defined on $\mathscr{A}^2(\Pi)$. In Section 2, we recall some preliminaries which will be useful throughout the article. In Section 3, we characterize the symbol φ so that the operator given by (2) is bounded on $\mathscr{A}^2(\Pi)$. Indeed, we prove the following result on $\mathscr{A}^2(\Pi)$.

Theorem 1 (Main Theorem). *Let* φ *be a function on* Π *such that* $\varphi((\cdot)-\overline{w}),\ \overline{\varphi(z-\overline{(\cdot)})} \in \mathscr{A}^2(\Pi)$ *for each* $z, w \in \Pi$. *Then the integral operator* S_φ *defined by* (2) *is bounded on* $\mathscr{A}^2(\Pi)$ *if and only if there exists* $\sigma \in L^\infty(\mathbb{R}_+)$ *such that*

$$\varphi(z) = \int_{\mathbb{R}_+} \xi\sigma(\xi) e^{iz\xi}\, d\xi,\ z \in \Pi. \tag{3}$$

Moreover, we have that

$$\left\|S_\varphi\right\|_{\mathscr{A}^2 \to \mathscr{A}^2} = \|\sigma\|_{L^\infty(\mathbb{R}_+)}.$$

Thus, we answer the *Question* for the kernels of the form $\pi^{-1}\varphi(z-\overline{w})$, where φ is a function on Π with $\varphi((\cdot)-\overline{w})$, $\varphi(z-\overline{(\cdot)})\in\mathscr{A}^2(\Pi)$ for each $z,w\in\Pi$. As a consequence of Theorem 1, we get that every vertical operator can be uniquely represented as an integral operator of the form (2) and vice-versa. Thus, the collection

$$\left\{S_\varphi\in\mathscr{B}\left(\mathscr{A}^2(\Pi)\right):\exists\,\sigma\in L^\infty(\mathbb{R}_+)\quad\text{and}\quad\varphi(z)=\int_{\mathbb{R}_+}\xi\sigma(\xi)e^{iz\xi}d\xi,\quad\forall\,z\in\Pi\right\}$$

gives all vertical operators in $\mathscr{B}(\mathscr{A}^2(\Pi))$. Also, we obtain various operator theoretic properties for the vertical operators such as compactness, normality, C^*-algebra properties, etc..

In mathematics, Toeplitz operators are one of the widely studied operators on holomorphic function spaces (Hardy space, Bergman space, Fock space, etc.). For a better understanding, these operators are studied by restricting the defining symbols to a particular class (For example, see [6, 7, 10–14, 17]). In [14], C^*-algebra generated by Toeplitz operators on $\mathscr{A}^2(\Pi)$ with vertical symbols from $L^\infty(\Pi)$ is described. As every Toeplitz operator $T_{\mathbf{a}}$ with vertical symbol $\mathbf{a}\in L^\infty(\Pi)$ is a vertical operator on $\mathscr{A}^2(\Pi)$, in Section 4, we represent $T_{\mathbf{a}}$ uniquely in the form (2) and explicitly give the operators in the C^*-algebra generated by Toeplitz operators with vertical symbols.

2. Notations and definitions

Let $\mathscr{H}$ be a Hilbert space and $\mathscr{B}(\mathscr{H})$ be the collection of all bounded operators on $\mathscr{H}$. Let $T\in\mathscr{B}(\mathscr{H})$, then the spectrum of T is defined by $\sigma(T)=\{\lambda\in\mathbb{C}:(T-\lambda I)^{-1}\notin\mathscr{B}(\mathscr{H})\}$ and the point spectrum of T is given by $\sigma_p(T)=\{\lambda\in\sigma(T):(T-\lambda I)\text{ is not injective}\}$. A number $\lambda\in\sigma(T)$ is an approximate eigenvalue of T if there exists a sequence (x_n) of unit vectors such that $(T-\lambda I)x_n\to 0$ as $n\to\infty$. The approximate point spectrum of T, denoted by $\sigma_a(T)$, consists of all approximate eigenvalues of T. Clearly, $\sigma_p(T)\subseteq\sigma_a(T)$. Let $\operatorname{ran}(T)=\{Tx:x\in\mathscr{H}\}$ and $\ker(T)=\{x\in X:Tx=0\}$. An operator $T\in\mathscr{B}(\mathscr{H})$ is said to be Fredholm if

(1) $\operatorname{ran}(T)$ is closed;
(2) $\ker(T)$ and $\ker(T^*)$ are finite dimensional.

The essential spectrum of T is defined by $\sigma_e(T)=\{\lambda\in\mathbb{C}:(T-\lambda I)\text{ is not Fredholm}\}$. For more details, we refer to [3, 5].

Let (X,M,ν) be a $\sigma-$finite measure space and $L^2(X,\nu):=L^2(X)$ be the Hilbert space of all $\nu-$measurable complex valued functions on X such that

$$\|f\|^2_{L^2(X)}=\int_X|f|^2\,d\nu<\infty.$$

The inner product on $L^2(X)$ is given by

$$\langle f,g\rangle_{L^2(X)}=\int_X f\overline{g}d\nu$$

for all $f,g\in L^2(X)$. Let f be a $\nu-$measurable complex valued function on X. Then the essential range of f, denoted by $\operatorname{ess}(f)$, is given by

$$\left\{a\in\mathbb{C}:\forall\,\epsilon>0,\quad\nu\left\{x\in X:\left|f(x)-a\right|<\epsilon\right\}>0\right\}.$$

Let $L^\infty(X,\nu):=L^\infty(X)$ be the collection of all essentially bounded $\nu-$measurable functions on X. It is a Banach space with the norm given by

$$\|f\|_{L^\infty(X)}=\sup\left\{|a|:a\in\operatorname{ess}\left(f\right)\right\}.$$

It is known that, the space $L^\infty(X)$ is a commutative C^*-algebra.

Let (X,M,ν) be a σ-finite measure space and m be a ν-measurable function on X. Let $\mathscr{D}_m\subseteq L^2(X)$ be the set of all $f\in L^2(X)$ such that $m\cdot f\in L^2(X)$. The operator $M_m:\mathscr{D}_m\to L^2(X)$ defined by $M_mf=m\cdot f$ for all $f\in\mathscr{D}_m$ is called a multiplication operator. It is well known that M_m is bounded on $L^2(X)$ if and only if $m \in L^\infty(X)$. If $\mathscr{M}(L^2(X)) = \{M_m : m \in L^\infty(X)\}$, then the map $\Lambda : L^\infty(X) \to \mathscr{M}(L^2(X))$ defined by $\Lambda(m) = M_m$ is a $\star-$isometric isomorphism.

Theorem 2 ([3,4]). *For all $m, m_1, m_2 \in L^\infty(X, M, \nu)$, we have*

(1) $M_m^* = M_{\overline{m}}$;
(2) $M_{m_1} M_{m_2} = M_{m_1 m_2} = M_{m_2 m_1} = M_{m_2} M_{m_1}$;
(3) *The collection $\mathscr{M}(L^2(X))$ is a maximal commutative C^*-subalgebra of $\mathscr{B}(L^2(X))$, where $\mathscr{B}(L^2(X))$ denote the set of all bounded linear operators on $L^2(X)$;*
(4) $\sigma(M_m) = \sigma_a(M_m) = \sigma_e(M_m) = \operatorname{ess}(m)$;
(5) *$\lambda \in \sigma_p(M_m)$ if and only if the Lebesgue measure of $\nu(\{x : m(x) = \lambda\})$ is positive.*

Theorem 3 ([15, Corollary 1.1]). *Let ν be a non-atomic σ-finite measure on X, and let $m \in L^\infty(X, M, \nu)$. Then M_m is compact if and only if $m = 0$ almost everywhere on X.*

Let $X = \mathbb{R}$ (or $\mathbb{R}_+$) and we denote the Lebesgue measure on $\mathbb{R}$ (or $\mathbb{R}_+$) by dx. Then the Hilbert spaces $L^2(\mathbb{R})$ and $L^2(\mathbb{R}_+)$ can be defined as above. For $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, its Fourier transform is given by

$$(\mathscr{F} f)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy} f(y)\, dy, \quad \forall f \in L^2(\mathbb{R}),\ x \in \mathbb{R}.$$

The transform $\mathscr{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is unitary. We refer to [9] for more information about the Fourier transform and it's various applications.

The following theorems are well known.

Theorem 4 ([14, Lemma 2.1]). *Let T be a bounded operator on $L^2(\mathbb{R})$ such that $TM_{e^{ix(\cdot)}} = M_{e^{ix(\cdot)}}T$ for all $x \in \mathbb{R}$, where $(M_{e^{ix(\cdot)}} f)(y) = e^{ixy} f(y)$ for all $y \in \mathbb{R}$. Then there exists $\sigma \in L^\infty(\mathbb{R})$ such that $T = M_\sigma$.*

Theorem 5 ([14, Lemma 2.2]). *Let T be a bounded operator on $L^2(\mathbb{R}_+)$ such that $TM^+_{e^{ix(\cdot)}} = M^+_{e^{ix(\cdot)}}T$ for all $x \in \mathbb{R}$, where $M^+_{e^{ix(\cdot)}}$ is the restriction of $M_{e^{ix(\cdot)}}$ to $L^2(\mathbb{R}_+)$. Then there exists $\sigma \in L^\infty(\mathbb{R}_+)$ such that $T = M_\sigma$.*

In [16], an integral operator $R : L^2(\mathbb{R}_+) \to \mathscr{A}^2(\Pi)$ defined by

$$(Rf)(z) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}_+} \sqrt{\xi} f(\xi) e^{iz\xi} d\xi,\ \forall f \in L^2(\mathbb{R}_+),\ z \in \Pi$$

is introduced and with the help of this transform, it was proved in [14] that the C^*-algebra generated by Toeplitz operators on $\mathscr{A}^2(\Pi)$ with vertical symbols is isomorphic to a C^*-subalgebra of $L^\infty(\mathbb{R}_+)$. Note that if $f \in L^2(\mathbb{R}_+)$, then for any $z = x + iy \in \Pi$, we have $\sqrt{\xi} f(\xi) e^{-y\xi} \in L^1(\mathbb{R}_+)$. Hence

$$\left|(Rf)(z)\right| \le \int_{\mathbb{R}_+} \left|\left(\sqrt{\xi} f(\xi) e^{-y\xi}\right) e^{ix\xi}\right| d\xi < \infty.$$

The operator R is shown to be an isometric isomorphism from $L^2(\mathbb{R}_+)$ onto the space $\mathscr{A}^2(\Pi)$ and its inverse is given by

$$(R^* F)(x) = (R^{-1} F)(x) = \frac{\sqrt{x}}{\sqrt{\pi}} \int_\Pi F(w) e^{-i\overline{w}x} d\mu(w),\ \forall F \in \mathscr{A}^2(\Pi),\ x \in \mathbb{R}_+.$$

Let $w = u + iv \in \Pi$, then for any $F \in \mathscr{A}^2(\Pi) \cap L^1(\Pi)$ we have

$$\begin{aligned} \left|(R^* F)(x)\right| &\le \frac{\sqrt{x}}{\sqrt{\pi}} \int_\Pi |F(w)| \left|e^{-i(u-iv)x}\right| d\mu(w) \le \frac{\sqrt{x}}{\sqrt{\pi}} \int_\Pi |F(w)| e^{-vx} d\mu(w) \\ &\le \frac{\sqrt{x}}{\sqrt{\pi}} \int_\Pi |F(w)| d\mu(w) < \infty. \end{aligned}$$

Thus the integral in the definition of R^* converges in the Lebesgue sense whenever $F \in \mathscr{A}^2(\Pi) \cap L^1(\Pi)$. The following result for the operator R is proved in [14].

Lemma 6. *For every $s \in \mathbb{R}$, we have $RM^+_{e^{is(\cdot)}}R^* = H_s$.*

We observe that the operator R has properties analogous to that of the Bargmann transform. We refer to [1, 2, 8, 18, 19] for more information about the Bargmann transform and its various applications.

3. Integral representation of vertical operators and their operator theoretic properties

In this section, we prove Theorem 1. As a consequence, we obtain various operator theoretic properties of the vertical operators. We start with some auxiliary results which will be useful in proving Theorem 1.

Lemma 7. *Let $\sigma \in L^\infty(\mathbb{R}_+)$. Then the function*

$$\phi_w(z) = \int_{\mathbb{R}_+} \xi\sigma(\xi)e^{i(z-\overline{w})\xi}d\xi,\ z \in \Pi,$$

is analytic on Π for each $w \in \Pi$.

Proof. Let $w = u + iv \in \Pi$ be fixed. For $z = x + iy \in \Pi$, we have

$$\begin{aligned}\left|\varphi_w(z)\right| &= \left|\int_{\mathbb{R}_+} \xi\sigma(\xi)e^{i(z-\overline{w})\xi}d\xi\right| \le \|\sigma\|_{L^\infty(\mathbb{R}_+)}\int_{\mathbb{R}_+}\left|\xi e^{i(x-u)\xi-(y+v)\xi}\right|d\xi \\ &\le \|\sigma\|_{L^\infty(\mathbb{R}_+)}\int_{\mathbb{R}_+}\xi e^{-(y+v)\xi}d\xi \le \|\sigma\|_{L^\infty(\mathbb{R}_+)}\int_{\mathbb{R}_+}\xi e^{-v\xi}d\xi < \infty.\end{aligned}$$

Now, we show that φ_w is continuous function on Π. We prove this with the help of dominated convergence theorem. Let $z = x + iy \in \Pi$ and $(z_n = x_n + iy_n)_{n\in\mathbb{N}}$ be a sequence in Π such that $z_n \to z$. For each $n \in \mathbb{N}$, define $f_n(\xi) = \xi e^{i(z_n-\overline{w})\xi}\sigma(\xi)$ and $f(\xi) = \xi e^{i(z-\overline{w})\xi}\sigma(\xi)$ for all $\xi \in \mathbb{R}_+$. Clearly, $(f_n - f)(\xi) \to 0$ pointwise a.e. on $\mathbb{R}_+$. Also

$$\begin{aligned}\left|\left(f_n - f\right)(\xi)\right| &= \left|\xi\sigma(\xi)\left(e^{iz_n\xi} - e^{iz\xi}\right)e^{-i\overline{w}\xi}\right| \le \|\sigma\|_{L^\infty(\mathbb{R}_+)}\xi e^{-v\xi}\left|e^{iz_n\xi} - e^{iz\xi}\right| \\ &\le \|\sigma\|_{L^\infty(\mathbb{R}_+)}\xi e^{-v\xi}\left(e^{-y_n\xi} + e^{-y\xi}\right) \le 2\|\sigma\|_{L^\infty(\mathbb{R}_+)}\xi e^{-v\xi}.\end{aligned}$$

Let $g(\xi) = \xi e^{-v\xi}$ for all $\xi \in \mathbb{R}_+$. Clearly, g is integrable function on $\mathbb{R}_+$. Therefore, by dominated convergence theorem, we have

$$\int_{\mathbb{R}_+}\left(f_n - f\right)(\xi)d\xi \to 0.$$

That is $\varphi_w(z_n) \to \varphi_w(z)$. Since (z_n) is any arbitrary sequence converging to z, it implies that φ_w is continuous at z. As $z \in \Pi$ is arbitrary, we get that φ_w is continuous on Π.

Let γ be a simple closed contour in Π. Then

$$\begin{aligned}\int_\gamma\int_{\mathbb{R}_+}\left|\xi\sigma(\xi)e^{i(z-\overline{w})\xi}\right|d\xi\left|d\mu(z)\right| &\le \|\sigma\|_{L^\infty(\mathbb{R}_+)}\int_\gamma\int_{\mathbb{R}_+}\xi\left|e^{iz\xi}e^{-i\overline{w}\xi}\right|d\xi\left|d\mu(z)\right| \\ &= \|\sigma\|_{L^\infty(\mathbb{R}_+)}\int_\gamma\int_{\mathbb{R}_+}\xi e^{-(y+v)\xi}d\xi\left|d\mu(z)\right| \le \|\sigma\|_{L^\infty(\mathbb{R}_+)}\int_{\mathbb{R}_+}\xi e^{-v\xi}d\xi\int_\gamma\left|d\mu(z)\right| < \infty.\end{aligned}$$

Therefore, by Fubini's theorem, we have

$$\begin{aligned}\int_\gamma\varphi_w(z)d\mu(z) &= \int_\gamma\int_{\mathbb{R}_+}\xi\sigma(\xi)e^{i(z-\overline{w})\xi}d\mu(z)d\xi = \int_{\mathbb{R}_+}\int_\gamma\xi\sigma(\xi)e^{i(z-\overline{w})\xi}d\xi d\mu(z) \\ &= \int_{\mathbb{R}_+}\xi\sigma(\xi)\int_\gamma e^{i(z-\overline{w})\xi}d\mu(z)d\xi = \int_{\mathbb{R}_+}\xi\sigma(\xi)e^{-\overline{w}\xi}(0)d\xi = 0.\end{aligned}$$

As γ is any arbitrary simple closed contour in Π, by Morera's theorem, we get that φ_w is analytic on Π. This proves the lemma. □

Lemma 8. *Let $\sigma \in L^{\infty}(\mathbb{R}_+)$. Then the function*

$$\phi_w(z) = \int_{\mathbb{R}_+} \xi\sigma(\xi)e^{i(z-\overline{w})\xi}\,d\xi,\ z \in \Pi,$$

belongs to the Bergman space $\mathscr{A}^2(\Pi)$ for each $w \in \Pi$.

Proof. Let $w(= u + iv) \in \Pi$ be fixed. By Lemma 7, φ_w is analytic on Π. Therefore, it is enough to show that $\|\varphi_w\|_{\mathscr{A}^2} < \infty$. Note that

$$\begin{aligned}\|\varphi_w\|^2_{\mathscr{A}^2} &= \int_\Pi |\varphi_w(z)|^2\,d\mu(z) = \int_\Pi \left|\int_{\mathbb{R}_+} \xi\sigma(\xi)e^{i(z-\overline{w})\xi}\,d\xi\right|^2 d\mu(z)\\ &= \int_\Pi \left|\int_{\mathbb{R}_+} \xi\sigma(\xi)e^{-(y+v)\xi}e^{i(x-u)\xi}\,d\xi\right|^2 dx\,dy.\end{aligned}$$

Define

$$\sigma_1(x) = \begin{cases}\sigma(x), & \text{if } x \geq 0\\ 0, & \text{otherwise.}\end{cases}$$

For $y, v \in \mathbb{R}_+$, we denote $f_{y,v}(\xi) = \xi\sigma_1(\xi)e^{-(y+v)\xi}$ for all $\xi \in \mathbb{R}$. Then we get

$$\begin{aligned}\|\varphi_w\|^2_{\mathscr{A}^2} &= \int_\Pi \left|\int_{\mathbb{R}} f_{y,v}(\xi)e^{i(x-u)\xi}\,d\xi\right|^2 dx\,dy\\ &= \int_\Pi \left|\left(\mathscr{F}^{-1}f_{y,v}\right)(x-u)\right|^2 dx\,dy\\ &= \int_{\mathbb{R}_+}\left(\int_{\mathbb{R}} \left|\left(\mathscr{F}^{-1}f_{y,v}\right)(x-u)\right|^2 dx\right)dy.\end{aligned}$$

We know that $L^2(\mathbb{R})$ is translation invariant. Therefore,

$$\begin{aligned}\|\varphi_w\|^2_{\mathscr{A}^2} &= \int_{\mathbb{R}_+}\left(\int_{\mathbb{R}} \left|\left(\mathscr{F}^{-1}f_{y,v}\right)(x)\right|^2 dx\right)dy\\ &= \int_{\mathbb{R}_+}\left(\int_{\mathbb{R}} \left|\left(f_{y,v}(\xi)\right)\right|^2 d\xi\right)dy = \int_{\mathbb{R}_+}\left(\int_{\mathbb{R}_+} e^{-2y\xi}\,dy\right)\xi^2\,|\sigma(\xi)|^2\,e^{-2v\xi}\,d\xi\\ &= \int_{\mathbb{R}_+}\left(\frac{0-1}{-2\xi}\right)\xi^2|\sigma(\xi)|^2e^{-2v\xi}\,d\xi = \frac{1}{2}\int_{\mathbb{R}_+} \xi\,|\sigma(\xi)|^2\,e^{-2v\xi}\,d\xi\\ &\leq \frac{1}{2}\|\sigma\|^2_{L^\infty(\mathbb{R}_+)}\int_{\mathbb{R}_+} \xi e^{-2v\xi}\,d\xi < \infty \qquad (\because v > 0).\end{aligned}$$

This proves the lemma. □

Lemma 9. *Let φ be a function defined on Π such that $\varphi((\cdot) - \overline{w})$ is analytic for each $w \in \Pi$. Then φ is analytic on Π.*

Proof. We show that φ is differentiable at each $z_0 = x_0 + iy_0 \in \Pi$. Let $\epsilon > 0$ such that $U(z_0,\epsilon) = \{z \in \Pi : |z - z_0| < \epsilon\} \subseteq \Pi$. Choose $w_0 = u_0 + iv_0$ such that $U(z_0,\epsilon) + \overline{w}_0 \subseteq \Pi$. Then for all $z \in U(z_0,\epsilon/4)$, we have

$$\lim_{z\to z_0}\frac{\varphi(z)-\varphi(z_0)}{z-z_0} = \lim_{z\to z_0}\frac{\varphi(z+\overline{w}_0-\overline{w}_0)-\varphi(z_0+\overline{w}_0-\overline{w}_0)}{z-z_0}$$

Let $\varphi_{w_0}(z) := \varphi(z - \overline{w_0})$ for all $z \in \Pi$, then

$$\lim_{z\to z_0}\frac{\varphi(z)-\varphi(z_0)}{z-z_0} = \lim_{z+\overline{w}_0\to z_0+\overline{w}_0}\frac{\varphi_{w_0}(z+\overline{w}_0)-\varphi_{w_0}(z_0+\overline{w}_0)}{(z+\overline{w}_0)-(z_0+\overline{w}_0)}.$$

As φ_{w_0} is analytic at $z_0 + \overline{w_0}$, it implies that φ is differentiable at z_0. Since $z_0 \in \Pi$ is arbitrary, the function φ is analytic on Π. □

Proposition 10. *Let $\sigma \in L^{\infty}(\mathbb{R}_+)$. Then the function*

$$\varphi(z) = \int_{\mathbb{R}_+} \xi\sigma(\xi)e^{iz\xi}d\xi, \quad z \in \Pi$$

is analytic on Π *and* $\varphi((\cdot)-\overline{w})$, $\overline{\varphi(z-\overline{(\cdot)})} \in \mathscr{A}^2(\Pi)$ *for each* $z, w \in \Pi$.

Proof. By lemmas 7, 8 and 9, it follows that the function φ defined by

$$\varphi(z) = \int_{\mathbb{R}_+} \xi\sigma(\xi)e^{iz\xi}d\xi, \quad z \in \Pi$$

is analytic on Π and $\varphi((\cdot)-\overline{w}) \in \mathscr{A}^2(\Pi)$ for all $w \in \Pi$. We notice that

$$\overline{\varphi(z-\overline{w})} = \int_{\mathbb{R}_+} \xi\overline{\sigma(\xi)}e^{i(w-\overline{z})\xi}d\xi, \quad z, w \in \Pi.$$

As $\overline{\sigma} \in L^{\infty}(\mathbb{R}_+)$, it follows that $\overline{\varphi(z-\overline{(\cdot)})} \in \mathscr{A}^2(\Pi)$. □

Now, we show that every bounded operator S_φ is of the form $RM_\sigma R^*$ for some $\sigma \in L^{\infty}(\mathbb{R}_+)$.

Lemma 11. *Let* φ *be a function on* Π *such that* $\varphi((\cdot)-\overline{w}), \overline{\varphi(z-\overline{(\cdot)})} \in \mathscr{A}^2(\Pi)$ *for each* $z, w \in \Pi$ *and* S_φ *given by* (2). *If* $S_\varphi \in \mathscr{B}(\mathscr{A}^2(\Pi))$, *then there exists* $\sigma \in L^{\infty}(\mathbb{R}_+)$ *such that* $S_\varphi = RM_\sigma R^*$.

Proof. We first show that every bounded S_φ is vertical. If $h \in \mathbb{R}$, then for every $f \in \mathscr{A}^2(\Pi)$ and $z \in \Pi$, we have

$$\left(S_\varphi H_h f\right)(z) = \frac{1}{\pi}\int_{\Pi}\left(H_h f\right)(w)\varphi(z-\overline{w})d\mu(w) = \frac{1}{\pi}\int_{\Pi} f(w-h)\varphi(z-\overline{w})d\mu(w)$$

Using the change of variable $w \mapsto w+h$ gives

$$\left(S_\varphi H_h f\right)(z) = \frac{1}{\pi}\int_{\Pi} f(w)\varphi((z-h)-\overline{w})d\mu(w) = \left(H_h S_\varphi f\right)(z).$$

Since $h \in \mathbb{R}$ is arbitrary, it follows that $S_\varphi H_h = H_h S_\varphi$ for all $h \in \mathbb{R}$. Combining Theorem 5 and Lemma 6, it follows that $S_\varphi = RM_\sigma R^*$ for some $\sigma \in L^{\infty}(\mathbb{R}_+)$. □

Lemma 12. *Let* $\sigma \in L^{\infty}(\mathbb{R}_+)$. *Then* $RM_\sigma R^* = S_\psi$, *where* ψ *and* σ *are related by*

$$\psi(z) = \int_{\mathbb{R}_+} \xi\sigma(\xi)e^{iz\xi}d\xi, \quad z \in \Pi.$$

Proof. For any $f \in \mathscr{A}^2(\Pi) \cap L^1(\Pi)$ and $z(=x+iy) \in \Pi$, we have

$$\begin{aligned}\left(RM_\sigma R^* f\right)(z) &= \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}_+}\sqrt{\xi}\left(M_\sigma R^* f\right)(\xi)e^{iz\xi}d\xi = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}_+}\sqrt{\xi}\sigma(\xi)\left(R^* f\right)(\xi)e^{iz\xi}d\xi \\ &= \frac{1}{\pi}\int_{\mathbb{R}_+}\left(\sqrt{\xi}\right)^2\sigma(\xi)\int_{\Pi} f(w)e^{-i\overline{w}\xi}d\mu(w)e^{iz\xi}d\xi \\ &= \frac{1}{\pi}\int_{\mathbb{R}_+}\int_{\Pi}\xi\sigma(\xi)f(w)e^{i(z-\overline{w})\xi}d\mu(w)d\xi.\end{aligned}$$

If $f \in \mathscr{A}^2(\Pi) \cap L^1(\Pi)$ and $z(=x+iy) \in \Pi$, then

$$\begin{aligned}\frac{1}{\pi}\int_{\mathbb{R}_+}\int_{\Pi}\left|\xi\sigma(\xi)f(w)e^{i(z-\overline{w})\xi}\right|d\mu(w)d\xi &\le \|\sigma\|_{L^{\infty}(\mathbb{R}_+)}\frac{1}{\pi}\int_{\mathbb{R}_+}\int_{\Pi}\xi\left|f(w)e^{i((x+iy)-(u-iv))\xi}\right|d\mu(w)d\xi \\ &\le \|\sigma\|_{L^{\infty}(\mathbb{R}_+)}\frac{1}{\pi}\int_{\mathbb{R}_+}\int_{\Pi}\xi\left|f(w)\right|e^{-y\xi}e^{-v\xi}d\mu(w)d\xi \\ &\le \|\sigma\|_{L^{\infty}(\mathbb{R}_+)}\frac{1}{\pi}\int_{\mathbb{R}_+}\int_{\Pi}\xi\left|f(w)\right|e^{-y\xi}d\mu(w)d\xi \\ &\le \|\sigma\|_{L^{\infty}(\mathbb{R}_+)}\frac{1}{\pi}\int_{\mathbb{R}_+}\left(\xi e^{-y\xi}\right)d\xi\int_{\Pi}\left|f(w)\right|d\mu(w) < \infty.\end{aligned}$$

Therefore, by Fubini's theorem, we get

$$\left(RM_\sigma R^* f\right)(z) = \frac{1}{\sqrt{\pi}} \int_\Pi f(w) \left(\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}_+} \xi \sigma(\xi) e^{i(z-\overline{w})\xi} d\xi \right) d\mu(w).$$

Define

$$\psi(z) = \int_{\mathbb{R}_+} \xi \sigma(\xi) e^{iz\xi} d\xi, \quad \forall\, z \in \Pi.$$

By Proposition 10, it follows that ψ is a well-defined analytic function on Π such that $\psi((\cdot) - \overline{w}), \overline{\psi(z - \overline{(\cdot)})} \in \mathscr{A}^2(\Pi)$ for each $z, w \in \Pi$. From above, we get $RM_\sigma R^* = S_\psi$ on $\mathscr{A}^2(\Pi) \cap L^1(\Pi)$.

Now we show that $RM_\sigma R^* = S_\psi$ on $\mathscr{A}^2(\Pi)$. Let $g \in \mathscr{A}^2(\Pi)$ and $\{g_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathscr{A}^2(\Pi) \cap L^1(\Pi)$ such that $g_n \to g$ in $\mathscr{A}^2(\Pi)$. For each $z \in \Pi$, let

$$h_z(w) := \overline{\psi(z - \overline{(w)})}, \quad w \in \Pi.$$

Then for each $z \in \Pi$, $h_z \in \mathscr{A}^2(\Pi)$ and $(S_\psi g_n)(z) = \langle g_n, h_z \rangle_{\mathscr{A}^2} \to \langle g, h_z \rangle_{\mathscr{A}^2} = (S_\psi g)(z)$. But $S_\psi g_n = RM_\sigma R^* g_n$ for all $n \in \mathbb{N}$. This implies that $(RM_\sigma R^* g_n)(z) \to (S_\psi g)(z)$ for all $z \in \Pi$. $RM_\sigma R^*$ is bounded on $\mathscr{A}^2(\Pi)$, we get $RM_\sigma R^* g_n \to RM_\sigma R^* g$ in $\mathscr{A}^2(\Pi)$. Since $\mathscr{A}^2(\Pi)$ is the reproducing kernel Hilbert space, $(RM_\sigma R^* g_n)(z) \to (RM_\sigma R^* g)(z)$ for all $z \in \Pi$. Hence $(RM_\sigma R^* g)(z) = (S_\psi g)(z)$ for all $z \in \Pi$ and $g \in \mathscr{A}^2(\Pi)$. That is, $RM_\sigma R^* g = S_\psi g$ for all $g \in \mathscr{A}^2(\Pi)$. Thus, we get $RM_\sigma R^* = S_\psi$ on $\mathscr{A}^2(\Pi)$. □

In the following lemma, we show that the representation of an operator in the form (2) is unique.

Lemma 13. *Let φ_1, φ_2 be functions on Π such that $\varphi_1((\cdot) - \overline{w})$, $\overline{\varphi_1(z - \overline{(\cdot)})}$, $\varphi_2((\cdot) - \overline{w})$, $\overline{\varphi_2(z - \overline{(\cdot)})} \in \mathscr{A}^2(\Pi)$ for each $z, w \in \Pi$ and $S_{\varphi_1}, S_{\varphi_2} \in \mathscr{B}(\mathscr{A}^2(\Pi))$. Then*

$$S_{\varphi_1} = S_{\varphi_2} \quad \textit{if and only if } \varphi_1 = \varphi_2.$$

Proof. Suppose $S_{\varphi_1} = S_{\varphi_2}$. Let $z_0 \in \Pi$ be fixed. Then for all $f \in \mathscr{A}^2(\Pi)$ we get,

$$\left(\left(S_{\varphi_1} - S_{\varphi_2}\right) f\right)(z_0) = 0 \implies \int_\Pi f(w) \left(\varphi_1 - \varphi_2\right)\left(z_0 - \overline{w}\right) d\mu(w) = 0. \tag{4}$$

Define $\Psi_{z_0}(w) = \overline{(\varphi_1 - \varphi_2)}(z_0 - \overline{w})$. As $\overline{\varphi_1(z - \overline{(\cdot)})}$, $\overline{\varphi_2(z - \overline{(\cdot)})} \in \mathscr{A}^2(\Pi)$ for each $z \in \Pi$, we get $\Psi_{z_0} \in \mathscr{A}^2(\Pi)$. Thus

$$\left(\left(S_{\varphi_1} - S_{\varphi_2}\right) f\right)(z_0) = 0 \implies \left\langle f, \Psi_{z_0} \right\rangle_{\mathscr{A}^2} = 0 \tag{5}$$

for all $f \in \mathscr{A}^2(\Pi)$. This implies that $\Psi_{z_0} = 0$. That is, $(\varphi_1 - \varphi_2)(z_0 - \overline{w}) = 0$ for all $w \in \Pi$. Since $z_0 \in \Pi$ is arbitrary, we get $\varphi_1(z - \overline{w}) = \varphi_2(z - \overline{w})$ for all $z, w \in \Pi$. Hence, we get $\varphi_1 = \varphi_2$.

Conversely, if $\varphi_1 = \varphi_2$, then it is easy to see that $S_{\varphi_1} = S_{\varphi_2}$. □

Now, we are set to give the proof of Theorem 1.

Proof of Theorem 1. Let φ be a function on Π such that $\varphi((\cdot) - \overline{w}), \overline{\varphi(z - \overline{(\cdot)})} \in \mathscr{A}^2(\Pi)$ for each $z, w \in \Pi$ and S_φ given by (2) be bounded on $\mathscr{A}^2(\Pi)$. By Lemma 11, it follows that there exists $\sigma \in L^\infty(\mathbb{R}_+)$ such that $S_\varphi = RM_\sigma R^*$. But, Lemma 12 implies $RM_\sigma R^* = S_\psi$, where ψ and σ satisfy (3), with ψ instead of φ. Thus, we have $S_\varphi = RM_\sigma R^* = S_\psi$. By Lemma 13, it follows that $\varphi = \psi$. That is,

$$\varphi(z) = \int_{\mathbb{R}_+} \xi \sigma(\xi) e^{iz\xi} d\xi, \quad \forall\, z \in \Pi.$$

Conversely, suppose (3) holds for some $\sigma \in L^\infty(\mathbb{R}_+)$. By Proposition 10, the function φ defined by (3) satisfies the required conditions. We know that $M_\sigma \in \mathscr{B}(L^2(\mathbb{R}_+))$ and $R : L^2(\mathbb{R}_+) \to \mathscr{A}^2(\Pi)$ is a unitary operator. Therefore, $RM_\sigma R^*$ is bounded on $\mathscr{A}^2(\Pi)$. By Lemma 12, $RM_\sigma R^* = S_\varphi$. Hence, $S_\varphi \in \mathscr{B}(\mathscr{A}^2(\Pi))$.

Also, $\|S_\varphi\|_{\mathscr{A}^2(\Pi)\to\mathscr{A}^2(\Pi)} = \|M_\sigma\|_{L^2(\mathbb{R}_+)\to L^2(\mathbb{R}_+)} = \|\sigma\|_{L^\infty(\mathbb{R}_+)}$. This proves the theorem. □

Let $\mathcal{V}$ be the collection of all vertical operators on $\mathscr{A}^2(\Pi)$. By combining lemmas 11, 12 and Theorem 1, we get

$$\mathcal{V} = \left\{ S_\varphi \in \mathscr{B}\left(\mathscr{A}^2(\Pi)\right) : \exists\, \sigma \in L^\infty(\mathbb{R}_+) \text{ and } \varphi(z) = \int_{\mathbb{R}_+} \xi\sigma(\xi)e^{iz\xi}d\xi, \quad \forall\, z \in \Pi \right\}.$$

3.1. *Operator theoretic properties of S_φ*

In this Section, we prove various operator theoretic properties of the operator S_φ given by (2). We first find the adjoint of S_φ.

Theorem 14 (Adjoint of S_φ). *Let φ be a function on Π such that $\varphi((\cdot)-\overline{w}), \overline{\varphi(z-\overline{(\cdot)})} \in \mathscr{A}^2(\Pi)$ for each $z, w \in \Pi$ and S_φ given by (2) be bounded on $\mathscr{A}^2(\Pi)$, then $S_\varphi^* = S_{\widetilde{\varphi}}$, where $\widetilde{\varphi}(z) = \overline{\varphi(-\overline{z})}$ for all $z \in \Pi$.*

Proof. Let φ be a function on Π such that $\varphi((\cdot)-\overline{w})$, $\overline{\varphi(z-\overline{(\cdot)})} \in \mathscr{A}^2(\Pi)$ for each $z, w \in \Pi$ and S_φ given by (2) be bounded on $\mathscr{A}^2(\Pi)$. Then by Theorem 1, there exists $\sigma \in L^\infty(\mathbb{R}_+)$ such that $S_\varphi = RM_\sigma R^*$, where φ and σ satisfy (3). Using Theorem 2, we get $S_\varphi^* = RM_{\overline{\sigma}}R^*$. Again by Theorem 1, $RM_{\overline{\sigma}}R^* = S_{\widetilde{\varphi}}$, where

$$\widetilde{\varphi}(z) = \int_{\mathbb{R}_+} \xi\overline{\sigma(\xi)}e^{iz\xi}d\xi = \overline{\int_{\mathbb{R}_+} \xi\sigma(\xi)e^{-i\overline{z}\xi}d\xi} = \overline{\varphi(-\overline{z})}, \quad \forall\, z \in \Pi.$$

This proves the theorem. □

By Theorem 1, we know that every bounded operator S_φ is of the form $RM_\sigma R^*$ for some $\sigma \in L^\infty(\mathbb{R}_+)$, where φ and σ satisfy (3). Using this, Theorem 2 and Theorem 3, it is easy to prove the following results. The proofs are left to the reader.

Theorem 15. *Let φ be a function on Π such that $\varphi((\cdot)-\overline{w}), \overline{\varphi(z-\overline{(\cdot)})} \in \mathscr{A}^2(\Pi)$ for each $z, w \in \Pi$ and S_φ given by (2) be bounded on $\mathscr{A}^2(\Pi)$, then*

(1) *S_φ is normal, that is, $S_\varphi S_\varphi^* = S_\varphi^* S_\varphi$.*
(2) *S_φ is compact if and only if $\varphi \equiv 0$*
(3) *The collection $\mathcal{V} = \{S_\varphi \in \mathscr{B}(\mathscr{A}^2(\Pi))\}$ is a maximal commutative C^*-subalgebra of $\mathscr{B}(\mathscr{A}^2(\Pi))$.*

Theorem 16 (Spectrum of S_φ). *Let φ be a function on Π such that $\varphi((\cdot)-\overline{w})$, $\overline{\varphi(z-\overline{(\cdot)})} \in \mathscr{A}^2(\Pi)$ for each $z, w \in \Pi$ and S_φ given by (2) be bounded on $\mathscr{A}^2(\Pi)$, then*

(1) *$\sigma(S_\varphi) = \sigma_a(S_\varphi) = \sigma_e(S_\varphi) = \text{ess}(m)$, where φ and m satisfy (3), with m instead of σ.*
(2) *$\lambda \in \sigma_p(S_\varphi)$ if and only if the Lebesgue measure of $\{x : m(x) = \lambda\}$ is positive, where φ and m satisfy (3), with m instead of σ.*

Now, we give structure of common reducing subspaces of operators in the collection $\mathcal{V}$. Before that, we recall some basic definitions and results.

Definition 17 ([5, Definition 4.41]). *Let $\mathscr{H}$ be a Hilbert space and $T \in \mathscr{B}(\mathscr{H})$. A closed subspace $\mathscr{M}$ of $\mathscr{H}$ is an invariant subspace of T if $T(\mathscr{M}) \subseteq \mathscr{M}$ and $\mathscr{M}$ is said to be a reducing subspace of T if it is invariant under both T and T^*.*

Lemma 18 ([5, Proposition 4.42]). *Let $\mathscr{H}$ be a Hilbert space and $T \in \mathscr{B}(\mathscr{H})$. Then $\mathscr{M}$ is invariant subspace of T if and only if $P_\mathscr{M} T P_\mathscr{M} = TP_\mathscr{M}$ and it is a reducing subspace of T if and only if $TP_\mathscr{M} = P_\mathscr{M} T$, where $P_\mathscr{M}$ is an orthogonal projection associated to $\mathscr{M}$.*

Theorem 19. *Let $\mathscr{M}$ be a subspace of the Bergman space $\mathscr{A}^2(\Pi)$. Then $\mathscr{M}$ is a reducing subspace of all the operators in $\mathcal{V}$ if and only if $\mathscr{M} = S_{\varphi_0}(\mathscr{A}^2(\Pi))$, where*

$$\varphi_0(z) = \int_{\mathbb{R}_+} \xi\chi_E(\xi)e^{iz\xi}\,d\xi \tag{6}$$

for all $z \in \Pi$, E is a measurable subset of $\mathbb{R}_+$ and χ_E is characteristic function associated to the set E.

Proof. Let $\mathscr{M}$ be a closed subspace of $\mathscr{A}^2(\Pi)$. By Lemma 18 and Theorem 1, $\mathscr{M}$ is reducing subspace of operators in $\mathcal{V} \iff S_\varphi P_{\mathscr{M}} = P_{\mathscr{M}} S_\varphi$ for all $S_\varphi \in \mathcal{V} \iff M_m(R^*P_{\mathscr{M}}R) = (R^*P_{\mathscr{M}}R)M_m$ for all $m \in L^\infty(\mathbb{R}_+)$. By Theorem 5, we get $(R^*P_{\mathscr{M}}R) = M_\sigma$ for some $\sigma \in L^\infty(\mathbb{R}_+)$.

As $M_\sigma (= R^*P_{\mathscr{M}}R)$ is an orthogonal projection on $L^2(\mathbb{R}_+)$, there exists a Lebesgue measurable set $E \subseteq \mathbb{R}_+$ such that $\sigma = \chi_E$ almost everywhere on $\mathbb{R}_+$ and $M_\sigma = M_{\chi_E}$. Hence, $P_{\mathscr{M}} = RM_{\chi_E}R^*$. By using Theorem 1, we get $P_{\mathscr{M}} = S_{\varphi_0}$, where

$$\varphi_0(z) = \int_{\mathbb{R}_+} \xi\chi_E(\xi)e^{iz\xi}\,d\xi, \ \forall\ z \in \Pi.$$

This proves the theorem. □

4. Toeplitz operators with vertical symbols

We know that $\mathscr{A}^2(\Pi)$ is a closed subspace of the Hilbert space $L^2(\Pi, d\mu)$. Let P denote the orthogonal projection on $L^2(\Pi, d\mu)$ with range $\mathscr{A}^2(\Pi)$. The operator P is an integral operator given by

$$(Pf)(z) = \langle f, K_z\rangle_{L^2(\Pi)} = -\frac{1}{\pi}\int_\Pi f(w)\frac{1}{(z-\overline{w})^2}\,d\mu(w), \quad f \in L^2(\Pi, d\mu).$$

For a function $\mathbf{a} \in L^\infty(\Pi, d\mu)$, the Toeplitz operator $T_{\mathbf{a}}$ on $\mathscr{A}^2(\Pi)$ is defined by $T_{\mathbf{a}}f = P(\mathbf{a}f)$. We say that the function $\mathbf{a} \in L^\infty(\Pi)$ is vertical if it is invariant under horizontal translations. That is, for each $h \in \mathbb{R}$, $\mathbf{a}((\cdot) - h) = \mathbf{a}(\cdot)$ almost everywhere on Π. If $\mathbf{a} \in L^\infty(\Pi)$ is a vertical function, then the Toeplitz operator $T_{\mathbf{a}}$ is also vertical operator. In fact, we have the following known result.

Theorem 20 ([14]). *Let $\mathbf{a} \in L^\infty(\Pi)$. Then $T_{\mathbf{a}}$ is vertical operator if and only if $\mathbf{a}$ is a vertical function.*

Let $\mathcal{V}_{top}$ denote the collection of all Toeplitz operators with vertical symbols and $\mathcal{V}\mathcal{T}(L^\infty)$ denote the C^*-algebra generated by $\mathcal{V}_{top}$. Note that $\mathcal{V}\mathcal{T}(L^\infty) \subset \mathcal{V}$. In this section, our aim is to give explicit representation of operators in $\mathcal{V}\mathcal{T}(L^\infty)$. We first recall some definitions and results from [14] which will be useful in this section.

Let $f : \mathbb{R}_+ \to \mathbb{C}$ be a bounded function. Then f is said to be very slowly oscillating function if the compositon $f \circ \exp$ is uniformly continuous with respect to the usual metric on $\mathbb{R}_+$. Let $\mathrm{VSO}(\mathbb{R}_+)$ denote the set of all very slowly oscillating functions.

Lemma 21 ([14, Proposition 4.2]). $\mathrm{VSO}(\mathbb{R}_+)$ *is a closed C^*-subalgebra of the C^*-algebra $C_b(\mathbb{R}_+)$ of all complex valued bounded continuous functions on $\mathbb{R}_+$ with pointwise operations.*

Since $C_b(\mathbb{R}_+)$ is closed C^*-subalgebra of $L^\infty(\mathbb{R}_+)$, it follows that $\mathrm{VSO}(\mathbb{R}_+)$ is a closed C^*-subalgebra of $L^\infty(\mathbb{R}_+)$. For Toeplitz operators with vertical symbols, the following result is known.

Lemma 22 ([14, Theorem 3.4]). *Let $\mathbf{a} \in L^\infty(\Pi)$ be a vertical function and $T_{\mathbf{a}}$ be the Toeplitz operator with defining symbol $\mathbf{a}$. Then there exists $\gamma_{\mathbf{a}} \in L^\infty(\mathbb{R}_+)$ such that $T_{\mathbf{a}} = RM_{\gamma_{\mathbf{a}}}R^*$, where*

$$\gamma_{\mathbf{a}}(x) = 2x\int_0^\infty \mathbf{a}(y)e^{-2xy}\,dy, \quad x \in \mathbb{R}_+. \tag{7}$$

Let $\mathscr{G}$ denote the collection of all $\gamma_{\mathbf{a}} \in L^\infty(\mathbb{R}_+)$, where $\mathbf{a} \in L^\infty(\Pi)$ is vertical function and $\gamma_{\mathbf{a}}$ is given by (7). In [14], the follwing result is proved.

Lemma 23. *The C^*-algebra generated by $\mathscr{G}$ is equal to $\overline{\mathscr{G}} = \mathrm{VSO}(\mathbb{R}_+)$.*

Now we give explicit integral representation of the form (2) for all the operators in the C^*-algebra generated by $\mathcal{V}_{top}$.

Theorem 24. *The C^*-algebra $\mathcal{V}\mathcal{T}(L^\infty)$ generated by $\mathcal{V}_{top}$ is given by*

$$\left\{S_\varphi \in \mathcal{V} : \exists\, \sigma \in \mathrm{VSO}(\mathbb{R}_+) \quad \textit{and} \quad \varphi(z) = \int_{\mathbb{R}_+} \xi\sigma(\xi)e^{iz\xi}d\xi, \quad \forall\, z \in \Pi\right\}.$$

Proof. Let $T_{\mathbf{a}} \in \mathcal{V}_{top}$. Then by Lemma 22, we get $T_{\mathbf{a}} = RM_{\gamma_{\mathbf{a}}}R^*$, where $\gamma_{\mathbf{a}} \in L^\infty(\mathbb{R}_+)$ is given by (7). By Theorem 1, we have $RM_{\gamma_{\mathbf{a}}}R^* = S_{\varphi_{\gamma_{\mathbf{a}}}}$, where $\varphi_{\gamma_{\mathbf{a}}}$ and $\gamma_{\mathbf{a}}$ satisfy

$$\varphi_{\gamma_{\mathbf{a}}}(z) = \int_{\mathbb{R}_+} \xi\gamma_{\mathbf{a}}(\xi)e^{iz\xi}d\xi, \quad \forall\, z \in \Pi.$$

This implies that $T_{\mathbf{a}} = S_{\varphi_{\gamma_{\mathbf{a}}}}$. Hence, we get $\mathcal{V}_{top} = \{S_{\varphi_{\gamma_{\mathbf{a}}}} : \gamma_{\mathbf{a}} \in \mathscr{G}\}$. Now, using Lemma 23, we get $\mathcal{V}\mathcal{T}(L^\infty) = \{S_\varphi \in \mathcal{V} : S_\varphi = RM_\sigma R^* \text{ for some } \sigma \in \mathrm{VSO}(\mathbb{R}_+)\}$. In fact, we have

$$\mathcal{V}\mathcal{T}(L^\infty) = \left\{S_\varphi \in \mathcal{V} : \exists\, \sigma \in \mathrm{VSO}(\mathbb{R}_+) \quad \text{and} \quad \varphi(z) = \int_{\mathbb{R}_+} \xi\sigma(\xi)e^{iz\xi}d\xi, \quad \forall\, z \in \Pi\right\}.$$

This proves the theorem. □

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