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Torus quotient of the Grassmannian $G_{n,2n}$

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Abstract. Let $G_{n,2n}$ be the Grassmannian parameterizing the n-dimensional subspaces of $\mathbb{C}^{2n}$. The Picard group of $G_{n,2n}$ is generated by a unique ample line bundle $\mathscr{O}(1)$. Let T be a maximal torus of $\mathrm{SL}(2n,\mathbb{C})$ which acts on $G_{n,2n}$ and $\mathscr{O}(1)$. By [10, Theorem 3.10, p. 764], 2 is the minimal integer k such that $\mathscr{O}(k)$ descends to the GIT quotient. In this article, we prove that the GIT quotient of $G_{n,2n}$ $(n \geq 3)$ by T with respect to $\mathscr{O}(2) = \mathscr{O}(1)^{\otimes 2}$ is not projectively normal when polarized with the descent of $\mathscr{O}(2)$.

Résumé. Soit $G_{n,2n}$ la Grassmannienne des sous-espaces de dimension n de $\mathbb{C}^{2n}$. Le groupe de Picard de $G_{n,2n}$ est engendré par un unique fibré en droites ample $\mathscr{O}(1)$. Fixons un tore maximal T du groupe $\mathrm{SL}(2n,\mathbb{C})$ qui agit sur $G_{n,2n}$ et $\mathscr{O}(1)$. D'après [10, Theorem 3.10, p. 764], 2 est l'entier minimal k tel que $\mathscr{O}(k)$ descende au quotient GIT. Dans cet article, nous prouvons que le quotient GIT de $G_{n,2n}$ $(n \geq 3)$ par T par rapport à $\mathscr{O}(2) = \mathscr{O}(1)^{\otimes 2}$ n'est pas projectivement normal lorsqu'il est polarisé avec la descente de $\mathscr{O}(2)$.

Keywords. Grassmannian, Line bundle, Semi-stable point, GIT-quotient, Projective normality.

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1. Introduction

A polarized variety $(X,\mathscr{L})$, where $\mathscr{L}$ is a very ample line bundle is said to be projectively normal if its homogeneous coordinate ring $\oplus_{m\in\mathbb{Z}_{\geq 0}} H^0(X,\mathscr{L}^{\otimes m})$ is integrally closed and it is generated as a $\mathbb{C}$-algebra by $H^0(X,\mathscr{L})$ (see [2, Chapter II, Exercise 5.14]). For example, the projective line $(\mathbb{P}^1,\mathscr{O}(1))$ is projectively normal. However, if we consider the rational twisted quartic curve in $\mathbb{P}^3$, i.e., image $X = \{[a^4 : a^3b : ab^3 : b^4] \in \mathbb{P}^3 : [a : b] \in \mathbb{P}^1\}$ of the embedding $i : \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ given by $[a : b] \mapsto [a^4 : a^3b : ab^3 : b^4]$, then $(X,\mathscr{O}_X(1)) = (\mathbb{P}^1,\mathscr{O}(3))$ is normal but not projectively normal as the affine cone of X inside $\mathbb{C}^4$ is not normal (see [2, Chapter I, Exercise 3.18]).

In [6], Kannan made an attempt to study projective normality of the GIT quotient of $G_{2,n}$ by a maximal torus T of $\mathrm{SL}(n,\mathbb{C})$ with respect to the descent of $\mathscr{O}(n)$ (n is odd). There it was proved that the homogeneous coordinate ring of the GIT quotient of $G_{2,n}$ by T with respect to the descent of $\mathscr{O}(n)$ is a finite module over the subring generated by the degree one elements. In [3], Howard et al. showed that the GIT quotient of $G_{2,n}$ by T with respect to the descent of $\mathscr{O}(\frac{n}{2})$ (respectively, $\mathscr{O}(n)$) is projectively normal if n is even (respectively, if n is odd). In [14], Nayek et al. used graph

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theoretic techniques to give a short proof of the projective normality of the GIT quotient of $G_{2,n}$ by T with respect to the descent of $\mathcal{O}(n)$ for any n.

To the best of our knowledge it is not known whether there is a suitable ample line bundle $\mathcal{L}$ on $G_{r,n}$ $(r \geq 3)$ such that the GIT quotient of $G_{r,n}$ by T with respect to the descent of the line bundle $\mathcal{L}$ is projectively normal (respectively, not projectively normal) with respect to the descent of $\mathcal{L}$.

In this article, we prove the following:

Theorem 1. *The GIT quotient of* $G_{n,2n}$ $(n \geq 3)$ *by a maximal torus* T *of* $\mathrm{SL}(2n,\mathbb{C})$ *with respect to the descent of* $\mathcal{O}(2)$ *is not projectively normal (for more precise see Corollary 6).*

The layout of the paper is as follows. In Section 2, we recall some preliminaries on algebraic groups, Standard Monomial Theory and Geometric Invariant Theory. In Section 3, we prove Theorem 1 (see Corollary 6).

2. Notation and Preliminaries

We refer to [4, 5, 11, 13, 15, 17] for preliminaries in algebraic groups, Lie algebras, Standard Monomial Theory and Geometric Invariant Theory.

Let $V = \mathbb{C}^{2n}$ and $(e_1, e_2, \ldots, e_{2n})$ be the standard basis of V. For a fixed integer r with $1 \leq r \leq 2n-1$, let $G_{r,2n}$ be the Grassmannian parameterizing the r-dimensional subspaces of $\mathbb{C}^{2n}$. Then there is a natural projective variety structure on $G_{r,2n}$ given by the Plücker embedding $\pi : G_{r,2n} \hookrightarrow \mathbb{P}(\wedge^r V)$ sending r-dimensional subspace to its r-th exterior power. The natural left action of $\mathrm{SL}(2n,\mathbb{C})$ on V induces an action of $\mathrm{SL}(2n,\mathbb{C})$ on $\wedge^r V$ and thus on $\mathbb{P}(\wedge^r V)$, moreover, π is $\mathrm{SL}(2n,\mathbb{C})$-equivariant. Let T be the maximal torus of $\mathrm{SL}(2n,\mathbb{C})$ consisting of diagonal matrices. Let $\mathcal{O}(1)$ denote the hyperplane line bundle on $G_{r,2n}$ given by the Plücker embedding π. Note that $\mathcal{O}(1)$ is $\mathrm{SL}(2n,\mathbb{C})$-linearized, in particular, T-linearized.

Let $I(r,2n)$ denote the indexing set $\{\underline{i} = (i_1, i_2, \ldots, i_r) | i_j \in \mathbb{Z} \text{ and } 1 \leq i_1 < i_2 < \cdots < i_r \leq 2n\}$. Let $e_{\underline{i}} = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r}$ for $\underline{i} = (i_1, i_2, \ldots, i_r) \in I(r,2n)$. Then $\{e_{\underline{i}} : \underline{i} \in I(r,2n)\}$ forms a basis of $\wedge^r V$. Let $\{p_{\underline{i}} : \underline{i} \in I(r,2n)\}$ be the basis of the dual space $(\wedge^r V)^*$, which is dual to $\{e_{\underline{i}} : \underline{i} \in I(r,2n)\}$, i.e., $p_{\underline{j}}(e_{\underline{i}}) = \delta_{\underline{i}\underline{j}}$. Note that $p_{\underline{i}}$'s are the $\underline{i}^{\text{th}}$ Plücker coordinates of $G_{r,2n}$.

In V, we fix a full flag $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{2n} = V$. For $w = (w_1, w_2, \ldots, w_r)$ in $I(r,2n)$, the Schubert variety in $G_{r,2n}$ associated to w is denoted by $X(w)$ and is defined by

$$X(w) = \left\{ W \in G_{r,2n} \,\middle|\, \begin{array}{l} \dim W \cap V_j \geq i, \text{ if } w_i \leq j < w_{i+1}, \\ \text{where } 1 \leq j \leq 2n, 0 \leq i \leq r \text{ and } w_0 := 0, w_{r+1} := 2n \end{array} \right\}.$$

The definition of a Schubert variety $X(w)$ depends on the choice of a full flag. However, given any two full flags $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{2n} = V$ and $\{0\} = V_0' \subset V_1' \subset \cdots \subset V_{2n}' = V$ in V, there exist an automorphism of V which takes V_i to V_i', which shows that $X(w)$ is well defined up to an automorphism of V. We note that $X(w)$ is a closed subvariety of $G_{r,2n}$ of dimension $\sum_{i=1}^{r} w_i - \frac{r(r+1)}{2}$.

There is a natural partial order on $I(r,2n)$, given as follows: for $v = (v_1, v_2, \ldots, v_r)$, $w = (w_1, w_2, \ldots, w_r)$, $v \leq w$ if and only if $v_i \leq w_i$ for all $1 \leq i \leq r$. For $v, w \in I(r,2n)$, $X(v) \subseteq X(w)$ if and only if $v \leq w$. Further, $p_v|_{X(w)} \neq 0$ if and only if $v \leq w$.

For $w \in I(r,2n)$, we also denote the restriction of the line bundle $\mathcal{O}(1)$ on $G_{r,2n}$ to $X(w)$ by $\mathcal{O}(1)$. The monomial $p_{\tau_1} p_{\tau_2} \ldots p_{\tau_m} \in H^0(X(w), \mathcal{O}(m))$, where $\tau_1, \tau_2, \ldots, \tau_m \in I(r,2n)$ is said to be standard monomial of degree m if $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_m \leq w$. The standard monomials of degree m

on $X(w)$ form a basis of $H^0(X(w),\mathcal{O}(m))$. The Grassmannian $G_{r,2n}\subseteq\mathbb{P}(\wedge^r V)$ is precisely the zero set of the following well known Plücker relations:

$$\sum_{h=1}^{r+1}(-1)^h p_{i_1,i_2,\dots,i_{r-1}j_h}p_{j_1,\dots,\hat{j_h},\dots,j_{r+1}}, \tag{1}$$

where $\{i_1,\dots,i_{r-1}\}$, $\{j_1,\dots,j_{r+1}\}$ are two subsets of $\{1,2,\dots,2n\}$ and $\widehat{j_h}$ means dropping the index j_h.

A point $p\in X(w)$ is said to be semi-stable with respect to the T-linearized line bundle $\mathcal{O}(1)$ if there is a T-invariant section $s\in H^0(X(w),\mathcal{O}(m))$ for some positive integer m such that $s(p)\neq 0$. We denote the set of all semi-stable points of $X(w)$ with respect to $\mathcal{O}(1)$ by $X(w)^{ss}_T(\mathcal{O}(1))$. A point p in $X(w)^{ss}_T(\mathcal{O}(1))$ is said to be stable if the T-orbit of p is closed in $X(w)^{ss}_T(\mathcal{O}(1))$ and the stabilizer of p in T is finite. We denote the set of all stable points of $X(w)$ with respect to $\mathcal{O}(1)$ by $X(w)^{s}_T(\mathcal{O}(1))$.

Let B $(\supset T)$ be the Borel subgroup of $\mathrm{SL}(2n,\mathbb{C})$ consisting of upper triangular matrices. For $1\le i\le 2n$, define $\varepsilon_i: T\to\mathbb{C}^\times$ by $\varepsilon_i(\mathrm{diag}(t_1,\dots,t_{2n}))=t_i$. Then $S:=\{\alpha_i:=\varepsilon_i-\varepsilon_{i+1}\,|\text{ for all } 1\le i\le 2n-1\}$ forms the set of simple roots of $\mathrm{SL}(2n,\mathbb{C})$ with respect to T and B. Let $\{\varpi_i\,|\,i=1,2,\dots,2n-1\}$ be the set of fundamental dominant weights corresponding to S.

For $\lambda=m\varpi_r$ $(m\ge 1)$, we associate a Young diagram (denoted by Γ) with λ_i number of boxes in the i-th column, where $\lambda_i:=m$ for $1\le i\le r$. It is also called Young diagram of shape λ.A Young diagram Γ associated to λ is said to be a Young tableau if the diagram is filled with integers $1,2,\dots,2n$. We also denote this Young tableau by Γ. A Young tableau is said to be standard if the entries along any column is non-decreasing from top to bottom and along any row is strictly increasing from left to right. Given a Young tableau Γ, let $\tau=\{i_1,i_2,\dots,i_r\}$ be a typical row in Γ, where $1\le i_1<\cdots<i_r\le 2n$. To the row τ, we associate the Plücker coordinate $p_{i_1,i_2,\dots,i_r}$. We set $p_\Gamma=\prod_\tau p_\tau$, where the product is taken over all the rows of Γ. Note that for $w\in I(r,2n)$, p_Γ is a standard monomial on $X(w)$ if Γ is standard and the bottom row of Γ is less than or equal to w. Further, p_Γ is also called standard monomial on $X(w)$ of shape λ. We use the notation p_Γ and Γ interchangeably.

Now we recall the definition of weight of a standard Young tableau Γ (see [12, Section 2, p. 336]). For a positive integer $1\le i\le 2n$, we denote by $c_\Gamma(i)$, the number of boxes of Γ containing the integer i. The weight of Γ is defined as $wt(\Gamma):=c_\Gamma(1)\varepsilon_1+\cdots+c_\Gamma(2n)\varepsilon_{2n}$.

We conclude this section by recalling the following key lemma about T-invariant monomials in $H^0(G_{r,2n},\mathcal{O}(m))$.

Lemma 2 ([14, Lemma 3.1, p. 4]). *A monomial $p_\Gamma\in H^0(G_{r,2n},\mathcal{O}(m))$ is T-invariant if and only if $c_\Gamma(i)=c_\Gamma(j)$ for all $1\le i,j\le 2n$.*

3. Main Theorem

First we recall that by [10, Theorem 3.10, p. 764], 2 is the minimal integer k such that the line bundle $\mathcal{O}(k)$ on $G_{n,2n}$ descends to the GIT quotient $T\backslash\backslash(G_{n,2n})^{ss}_T(\mathcal{O}(2))$. In this section, we prove that there exists a Schubert subvariety $X(v)$ of $G_{n,2n}$ admitting semi-stable points such that the GIT quotient $T\backslash\backslash(X(v))^{ss}_T(\mathcal{O}(2))$ with respect to the descent of $\mathcal{O}(2)$ is not projectively normal (see Theorem 5). As a consequence, we conclude that any Schubert variety $X(w)$ containing $X(v)$, the GIT quotient $T\backslash\backslash(X(w))^{ss}_T(\mathcal{O}(2))$ with respect to the descent of $\mathcal{O}(2)$ is not projectively normal. In particular, $T\backslash\backslash(G_{n,2n})^{ss}_T(\mathcal{O}(2))$ is not projectively normal.

Recall that $2\varpi_n=2\varepsilon_1+2\varepsilon_2+\cdots+2\varepsilon_n$ (see [4, Table 1, p. 69]). For $u=(u_1,u_2,\dots,u_n)\in I(n,2n)$, we define $u(2\varpi_n)=2\varepsilon_{u_1}+2\varepsilon_{u_2}+\cdots+2\varepsilon_{u_n}$.

Recall that by [9, Corollary 1.9, p. 85], there exists a unique minimal element $w_1\in I(n,2n)$ such that $w_1(2\varpi_n)\le 0$, i.e., $-w_1(2\varpi_n)$ is a non-negative linear combination of simple roots. Consider

$w=(2,4,6,\dots,2n-4,2n-2,2n)$. Then $-w(2\varpi_n)=-(2\varepsilon_2+2\varepsilon_4+\cdots+2\varepsilon_{2n})=\alpha_1+\alpha_3+\cdots+\alpha_{2n-1}$, as $\sum_{i=1}^{2n}\varepsilon_i=0$. Thus, $w(2\varpi_n)\le 0$. On the other hand for any $v\le w$ such that $\sum_{i=1}^{n}(w_i-v_i)=1$, we have $v=(2,4,6,\dots,2i-2,2i-1,2i+2,\dots,2n-4,2n-2,2n)$ for some $1\le i\le n$. Then $-v(2\varpi_n)=-(2\varepsilon_2+\cdots+2\varepsilon_{2i-2}+2\varepsilon_{2i-1}+2\varepsilon_{2i+2}+\cdots+2\varepsilon_{2n})$. Since $\sum_{i=1}^{2n}\varepsilon_i=0$, we have $-v(2\varpi_n)=(\alpha_1+\alpha_3+\cdots+\alpha_{2i-3}+\alpha_{2i+1}+\alpha_{2i+3}+\cdots+\alpha_{2n-1})-\alpha_{2i-1}$. Thus, $v(2\varpi_n)\not\le 0$. Therefore, $w=w_1$.

Now we consider the following w_i's such that $w_1\le w_i$ for all $2\le i\le 5$:

- $w_2=(2,4,6,\dots,2n-6,2n-3,2n-2,2n)$
- $w_3=(2,4,6,\dots,2n-6,2n-4,2n-1,2n)$
- $w_4=(2,4,6,\dots,2n-6,2n-3,2n-1,2n)$
- $w_5=(2,4,6,\dots,2n-6,2n-2,2n-1,2n)$.

Note that $\{w_1,w_2,w_3,w_4,w_5\}$ is precisely the set $\{w\in I(n,2n): w_1\le w\le w_5\}$. Further, note that w_2 and w_3 are non-comparable and $w_2,w_3\le w_4\le w_5$. Since $w_1\le w_i$ and $w_1(2\varpi_n)\le 0$, we have $w_i(2\varpi_n)\le 0$ for all $2\le i\le 5$. Thus, by [8, Lemma 2.1, p. 470], $X(w_i)^{ss}_T(\mathcal{O}(2))$ is non-empty for all $1\le i\le 5$.

Let $X=T\backslash\backslash(X(w_5))^{ss}_T(\mathcal{O}(2))$. Then we have $X=\mathrm{Proj}(R)$, where $R=\bigoplus_{k\in\mathbb{Z}_{\ge0}}R_k$ and $R_k=H^0(X(w_5),\mathcal{O}(2k))^T$. Note that R_k's are finite dimensional vector spaces.

Let us consider the following standard monomials

$$X_1=\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&3&5&\cdots&2n-7&2n-5&2n-3&2n-1\\\hline 2&4&6&\cdots&2n-6&2n-4&2n-2&2n\\\hline\end{array}\qquad X_2=\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&3&5&\cdots&2n-7&2n-5&2n-4&2n-1\\\hline 2&4&6&\cdots&2n-6&2n-3&2n-2&2n\\\hline\end{array}$$

$$X_3=\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&3&5&\cdots&2n-7&2n-5&2n-3&2n-2\\\hline 2&4&6&\cdots&2n-6&2n-4&2n-1&2n\\\hline\end{array}\qquad X_4=\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&3&5&\cdots&2n-7&2n-5&2n-4&2n-2\\\hline 2&4&6&\cdots&2n-6&2n-3&2n-1&2n\\\hline\end{array}$$

$$X_5=\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&3&5&\cdots&2n-7&2n-5&2n-4&2n-3\\\hline 2&4&6&\cdots&2n-6&2n-2&2n-1&2n\\\hline\end{array}$$

$$Y_1=\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&3&5&\cdots&2n-7&2n-5&2n-4&2n-3\\\hline 1&3&5&\cdots&2n-7&2n-5&2n-2&2n-1\\\hline 2&4&6&\cdots&2n-6&2n-4&2n-2&2n\\\hline 2&4&6&\cdots&2n-6&2n-3&2n-1&2n\\\hline\end{array}\qquad Y_2=\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&3&5&\cdots&2n-7&2n-5&2n-4&2n-2\\\hline 1&3&5&\cdots&2n-7&2n-5&2n-3&2n-1\\\hline 2&4&6&\cdots&2n-6&2n-4&2n-3&2n\\\hline 2&4&6&\cdots&2n-6&2n-2&2n-1&2n\\\hline\end{array}.$$

Remark 3. Let $\mathcal{M}$ denote the descent of the line bundle $\mathcal{O}(2)$ to X. Then by using *Quantization commutes with reduction* we have $H^0(X(w_5),\mathcal{O}(2k))^T=H^0(X,\mathcal{M}^{\otimes k})$ for $k\in\mathbb{Z}_{\ge0}$ (see [18, Theorem 3.2.a., p. 11] or [16, Theorem 4.1 (ii), p. 526]).

Remark 4.

(i) Note that the set of T-invariant standard monomials of shape $2\varpi_n$ on $X(w_5)$ is $\{X_i: 1\le i\le 5\}$. Thus by [11, Theorem 12.4.8, p. 207], the set $\{X_i: 1\le i\le 5\}$ forms standard monomial basis of R_1.

(ii) The set of T-invariant standard monomials of shape $4\varpi_n$ on $X(w_5)$ is $\{Y_1,Y_2\}\cup\{X_iX_j: 1\le i\le j\le 5\}\setminus\{X_2X_3\}$. Therefore, by [11, Theorem 12.4.8, p. 207], the set $\{Y_1,Y_2\}\cup\{X_iX_j: 1\le i\le j\le 5\}\setminus\{X_2X_3\}$ forms standard monomial basis of R_2.

Theorem 5. *The GIT quotient X with respect to the descent of $\mathcal{O}(2)$ is not projectively normal.*

Proof. Consider the natural map $f: R_1\otimes R_1\to R_2$ of vector spaces given by $X_i\otimes X_j\mapsto X_iX_j$ for $1\le i,j\le 5$. Then f factors through second symmetric power S^2R_1 of the vector space R_1. For simplicity we also denote the factor map $S^2R_1\to R_2$ by f. By Remark 4, we have $\dim(R_1)=5$

and $\dim(R_2) = 16$. So, the map $f : S^2R_1 \to R_2$ cannot be surjective, since $\dim(S^2R_1) = 15 < \dim(R_2)$. $\square$

Corollary 6. *The GIT quotient $T\backslash\backslash(X(w))^{ss}_T(\mathscr{O}(2))$ with respect to the descent of $\mathscr{O}(2)$ is not projectively normal for $w \in I(n,2n)$ such that $w_5 \le w$. In particular, the GIT quotient $T\backslash\backslash(G_{n,2n})^{ss}_T(\mathscr{O}(2))$ with respect to the descent of $\mathscr{O}(2)$ is not projectively normal.*

Proof. By [1, Theorem 3.1.1 (b), p. 85], the restriction map

$$\phi : H^0(X(w),\mathscr{O}(2k)) \to H^0(X(w_5),\mathscr{O}(2k))$$

is surjective. Further, since T is linearly reductive, the restriction map $\phi : H^0(X(w),\mathscr{O}(2k))^T \to H^0(X(w_5),\mathscr{O}(2k))^T$ is surjective for all $k \ge 1$. So, by Theorem 5, $T\backslash\backslash(X(w))^{ss}_T(\mathscr{O}(2))$ with respect to the descent of $\mathscr{O}(2)$ is not projectively normal. $\square$

Lemma 7. *The homogeneous coordinate ring of X is generated by elements of degree at most two.*

Proof. Let $f \in R_k$ be a standard monomial. We claim that $f = f_1f_2$, where f_1 is in R_1 or R_2. The Young diagram associated to f has the shape $(\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n) = (\underbrace{2k,2k,\ldots,2k}_{n})$. So the Young tableau Γ associated to f has $2k$ rows and n columns with strictly increasing rows and non-decreasing columns. Since f is T-invariant, by Lemma 2, we have $c_\Gamma(t) = k$ for all $1 \le t \le 2n$. Let r_i be the i-th row of the tableau. Let $E_{i,j}$ be the (i,j)-th entry of the tableau Γ and $N_{t,j}$ is the number of boxes in the j-th column of Γ containing the integer t.

Recall that w_5 is $(2,4,6,\ldots,2n-6,2n-2,2n-1,2n)$. Since $r_{2k} \le w_5$, we have $E_{2k,j} \le 2j$ for all $1 \le j \le n-3$. Note that for $1 \le j \le n-3$, we have $E_{i,j} = 2j$ for all $k+1 \le i \le 2k$. Thus, for $1 \le j \le n-2$, we have $E_{i,j} = 2j-1$ for all $1 \le i \le k$. Thus, the following rows are the possibilities for r_{2k}:

- $(2,4,6,\ldots,2n-6,2n-4,2n-2,2n)$
- $(2,4,6,\ldots,2n-6,2n-3,2n-2,2n)$
- $(2,4,6,\ldots,2n-6,2n-4,2n-1,2n)$
- $(2,4,6,\ldots,2n-6,2n-3,2n-1,2n)$
- $(2,4,6,\ldots,2n-6,2n-2,2n-1,2n)$.

Case I. Assume that $r_{2k} = (2,4,6,\ldots,2n-6,2n-4,2n-2,2n)$. Then for $1 \le j \le n$, we have $E_{i,j} = 2j-1$ (resp. $E_{i,j} = 2j$) for all $1 \le i \le k$ (resp. for all $k+1 \le i \le 2k$). Therefore, $r_1 = (1,3,5,\ldots,2n-7,2n-5,2n-3,2n-1)$. Hence, r_1, r_{2k} together give a factor X_1 of f.

Case II. Assume that $r_{2k} = (2,4,6,\ldots,2n-6,2n-3,2n-2,2n)$. Since $N_{2n-3,n-2} \ge 1$, we have $N_{2n-4,n-2} \le k-1$. Thus, $E_{1,n-1} = 2n-4$. Since $E_{2k,n-1} = 2n-2$, we have $E_{i,n} = 2n-1$ for all $1 \le i \le k$. Therefore, $r_1 = (1,3,5,\ldots,2n-7,2n-5,2n-4,2n-1)$. Hence, r_1, r_{2k} together give a factor X_2 of f.

Case III. Assume that $r_{2k} = (2,4,6,\ldots,2n-6,2n-4,2n-1,2n)$. Then $E_{i,n-2} = 2n-4$ for all $k+1 \le i \le 2k$ and $E_{i,n-1} = 2n-3$ for all $1 \le i \le k$. Since $N_{2n-1,n-1} \ge 1$, we have $N_{2n-2,n-1} \le k-1$. Thus, $E_{1,n} = 2n-2$. Therefore, $r_1 = (1,3,5,\ldots,2n-7,2n-5,2n-3,2n-2)$. Hence, r_1, r_{2k} together give a factor X_3 of f.

Case IV. Assume that $r_{2k} = (2,4,6,\ldots,2n-6,2n-3,2n-1,2n)$. Since $N_{2n-3,n-2} \ge 1$, we have $N_{2n-4,n-2} \le k-1$. Thus, $E_{1,n-1} = 2n-4$. Since $E_{2k,n-1} = 2n-1$, we have $E_{1,n} \le 2n-2$. Thus, r_1 is either $(1,3,5,\ldots,2n-7,2n-5,2n-4,2n-3)$ or $(1,3,5,\ldots,2n-7,2n-5,2n-4,2n-2)$. If $r_1 = (1,3,5,\ldots,2n-7,2n-5,2n-4,2n-2)$, then r_1, r_{2k} together give a factor X_4 of f. If $r_1 = (1,3,5,\ldots,2n-7,2n-5,2n-4,2n-3)$, then $E_{k,n-1} = 2n-2$. Otherwise, if $E_{k,n-1} = 2n-4$ or $2n-3$, then $\sum_{t=2n-4}^{2n-3}(N_{t,n-2}+N_{t,n-1}+N_{t,n}) \ge 2k+1$, which is a contradiction. Therefore, $E_{k,n} = 2n-1$. Thus, $E_{k+1,n-1} = 2n-2$. Therefore, $r_k = (1,3,5,\ldots,2n-7,2n-5,2n-2,2n-1)$ and $r_{k+1} = (2,4,6,\ldots,2n-6,2n-4,2n-2,2n)$. Hence, $r_1, r_k, r_{k+1}, r_{2k}$ together give a factor Y_1 of f.

Case V. Assume that $r_{2k} = (2,4,6,\ldots,2n-6,2n-2,2n-1,2n)$. Since $E_{2k,n-2} = 2n-2$, we have $E_{1,n-1} = 2n-4$. Thus, $E_{1,n} \geq 2n-3$. If $E_{1,n} = 2n-1$, then $N_{2n-1,n-1} + N_{2n-1,n} \geq k+1$, which is a contradiction. Thus, $E_{1,n}$ is either $2n-3$ or $2n-2$. Thus, r_1 is either $(1,3,5,\ldots,2n-7,2n-5,2n-4,2n-3)$ or $(1,3,5,\ldots,2n-7,2n-5,2n-4,2n-2)$. If $r_1 = (1,3,5,\ldots,2n-7,2n-5,2n-4,2n-3)$, then r_1, r_{2k} together give a factor X_5 of f. If $r_1 = (1,3,5,\ldots,2n-7,2n-5,2n-4,2n-2)$, then $2n-4 \leq E_{k,n-1} \leq 2n-1$. If $E_{k,n-1} = 2n-1$, then $N_{2n-1,n-1} \geq k+1$, which is a contradiction. If $E_{k,n-1} = 2n-2$, then $\sum_{t=2n-2}^{2n-1}(N_{t,n-2} + N_{t,n-1} + N_{t,n}) \geq 2k+2$, which is a contradiction. If $E_{k,n-1} = 2n-4$, then $E_{i,n-2} = 2n-3$ for all $k+1 \leq i \leq 2k-1$. Thus, $c_\Gamma(2n-3) \leq k-1$, which is a contradiction. Therefore, $E_{k,n-1} = 2n-3$. Thus, $E_{k,n} = 2n-1$. Then $2n-3 \leq E_{k+1,n-1} \leq 2n-1$. If $E_{k+1,n-1} = 2n-1$, then $N_{2n-1,n-1} + N_{2n-1,n} \geq k+1$, which is a contradiction. If $E_{k+1,n-1} = 2n-2$, then $\sum_{t=2n-2}^{2n-1}(N_{t,n-2} + N_{t,n-1} + N_{t,n}) \geq 2k+1$, which is a contradiction. Thus, $E_{k+1,n-1} = 2n-3$. Therefore, $r_k = (1,3,5,\ldots,2n-7,2n-5,2n-3,2n-1)$ and $r_{k+1} = (2,4,6,\ldots,2n-6,2n-4,2n-3,2n)$. Therefore, $r_1, r_k, r_{k+1}, r_{2k}$ together give a factor Y_2 of f. □

Lemma 8. *X_i's $(1 \leq i \leq 5)$, and Y_j's $(1 \leq j \leq 2)$ satisfy the following relation in R_2 :* $X_2X_3 = X_1X_4 - Y_2 - Y_1 + X_5(X_1 - X_2 - X_3 + X_4 - X_5)$.

Proof. Note that

$$X_2X_3 = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 3 & 5 & \cdots & 2n-7 & 2n-5 & 2n-4 & 2n-1 \\ \hline 1 & 3 & 5 & \cdots & 2n-7 & 2n-5 & 2n-3 & 2n-2 \\ \hline 2 & 4 & 6 & \cdots & 2n-6 & 2n-3 & 2n-2 & 2n \\ \hline 2 & 4 & 6 & \cdots & 2n-6 & 2n-4 & 2n-1 & 2n \\ \hline \end{array}.$$

By using (1), we have the following straightening laws in $X(w_5)$

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 3 & 5 & \cdots & 2n-7 & 2n-5 & 2n-4 & 2n-1 \\ \hline 1 & 3 & 5 & \cdots & 2n-7 & 2n-5 & 2n-3 & 2n-2 \\ \hline \end{array}$$
$$= \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 3 & 5 & \cdots & 2n-7 & 2n-5 & 2n-4 & 2n-2 \\ \hline 1 & 3 & 5 & \cdots & 2n-7 & 2n-5 & 2n-3 & 2n-1 \\ \hline \end{array} - \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 3 & 5 & \cdots & 2n-7 & 2n-5 & 2n-4 & 2n-3 \\ \hline 1 & 3 & 5 & \cdots & 2n-7 & 2n-5 & 2n-2 & 2n-1 \\ \hline \end{array}$$

and

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline 2 & 4 & 6 & \cdots & 2n-6 & 2n-3 & 2n-2 & 2n \\ \hline 2 & 4 & 6 & \cdots & 2n-6 & 2n-4 & 2n-1 & 2n \\ \hline \end{array}$$
$$= \begin{array}{|c|c|c|c|c|c|c|c|}\hline 2 & 4 & 6 & \cdots & 2n-6 & 2n-4 & 2n-2 & 2n \\ \hline 2 & 4 & 6 & \cdots & 2n-6 & 2n-3 & 2n-1 & 2n \\ \hline \end{array} - \begin{array}{|c|c|c|c|c|c|c|c|}\hline 2 & 4 & 6 & \cdots & 2n-6 & 2n-4 & 2n-3 & 2n \\ \hline 2 & 4 & 6 & \cdots & 2n-6 & 2n-2 & 2n-1 & 2n \\ \hline \end{array}.$$

Therefore, by using the above straightening laws we have

$$X_2X_3 = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 3 & 5 & \cdots & 2n-7 & 2n-5 & 2n-4 & 2n-2 \\ \hline 1 & 3 & 5 & \cdots & 2n-7 & 2n-5 & 2n-3 & 2n-1 \\ \hline 2 & 4 & 6 & \cdots & 2n-6 & 2n-4 & 2n-2 & 2n \\ \hline 2 & 4 & 6 & \cdots & 2n-6 & 2n-3 & 2n-1 & 2n \\ \hline \end{array} - \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 3 & 5 & \cdots & 2n-7 & 2n-5 & 2n-4 & 2n-2 \\ \hline 1 & 3 & 5 & \cdots & 2n-7 & 2n-5 & 2n-3 & 2n-1 \\ \hline 2 & 4 & 6 & \cdots & 2n-6 & 2n-4 & 2n-3 & 2n \\ \hline 2 & 4 & 6 & \cdots & 2n-6 & 2n-2 & 2n-1 & 2n \\ \hline \end{array}$$

$$- \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 3 & 5 & \cdots & 2n-7 & 2n-5 & 2n-4 & 2n-3 \\ \hline 1 & 3 & 5 & \cdots & 2n-7 & 2n-5 & 2n-2 & 2n-1 \\ \hline 2 & 4 & 6 & \cdots & 2n-6 & 2n-4 & 2n-2 & 2n \\ \hline 2 & 4 & 6 & \cdots & 2n-6 & 2n-3 & 2n-1 & 2n \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 3 & 5 & \cdots & 2n-7 & 2n-5 & 2n-4 & 2n-3 \\ \hline 1 & 3 & 5 & \cdots & 2n-7 & 2n-5 & 2n-2 & 2n-1 \\ \hline 2 & 4 & 6 & \cdots & 2n-6 & 2n-4 & 2n-3 & 2n \\ \hline 2 & 4 & 6 & \cdots & 2n-6 & 2n-2 & 2n-1 & 2n \\ \hline \end{array}$$

$$= X_1X_4 - Y_2 - Y_1 + X_5Z,$$

where

$$Z = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 3 & 5 & \cdots & 2n-7 & 2n-5 & 2n-2 & 2n-1 \\ \hline 2 & 4 & 6 & \cdots & 2n-6 & 2n-4 & 2n-3 & 2n \\ \hline \end{array}.$$

By using $(*)$ (see Appendix) we have $Z = X_1 - X_2 - X_3 + X_4 - X_5$. Therefore, $X_2X_3 = X_1X_4 - Y_2 - Y_1 + X_5(X_1 - X_2 - X_3 + X_4 - X_5)$. □

Proposition 9. *We have*

(i) *The GIT quotient $T\backslash\backslash(X(w_1))^{ss}_T(\mathcal{O}(2))$ with respect to the descent of $\mathcal{O}(2)$ is projectively normal and isomorphic to point.*

(ii) *The GIT quotient $T\backslash\backslash(X(w_2))^{ss}_T(\mathcal{O}(2))$ with respect to the descent of $\mathcal{O}(2)$ is projectively normal and isomorphic to $\mathbb{P}^1$ polarized with $\mathcal{O}(1)$.*

(iii) *The GIT quotient $T\backslash\backslash(X(w_3))^{ss}_T(\mathcal{O}(2))$ with respect to the descent of $\mathcal{O}(2)$ is projectively normal and isomorphic to $\mathbb{P}^1$ polarized with $\mathcal{O}(1)$.*

(iv) *The GIT quotient $T\backslash\backslash(X(w_4))^{ss}_T(\mathcal{O}(2))$ with respect to the descent of $\mathcal{O}(2)$ is projectively normal and isomorphic to $\mathbb{P}^3$ polarized with $\mathcal{O}(1)$.*

Proof. Note that $Y_2 = 0$ on $X(w_i)$ for all $1 \le i \le 4$.

Proof of (iv). By [1, Theorem 3.1.1(b), p. 85], the restriction map $H^0(X(w_5), \mathcal{O}(2k)) \to H^0(X(w_4), \mathcal{O}(2k))$ is surjective. Further, since T is linearly reductive, the restriction map $H^0(X(w_5), \mathcal{O}(2k))^T \to H^0(X(w_4), \mathcal{O}(2k))^T$ is surjective for all $k \ge 1$. Note that $X_5 = 0$ on $X(w_4)$. Consider X_i's $(1 \le i \le 4)$ as elements of $H^0(X(w_4), \mathcal{O}(2))^T$. Recall that $(X_1, \ldots, X_4)$ is a basis of $H^0(X(w_4), \mathcal{O}(2))^T$.

We claim that any relation among X_i's $(1 \le i \le 4)$ given by a homogeneous polynomials of degree k is identically zero. Suppose

$$\sum c_{\underline{m}} X^{\underline{m}} = 0 \tag{2}$$

where $\underline{m} = (m_1, m_2, m_3, m_4)$ are tuples of non-negative integers such that $m_1 + m_2 + m_3 + m_4 = k$, $X^{\underline{m}} = X_1^{m_1} X_2^{m_2} X_3^{m_3} X_4^{m_4}$ and $c_{\underline{m}}$'s are non-zero scalars. Then rewriting (2) as

$$\sum_{m_2 \le m_3} c_{\underline{m}} X_1^{m_1} (X_2X_3)^{m_2} X_3^{m_3 - m_2} X_4^{m_4} + \sum_{m_2 > m_3} c_{\underline{m}} X_1^{m_1} X_2^{m_2 - m_3} (X_2X_3)^{m_3} X_4^{m_4} = 0.$$

Recall that by Lemma 8, we have $X_2X_3 = X_1X_4 - Y_1$. Replacing X_2X_3 by $X_1X_4 - Y_1$ in the above equation we get

$$\sum_{m_2 \le m_3} c_{\underline{m}} X_1^{m_1} (X_1X_4 - Y_1)^{m_2} X_3^{m_3 - m_2} X_4^{m_4} + \sum_{m_2 > m_3} c_{\underline{m}} X_1^{m_1} X_2^{m_2 - m_3} (X_1X_4 - Y_1)^{m_3} X_4^{m_4} = 0.$$

Note that any monomial in X_1, X_3, X_4, Y_1 (respectively, in X_1, X_2, X_4, Y_1) is standard. Hence, $c_{\underline{m}} = 0$ for all $\underline{m}$. Thus, X_1, X_2, X_3, X_4 are algebraically independent. Further, by Lemma 7, and above surjectivity, the homogeneous coordinate ring of $T\backslash\backslash(X(w_4))^{ss}_T(\mathcal{O}(2))$ is generated by X_1, X_2, X_3, X_4. On the other hand, by [1, Theorem 3.2.2, p. 92], $X(w_4)$ is normal. As $T\backslash\backslash(X(w_4))^{ss}_T(\mathcal{O}(2))$ is an open subset of $X(w_4)$, it is also normal. Hence, by Remark 3, the GIT quotient $T\backslash\backslash(X(w_4))^{ss}_T(\mathcal{O}(2))$ with respect to the descent of $\mathcal{O}(2)$ is projectively normal and isomorphic to $\mathrm{Proj}(\mathbb{C}[X_1, X_2, X_3, X_4]) = \mathbb{P}^3$ polarized with $\mathcal{O}(1)$.

By [1, Theorem 3.1.1(b), p. 85], it follows that the restriction map $H^0(X(w_4), \mathcal{O}(2k)) \to H^0(X(w_i), \mathcal{O}(2k))$ is surjective for all $1 \le i \le 3$. Further, since T is linearly reductive, the restriction map $H^0(X(w_4), \mathcal{O}(2k))^T \to H^0(X(w_i), \mathcal{O}(2k))^T$ is surjective for all $k \ge 1$. On the other hand, by [1, Theorem 3.2.2, p. 92], $X(w_i)$ is normal. As $T\backslash\backslash(X(w_i))^{ss}_T(\mathcal{O}(2))$ is an open subset of $X(w_i)$, it is also normal. Hence, by (iv) the GIT quotient $T\backslash\backslash(X(w_i))^{ss}_T(\mathcal{O}(2))$ with respect to the descent of $\mathcal{O}(2)$ is also projectively normal for all $1 \le i \le 3$.

Proof of (iii). Note that X_2 and X_4 are identically zero on $X(w_3)$. Since the restriction map $H^0(X(w_4), \mathcal{O}(2k))^T \to H^0(X(w_3), \mathcal{O}(2k))^T$ is surjective for all $k \ge 1$, by (iv) any standard monomial in $H^0(X(w_3), \mathcal{O}(2k))^T$ is of the form $X_1^{k_1} X_3^{k_2}$, where $k_1 + k_2 = k$. Hence, the GIT quotient $T\backslash\backslash(X(w_3))^{ss}_T(\mathcal{O}(2))$ is isomorphic to $\mathrm{Proj}(\mathbb{C}[X_1, X_3]) = \mathbb{P}^1$ polarized with $\mathcal{O}(1)$.

Proof of (ii). Note that X_3 and X_4 are identically zero on $X(w_2)$. Since the restriction map $H^0(X(w_4), \mathcal{O}(2k))^T \to H^0(X(w_2), \mathcal{O}(2k))^T$ is surjective for all $k \ge 1$, by (iv) any standard monomial in $H^0(X(w_2), \mathcal{O}(2k))^T$ is of the form $X_1^{k_1} X_2^{k_2}$, where $k_1 + k_2 = k$. Hence, the GIT quotient $T\backslash\backslash(X(w_2))^{ss}_T(\mathcal{O}(2))$ is isomorphic to $\mathrm{Proj}(\mathbb{C}[X_1, X_2]) = \mathbb{P}^1$ polarized with $\mathcal{O}(1)$.

Proof of (i). Note that X_2, X_3 and X_4 are identically zero on $X(w_1)$. Since the restriction map $H^0(X(w_4), \mathcal{O}(2k))^T \to H^0(X(w_1), \mathcal{O}(2k))^T$ is surjective for all $k \ge 1$, by (iv) any standard monomial in $H^0(X(w_1), \mathcal{O}(2k))^T$ is of the form X_1^k. Hence, the GIT quotient $T\backslash\backslash(X(w_1))^{ss}_T(\mathcal{O}(2))$ is isomorphic to $\mathrm{Proj}(\mathbb{C}[X_1])$. $\square$

Remark 10.

(i) The GIT quotient $T\backslash\backslash(G_{1,2})^{ss}_T(\mathcal{O}(2))$ with respect to the descent of $\mathcal{O}(2)$ is projectively normal and isomorphic to point.

(ii) The GIT quotient $T\backslash\backslash(G_{2,4})^{ss}_T(\mathcal{O}(2))$ with respect to the descent of $\mathcal{O}(2)$ is projectively normal (see [3, Theorem 2.3, p. 182]) and isomorphic to $(\mathbb{P}^1, \mathcal{O}(1))$ (see [7, Proposition 3.5, p. 277]).

Now we prove that the GIT quotient of $X(w_5)$ by T with respect to the descent of $\mathcal{O}(4)$ is projectively normal.

Theorem 11. *The homogeneous coordinate ring of $T\backslash\backslash(X(w_5))^{ss}_T(\mathcal{O}(4))$ is generated by elements of degree one.*

Proof. Let $f \in H^0(X(w_5), \mathcal{O}(4)^{\otimes k})^T = H^0(X(w_5), \mathcal{O}(4k))^T$. Then by Lemma 7, we have

$$f = \sum a_{(\underline{m},\underline{n})} X^{\underline{m}} Y^{\underline{n}},$$

where $\underline{m} = (m_1, m_2, m_3, m_4, m_5)$, $\underline{n} = (n_1, n_2)$, are tuples of non-negative integers such that $m_1 + m_2 + m_3 + m_4 + m_5 + 2n_1 + 2n_2 = 2k$, $X^{\underline{m}} = X_1^{m_1} X_2^{m_2} X_3^{m_3} X_4^{m_4} X_5^{m_5}$, $Y^{\underline{n}} = Y_1^{n_1} Y_2^{n_2}$, and $a_{(\underline{m},\underline{n})}$'s are non-zero scalars.

Now to prove that the homogeneous coordinate ring of $T\backslash\backslash(X(w_5))^{ss}_T(\mathcal{O}(4))$ is generated by $H^0(X(w_5), \mathcal{O}(4))^T$ as a $\mathbb{C}$-algebra, it is enough to show that for each f as above and each monomial appearing in the expression of f is in the image of $S^k H^0(X(w_5), \mathcal{O}(4))^T$, under the natural map $S^k H^0(X(w_5), \mathcal{O}(4))^T \to H^0(X(w_5), \mathcal{O}(4k))^T$, where $S^k H^0(X(w_5), \mathcal{O}(4))^T$ denotes the k^{th} symmetric power of the vector space $H^0(X(w_5), \mathcal{O}(4))^T$.

Consider the monomial $X^{\underline{m}}Y^{\underline{n}}$ appearing in the expression of f. Since $m_1+m_2+m_3+m_4+m_5$ is even integer, $X^{\underline{m}}$ can be written as $\prod_{(i,j)} X_iX_j$, where the number of pairs (i,j) is $(k-n_1-n_2)$ and repetition of X_i's are allowed. Thus, $X^{\underline{m}}$ is a product of $(k-n_1-n_2)$ number of monomials in $H^0(X(w_5),\mathcal{O}(4))^T$. Note that $Y_1, Y_2 \in H^0(X(w_5),\mathcal{O}(4))^T$. Therefore, $X^{\underline{m}}Y^{\underline{n}}$ is in the image of $S^k(H^0(X(w_5),\mathcal{O}(4))^T)$ under the natural map. □

Corollary 12. *The GIT quotient $T\backslash\backslash(X(w_5))^{ss}_T(\mathcal{O}(4))$ is projectively normal with respect to the descent of $\mathcal{O}(4)$.*

Proof. Follows from Theorem 11. □

Corollary 13. *The GIT quotient $T\backslash\backslash(G_{3,6})^{ss}_T(\mathcal{O}(4))$ is projectively normal with respect to the descent of $\mathcal{O}(4)$.*

Proof. Note that for $n=3$, $w_5=(4,5,6)$. So, we have $X(w_5)=G_{3,6}$. Therefore, proof immediately follows from Theorem 11. □

In the view of the above results the following question is open:

Problem. Is the GIT quotient of $G_{n,2n}$ $(n\geq 4)$ by T with respect to the descent of $\mathcal{O}(4)$ projectively normal?

Appendix

Here, we prove the following straightening law on $X(w_5)$ that we used in the proof of Lemma 8.

$$\begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 3 & 5 & \cdots & 2n-7 & 2n-5 & 2n-2 & 2n-1 \\ \hline 2 & 4 & 6 & \cdots & 2n-6 & 2n-4 & 2n-3 & 2n \\ \hline \end{array} = X_1 - X_2 - X_3 + X_4 - X_5. \qquad (*)$$

Proof. Let $\underline{i}=\{1,3,\ldots,2n-7\}$ and $\underline{j}=\{2,4,6,\ldots,2n-6\}$. Let $I=\{1,3,\ldots,2n-7,2n-5,2n-4\}$ and $J=\{2,4,\ldots,2n-6,2n-3,2n-2,2n-1,2n\}$ be two subsets of $\{1,2,\ldots,2n\}$. Then by using (2.1) the following straightening law holds in $X(w_5)$

$$p_{\underline{i},2n-5,2n-4,2n-3}p_{\underline{j},2n-2,2n-1,2n} - p_{\underline{i},2n-5,2n-4,2n-2}p_{\underline{j},2n-3,2n-1,2n} + p_{\underline{i},2n-5,2n-4,2n-1}p_{\underline{j},2n-3,2n-2,2n} - p_{\underline{i},2n-5,2n-4,2n}p_{\underline{j},2n-3,2n-2,2n-1} = 0. \qquad (3)$$

Let $I=\{1,3,\ldots,2n-7,2n-5,2n-3\}$ and $J=\{2,4,\ldots,2n-6,2n-4,2n-2,2n-1,2n\}$. Then by using (2.1) the following straightening law holds in $X(w_5)$

$$p_{\underline{i},2n-5,2n-4,2n-3}p_{\underline{j},2n-2,2n-1,2n} + p_{\underline{i},2n-5,2n-3,2n-2}p_{\underline{j},2n-4,2n-1,2n} - p_{\underline{i},2n-5,2n-3,2n-1}p_{\underline{j},2n-4,2n-2,2n} + p_{\underline{i},2n-5,2n-3,2n}p_{\underline{j},2n-4,2n-2,2n-1} = 0. \qquad (4)$$

Let $I=\{1,3,\ldots,2n-7,2n-5,2n-2\}$ and $J=\{2,4,\ldots,2n-6,2n-4,2n-3,2n-1,2n\}$. Then by using (2.1) the following straightening law holds in $X(w_5)$

$$p_{\underline{i},2n-5,2n-4,2n-2}p_{\underline{j},2n-3,2n-1,2n} - p_{\underline{i},2n-5,2n-3,2n-2}p_{\underline{j},2n-4,2n-1,2n} - p_{\underline{i},2n-5,2n-2,2n-1}p_{\underline{j},2n-4,2n-3,2n} + p_{\underline{i},2n-5,2n-2,2n}p_{\underline{j},2n-4,2n-3,2n-1} = 0. \qquad (5)$$

Let $I=\{1,3,\ldots,2n-7,2n-5,2n-1\}$ and $J=\{2,4,\ldots,2n-6,2n-4,2n-3,2n-2,2n\}$. Then by using (2.1) the following straightening law holds in $X(w_5)$

$$p_{\underline{i},2n-5,2n-4,2n-1}p_{\underline{j},2n-3,2n-2,2n} - p_{\underline{i},2n-5,2n-3,2n-1}p_{\underline{j},2n-4,2n-2,2n} + p_{\underline{i},2n-5,2n-2,2n-1}p_{\underline{j},2n-4,2n-3,2n} + p_{\underline{i},2n-5,2n-1,2n}p_{\underline{j},2n-4,2n-3,2n-2} = 0. \qquad (6)$$

Let $I = \{1,3,\ldots,2n-7,2n-5,2n\}$ and $J = \{2,4,\ldots,2n-6,2n-4,2n-3,2n-2,2n-1\}$. Then by using (2.1) the following straightening law holds in $X(w_5)$

$$p_{\underline{i},2n-5,2n-4,2n}p_{\underline{j},2n-3,2n-2,2n-1} - p_{\underline{i},2n-5,2n-3,2n}p_{\underline{j},2n-4,2n-2,2n-1} + p_{\underline{i},2n-5,2n-2,2n}p_{\underline{j},2n-4,2n-3,2n-1} - p_{\underline{i},2n-5,2n-1,2n}p_{\underline{j},2n-4,2n-3,2n-2} = 0. \quad (7)$$

Thus by using (5), we have

$$p_{\underline{i},2n-5,2n-2,2n-1}p_{\underline{j},2n-4,2n-3,2n} = p_{\underline{i},2n-5,2n-4,2n-2}p_{\underline{j},2n-3,2n-1,2n} - p_{\underline{i},2n-5,2n-3,2n-2}p_{\underline{j},2n-4,2n-1,2n} + p_{\underline{i},2n-5,2n-2,2n}p_{\underline{j},2n-4,2n-3,2n-1}.$$

By using (7) we have

$$\begin{aligned} p_{\underline{i},2n-5,2n-2,2n-1}p_{\underline{j},2n-4,2n-3,2n} &= p_{\underline{i},2n-5,2n-4,2n-2}p_{\underline{j},2n-3,2n-1,2n} - p_{\underline{i},2n-5,2n-3,2n-2}p_{\underline{j},2n-4,2n-1,2n} \\ &\quad - p_{\underline{i},2n-5,2n-4,2n}p_{\underline{j},2n-3,2n-2,2n-1} + p_{\underline{i},2n-5,2n-3,2n}p_{\underline{j},2n-4,2n-2,2n-1} \\ &\quad + p_{\underline{i},2n-5,2n-1,2n}p_{\underline{j},2n-4,2n-3,2n-2}. \end{aligned}$$

Further, by using (3), (4) and (6), $(*)$ follows. □

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