Journal of Advanced Zoology

ISSN: 0253-7214

*Volume **44** Issue **S-2 Year 2023** Page **2021:2030***

THE CHARACTERIZED CONCEPT OF VAGUE CO-FRAMES

R Panduranga rao[1] , Y.V.Seshagiri Rao[2*], G. Y. Sagar[3], B. Nageswara Rao[4]

[1,2]Department of Mathematics, Vignan Institute of Technology and Science(A), Near Ramoji Film City, Vignan Hills, Deshmukhi (V), Pochampally (M), Yadadri Bhuvanagiri (D) - 508284, Telangana, India

[3]Department of Statistics, College of Natural & Computational Sciences, Gambella University, Ethiopia, P.O.Box:126

[4]Department of Mathematics, Lendi Institute of Engineering and Technology, Jonanda, Vizinagaram- Dist,pin:535005, Andhra Pradesh , India.

Article History

Received: 12 July 2023

Revised: 10 September 2023

Accepted:27 October 2023

Abstract:-

In this paper we introduce vague frame, vague co- frames, vague frame generated by frame and investigate some of its basic properties. In particular, some interesting characterizations closely related to the vague co- frames and vague cut sets on vague co-frame are given also studied their properties. Further we investigate the development of some important results and theorems about vague cut sets on vague co-frames.

Keywords: - Vague set, Vague Frame, vague cut-set, Vague co-frame,

Mathematics Subject Classification (2000): 08A73

I. **Introduction**

The concept of frames has been studied by many mathematicians including Banaschewski, Diwker,and Johnstone,For details one can refer to [2-3].In 1965 Also the concept proposed by Zadeh.L.A.[12] defining a fuzzy subset A of a given universe X characterizing the membership of an element x of X belonging to A by means of a membership maping $m_A(x)$ defined from X in to [0 1] has revolutionized the theory of Mathematical modeling. Decision making etc.,in handling the imprecise real life situations mathematically. Now several branches of fuzzy mathematics like fuzzy algebra,fuzzy topology ,fuzzy control theory ,fuzzy measure theory etc.,haveemerged. But in the decision making, the fuzzy theory takes care of membership of an element x only, that is the evidence against x belonging to A .It is felt by several decision makers and researchers that in proper decision making,the evidence belongs to A and evidence not belongs to A are both necessary and how much X belongs to A or how much x does not belongs to A are necessary. Several generalizations of Zadeh's fuzzy set theory have been proposed, such as L-fuzzy sets [6]. Interval valued fuzzy sets,Intuitionistic fuzzy sets by Atanassov.K.T [1] ,Vague sets [5] are mathematically equivalent. Any such set A of a given Universe X can be charactrierized by means of a pair of function (m_A , n_A)

where m_A : X to [0 1] and f_A : X *to* [0, 1] such that $0 \leq m_A(x) + n_A(x) \leq 1$ for all x in X.The

set $m_A(x)$ is called the membership function and the set $n_A(x)$ is called non membership function and $m_A(x)$ gives the evidence of how much x belongs to A, $n_A(x)$ gives the

evidence of how much x does not belongs to A.These concepts are being applied in several areas like decision-making, fuzzy control, knowledge discovery and fault diagnosis etc.It is believed the vague sets (or equivalently intuitionistic fuzzy sets) will more useful in decision making, and other areas of Mathematical modeling. Through Atanassov's intuitionisticfuzzysets, [4] Gahu.w and Buehrer.D.J and some other areas of Mathematical modeling. Since then the theory of fuzzy sets developed extensively and embraced almost all subjects like engineering science and technology. But the membership function m_A (x) gives only a approximation belong to A .To avoid this and obtain a better estimation and analysis of data decision making.Gahu.W. and Bueher D.J. [4] have initiated the study ofvague sets with the hope that they form a better tool to understand, interpret and solve real life problems which are in general vague,than the theory of vague sets do.The objective of this paper is to contribute further to the study of vague algebra by introducing the concepts of vague frames, vague co-frames and vague cut-set on vague frames, vague co-frames etc on vague frames with suitable examples.

2. Preliminaries

In this section we briefly present the necessary material on frame, fuzzy frames, lattices, complete lattices and illustrate with examples.

Definition 2.1 [4]

A frame is a complete lattice L satisfying the distributive law x⋀(VS)= V{x⋀s:s∈S} for all x∈ L and S⊆ L, ⋀ denotes arbitrary meet and V denotes arbitrary join

Definition 2.1 [4]

A co-frame is a complete lattice F satisfying the infinite distributive lattice L satisfying the distributive law x⋀(VS)= V{x⋀s:s∈S} for all x∈ L and S⊆ L, ⋀ denotes arbitrary meet and V denotes arbitrary join

Definition 2.1 [4] For any co-frame F, a sub set p of S is a filter if x, y x ∈ L

(x⋀y) ∈ L and x∈ L,y≥x implies y≥P ∈ L

Definition 2.2 Let F be a fuzzy frame, then a fuzzy set μ:F→[0 1] of F is said to be a fuzzy frame if it satisfies the following properties.

1. μ(VS)≥min{μ(a):a∈ S} for every S⊂ F
2. μ(a⋀b)≥ min{ μ(a), μ(b)} for all a,b∈ F
3. $\mu(e_f)= \mu(o_f) \geq \mu(a)$ for all a∈ F,where e_f and o_f respectively the unit and zero element of the frame F.

Definition 2.9: Vague sets A and B are equal, written as $A = B$, *iff* $A \subseteq B$ *and* $B \subseteq A$. *i.e.,* $t_A = t_B$,$f_A = f_B$.

3. Vague Co-frames: In this section we introduce vague frames, vague co-frames and complement of vague co frames etc.

Definition 3.1 Let F be a co-frame, Let $A=(t_A, f_A)$be a vague set of F is said to be vague frame if,

1. $t_A(x\wedge y)\geq \min\{t_A(x), t_A(y)\}$
2. $f_A(x\wedge y)\leq \max\{f_A(x), f_A(y)\}$
3. $t_A(x\vee y)\geq \max\{t_A(x), t_A(y)\}$
4. $f_A(x\vee y)\leq \min\{f_A(x), f_A(y)\}\ \forall x, y\in f$

Lemma 3.2 If A and B are any two vague frames of F. Then $(A\cap B)$ is also vague frame of F and its union is need not be a vague frame.

Definition 3.3 Let A be a vague frame set and $\langle A\rangle=\cap\{B: A\subseteq B\}$,B is a vague frame of F where $A\subseteq B$ means $t_A(x)\leq t_B(x)$ and $f_A(x)\geq f_B(x)\ \forall x\in F$. Then $\langle A\rangle$is called the vague frame generated by A.

Definition 3.4 The operation of meet and join on a coframe F can be extended to vague sets of F as follows.

$(i)\ (A\barwedge B)=\{(x, t_{(A\barwedge B)}(x), f_{(A\barwedge B)}(x)); x\in f\}$

$(ii)\ (A\bar\vee B)=\{(x, t_{(A\bar\vee B)}(x), f_{(A\bar\vee B)}(x)); x\in f\}$

Where $t_{(A\barwedge B)}(x)=\max_{u\wedge v=x}\{t_A(u)\wedge t_B(v)\}\ and\ f_{(A\barwedge B)}(x)=\min_{u\wedge v=x}\{f_A(u)\vee f_B(v)\}$

$t_{(A\bar\vee B)}(x)=\max_{u\vee v=x}\{t_A(u)\wedge t_B(v)\}\ and\ f_{(A\bar\vee B)}(x)=\min_{u\vee v=x}\{t_A(u)\vee t_B(v)\}$

The operations u and v an co-frame f can be retrieved from $\barwedge$ and $\bar\vee$ by embedding F into vague set as the set of all vague singletons of which is an vague set

$F_x=\{(y, t_A(y), f_A(y); y\in F)\}$

Where $t_A(y)=\begin{cases}1 & ; x=y\\ 0 & ; x\neq y\end{cases}\quad and\quad f_A(y)=\begin{cases}0 & ; x=y\\ 1 & ; x\neq y\end{cases}$

3.5 Theorem. If every non-empty vague cut-set $A_{(\alpha,\beta)}$ of a vague set A is a vague frame of f then A is a vague frame on f .

Proof: Let $A_{(\alpha,\beta)}=\{x\in f; t_A(x)\geq\alpha, f_A(x)\leq\beta\}\ \forall\alpha,\beta\in[0,1]$

Hence $f_A(e_f)=f_A(o_f)\leq\beta$ and $t_A(e_f)=t_A(o_f)\ \forall\alpha\in[0,1]$

In particular we have o_f and e_f belongs to both $(t_A)_{T_1}$ and $(f_A)_{T_1}$ also $(t_A)_{T_2}$ and $(f_A)_{T_2}$.

Where T_1 and T_2 are largest and smallest elements of $[0,1]$ such that $(t_A)_{T_1}, (f_A)_{T_1}, (t_A)_{T_2}$ and $(f_A)_{T_2}$ are non-empty.

Hence $t_A(e_f)=f_A(o_f)\geq t_A(x)\ and\ f_A(e_f)=f_A(o_f)\leq t_A(x)\ \forall x\in f$

Now let S be an arbitrary subset of f .

Then $(i)\ t_A(\vee S)\geq\min\{t_A(x); x\in S\}$

$f_A(\vee S)\leq\max\{f_A(x); x\in S\}$

Observe there exists some $S_0\subset F$ such that

$t_A(\vee S)\leq\min\{t_A(x); x\in S_0\}$ or $f_A(\vee S)\geq\max\{f_A(x); x\in S_0\}$

Taking $A_{(\alpha,\beta)} = \frac{1}{2}\{t_A(\vee S_0) + \min\{t_A(x); x \in S_0\}\}$

We have $t_A(\vee S_0) < t_0 < \min\{t_A(x); x \in S_0\}$

Hence $\alpha \in t_{A_\alpha}$ $\forall x \in S_0$ and hence $S_0 \subseteq t_{A_\alpha}$

Since $(t_A)_{A_{(\alpha_0,\beta_0)}}$ is a frame and we have $(\vee S_0) \in (t_A)_{A_{(\alpha,\beta)}}$

$\therefore$ $t_A(\vee S_0) > t_{A_{(\alpha,\beta)}} > t_{A_{(\alpha_0,\beta_0)}}$ $\forall \alpha_0, \beta_0 \in [0,1]$

Which is contradiction.

Similarly by taking $t_0 = \frac{1}{2}\{f_A(\vee S_0) + \max\{f_A(x); x \in S_0\}\}$ and we have $f_A(\vee S_0) > t_{A_{(\alpha_0,\beta_0)}} > \max\{f_A(x); x \in S_0\}$,Hence $t_{A_{(\alpha_0,\beta_0)}} > f_A(x)$ $\forall x \in S_0$

$\therefore$ $x \in f_{A_{(\alpha_0,\beta_0)}}$ $\forall x \in S_0$ and hence $S_0 \subseteq f_{A_\alpha}$

$\therefore$ $x \in A_{(\alpha_0,\beta_0)}$ is a vague frame.

3.6 Theorem. Let A and B are vague frames then $(A \bar{\nabla} B) = (A \cap B)$

Proof: we have $x \in F$, $t_{(A \bar{\nabla} B)}(x) = \max_{u \vee v = x}\{t_A(u) \wedge t_B(v)\}$

$\geq t_A(x) \wedge t_B(x) \geq t_{A \cap B}(x)$

And $f_{(A \bar{\nabla} B)}(x) = \min_{u \vee v = x}\{f_A(u) \vee f_B(v)\}$

$\leq f_A(x) \vee f_B(x) \leq f_{A \cap B}(x)$

Hence $(A \cap B) \subseteq (A \bar{\nabla} B)$,Now since $t_{A_i}(u \vee v) = t_{A_i}(x) \geq t_{A_i}(u) \vee t_{A_i}(v)$

And hence we have $t_{A_i}(x) \geq t_{A_i}(u)$ and $t_{A_i}(x) \geq t_{A_i}(v)$ for $A_i = B_i$

$t_{(A \bar{\nabla} B)}(x) = \max_{u \vee v = x}\{t_A(u) \wedge t_B(v)\}$

$\leq \max_{u \vee v = x}\{t_A(x) \wedge t_B(x)\} \leq t_A(x) \wedge t_B(x) \leq t_{(A \cap B)}(x)$

$f_{(A \bar{\nabla} B)}(x) = \min_{u \vee v = x}\{f_A(u) \vee f_B(v)\}$

$\geq \min_{u \vee v = x}\{f_A(x) \vee f_B(x)\} \geq f_A(x) \vee f_B(x) \geq f_{(A \cap B)}(x)$

$(A \bar{\nabla} B) \subseteq (A \cap B)$

Hence $(A \bar{\nabla} B) = (A \cap B)$.

Theorem.3.7 If A is a vague frame of a frame f. Then every vague cut-set $A_{(\alpha,\beta)}$ of A is a vague frame of f.

Proof: Let $A = (x, t_A, f_A)$ be a vague frame of f.

Let $x, y \in A_{(\alpha,\beta)}$ then $t_A(x) \geq \alpha$ and $f_A(x) \leq \beta$

$(i)\, t_A(x \wedge y) \geq \min\{t_A(x), t_A(y)\} \geq \alpha$

$\Rightarrow t_A(x \wedge y) \geq \alpha$

$(ii)\; f_A(x \wedge y) \leq \max\{f_A(x), f_A(y)\}$

$\leq \max\{\beta, \beta\}$

$\leq \beta$

$\Rightarrow f_A(x \wedge y) \leq \beta$

$(iii)\, t_A(\vee A_{(\alpha,\beta)}) \geq \min\{t_A(x); x \in A_{(\alpha,\beta)}\}$

$f_A(\vee A_{(\alpha,\beta)}) \leq \max\{f_A(x); x \in A_{(\alpha,\beta)}\}$

$\Rightarrow o_f$ *and* $e_f \in A_{(\alpha,\beta)}$

$\therefore\; A_{(\alpha,\beta)}$ is a vague frame of f .

Theorem 3.8 Given an arbitrary collection $(A_\alpha)_{\alpha\in\Lambda}$ and A is a vague frame of f then

$$A \cap \left(\underset{\alpha\in\Lambda}{\barwedge} A_\alpha \right) = \underset{\alpha\in\Lambda}{\barwedge} (A \cap A_\alpha)$$

Proof: we have for $w \in F, t_{\left[A \cap \left(\underset{\alpha\in\Lambda}{\barwedge} A_\alpha\right)\right]}(w) = t_A(w) \wedge \underset{\alpha\in\Lambda}{\barwedge}(t_{A_\alpha})(w)$

$= t_A(w) \wedge \max_{a_\alpha \in A = w} \{\wedge_{\alpha\in\Lambda}(t_{A_\alpha}(a_\alpha))\}$

$= \max\{\wedge_{\alpha\in\Lambda}(t_A(w) \wedge t_{A_\alpha}(a_\alpha))\}$

$\leq \max\{\wedge_{\alpha\in\Lambda}(t_A(a_\alpha \vee w) \wedge t_{A_\alpha}(a_\alpha))\}$

$= \max_{a_\alpha \in \wedge a_\alpha = w}\{\wedge_{\alpha\in\Lambda}(t_A(a_\alpha) \wedge t_{A_\alpha}(a_\alpha))\}$

$= \max\{\wedge_{\alpha\in\Lambda}(t_A \cap t_{A_\alpha})(\alpha)\}$

$= \{t_{\underset{\alpha\in A}{\barwedge}}(A \cap A_\alpha)(w), f_{A \cap A_\alpha \in A}(w)\}$

$= f_A(w) \wedge (\underset{\alpha\in A}{\barwedge} f_{A_\alpha})(w)$

$= f_A(w) \wedge \max_{\wedge_\alpha \in \wedge a_\alpha = w}(\vee_{\alpha\in A} f_{A_\alpha}(a_\alpha))$

$= \min_{\wedge_\alpha \in \wedge a_\alpha = w}(\vee_{\alpha\in A}(f_A(w) \wedge f_{A_\alpha}(a_\alpha)))$

$= \min_{\wedge_\alpha \in \wedge a_\alpha = w}(\vee_{\alpha\in A}(f_A(a_\alpha \vee w) \wedge f_{A_\alpha}(a_\alpha)))$

$= \min_{\wedge_\alpha \in \wedge a_\alpha = w}(\vee_{\alpha\in A}(f_A(a_\alpha) \wedge f_{A_\alpha}(a_\alpha)))$

$= \min_{\wedge_\alpha \in \wedge a_\alpha = w}(\vee_{\alpha\in A}(f_A \cap f_{A_\alpha})(a_\alpha))$

$= f_{\wedge_\alpha \in A}(A \cap A_\alpha)(w)$

$\therefore A \cap (\underset{\alpha\in A}{\barwedge}(A_\alpha)) \subseteq \underset{\alpha\in A}{\barwedge}(A \cap A_\alpha)$ and also $(A \cap A_\alpha) \subseteq A_\alpha$ and $(A \cap A_\alpha) \subseteq A\ \forall \alpha \in \Lambda$

Hence $\underset{\alpha\in A}{\barwedge}(A \cap A_\alpha) \subseteq \underset{\alpha\in A}{\barwedge} A_\alpha$ and $\underset{\alpha\in A}{\barwedge}(A \cap A_\alpha) \subseteq A$

$\therefore \underset{\alpha\in A}{\barwedge}(A \cap A_\alpha) = A \cap (\underset{\alpha\in A}{\barwedge} A_\alpha)$

Lemma 3.9 Let A, B and C be any three vague frame of f then

$(i)\ A \barvee (B \barwedge C) \subseteq (A \barvee B) \barwedge (A \barvee C)\ \ (ii) A \subseteq B \Rightarrow A \barvee C \subseteq B \barvee C$

Theorem 3.10 Let A and B be any two vague co-frames then $(A \barwedge B)$ is a vague co-frame and $(A \barwedge B) = \langle A \cup B \rangle$.

Proof: Let $x \in f$ then

$(i)(A \barwedge B)(x) = \{(x, \max_{u \wedge v = x}\{t_A(u) \wedge t_B(v)\}, \min_{u \wedge v = x}\{f_A(u) \wedge f_B(v)\}; x \in f)\}$

$\supseteq \{(x, \{t_A(x) \wedge t_B(e_f)\}, \{f_A(x) \wedge f_B(e_f)\})\}$

$\supseteq (x, t_A(x), f_A(x))$

$\therefore (A \barwedge B) \supseteq A$ Similarly we can prove that $(A \barwedge B) \supseteq B$ Thus $(A \barwedge B) \supseteq (A \cup B)$

$(ii) t_{(A \barwedge B)}(x \wedge y) = \max_{u \wedge v = x \wedge y}\{t_A(u) \wedge t_B(v)\}$

$\geq \max\{t_A(u_1 \wedge v_1) \wedge t_B(u_2 \wedge v_2)\}$

For $u_1 \wedge u_2 = x$ and $v_1 \wedge v_2 = y$; $u, v \in f$

$\geq \max\{\{t_A(u_1) \wedge t_A(v_1)\} \wedge \{t_B(u_2) \wedge t_B(v_2)\}\}$

$\geq \max\{\{t_A(u_1) \wedge t_B(u_2)\} \wedge \{t_A(u_1) \wedge t_B(u_2)\}\}$

$\geq \max\{\{t_A(u_1)\} \wedge \{t_B(u_2)\}; u_1 \wedge u_2 = x, u_1, u_2 \in F\}$

$\wedge \max\{\{t_A(v_1)\} \wedge \{t_B(v_2)\}; v_1 \wedge v_2 = y, v_1, v_2 \in F\}$

$= \{t_{(A \barwedge B)}(x) \wedge t_{(A \barwedge B)}(y)\}; \forall x, y \in F$

$f_{(A \barwedge B)}(x \wedge y) = \min_{u \wedge v = x \wedge y}\{f_A(u) \vee f_B(v)\}$

$\leq \min\{f_A(u_1 \wedge v_1) \vee f_B(u_2 \wedge v_2)\}$

$\leq \min\{\{f_A(u_1) \wedge f_A(v_1)\} \vee \{f_B(u_2) \wedge f_B(v_2)\}\}$

$\leq \min\{\{f_A(u_1) \vee f_B(u_2)\} \vee \{f_A(v_1) \vee f_B(v_2)\}\}$

$\vee \min\{\{f_A(v_1)\} \vee \{f_B(v_2)\}; u \wedge v = x, u, v \in F\}$

$= \{f_{(A \barwedge B)}(x) \vee f_{(A \barwedge B)}(y)\}; \forall x, y \in F$

$(iii) t_{(A \barwedge B)}(x \wedge y) = \max_{u \wedge v = x \vee y}\{t_A(u) \wedge t_B(v)\}$

$= \max\{t_A(y \vee u) \wedge t_B(y \vee v)\}; u \wedge v = x, u, v \in F (\because (y \vee u) \wedge (y \vee v) = x \vee y)$

$\geq \max\{t_A(u) \wedge t_B(y)\}; u \wedge v = x, u, v \in F$

$= t_{(A \barwedge B)}(x) \forall x, y \in F$

Similarly we have $t_{(A \barwedge B)}(x \vee y) \geq t_{(A \barwedge B)}(y); \forall x, y \in F$

$t_{(A \barwedge B)}(x \vee y) \geq t_{(A \barwedge B)}(x) \vee t_{(A \barwedge B)}(y); \forall x, y \in F$

$(iv) f_{(A \barwedge B)}(x \vee y) = \min_{u \wedge v = x \vee y}\{f_A(u) \wedge f_B(v)\}$

$\leq \min\{f_A(y \vee u) \vee f_B(y \vee v)\}; u \wedge v = x, u, v \in F(\because (y \vee u) \wedge (y \vee v) = x \vee y)$

$\leq \wedge \min\{f_A(u) \vee f_B(y)\}; u \wedge v = x, u, v \in F$

$= f_{(A \barwedge B)}(x) \forall x, y \in F$

Similarly we have $f_{(A \barwedge B)}(x \vee y) \leq f_{(A \barwedge B)}(y); \forall x, y \in F$

$\therefore f_{(A \barwedge B)}(x \vee y) \leq f_{(A \barwedge B)}(x) \wedge f_{(A \barwedge B)}(y); \forall x, y \in F$

$(v)\ \because t_A(e_f) = f_B(e_f) = 1$

Clearly $t_{(A \barwedge B)}(e_f) = \max_{u \wedge v = e_f}\{t_A(u) \wedge t_B(v)\}$

$\geq t_A(e_f) \wedge t_B(e_f) = 1$

And since $f_A(e_f) = f_B(e_f) = 0$

$f_{(A \barwedge B)}(e_f) = \min_{u \wedge v = e_f}\{t_A(u) \wedge t_B(v)\}$

$\leq f_A(e_f) \wedge f_B(e_f) = 0$

Remark. If we interchange the role of $\barwedge$ and $\cap$ in the above theorem, only one sided inequality is holds $A \barwedge \left(\bigcap_{\alpha \in \Lambda} A_\alpha\right) \subseteq \bigcap_{\alpha \in \Lambda}(A \barwedge A_\alpha)$

Theorem 3.11 The set $I_F(F)$ of all vague filters of the co-frame F is a co-frame.

Proof: Let $I_F(F)$ is a complete lattice, which is bounded above by $A_\chi(F) = \{(x, t_{A_{\chi F}}(x), f_{A_{\chi F}}(x)) / x \in F\}$,Where $t_{A_{\chi F}}(x) = 1$ and $f_{A_{\chi F}}(x) = 0$

And bounded bellow by $A_{e_F} = \{(x, t_{A_{\chi F}}(x), f_{A_{\chi F}}(x)) / x \in F\}$, Where $t_{e_f}(x) = \begin{cases} 1; & if\ x = e_f \\ 0; & otherwise \end{cases}$ and $f_{e_f}(x) = \begin{cases} 0; & if\ x = e_f \\ 1; & otherwise \end{cases}$ also $\barwedge_{\alpha \in \Lambda}(A \veebar A_\alpha) = A \vee \left(\barwedge_{\alpha \in \Lambda} A_\alpha\right)$

For arbitrary collection $(A_\alpha)_{\alpha \in \Lambda}$ and A of a vague frame of f then by the above theorems $I_F(F)$ is a co-frame.

Theorem 3.12 Let A and B be any two vague frames of a frame F then $(A \barwedge B)$ is a vague frame of F and $(A \barwedge B) = \langle A \cup B \rangle$

Proof: Let $x \in F$

$(i)(A \barwedge B)(x) = \{(x, \max_{u \wedge v = x}\{t_A(u) \wedge t_B(v)\}, \min_{u \wedge v = x}\{f_A(u) \wedge f_B(v)\}; x \in f)\}$

$\supseteq \{(x, \{t_A(x) \wedge t_B(e_f)\}, \{f_A(x) \vee f_B(e_f)\})\}$

$\supseteq (x, t_A(x), f_A(x))$

$\therefore (A \barwedge B) \supseteq A$ Similarly we can prove that $(A \barwedge B) \supseteq B$ Thus $(A \barwedge B) \supseteq (A \cup B)$

$t_{(A \barwedge B)}(x \wedge y) = \max_{u \wedge v = x \wedge y}\{t_A(u) \wedge t_B(v)\}$

$\geq \vee\{t_A(u_1 \wedge v_1) \wedge t_B(u_2 \wedge v_2)\}$

For $u_1 \wedge u_2 = x$ and $v_1 \wedge v_2 = y$; $u, v \in f$

$\geq \vee\{\{t_A(u_1)\wedge t_A(v_1)\}\wedge\{t_B(u_2)\wedge t_B(v_2)\}\}$

$\geq \vee\{\{t_A(u_1)\wedge t_B(u_2)\}\wedge\{t_A(v_1)\wedge t_B(v_2)\}\}$

$\geq \vee\{\{t_A(u_1)\}\wedge\{t_B(u_2)\}; u_1\wedge u_2 = x, u_1, u_2\in F\}\wedge\vee\{\{t_A(v_1)\}\wedge\{t_B(v_2)\}; v_1\wedge v_2 = y, v_1, v_2\in F\}$

$=\{t_{(A\barwedge B)}(x)\wedge t_{(A\barwedge B)}(y)\};\forall x, y\in F$

$f_{(A\barwedge B)}(x\wedge y)=\min_{u\wedge v=x\wedge y}\{f_A(u)\vee f_B(v)\}$

$\leq\{f_A(u_1\wedge v_1)\vee f_B(u_2\wedge v_2)\}$

$\leq\wedge\{\{f_A(u_1)\vee f_A(v_1)\}\vee\{f_B(u_2)\vee f_B(v_2)\}\}$

$\leq\wedge\{\{f_A(u_1)\vee f_A(v_1)\}\vee\{f_B(u_2)\vee f_B(v_2)\}\}$

$=\wedge\{\{f_A(u_1)\vee f_B(u_2)\}\vee\{f_A(v_1)\vee f_B(v_2)\}\}$

$\wedge\{f_A(u_1)\vee f_B(u_2): u_1\wedge u_2 = x: u_1, u_2\in F\}\vee\wedge\{f_A(v_1)\vee f_B(v_2): v_1\wedge v_2 = y: v_1, v_2\in F\}$

$=\{f_{(A\barwedge B)}(x)\vee f_{(A\barwedge B)}(y)\};\forall x, y\in F$

$(ii) t_{(A\barwedge B)}(x\vee y)=\max_{u\wedge v=x\vee y}\{t_A(u)\wedge t_B(v)\}$

$\geq\vee\{t_A(y\vee u)\wedge t_B(y\vee v)\}; u\wedge v = x, u, v\in F(\because (y\vee u)\wedge(y\vee v)=x\vee y)$

$\geq\vee\{t_A(u)\wedge t_B(v)\}; u\wedge v = x, u, v\in F$

$=t_{(A\barwedge B)}(x)\forall x, y\in F$

Similarly we have $t_{(A\barwedge B)}(x\vee y)\geq t_{(A\barwedge B)}(y);\forall x, y\in F$

$t_{(A\barwedge B)}(x\vee y)\geq t_{(A\barwedge B)}(x)\vee t_{(A\barwedge B)}(y);\forall x, y\in F$

$f_{(A\barwedge B)}(x\vee y)=\min_{u\wedge v=x\vee y}\{f_A(u)\vee f_B(v)\}$

$\leq\wedge\{f_A(y\vee u)\vee f_B(y\vee v)\}; u\wedge v = x, u, v\in F(\because (y\vee u)\wedge(y\vee v)=x\vee y)$

$\leq\wedge\{f_A(u)\vee f_B(v)\}; u\wedge v = x, u, v\in F$

$= f_{(A\barwedge B)}(x)\forall x, y\in F$

Similarly we have $f_{(A\barwedge B)}(x\vee y)\leq f_{(A\barwedge B)}(y);\forall x, y\in F$

$\therefore f_{(A\barwedge B)}(x\vee y)\leq f_{(A\barwedge B)}(x)\wedge f_{(A\barwedge B)}(y);\forall x, y\in F$

$(v)\ \because t_A(e_f)= f_B(e_f)=1$

Clearly $t_{(A\barwedge B)}(e_f)=\max_{u\wedge v=e_f}\{t_A(u)\wedge t_B(v)\}$

$\geq t_A(e_f)\wedge t_B(e_f)=1$

And since $f_A(e_f)= f_B(e_f)=0$

$f_{(A\barwedge B)}(e_f)=\min_{u\wedge v=e_f}\{t_A(u)\wedge t_B(v)\}$

$\leq f_A(e_f)\wedge f_B(e_f)=0$

Thus if A and B are vague frame of F then $(A\barwedge B)$ is a vague frame of F.

Now let C any vague frame of F such that $(A\cup B)\subseteq C$

Then $t_{(A\barwedge B)}(x) = \max_{u\wedge v=x}\{t_A(u)\wedge t_B(v)\}$
$\leq \max\{t_C(u)\wedge t_C(v)\}$
$\leq \max_{u\wedge v=x}\{t_C(u\wedge v)\} = t_C(x)$
$f_{(A\barwedge B)}(x) = \min_{u\wedge v=x}\{t_A(u)\wedge t_B(v)\}$
$\geq \min_{u\wedge v=x}\{f_C(u)\wedge f_C(v)\}$
$\geq \min_{u\wedge v=x}\{f_C(u\wedge v)\} = f_C(x)$
Hence $(A\barwedge B)\subseteq C$
Thus $(A\barwedge B)$ is the smallest filter of F such that $(A\cup B)\subseteq(A\barwedge B)$.
Hence $(A\barwedge B)=(A\cup B)$.

Theorem 3.13 Let A is a vague co-frame of F and B is a vague set of S with $B\subseteq A$ then B is a vague frame of A if and only if
$(i)\, t_B(x\wedge y)\geq t_B(x)\wedge t_B(y), f_B(x\wedge y)\leq f_B(x)\vee f_B(y) \forall x,y\in F$
$(ii)\,(A\veebar B)\subseteq B\;(iii)\, t_B(e_f)=t_A(e_f)$ *and* $f_B(e_f)=t_A(e_f)$

Proof: Suppose the above conditions are hold then we have to prove that B is vague frame of A. Since $(A\veebar B)\subseteq B$ then $t_B(x\vee y)\geq t_{(A\veebar B)}(x\vee y) = \max_{u\vee v=x\vee y}\{t_A(u)\wedge t_B(v)\}$
$\geq \min\{t_A(x), t_B(y)\}, \forall x,y\in F$
$f_B(x\vee y)\leq f_{(A\veebar B)}(x\vee y) = \max_{u\vee v=x\vee y}\{f_A(u)\vee f_B(v)\}$
$\leq \min\{f_A(x), f_B(y)\}, \forall x,y\in F$
Also $t_B(x\vee y)\geq \min\{t_A(y), t_B(x)\}, \forall x,y\in F$ and $f_B(x\vee y)\leq \min\{f_A(y), f_B(x)\}, \forall x,y\in F$
Hence $t_B(x\vee y)\geq \max\{\min\{t_A(y), t_B(x)\}, \min\{t_B(x), t_A(y)\}\}$ and
$f_B(x\vee y)\leq \min\{\max\{f_A(x), f_B(y)\}, \max\{f_B(x), f_A(y)\}\}$
Therefore B is a vague frame of A.
Conversely, If B is a vague frame of A, then we have
$t_B(x\wedge y)\geq \min\{t_B(x), t_B(y)\}; \forall x,y\in F$ and $f_B(x\wedge y)\leq \max\{f_B(x), f_B(y)\}; \forall x,y\in F$ and
$t_B(e_f)=t_A(e_f)$, $f_B(e_f)=t_A(e_f)$
Now for all $Z\in F$ with $Z=(x\vee y)$, $\max_{x\vee y=z}\min\{t_A(x), t_B(y)\} = t_{(A\veebar B)}(z)$ and
$f_B(z)\leq \min_{x\vee y=z}\max\{f_A(x), f_B(y)\} = f_{(A\veebar B)}(z)$
Hence $(A\veebar B)\subseteq B$.

Theorem 3.14 Let A is a vague co-frame of F and let B,C two vague frames of A then $(B\barwedge C)$ is a vague frame of A and $B\subseteq(B\barwedge C), C\subseteq(B\barwedge C)$.

Proof: (i) we have $t_{(B\barwedge C)}(x\wedge y)\geq \min\{t_{(B\barwedge C)}(x), t_{(B\barwedge C)}(y)\}$ and
$f_{(B\barwedge C)}(x\wedge y)\leq \max\{f_{(B\barwedge C)}(x), f_{(B\barwedge C)}(y)\}(\because (A\barwedge B)=(A\cup B))$
(ii) we have B,C are vague frames of a frame $(A\veebar B)\subseteq B$ and $(A\veebar C)\subseteq C$

Hence $A \bar{\vee} (B \bar{\wedge} C) \subseteq (A \bar{\vee} B) \bar{\wedge} (A \bar{\vee} C) \subseteq (B \bar{\vee} C)$

(iii) Also clearly $t_{(B \bar{\wedge} C)}(e_f) = t_A(e_f)$ and $f_{(B \bar{\wedge} C)}(e_f) = f_A(e_f)$

Hence $(B \bar{\wedge} C)$ is a vague frame of A.

Again $t_{(B \bar{\wedge} C)}(x) = \max_{u \wedge v = x} \min\{t_B(u), t_C(v)\} \geq \min\{t_B(x), t_C(e_f)\} = t_B(x)$ and

$f_{(B \bar{\wedge} C)}(x) = \min_{u \wedge v = x} \max\{f_B(u), f_C(v)\} \leq \max\{f_B(x), f_C(x)\} = f_B(x); \forall x \in F$

Hence $B \subseteq (B \bar{\wedge} C)$, similarly we can prove that $C \subseteq (B \bar{\wedge} C)$

References:

[1] Atanassov.K.T,Intuitionisticfuzzy sets,Fuzzy sets and systems,33(1989),37-45.

[2] Banasschewski.B Lecturers on frame,University of cape Town 1998

[3] C.H.Dowker and D.papert,"Quotient frames and subspaces" proceedings of the London Mathematical society,vol.16,pp.275-296,1996.

[3] Dowker.C.H and Papert.D "quotient frames and subspaces proceedings of the London mathematics society,vol.16,pp:275-296,1966.

[4] Gahu.w.L.Buehrer.D.J.,vague sets IEEETransactions on systems,Man and cybernetics vol.23(1993),610-614.

[5] Goguen.,A,L-Fuzzy sets ,J.Math.Anal.,Applic.,18(1967),145-174.

[6] Nageswara Rao.B,Ramakrishana.N,Eswarlal.T,L-vague semirings of L-semring,International journal of Mathematics Trends and Technology,Vol57,Issue5-May 2018.

[7] Nageswara Rao.B,Bulli Babu.R,Eswarlal.T,Vague Join semi lattices ,International journal of Mathematics Trends and Technology,Vol66,Issue2,Feb 2020.

[8] Ramakrishana.N,Nageswararao.B,Eswarlal.Tandsatyanarayana.ch,Anti-homomorphisms invague groups(IJMSEA)ISSN 0973-9424,vol.6, No.2.(March,2012),PP.449-459.

[9] Ramakrishna.N,Vague groups and vague weights,Int.journal of computational congnition, Vol.66 No.1 (2009),913-916.

[10] Rosenfeld A. Fuzzy groups.Jon.Maths.Anal.Appli.35 (1971)512-517.

[11] Vimala.J,Fuzzy lattice orderdgroup,International journal of science and engineering Research Volume 5,Issue 9,september 2.

[12] Zadeh,L.A., Fuzzy Sets1, Infor and Control, Volume 8 (1965) 338-353.