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A Stochastic Optimal Control of DC-Pension Fund Optimization Driven By Fractional Brownian motion with Hurst Parameter $> \alpha = \frac{1}{2}$

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ABSTRACT

We considered the goal of the pension fund manager as to minimize the utility loss and maximizeutility at exit from the scheme, the noises involved in the dynamics of the process are fractional Brownian motions with short-range dependence. We use the DPP method to derive the HJB equation for the value function, and obtain the optimal strategy explicitly driven by fractional Brownian motion with Hurst parameter H $\in (0,1)$

KEYWORDS

DC-Pension Scheme, fractional Brownian motion, non-random optimal control, Stochastic Optimal Control

INTRODUCTION

There are two different methods to implement pension funds policies: Defined-Benefit plan (DB) and Defined-Contribution plan (DC). In DB, the benefits are fixed. The benefits accruable are defined in advanced and the contributions are systematically adjusted to guarantee that the fund balance is maintained in line with government policies. The participant of this type of scheme is responsible for all the associated financial risks. The defined-contribution scheme is designed so that the contributions are defined in advance and benefits on the return on the assets of the fund, all the associated financial risks are borne by the beneficiary. The demographic advancement that affects the sustainability of retirement income and the subsequent evolution of equity market has increased adoption of the subject. The benefit payments depend on the fund portfolio and the efficiency of the investment strategy. In the classical work of Merton dynamics portfolio selection model, return rates and volatilities of risk assets are all assumed to be deterministic [1] and[2]. However, many scholars started to investigate the problems of investment and consumption under different market environments [3],[4], [5], [6],[7],[8], and [9]. In these works, the optimal control theory is a dominant approach in solving problems associated to asset allocation and optimization for the pension scheme.

However, in the real world, the market tends to support the stochastic volatility. Recently, the studies on the stochastic volatility (SV) model is a useful tool in examining the stock price. See the works of [10], and [11] all used Heston's volatility model to describe the price of the risky asset. By extension, Guan and Liang [12] and Hao and Xue-Yan [13] derived the optimal investment and consumption problem of a DC pension fund with combined stochastic affine interest rate and stochastic volatility. As the pension management and the plan members are given more flexibilities to select without restrictions appropriate benefit outgo, the optimal benefit outgo is described as control variable in recent literatures. In Kapu and Orszag [14], the benefit outgo is dynamically chosen by the PFM to achieve the plan member's objectives. In DiGiacinto et al [15] solve the stochastic optimal problem with constraints on the control policies with some defined objective functions. The benefit payment policy has not gained popularity in a DC pension fund study. For the benefit payment policy, the retirements depend on traditional DC scheme as demonstrated in actuarial principles and concepts. Furthermore, this study will explore the optimal investment and benefit payments policies problem for a DC pension fund scheme under the income drawdown option associated with stochastic interest rates with the introduction of the fractional Brownian motion as the tool needed to evaluate the evolving dynamic process. In a defined contribution (DC) pension plan, the financial risk is borne by the member: contributions are fixed in advance, and the benefits provided by the plan depend on the investment performance experienced during the active membership and on the price of the annuity at retirement, in the case that the benefits are given in the form of an annuity. Therefore, the financial risk can be split into two parts: investment risk, during the accumulation phase, and annuity risk, focused at retirement. Recently, due to the demographic evolution and the development of the equity market, DC schemes have become popular in global pension market. A successful DC scheme will deliver good annuity a tretirement, so the investment strategy for the accumulation phase in DC schemes is very critical. Literature about investment strategy of DC pension funds is prolific and from a methodological point of view, two approaches are exploited. The first one is stochastic control, used for the first time by (see [16]). Among the recent applications of this theory to DC pension fund portfolio, (see, [17-19]). The second method, also called the martingale method, was developed by[20]in the setting of complete markets and relies on the theory. The novel of this work is to extend the Brownian motion about some noises involved in the dynamics of wealth to fractional Brownian motion (fBm) with short range dependence in pension fund. Instead of using the classical tool of optimal control as optimization engine , we convert the stochastic optimal control problem into a non-random optimization, and try to find explicit solutions under the minimization of the expected utility loss function. We analyse the optimal investment and benefit payments strategies with exact solution under Power utility function. The paper is organized as follows. In section two (2), we setup the dynamics of the fund wealth process model. In section three (3), we setup the Optimization framework for the value function, given the investor (pension fund manager) option on the proportion of wealth to invest in risky asset and amount to withdraw from the pool on exit from the scheme and derive the HJB equation. In section four (4), we introduced the fBm tools required for the Hurst parameter for the predictability and investment returns. Section five (5) explore the optimal control of DC-Pension fund with fBm and finally the explicit solution for the value function and optimal investment strategy.

MATHEMATICAL MODEL

In this section, we consider the market structure and define the stochastic dynamics of the asset values and the contributions. We consider a complete and frictionless financial market which is continuously open over the fixed time interval $[0, T], where\ T > 0$denotes the retirement time. We also consider the uncertainty involved in the financial market is defined and shaped by a complete filtered probability space$(\Omega, \mathcal{F}, \mathcal{P})\ and\ \mathcal{F} = f(t)_{t\geq 0}$, represents the information available before time t in the market, and adapted from $\mathcal{F}(t)_{t\geq 0}$.

The Financial Market:

We suppose that the market is composed of three kinds of financial assets namely: fixed income securities (bond/ bank account), equities (stocks) and real estate securities (property). These assets operate independently with their associated risks depending on market environments.

Fixed income securities:

The first asset in a financial market is the risk-free asset (bond/ bank account) denoted by$\mathscr{B}(t)$ and the price of this asset at time t evolves according to the dynamics of differential

equations: $d\mathscr{B}(t) = \mathscr{B}(t)t(t)dt\ \ ,\ \mathscr{B}(0) = \mathscr{B}_0, r(t) > 0.$ (1)
where $\mathscr{B}_0$ is the initial price of the risk-free asset, and $r(t)$ is the instantaneous rate of interest follows Hull-White model (1990). On the historical probability measure $\mathbb{P}$, the dynamics of $r(t)$ is given by the mean-reverting stochastic differential equation (SDE) as :

$$dr(t) = (\theta(t) - \beta r(t))dt + \sigma dW(t), r(0) = r_0. \quad (2)$$

Where $\sigma > 0,\ \theta(t),\ and\ \beta$ denotes the interest rate volatility, the mean-reversion which is time dependent, and the reversion rate. $W(t)$ is a standard Brownian motion.

The Stock: The stock is denoted by $S(t)$ whose dynamic follows SDE governed by

$$dS(t) = S(t)\theta_s dt + \sigma_s dW_s(t);\ \ S(0) = S_0 \quad (3)$$

where$S_0, \theta_s(t),\ and\ \sigma_s$ denote; the initial stock price, the expected rate of return, and the stock volatility rate respectively.

Contributions to the Funds:

In the defined contribution (DC) management, the members will be continuously contributing the part/ proportion of their salaries to the retirement time T. The contributions process at time t is given by the SDE.

$$dC(t) = C(t)\theta_c(t)dt + \sigma_C W_C(t), C(0) = C_0. \quad (4)$$

Where $\theta_C > 0, \sigma_C > 0,\ and\ C_0 > 0$ are respectively the rates of contribution, contributions volatility, and the initial contribution.

Pension Wealth Process:

We consider a complete probability space $(\Omega, \mathscr{F}, \mathcal{P})$ with a filtration$\{\mathscr{F}_t\}_{t\geq 0}$, $t \geq 0$.
The filtration $\{\mathscr{F}_t\}_{t\geq 0}$ is generated by the trajectories of a 1-dimensional standard Brownian motion$B(t)$, $t \geq 0$. Given the financial market composed of two types of assets: riskless and risky assets respectively. The riskless asset $S_0(t), t \geq 0$; evolves according to the dynamics.

$$dS_0(t) = rS_0(t)dt, S_0(0) = 1 \tag{5}$$

Where $r \geq 0$ is the instantaneous spot rate of return. The price of the risky asset $S_0(t), t \geq 0$ follows the Itô's process evolves and satisfies the SDE:

$$dS_1(t) = rS_1(t)dt + \sigma S_1(t)dB(t), \quad S_1(0) = S_0 \tag{6}$$

r is the rate of expected return and $\sigma > 0$ is the instantaneous rate of volatility.

We assume that the financial market consists of two assets namely: risky and risk-free. Risk-free is usually at time t evolves as ODE

$$d\mathscr{B}(t) = \mathscr{B}(t)r(t)dt \quad , \; r(0) = r_0 > 0 \tag{7}$$

The price of the risk-free asset is controlled by the ODE as stated in (5). The dynamics of a risky asset price evolve according to the combined models of SDEs as sated as:

The Optimization Framework

The objectives of Pension Fund Manager (PFM) among others is to ensure the best strategy in the investment process, with a define target of fund solvency. Again, the PFM is to select investment strategy to minimize the expected value of utility loss function.

The optimization framework is defined as

$$\hat{Q} = (t, v, r, x) = \inf_{\pi, p} E\left[\int_0^\infty \exp(-\beta t)\, U(p(t), X(t))\, dt\right] \tag{8}$$

where β is the discount rate, $p(t)$ is the payment at a given time t, and π, p are the parameters for both investment and outputs strategies.However, during the transitory stage for $(s, x) \in C$ the problem consists of maximizing over the set of admissible strategies above the solvency level$l(t) \equiv l :=$ $l(T)\;\; for\; t \geq T$.

Given

$$\Theta_{ad}(S, x) = \left\{\theta : [S, T] \times \Omega \to [0,1] - prog.meas.wrt\, \{\mathscr{F}_t\}_{t\in[S,T]} \,\middle|\, \begin{matrix} X(t; s, x, \theta(\cdot) \geq l(t), \\ t \in [S, T] \end{matrix}\right\}$$

(9)

Proposition 1: *Let* $(s, x) \in C$ *and let* $X(t) = X(t; s, x, 0)$;

$$then\; X(t) - l(t) \geq \big(x - l(s)\big) \exp\big(r(t - s)\big), t \in [S, T] \tag{10}$$

Recall that for $(s, x) \in C, \theta(\cdot) \in \Theta_{ad}(S, x)$, we define

$$\widehat{Q}\,\big(s, x;\, \theta(\cdot)\big) = E\left[\int_0^\infty \exp(-\rho t)\, U\big(t, X\big(t; s, x, \theta(\cdot)\big)dt + f(X(T; s, x, \theta(\cdot)\right] \tag{11}$$

The value function for $(s, x) \in C$ is defined as $V(s, x) = \sup\limits_{\theta(\cdot)\in\Theta_{ad}(s,x)} \hat{Q}(s, x; \theta(\cdot))$

Fractional Brownian motion (*fBm*)

The pioneering work of Mandelbrot and Van Ness (see [21]), fBm was proposed as a tool to model various areas such as population dynamics , mathematical finance , random dynamical systems and telecommunications. However , fractional Brownian motion (fBm) of Hurst parameter $H \in (0,1)$ was introduced as a Gaussian process $\mathscr{B}^{\mathscr{H}} = \{\mathscr{B}_t^{\mathscr{H}}, t \geq 0\}$. The Hurst parameter $H \in (0,1)$ is a measure for The predictability of investments, especially the risky assets which depends on the Hurst parameter that permits for modeling the autocorrelation of asset returns. Some studies have shown long-range dependence exists between the asset returns in some markets. Hence, a proposed shift from Brownian motion in financial modeling to fractional Brownian motion expressed as $\mathscr{B}^{\mathscr{H}} = \{\mathscr{B}_t^{\mathscr{H}}, t \geq 0\}$.

Previous work of some authors has shown that Hurst exponent greater than $1/2$within the range of $0 \leq \mathscr{H} \leq 1$. We are interested in $\mathscr{H} > 1/2$ since if $0 \leq \mathscr{H} < 1/2$ the series exhibit mean reverting and if $1/2 < H \leq 1$, it implies the existence of long-range dependence, which is the same as along memory process.

Definition 1: A stochastic function $\{X(t)\}_{t\geq 0}$ is self-similar if for every $a > 0$, there Exists$b > 0$ such that $Law(X(at)) = Law(bX(t))$

Definition 2: A stationary sequence $(X(n))_{n\in N}$ exhibits long-memory dependence if the auto-covariance function $\rho(n) = cov\big(X(k), X(k+n)\big)$satisfies$\lim\limits_{n\to\infty} \frac{\rho_n}{C_n^{-\infty}} = 1$.

Definition 3: Fractional Brownian motion (fBm) of the Hurst parameter is defined as:

Let the Hurst parameter $\mathscr{H}$ be a constant with $0 < \mathscr{H} < 1$. A continuous and centered Gaussian process $\{\mathscr{B}^{\mathscr{H}}(t)\}_{t\geq 0}$ for all $t, s \in \mathscr{R}$ is a fractional Brownian motion (*fBm*) given its covariance function as

$$\mathrm{E}\big[\mathscr{B}^{\mathscr{H}}(t)\mathscr{B}^{\mathscr{H}}(t)\big] = \frac{1}{2}(|t|^{2\mathscr{H}} + |s|^{2\mathscr{H}} - |t-s|^{2\mathscr{H}}) \tag{12}$$

With the following properties:

P1: $\mathscr{B}^{\mathscr{H}}(0) = 1$

P2: $\mathrm{E}\big[\mathscr{B}^{\mathscr{H}}(t)\big] = 0 \ \forall\ t > 0$

P3: $Var\big[\mathscr{B}^{\mathscr{H}}(t)\big] = \frac{1}{2}(|t|^{2\mathscr{H}} + |s|^{2\mathscr{H}} - |t-s|^{2\mathscr{H}}) = t^{2\mathscr{H}}$

P4: when $\mathscr{H} = 1/2$, fBm= classical Brownian motion.

Consider, for $0 < \alpha < 1$, define fractional integrals of Riemann-Liouville type as

$$\big(I_{\pm}^{\alpha} f\big)(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} f(x \mp t) t^{\alpha-1} dt \tag{13}$$

Let $I^{\alpha}(L^P)$ be a space, and Let$\in (0,1), 1 \leq p < \alpha^{-1}$, $and\ \varepsilon > 0\ and\ f \in I^{\alpha}(L^P)$, then

the fractional derivatives of Marchaud's type are stated as:

$$\big(I_{\pm,\varepsilon}^{-\alpha} f\big)(x) = \lim_{\varepsilon\to 0^+} \frac{\alpha}{\Gamma(1-\alpha)} \int_{\varepsilon}^{\infty} \frac{f(x) - f(x \mp t)}{t^{\alpha+1}} dt = L^P(\mathbb{R}) \tag{14}$$

Lemma 1: $$\mathrm{E}\left[\left(\mathscr{B}^{\mathscr{H}}_{t+T}-\mathscr{B}^{\mathscr{H}}_{t}\right)^2\right]=T^{2\mathscr{H}} \tag{15}$$

Proof: Recall that by definition *fBm* as a process is given by the Wiener integral for $H \in (0,1)$

$$\mathscr{B}^{\mathscr{H}}_{t}=\frac{K_H}{\Gamma\left(H+\frac{1}{2}\right)}\int_R (t-s)_+^{H-\frac{1}{2}}-(-s)_+^{H-\frac{1}{2}}\,d\mathscr{B}_s \tag{16}$$

With the normalizing constant K_H defined by

$$K_H=\Gamma\left(H+\frac{1}{2}\right)\left(\left(\int_R (1+s)^{H-\frac{1}{2}}-s^{H-\frac{1}{2}}\right)^2 ds+\frac{1}{2H}\right)^{-\frac{1}{2}}, Let\ \alpha=H-\frac{1}{2} \tag{17}$$

then , $\mathrm{E}\left[\left(\mathscr{B}^{\mathscr{H}}_{t+T}-\mathscr{B}^{\mathscr{H}}_{t}\right)^2\right]=$

$$\mathrm{E}\left[\left(\frac{K_H}{\Gamma(\alpha+1)}\langle\cdot,(t+T-s)_+^{\alpha}\quad-(-s)+^{\alpha}\rangle-\frac{K_H}{\Gamma(\alpha+1)}\langle\cdot,(t-s)_+^{\alpha}-\quad(-s)+^{\alpha}\rangle\right)^2\right] \tag{18}$$

By Itô's isometry it follows that,

$$\mathrm{E}\left[\left(\mathscr{B}^{\mathscr{H}}_{t+T}-\mathscr{B}^{\mathscr{H}}_{t}\right)^2\right]=\left(\frac{K_H}{\Gamma(\alpha+1)}\right)^2\int_R ((t+T-s)_+^{\alpha}-(t-s)_+^{\alpha})^2 ds \tag{19}$$

By change of variable, we have the LHS of (19) as: $t-s \rightsquigarrow Tv$

$$\left(\frac{K_H}{\Gamma(\alpha+1)}\right)^2 T^{2\alpha+1}\left(\int_{-\infty}^{0}((1-v)^{\alpha}-(-v)^{\alpha})^2 dv+\int_0^1(1-v)^{2\alpha}dv\right) \tag{20}$$

But $\int_0^1(1-v)^{2\alpha}dv=\frac{1}{2H}$,

Then, by conjunction, the LHS of (19) is expressed as

$$T^{2H}\left(\frac{K_H}{\Gamma(\alpha+1)}\right)^2\left(\int_0^{\infty}((1+s)^{\alpha}-s^{\alpha})^2 ds+\frac{1}{2H}\right)=T^{2H} \tag{21}$$

We now introduce and apply the work of Hu and Øksendal's *fBm* (see[24])required for the evaluation of the financial assets.

Consider a fractional market with an investment in risky and riskless assets driven by fBm in a continuous time $t \in [0,T]$. The dynamics of the riskless asset is governed by

$$dR(t)=rR(t)dt,\ \ R(0)=1 \tag{22}$$

and the dynamics of the risky asset governed by

$$dS(t)=\mu S(t)dt+\sigma S(t)d\mathscr{B}^{\mathscr{H}}(t), s(0)=s>0. \tag{23}$$

The Wick calculus solution (see Hu and Øksendal (2000)) is given as

$$S(t)=s\exp\left(\sigma\mathscr{B}^{\mathscr{H}}(t)+\mu t-\frac{1}{2}\sigma^2 t^{2\mathscr{H}}\right) \tag{24}$$

Optimal Control of DC-Pension fund with fBm.

We consider the general properties of fBm ,then examine the associated optimization problem driven by fBm for the assets.

$Let\ (\Omega, \mathscr{F}, P)$ be the probability space of one-dimensional Brownian motion

$\mathfrak{B} = \{\mathfrak{B}_t, t \in [0, T]\}$. $Let\ \mathscr{H}\ be\ the\ Hilbert\ space\ \mathscr{L}^2([0,T])$. For any $\hbar \in \mathscr{H}$, the Wiener integral $\mathfrak{B}(h) = \int_0^T h(t) d\mathfrak{B}_t$ Let Λbe the smooth and regular variables of the sequence:

$$F = f(\mathfrak{B}(h_1), \dots \mathfrak{B}(h_n), where\ n \geq 1,\ f \in C_d^{\infty}(R^n). \tag{25}$$

From (25), the stochastic process $\{\mathcal{D}_t F, t \in [0,T]\}$ is given by

$$\mathcal{D}_t F = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} (\mathfrak{B}(h_1), \dots \mathfrak{B}(h_n))\ h_k(t), t \in [0,T] \tag{26}$$

On the combination of fractional integrals and its derivatives, we write the fractional

Riemann-Liouville integral of f of order $\alpha\ on\ [0,T]\ as$

$$\mathrm{I}_{T^-}^{\alpha} f(s) = \frac{1}{\Gamma(\alpha)} \int_s^T (r-s)^{\alpha-1} f(r) dr \tag{27}$$

Proposition 2:

$Let\ u \in \mathscr{L}^{1,2}\quad and\ \mathscr{F} \in \mathbb{D}^{,2}\ such\ that\ \mathbb{E}\left[\mathscr{F}^2 \int_0^T u_t^2 dt\right] < \infty$, then $\mathscr{F}u$ is integrable

$$\text{and } \int_0^T F u_t d\mathfrak{B}_t = F \int_0^T u_t d\mathfrak{B}_t - \int_0^T \mathcal{D}_t F u_t dt \tag{28}$$

Provided the RHS of (28) is integrable.

Definition 4: By the definition of fBm on Hurst parameter, the process $\mathfrak{B}(t, \infty), t \geq 0$ defined on the probability space is a fBm of order α provided

(a) $p\{\mathfrak{B}(0, \alpha) = 0\} = 1$

(b) $For\ t \in \mathbb{R}_+^n, \mathfrak{B}(t, \alpha)\ is\ \mathscr{F} - measurable\ random\ variable$

(c) $\mathscr{F}or\ t, \tau \in \mathbb{R}_+, \mathbb{E}[\mathfrak{B}(t,\alpha)\mathfrak{B}(\tau,\alpha)] = \frac{\sigma^2}{2}(t^{2\alpha} + \tau^{2\alpha} - |t-\tau|^{2\alpha})$ (29)

By Kolmogorov's continuity criterion, for $\alpha > \frac{1}{2}, then\ \mathfrak{B}(t, \alpha)$ are continuous if $\alpha = \frac{1}{2}$ in (29), we recover the standard Brownian motion.

Mandelbrot and Vas Ness proposed the construction of fBm by the introduction of gamma function

$$\mathfrak{B}(t) = \Gamma\left(\alpha + \frac{1}{2}\right)^{-1} \int_0^t (t-\tau)^{\alpha - \frac{1}{2}} d\mathfrak{B}(t) \tag{30}$$

Definition 5: $Let\ f: \mathbb{R} \to \mathbb{R}, x \to f(x)$be continuous function. The fractional derivative of order $\alpha$is given by: $f^{\alpha}(x) = \Gamma(-\alpha)^{-1} \int_0^x (x-\varepsilon)^{-\alpha-1} f(\varepsilon) d\varepsilon, \alpha < 0$ (31)

For $\alpha > 0$, then , $f^{\alpha}(x) = (f^{\alpha-\varphi})^{\varphi},\ \ \varphi - 1 < \alpha < \varphi$.

From $\mathfrak{B}(\rho t, \alpha) = \rho^{\alpha}\mathfrak{B}(t, \alpha),\ \rho > 0$ (32)

(30) and (32) can be written as : $\mathfrak{B}(t,\alpha) = \frac{1}{\mathcal{D}^{\left(\alpha+\frac{1}{2}\right)}}\mathfrak{B}(t)$

Lemma 2:

Let $f(t)$ be a continuous function, then

$$\int_0^t f(\tau)\cdot(d\tau)^\alpha = \alpha\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau\,, \alpha \in (0,1) \tag{33}$$

Proof:

Let $x^\alpha(t) = f(t), \alpha \in (0,1)$ *be fractional DE with solution* $x(t) = \mathcal{D}^{-\alpha} f(x)$

Then $x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau$ (34)

By Taylor's expansion:

$$f^\alpha(x) = \lim_{h\to 0}\frac{\delta^\alpha f(x)}{h^\alpha} = \Gamma(1+\alpha)\lim_{h\to 0}\frac{\delta f(x)}{h^\alpha}\,,\ \ \alpha \in (0,1) \Rightarrow \mathcal{D}^\alpha\Gamma(1+\alpha)\,\mathcal{D}f \tag{35}$$

Using $x^\alpha(t) = f(t)\,, and\ \mathcal{D}x^\alpha = f(t)(\mathcal{D}t)^\alpha$ (36)

Then $\Gamma(1+\alpha)\,\mathcal{D}x^\alpha = f(t)(\mathcal{D}t)^\alpha$ (37)

Such that $x(t) = \Gamma(1+\alpha)^{-1}\int_0^t f(\tau)\,\mathcal{D}\tau$ (38)

$$\Rightarrow \Gamma(1+\alpha)^{-1}\int_0^t f(\tau)\,d\tau = \Gamma(\alpha)^{-1}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathcal{D}\tau \tag{39}$$

It follows then, that

$$\int_0^t f(\tau)\,\mathcal{D}\tau = \alpha\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathcal{D}\tau\ ,\ \ \alpha \in (0,1) \tag{40}$$

Given the evolving dynamics of the riskless asset by

$$dS_0(t) = rS_0(t) \tag{41}$$

And the dynamics of the risky asset by

$$dS(t) = S(t)(r+\mu)dt + \sigma_1 d\mathfrak{B}^{(t)} + \sigma_2 d\mathfrak{B}(t,a) \tag{42}$$

Where μ is the rate of return of the risky asset, $\sigma_1$is the covariance of the risky asset and under the classical Brownian motion and σ_2 is the covariance of the risky asset under the fractional Brownian motion.

The objective of the PFM is to formulate portfolio strategies as to minimize the utility loss function. From (34) above , we seek the strategy π^* minimizing the utility function

$$\min_{\pi^*} E\left[\int_0^T e^{-\rho t}U(\pi(t) - x(t))dt\right] \tag{43}$$

The value function of (43) is

$$\mathcal{V}(t,x) = E\left[\int_0^T e^{-\rho t}U(\pi(s) - x(s))ds,\ \ x_t = x\right] \tag{44}$$

$$\Rightarrow \mathcal{V}(t,x) = \int_t^T \left[e^{-\rho t} X^{\varphi}(s) E\left(x^{\varphi}(s)\right) ds,\ \ x_t = 0\right] \tag{45}$$

$Let\ \mathfrak{B}(t) = E(x^{\varphi}(t))$ be a state variable such that

$$\mathfrak{B}(t) = d\mathfrak{B}(t) = \ \varphi\mathfrak{B}(t)dt + \frac{\varphi(\varphi-1)}{2}\mathfrak{B}(t)\sigma_1^2 dt + \frac{\varphi(\varphi-1)}{2}\mathfrak{B}(t)\sigma_2^2 (dt)^2,\ \ at\ \mathfrak{B}(0) = x_0^{\varphi} \tag{46}$$

We now initiate the process of deriving the explicit solution for the value function and optimal investment strategy Our initial ***SOCP*** (43) now reset as a non-random optimal control in (46) from which the optimal strategy is obtained. It's important to note that, when fractional Brownian motion with Hurst value less than0.5, its variance is an infinitesimal value as$dt \to 0$, with optimal portfolio given by

$$\Lambda = \frac{(\sigma')^{-1}\sigma^{-1}U}{1-\nu} \tag{47}$$

using (43) and with the fBm model to derive the closed-form solution. By the application of DPP, we derive the HJB equation for the value function and obtain the optimal investment policies defined by the maximization of the expected utility of the terminal wealth using power utility of

$$\sup_{\pi\in\Pi} E_t[U(X(T))] \tag{48}$$

by using the evolving dynamics stated in (42) equivalently as :

$$dX_t = X_t r dt + X_t \pi_t \sigma_1 \theta dt + X_t \pi_t \sigma_2 \mathscr{B}_{\mathscr{H}}(t) \tag{49}$$

with its fBm components, where $\theta = \frac{\mu-r}{\sigma}$ is the value of the predictable process.

The PFM is to minimize loss function or maximize the terminal wealth of the investor, we define the value function of (48):$\Lambda(t,x) = \underset{\pi\in\Pi_t}{Sup} E_t[U(X(t))]$ (50)

where the expectation and utility function $\Lambda(T,x) = U(x), \forall x \geq 0$ of (50) satisfies the optimal value function

$$(\Lambda_t + rX\Lambda_x)(4Ht^{2\mathscr{H}-1}\Lambda_{xx}) = \Theta^2\Lambda_x^2\ \ , on\ [0,T] \times [0,\infty) \tag{51}$$

Lemma 3:$Let\ \psi(x,s){:}\ \mathbb{R} \times \mathbb{R} \to \mathbb{R} \in \Lambda^{1,2}(\mathbb{R} \times \mathbb{R})\ and\ Let\ \psi\left(t, \mathscr{B}_{\mathscr{H}}(t)\right),$

$$\int_0^t \frac{\partial\psi}{\partial s}\left(s, \mathscr{B}_{\mathscr{H}}(s)\right) ds\ , \int_0^t \frac{\partial^2\psi}{\partial x^2}\left(s, \mathscr{B}_{\mathscr{H}}(s)\right) s^{2\mathscr{H}-1} ds \tag{52}$$

be the class of $\mathscr{L}^2(\Theta)$, then

$$\psi\left(t, \mathscr{B}_{\mathscr{H}}(t)\right) = \psi(0,0) + \int_0^t \frac{\partial\psi}{\partial s}\left(s, \mathscr{B}_{\mathscr{H}}(s)\right) ds + \mathscr{H}\int_0^t \frac{\partial^2\psi}{\partial x^2}\left(s, \mathscr{B}_{\mathscr{H}}(s)\right) s^{2\mathscr{H}T} ds$$

$$+ \int_0^t \frac{\partial\psi}{\partial x}\left(s, \mathscr{B}_{\mathscr{H}}(s)\right) d\mathscr{B}_{\mathscr{H}}(s) \tag{53}$$

(See [26]and [27] for proofs)

Consider the power utility function defined by $U(x) = \frac{x^{\gamma}}{\gamma}\ \ ,\ \gamma \in (-\infty, 1)$ (54)

Usually motivated by the following factors:

(i) The space defined by the investors is large and the investment policy depends on the solvency level.

(ii) The PFM ensures that funds are not less than zero (0) as minimum guarantee level.

If $U(x) = \frac{x^{\gamma}}{\gamma}$, $\gamma \in (-\infty, 1), \gamma \neq 0$, $with\ BC\ \Lambda(T, x) = U(x)$, we suggest that the value

function is of the form: $\Lambda(T, x) = \psi(t)\frac{x^{\gamma}}{\gamma}$ with terminal condition $\psi(T) = 1$ (55)

$$\Rightarrow \frac{\partial \Lambda}{\partial t} = \frac{d\psi}{dt} \cdot \frac{x^{\gamma}}{\gamma} \cdot \frac{\partial \Lambda}{\partial x} = \psi(x)x^{\gamma-1} \tag{56}$$

$$\Rightarrow \frac{\partial^2 \Lambda}{\partial x^2} = \psi(t)(\gamma - 1)x^{\gamma-2} \tag{57}$$

Recall that optimal value function $\Lambda(t, x)$implies that

$$\frac{d\psi}{dt} \cdot \frac{x^{\gamma}}{\gamma} + r\psi(t)x - \frac{\theta^2}{(4\mathscr{H}t^{2\mathscr{H}-1})} \cdot \frac{\psi(t)x^{\gamma}}{(\gamma-1)} = 0 \tag{58}$$

$$(4\mathscr{H}t^{2\mathscr{H}-1})(\gamma - 1)\left[\frac{d\psi}{dt}\frac{x^{\gamma}}{\gamma} + r\psi(t)(x^{\gamma})\right] = \theta^2\psi(t)(x^{\gamma}) \tag{59}$$

$$\Rightarrow \frac{d\psi}{dt} = \psi(t)\left[\frac{\theta^2\gamma}{4\mathscr{H}t^{2\mathscr{H}-1}(\gamma-1)} - r\gamma\right] \tag{60}$$

Set $\psi(t) = 1,\ \psi(t) = e^{\left[r\gamma(T-t) - \frac{\theta^2\gamma\left(T^{2(1-\mathscr{H})} - t^{2(1-\mathscr{H})}\right)}{8\mathscr{H}(1-\mathscr{H})(\gamma-1)}\right]}$ (61)

Substitute (61) into (55) , this yields the required optimal investment strategy:

$$\pi^* = \frac{\theta}{2\mathscr{H}\sigma(1-\gamma)t^{2\mathscr{H}-1}} \tag{62}$$

Conclusion:

The PFM goal is to maximize the utility of an investor in the pension scheme. By the application of *fBm*, we explore the investment and benefit payments policies for DC-Pension under the drawdown option associated with stochastic interest rates to model and evaluate the dynamic process using the Hurst parameter. In future research, the direction would be the pension fund investment to investigate Hurst parameter greater than $1/2$, that of long-range dependence.

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