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MECHANICAL AND AERONAUTICAL ENGINEERING, THERMODYNAMICS

Comparison of main geometric characteristics of deformed sphere and standard spheroid

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Abstract. In the paper we compare the geometric descriptions of the deformed sphere (i.e., the so-called λ-sphere) and the standard spheroid (namely, World Geodetic System 1984's reference ellipsoid of revolution). Among the main geometric characteristics of those two surfaces of revolution embedded into the three-dimensional Euclidean space we consider the semi-major (equatorial) and semi-minor (polar) axes, quarter-meridian length, surface area, volume, sphericity index, and tipping (bifurcation) point for geodesics. Next, the RMS (Root Mean Square) error is defined as the square-rooted arithmetic mean of the squared relative errors for the individual pairs of the discussed six main geometric characteristics. As a result of the process of minimization of the RMS error, we have obtained the proposition of the optimized value of the deformation parameter of the λ-sphere, for which we have calculated the absolute and relative errors for the individual pairs of the discussed main geometric characteristics of λ-sphere and standard spheroid (the relative errors are of the order of 10^{-6} – 10^{-9}). Among others, it turns out that the value of the flattening factor of the spheroid is quite a good approximation for the corresponding value of the deformation parameter of the λ-sphere (the relative error is of the order of 10^{-4}).

Key words: deformed sphere; standard spheroid; sphericity index; tipping (bifurcation) point for geodesics; elliptic integrals and functions.

1. INTRODUCTION

The λ-spheres (as deformations of the usual sphere) have been introduced by Faridi and Schucking in [1] as an alternative to the spheroids (ellipsoids of revolution). In our previous papers, using the mechanical approach developed in [2, 3], we have obtained the general form of the solutions of the geodesic [4] and geodetic (i.e., without any potential energy) [5] equations of motion for the incompressible test bodies moving on the deformed spheres with the parameter of deformation $\lambda < 1/3$ (then the Gaussian curvature of the surface is strictly positive).

It turns out that the geodesics on λ-spheres can be expressed through the well-known analytical functions (inverse tangent), whereas the geodesics on the spheroids are expressed through the incomplete elliptic integrals of the first and third kind (see Section 3). The above observation justifies our idea to propose a new reference model for the geoid (i.e., the surface that approximates the actual shape of the Earth [6]) that is based on the λ-spheres (see [4, 7, 8]) alternatively to the standard reference models that are based on the rotational ellipsoids.

It is worth mentioning here that there are some propositions about the triaxial (nonrotational) ellipsoid reference models as more accurate approximations of the shape of the Earth than the biaxial (rotational) reference ellipsoids (see, e.g., [9]). In future research, it would be also interesting to confront the proposed reference model based on the deformed spheres with the existing proposals of the triaxial terrestrial ellipsoidal models.

In the present article we have constructed the appropriate schemes that allow us to compare the geometric characteristics of the deformed sphere and the standard spheroid. For this comparison we use the ellipsoidal reference model for the geoid within the World Geodetic System 1984 (WGS 84) [10].

As a result, performing the process of minimization of the RMS (root mean square) error, in Section 4 we propose the optimized value of the deformation parameter that can be next used for calculations (e.g., of the solutions of the direct and inverse geodetic problems [8] in geodesy or navigation) done within the new reference model for the geoid based on the λ-sphere that is well-suited to the WGS84's standard spheroid.

2. GEOMETRIC DESCRIPTION OF λ-SPHERE

Let us consider a deformed sphere of the equatorial radius R with the deformation parameter λ being in the range from 0 to $1/3$, where the case $\lambda = 0$ corresponds to the usual, non-deformed sphere of the radius R. The above condition on the deformation parameter ensures that the shape of such a surface is oblate and its Gaussian curvature is strictly positive. Then the parametrization of the λ-sphere surface embedded into the three-dimensional Euclidean space is given as (see [5])

$$x(u,v) = Ru\cos v, \qquad y(u,v) = Ru\sin v, \tag{1}$$

$$z(u) = \pm R\left(\frac{E(\theta,k)}{1-k} - F(\theta,k) - \frac{1-k}{r^2}\,\Pi(n,\theta,k)\right), \tag{2}$$

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where $u \in [0,1]$, $v \in [-\pi,\pi]$ define the local latitude and longitude variables respectively, $r = 1/\sqrt{1-\lambda} \in [1,\sqrt{3/2})$ since $\lambda \in [0,1/3)$, whereas $F(\theta,k)$, $E(\theta,k)$, and $\Pi(n,\theta,k)$ denote the incomplete elliptic integrals of the first, second, and third kind (see [11–13]) with $n = k/r^2$ being the characteristic,

$$k = \frac{1}{2}\left(r^2+1-\sqrt{\left(r^2+1\right)^2-4}\right) \tag{3}$$

being the elliptic modulus that fulfills identically the equation

$$k(r^2-1) = (1-k)^2 \tag{4}$$

and $\theta(u) = \arcsin\left((r^2-k)\sqrt{1-u^2}\right)$ being the so-called Jacobian amplitude. The signs $\pm$ in (2) correspond to the Northern or Southern Hemisphere of the λ-sphere, respectively, whereas $u = 1$ corresponds to the Equator and $u = 0$ corresponds to the North/South Pole.

The connection between the local latitude u_P of the point P on the λ-sphere and corresponding geodetic latitude ϕ_P, which is defined through the angle between the normal vector to the surface at the point P and the equatorial plane, is given as [8]

$$\tan\phi_P = -\frac{R}{z'(u_P)} = \pm\frac{\alpha(u_P)}{\sqrt{1-\alpha^2(u_P)}} \tag{5}$$

or equivalently $\sin\phi_P = \pm\alpha(u_P)$, where the signs $\pm$ correspond to the localization of P in the Northern/Southern Hemisphere, $z'(u)$ denotes the derivative of (2) with respect to u and

$$\alpha(u) = \left(1+\left(r^2-1\right)u^2\right)\sqrt{1-u^2}. \tag{6}$$

Therefore, for any chosen point P on the deformed sphere the geodetic latitude $\phi(u_P) = \pm\arcsin\alpha(u_P)$ defines the vertical (normal to the surface) direction and can be used in order to calculate the vertical distances (heights) above the surface within the new proposed reference model for the geoid.

Actually, it is planed in the future study to find out the explicit formula for the λ-sphere heights (i.e., the distances along the vertical direction between a chosen point on the Earth's surface and the corresponding point on the deformed sphere surface) and compare them to the ellipsoidal heights (i.e., the similar distances determined relative to the ellipsoid of revolution).

Similarly, there could be also investigated the spatial pattern of differences between two discussed reference surfaces, i.e., the deformed sphere and the WGS84 ellipsoid, using either the orthogonal (vertical) projection of the corresponding surface points on them or the one-to-one relationship between the grids of meridians and parallels on those surfaces.

3. COMPARISON OF λ-SPHERE TO SPHEROID

The oblate $(a > b)$ spheroid (ellipsoid of revolution) is defined through the equation

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2} = 1, \tag{7}$$

where a and b are called the semi-major and semi-minor axes (or equivalently, the equatorial and polar radii).

The parameterization of such a surface embedded into the three-dimensional Euclidean space is given as

$$x = a\sin\phi\cos v, \qquad y = a\sin\phi\sin v, \qquad z = b\cos\phi, \tag{8}$$

where $\phi \in [0,\pi]$ and $v \in [-\pi,\pi]$ are the latitude and longitude variables on the spheroid respectively.

We can also define the first and second eccentricities and the flattening factor describing the spheroid (7) as

$$\varepsilon = \sqrt{1-\frac{b^2}{a^2}}, \qquad \varepsilon' = \sqrt{\frac{a^2}{b^2}-1}, \qquad f = 1-\frac{b}{a}. \tag{9}$$

Therefore, we see that all geometric characteristics of the spheroid can be defined as functions of only two chosen parameters. For instance, the ellipsoidal reference model for the geoid within the WGS 84 is based on two geometric parameters (see [10, 14]) $a = 6\ 378\ 137$ m, $1/f = 298.257\ 223\ 563$, i.e., the Earth's equatorial radius and the reciprocal of the flattening factor of the Earth, respectively.

Similarly, all the geometric characteristics of the λ-sphere can be determined as functions of two parameters, e.g., the equatorial radius R and the deformation parameter λ.

This means that in order to compare those two surfaces one to another we need to construct some correspondence between the above-defined pairs of geometric parameters. Of course, it could be done in many different ways, some of which are discussed in detail in the following sections of the article.

3.1. Semi-major axis and semi-height or flattening factor

Let us identify the semi-major and semi-minor axes a and b of the spheroid (7) with the equatorial radius R and the semi-height H_λ of the λ-sphere at the level of the North Pole (or, in other words, the polar radius of the λ-sphere), i.e., $R \equiv a$ and

$$H_\lambda = R\left(\frac{E(\theta_0,k)}{1-k} - F(\theta_0,k) - \frac{1-k}{r^2}\,\Pi(n,\theta_0,k)\right) \equiv b \tag{10}$$

with $\theta_0 = \arcsin\left(r^2-k\right)$. From (10) we see that the general dependency of the scaled semi-height H_λ/R on the deformation parameter λ is a one-to-one function in the range $\lambda \in [0,1/3)$ with values smaller than or equal to 1 (see Fig. 1). By the way, this justifies the statement that the shape of λ-sphere is oblate.

Equivalently, we can rewrite the above identity (10) as an identity imposed on the flattening factors of the λ-sphere and spheroid that are defined by the last formula in (9), i.e.,

$$f_\lambda = 1 + F(\theta_0,k) - \frac{E(\theta_0,k)}{1-k} + \frac{1-k}{r^2}\,\Pi(n,\theta_0,k) \equiv f. \tag{11}$$

Again, the general dependency of the flattening factor f_λ on the deformation parameter λ is a one-to-one function in the range $\lambda \in [0,1/3)$ (see Fig. 2).

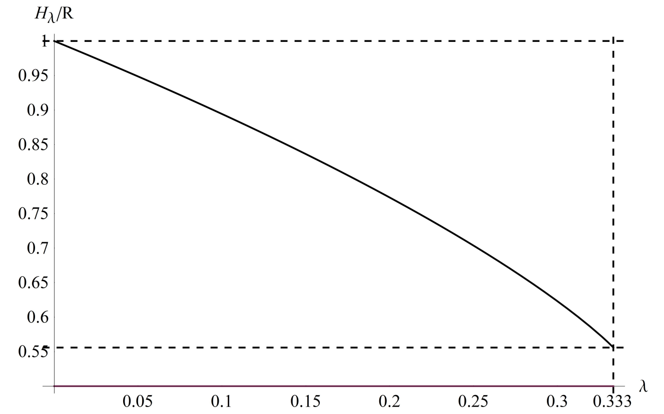

Fig. 1. Dependence of semi-height H_λ scaled by equatorial radius R on deformation parameter $\lambda \in [0,1/3)$. Gridlines (dashed) are $\lambda = 1/3$, $H_\lambda/R = 0.555\,947\,286\ldots$, and $H_\lambda/R = 1$

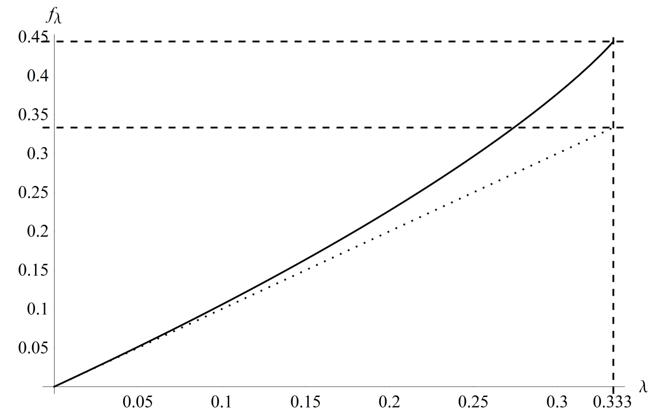

Fig. 2. Dependence of flattening factor f_λ (continuous line) on deformation parameter $\lambda \in [0,1/3)$. For comparison there is also shown the graph of $f_\lambda = \lambda$ (dotted line). Gridlines (dashed) are $\lambda = 1/3$, $f_\lambda = 1/3$, and $f_\lambda = 0.444\,052\,714\ldots$

It can be also shown that for small values of λ we have that k is close to 1, therefore, using the identities (see, e.g., [11])

$$F(\theta_0, 1) = \operatorname{arctanh}(\sin\theta_0), \qquad E(\theta_0, 1) = \sin\theta_0, \tag{12}$$

$$\Pi(n, \theta_0, 1) = \frac{\sqrt{n}\operatorname{arctanh}(\sqrt{n}\sin\theta_0) - \operatorname{arctanh}(\sin\theta_0)}{n-1}, \tag{13}$$

and taking into account that $\theta_0 = \arcsin(r^2 - k) \approx r^2 - k$ when $r^2 - k$ is close to 0, we obtain that

$$f_\lambda \approx 1 + r^2 - k - \frac{r^2 - k}{1-k} + \frac{1-k}{r^2}\left(r^2 - k\right) = \frac{r^2 - 1}{r^2} = \lambda \tag{14}$$

since $1 + r^2 - k = 1/k$, $r^2 - k = (1-k)/k$, and from (4) we have that $(1-k)(r^2-k) = (1-k)^2/k = r^2 - 1$.

Finally, from (10) or (11) we can estimate the value of the deformation parameter for the λ-sphere corresponding to the spheroid (ellipsoid of revolution) of WGS84 (see, e.g., [4]) as

$$\lambda_1 \approx 0.003\,347\,187\ldots \tag{15}$$

For the above value of the deformation parameter, we obtain that the semi-height H_λ and the flattening factor f_λ of the λ-sphere (or, equivalently, the semi-minor axis b and the flattening factor f of the spheroid) can be estimated as

$$H_\lambda = b \approx 6\,356\,752.314\,245\,176\ldots \text{ m}, \tag{16}$$

$$f_\lambda = f \approx 0.003\,352\,811\ldots \tag{17}$$

As it was shown above, we have that $f_\lambda = f \approx \lambda_1$ (with the relative error $\delta = |(f - \lambda_1)/f|$ being of the order 1.677×10^{-3}). This also means that we can estimate the eccentricity of the spheroid as $\varepsilon = \sqrt{f(2-f)} \approx \sqrt{\lambda(2-\lambda)} = \sqrt{r^4 - 1}/r^2$.

3.2. Semi-major axis and quarter-meridian length

Let us identify the semi-major axis a and the quarter-meridian length (defined as the arc length along the meridian between the Equator and North Pole) m_P of the spheroid (7) with the equatorial radius R and the quarter-meridian length m_λ of the λ-sphere, i.e., $R \equiv a$ and $m_\lambda \equiv m_P$, where (see, e.g., [7, 8, 15])

$$m_\lambda = \frac{\pi R}{2r}, \qquad m_P = aE(\varepsilon), \tag{18}$$

and $E(\varepsilon)$ is the complete elliptic integral of the second kind (i.e., $E(\varepsilon) = E(\pi/2, \varepsilon)$) with the eccentricity $\varepsilon = \sqrt{f(2-f)}$.

From the first equation in (18) we can see that the general dependency of the quarter-meridian length m_λ of the λ-sphere scaled by the quarter-meridian length of the equivalent sphere with the radius R (i.e., $m_{\text{sphere}} = \pi R/2$) on the deformation parameter λ is again a one-to-one function in the range $\lambda \in [0,1/3)$ (see Fig. 3).

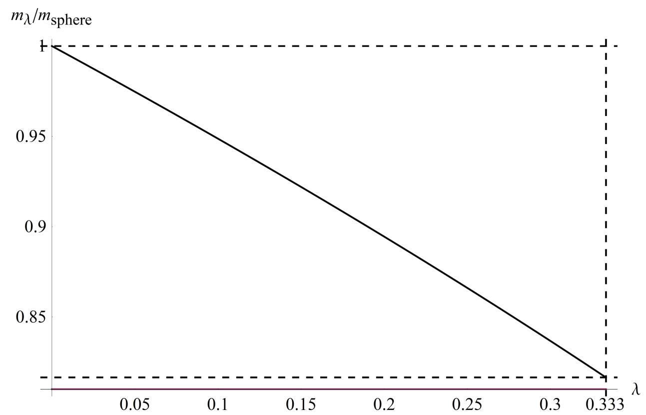

Fig. 3. Dependence of quarter-meridian length m_λ of deformed sphere scaled by quarter-meridian length $m_{\text{sphere}} = \pi R/2$ of equivalent sphere with radius R on deformation parameter $\lambda \in [0,1/3)$. Gridlines (dashed) are $\lambda = 1/3$, $m_\lambda/m_{\text{sphere}} = 0.816\,496\,581\ldots$, and $m_\lambda/m_{\text{sphere}} = 1$

Therefore, we have the condition for estimation (on the basis of WGS84) of the deformation parameter value given as

$$\lambda_2 = 1 - \frac{4E^2(\varepsilon)}{\pi^2} \approx 0.003\,348\,595\ldots \tag{19}$$

For the above value of the deformation parameter, we obtain that the quarter-meridian length m_λ for the λ-sphere (or, equivalently, m_P for the spheroid) can be estimated as

$$m_\lambda = m_P \approx 10\,001\,965.729\,312\,722\ldots \text{ m}. \tag{20}$$

3.3. Semi-major axis and surface area

The surface area of the λ-sphere is given as

$$S_\lambda = 2\pi R \int_0^1 \sqrt{R^2 + (z'(u))^2}\, \mathrm{d}u^2 = 2\pi R^2 \int_0^1 \frac{\mathrm{d}u^2}{\alpha(u)}. \tag{21}$$

Substituting $\alpha(u)$ given by (6) into the above expression, we can calculate that (see, e.g., [1])

$$\frac{S_\lambda}{4\pi R^2} = \frac{1-\lambda}{\sqrt{\lambda}} \operatorname{arcsinh} \sqrt{\frac{\lambda}{1-\lambda}} = \frac{1-\lambda}{\sqrt{\lambda}} \operatorname{arctanh} \sqrt{\lambda}. \tag{22}$$

On the other side, for the surface area of the spheroid we have that (see, e.g., [16] and [13, p. 102])

$$\frac{S}{2\pi a^2} - 1 = \frac{1-\varepsilon^2}{\varepsilon} \operatorname{arcsinh} \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} = \frac{1-\varepsilon^2}{2\varepsilon} \ln \frac{1+\varepsilon}{1-\varepsilon}. \tag{23}$$

Again, from (22) we obtain that the general dependency of the surface area S_λ of the λ-sphere being scaled by the surface area of the equivalent sphere with the radius R (i.e., $S_{\text{sphere}} = 4\pi R^2$) on the deformation parameter λ is a one-to-one function in the range $\lambda \in [0, 1/3)$ (see Fig. 4).

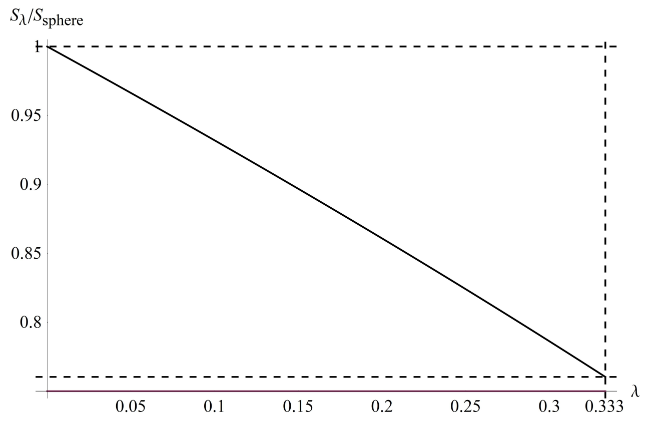

Fig. 4. Dependence of surface area S_λ of deformed sphere scaled by surface area $S_{\text{sphere}} = 4\pi R^2$ of equivalent sphere with radius R on deformation parameter $\lambda \in [0, 1/3)$. Gridlines (dashed) are $\lambda = 1/3$, $S_\lambda/S_{\text{sphere}} = 0.760\,345\,996\ldots$, and $S_\lambda/S_{\text{sphere}} = 1$

Therefore, let us identify the semi-major axis a with the equatorial radius R, i.e., $R \equiv a$, whereas the surface area of the spheroid is identified with the surface area of the λ-sphere, i.e., $S_\lambda \equiv S$. Then on the basis of WGS84 the estimated value of the deformation parameter can be now found as

$$\lambda_3 \approx 0.003\,349\,437\ldots \tag{24}$$

For the above value of the deformation parameter, we obtain that the surface area S_λ of the λ-sphere (or, equivalently, the surface area S of the spheroid) can be estimated as

$$S_\lambda = S \approx 510\,065\,621.724\,088\ldots \text{ km}^2. \tag{25}$$

3.4. Semi-major axis and volume

Next, using integration by parts and taking into account that $\left[u^2 z(u)\right]_0^1 = 0$, the volume of the λ-sphere can be calculated as

$$V_\lambda = 4\pi R^2 \int_0^1 u z(u)\mathrm{d}u = -\pi R^2 \int_0^1 u z'(u)\mathrm{d}u^2 = \pi R^3 \int_0^1 u \frac{\sqrt{1-\alpha^2(u)}}{\alpha(u)}\, \mathrm{d}u^2. \tag{26}$$

Substituting $\alpha(u)$ given by (6) into the above expression, we can calculate this integral as (see Appendix A.1 for details)

$$V_\lambda = \frac{2}{3}\pi R^3 \left(\frac{\sqrt{k(2k-1)(2-k)} + (2k-1)F(\theta_0,k)}{(1-k)^2} + \frac{k\left(r^2-2\right)}{(1-k)^3} E(\theta_0,k) + \frac{3n\Pi(n,\theta_0,k)}{1-k} \right). \tag{27}$$

On the other side, the volume of the spheroid is given as

$$V = \frac{4}{3}\pi a^2 b = \frac{4}{3}\pi a^3 (1-f). \tag{28}$$

Similarly to the previous cases, from (27) we obtain that the general dependency of the volume V_λ of the λ-sphere being scaled by the volume of the equivalent sphere with the radius R (i.e., $V_{\text{sphere}} = (4/3)\pi R^3$) on the deformation parameter λ is a one-to-one function in the range $\lambda \in [0, 1/3)$ (see Fig. 5).

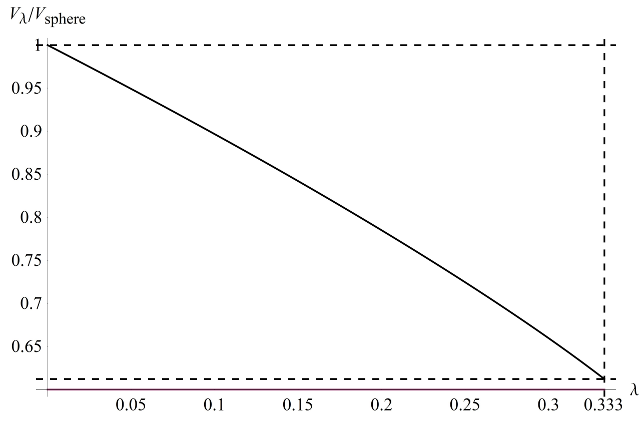

Fig. 5. Dependence of volume V_λ of deformed sphere scaled by volume $V_{\text{sphere}} = (4/3)\pi R^3$ of equivalent sphere with radius R on deformation parameter $\lambda \in [0, 1/3)$. Gridlines (dashed) are $\lambda = 1/3$, $V_\lambda/V_{\text{sphere}} = 0.612\,218\,125\ldots$, and $V_\lambda/V_{\text{sphere}} = 1$

Therefore, when we identify the semi-major axis a with the equatorial radius R and the volumes of the spheroid and λ-sphere, i.e., $R \equiv a$ and $V_\lambda \equiv V$, then the estimated value of the deformation parameter (on the basis of WGS84) is found as

$$\lambda_4 \approx 0.003\,349\,435\ldots \tag{29}$$

For the above value of the deformation parameter, we obtain that the volume V_λ of the λ-sphere (or, equivalently, the volume V of the spheroid) can be estimated as

$$V_\lambda = V \approx 1\,083\,207\,319\,801.412\ldots\ \mathrm{km}^3. \tag{30}$$

Let us also note that we can rewrite the pair of identities $R \equiv a$ and $V_\lambda \equiv V$ as an identity on the flattening factor f of the spheroid and the corresponding expression dependent on the deformation parameter λ, i.e.,

$$1 - \frac{1}{2}\left(\frac{\sqrt{k(2k-1)(2-k)} + (2k-1)F(\theta_0,k)}{(1-k)^2} + \frac{k\left(r^2-2\right)}{(1-k)^3}E(\theta_0,k) + \frac{3n\Pi(n,\theta_0,k)}{1-k}\right) \equiv f. \tag{31}$$

Please compare the above expression with the identity (11) that was obtained in Section 3.1.

3.5. Semi-major axis and sphericity index

Let us consider the sphericity index defined by Wadell in [17] as a measure of how closely the shape of an object resembles that of a perfect sphere. It is generally defined as the ratio of the surface area of an equivalent sphere with the same volume as the given object to the surface area of the object, i.e.,

$$\Psi = \frac{\pi^{1/3}(6V)^{2/3}}{S}. \tag{32}$$

For a sphere, the sphericity index Ψ is identically equal to one.

Next, using (32) in order to obtain the general dependency of the sphericity index Ψ_λ of the λ-sphere on the deformation parameter λ, we see that it is again given as a one-to-one function in the range $\lambda \in [0, 1/3)$ (see Fig. 6).

Therefore, when we identify the semi-major axis a and the sphericity index Ψ of the spheroid with the equatorial radius R and the sphericity index Ψ_λ of the λ-sphere, then we obtain that the estimation of the deformation parameter (on the basis of WGS84) is given as

$$\lambda_5 \approx 0.003\,347\,992\ldots \tag{33}$$

For the above value of the deformation parameter, we obtain that the sphericity index Ψ_λ of the λ-sphere (or, equivalently, the sphericity index Ψ of the spheroid) can be estimated as

$$\Psi_\lambda = \Psi \approx 0.999\,997\,993\,753\ldots \tag{34}$$

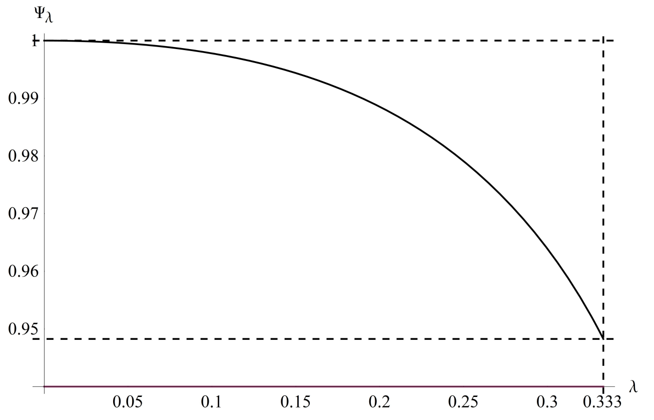

Fig. 6. Dependence of sphericity index Ψ_λ of deformed sphere on deformation parameter $\lambda \in [0, 1/3)$. Gridlines (dashed) are $\lambda = 1/3$, $\Psi_\lambda = 0.948\,257\,294\ldots$, and $\Psi_\lambda = 1$

3.6. Semi-major axis and tipping (bifurcation) point

In our previous papers [4, 5, 8] we obtained that the differential formula for the geodesic on the λ-sphere is given as

$$\mathrm{d}v = \pm \frac{\mathrm{d}u}{u\alpha(u)\sqrt{\delta^2u^2-1}}, \tag{35}$$

where $\delta = R/C$ is the inverse of the dimensionless Clairaut's constant. Let us recall that the Clairaut's relation states that for a geodesic on any surface of revolution the product of the radius of the parallel Ru and the sine of the azimuth α has a constant value C called the Clairaut's constant: $Ru\sin\alpha = C$.

Integrating the expression (35) we obtain that [4, 8]

$$v(u) = v_0 \pm \arctan\sqrt{\frac{1-u^2}{\delta^2u^2-1}} \mp \frac{\lambda}{\gamma}\arctan\left(\gamma\sqrt{\frac{1-u^2}{\delta^2u^2-1}}\right), \tag{36}$$

where $\gamma = (1/r)\sqrt{\delta^2+r^2-1}$.

In Fig. 7 there is shown an exemplary dependency $v(u)$ of the geodesic (36) on the unit ($R = 1$) λ-sphere with the chosen value of the deformation parameter $\lambda = f$ and the Clairaut's constant $C = \sin(\pi/4) = 1/\sqrt{2}$, i.e., with $\delta = \sqrt{2}$.

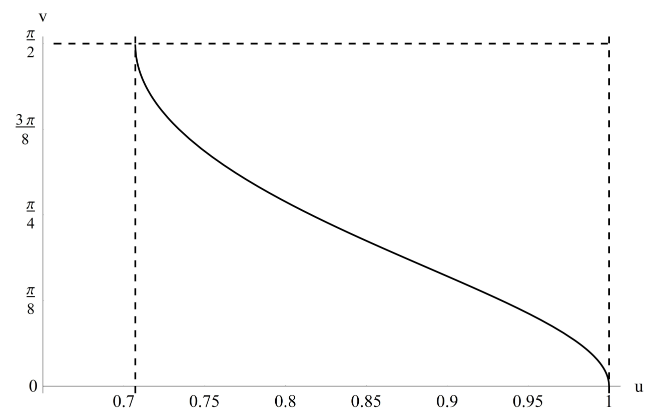

Fig. 7. Dependence of longitude v on local latitude u for geodesic on deformed sphere of unit equatorial radius $R = 1$ with deformation parameter $\lambda = f \approx 0.003\,352\,811\ldots$ and Clairaut's constant $C = \sin(\pi/4) = 1/\sqrt{2}$. Gridlines (dashed) are $u = \sqrt{2}$, $u = 1$, and $v = \pi/2$

From the Clairaut's relation we can deduce that any geodesic on a surface of revolution is sinusoidally changes in the strip around the Equator where the radius of the parallel Ru is greater than or equal to the Clairaut's constant $|C|$, since $|\sin\alpha| = |C|/Ru \in [0,1]$. This means that the local latitude u is restricted to the range from $u_{\min} = 1/|\delta|$ to $u_{\max} = 1$.

Therefore, the half-period $T_\lambda^{1/2}$ in the longitude variable v for the chosen geodesic (36) described with the help of the inverse dimensionless Clairaut's constant δ can be obtained as

$$T_\lambda^{1/2}(\delta) = \pi\left(1 - \frac{\lambda r}{\sqrt{\delta^2 + r^2 - 1}}\right). \tag{37}$$

From the above expression we can obtain that for two points $P_1 = (1,0)$ and $P_2 = (1, v_2)$ laying on the Equator there exists the tipping (bifurcation) point (see, e.g., [8, 18]) defined as $T_\lambda^{1/2}(\delta = 1) = \pi(1 - \lambda)$ which distinguishes between the region where the solution of the inverse geodetic problem (IGP) is unique, i.e., then we have that $v_2 \leq T_\lambda^{1/2}$, and the region where the solution of IGP is not unique, i.e., then we have that $v_2 > T_\lambda^{1/2}$. In the second case there are two distinct geodesics with the equal lengths but different azimuths (one of them is ascending to the north, whereas another is descending to the south). Eventually, when $C = 0$ (i.e., $\delta \to \infty$) the above two geodesics become the meridian arcs connecting the starting point P_1 and the destination point P_2 that pass through the North/South Pole, respectively.

Similarly, for the spheroid (ellipsoid of revolution) defined using the parametrization (8) we can obtain that the differential formula for the geodesic is given as

$$\mathrm{d}v = \sqrt{\frac{1 - \varepsilon^2 \sin^2\phi}{\delta^2 \sin^2\phi - 1}}\,\frac{\mathrm{d}\phi}{\sin\phi}, \tag{38}$$

where $\delta = a/C$ is again defined as the inverse of the dimensionless Clairaut's constant on the spheroid. Integrating the above expression (see Appendix A.2 for details), we obtain that

$$v(u) = v_0 \pm \frac{\delta^2 \Pi\left(1 - \delta^2, \varphi, \kappa\right) - \varepsilon^2 F(\varphi, \kappa)}{\sqrt{\delta^2 - \varepsilon^2}}, \tag{39}$$

where the amplitude and elliptic modulus of the incomplete elliptic integrals of the first and third kind are given as

$$\varphi = \arcsin\sqrt{\frac{\delta^2 \sin^2\phi - 1}{\delta^2 - 1}}, \qquad \kappa = \varepsilon\sqrt{\frac{\delta^2 - 1}{\delta^2 - \varepsilon^2}}. \tag{40}$$

Figure 8 exemplifies the dependency $v(\phi)$ of the geodesic (39) on the unit ($a = 1$) spheroid with the Clairaut's constant $C = \sin(\pi/4) = 1/\sqrt{2}$, i.e., with $\delta = \sqrt{2}$.

From the Clairaut's relation we can again obtain that on the spheroid the radius of the parallel $a\sin\phi$ is greater than or equal to the Clairaut's constant C, since $|\sin\alpha| = |C|/a\sin\phi \in [0,1]$. This means that the latitude ϕ is restricted to the range from $\phi_{\min} = \arcsin(1/|\delta|)$ to $\phi_{\max} = \pi/2$ for the Northern Hemisphere. Therefore, the half-period $T^{1/2}$ in the longitude variable v for the chosen geodesic (39) can be obtained as

$$T^{1/2}(\delta) = \frac{2}{\sqrt{\delta^2 - \varepsilon^2}}\left(\delta^2 \Pi\left(1 - \delta^2, \kappa\right) - \varepsilon^2 K(\kappa)\right), \tag{41}$$

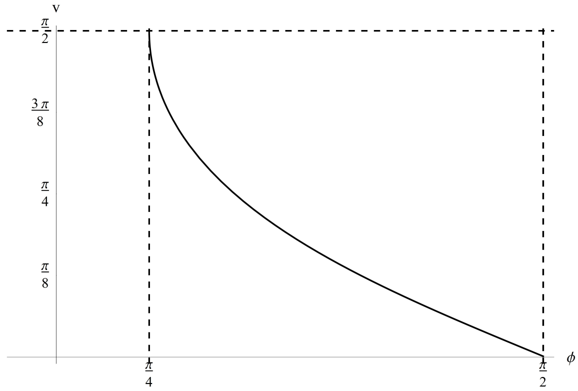

Fig. 8. Dependence of longitude v on latitude ϕ for geodesic on spheroid with unit equatorial radius $a = 1$ and Clairaut's constant $C = \sin(\pi/4) = 1/\sqrt{2}$. Gridlines (dashed) are $\phi = \pi/4$, $\phi = \pi/2$, and $v = \pi/2$

where $K(\kappa)$ and $\Pi(n,\kappa)$ are the complete elliptic integrals of the first and third kind that are defined through the equalities $K(\kappa) = F(\pi/2, \kappa)$ and $\Pi(n,\kappa) = \Pi(n, \pi/2, \kappa)$. Then we have that the tipping (bifurcation) point on the spheroid is given as

$$T^{1/2}(\delta = 1) = \pi\sqrt{1 - \varepsilon^2} = \pi(1 - f). \tag{42}$$

Finally, let us identify the semi-major axis a with the equatorial radius R as well as the tipping points for geodesics on the spheroid and λ-sphere, i.e., $R \equiv a$ and $T_\lambda^{1/2}(\delta = 1) \equiv T^{1/2}(\delta = 1)$. The last condition gives us equivalently that $\lambda \equiv f$.

Therefore, the estimated value (on the basis of WGS84) of the deformation parameter is given as

$$\lambda_6 = f \approx 0.003\,352\,811\ldots \tag{43}$$

For the above value of the deformation parameter, we obtain that the tipping point for geodesics on the λ-sphere (or, equivalently, the tipping point for geodesics on the spheroid) can be estimated as

$$T_\lambda^{1/2}(\delta = 1) = T^{1/2}(\delta = 1) \approx 179^\circ 23' 47.38\ldots''. \tag{44}$$

4. MINIMIZATION OF ROOT MEAN SQUARE ERROR

The formula for the RMS (Root Mean Square) error, i.e., the square root of the arithmetic mean of the squares of relative errors $\{\delta_H, \delta_M, \delta_S, \delta_V, \delta_\Psi, \delta_T\}$, is defined as

$$\mathrm{RMS} = \sqrt{\frac{1}{6}\left(\delta_H^2 + \delta_M^2 + \delta_S^2 + \delta_V^2 + \delta_\Psi^2 + \delta_T^2\right)}. \tag{45}$$

Finding the minimum of the above functional dependency of the RMS error on the deformation parameter λ, we obtain that the optimized value of the deformation parameter for the deformed sphere is given as (see Fig. 9)

$$\lambda_{\mathrm{RMS}} \approx 0.003\,349\,672\ldots \tag{46}$$

For the above value of the deformation parameter, we obtain that the minimum value of the RMS error is given as

$$\mathrm{RMS} \approx 1.660\,691\,537\ldots \times 10^{-6} \tag{47}$$

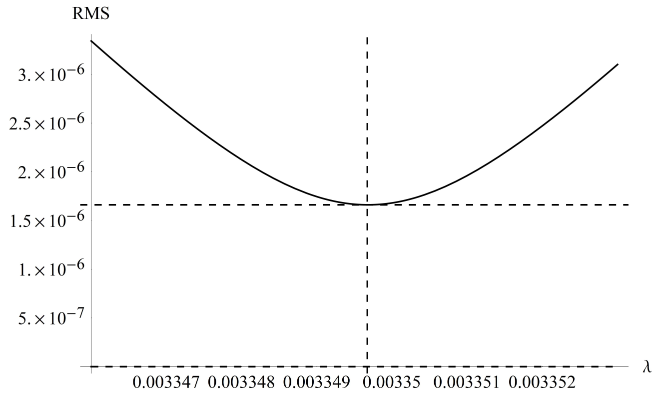

Fig. 9. Dependence of RMS error given by (45) on deformation parameter $\lambda \in [0.003\,346, 0.003\,353]$. Gridlines (dashed) correspond to minimum with $\lambda_{\mathrm{RMS}} = 0.003\,349\,672\ldots$ and $\mathrm{RMS} = 1.660\,691\,537\ldots \times 10^{-6}$

whereas the absolute and relative errors for the individual pairs of the discussed six geometric characteristics (i.e., semi-minor axes, quarter-meridian lengths, surface areas, volumes, sphericity indices, and tipping points) as well as the absolute and relative errors for matching the flattening factor f with the deformation parameter λ_{RMS} are given in Table 1.

Table 1

Absolute and relative errors for individual pairs of discussed main geometric characteristics for λ-sphere and standard spheroid

	b	$6\,356\,752.314\ldots$ m
ΔH	$b - H_\lambda$	$15.897\ldots$ m
δ_H	$(b - H_\lambda)/b$	$2.501\ldots \times 10^{-6}$
	m_P	$10\,001\,965.729\ldots$ m
Δm	$m_P - m_\lambda$	$5.401\ldots$ m
δ_M	$(m_P - m_\lambda)/m_P$	$5.400\ldots \times 10^{-7}$
	S	$510\,065\,621.724\ldots$ km^2
ΔS	$S - S_\lambda$	$79.925\ldots$ km^2
δ_S	$(S - S_\lambda)/S$	$1.567\ldots \times 10^{-7}$
	V	$1\,083\,207\,319\,801.412\ldots$ km^3
ΔV	$V - V_\lambda$	$257\,883.134\ldots$ km^3
δ_V	$(V - V_\lambda)/V$	$2.381\ldots \times 10^{-7}$
	Ψ	$0.999\,997\,993\,753\ldots$
$\Delta\Psi$	$\Psi - \Psi_\lambda$	$0.000\,000\,002\,020\ldots$
δ_Ψ	$(\Psi - \Psi_\lambda)/\Psi$	$2.020\ldots \times 10^{-9}$
	$T^{1/2}$	$179°23'47.38\ldots''$
$\Delta T^{1/2}$	$T^{1/2} - T_\lambda^{1/2}$	$-0°00'02.03\ldots''$
δ_T	$(T^{1/2} - T_\lambda^{1/2})/T^{1/2}$	$-3.150\ldots \times 10^{-6}$
	f	$0.003\,352\,811\ldots$
Δf	$f - \lambda_{\mathrm{RMS}}$	$0.000\,003\,139\ldots$
δf	$(f - \lambda_{\mathrm{RMS}})/f$	$9.362\ldots \times 10^{-4}$

5. CONCLUSIONS

In the present paper we have proposed a few different schemes of matching the deformed sphere with the standard spheroid within the WGS 84 reference model for the geoid. For instance, this matching can be done by identifying the corresponding values of the main geometric characteristics for both discussed surfaces of revolution. In such a way we have obtained six propositions for the estimated value of the deformation parameter $\lambda \in [0.003\,347,\ 0.003\,353]$.

It has turned out that one of the above estimations is especially simple and geometrically appealing. Namely, we can match the deformed sphere with the standard spheroid (e.g., the one within the WGS84's reference model for the geoid) simply identifying the pairs of geometric characteristics that define them, i.e., the equatorial radii of both surfaces $R = a = 6\,378\,137$ m as well as the deformation parameter of the deformed sphere and the flattening factor of the standard spheroid $\lambda_6 = f = 1/298.257\,223\,563$.

Additionally, using the process of minimization of the RMS error defined as the square-rooted arithmetic mean of the squared relative errors for the individual pairs of the discussed six main geometric characteristics, we have obtained the optimized value $\lambda_{\mathrm{RMS}} \approx 0.003\,349\,672\ldots$ which is placed almost perfectly in the center of the interval $[0.003\,347,\ 0.003\,353]$.

The main practical advantage of using the deformed spheres to approximate the shape of the Earth is that the geodesics on λ-spheres can be expressed through the well-known analytical functions (inverse tangent), whereas the geodesics on the spheroids are expressed through the incomplete elliptic integrals of the first and third kind. In such a way, we offer an easier, more friendly, and faster computational model for numerical analysis of this kind of geometrical quantities that are extremely important in geodesy and navigation problems.

Finally, in future research, we plan to use the obtained estimations of the deformation parameter (especially λ_6) or the optimized value λ_{RMS} for some practical calculations in geodesy or navigation problems, which can be done within the new reference model for the geoid based on the deformed sphere that is an alternative to the standard biaxial (rotational) ellipsoidal reference models for the geoid (e.g., WGS 84).

As a reference surface, the deformed sphere should fulfil the same conditions as the terrestrial ellipsoid, namely: 1) the deformed sphere center and its equatorial plane should coincide with the Earth's center of gravity and the plane of the Earth's equator, 2) the deformed sphere volume should be equal to the geoid's volume, 3) the sum of the squares of the deviations of the geoid from the deformed sphere should be minimal.

The presented schemes allow us also to easily re-obtain the estimations of the deformation parameter λ for different given values of two geometric parameters describing the standard spheroid, e.g., for World Geodetic Datum 2000 where we have that the semi-major and semi-minor axes are defined as

$$a = 6\,378\,136.572 \pm 0.053 \text{ m}, \tag{48}$$

$$b = 6\,356\,751.920 \pm 0.052 \text{ m}, \tag{49}$$

for the tide-free ellipsoidal reference model of the geoid or as

$$a = 6\ 378\ 136.602 \pm 0.053 \text{ m}, \tag{50}$$
$$b = 6\ 356\ 751.860 \pm 0.052 \text{ m}, \tag{51}$$

for the zero-frequency tide geoid of reference specified in [19].

APPENDIX

A.1. Calculation of expression (27)

Let us calculate the integral

$$I_1 = \int u \frac{\sqrt{1-\alpha^2(u)}}{\alpha(u)} \, du^2. \tag{52}$$

Introducing $\xi^2 = (r^2-1)u^2$ we can rewrite it as

$$I_1 = \frac{1}{(r^2-1)^{3/2}} \int \frac{\xi^2}{1+\xi^2} \sqrt{\frac{\xi^4 + A\xi^2 + B}{r^2-1-\xi^2}} \, d\xi^2, \tag{53}$$

where $A = 3 - r^2$, $B = 3 - 2r^2$.

Next substituting $\zeta^2 = r^2 - 1 - \xi^2$ we obtain that

$$I_1 = \frac{-2}{(r^2-1)^{3/2}} \int \left(\zeta^4 - r^2\zeta^2 + \frac{r^2-1}{r^2-\zeta^2} \right) \frac{d\zeta}{\sqrt{P_2(\zeta^2)}}, \tag{54}$$

where $P_2(\zeta^2) = \zeta^4 + (r^2+1)\zeta^2 + 1$ is the second-order polynomial with respect to the variable ζ^2 which has the roots

$$\zeta_\pm^2 = \frac{1}{2}\left(r^2 + 1 \pm \sqrt{(r^2+1)^2 - 4} \right). \tag{55}$$

Let us denote $\zeta_-^2 = k$. Then using the relations $\zeta_+^2 \zeta_-^2 = 1$ and $\zeta_-^2 \leq \zeta_+^2$ we obtain that $\zeta_+^2 = 1/k$ and $k^2 \leq 1$. Moreover, using the relation $\zeta_+^2 + \zeta_-^2 = r^2 + 1$ we obtain that $r^2 = k + 1/k - 1$.

Then, substituting $\zeta = \sqrt{k} \sin\theta$ we rewrite (54) as

$$\begin{aligned} I_1 = \frac{-2k^2}{(1-k)^3} \Bigg(& \lambda \int \frac{d\theta}{(1 - n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} \\ & - kr^2 \int \frac{\sin^2\theta \, d\theta}{\sqrt{1-k^2\sin^2\theta}} + k^2 \int \frac{\sin^4\theta \, d\theta}{\sqrt{1-k^2\sin^2\theta}} \Bigg), \end{aligned} \tag{56}$$

where $n = k/r^2$. Therefore, using the identities (see, e.g., [11])

$$\int \frac{\sin^2\theta \, d\theta}{\sqrt{1-k^2\sin^2\theta}} = \frac{1}{k^2}\Big(F(\theta,k) - E(\theta,k) \Big), \tag{57}$$

$$\begin{aligned} \int \frac{\sin^4\theta \, d\theta}{\sqrt{1-k^2\sin^2\theta}} &= \frac{2+k^2}{3k^4} F(\theta,k) - \frac{2(1+k^2)}{3k^4} E(\theta,k) \\ &\quad + \frac{\sin\theta\cos\theta\sqrt{1-k^2\sin^2\theta}}{3k^2} \end{aligned} \tag{58}$$

and the definition of the incomplete elliptic integral of the third kind, we obtain that

$$\begin{aligned} I_1 = -\frac{2}{3} \Bigg(& \frac{3n\lambda\Pi(n,\theta,k)}{1-k} + \frac{k^2\sin\theta\cos\theta\sqrt{1-k^2\sin^2\theta}}{(1-k)^3} \\ & + \frac{(2k-1)F(\theta,k)}{(1-k)^2} + \frac{k(r^2-2)E(\theta,k)}{(1-k)^3} \Bigg), \end{aligned} \tag{59}$$

where the amplitude is given as

$$\theta = \arcsin\sqrt{\frac{r^2-1-\xi^2}{k}} = \arcsin\left((r^2-k)\sqrt{1-u^2} \right). \tag{60}$$

Finally, substituting the obtained antiderivative $I_1(u)$ into the definite integral in (26) we obtain the expression (27).

A.2. Calculation of expression (39)

Let us calculate the integral

$$I_2 = \int \sqrt{\frac{1-\varepsilon^2\sin^2\phi}{\delta^2\sin^2\phi - 1}} \frac{d\phi}{\sin\phi}. \tag{61}$$

Denoting $\zeta^2 = \delta^2\sin^2\phi - 1$ we can rewrite it as

$$I_2 = \int \sqrt{\frac{\delta^2 - \varepsilon^2 - \varepsilon^2\zeta^2}{\delta^2 - 1 - \zeta^2}} \frac{d\zeta}{1+\zeta^2}. \tag{62}$$

Next substituting $\eta = \zeta/\sqrt{\delta^2-1}$ we obtain that

$$I_2 = \int \frac{\sqrt{\delta^2 - \varepsilon^2 - \varepsilon^2(\delta^2-1)\eta^2}}{1-(1-\delta^2)\eta^2} \frac{d\eta}{\sqrt{1-\eta^2}}. \tag{63}$$

Then introducing the new variable $\varphi = \arcsin\eta$ we have

$$I_2 = \int \frac{\sqrt{\delta^2 - \varepsilon^2 - \varepsilon^2(\delta^2-1)\sin^2\varphi}}{1-(1-\delta^2)\sin^2\varphi} \, d\varphi. \tag{64}$$

Finally, when we denote

$$n = 1 - \delta^2, \qquad \kappa^2 = \frac{\varepsilon^2(\delta^2-1)}{\delta^2 - \varepsilon^2} \tag{65}$$

the above integral can be rewritten as

$$\begin{aligned} I_2 = & \frac{\delta^2}{\sqrt{\delta^2-\varepsilon^2}} \int \frac{d\varphi}{(1-n\sin^2\varphi)\sqrt{1-\kappa^2\sin^2\varphi}} \\ & - \frac{\varepsilon^2}{\sqrt{\delta^2-\varepsilon^2}} \int \frac{d\varphi}{\sqrt{1-\kappa^2\sin^2\varphi}}. \end{aligned} \tag{66}$$

Therefore, using the definitions of the incomplete elliptic integrals of the first and third kind (see, e.g., [11]) we obtain that

$$I_2 = \frac{1}{\sqrt{\delta^2-\varepsilon^2}} \left(\delta^2\Pi(n,\varphi,\kappa) - \varepsilon^2 F(\varphi,\kappa) \right), \tag{67}$$

where the amplitude φ as a function of the original variable ϕ is given as

$$\varphi = \arcsin\frac{\zeta}{\sqrt{\delta^2-1}} = \arcsin\sqrt{\frac{\delta^2\sin^2\phi - 1}{\delta^2 - 1}}. \tag{68}$$

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