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*Research article*

# Reversible codes and applications to DNA codes over $F_{4^{2t}}[u]/(u^2 - 1)$

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**Abstract:** Let $n \geq 1$ be a fixed integer. Within this study, we present a novel approach for discovering reversible codes over rings, leveraging the concept of $r$-glifted polynomials. This technique allows us to achieve optimal reversible codes. As we extend our methodology to the domain of DNA codes, we establish a correspondence between $4t$-bases of DNA and elements within the ring $R_{2t} = F_{4^{2t}}[u]/(u^2 - 1)$. By employing a variant of $r$-glifted polynomials, we successfully address the challenges of reversibility and complementarity in DNA codes over this specific ring. Moreover, we are able to generate reversible and reversible-complement DNA codes that transcend the limitations of being linear cyclic codes generated by a factor of $x^n - 1$.

**Keywords:** reversible code; DNA code; $r$-glifted polynomial
**Mathematics Subject Classification:** 94B05, 94B60, 94B65

## 1. Introduction

DNA, the genetic information of living organisms, consists of four bases (nucleotides): adenine (A), guanine (G), thymine (T), and cytosine (C). These bases form strands in a double helix structure, linked according to the Watson-Crick model. The Watson-Crick complement (WCC) pairs are A-T and G-C. This is denoted as $A^c = T, T^c = A, G^c = C$, and $C^c = G$. Adleman's groundbreaking work [1] demonstrated the successful use of DNA molecules to solve a combinatorial problem known as the directed salesman problem. This approach relied on the Watson Crick Complement (WCC)-property for DNA strands. Boneh et al. [2] and Adleman et al. [3] independently presented a molecular program for breaking the Data Encryption Standard (DES) algorithm using DNA. The potential of DNA molecules as a storage medium was explored in [4]. The connection between DNA and error-correcting codes has been a subject of interest for researchers. Liebovitch et al. [5] searched for an error-correcting code in real DNA sequences, and Brandao et al. [6] further investigated this area. DNA codes often involve constraints such as the Hamming distance constraint, reverse-complement constraint, reverse constraint, and fixed GC-content constraint, as seen in the literature (see [7–12] for details). The general definition of DNA codes in the literature considers codes $C$ over $R^n$ (where $R$ is a ring) that satisfy at least one constraint. DNA corresponding to code $C$ preserves the properties of being reversible or having a reversible complement. However, applying constraints to DNA correspondence of $C$ when $|R| > 4$ remains an open problem. The first solution of this problem for DNA codes over $F_{16}$ was discussed in [13] and the generalization of the solution was presented in [14] by using lifted polynomials.

Researchers also focused on constructing large sets of DNA codewords with prescribed minimum Hamming distance (e.g., see [8, 15, 16] for details). Further, improvements and constructions for DNA codes have been given in [17–19]. Some researchers explored four-element sets with algebraic structures due to the four-letter DNA alphabet. Abualarub et al. [15] studied DNA codes over finite fields with four elements, while DNA codes over the finite ring $F_2[u]/(u^2 - 1)$ were examined in [20, 21]. Some previous studies used $Z_4$, $F_4$, $F_2$ for base fields for ring extension that correspond to DNA bases (cf.; [22–27]).

In the present study, we use a base field greater than $F_4$ for the ring as in different previous studies. Thus, the reversibility problem has become a two step reversibility problem. We solve this problem by defining new polynomials and generation methods for DNA codes over the ring $R_{2t} = F_{4^{2t}} + uF_{4^{2t}} = F_{4^{2t}}[u]/(u^2 - 1)$. We introduce "$r$-glifted polynomial" and a construction methods to facilitate the construction of reversible codes, some of which are optimal, over the ring $H_q = F_q[u]/(u^2 - 1)$ and finite fields $F_q$. These constructions offer direct methods compared to previous works, where codes are generated by a factor or combination of factors of $x^n - 1$. We use the factors of $x^n - 1$ only to design the generator limit and "$r$-glifted polynomial". By employing special polynomials, we extend the study of DNA codes by defining "$4^t - r$-lifted polynomial" over an extension ring of $R_{2t}$, identifying sequences of DNA bases with elements from the extension ring and defining the reverse complement property of DNA within the ring.

The significance of this approach lies in the ability to view any $4t$-base DNA strand as a ring element, providing more structural insight into DNA than the traditional four-letter alphabet allows. For instance, in previous works, the correspondence between elements of $Z_4$ and DNA nucleotides was not fixed, making it challenging to precisely identify DNA codewords in certain scenarios [28]. The use of DNA $4t$-bases, in this work allows for a direct mapping of DNA sequences to elements of the ring $R_{2t}$.

The rest of the paper is organized as follows: Section 2 provides background on the studied ring and basic notions related to DNA codes. Section 3 introduces $r$-glifted polynomials and uses them to construct reversible codes over rings. In Section 4, $4^t - r$-lifted polynomials are applied to codes over $R_{2t}$, resulting in reversible and reversible-complement DNA codes. The paper concludes with final remarks and potential future research directions.

## 2. Preliminaries

Each codeword $(f_0, f_1, \cdots, f_{n-1})$ corresponds the polynomial $f(x) = f_0 + f_1 x + \cdots + f_{n-1}x^{n-1}$ where $f_i \in F$. Let

$$\begin{aligned} \gamma : R[x]/(x^n - 1) &\rightarrow C \\ f(x) = f_0 + f_1 x + \cdots + f_{n-1}x^{n-1} &\rightarrow (f_0, f_1, \cdots, f_{n-1}). \end{aligned} \tag{2.1}$$

For each codeword $c = (f_0, f_1, \cdots, f_{n-1})$, we define the reverse of $c$ to be $c^r = (f_{n-1}, f_{n-2}, \cdots, f_0)$. Moreover, $H_q = F_q[u]/(u^2 - 1) = \{h_0 + h_1 u \mid h_0, h_1 \in F_q, u^2 = 1\}$, which is a commutative ring of size $q^2$, where $q$ is a prime power. The reciprocal of a polynomial $f(x) = f_0 + f_1 x + \cdots + f_s x^s$ with $f_s \neq 0$ is defined to be the polynomial $f^*(x) = x^s f(1/x) = f_s + f_{s-1}x + \cdots + f_0 x^s$. Acording to the reciprocal property, $\deg(f^*(x)) \leq \deg(f(x))$, but if $f_0 \neq 0$ then $\deg(f(x)) = \deg(f^*(x))$.

The lifted polynomial defined by [13, 14] and its general form was introduced by [29]. These definitions are not sufficient for our structures.

**Definition 1.** *( [14]) Let $h(x) = b_0 + b_1 x + ... + a_s x_s$ be a self reciprocal polynomial over $Z_p$ (prime field with p (prime) elements) and $h(x)|(x^n - 1)(mod p)$. A lifted polynomial of $h(x)$ is denoted by $g(x) \in F_q[x]$ and is defined as follows. If s is odd, then*

$$\ell_h(x) = \sum_{i=0}^{\frac{s-1}{2}} \begin{cases} \beta_i x^i + \beta_i x^{s-i} & , b_i \neq 0 \\ 0 & , b_i = 0. \end{cases} \tag{2.2}$$

*If s is even, then*

$$\ell_h(x) = \sum_{i=0}^{\frac{s}{2}} \begin{cases} \beta_i x^i + \beta_i x^{s-i} & , b_i \neq 0, i \neq s/2 \\ 0 & , b_i = 0 \\ \beta_{s/2} x^{s/2} & , b_i \neq 0, i = s/2 \end{cases} \tag{2.3}$$

*where $\beta_i \in F_q - \{0\}$.*

In this paper, we define new structures to generate reversible codes and solve the reversibility problem for DNA codes. We define the following definitions:

**Definition 2.** *Let $f(x) = f_0 + f_1 x + \cdots + f_s x^s$ be a factor of $x^n - 1$ over $F_p$ (p is a prime and $q = p^d$). Let $nc(f(x))$ denotes the number of nonzero coefficients and $\{cf(f(x))\}$ denotes the set of all coefficients of $f(x)$. A restricted lifted polynomial is $\ell_f^-(x) \in F_q[x]$ and is defined as follows:*

$$\ell_f^-(x) = \sum_{i=0}^{s} \begin{cases} \rho_i x^i & , f_i \neq 0, \rho_i \neq \rho_j \neq 0, j < i, \text{ if } q > nc(f) \\ \rho_i x^i & , f_i \neq 0, |\{cf(f(x))\} - 0| = q - 1, \text{ if } q \leq nc(f) \\ 0 & , a_i = 0 \end{cases} \tag{2.4}$$

*where $\rho_i \in F_q - \{0\}$ and $\rho_i$ is chosen arbitrarily.*

The restricted lifted polynomial provides the variation between coefficients of a polynomial due to its structure. It helps to keep the minimum distance in the codes high.

**Definition 3.** *Let $h(x)$ be a polynomial over a commutative ring $\mathbb{R}$ such that $\deg(h(x)) = s$. Generator set of $h(x)$ is given as follows:*

$$T^n_{h(x)} = \{h(x), xh(x), \cdots, x^{n-s-1}h(x)\}. \tag{2.5}$$

*$\langle T^n_{h(x)}\rangle$ is a $\mathbb{R}$-module (for finite field $F_q$ it is a $F_q$ vector space). The matrix form of $\langle T^n_{h(x)}\rangle$ is*

$$G_h = \begin{bmatrix} z_0 & z_1 & \cdots & z_{s-1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & z_0 & z_1 & \cdots & z_{s-1} & 0 & 0 & \cdots & 0 \\ \ddots & & & & \ddots & & & & \ddots \\ 0 & \cdots & 0 & 0 & 0 & z_0 & z_1 & \cdots & z_{s-1} \end{bmatrix},$$

*where $h(x) = z_0 + z_1 x + \cdots + z_{s-1}x^s$ , $z_i \in \mathbb{R}$ $(0 \le i \le s-1)$.*

## 3. Reversible codes over $\mathcal{R}$

In this section, we give a new approach to generate reversible codes over a ring $\mathcal{R}$. Moreover, we obtain optimal codes. We define $\varrho$ as a constant number for Definition 4 as follows:

$$\varrho = \begin{cases} 1 & , n-s \ge 3 \\ 0 & , n-s < 3 \end{cases}$$

where $n$ is length of code and $\deg(f(x)) = s$.

We give a definition named "reversible general lifted" polynomial ($r$-glifted). We begin with the following definition, which will be use to generate reversible codes over rings:

**Definition 4.** *Let $R$ be a ring and $\mathcal{R}$ be an extension of $R$ as $char(R) = char(\mathcal{R})$. Let $f(x) = f_0 + f_1 x + \cdots + f_s x^s$ be a reciprocal polynomial over $R$ and $f(x) \mid (x^n - 1)$. An $r$-glifted polynomial of $f(x)$ is denoted by $f^{r\lambda}(x) \in \mathcal{R}$ and is defined as follows:*

*If $n$ and $s$ are odd, then*

$$\begin{aligned} f^{r\lambda}(x) = &\sum_{i=0}^{\frac{s-1}{2}} \begin{cases} \theta_i x^i + \theta_i x^{s-i}, f_i \in U_R \\ \zeta_i x^i + \zeta_i x^{s-i}, f_i \in Z_R \end{cases} \\ &+ \varrho\left(\sum_{j=0}^{\frac{n-s-2}{2}} \begin{cases} \zeta_{s+1+j}x^{s+1+j} + \zeta_{s+1+j}x^{n-1-j}, \ j \ne (n-s-2)/2 \\ \zeta_{\frac{n+s}{2}} x^{\frac{n+s}{2}}, \ j = (n-s-2)/2 \end{cases}\right). \end{aligned} \tag{3.1}$$

*If $n$ is odd and $s$ is even, then*

$$f^{r\lambda}(x) = \sum_{i=0}^{\frac{s}{2}} \begin{cases} \theta_i x^i + \theta_i x^{s-i} & , f_i \in U_R, \ i \ne s/2 \\ \zeta_i x^i + \zeta_i x^{s-i} & , f_i \in Z_R, \ i \ne s/2 \\ \theta_i x^{s/2} & , f_i \in U_R, \ i = s/2 \\ \zeta_i x^{s/2} & , f_i \in Z_R, \ i = s/2 \end{cases} + \varrho\left(\sum_{j=0}^{\frac{n-s-3}{2}} \zeta_{s+1+j}x^{s+1+j} + \zeta_{s+1+j}x^{n-1-j}\right). \tag{3.2}$$

*If $n$ is even and $s$ is odd, then*

$$f^{r\lambda}(x)=\sum_{i=0}^{\frac{s-1}{2}}\begin{cases}\theta_i x^i+\theta_i x^{s-i} & ,f_i\in U_R\\ \zeta_i x^i+\zeta_i x^{s-i} & ,f_i\in Z_R\end{cases}+\varrho\left(\sum_{j=0}^{\frac{n-s-3}{2}}\zeta_{s+1+j}x^{s+1+j}+\zeta_{s+1+j}x^{n-1-j}\right). \tag{3.3}$$

*If n is even and s is even, then*

$$\begin{aligned}f^{r\lambda}(x)=&\sum_{i=0}^{\frac{s}{2}}\begin{cases}\theta_i x^i+\theta_i x^{s-i}, f_i\in U_R,\ i\neq s/2\\ \zeta_i x^i+\zeta_i x^{s-i}, f_i\in Z_R,\ i\neq s/2\\ \theta_i x^{s/2}, f_i\in U_R,\ i=s/2\\ \zeta_i x^{s/2}, f_i\in Z_R,\ i=s/2\end{cases}\\ &+\varrho\left(\sum_{j=0}^{\frac{n-s-2}{2}}\begin{cases}\zeta_{s+1+j}x^{s+1+j}+\zeta_{s+1+j}x^{n-1-j},\ j\neq(n-s-2)/2\\ \zeta_{\frac{n+s}{2}}x^{\frac{n+s}{2}},\ j=(n-s-2)/2\end{cases}\right),\end{aligned} \tag{3.4}$$

*where $U_R$ is set of units and $Z_R$ is set of zeros and zero divisors. $\theta_i\in U_R$ and $\zeta_i\in Z_R$ such that $i\in\{0,1,\cdots,n-1\}$.*

**Example 1.** *Let $n=10$ and $f(x)=1+2x+x^2+2x^3+x^4\mid(x^{10}-1)$ over $F_3$ and $f(x)$ is a self-reciprocal polynomial. Some r-glifted polynomials of $f(x)$ can be written as follows:*

$$\begin{aligned}f_1^{r\lambda}=&(1+w^3u)+(w^{124}+w^{241}u)x+(2+w^{23}u)x^2+(w^{124}+w^{241}u)x^3+(1+w^3u)x^4\\&+(w^{35}+w^{35}u)x^5+(w^{98}+w^{98}u)x^6+(w^{211}+w^{211}u)x^7+(w^{98}+w^{98}u)x^8\\&+(w^{35}+w^{35}u)x^9,\end{aligned}$$

$$\begin{aligned}f_2^{r\lambda}=&(w+w^5u)+(w^{14}+w^{41}u)x+(w^7+w^{47}u)x^2+(w^{14}+w^{41}u)x^3+(w+w^5u)x^4\\&+(w^5+w^5u)x^5+(w^8+w^8u)x^6+(w^{21}+w^{21}u)x^7+(w^8+w^8u)x^8+(w^5+w^5u)x^9,\end{aligned}$$

$$\begin{aligned}f_3^{r\lambda}=&w+(w^{156}+w^{89}u)x+2x^2+(w^{156}+w^{89}u)x^3+wx^4+(1+u)x^5+(w^{81}+w^{81}u)x^6\\&+(w^{221}+w^{221}u)x^7+(w^{81}+w^{81}u)x^8+(1+u)x^9,\end{aligned}$$

*where $a+ub\in F_{3^5}/(u^2-1)$ and $a,b,w^i\in F_{3^5}$ s.t. $0\leq i\leq 3^5-2$.*

The following theorem provides the criterion for determining the reversibility of a cyclic code within the field $F_q$:

**Theorem 1.** *[30] The cyclic code generated by the monic polynomial $g(x)=g_0+g_1x+\cdots+g_sx^s$ is reversible if and only if $g(x)$ is self-reciprocal, where $g(x)\mid(x^n-1)$.*

We will use the r-glifted polynomial to obtain reversible codes. In Theorem 2 below, we provide a general definition for lifting from $F_q$ to $H_q$. These algebraic structures can be extended to $R$ and $\mathcal{R}$, respectively. In this study, we focus on reversible codes and reversible DNA codes over the ring $F_{4^{2t}}/(u^2-1)$. Therefore, we utilize the special rings for the next theorems and definitions.

**Theorem 2.** *Let $n\geq 1$ be a fixed integer and $f(x)\mid(x^n-1)$ be a reciprocal polynomial over $F_q$ and $f^{r\lambda}(x)$ over $H_q$. Then, $C=\langle T^n_{f^{r\lambda}(x)}\rangle$ is a reversible code over $H_q$ of length n.*

*Proof.* Because of the structure of the generator and Definition 4, each generator component has its reverse in code. In set $T^n_{f^{r\lambda}(x)}$, the reverse of each $\gamma(x^i f^{r\lambda}(x))$ is $\gamma(x^{n-s-1-i} f^{r\lambda}(x))$. Then, the generator set satisfy a reversible vector set. Thus, $C$ is a reversible code over $\mathcal{R}$. □

**Example 2.** *Let* $n = 16$ *and* $f(x) = x^{12} + x^8 + x^4 + 1 \mid (x^{16} - 1)$ *over* $F_3$ *and* $f(x) = (x^2 + x + 2)(x^2 + 2x + 2)(x^4 + x^2 + 2)(x^4 + 2x^2 + 2) = x^{12} + x^8 + x^4 + 1$ *be a self-reciprocal polynomial. Let r-glifted polynomials of* $f(x)$ *be*

$$\begin{aligned} f^{r\lambda} &= (1 + w^3u) + (w^4 + w^4u)x + (2w^2 + w^2u)x^2 + (w^5 + w^5u)x^3 + (1 + w^6u)x^4 \\ &+ (w^5 + w^5u)x^5 + (w^9 + w^9u)x^6 + (w^5 + w^5u)x^7 + (1 + w^6u)x^8 \\ &+ (w^5 + w^5u)x^9 + (2w^2 + w^2u)x^{10} + (w^4 + w^4u)x^{11} + (1 + w^3u)x^{12} \\ &+ (w^7 + w^7u)x^{13} + (2w^2 + w^2u)x^{14} + (w^7 + w^7u)x^{15}, \end{aligned}$$

*where* $a+ub \in F_{3^2}[u]/(u^2-1)$ *and* $a, b, w^i \in F_{3^2}$ *s.t.* $0 \le i \le 3^2 - 2$. *Moreover,* $C = \langle f(x) \rangle$ *is a* $[16, 4, 4]$ *reversible code over* $F_3$. $C' = \langle T^{16}_{f^{r\lambda}(x)} \rangle$ *is a* $[16, 4, 6]$ *reversible code over* $F_{3^2}[u]/(u^2 - 1)$.

*Thus, we generate a reversible codes without finding a factor of* $x^{16} - 1$ *over* $F_{3^2}[u]/(u^2 - 1)$.

According to Theorem 1, a reciprocal polynomial is used to generate reversible code. We show that reversible codes can be generated by using any polynomials, in Theorems 3 and 4 below.

**Theorem 3.** *Let* $f(x) \mid (x^n - 1)$ *over* $F_p$ *and* $t_f(x) = \ell^-_f(x) + (\ell^-_f(x))^*$ *over* $F_q$. *Then,* $C = \langle T^n_{t_f(x)} \rangle$ *is a reversible code over* $F_q$ *of length n.*

*Proof.* Let $\ell^-_f(x) = a_0 + a_1x + ... + a_{s-1}x^{s-1} + a_sx^s$ be a restricted lifted polynomial over $F_q$. $(\ell^-_f(x))^* = a_s + a_{s-1}1x + ... + a_x^{s-1} + a_0x^s$ is reciprocal of $\ell^-_f(x)$. $t_f(x) = \ell^-_f(x) + (\ell^-_f(x))^* = (a_0 + a_s) + (a_1 + a_{s-1})x + ... + (a_{s-1} + a_1)x^{s-1} + (a_s + a_0)x^s$ is obtained. In this polynomial, coefficients satisfy the symmetry as: the coefficient of $x^i = a_i + a_{s-i}$ equals coefficient of $x^{s-i} = a_{s-i} + a_i$. After that, we obtain a generator that all polynomials (or rows) has its reverse according to the structure of $\langle T^n_{t_f(x)} \rangle$ in Definition 3. If all rows have its reverse in the generator, this generator generates reversible code. □

In Theorem 3, we consider $\ell^-_f(x)$ instead of $f(x)$ to obtain the reciprocal polynomial. It is explained why we use $\ell^-_f(x)$ in the following example:

**Example 3.** *Let* $f(x) = x^5 + x^4 + 2x^3 + x^2 + 2 \mid (x^{11} - 1)$ *over* $F_3$. $f^*(x) = 2x^5 + x^3 + 2x^2 + x + 1$. $t_1(x) = f(x) + f^*(x) = x^4 + x$. $t_1(x)$ *is a reciprocal polynomial. But it has limited component for lifting and effecting the distance over a finite field. If we choose* $\ell^-_f(x) = w^6x^5 + w^{21}x^4 + w^{47}x^3 + w^{63}x^2 + w^{79}$ *over* $F_{81}$, *then* $(\ell^-_f(x))^* = w^{79}x^5 + w^{63}x^3 + w^{47}x^2 + w^{21}x + w^6$ *and* $t_f(x) = \ell^-_f(x) + (\ell^-_f(x))^* = w^{61}x^5 + w^{21}x^4 + w^5x^3 + w^5x^2 + w^{21}x + w^{61}$ *can be obtained. Therefore,* $t_f(x)$ *has more component to effect the distance instead of* $t_1(x)$.

Moreover, it satisfies variety for reversible code and $t_f(x)$ can be more protective for distance then $f(x) + f^*(x)$.

The following example gives an optimal code according to Griesmer bound. Griesmer bound is $n \ge \sum_{i=0}^{k-1} \lceil \frac{d}{q^i} \rceil$ for a $[n, k, d]$ code over $F_q$ (see [31] for details).

**Example 4.** *Let* $f(x) = x^3 + x^2 + 1 \mid (x^7 - 1)$ *over* $F_2$ *and* $C = \langle f(x) \rangle$ *is an* $[7, 4, 3]$ *optimal code over* $F_2$. *It is not a reversible code.*

*Let us choose* $\ell_f^-(x) = w^5x^3 + w^{12}x^2 + w^8$ *over* $F_{16}$ *and* $(\ell_f^-(x))^* = w^8x^3 + w^{12}x + w^5$, $t_f(x) = \ell_f^-(x) + (\ell_f^-(x))^* = w^4x^3 + w^{12}x^2 + w^{12}x + w^4$. *Then,* $C = \langle T^7_{t_f(x)} \rangle$ *is a* $[7, 4, 4]$ *optimal code. It is also a reversible code over* $F_{16}$.

We give a map as follows:

$$\begin{aligned} \eta : H_q &\to F_q \times F_q \\ a + bu &\to (a, b). \end{aligned} \tag{3.5}$$

**Theorem 4.** *Let* $f(x) \mid (x^n - 1)$ *over* $F_p$. $y_f(x) = \ell_f^-(x) + u(\ell_f^-(x))^*$ *over* $H_q$. $C = \langle T^n_{y_f(x)} \rangle$ *is a code over* $H_q$. *Then,* $\eta(C)$ *is a* $[2n, 2k, d]$ *reversible code over* $F_q$. *The generator of* $\eta(C)$ *as follows:*

$$G = \begin{bmatrix} G_{\ell_f^-} & G_{(\ell_f^-)^*} \\ G_{(\ell_f^-)^*} & G_{\ell_f^-} \end{bmatrix}.$$

*Proof.* $\ell_f^-(x)+u(\ell_f^-(x))^*$ creates a kind of $f^{r\lambda}(x)$. Then the proof is similar to the proof of Theorem 2. □

**Example 5.** *Let* $f(x) = x^3 + x^2 + 1 \mid (x^7 - 1)$ *over* $F_2$. *Let us choose* $\ell_f^-(x) = w^2x^3 + w^5x^2 + 1$ *over* $F_8$. $(\ell_f^-(x))^* = x^3+w^5x+w^2$. $y_f(x) = w^2x^3+w^5x^2+1+u(x^3+w^5x+w^2) = (w^2+u)x^3+w^5x^2+w^5ux+(1+w^2u)$. *Then,* $C = \langle T^n_{y_f(x)} \rangle$ *is a* $[7, 4, 4]_{H_8}$ *code and it is an optimal code. The Grismer bound for rings was defined in [32]. Hence,* $\eta(C)$ *is a reversible* $[4, 8, 14]$ *code over* $F_8$.

## 4. Reversible DNA codes over $R_{2t} = F_{4^{2t}}[u]/(u^2 - 1)$

In this section, we modify the definitions and theorems of Section 3 to generate reversible DNA codes. In [14], DNA correspondence to $F_{4^{2t}}$ was given for DNA strands of $2t$ lengths. For each DNA base of length $2t$, an algorithm is given such that the corresponding element is $w \in F_{4^{2t}}$ and its DNA reverse is $w^{4^t} \in F_{4^{2t}}$. Let's denote the bijective map giving the DNA correspondence of the element of the field by $\tau$. For example, $\tau(w^{12}) \to AACA$ according to the algorithm of [14]. In this paper, we use the DNA correspondence tables of [14] and [13] for examples over finite fields. Then DNA reverse of $w^{12}$ is $\tau(w^{12^{4^2}}) = \tau(w^{192}) \to ACAA$. $\tau$ can be extended for codewords such that $\tau(x_0, x_2, \cdots, x_{n-1})) = (\tau(x_0), \tau(x_2), \cdots, \tau(x_{n-1}))$, where $x_i \in F_{4^{2t}}$, $i \in [0, n-1]$.

The main problem while using a higher than four element structure is the reversibility problem. This problem was defined and solved in [13] and [14] for DNA codes with two bases DNA, 2t-bases DNA and over $F_{16}$, $F_{256}$, respectively. Let $(\alpha_1, \alpha_2, \alpha_3)$ be a codeword that corresponds to ATGCAC as a DNA strand. The reverse of $(\alpha_1, \alpha_2, \alpha_3)$ is $(\alpha_3, \alpha_2, \alpha_1)$ and $(\alpha_3, \alpha_2, \alpha_1)$ corresponds to ACGCAT. But ACGCAT is not the actual reverse of ATGCAC. This is the first step of the reversibility problems. The second step is started by using a ring that is an extension of $F_{4^{2t}}$. Let $(a_1 + b_1u, a_2 + b_2u, a_3 + b_3u)$ be a codeword that corresponds to $(a_1, b_1, a_2, b_2, a_3, b_3)$ and ATAG ACGC CATG. The reverse of $(a_1 + b_1u, a_2 + b_2u, a_3 + b_3u)$ is $(a_3 + b_3u, a_2 + b_2u, a_1 + b_1u)$ that corresponds to $(a_3, b_3, a_2, b_2, a_1, b_1)$ and CATG ATAG ACGC. Both $(a_1 + b_1u, a_2 + b_2u, a_3 + b_3u)$ and $(a_1, b_1, a_2, b_2, a_3, b_3)$ have reversibility problems, because of $((a_1, b_1, a_2, b_2, a_3, b_3))^r \neq (a_3, b_3, a_2, b_2, a_1, b_1)$ and CATG ATAG ACGC is not reverse of ATAG ACGC CATG. In this section, we solve this double step reversibility problem by defining special polynomials.

We give the following definition for an extended correspondence map between $4tn$-bases DNA and codewords over elements of $R_{2t}^n$.

**Definition 5.** *Let $c = (c_0, c_1, \cdots, c_{n-1})$ be a codeword of code $C$ over $R_{2t}$. The DNA correspondence of $c$ is*

$$\gamma^\eta(c) : C \to (\tau(\eta(c_0)), \tau(\eta(c_1)), \cdots, \tau(\eta(c_{n-1}))). \tag{4.1}$$

*Then, $\gamma^\eta(c)$ is a DNA strand of length $4tn$.*

For instance, $\gamma^\eta(c_0, c_1) = (\omega^{23} + u\omega^{34}, \omega^{54} + u\omega^{70}) \to (\tau(\eta(\omega^{23} + u\omega^{34})), \tau(\eta(\omega^{54} + u\omega^{70})) = (\tau(\omega^{23}), \tau(\omega^{34}), \tau(\omega^{54}), \tau(\omega^{70})) = (ATTCAGGAAGGCTTAG)$ in $F_{256}$.

**Definition 6.** *Let $f(x) = f_0 + f_1 x + \cdots + f_s x^s$ be a reciprocal polynomial over $F_{4^{2t}}$ and $f(x)|(x^n - 1)$. A $4^t$-r-glifted polynomial of $f(x)$ is denoted by $f^{r\sigma}(x) \in R_{2t}$ and is defined as follows:*

*If $n$ and $s$ are odd, then*

$$\begin{aligned} f^{r\sigma}(x) = &\sum_{i=0}^{\frac{s-1}{2}} \begin{cases} \theta_i x^i + \theta_i'^{4^t} x^{s-i}, f_i \in U_R \\ \zeta_i x^i + \zeta_i'^{4^t} x^{s-i}, f_i \in Z_R \end{cases} \\ &+ \varrho \left( \sum_{j=0}^{\frac{n-s-2}{2}} \begin{cases} \zeta_{s+1+j} x^{s+1+j} + \zeta_{s+1+j}'^{4^t} x^{n-1-j}, \ j \neq (n-s-2)/2 \\ \zeta_{\frac{n+s}{2}} x^{\frac{n+s}{2}}, \ j = (n-s-2)/2, \zeta_{\frac{n+s}{2}}'^{4^t} = \zeta_{\frac{n+s}{2}} \end{cases} \right). \end{aligned} \tag{4.2}$$

*If $n$ is odd and $s$ is even, then*

$$\begin{aligned} f^{r\sigma}(x) = &\sum_{i=0}^{\frac{s}{2}} \begin{cases} \theta_i x^i + \theta_i'^{4^t} x^{s-i}, f_i \in U_R, \ i \neq s/2 \\ \zeta_i x^i + \zeta_i'^{4^t} x^{s-i}, f_i \in Z_R, \ i \neq s/2 \\ \theta_i x^{s/2}, f_i \in U_R, \ i = s/2, \theta_i x^{s/2} = \theta_i'^{4^t} x^{s/2} \\ \zeta_i x^{s/2}, f_i \in Z_R, \ i = s/2, \theta_i x^{s/2} = \theta_i'^{4^t} x^{s/2} \end{cases} \\ &+ \varrho \left( \sum_{j=0}^{\frac{n-s-3}{2}} \zeta_{s+1+j} x^{s+1+j} + \zeta_{s+1+j}'^{4^t} x^{n-1-j} \right). \end{aligned} \tag{4.3}$$

*If $n$ is even and $s$ is odd, then*

$$f^{r\sigma}(x) = \sum_{i=0}^{\frac{s-1}{2}} \begin{cases} \theta_i x^i + \theta_i'^{4^t} x^{s-i} & , f_i \in U_R \\ \zeta_i x^i + \zeta_i'^{4^t} x^{s-i} & , f_i \in Z_R \end{cases} + \varrho \left( \sum_{j=0}^{\frac{n-s-3}{2}} \zeta_{s+1+j} x^{s+1+j} + \zeta_{s+1+j}'^{4^t} x^{n-1-j} \right). \tag{4.4}$$

*If $n$ is even and $s$ is even, then*

$$\begin{aligned} f^{r\sigma}(x) = &\sum_{i=0}^{\frac{s}{2}} \begin{cases} \theta_i x^i + \theta_i'^{4^t} x^{s-i}, f_i \in U_R, \ i \neq s/2 \\ \zeta_i x^i + \zeta_i'^{4^t} x^{s-i}, f_i \in Z_R, \ i \neq s/2 \\ \theta_i x^{s/2}, f_i \in U_R, \ i = s/2, \theta_i x^{s/2} = \theta_i'^{4^t} x^{s/2} \\ \zeta_i x^{s/2}, f_i \in Z_R, \ i = s/2, \theta_i x^{s/2} = \theta_i'^{4^t} x^{s/2} \end{cases} \\ &+ \varrho \left( \sum_{j=0}^{\frac{n-s-2}{2}} \begin{cases} \zeta_{s+1+j} x^{s+1+j} + \zeta_{s+1+j}'^{4^t} x^{n-1-j}, \ j \neq (n-s-2)/2 \\ \zeta_{\frac{n+s}{2}} x^{\frac{n+s}{2}}, \ j = (n-s-2)/2, \zeta_{\frac{n+s}{2}}'^{4^t} = \zeta_{\frac{n+s}{2}} \end{cases} \right), \end{aligned} \tag{4.5}$$

*where $U_{H_q}$ is a set of units and $Z_{H_q}$ is a set of zero and zero divisors. $\theta_i = a_i + b_i u \in U_{H_q}$ and $\zeta_i = z_i + v_i u \in Z_{H_q}$ such that $i \in \{0, 1, \cdots, n-1\}$. In here, we consider that $(\theta_i)^{'4^t} = (a_i)^{4^t} + (b_i)^{4^t} u$ and $(\zeta_i)^{'4^t} = (z_i)^{4^t} + (v_i)^{4^t} u$.*

Taking the $'4^t$ power of a polynomial means taking the $'4^t$ power of a coefficient only such that $p(x) = r_0 + r_1 x + \cdots + r_s x_s$ and $p(x)^{'4^t} = r_0^{'4^t} + r_1^{'4^t} x + \cdots + r_s^{'4^t} x_s$. In short, only elements of finite fields are affected by taking the $'4^t$ power of the element. By the following theorem, we obtain reversible DNA codes over $R_{2t}$ by using $4^t$-r-glifted polynomials.

**Lemma 1.** *For each DNA base of length $4t$, an algorithm is given such that the corresponding element is $a + bu \in R_{2t}$ and its DNA reverse is $(u(a + ub))^{'4^t} \in R_{2t}$.*

For example, $w^{12} + w^{14}u \in R_4$. $\tau(\eta(w^{12} + w^{14}u)) = \tau(w^{12}, w^{14}) = (\tau(w^{12}),$ $\tau(w^{14})) = AACA\ AACG$. DNA reverse of $w^{12} + w^{14}u$ is $(u(w^{12} + w^{14}u))^{'4^2}$. $\tau(\eta((u(w^{12} + w^{14}u))^{'4^2})) = \tau(\eta((w^{14} + w^{12}u)^{'4^2})) = \tau(\eta(w^{224} + w^{192}u)) = \tau((w^{224}), (w^{192})) = (\tau(w^{224}), \tau(w^{192})) = GCAA\ ACAA$.

**Lemma 2.** *For each DNA base of length $4tn$, an algorithm is given such that the corresponding vector is $(a_0 + b_0 u, a_1 + b_1 u, \cdots, a_{n-1} + b_{n-1} u) \in R_{2t}^n$ and its DNA reverse is $((u(a_0 + b_0 u, a_1 + b_1 u, \cdots, a_{n-1} + b_{n-1} u))^{'4^t})^r = ((u(a_{n-1} + ub_{n-1}))^{'4^t}, \cdots, (u(a_1 + ub_1))^{'4^t}, (u(a_0 + ub_0))^{'4^t}) \in R_{2t}$.*

**Theorem 5.** *Let $f(x) \mid (x^n - 1)$ be a reciprocal polynomial over $F_{4^{2t}}$ and $f^{r\sigma}(x)$ over $R_{2t}$. Then, $C = \langle T^n_{f^{r\sigma}(x)} \rangle$ is a code over $R_{2t}$ of length $n$ and $\gamma^\eta(C)$ is a reversible DNA code of length $4tn$.*

*Proof.* $\gamma^\eta(\sum_i x^i f^{r\sigma}(x))$ determines DNA codewords of $\gamma^\eta(C)$. Reverses of DNA codewords are denoted as follows because $4^t$-r-glifted polynomials are used to obtain a generator.

$$\gamma^\eta(\sum_i (a + ub) x^i f^{r\sigma}(x))^r = \gamma^\eta(\sum_i x^{n-t-1-i} u((a^{4^t} + b^{4^t} u) f^{r\sigma}(x))^{4^t}),$$

where $i \in \{0, 1, \cdots, n-t-1\}$ and $a, b \in F_{4^{2t}}$. By Lemma 2, the DNA reverse of each codeword is in $C$. Then, $\gamma^\eta(C)$ is a reversible DNA code. □

**Example 6.** *Let $f(x) = x^7 + x^6 + x^4 + x^3 + x + 1 | (x^9 - 1)$ be a reciprocal polynomial over $F_2$. Let us choose $f^{r\sigma}(x) = (w^7 + uw^3) + (w^2 + uw^5)x + (w^8 + uw^8)x^2 + (w^9 + uw^4)x^3 + ((w^9)^4 + u(w^4)^4)x^4 + ((w^8)^4 + u(w^8)^4)x^5 + ((w^2)^4 + u(w^5)^4)x^6 + ((w^7)^4 + u(w^3)^4)x^7 + 0x^8$. Then $C = \langle T^n_{f^{r\sigma}(x)} \rangle$ is a $[9, 2, 6]_{R_2}$ code. Thus, $|C| = 256^2$ and hence $\eta(C)$ is a reversible DNA code with length of 36.*

We can generate a reversible DNA code by using any polynomial, which satisfies a variety of DNA codes. This approach offers the advantage of allowing us to freely choose a polynomial for generating reversible codes, as opposed to choosing reciprocal polynomials for the code design. Then, we introduce Theorem 6 for $F_{4^{2t}}$ and Theorem 7 for $R_{2t}$.

**Theorem 6.** *Let $f(x) \mid (x^n - 1)$ over $F_{4^{2t}}$ and $t_f(x) = \ell_f^-(x) + ((\ell_f^-(x))^{'4^t})^*$ over $F_{4^{2t}}$. Then, $C = \langle T^n_{t_f(x)} \rangle$ is a linear code over $F_{4^{2t}}$ and $\tau(C)$ is a reversible DNA code of length $2tn$.*

*Proof.* $\ell_f^-(x) + ((\ell_f^-(x))^{'4^t})^*$ creates a kind of $4^t$-lifted polynomial. Then, the proof is similar to proof of Theorem 4.6 in [14]. □

**Theorem 7.** *Let $f(x) \mid (x^n - 1)$ over $F_{4^{2t}}$. $t_f(x) = \ell_f^-(x) + u((\ell_f^-(x))^{'4^t})^*$ over $R_{2t}$. Then, $C = \langle T^n_{t_f(x)} \rangle$ is a code over $H_q$ and $\gamma^\eta(C)$ is a reversible DNA code of length $4tn$.*

*Proof.* $\ell_f^-(x) + u((\ell_f^-(x))'^{4^t})^*$ creates a kind of $f^{r\sigma}(x)$. Then the proof is similar to proof of Theorem 5. □

**Example 7.** *Let* $f(x) = x^4 + x^2 + x + 1|(x^7 - 1)$ *be a polynomial over* $F_2$.

*Let us choose* $\ell_f^-(x) = w^7 + w^2x + w^8x^2 + w^9x^4$ *and* $(\ell_f^-(x))'^{4^t} = w^{13} + w^8x + w^2 * x^2 + w^6x^4$. $(\ell_f^-(x))'^{4^t})* = w^6 + w^2x^2 + w^8x^3 + w^{13}x^4$. $t_f(x) = \ell_f^-(x) + u((\ell_f^-(x))'^{4^t})^* = (w^7 + w^6u) + w^2x + (w^8 + w^2 * u)x^2 + w^8ux^3 + (w^9 + w^{13}u)x^4$. *Thus,* $C = \langle T^n_{t_f(x)} \rangle$ *is a* $[7, 3, 4]_{R_2}$ *code and hence* $\eta(C)$ *is a reversible DNA code with a length of 28.*

The following corollaries provide the properties of being reversible complement codes:

**Corollary 1.** *If* $C = \langle T^n_{f(x)} \rangle$ *is a code over* $R_{2t}$ *and* $(1 + x + x^2 + \cdots + x^n - 1) \in C$*, then* $\gamma^n(C)$ *is a reversible complement DNA code of length* $4tn$.

**Corollary 2.** *If* $C = \langle T^n_{f(x)} \rangle$ *is a code over* $R_{2t}$ *and* $\gamma^n(C)$ *is a reversible DNA code of length* $4tn$*, then* $\overline{C} = \langle T^n_{f(x)}, (1 + u)(1 + x + x^2 + \cdots + x^n - 1) \rangle$ *generates reversible complement DNA code with a length of* $4tn$ *and denoted by* $\gamma^n(\overline{C})$.

## 5. Conclusions

In the present study, we introduced special polynomials and generators to construct reversible codes over rings, employing $r$-glifted polynomials. This approach offers the advantage of generating numerous reversible codes, including some that are optimal. It liberates us from the constraints imposed by the classical method of generating reversible codes using self-reciprocal divisors of $x^n - 1$, a generally challenging task. Much of the existing DNA code research revolves around alphabets of size 4 or uses four-element base ring extension, where each basic DNA nucleotide corresponds to an alphabet element. Consequently, studies often focus on reversible and complementary properties. However, the limited alphabet size and one-to-one mapping restrict the range of achievable outcomes.

In this work, we used the ring $R_{2t} = F_{4^{2t}}[u]/(u^2 - 1)$. This ring allows us to associate any $4t$-base DNA sequence with a ring element, effectively situating DNA segments within the ring. The issue of reversibility is addressed through $4^t - r$-lifted polynomials, as previously detailed. Similarly, solving the complement problem within the ring is straightforward. Consequently, we consider a broader range of DNA codes as codes over this ring, endowed with specific properties. Notably, we accomplished the generation of reversible and complement codes that need not be cyclic linear codes over $R$. However, they still possess an inherited algebraic structure from the ring. This expands the diversity of DNA codes achievable via this ring. Potential avenues for future research could involve exploring DNA codes with respect to particular bounds and metrics. Another intriguing problem might involve constructing the dual codes of those generated by $r$-glifted polynomials.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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**Conflict of interest**

The authors declare that they have no conflicts of interest.

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