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Research article

Stability analysis of new generalized mean-square stochastic fractional differential equations and their applications in technology

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Abstract: Stability theory has significant applications in technology, especially in control systems. On the other hand, the newly defined generalized mean-square stochastic fractional (GMSF) operators are particularly interesting in control theory and systems due to their various controllable parameters. Thus, the combined study of stability theory and GMSF operators becomes crucial. In this research work, we construct a new class of GMSF differential equations and provide a rigorous proof of the existence of their solutions. Furthermore, we investigate the stability of these solutions using the generalized Ulam-Hyers-Rassias stability criterion. Some examples are also provided to demonstrate the effectiveness of the proposed approach in solving fractional differential equations (FDEs) and evaluating their stability. The paper concludes by discussing potential applications of the proposed results in technology and outlining avenues for future research.

Keywords: fractional differential equations; stability analysis; mean-square stochastic calculus; nonlinear systems, control theory; signal processing
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1. Introduction

Fractional differential equations (FDEs) have gained significant importance in recent years due to their ability to model complex phenomena in physics, engineering, technology and biology that cannot be adequately captured by traditional integer-order differential equations [1]. FDEs have been used to model a wide range of phenomena, such as the behavior of viscoelastic materials, the spread of diseases, and the dynamics of financial markets. They also have numerous applications in areas such as control theory, signal processing, and image analysis [2].

One of the fundamental concepts in the study of FDEs is stability, which refers to the behavior of a solution under small perturbations. Stability analysis is crucial in determining the long-term behavior of a system and in designing control strategies for engineering applications. In the context of FDEs, the concept of stability takes on new meanings, and new techniques have been developed to analyze it [3]. Stability analysis is a critical component in FDEs because it helps to ensure the reliability and accuracy of numerical methods used to solve these equations. Stability in FDEs refers to the behavior of the numerical solution as the time step size approaches zero. In other words, a stable numerical method for FDEs should produce solutions that do not diverge or oscillate as the time step size decreases. Various stability criteria and techniques have been developed for FDEs [4, 5]. These stability criteria and techniques are essential for ensuring that the solutions obtained from a method are reliable, accurate, and consistent with the underlying physical phenomena being modeled by the FDEs.

1.1. Preliminaries

We start this section with the definition of GMSF integral operators. The GMSF operators are a class of newly proposed operators that extends the conventional idea of fractional calculus of deterministic functions to the calculus of probabilistic stochastic processes. Before continuing, it is important to note that for an interval $[s, t] = \mathcal{I} \subseteq \mathbb{R}$, $\mathcal{L}_2(\mathcal{I})$ denotes the Banach space of second-order mean-square (m.s.) Riemann integrable (or continuous) stochastic processes with the norm defined as: $\|y\|_2 = \sqrt{\mathbb{E}[y^2]}$. Also, $(\Delta, \mathcal{A}, \mathcal{P})$ is a probability space with the sample space Δ, σ-algebra $\mathcal{A}$ and probability measure $\mathcal{P}$. The following definition of GMSF integrals was given by [6].

Definition 1. *Suppose $y \in \mathcal{L}_2(\mathcal{I})$. The left- and right-sided GMSF integrals ${}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{s^+}$ and ${}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{t^-}$ of order $\theta > 0$ with $\varsigma \in (0, 1]$, $\nu \in \mathbb{R}$ such that $\nu + \varsigma \neq 0$ are defined by:*

$$
{}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{s^+}y(r, .) = \frac{1}{\Gamma(\theta)} \int_{s}^{r} \left(\frac{r^{\nu+\varsigma} - w^{\nu+\varsigma}}{\nu + \varsigma} \right)^{\theta-1} y(w, .)w^{\nu+\varsigma-1}dw, \quad r > s, \tag{1.1}
$$

and

$$
{}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{t^-}y(r, .) = \frac{1}{\Gamma(\theta)} \int_{r}^{t} \left(\frac{w^{\nu+\varsigma} - r^{\nu+\varsigma}}{\nu + \varsigma} \right)^{\theta-1} y(w, .)w^{\nu+\varsigma-1}dw, \quad t > r, \tag{1.2}
$$

respectively, and ${}^{\nu}_{\varsigma}\mathcal{K}^{0}_{s^+}y(r, .) = {}^{\nu}_{\varsigma}\mathcal{K}^{0}_{t^-}y(r, .) = y(r, .)$.

Some properties of the GMSF integral operators are given below:

Theorem 1. ***Index and Semigroup Property*** *[6] For any $y \in \mathcal{L}_2(\mathcal{I})$, we have:*

$$
\lim_{\theta\to 0} {}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{s^+}y(r, .) = {}^{\nu}_{\varsigma}\mathcal{K}^{0}_{s^+}y(r, .) = y(r, .), \quad \lim_{\theta\to 0} {}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{t^-}y(r, .) = {}^{\nu}_{\varsigma}\mathcal{K}^{0}_{t^-}y(r, .) = y(r, .). \tag{1.3}
$$

In addition, for θ_1, $\theta_2 > 0$ we have:

$$
{}^{\nu}_{\varsigma}\mathcal{K}^{\theta_1}_{s^+}{}^{\nu}_{\varsigma}\mathcal{K}^{\theta_2}_{s^+}y(r, .) = {}^{\nu}_{\varsigma}\mathcal{K}^{\theta_1+\theta_2}_{s^+}y(r, .), \quad {}^{\nu}_{\varsigma}\mathcal{K}^{\theta_1}_{t^-}{}^{\nu}_{\varsigma}\mathcal{K}^{\theta_2}_{t^-}y(r, .) = {}^{\nu}_{\varsigma}\mathcal{K}^{\theta_1+\theta_2}_{t^-}y(r, .). \tag{1.4}
$$

Theorem 2. ***Linearity and Boundedness*** *[6] The GMSF integral operators ${}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{s^+}$ and ${}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{t^-}$ of order $\theta > 0$ are linear on $\mathcal{L}_2(\mathcal{I})$. That is,*

$$ {}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{s^+},\ {}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{t^-} : \mathcal{L}_2(\mathcal{I}) \to \mathcal{L}_2(\mathcal{I}). $$

Then, for any $y_1,\ y_2 \in \mathcal{L}_2(\mathcal{I})$ and $\vartheta_1,\ \vartheta_2 \in \mathbb{R}$,

$$ {}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{s^+}(\vartheta_1 y_1 + \vartheta_2 y_2) = \vartheta_1 {}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{s^+} y_1 + \vartheta_2 {}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{s^+} y_2, \tag{1.5} $$

$$ {}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{t^-}(\vartheta_1 y_1 + \vartheta_2 y_2) = \vartheta_1 {}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{t^-} y_1 + \vartheta_2 {}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{t^-} y_2. \tag{1.6} $$

In addition, the operators ${}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{s^+}$ and ${}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{t^-}$ are bounded on $\mathcal{L}_2(\mathcal{I})$. That is,

$$ \left\| {}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{s^+} y \right\| \le M \|y\|,\ \left\| {}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{t^-} y \right\| \le M \|y\|, \tag{1.7} $$

where $\|y\| = \max_{r \in [s,t]} \|y(r,.)\|_2$, $M = \frac{(\nu+\varsigma)^{-\theta}}{\theta+1}(t^{\nu+\varsigma} - s^{\nu+\varsigma})^{\theta}$.

Associated with the GMSF integral operators, the left- and right-sided GMSF derivative operators are defined as follows [6]:

Definition 2. *Let $y \in \mathcal{L}_2(\mathcal{I})$. The left- and right-sided GMSF derivative operators ${}^{\nu}_{\varsigma}\mathcal{T}^{\theta}_{s^+}$ and ${}^{\nu}_{\varsigma}\mathcal{T}^{\theta}_{t^-}$ of order $0 < \theta < 1$ are defined by:*

$$ {}^{\nu}_{\varsigma}\mathcal{T}^{\theta}_{s^+} y(r,.) = \frac{r^{1-\varsigma-\nu}}{\Gamma(1-\theta)} \frac{d}{dr} \int_s^r \left(\frac{r^{\nu+\varsigma} - w^{\nu+\varsigma}}{\nu+\varsigma} \right)^{-\theta} y(w,.) w^{\nu+\varsigma-1} dw, \quad r > s, \tag{1.8} $$

and

$$ {}^{\nu}_{\varsigma}\mathcal{T}^{\theta}_{t^-} y(r,.) = \frac{r^{1-\varsigma-\nu}}{\Gamma(1-\theta)} \frac{d}{dr} \int_r^t \left(\frac{r^{\nu+\varsigma} - w^{\nu+\varsigma}}{\nu+\varsigma} \right)^{-\theta} y(w,.) w^{\nu+\varsigma-1} dw, \quad t > r, \tag{1.9} $$

respectively, where ${}^{\nu}_{\varsigma}\mathcal{T}^{0}_{s^+} y(r,.) = {}^{\nu}_{\varsigma}\mathcal{T}^{0}_{t^-} y(r,.) = y(r,.)$. Also, $\frac{d}{dr}$ denotes mean-square stochastic derivative.

Some properties of the GMSF derivative operators are given as below [6]:

Theorem 3. ***Inverse Property*** *For any $y \in \mathcal{L}_2(\mathcal{I})$ in the domain of ${}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{s^+}$, ${}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{t^-}$, ${}^{\nu}_{\varsigma}\mathcal{T}^{\theta}_{s^+}$ and ${}^{\nu}_{\varsigma}\mathcal{T}^{\theta}_{t^-}$ we have*

$$ {}^{\nu}_{\varsigma}\mathcal{T}^{\theta}_{s^+} {}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{s^+} y(r,.) = y(r,.);\ \ {}^{\nu}_{\varsigma}\mathcal{T}^{\theta}_{t^-} {}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{t^-} y(r,.) = y(r,.). \tag{1.10} $$

$$ {}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{s^+} {}^{\nu}_{\varsigma}\mathcal{T}^{\theta}_{s^+} y(r,.) = y(r,.);\ \ {}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{t^-} {}^{\nu}_{\varsigma}\mathcal{T}^{\theta}_{t^-} y(r,.) = y(r,.). \tag{1.11} $$

Theorem 4. ***Linearity and Semigroup Property*** *Let $y_1,\ y_2 \in \mathcal{L}_2(\mathcal{I})$ and $\mu_1,\ \mu_2 \in \mathbb{R}$. Then,*

$$ {}^{\nu}_{\varsigma}\mathcal{T}^{\theta}_{s^+}(\mu_1 y_1 + \mu_2 y_2) = \mu_1 {}^{\nu}_{\varsigma}\mathcal{T}^{\theta}_{s^+} y_1 + \mu_2 {}^{\nu}_{\varsigma}\mathcal{T}^{\theta}_{s^+} y_2, \tag{1.12} $$

$$ {}^{\nu}_{\varsigma}\mathcal{T}^{\theta}_{t^-}(\mu_1 y_1 + \mu_2 y_2) = \mu_1 {}^{\nu}_{\varsigma}\mathcal{T}^{\theta}_{t^-} y_1 + \mu_2 {}^{\nu}_{\varsigma}\mathcal{T}^{\theta}_{t^-} y_2. \tag{1.13} $$

Also, for any $y \in \mathcal{L}_2(\mathcal{I})$ and $0 < \theta_1 < 1,\ 0 < \theta_2 < 1$ we have:

$$ {}^{\nu}_{\varsigma}\mathcal{T}^{\theta_1}_{s^+} {}^{\nu}_{\varsigma}\mathcal{T}^{\theta_2}_{s^+} y(r,.) = {}^{\nu}_{\varsigma}\mathcal{T}^{\theta_1+\theta_2}_{s^+} y(r,.),\ \ {}^{\nu}_{\varsigma}\mathcal{T}^{\theta_1}_{t^-} {}^{\nu}_{\varsigma}\mathcal{T}^{\theta_2}_{t^-} y(r,.) = {}^{\nu}_{\varsigma}\mathcal{T}^{\theta_1+\theta_2}_{t^-} y(r,.). \tag{1.14} $$

The following two theorems and the definition relate directly to obtaining our main results [7].

Theorem 5. *Consider a separable Banach space $\mathcal{S}$ and its nonempty subset $\mathcal{B}$ which is also closed and bounded. Also, let $M : \Delta \times \mathcal{B} \to \mathcal{B}$ be a continuous and compact random (stochastic) operator. Then, the equation $M(w)y = y$ has a stochastic solution.*

Theorem 6. ***Arzelà-Ascoli*** *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be metric spaces, and let $\mathcal{K}$ be a compact subset of $\mathcal{S}_1$. A subset $\mathcal{F}$ of the space $C(\mathcal{K}, \mathcal{S}_2)$ (of continuous functions from $\mathcal{K}$ to $\mathcal{S}_2$) is compact iff it is equicontinuous and uniformly bounded.*

Definition 3. ***Random Carathéodory Function*** *A function $g : \mathcal{I} \times F \times \Delta \to F$ is said to be a random Carathéodory function if the map $(\eta, w) \to g(\eta, y, w)$ is jointly measurable for all $y \in F$, and also for almost all $\eta \in \mathcal{I}$ and $w \in \Delta$, $y \to g(\eta, y, w)$ is continuous.*

1.2. Previous contributions and related work

In recent years, there has been a surge in interest in using FDEs for the stability analysis. For example, in [8], various kinds of stability of differential equations with a distinct kind of general conformable derivative have been examined in a novel approach by the authors. They used some kind of Banach fixed-point theory in their analysis. They generalized a number of intriguing past findings in this manner. In [9], for a nonlinear FDE with three point integral boundary conditions, its Hyers-Ulam stability and Ulam-Hyers-Rassias stability were examined. Standard methods for solving the Hyers-Ulam Mittag-Leffler problem using nonlinear fractional integrals and derivatives have been explored in [10]. For the purpose of solving the linear FDE using the fractional Fourier transform, the authors of this study have provided a brief overview of the Hyers-Ulam Mittag Leffler problem approach and mentioned the limitations of its applicability. Additionally, they developed a theory that describes the Hyers-Ulam-type Mittag-Leffler problem's structure for linear two terms equations. Under appropriate circumstances, Ulam, Hyers, and Rassias' stability results for the fractionally nonlinear Fredholm and Volterra integral equations with delay have been investigated in [11]. Additionally, these stability results have been applied to fractional integral equations with an unbounded interval for the integration domain. In [12] the authors have examined stability of the fractional stochastic impulsive differential equations. According to the results obtained in [13], the stability problem has been studied using the well-known fixed point theorem, and the Ulam-Hyers stability was also demonstrated for the class of FDEs. Exponential stability of a class of neutral inertial neural networks with multi-proportional delays and leakage delays has been studied in [14]. The research work described in [15] involved the construction of a new ψ-Hilfer differential equation with integral-type subsidiary conditions. Additionally, stability analysis as defined by Ulam-Hyers Mittag-Leffler has been explored. Some Ulam-Hyers stability results for matrix-valued FDEs have been found in [16], and the authors have also defined some necessary criteria for the stability of these equations. In [17], almost sure exponential stability of uncertain stochastic Hopfield neural networks based on subadditive measures is discussed. The authors in [18–21] have also examined various solutions for their stability. The analyses in these were conducted using traditional nonlinear functional analysis methods. Some other developments on the present topic can be found in [22–27].

In summary, the stability analysis of FDEs has been a highly active area of research, with significant contributions from a diverse range of fields including physics, engineering, biology, and technology.

The previous work has demonstrated the effectiveness of fractional calculus in modeling and analyzing complex systems, and has laid a strong foundation for further developments in the field.

2. Main results

We initiate the present work with the following class of GMSF differential equations:

$$ {}^{\nu}_{\varsigma}\mathcal{T}^{\theta}_{s^+}y(\eta,w) = f\left(\eta, y(\eta,w), w\right), \tag{2.1} $$

with the terminal condition

$$ y(R,w) = y_R(w), \tag{2.2} $$

where $\eta \in \mathcal{I} := [0,R]$, $w \in \Delta$. As discussed above, $(\Delta, \mathcal{A}, \mathcal{P})$ is a probability space with the sample space Δ, σ-algebra $\mathcal{A}$ and probability measure $\mathcal{P}$. In addition, $\varsigma \in (0,1]$, $\nu \in \mathbb{R}$, such that $\nu + \varsigma \neq 0$. Also, $y_R : \Delta \to E$ is a measurable stochastic process, and $f : \mathcal{I} \times E \times \Delta \to E$, where E is a Banach space. Also, ${}^{\nu}_{\varsigma}\mathcal{T}^{\theta}_{s^+}$ denotes the GMSF derivative operator of order θ. Some other notations that we will use in the development of our main results, are explained as follows.
Let $y(\eta,w) = \{y(\eta,.),\ \eta \in \mathcal{I},\ w \in \Delta\}$ be a second-order stochastic process, that is, $E(y^2(\eta,.)) < \infty$, for all $\eta \in \mathcal{I}$. The notation $C(\mathcal{I} \times \Delta, E)$ or $C(\mathcal{I})$ represents the Banach space of second-order stochastic processes $y : \mathcal{I} \times \Delta \to E$ which are continuous with the norm

$$ \|y\|_{\infty} = \sup_{\eta \in \mathcal{I}} \|y(\eta,.)\|. \tag{2.3} $$

In addition, $C_{\theta,\nu+\varsigma}(\mathcal{I})$ is the weighted space having continuous stochastic processes and defined by

$$ C_{\theta,\nu+\varsigma}(\mathcal{I}) = \left\{ y : \mathcal{I} \times \Delta \to E; \eta^{(\nu+\varsigma)(1-\theta)} y(\eta,.) \in C(\mathcal{I}) \right\}, \tag{2.4} $$

with the norm

$$ \|y\|_{C} = \sup_{\eta \in \mathcal{I}} \|\eta^{(\nu+\varsigma)(1-\theta)} y(\eta,.)\|. \tag{2.5} $$

The following two definitions will be used in our mains results.

Definition 4. *[8] For the jointly measurable stochastic process* $\Psi : \mathcal{I}\times\Delta \to [0,\infty)$*, and the inequality:*

$$ \|{}^{\nu}_{\varsigma}T^{\theta}_{0^+}y(\eta,w) - f\left(\eta, y(\eta,w), w\right)\| \leq \Psi(\eta,w),\ \ \eta \in \mathcal{I},\ w \in \Delta, \tag{2.6} $$

the problem (2.1)-(2.2) is said to be generalized Ulam-Hyers-Rassias stable *w.r.t.* Ψ *if* $\exists\ d_{f,\Psi} > 0$ *such that* $\forall\ y(\eta,w) \in C_{\theta,\nu+\varsigma}(\mathcal{I})$ *satisfying the inequality (2.6),* $\exists\ v(\eta,w) \in C_{\theta,\nu+\varsigma}(\mathcal{I})$ *that satisfies the problem (2.1)-(2.2) with*

$$ \|\eta^{(\nu+\varsigma)(1-\theta)} y(\eta,w) - \eta^{(\nu+\varsigma)(1-\theta)} v(\eta,w)\| \leq d_{f,\Psi}\Psi(\eta,w),\ \ \eta \in \mathcal{I},\ w \in \Delta. \tag{2.7} $$

As our research is related to the investigation of random solutions, it is crucial to begin by providing a definition of what we mean by a random solution.

Definition 5. ***Random Solution*** *By a random (stochastic) solution of the problem (2.1)-(2.2), we mean a second-order stochastic process* $y(\eta,\cdot) \in C_{\theta,\nu+\varsigma}(\mathcal{I})$ *that satisfies the problem (2.1)-(2.2) [8].*

Next, we need to establish the following lemma.

Lemma 1. *The problem*

$$\begin{cases} {}^{\nu}_{\varsigma}\mathcal{T}^{\theta}_{0^+}y(r,.) = h(r,.), \quad r \in [0,R) \\ y(R,.) = y_R \end{cases} \tag{2.8}$$

has the solution

$$y(r,.) = \frac{1}{\Gamma(\theta)} \int_0^r \left(\frac{r^{\nu+\varsigma} - w^{\nu+\varsigma}}{\nu+\varsigma} \right)^{\theta-1} h(w,.) w^{\nu+\varsigma-1} dw, \tag{2.9}$$

where

$$y_R = \frac{1}{\Gamma(\theta)} \int_0^R \left(\frac{R^{\nu+\varsigma} - w^{\nu+\varsigma}}{\nu+\varsigma} \right)^{\theta-1} h(w,.) w^{\nu+\varsigma-1} dw. \tag{2.10}$$

Proof. Let $r \in [0,R)$. Then, solving the equation:

$${}^{\nu}_{\varsigma}\mathcal{T}^{\theta}_{0^+}y(r,.) = h(r,.)$$

by applying ${}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{0^+}$ from the left side, using the relation (1.11), we obtain:

$$y(r,.) = {}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{0^+}h(r,.) = \frac{1}{\Gamma(\theta)} \int_0^r \left(\frac{r^{\nu+\varsigma} - w^{\nu+\varsigma}}{\nu+\varsigma} \right)^{\theta-1} h(w,.) w^{\nu+\varsigma-1} dw,$$

where we get that

$$y_R = {}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{0^+}h(R,.).$$

Thus, we get the required solution.
The following corollary is the direct consequence of the above Lemma. □

Corollary 1. *The second-order stochastic process y is the random solution of (2.1)-(2.2) if it satisfies*

$$y(\eta,w) = \frac{(\nu+\varsigma)^{1-\theta}}{\Gamma(\theta)} \int_0^{\eta} \frac{s^{\nu+\alpha-1}}{(\eta^{\nu+\varsigma} - s^{\nu+\varsigma})^{1-\theta}} f(s,y,w) ds, \tag{2.11}$$

and

$$y_R = \frac{1}{\Gamma(\theta)} \int_0^R \left(\frac{R^{\nu+\varsigma} - s^{\nu+\varsigma}}{\nu+\varsigma} \right)^{\theta-1} f(s,y,w) s^{\nu+\varsigma-1} ds. \tag{2.12}$$

Proof. The proof is simple by just applying the operator ${}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{t^-}$ to (2.1)-(2.2) from the left side utilizing the relation (1.11). □

The following Theorem 7 provides a sufficient condition for the existence of a random solution to the GMSFDE (2.1) with terminal condition (2.2).

Theorem 7. *Suppose f is a random Carathéodory function such that there exists essentially bounded and measurable stochastic processes m_1 and m_2 such that*

$$\|f(r,y,w)\| \leq m_1(r,w) + m_2(r,w)r^{(\nu+\varsigma)(1-\theta)}\|y\| \tag{2.13}$$

for all $y \in E$ and $r \in \mathcal{I}$, and $\frac{(\nu+\varsigma)^{-\theta}R^{\nu+\varsigma}}{\Gamma(1+\theta)}m_2^*(w) < 1$, where $m_i^*(w) = \sup_{r\in\mathcal{I}} m_i(r,w)$, $i = 1,2$. Then, for the problem (2.1)-(2.2), a random (stochastic) solution exists.

Proof. Let us consider the operator M, defined by

$$My(r,w) = \frac{1}{\Gamma(\theta)}\int_0^r \left(\frac{r^{\nu+\varsigma} - s^{\nu+\varsigma}}{\nu+\varsigma}\right)^{\theta-1} f(s,y(s,w),w)s^{\nu+\varsigma-1}ds, \tag{2.14}$$

and set

$$\mathcal{E}(w) > \frac{\frac{(\nu+\varsigma)^{-\theta}R^{(\nu+\varsigma)}m_1^*(w)}{\Gamma(1+\theta)}}{1-\frac{(\nu+\varsigma)^{-\theta}R^{(\nu+\varsigma)}m_2^*(w)}{\Gamma(1+\theta)}}; \quad w \in \Delta. \tag{2.15}$$

Define the ball

$$B_{\mathcal{E}} = B(0,\mathcal{E}(w)) := \{y \in C_{\theta,\nu+\varsigma}(\mathcal{I}) : \|y\|_C \leq \mathcal{E}(w)\}. \tag{2.16}$$

From (2.14), we have

$$\begin{aligned}
&\|r^{(\nu+\varsigma)(1-\theta)}My(r,w)\| \\
= \;& \|\frac{r^{(\nu+\varsigma)(1-\theta)}}{\Gamma(\theta)}\int_0^r \left(\frac{r^{\nu+\varsigma} - s^{\nu+\varsigma}}{\nu+\varsigma}\right)^{\theta-1} f(s,y(s,w),w)s^{\nu+\varsigma-1}ds\| \\
\leq \;& \frac{R^{(\nu+\varsigma)(1-\theta)}}{\Gamma(\theta)}\int_0^r \left(\frac{r^{\nu+\varsigma} - s^{\nu+\varsigma}}{\nu+\varsigma}\right)^{\theta-1} \|m_1(s,w)\|s^{\nu+\varsigma-1}ds \\
&+\frac{R^{(\nu+\varsigma)(1-\theta)}}{\Gamma(\theta)}\int_0^r \left(\frac{r^{\nu+\varsigma} - s^{\nu+\varsigma}}{\nu+\varsigma}\right)^{\theta-1} \|s^{(\nu+\varsigma)(1-\theta)}m_2(s,w)y(s,w)\|s^{\nu+\varsigma-1}ds \\
\leq \;& \frac{(\nu+\varsigma)^{1-\theta}R^{(\nu+\varsigma)(1-\theta)}R^{(\nu+\varsigma)(\theta)}m_1^*(w)}{\theta(\nu+\varsigma)\Gamma(\theta)} \\
&+\frac{m_2^*(w)R^{(\nu+\varsigma)(1-\theta)}}{\Gamma(\theta)}\int_0^r \left(\frac{r^{\nu+\varsigma} - s^{\nu+\varsigma}}{\nu+\varsigma}\right)^{\theta-1} \|s^{(\nu+\varsigma)(1-\theta)}y(s,w)\|s^{\nu+\varsigma-1}ds \\
\leq \;& \frac{(\nu+\varsigma)^{-\theta}R^{(\nu+\varsigma)}m_1^*(w)}{\Gamma(1+\theta)} + \frac{(\nu+\varsigma)^{-\theta}R^{(\nu+\varsigma)}m_2^*(w)}{\Gamma(1+\theta)}\|y\|_C \quad \text{(using (2.5))} \\
\leq \;& \frac{(\nu+\varsigma)^{-\theta}R^{(\nu+\varsigma)}m_1^*(w)}{\Gamma(1+\theta)} + \frac{(\nu+\varsigma)^{-\theta}R^{(\nu+\varsigma)}m_2^*(w)}{\Gamma(1+\theta)}\mathcal{E}(w) \\
\leq \;& \mathcal{E}(w),
\end{aligned} \tag{2.17}$$

that is,

$$\|(M)(w)y\|_C \leq \mathcal{E}(w).$$

Hence, $(M)(w)B_{\mathcal{E}} \subset B_{\mathcal{E}}$. We will prove that $M : \Delta \times B_{\mathcal{E}} \to B_{\mathcal{E}}$ satisfies the assumptions of Theorem 5.
First, $M(w)$ is a random operator.
Since f is a randomly Carathéodory function, $w \to f(r, y, w)$ is measurable. Also, the map

$$w \to \frac{1}{\Gamma(\theta)} \int_0^r \left(\frac{r^{\nu+\varsigma} - s^{\nu+\varsigma}}{\nu+\varsigma} \right)^{\theta-1} f(s, y(s, w), w) s^{\nu+\varsigma-1} ds$$

is measurable due to the integral being equal to the limit of a finite sum of measurable stochastic processes.
Second, $M(w)$ is continuous.
For this, let us take the sequence y_n with $y_n \to y$ in $C_{\theta,\nu+\varsigma}$.
Set

$$v_n(r, w) = r^{(\nu+\varsigma)(1-\theta)} M y_n(r, w), \tag{2.18}$$

and

$$v(r, w) = r^{(\nu+\varsigma)(1-\theta)} M y(r, w). \tag{2.19}$$

Then,

$$\begin{aligned} &\|v_n(r, w) - v(r, w)\| \\ \leq\ & \frac{R^{(\nu+\varsigma)(1-\theta)}}{\Gamma(\theta)} \int_0^r \left(\frac{r^{\nu+\varsigma} - s^{\nu+\varsigma}}{\nu+\varsigma} \right)^{\theta-1} \|f(s, y_n(s, w), w) - f(s, y(s, w), w)\| s^{\nu+\varsigma-1} ds. \end{aligned} \tag{2.20}$$

Since we have that f is a random Carathéodory function,

$$\|v_n(r, w) - v(r, w)\| \to 0,\ n \to \infty. \tag{2.21}$$

Thus, $M(w)$ is continuous.
Next, we prove that $M(w)B_{\mathcal{E}}$ is equicontinuous. For $1 \leq r_1 \leq r_2 \leq R$, and $y \in B_{\mathcal{E}}$ we have:

$$\begin{aligned} &\|r_2^{(\nu+\varsigma)(1-\theta)} M y(r_2, w) - r_1^{(\nu+\varsigma)(1-\theta)} M y(r_1, w)\| \\ \leq\ & \Bigg\| \frac{r_2^{(\nu+\varsigma)(1-\theta)}}{\Gamma(\theta)} \int_0^{r_2} \left(\frac{r_2^{\nu+\varsigma} - s^{\nu+\varsigma}}{\nu+\varsigma} \right)^{\theta-1} f(s, y(s, w), w) s^{\nu+\varsigma-1} ds \\ & - \frac{r_1^{(\nu+\varsigma)(1-\theta)}}{\Gamma(\theta)} \int_0^{r_1} \left(\frac{r_1^{\nu+\varsigma} - s^{\nu+\varsigma}}{\nu+\varsigma} \right)^{\theta-1} f(s, y(s, w), w) s^{\nu+\varsigma-1} ds \Bigg\| \\ \leq\ & \Bigg\| \frac{r_2^{(\nu+\varsigma)(1-\theta)}}{\Gamma(\theta)} \int_{r_1}^{r_2} \left(\frac{r_2^{\nu+\varsigma} - s^{\nu+\varsigma}}{\nu+\varsigma} \right)^{\theta-1} f(s, y(s, w), w) s^{\nu+\varsigma-1} ds \end{aligned}$$

$$
\begin{aligned}
& -\frac{r_1^{(\nu+\varsigma)(1-\theta)}}{\Gamma(\theta)} \int_0^{r_1} \left(\frac{r_1^{\nu+\varsigma} - s^{\nu+\varsigma}}{\nu+\varsigma} \right)^{\theta-1} f(s, y(s,w), w) s^{\nu+\varsigma-1} ds \\
& + \frac{r_2^{(\nu+\varsigma)(1-\theta)}}{\Gamma(\theta)} \int_0^{r_1} \left(\frac{r_2^{\nu+\varsigma} - s^{\nu+\varsigma}}{\nu+\varsigma} \right)^{\theta-1} f(s, y(s,w), w) s^{\nu+\varsigma-1} ds \Big\| \\
\leq \; & \frac{R^{(\nu+\varsigma)(1-\theta)}}{\Gamma(\theta)} \int_{r_1}^{r_2} \left(\frac{r_2^{\nu+\varsigma} - s^{\nu+\varsigma}}{\nu+\varsigma} \right)^{\theta-1} \Big\| f(s, y(s,w), w) \Big\| s^{\nu+\varsigma-1} ds \\
& - \frac{R^{(\nu+\varsigma)(1-\theta)}}{\Gamma(\theta)} \int_0^{r_1} \left(\frac{r_1^{\nu+\varsigma} - s^{\nu+\varsigma}}{\nu+\varsigma} \right)^{\theta-1} \Big\| f(s, y(s,w), w) \Big\| s^{\nu+\varsigma-1} ds \\
& + \frac{R^{(\nu+\varsigma)(1-\theta)}}{\Gamma(\theta)} \int_0^{r_1} \left(\frac{r_2^{\nu+\varsigma} - s^{\nu+\varsigma}}{\nu+\varsigma} \right)^{\theta-1} \Big\| f(s, y(s,w), w) \Big\| s^{\nu+\varsigma-1} ds \\
\leq \; & \frac{r_2^{\theta(\nu+\varsigma)} + r_1^{\theta(\nu+\varsigma)} + 2(r_2^{\nu+\varsigma} - r_1^{\nu+\varsigma})^\theta}{(\nu+\varsigma)^\theta \Gamma(1+\theta)} R^{(1-\theta)(\nu+\varsigma)} (m_1^*(w) + m_2^*(w)\mathcal{E}(w)) \\
\to \; & 0, \text{as } r_2 \to r_1. \qquad (2.22)
\end{aligned}
$$

The Arzelá-Ascoli theorem implies that M is compact and continuous. Hence, from Theorem 5, we establish that a random solution to the problem (2.1)-(2.2) exists. □

The following result establishes the criteria for generalized Ulam-Hyers-Rassias stability of (2.1)-(2.2).

Theorem 8. *Suppose f is a random Carathéodory function such that for any $w \in \Delta$ and $\Psi(r,.) \in C(\mathcal{I})$ there exists an essentially bounded and measurable stochastic process $z : \mathcal{I} \times \Delta \to C(\mathcal{I}, [0,\infty))$, such that*

$$
\|f(r, x(r,w), w) - f(r, y(r,w), w)\| \leq \frac{z(r,w)\Psi(r,w) r^{(\nu+\varsigma)(1-\theta)} \|x-y\|}{(1 + \|x-y\|)}.
$$

Also, let $\lambda_\Psi > 0$ be such that ${}_{\varsigma}^{\nu}\mathcal{K}_{0^+}^{\theta} \Psi(r,w) \leq \lambda_\Psi \Psi(r,w)$, and

$$
\frac{(\nu+\varsigma)^{-\theta} R^{\nu+\varsigma}}{\Gamma(1+\theta)} \Psi^*(w) z^*(w) < 1,
$$

where

$$
\Psi^*(w) = \sup_{r \in \mathcal{I}} \Psi(r,w), \;\; z^*(w) = \sup_{r \in \mathcal{I}} z(r,w). \qquad (2.23)
$$

Then, the problem (2.1)-(2.2) has at least one random solution that is generalized Ulam-Hyers-Rassias stable.

Proof. In light of Theorem 7, first we show that the problem (2.1)-(2.2) has at least one stochastic solution y. We show that all conditions in the hypothesis of Theorem 7 hold true.

f is a random Carathéodory function, and for any $w \in \Delta$, $\Psi(r,.) \in \mathcal{L}_2(I)$, and there exists an essentially bounded and measurable stochastic process $z(r,w)$ such that

$$\|f(r,x(r,w),w) - f(r,y(r,w),w)\| \leq \frac{z(r,w)\Psi(r,w)r^{(\nu+\varsigma)(1-\theta)}\|x-y\|}{(1+\|x-y\|)}.$$

This implies the relation (2.13) with:

$$m_1(w,r) = f(r,0,w), \;\; m_2(w) = z(r,w)\Psi(r,w). \tag{2.24}$$

Thus the problem (2.1)-(2.2) has (using Theorem 7) at least one stochastic solution y. Then

$$y(r,w) = \frac{1}{\Gamma(\theta)}\int_0^r \left(\frac{r^{\nu+\varsigma} - s^{\nu+\varsigma}}{\nu+\varsigma}\right)^{\theta-1} f(s,y(s,w),w)s^{\nu+\varsigma-1}ds. \tag{2.25}$$

To check whether the problem (2.1)-(2.2) is generalized Ulam-Hyers-Rassias stable, we proceed in the light of Definition 4. Suppose y is a stochastic solution of (2.6). We obtain

$$\begin{aligned} &\|r^{(\nu+\varsigma)(1-\theta)}x(r,w) - \frac{r^{(\nu+\varsigma)(1-\theta)}}{\Gamma(\theta)}\int_0^r \left(\frac{r^{\nu+\varsigma} - s^{\nu+\varsigma}}{\nu+\varsigma}\right)^{\theta-1} f(s,x(s,w),w)s^{\nu+\varsigma-1}ds\| \\ \leq\;& R^{(\nu+\varsigma)(1-\theta)}\,{}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{0^+}\Psi(r,w). \end{aligned} \tag{2.26}$$

Considering $\|r^{(\nu+\varsigma)(1-\theta)}x(r,w) - r^{(\nu+\varsigma)(1-\theta)}y(r,w)\|$, adding and subtracting the mid terms and putting the value of $y(r,w)$ from (2.25),

$$\begin{aligned} &\|r^{(\nu+\varsigma)(1-\theta)}x(r,w) - r^{(\nu+\varsigma)(1-\theta)}y(r,w)\| \\ \leq\;& \|r^{(\nu+\varsigma)(1-\theta)}x(r,w) - \frac{r^{(\nu+\varsigma)(1-\theta)}}{\Gamma(\theta)}\int_0^r \left(\frac{r^{\nu+\varsigma} - s^{\nu+\varsigma}}{\nu+\varsigma}\right)^{\theta-1} f(s,x(s,w),w)s^{\nu+\varsigma-1}ds\| \\ &+\|\frac{r^{(\nu+\varsigma)(1-\theta)}}{\Gamma(\theta)}\int_0^r \left(\frac{r^{\nu+\varsigma} - s^{\nu+\varsigma}}{\nu+\varsigma}\right)^{\theta-1} f(s,x(s,w),w)s^{\nu+\varsigma-1}ds \\ &-\frac{r^{(\nu+\varsigma)(1-\theta)}}{\Gamma(\theta)}\int_0^r \left(\frac{r^{\nu+\varsigma} - s^{\nu+\varsigma}}{\nu+\varsigma}\right)^{\theta-1} f(s,y(s,w),w)s^{\nu+\varsigma-1}ds\| \\ \leq\;& R^{(\nu+\varsigma)(1-\theta)}\,{}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{0^+}\Psi(r,w) \\ &+\frac{R^{(\nu+\varsigma)(1-\theta)}}{\Gamma(\theta)}\int_0^r \left(\frac{r^{\nu+\varsigma} - s^{\nu+\varsigma}}{\nu+\varsigma}\right)^{\theta-1} \|f(s,x(s,w),w) - f(s,y(s,w),w)\|s^{\nu+\varsigma-1}ds \\ \leq\;& R^{(\nu+\varsigma)(1-\theta)}\,{}^{\nu}_{\varsigma}\mathcal{K}^{\theta}_{0^+}\Psi(r,w) \\ &+\frac{R^{(\nu+\varsigma)(1-\theta)}}{\Gamma(\theta)}\int_0^r \left(\frac{r^{(\nu+\varsigma)(1-\theta)} - s^{\nu+\varsigma}}{\nu+\varsigma}\right)^{\theta-1} z^*(w)\Psi(s,w)s^{(\nu+\varsigma)(1-\theta)}s^{\nu+\varsigma-1}\frac{\|x-y\|}{1+\|x-y\|}ds \end{aligned}$$

$$\leq \quad R^{(\nu+\varsigma)(1-\theta)}\lambda_\Psi \Psi(r,w) + R^{2(\nu+\varsigma)(1-\theta)}\lambda_\Psi z^*(w). \tag{2.27}$$

Thus, we get

$$\begin{aligned} \|r^{(\nu+\varsigma)(1-\theta)}x(r,w) - r^{(\nu+\varsigma)(1-\theta)}y(r,w)\| &\leq (1 + R^{(\nu+\varsigma)(1-\theta)}z^*(w))(R^{(\nu+\varsigma)(1-\theta)}\lambda_\Psi \Psi(r,w)) \\ &:= d_{f,\Psi}\Psi(r,w). \end{aligned} \tag{2.28}$$

Hence, by Definition 4, the problem (2.1)-(2.2) is generalized Ulam-Hyers-Rassias stable.
This completes the proof. □

Examples

To demonstrate the effectiveness of our approach in practical scenarios, we provide two numerical examples corresponding to our main results Theorems 7 and 8. By analyzing these examples, we can gain insight into the behavior of the solutions of the considered generalized m.s. FDEs and their stability properties.

Example 1. *Consider the following GMSF differential equation with terminal condition:*

$$ {}^{\frac{3}{2}}_{0}\mathcal{T}^{\frac{1}{2}}_{0^+}y(\eta,w) = \frac{\eta}{2}y(\eta,w) + \cos(w\ln\eta), \quad y(1,w) = 1, \tag{2.29}$$

where $\eta \in [0,1]$, $\|y\| = 1$, and $w \in [0,2\pi]$. We want to show that the problem has a random solution. To apply Theorem 7, it is straightforward that

$$f(\eta,y,w) = \frac{\eta}{2}y(\eta,w) + \cos(w\ln\eta)$$

satisfies the Carathéodory condition. Also, to verify (2.13), we have

$$\begin{aligned} \|f(\eta,y,w)\| &= \left\|\frac{\eta}{2}y + \cos(w\ln\eta)\right\| \\ &\leq \frac{1}{2} + 1 = \frac{3}{2} \\ &\leq m_1(w) + m_2(w)\eta^{\frac{3}{2}\cdot(1-\frac{1}{2})}\|y\| \end{aligned}$$

where $m_1(w) = \frac{1}{2}$ and $m_2(w) = 1$ for all $w \in [0,2\pi]$. Thus, this satisfies the given condition (2.13). Moreover,

$$\frac{(\nu+\varsigma)^{-\theta}R^{\nu+\varsigma}}{\Gamma(1+\theta)}m_2^*(w) = \frac{1}{(\frac{3}{2})^{\frac{1}{2}}\Gamma(1+\frac{1}{2})} < 1.$$

Thus, by Theorem 7, there exists a random solution for the given problem (2.29).

Example 2. *Consider $\Delta = (-\infty, 0)$ with the usual σ-algebra containing those subsets of Δ which are Lebesgue measurable. We take*

$$l^1 = \left\{ y = (y_1, y_2, y_3, ...), \sum_{n=1}^{\infty} |y_n| < \infty \right\}, \tag{2.30}$$

which is a Banach space with the norm

$$\|y\| = \sum_{n=1}^{\infty} |y_n|. \tag{2.31}$$

We consider the GMSF differential equation:

$$ {}_{\varsigma}^{\nu}\mathcal{T}_{0^+}^{\theta} y_n(r,w) = f_n(r, y(r,w), w),\ r \in [0,1],\ w \in \Delta \tag{2.32}$$

with the terminal condition

$$y(1,w) = \left((1+w^2)^{-1}, 0, 0, ...\right) \tag{2.33}$$

with $y = (y_1, y_2, y_3, ...)$, $f = (f_1, f_2, f_3, ...)$,

$$ {}_{\varsigma}^{\nu}\mathcal{T}_{0^+}^{\theta} y = ({}_{\varsigma}^{\nu}\mathcal{T}_{0^+}^{\theta} y_1, {}_{\varsigma}^{\nu}\mathcal{T}_{0^+}^{\theta} y_2, ..., {}_{\varsigma}^{\nu}\mathcal{T}_{0^+}^{\theta} y_n, ...),$$

and

$$f_n(r, y(r,w), w) = \frac{w^2 r^{(\nu+\varsigma)(1-\theta)}(2^{-n} + y_n(r,w))}{2(1+w^2)(1+\|y\|)} \left(e^{-7-w^2} + \frac{1}{e^{r+5}}\right),\ r \in [0,1],\ w \in \Delta. \tag{2.34}$$

We have

$$\|f(r,y,w) - f(r,v,w)\| \leq \left(e^{-7-w^2} + \frac{1}{e^{r+5}}\right) \frac{w^2 r^{(\nu+\varsigma)(1-\theta)}\|y-v\|}{1+\|y-v\|}. \tag{2.35}$$

Hence, the hypotheses in Theorem 8 hold true with:

$$z(r,w) = \left(e^{-7-w^2} + \frac{1}{e^{r+5}}\right),\ \ \Psi(r,w) = w^2. \tag{2.36}$$

Hence, by Theorem 8, the problem (2.32)-(2.33) has a generalized Ulam-Hyers-Rassias stable random solution.

3. Applications of the obtained results in technology

The obtained results are significant in various senses. The main point of concern is that they contain the newly-defined GMSF operators having various parameters defined over the meaningful intervals of real numbers. Their possible applications in various fields of technology are discussed as below.

3.1. Control theory

One of the primary objectives of control theory is to create controllers to stabilize a particular system [28]. By creating control signals that reflect the system's current state, the feedback control is a common technique for guiding a system to the desired state [29]. The system that has to be controlled may commonly be represented as a differential equation with possibly fractional derivatives. The difficulty of system stabilization then is found in identifying the solution that satisfies certain stability constraints [30].

In the context of the present work, taking Eq (2.1) with the terminal condition (2.2), we may interpret $y(\eta, w)$ as the state of the system at time η with parameters w and $f(\eta, y(\eta, w), w)$ as the dynamics of the system. Finding a function $y(\eta, w)$ that fullfils the provided equation and terminal condition as well as a number of stability criteria is the aim of control theory.

One stability criterion, for instance, would be that the entire system must remain stable with respect to disturbances, i.e., that slight changes in the parameters or initial state should not result in significant modifications to the system's behavior. Another requirement can be that the output of the system not be unduly sensitive to random fluctuations in the input, or that the system must be stable with regard to noise.

Therefore, control theory can be benefited from studying the stability of systems that can be described by GMSF differential equations (2.1) with the terminal condition (2.2) because it offers a framework for modeling and analyzing such stability. The goal is to identify solutions that satisfy stability criteria and may be used to develop controllers that stabilize the system.

3.2. Control systems

Control systems are essential to many technical systems, such as robots, airplanes, and automobiles [31]. The design and implementation of control systems that can assure the stability of these systems can be improved by the stability analysis of GMSF differential equations. For instance, the stability analysis of these equations in robotics is useful in the development of control algorithms that guarantee the motion stability of the robot, resulting in precise and accurate motions [32].

The idea of state-space representation helps to clarify how Eqs (2.1) and (2.2) can be applied to control systems. Equation (2.1) may be thought of as a state equation that explains the dynamics of the system, with $y(\eta, w)$ standing for the system's state at time xi and input w. The link between the state of the system, the output, and the input is represented by the function $f(\eta, y(\eta, w), w)$.

As a result, it is possible to describe and analyze the behavior of a control system using the Eqs (2.1) and (2.2), where the input w is the control action that is applied to the system, and the output is the system's reaction to this control action. One can determine whether a control action will get a desired response from the system by examining the stability of the system given by these equations.

3.3. Signal processing

Equations (2.1) and (2.2) could potentially be used in signal processing. They can be used to model certain kinds of signals and create filters for those signals [33].

For example, let $y(t)$ be a signal that is sampled at discrete time intervals. The rate at which change occurs in the signal (at each point of time) may be modeled using the GMSF derivative operator, and the Eq (2.1) may be utilized for modeling the signal's overall behavior throughout the time. By solving the Eq (2.1) with condition (2.2), one can acquire information about the signal's properties, including its stability, frequency content, and reaction to external inputs.

Additionally, the signal's filters may be designed using the Eq (2.1). By selecting the right function $f(\eta, y(\eta, w), w)$, one may create a filter that selectively eliminates some frequencies or components of the signal while maintaining others. Several signal processing applications, including audio and picture processing, where it is frequently desirable to eliminate noise or other undesirable aspects from the signal, can benefit from this [34].

In a nutshell, the GMSF Eqs (2.1) and (2.2) have various applications in the field of signal processing, including modeling signal behavior and creating filters to handle that behavior.

4. Conclusions and future works

This study examined the existence and stability of the solutions of fractional differential equations using the GMSF operators. The GMSF operators are a recently defined new class of fractional operators that extend the conventional fractional calculus of deterministic functions to the m.s. stochastic calculus of probabilistic processes. The present investigation verifies the generalized Ulam-Hyers-Rassias stability criteria and offers a rigorous proof for the existence of solutions for the class of the GMSF differential equations. The numerical examples show how our established findings for solving FDEs and assessing their stability work accurately and effectively. Additionally, the applicability of the obtained results in various technological fields, including control systems, control theory, and signal processing, has also been focused on.

This research work has a number of benefits over the ones already in use. First, it increases the applicability of the generalized m.s. fractional derivative as a tool for solving FDEs by extending it to a wider class of functions. Second, it offers a thorough demonstration of the presence of solutions for a class of FDEs with the GMSF operators, which is a crucial first step in the development of trustworthy numerical techniques for resolving these equations. Finally, the generalized Ulam-Hyers-Rassias stability criterion, which is essential for analyzing the behavior of solutions over time, can be used to evaluate stability using the proposed method.

There are numerous exciting directions that future research in the fields of control theory, signal processing, and control systems can go. One area that has promise is the development of new algorithms and methods that make use of the GMSF derivative to improve the effectiveness of these systems. To fully comprehend these algorithms' theoretical properties, such as stability and convergence, further research is also needed. The GMSF derivative may be helpful for modeling and analyzing complicated systems, especially those containing non-Newtonian fluids or visco-elastic materials, in the field of material science. Overall, the GMSF derivative is a potent instrument with a wide range of potential applications in numerous technological domains, and we expect the development and growth that will be accomplished in the ensuing years.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest in this paper.

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