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Research article

Exploring optical soliton solutions of the time fractional q-deformed Sinh-Gordon equation using a semi-analytic method

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Abstract: The q-deformed Sinh-Gordon equation extends the classical Sinh-Gordon equation by incorporating a deformation parameter q. It provides a framework for studying nonlinear phenomena and soliton dynamics in the presence of quantum deformations, leading to intriguing mathematical structures and potential applications in diverse areas of physics. In this work, we imply the homotopy analysis method, to obtain approximate solutions for the proposed equation, the error estimated from the obtained solutions reflects the efficiency of the solving method. The solutions were presented in the form of 2D and 3D graphics. The graphics clarify the impact of a set of parameters on the solution, including the deformation parameter q, as well as the effect of time and the fractional order derivative.

Keywords: time-fractional q-deformed Sinh-Gordon equation; homotopy analysis method; Adomian polynomials; Caputo fractional derivatives
Mathematics Subject Classification: 26Axx, 78Axx

1. Introduction

The q-deformed Sinh-Gordon equation (SGE) is a generalized version of the SGE, incorporating the deformation parameter q. The SGE is a nonlinear partial differential equation that arises in mathematical physics, particularly in the study of soliton theory. The q-deformed SGE exhibits rich mathematical properties and has connections to various areas of theoretical physics, including integrable systems, quantum field theory, and string theory. It is known for supporting soliton solutions, which are localized, stable wave-like structures that propagate without dispersion. Solitons are of great interest in the study of nonlinear phenomena and have applications in fields such as optics, condensed matter physics, and particle physics, see [1].

q-deformed SGE has been handled several times in many papers [2–5], until it reached its recently general modification form:

$$\frac{\partial^2 \mathsf{v}}{\partial \varkappa^2} - \frac{\partial^2 \mathsf{v}}{\partial \mathsf{t}^2} = e^{\alpha \mathsf{v}}[sinh_{\mathsf{q}}(\mathsf{v}^{\gamma})]^p - \varrho, \tag{1.1}$$

where $sinh_{\mathsf{q}}(\mathsf{v})$ is a q-deformed function given by the relation

$$sinh_{\mathsf{q}}(\mathsf{v}) = \frac{e^{\mathsf{v}} - \mathsf{q}e^{-\mathsf{v}}}{2}, \tag{1.2}$$

where $\mathsf{v}(\varkappa, \mathsf{t})$ represents the unknown function and it is a function of space $\varkappa$ and time t, q is the deformation parameter, $\alpha, \gamma, p, \varrho$ are constants $\in R$.

Since Eq (1.1) is a recent modification version, it has been solved only a few times. Khalid K. Ali et al. investigate its soliton solutions using Kudryashov method [6]. Syeda Sarwat Kazmi et al. used Lie symmetry approach and similarity reduction to examine its solutions, [7]. However, Eq (1.1) has not been solved or addressed in fractional form. Therefore, in this study, we will present its solutions in fractional form for the first time, this is due to the significance of fractional calculus in several scientific fields. Fractional calculus has gained significant attention among scientists and engineers due to its practicality in modeling applications across various domains, including physics, fluid and quantum mechanics, wave phenomena [8], biomedical fields [9], and optical fiber technology. In numerous scientific disciplines, fractional differential equations play a pivotal role in simulating phenomena [10, 11], and even systems of equations are employed for modeling physical systems exhibiting anomalous diffusion, memory effects, and long-range interactions-phenomena that cannot be adequately explained using conventional integer-order differential equations. Engineers make extensive use of fractional partial differential equations to innovate materials with tailored mechanical properties, control heat transfer in microdevices, optimize signal processing algorithms, investigate aspects like shallow water waves [12], analyze financial models [13] and analyze projectile motion, see [14–17].

In this article, we will solve the time-fractional q-deformed SGE in the form:

$$\frac{\partial^2 \mathsf{v}}{\partial \varkappa^2} - \frac{\partial^{2\wp} \mathsf{v}}{\partial \mathsf{t}^{2\wp}} = e^{\alpha \mathsf{v}}[sinh_{\mathsf{q}}(\mathsf{v}^{\gamma})]^p - \varrho, \tag{1.3}$$

constrained by the initial guess:

$$\begin{aligned} \mathsf{v}(\varkappa, 0) &= \mathsf{v}_0(\varkappa, \mathsf{t}), \\ \mathsf{v}_{\mathsf{t}}(\varkappa, 0) &= \mathsf{v}_{0\mathsf{t}}(\varkappa, \mathsf{t}). \end{aligned}$$

Where $\wp$ is the fractional order derivative $0 < \wp \leq 1$ in Caputo sense, using homotopy analysis method (HAM). The HAM is a valuable semi-analytical technique employed for solving nonlinear equations and systems. It combines elements from perturbation theory and homotopy continuation methods to obtain approximate solutions. It was first devised in 1992 by Liao Shijun [18], then the method was further modified to encompass a wider range of nonlinear differential equations and provide high accuracy, [19]. HAM has been applied to solve fractional partial differential equations, and the results obtained ensures the importance and efficiency of this method. HAM is a valuable mathematical tool for approximating solutions to nonlinear differential equations, offering flexibility and analytical insight. However, it has limitations related to complexity in calculating higher terms, and sensitivity to initial conditions, further details and applications, see [20–22].

This article is organized as follow: Section 2 presents the fundamentals utilized to solve the proposed equation, such as fractional derivatives and integrals, and Adomian polynomials. Section three introduces the structure of the HAM as a technique for the solution and the convergence, existence, and the uniqueness of the solution discussed. A brief explanation of the solution of the q-deformed Sinh-Gordon equation is given in section four. The results obtained in section four are visually represented in section five. Section six includes the concluding remarks of the present study.

2. Notaions on basic definitions

2.1. Fractional derivatives

The majority of research in fractional calculus centers around the exploration of two different types of fractional derivatives: the Riemann-Liouville and Caputo derivatives, each with unique definitions. The Caputo derivative has gained greater traction in modeling real-world problems due to two distinct advantages it provides. Firstly, it assigns a value of zero to the derivative of a constant, ensuring the Caputo fractional derivative (CFD) remains bounded. Secondly, it allows for the expression of initial conditions in the form of an integer-order derivative. It's important to highlight that Caputo's definition is applicable solely to functions that are differentiable, see [23–26].

In this research, our objective is to address time fractional q-deformed SGE utilizing the Caputo fractional derivative definition.

Definition 2.1. *[23] The CFD is stated as:*

$$
{}^{\mathrm{C}}\mathrm{D}_{\varphi}^{\wp} f(\varphi) = \begin{cases} \jmath^{\mathrm{n}-\wp} \dfrac{d^{\mathrm{n}}}{d\varphi^{\mathrm{n}}} f(\varphi), & \mathrm{n}-1 < \wp < \mathrm{n}, \\ \\ \dfrac{d^{\mathrm{n}}}{d\varphi^{\mathrm{n}}} f(\varphi), & \wp = \mathrm{n}. \end{cases} \tag{2.1}
$$

Where $\jmath^{\mathrm{n}-\wp}$ represents the Riemann-Liouville fractional integral which can be stated as:

$$
\jmath^{\wp} f(\varphi) = \frac{1}{\Gamma(\wp)} \int_0^{\varphi} (\varphi - \Upsilon)^{(\wp-1)} f(\Upsilon)\, d\Upsilon, \quad \varphi > 0, \quad \wp \in R^+, \tag{2.2}
$$

here, R^+ represents the set of positive real numbers, and $\Gamma(.)$ is the known Gamma function. The operator $\jmath^{\wp}$ satisfy the following properties for $\delta, \gamma \geq -1$:

$$
\jmath^{\gamma} \jmath^{\delta} f(\varphi) = \jmath^{\gamma+\delta} f(\varphi), \tag{2.3}
$$

$$
\jmath^{\gamma} \jmath^{\delta} f(\varphi) = \jmath^{\delta} \jmath^{\gamma} f(\varphi), \tag{2.4}
$$

$$
\jmath^{\gamma} \varphi^{\mathrm{m}} = \frac{\Gamma(\mathrm{m}+1)}{\Gamma(\mathrm{m}+1+\gamma)} \varphi^{\mathrm{m}+\gamma}. \tag{2.5}
$$

The CFD satisfy the following properties:

$$
{}^{\mathrm{C}}\mathrm{D}_{\varphi}^{\wp} \Big[\jmath^{\wp} f(\varphi) \Big] = f(\varphi), \tag{2.6}
$$

$$J^{\wp}\left[{}^{\mathrm{C}}\mathrm{D}_{\varphi}^{\wp}f(\varphi)\right] = f(\varphi) - \sum_{\imath=0}^{\mathrm{n}-1} f^{\imath}(0)\frac{\varphi^{\imath}}{\imath!}, \quad \varphi > 0, \tag{2.7}$$

$${}^{\mathrm{C}}\mathrm{D}_{\varphi}^{\wp}\varphi^{\mathrm{m}} = \frac{\Gamma(\mathrm{m}+1)}{\Gamma(\mathrm{m}+1-\wp)}\varphi^{\mathrm{m}-\wp}. \tag{2.8}$$

2.2. *Adomian polynomials*

The Adomian decomposition method has established the notion that the unknown linear function u can be expressed using a series of decompositions:

$$\mathrm{u} = \sum_{\imath=0}^{\infty} \mathrm{u}_{\imath}, \tag{2.9}$$

where the components $\mathrm{u}_{\imath}$can be computed recursively. the nonlinear term $\mathrm{F}(\mathrm{u})$, such as $\mathrm{u}^2, \mathrm{u}^3$, $sin\mathrm{u}$, e^{u}, etc. can be expressed by Adomian polynomials (AP) $\mathrm{A}_{\imath}$ in the configuration:

$$\mathrm{F}(\mathrm{u}) = \sum_{\imath=0}^{\infty} \mathrm{A}_{\imath}(\mathrm{u}_0, \mathrm{u}_1, ..., \mathrm{u}_{\imath}). \tag{2.10}$$

The AP can be calculated to address various types of nonlinearity. Adomian [27], proposed a methodology for the computation of AP, which was subsequently validated through formal justifications. Alternative approaches have been devised founded upon Taylor series, see [28,29].

The computation of AP $\mathrm{A}_{\imath}$ for the nonlinear term $\mathrm{F}(\mathrm{u})$ can be accomplished by utilizing the general expression:

$$\mathrm{A}_{\imath} = \frac{1}{\imath\,!}\frac{d^{\imath}}{d\vee^{\imath}}\left[\mathrm{F}\left(\sum_{\ell=0}^{\imath} \vee^{\ell}\mathrm{u}_{\ell}\right)\right]_{\vee=0}, \qquad \imath = 0, 1, 2, ... \tag{2.11}$$

The general expression (2.11) can be condensed in the following manner:

$$\begin{aligned}
\mathrm{A}_0 &= \mathrm{F}(\mathrm{u}_0),\\
\mathrm{A}_1 &= \mathrm{u}_1\mathrm{F}'(\mathrm{u}_0),\\
\mathrm{A}_2 &= \mathrm{u}_2\mathrm{F}'(\mathrm{u}_0) + \frac{1}{2!}\mathrm{u}_1^2\mathrm{F}''(\mathrm{u}_0),\\
\mathrm{A}_3 &= \mathrm{u}_3\mathrm{F}'(\mathrm{u}_0) + \mathrm{u}_1\,\mathrm{u}_2 F''(\mathrm{u}_0) + \frac{1}{3!}\mathrm{u}_1^3\mathrm{F}'''(\mathrm{u}_0),\\
&\vdots
\end{aligned} \tag{2.12}$$

It is obvious that A_0 depends only on u_0, A_1 depends only u_0 and u_1, A_2 depends only on u_0,u_1 and u_2, etc.

3. Analysis of homotopy analysis method

The HAM, introduced by Liao [18], offers a versatile analytical approach for obtaining series solutions to a wide range of nonlinear equations. These encompass algebraic equations, ordinary differential equations (ODEs), partial differential equations (PDEs), and coupled equations involving them. HAM relies on the fundamental concept of homotopy from topology in which it develop a link between linear and nonlinear DEs of significance. The following steps illustrates the procedure to apply the HAM for nonlinear fractional partial differential equation (NFPDE), [30, 31].

Consider the NFPDE:

$$\mathsf{NFP}\{\mathsf{v}(\varkappa,\mathsf{t})\} = 0, \tag{3.1}$$

where $\mathsf{v}(\varkappa,\mathsf{t})$ is the unknown function, $\varkappa,\mathsf{t}$ are the independent variables, NFP is the nonlinear operator of the FPDE. Construct the zero order deformation equation:

$$(1-\mathsf{p})\pounds\{\upsilon(\varkappa,\mathsf{t},\mathsf{p}) - \mathsf{v}_0(\varkappa,\mathsf{t})\} = \mathsf{p}\,\mathsf{h}\,\mathsf{NFP}(\mathsf{v}(\varkappa,\mathsf{t},\mathsf{p})), \tag{3.2}$$

in this context, p represents the embedding parameter, which is constrained to the interval $[0,1]$. h refers to a non-zero auxiliary or supplementary parameter, and £ is an auxiliary operator.

Clearly, when $\mathsf{p}=0$, we have $\upsilon(\varkappa,\mathsf{t},0) = \mathsf{v}_0(\varkappa,\mathsf{t})$, and when $\mathsf{p}=1$, we find $\upsilon(\varkappa,\mathsf{t},1) = \mathsf{v}(\varkappa,\mathsf{t})$, this implies that, as the parameter p smoothly transitions from 0 to 1, the solution undergoes a continuous change from the initial approximation $\mathsf{v}_0(\varkappa,\mathsf{t})$ to the actual solution $\mathsf{v}(\varkappa,\mathsf{t})$.

Expanding $\mathsf{v}(\varkappa,\mathsf{t},\mathsf{p})$ around p by employing Taylor series, we get

$$\mathsf{v}(\varkappa,\mathsf{t},\mathsf{p}) = \mathsf{v}_0(\varkappa,\mathsf{t}) + \sum_{\mathsf{m}=1}^{\infty}\mathsf{v}_{\mathsf{m}}(\varkappa,\mathsf{t})\mathsf{p}^{\mathsf{m}}, \tag{3.3}$$

where

$$\mathsf{v}_{\mathsf{m}}(\varkappa,\mathsf{t}) = \frac{1}{\mathsf{m}!}\frac{\partial^{\mathsf{m}}\mathsf{v}(\varkappa,\mathsf{t},\mathsf{p})}{\partial\mathsf{p}^{\mathsf{m}}}|_{\mathsf{p}=0}.$$

With appropriate selections for the supplementary parameter h, the initial approximation, and $\mathsf{v}_0(\varkappa,\mathsf{t})$, the series described in Eq (3.3) achieves convergence when $\mathsf{p}=1$, hence,

$$\mathsf{v}(\varkappa,\mathsf{t}) = \mathsf{v}_0(\varkappa,\mathsf{t}) + \sum_{\mathsf{m}=1}^{\infty}\mathsf{v}_{\mathsf{m}}(\varkappa,\mathsf{t}). \tag{3.4}$$

Define the vector

$$\vec{\mathsf{v}_{\mathsf{m}}}(\varkappa,\mathsf{t}) = \{\mathsf{v}_0(\varkappa,\mathsf{t}),\mathsf{v}_1(\varkappa,\mathsf{t}),\mathsf{v}_2(\varkappa,\mathsf{t}),...,\mathsf{v}_{\mathsf{m}}(\varkappa,\mathsf{t})\}.$$

By differentiating Eq (3.1) mth time with respect to p and putting $\mathsf{p}=0$, we derive the equation with a deformation order of m,

$$\pounds\{\mathsf{v}_{\mathsf{m}}(\varkappa,\mathsf{t}) - \mathcal{X}_{\mathsf{m}}\mathsf{v}_{\mathsf{m}-1}(\varkappa,\mathsf{t})\} = \mathsf{h}\,\mathsf{NFP}(\vec{\mathsf{v}_{\mathsf{m}-1}}(\varkappa,\mathsf{t})), \tag{3.5}$$

where

$$\mathcal{X}_{\mathsf{m}} = \begin{cases} 0, & \mathsf{m} \le 1, \\ 1, & \mathsf{m} > 1. \end{cases}$$

The mth order deformation equation (3.5) can be easily solved with the aid of Mathematica program.

3.1. Convergence analysis

To prove that, the approximate series solution (3.3) converges to the exact series (3.4), consider the following theorem:

Theorem 3.1. *Assume* $\mathsf{B} \subset \mathsf{R}$ *be a Banach space with suitable* $\|.\|$*, in which the sequence* $\mathsf{v}_{\mathsf{m}}(\varkappa, \mathsf{t})$ *is defined for a definite value of* h*, for a constant* $\mathsf{L} \in \mathsf{R}$*, then:*
if $\|\mathsf{v}_{\mathsf{m}+1}(\varkappa, \mathsf{t})\| \leq \|\mathsf{v}_{\mathsf{m}}(\varkappa, \mathsf{t})\|\ \forall \mathsf{m}$*, then the series solution* $\mathsf{v}(\varkappa, \mathsf{t}) = \sum_{\mathsf{m}=0}^{\infty} \mathsf{v}_{\mathsf{m}}(\varkappa, \mathsf{t})\mathsf{p}^{\mathsf{m}}$ *converges absolutely to* $\mathsf{v}(\varkappa, \mathsf{t}) = \mathsf{L} \sum_{\mathsf{m}=0}^{\infty} \mathsf{v}_{\mathsf{m}}(\varkappa, \mathsf{t})$.

Proof. By implying the ratio test of the power series, if S_ℓ is sequence of partial sum of the series $\mathsf{v}(\varkappa, \mathsf{t}) = \sum_{\mathsf{m}=0}^{\infty} \mathsf{v}_{\mathsf{m}}(\varkappa, \mathsf{t})$, we want to prove that S_ℓ is a Cauchy sequence in B, consider

$$\|\mathsf{S}_{\ell+1}(\varkappa, \mathsf{t}) - \mathsf{S}_\ell(\varkappa, \mathsf{t})\| = \|\mathsf{v}_{\ell+1}(\varkappa, \mathsf{t})\| \leq \mathsf{L}\|\mathsf{v}_\ell(\varkappa, \mathsf{t})\| \leq \mathsf{L}^2\|\mathsf{v}_{\ell-1}(\varkappa, \mathsf{t})\| ... \leq \mathsf{L}^{\ell+1}\|\mathsf{v}_0(\varkappa, \mathsf{t})\|. \tag{3.6}$$

For every $\ell, \mathcal{E} \in \mathsf{N},\ \ell > \mathcal{E}$, using the relation (3.6) side by side with the triangle inequality, we have

$$\begin{aligned}\|\mathsf{S}_\ell(\varkappa, \mathsf{t}) - \mathsf{S}_{\mathcal{E}}(\varkappa, \mathsf{t})\| &= \|(\mathsf{S}_\ell(\varkappa, \mathsf{t}) - \mathsf{S}_{\ell-1}(\varkappa, \mathsf{t})) + (\mathsf{S}_{\ell-1}(\varkappa, \mathsf{t}) - \mathsf{S}_{\ell-2}(\varkappa, \mathsf{t})) + ... + (\mathsf{S}_{\mathcal{E}+1}(\varkappa, \mathsf{t}) - \mathsf{S}_{\mathcal{E}}(\varkappa, \mathsf{t}))\| \\ &\leq \frac{1 - \mathsf{L}^{\ell+\mathcal{E}}}{1 - \mathsf{L}}\mathsf{L}^{\mathcal{E}+1}\|\mathsf{v}_0(\varkappa, \mathsf{t})\|.\end{aligned} \tag{3.7}$$

For $0 < \mathsf{L} < 1$, the relation (3.7) implies

$$\lim_{\ell, \mathcal{E} \longrightarrow \infty} \|\mathsf{S}_\ell(\varkappa, \mathsf{t}) - \mathsf{S}_{\mathcal{E}}(\varkappa, \mathsf{t})\| = 0. \tag{3.8}$$

Therefore, $\mathsf{S}_\ell(\varkappa, \mathsf{t})$ is a Cauchy sequence in the Banach space B, hence, the series $\mathsf{v}(\varkappa, \mathsf{t}) = \sum_{\mathsf{m}=0}^{\infty} \mathsf{v}_{\mathsf{m}}(\varkappa, \mathsf{t})$ is convergent. □

3.2. The existence and the uniqueness of the solution

The following definitions and theorems are important in proving the existance and the uniqueness of the solution.

Definition 3.1. *[32] In a normed space* $(\mathsf{v}, \|\ .\ \|)$*, a contraction of* v *is a mapping* $\mathsf{T} : \mathsf{v} \longrightarrow \mathsf{v}$ *that meets the following condition for any* v_1 *and* v_2 *within* v*:* $\| \mathsf{T}(\mathsf{v}_1) - \mathsf{T}(\mathsf{v}_2) \| \leq \delta \| \mathsf{v}_1 - \mathsf{v}_2 \|$ *Here,* δ *is a real value with the constraint* $0 \leq \delta < 1$.

Theorem 3.2. *[33] Every contraction mapping on a complete metric space has a unique fixed point.*

Let the q-deformed SGE in the form:

$$\frac{\partial^{2\wp}\mathsf{v}}{\partial \mathsf{t}^{2\wp}} = \frac{\partial^2 \mathsf{v}}{\partial \varkappa^2} + \mathsf{F}(\varkappa, \mathsf{t}), \quad 0 < \wp \leq 1. \tag{3.9}$$

Denote B as a Banach space of all continuous functions C in R with norm defined by $\|.\|_\infty$ defined by $\|\mathsf{v}\|_\infty = Sup\{|\mathsf{v}|, \varkappa, \mathsf{t} \in \mathsf{C}\}$.

Theorem 3.3. *If there exist a constant* L *such that* $|\mathsf{v}_{1\varkappa\varkappa} - \mathsf{v}_{2\varkappa\varkappa}| \leq \mathsf{L}|\mathsf{v}_1 - \mathsf{v}_2|$ *for all* $(\varkappa, \mathsf{t}) \in \mathsf{C}$*. If* $\dfrac{\mathsf{t}^{2\wp}}{\Gamma(2\wp + 1)}\mathsf{L} < 1$*, then the Eq* (3.9) *has a unique solution on* $\mathsf{B}(\mathsf{C})$.

Proof. We will transform the investigated equation into a fixed point equation, consider the operator $\Xi : \mathsf{B}(\mathsf{C},\mathsf{R}) \longrightarrow \mathsf{B}(\mathsf{C},\mathsf{R})$, and employing the property defined in Eq (2.7), we obtain:

$$\Xi\Big(\mathsf{v}(\varkappa,\mathsf{t})\Big) = \mathsf{v}(\varkappa,0) + \mathsf{t}\mathsf{v}_{\mathsf{t}}(\varkappa,0) + J^{2\wp}\Big(\frac{\partial^2 \mathsf{v}}{\partial \varkappa^2} + \mathsf{F}(\varkappa,\mathsf{t})\Big), \quad 0 < \wp \le 1.$$

Let $\mathsf{v}_1, \mathsf{v}_2 \in \mathsf{B}(\mathsf{C},\mathsf{R})$, then for each $(\varkappa,\mathsf{t}) \in \mathsf{C}$, we have:

$$\begin{aligned}
|\Xi\mathsf{v}_1(\varkappa,\mathsf{t}) - \Xi\mathsf{v}_2(\varkappa,\mathsf{t})| &= J^{2\wp}|\mathsf{v}_{1\varkappa\varkappa} - \mathsf{v}_{2\varkappa\varkappa}| \\
&\le \frac{\mathsf{t}^{2\wp}}{\Gamma(2\wp+1)}\mathsf{L}|\mathsf{v}_1 - \mathsf{v}_2| \\
&\le \frac{\mathsf{t}^{2\wp}}{\Gamma(2\wp+1)}\mathsf{L}\ Sup|\mathsf{v}_1 - \mathsf{v}_2| \\
&\le \frac{\mathsf{t}^{2\wp}}{\Gamma(2\wp+1)}\mathsf{L}||\mathsf{v}_1 - \mathsf{v}_2||.
\end{aligned}$$

□

These steps for proving two steps of the solution, we can continue to find the proof for three steps and more.

4. Solution of the proposed model

In this section, the HAM is applied to explore the solutions of the time-fractional q-deformed SGE under three cases of initial conditions. Write Eq (1.3) in the form:

$$\frac{\partial^{2\wp}\mathsf{v}}{\partial \mathsf{t}^{2\wp}} = \frac{\partial^2 \mathsf{v}}{\partial \varkappa^2} - e^{\alpha \mathsf{v}}[sinh_{\mathsf{q}}(\mathsf{v}^{\gamma})]^{p} + \varrho. \tag{4.1}$$

Equation (4.1) will be handled subject to three cases of initial conditions [6]:
Case I : For $\alpha = \gamma = p = 1$ and $\varrho = -\frac{\mathsf{q}}{2}$.

By applying the relation presented in Eq (1.2), Eq (4.1) can be simplified to the form:

$$\frac{\partial^{2\wp}\mathsf{v}}{\partial \mathsf{t}^{2\wp}} = \frac{\partial^2 \mathsf{v}}{\partial \varkappa^2} - \frac{1}{2}e^{2\mathsf{v}}, \tag{4.2}$$

constrained by the initial guess:

$$\begin{aligned}
\mathsf{v}(\varkappa,0) &= \frac{1}{2}ln\left(\frac{0.112}{1+e^{(0.214476i)\varkappa}} - \frac{0.112}{(1+e^{(0.214476i)\varkappa})^2}\right), \\
\mathsf{v}_{\mathsf{t}}(\varkappa,0) &= \frac{-0.05 + 0.05e^{(0.214476i)\varkappa}}{1+e^{(0.214476i)\varkappa}}.
\end{aligned}$$

When applying the steps of HAM presented in section three, define the nonlinear operator

$$\mathsf{NFP}(\mathsf{v}(\varkappa,\mathsf{t},\mathsf{p})) = \frac{\partial^{2\wp}\mathsf{v}}{\partial \mathsf{t}^{2\wp}} - \frac{\partial^2 \mathsf{v}}{\partial \varkappa^2} + \frac{1}{2}e^{2\mathsf{v}}. \tag{4.3}$$

The nonlinear term can be treated using Adomian polynomials discussed in section (2.2), hence according to Eq (2.12),

$$\begin{aligned}
&\mathsf{F}(\mathsf{v}) = \frac{1}{2}e^{2\mathsf{v}}, \quad \mathsf{A}_0 = \frac{1}{2}e^{2\mathsf{v}_0},\\
&\mathsf{A}_1 = \mathsf{v}_1 e^{2\mathsf{v}_0}, \quad \mathsf{A}_2 = \mathsf{v}_2 e^{2\mathsf{v}_0} + \frac{1}{2!}\mathsf{v}_1^2\, 2e^{2\mathsf{v}_0},\\
&\vdots
\end{aligned} \tag{4.4}$$

One can obtain the mth order deformation equation:

$$\mathsf{v}_\mathsf{m}(\varkappa,\mathsf{t}) = \mathcal{X}_\mathsf{m}\mathsf{v}_{\mathsf{m}-1}(\varkappa,\mathsf{t}) + \mathsf{h}\,£^{-1}\,\mathsf{NFP}(\mathsf{v}_{\mathsf{m}-1}(\varkappa,\mathsf{t})), \tag{4.5}$$

where

$$\mathsf{NFP}(\mathsf{v}_{\mathsf{m}-1}(\varkappa,\mathsf{t})) = \mathsf{D}_\mathsf{t}^{2\wp}\ \mathsf{v}_{\mathsf{m}-1}(\varkappa,\mathsf{t}) - \frac{\partial^2\mathsf{v}_{\mathsf{m}-1}}{\partial\varkappa^2} + \frac{1}{2}e^{2\mathsf{v}_{\mathsf{m}-1}}. \tag{4.6}$$

From Eqs (4.5) and (4.6), we obtain the recursive equations:

$$\begin{aligned}
&\mathsf{v}_0(\varkappa,\mathsf{t}) = \mathsf{v}(\varkappa,0) + \mathsf{t}\,\mathsf{v}_\mathsf{t}(\varkappa,0),\\
&\mathsf{v}_1(\varkappa,\mathsf{t}) = \mathsf{h}\,£^{-1}\Big(\mathsf{D}_\mathsf{t}^{2\wp}\ \mathsf{v}_0(\varkappa,\mathsf{t}) - \frac{\partial^2\mathsf{v}_0}{\partial\varkappa^2} + \frac{1}{2}e^{2\mathsf{v}_0}\Big),\\
&\mathsf{v}_2(\varkappa,\mathsf{t}) = \mathsf{v}_1(\varkappa,\mathsf{t}) + \mathsf{h}\,£^{-1}\Big(\mathsf{D}_\mathsf{t}^{2\wp}\ \mathsf{v}_1(\varkappa,\mathsf{t}) - \frac{\partial^2\mathsf{v}_1}{\partial\varkappa^2} + \mathsf{v}_1 e^{2\mathsf{v}_0}\Big),\\
&\mathsf{v}_3(\varkappa,\mathsf{t}) = \mathsf{v}_2(\varkappa,\mathsf{t}) + \mathsf{h}\,£^{-1}\Big(\mathsf{D}_\mathsf{t}^{2\wp}\ \mathsf{v}_2(\varkappa,\mathsf{t}) - \frac{\partial^2\mathsf{v}_2}{\partial\varkappa^2} + \mathsf{v}_2 e^{2\mathsf{v}_0} + \frac{1}{2!}\mathsf{v}_1^2\, 2e^{2\mathsf{v}_0}\Big),\\
&\vdots
\end{aligned} \tag{4.7}$$

To enhance the computational efficiency of the nonlinear term, the scheme presented in Eq (4.7) has been altered as follows:

$$\begin{aligned}
&\mathsf{v}_0(\varkappa,\mathsf{t}) = \mathsf{v}(\varkappa,0),\\
&\mathsf{v}_1(\varkappa,\mathsf{t}) = \mathsf{t}\,\mathsf{v}_\mathsf{t}(\varkappa,0) + \mathsf{h}\,J^{2\wp}\Big(\mathsf{D}_\mathsf{t}^{2\wp}\ \mathsf{v}_0(\varkappa,\mathsf{t}) - \frac{\partial^2\mathsf{v}_0}{\partial\varkappa^2} + \frac{1}{2}e^{2\mathsf{v}_0}\Big),\\
&\mathsf{v}_2(\varkappa,\mathsf{t}) = \mathsf{h}\,J^{2\wp}\Big(\mathsf{D}_\mathsf{t}^{2\wp}\ \mathsf{v}_1(\varkappa,\mathsf{t}) - \frac{\partial^2\mathsf{v}_1}{\partial\varkappa^2} + \mathsf{v}_1 e^{2\mathsf{v}_0}\Big),\\
&\mathsf{v}_3(\varkappa,\mathsf{t}) = \mathsf{v}_2(\varkappa,\mathsf{t}) + \mathsf{h}\,J^{2\wp}\Big(\mathsf{D}_\mathsf{t}^{2\wp}\ \mathsf{v}_2(\varkappa,\mathsf{t}) - \frac{\partial^2\mathsf{v}_2}{\partial\varkappa^2} + \mathsf{v}_2 e^{2\mathsf{v}_0} + \frac{1}{2!}\mathsf{v}_1^2\, 2e^{2\mathsf{v}_0}\Big),\\
&\vdots
\end{aligned} \tag{4.8}$$

By inserting the initial conditions and made the computations with the aid of Mathematica program, we obtain:

$$\mathsf{v}_0(\varkappa,\mathsf{t}) = \frac{1}{2}ln\left(\frac{0.112}{1+e^{(0.214476i)\varkappa}} - \frac{0.112}{(1+e^{(0.214476i)\varkappa})^2}\right), \tag{4.9}$$

$$\begin{aligned}
\mathsf{v}_1(\varkappa,\mathsf{t}) = {}&\mathsf{t}\frac{\left(-0.05+0.05e^{(0.214476i)\varkappa}\right)}{1+e^{(0.214476i)\varkappa}} + \frac{\mathsf{h}}{2}ln\left(\frac{0.112}{1+e^{(0.214476i)\varkappa}} - \frac{0.112}{(1+e^{(0.214476i)\varkappa})^2}\right)\\
&+\mathsf{h}\frac{\left(0.01e^{(0.214476i)\varkappa} + (5.1070256)^{-18}e^{(0.428952i)\varkappa} + (2.5535128)^{-18}\right)}{(2e^{(0.214476i)\varkappa} + e^{(0.428952i)\varkappa} + 1\,)}\frac{\mathsf{t}^{2\wp}}{\Gamma(2\wp+1)},
\end{aligned} \tag{4.10}$$

$$\begin{aligned}\mathsf{v}_2(\varkappa,\mathsf{t}) = \mathsf{v}_1(\varkappa,\mathsf{t})(\mathsf{h}) + \mathsf{h}\Bigg(& -\frac{\left((0.00115\,i)\mathsf{h}e^{(0.428952i)\varkappa}\right)}{\left((1+e^{(0.214476i)\varkappa})^6\left(\frac{0.112}{1+e^{(0.214476i)\varkappa}}-\frac{0.112}{\left(1+e^{(0.214476i)\varkappa}\right)^2}\right)^2\right)}\frac{\mathsf{t}^{2\wp}}{\Gamma(2\wp+1)} \\ & + \frac{(0.001154)\mathsf{h}e^{(0.428952i)\varkappa}}{(1+e^{(0.214476i)\varkappa})^5\left(\frac{0.112}{1+e^{(0.214476i)\varkappa}}-\frac{0.112}{\left(1+e^{(0.214476i)\varkappa}\right)^2}\right)^2}\frac{\mathsf{t}^{2\wp}}{\Gamma(2\wp+1)} + \ldots \\ & + \frac{\left((9.3969)^{-19}\mathsf{h}e^{(0.428952i)\varkappa}\right)}{\left((2.e^{(0.214476i)\varkappa}+e^{(0.428952i)\varkappa}+1.)^3\right)}\frac{\mathsf{t}^{4\wp}}{\Gamma(4\wp+1)} + \ldots \\ & + \frac{\left(0.00112\mathsf{h}e^{(0.214476i)\varkappa}\right)}{((1+e^{(0.214476i)\varkappa})\,(2.e^{(0.214476i)\varkappa}+e^{(0.428952i)\varkappa}+1\,))}\frac{\mathsf{t}^{4\wp}}{\Gamma(4\wp+1)} + \ldots\Bigg). \end{aligned} \tag{4.11}$$

We can continue in the same manner to get high accuracy, but due to huge calculations, we stop at evaluating three terms approximate series in the form:

$$\mathsf{v}(\varkappa,\mathsf{t}) = \mathsf{v}_0(\varkappa,\mathsf{t}) + \mathsf{v}_1(\varkappa,\mathsf{t}) + \mathsf{v}_2(\varkappa,\mathsf{t}). \tag{4.12}$$

Table 1 represents a contrast between the analytical solution presented in [6] and the results obtained in the present study using HAM at different values of h for $\wp = 1$.

Table 1. A comparison between the analytical solution presented in [6] and the obtained solution using HAM for the q-deformed SGE at $\wp = 1$.

t	$\varkappa$	Analytical solution in [6]	HAM at $\mathsf{h} = -1$	Abs. error	HAM at $\mathsf{h} = -0.01$	Abs. error
	-10	1.04980	1.0506	7.99884×10^{-4}	1.04869	1.10328×10^{-3}
	-7	1.47479	1.47525	4.62934×10^{-4}	1.47329	1.49662×10^{-3}
	-4	1.69282	1.69316	3.39439×10^{-4}	1.69112	1.70621×10^{-3}
0.1	-1	1.78203	1.78232	2.97814×10^{-4}	1.78023	1.79286×10^{-3}
	2	1.76461	1.76492	3.05566×10^{-4}	1.76283	1.77591×10^{-3}
	5	1.63658	1.63695	3.68234×10^{-4}	1.63493	1.65181×10^{-3}
	8	1.36321	1.36374	5.39122×10^{-4}	1.36181	1.39098×10^{-3}
	-10	1.05008	1.05328	3.19955×10^{-3}	1.04882	1.26722×10^{-3}
	-7	1.47488	1.47673	1.85169×10^{-3}	1.47331	1.56669×10^{-3}
0.2	-4	1.69287	1.69423	1.35774×10^{-3}	1.69112	1.75151×10^{-3}
	-1	1.78207	1.78326	1.19125×10^{-3}	1.78023	1.83076×10^{-3}
	2	1.76465	1.76587	1.22226×10^{-3}	1.76284	1.81515×10^{-3}
	5	1.63664	1.63811	1.47292×10^{-3}	1.63493	1.70251×10^{-3}
	8	1.36333	1.36549	2.15644×10^{-3}	1.36185	1.47857×10^{-3}

Case II : For $\alpha = -1$, $\gamma = p = 1$ and $\varrho = \frac{1}{2}$.

Equation (4.1) will be in the form

$$\frac{\partial^{2\wp} v}{\partial t^{2\wp}} = \frac{\partial^2 v}{\partial \varkappa^2} + \frac{q}{2} e^{-2v}, \tag{4.13}$$

subject to:

$$v(\varkappa, 0) = \frac{1}{2} ln \left(\frac{1}{4} \left(25\sqrt{-q}(0.07\varkappa) + 1.4 \right)^2 \right), \quad v_t(\varkappa, 0) = -\frac{2.25\sqrt{-q}}{1.4 - 2.25\sqrt{-q}(-0.77778\varkappa)}.$$

By applying the same steps that presented briefly in case I, the mth order deformation equation will be in the form:

$$v_m(\varkappa, t) = \mathcal{X}_m v_{m-1}(\varkappa, t) + h\, £^{-1}\, NFP(v_{m-1}(\varkappa, t)), \tag{4.14}$$

where

$$NFP(v_{m-1}(\varkappa, t)) = D_t^{2\wp}\ v_{m-1}(\varkappa, t) - \frac{\partial^2 v_{m-1}}{\partial \varkappa^2} - \frac{q}{2} e^{-2v_{m-1}}. \tag{4.15}$$

By implying the AP into the nonlinear term, therefore, the recursive terms in the power series will be in the form,

$$\begin{aligned}
&v_0(\varkappa, t) = v(\varkappa, 0), \quad v_1(\varkappa, t) = t\, v_t(\varkappa, 0) + h\, J^{2\wp}\Big(D_t^{2\wp}\ v_0(\varkappa, t) - \frac{\partial^2 v_0}{\partial \varkappa^2} - \frac{q}{2} e^{-2v_0} \Big), \\
&v_2(\varkappa, t) = h\, J^{2\wp}\Big(D_t^{2\wp}\ v_1(\varkappa, t) - \frac{\partial^2 v_1}{\partial \varkappa^2} + q v_1 e^{-2v_0} \Big), \\
&v_3(\varkappa, t) = v_2(\varkappa, t) + h\, J^{2\wp}\Big(D_t^{2\wp}\ v_2(\varkappa, t) - \frac{\partial^2 v_2}{\partial \varkappa^2} - \frac{q}{2}(-2v_2 e^{-2v_0} + \frac{1}{2!} v_1^2\, 4e^{-2v_0}) \Big), \\
&\vdots
\end{aligned} \tag{4.16}$$

Apply the initial conditions to the recursive Eq (4.16):

$$v_0(\varkappa, t) = \frac{1}{2} ln \left(\frac{1}{4} \left(25\sqrt{-q}(0.07\varkappa) + 1.4 \right)^2 \right), \tag{4.17}$$

$$\begin{aligned}
v_1(\varkappa, t) = {} & t\Big(-\frac{2.25\sqrt{-q}}{1.4 - 2.25\sqrt{-q}(-0.77778\varkappa)} \Big) + \frac{h}{2} ln \left(\frac{1}{4} \left(25\sqrt{-q}(0.07\varkappa) + 1.4 \right)^2 \right) \\
& + h\Big(-\frac{5.0625q}{(1.75\sqrt{-q}\varkappa + 1.4)^2} \Big) \frac{t^{2\wp}}{\Gamma(2\wp + 1)},
\end{aligned} \tag{4.18}$$

$$\begin{aligned}
v_2(\varkappa, t) = {} & v_1(\varkappa, t)(h) + h\Big[\Big(\frac{2hq\, ln\left(\frac{1}{4}(1.75\sqrt{-q}\varkappa + 1.4)^2 \right)}{(1.75\sqrt{-q}\varkappa + 1.4)^2} - \frac{3.0625hq}{(1.75\sqrt{-q}\varkappa + 1.4)^2} \Big) \frac{t^{2\wp}}{\Gamma(2\wp + 1)} \\
& + \Big(-\frac{113.273hq^2}{(1.75\sqrt{-q}\varkappa + 1.4)^4} \Big) \frac{t^{4\wp}}{\Gamma(4\wp + 1)} \\
& + \Big(-\frac{9\, q\sqrt{-q}}{(1.75\sqrt{-q}\varkappa + 1.4)^2 (1.75\sqrt{-q}\varkappa + 1.4)} - \frac{13.7813q\sqrt{-q}}{(1.75\sqrt{-q}\varkappa + 1.4)^3} \Big) \frac{t^{2\wp + 1}}{\Gamma(2\wp + 2)} \Big],
\end{aligned} \tag{4.19}$$

$$
\begin{aligned}
\mathsf{v}_3(\varkappa,\mathsf{t}) =&\mathsf{v}_2(\varkappa,\mathsf{t})(1+\mathsf{h})+\mathsf{h}\Bigg[\Bigg(-\frac{\left(3.0625\mathsf{h}^2\mathsf{q}\right)}{(1.75\sqrt{-\mathsf{q}}\varkappa+1.4)^2}+\frac{2\mathsf{h}^2\mathsf{q}\,ln\left(\frac{1}{4}(1.75\sqrt{-\mathsf{q}}\varkappa+1.4)^2\right)}{(1.75\sqrt{-\mathsf{q}}\varkappa+1.4)^2}\\
&-\frac{\mathsf{h}^2\mathsf{q}^2\,ln\left(\frac{1}{4}(1.75\sqrt{-\mathsf{q}}\varkappa+1.4)^2\right)}{(1.75\sqrt{-\mathsf{q}}\varkappa+1.4)^2}\Bigg)\frac{\mathsf{t}^{2\wp}}{\Gamma(2\wp+1)}+\Bigg(-\frac{(13.7813\mathsf{h}\sqrt{-\mathsf{q}}\mathsf{q})}{(1.75\sqrt{-\mathsf{q}}\varkappa+1.4)^3}\\
&-\frac{9\mathsf{h}\sqrt{-\mathsf{q}}\mathsf{q}}{(1.75\sqrt{-\mathsf{q}}\varkappa+1.4)^2(1.75\sqrt{-\mathsf{q}}\varkappa+1.4)}\Bigg)\frac{\mathsf{t}^{2\wp+1}}{\Gamma(2\wp+2)}\\
&+\frac{20.25\mathsf{q}^2}{(1.75\sqrt{-\mathsf{q}}\varkappa+1.4)^2(1.75\sqrt{-\mathsf{q}}\varkappa+1.4)^2}\frac{2\mathsf{t}^{2\wp+2}}{\Gamma(2\wp+3)}\\
&-\frac{102.516\mathsf{h}^2\mathsf{q}^3}{\Gamma(2\wp+1)^2(1.75\sqrt{-\mathsf{q}}\varkappa+1.4)^6}\frac{\Gamma(4\wp+1)\mathsf{t}^{6\wp}}{\Gamma(6\wp+1)}\\
&-\frac{7391.09\mathsf{h}^2\mathsf{q}^3}{(1.75\sqrt{-\mathsf{q}}\varkappa+1.4)^6}\frac{\mathsf{t}^{6\wp}}{\Gamma(6\wp+1)}\\
&-\frac{91.125\mathsf{h}\sqrt{-\mathsf{q}}\mathsf{q}^2}{\Gamma(2\wp+1)(1.75\sqrt{-\mathsf{q}}\varkappa+1.4)^4(1.75\sqrt{-\mathsf{q}}\varkappa+1.4)}\frac{\Gamma(2\wp+2)\mathsf{t}^{4\wp+1}}{\Gamma(4\wp+2)}\\
&+\Bigg(-\frac{366.75\mathsf{h}\sqrt{-\mathsf{q}}\mathsf{q}^2}{(1.75\sqrt{-\mathsf{q}}\varkappa+1.4)^5}-\frac{506.467h\sqrt{-\mathsf{q}}\mathsf{q}^2}{(1.75\sqrt{-\mathsf{q}}\varkappa+1.4)^5}\\
&-\frac{55.1253\mathsf{h}\sqrt{-\mathsf{q}}\mathsf{q}^2}{(1.75\sqrt{-\mathsf{q}}\varkappa+1.4)^2(1.75\sqrt{-\mathsf{q}}\varkappa+1.4)^3}\Bigg)\frac{\mathsf{t}^{4\wp+1}}{\Gamma(4\wp+2)}\\
&+\Bigg(\frac{65\mathsf{h}^2\mathsf{q}^2\,ln\left(\frac{1}{4}(1.75\sqrt{-\mathsf{q}}\varkappa+1.4)^2\right)}{(1.75\sqrt{-\mathsf{q}}\varkappa+1.4)^4}\\
&-\frac{243.047\mathsf{h}^2\mathsf{q}^2}{(1.75\sqrt{-\mathsf{q}}\varkappa+1.4)^4}\Bigg)\frac{\mathsf{t}^{4\wp}}{\Gamma(4\wp+1)}\Bigg].
\end{aligned}
\tag{4.20}
$$

If we truncate at the term $\mathsf{v}_3(\varkappa,\mathsf{t})$, the approximate series solution will be in the form:

$$
\mathsf{v}(\varkappa,\mathsf{t})=\mathsf{v}_0(\varkappa,\mathsf{t})+\mathsf{v}_1(\varkappa,\mathsf{t})+\mathsf{v}_2(\varkappa,\mathsf{t})+\mathsf{v}_3(\varkappa,\mathsf{t}). \tag{4.21}
$$

Table 2 represents the results in [6] and the results that we obtained from solving q-deformed SGE under initial conditions defined in Case II using HAM at $\wp = 1$ and at different values of q. The results include evaluating the absolute error at each case.

From Table 2, we notice that, varying the deformable factor q leads to variation of the result obtained, it gives good accuracy as the value of q within the specific range $(0,1]$. For $\mathsf{q} > 1$, the quantum deformation affects the accuracy and lead to less accuracy.

Table 2. The results obtained in [6] and the result obtained in the present study for solving the q-deformed SGE under initial conditions defined in case II at $\wp = 1$, and $\mathsf{h} = -0.01$.

t	$\varkappa$	q = 0.04			q = 0.4			q = 3
		Exact sol. from [6]	Approx. sol. HAM	Abs. error	Exact sol. from [6]	Approx. sol. HAM	Abs. error	Abs. error
0.1	-10	1.35760	1.35632	1.2814×10^{-3}	2.25618	2.25401	2.1728×10^{-3}	3.11×10^{-2}
	-7	1.11831	1.11732	9.8348×10^{-4}	1.96797	1.96616	1.8069×10^{-3}	2.78×10^{-2}
	-4	0.80123	0.80070	5.3006×10^{-4}	1.54293	1.54184	1.0893×10^{-3}	2.27×10^{-3}
	2	0.50784	0.50773	1.1094×10^{-4}	1.00160	1.00205	4.4967×10^{-4}	1.57×10^{-2}
	5	0.88874	0.88804	7.0064×10^{-4}	1.66826	1.66686	1.3987×10^{-3}	2.48×10^{-2}
	8	1.18353	1.18243	1.0937×10^{-3}	2.04827	2.04632	1.9466×10^{-3}	2.90×10^{-2}
0.2	-10	1.36659	1.36551	1.0817×10^{-3}	2.26671	2.26475	1.9567×10^{-3}	3.08×10^{-2}
	-7	1.12992	1.12930	6.1825×10^{-4}	1.98238	1.98101	1.3735×10^{-3}	2.74×10^{-2}
	-4	0.81682	0.81705	2.2528×10^{-4}	1.56591	1.56608	1.7267×10^{-4}	2.14×10^{-2}
	2	0.49154	0.49269	1.1490×10^{-3}	0.95916	0.96425	5.0938×10^{-3}	9.91×10^{-3}
	5	0.87404	0.87395	8.4980×10^{-5}	1.64778	1.64726	5.1361×10^{-4}	2.38×10^{-2}
	8	1.17255	1.17177	7.8891×10^{-4}	2.03487	2.03327	1.5977×10^{-3}	2.87×10^{-2}

Case III: For $\alpha = 2$, $p = 2$, $\gamma = 1$ and $\varrho = (\frac{\mathsf{q}}{2})^2$.
Equation (4.1) transformed into:

$$\frac{\partial^{2\wp}\mathsf{v}}{\partial \mathsf{t}^{2\wp}} = \frac{\partial^2\mathsf{v}}{\partial\varkappa^2} - \frac{1}{4}e^{4\mathsf{v}} + \frac{1}{2}\mathsf{q}e^{2\mathsf{v}}, \tag{4.22}$$

subject to:

$$\mathsf{v}(\varkappa,0) = \frac{1}{2}\,ln\left(\frac{2\mathsf{q}}{e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}+1}\right),$$
$$\mathsf{v}_\mathsf{t}(\varkappa,0) = \frac{0.125\,e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}}{e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}+1}.$$

Following the steps of HAM to find the approximate series solution of Eq (4.2) under Case III of initial conditions, the mth order deformation equation will be in the form:

$$\mathsf{v}_\mathsf{m}(\varkappa,\mathsf{t}) = \mathcal{X}_\mathsf{m}\mathsf{v}_{\mathsf{m}-1}(\varkappa,\mathsf{t}) + \mathsf{h}\,\pounds^{-1}\,\mathsf{NFP}(\mathsf{v}_{\mathsf{m}-1}(\varkappa,\mathsf{t})), \tag{4.23}$$

where

$$\mathsf{NFP}(\mathsf{v}_{\mathsf{m}-1}(\varkappa,\mathsf{t})) = \mathsf{D}_\mathsf{t}^{2\wp}\,\mathsf{v}_{\mathsf{m}-1}(\varkappa,\mathsf{t}) - \frac{\partial^2\mathsf{v}_{\mathsf{m}-1}}{\partial\varkappa^2} + \frac{1}{4}e^{4\mathsf{v}_{\mathsf{m}-1}} - \frac{1}{2}\mathsf{q}e^{2\mathsf{v}_{\mathsf{m}-1}}. \tag{4.24}$$

By expanding Eqs (4.23) and (4.24), then apply the initial conditions to the recursive equation we obtain:

$$\mathsf{v}_0(\varkappa,\mathsf{t}) = \frac{1}{2}\,ln\left(\frac{2\mathsf{q}}{e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}+1}\right), \tag{4.25}$$

$$
\begin{aligned}
v_1(\varkappa,t) =& t\left(\frac{0.125\, e^{\sqrt{2q^2+0.0625}\varkappa}}{e^{\sqrt{2q^2+0.0625}\varkappa}+1}\right) + \frac{h}{2} ln\left(\frac{2q}{e^{\sqrt{2q^2+0.0625}\varkappa}+1}\right) \\
&+ h\Bigg[-\frac{q}{e^{\sqrt{2q^2+0.0625}\varkappa}+1} + \frac{0.03125 e^{\sqrt{2q^2+0.0625}\varkappa}}{e^{\sqrt{2q^2+0.0625}\varkappa}+1} \\
&- \frac{0.03125 e^{2\sqrt{2q^2+0.0625}\varkappa}}{\left(e^{\sqrt{2q^2+0.0625}\varkappa}+1\right)^2} \frac{q^2 e^{\sqrt{2q^2+0.0625}\varkappa}}{e^{\sqrt{2q^2+0.0625}\varkappa}+1} \\
&- \frac{q^2 e^{2\sqrt{2q^2+0.0625}\varkappa}}{\left(e^{\sqrt{2q^2+0.0625}\varkappa}+1\right)^2} + \frac{q^2}{\left(e^{\sqrt{2q^2+0.0625}\varkappa}+1\right)^2}\Bigg]\frac{t^{2\wp}}{\Gamma(2\wp+1)},
\end{aligned}
\tag{4.26}
$$

$$
\begin{aligned}
v_2(\varkappa,t) =& v_1(\varkappa,t)(h) + h\Bigg[\Bigg(-\frac{hq^2 e^{2\sqrt{2q^2+0.0625}\varkappa}}{\left(e^{\sqrt{2q^2+0.0625}\varkappa}+1\right)^2} + \frac{0.03125 h e^{\sqrt{2q^2+0.0625}\varkappa}}{e^{\sqrt{2q^2+0.0625}\varkappa}+1} \\
&- \frac{0.03125 h e^{2\sqrt{2q^2+0.0625}\varkappa}}{\left(e^{\sqrt{2q^2+0.0625}\varkappa}+1\right)^2} + \frac{hq^2 e^{\sqrt{2q^2+0.0625}\varkappa}}{e^{\sqrt{2q^2+0.0625}\varkappa}+1} + \frac{2hq^2\, ln\left(\frac{2q}{e^{\sqrt{2q^2+0.0625}\varkappa}+1}\right)}{\left(e^{\sqrt{2q^2+0.0625}\varkappa}+1\right)^2} \\
&- \frac{hq^2\, ln\left(\frac{2q}{e^{\sqrt{2q^2+0.0625}\varkappa}+1}\right)}{e^{\sqrt{2q^2+0.0625}\varkappa}+1}\Bigg)\frac{t^{2\wp}}{\Gamma(2\wp+1)} + \Bigg(\frac{0.0117188 h e^{4\sqrt{2q^2+0.0625}\varkappa}}{\left(e^{\sqrt{2q^2+0.0625}\varkappa}+1\right)^4} \\
&- \frac{0.0234375 h e^{3\sqrt{2q^2+0.0625}\varkappa}}{\left(e^{\sqrt{2q^2+0.0625}\varkappa}+1\right)^3} + \frac{0.0136719 h e^{2\sqrt{2q^2+0.0625}\varkappa}}{\left(e^{\sqrt{2q^2+0.0625}\varkappa}+1\right)^2} \\
&- \frac{0.00195313 h e^{\sqrt{2q^2+0.0625}\varkappa}}{e^{\sqrt{2q^2+0.0625}\varkappa}+1} + \frac{0.125 hq e^{2\sqrt{2q^2+0.0625}\varkappa}}{\left(e^{\sqrt{2q^2+0.0625}\varkappa}+1\right)^3} \\
&- \frac{0.0625 hq e^{\sqrt{2q^2+0.0625}\varkappa}}{\left(e^{\sqrt{2q^2+0.0625}\varkappa}+1\right)^2} + \frac{0.25 hq^2 e^{\sqrt{2q^2+0.0625}\varkappa}}{\left(e^{\sqrt{2q^2+0.0625}\varkappa}+1\right)^3} + \frac{0.75 hq^2 e^{4\sqrt{2q^2+0.0625}\varkappa}}{\left(e^{\sqrt{2q^2+0.0625}\varkappa}+1\right)^4} \\
&- \frac{0.5 hq^2 e^{2\sqrt{2q^2+0.0625}\varkappa}}{\left(e^{\sqrt{2q^2+0.0625}\varkappa}+1\right)^4} + \frac{0.0625 hq^2 e^{2\sqrt{2q^2+0.0625}\varkappa}}{\left(e^{\sqrt{2q^2+0.0625}\varkappa}+1\right)^3} - \frac{0.0625 hq^2 e^{\sqrt{2q^2+0.0625}\varkappa}}{\left(e^{\sqrt{2q^2+0.0625}\varkappa}+1\right)^2} \\
&- \frac{1.5 hq^2 e^{3\sqrt{2q^2+0.0625}\varkappa}}{\left(e^{\sqrt{2q^2+0.0625}\varkappa}+1\right)^3} - \frac{0.125 hq^2 e^{\sqrt{2q^2+0.0625}\varkappa}}{e^{\sqrt{2q^2+0.0625}\varkappa}+1} + \frac{0.875 hq^2 e^{2\sqrt{2q^2+0.0625}\varkappa}}{\left(e^{\sqrt{2q^2+0.0625}\varkappa}+1\right)^2} \\
&- \frac{4hq^3}{\left(e^{\sqrt{2q^2+0.0625}\varkappa}+1\right)^3} + \frac{2hq^3}{\left(e^{\sqrt{2q^2+0.0625}\varkappa}+1\right)^2}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{2\mathsf{h}\mathsf{q}^3e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}}{\left(e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}+1\right)^2}+\frac{4h\mathsf{q}^3e^{2\sqrt{2\mathsf{q}^2+0.0625}\varkappa}}{\left(e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}+1\right)^3}\\
&+\frac{12\mathsf{h}\mathsf{q}^4e^{4\sqrt{2\mathsf{q}^2+0.0625}\varkappa}}{\left(e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}+1\right)^4}+\frac{4\mathsf{h}\mathsf{q}^4}{\left(e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}+1\right)^4}-\frac{16\mathsf{h}\mathsf{q}^4e^{2\sqrt{2\mathsf{q}^2+0.0625}\varkappa}}{\left(e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}+1\right)^4}\\
&+\frac{8\mathsf{h}\mathsf{q}^4e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}}{\left(e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}+1\right)^3}+\frac{2\mathsf{h}\mathsf{q}^4e^{2\sqrt{2\mathsf{q}^2+0.0625}\varkappa}}{\left(e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}+1\right)^3}-\frac{2\mathsf{h}\mathsf{q}^4}{\left(e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}+1\right)^3}\\
&+\frac{14\mathsf{h}\mathsf{q}^4e^{2\sqrt{2\mathsf{q}^2+0.0625}\varkappa}}{\left(e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}+1\right)^2}-\frac{2\mathsf{h}\mathsf{q}^4e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}}{\left(e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}+1\right)^2}-\frac{24\mathsf{h}\mathsf{q}^4e^{3\sqrt{2\mathsf{q}^2+0.0625}\varkappa}}{\left(e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}+1\right)^3}\\
&-\frac{2\mathsf{h}\mathsf{q}^4e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}}{e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}+1}\Bigg)\frac{\mathsf{t}^{4\wp}}{\Gamma(4\wp+1)}+\Bigg(\frac{0.0234375e^{2\sqrt{2\mathsf{q}^2+0.0625}\varkappa}}{\left(e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}+1\right)^2}\\
&-\frac{0.015625e^{3\sqrt{2\mathsf{q}^2+0.0625}\varkappa}}{\left(e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}+1\right)^3}-\frac{0.0078125e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}}{e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}+1}\\
&+\frac{0.5\mathsf{q}^2e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}}{\left(e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}+1\right)^3}-\frac{0.5\mathsf{q}^2e^{3\sqrt{2\mathsf{q}^2+0.0625}\varkappa}}{\left(e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}+1\right)^3}-\frac{0.25\mathsf{q}^2e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}}{e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}+1}\\
&-\frac{0.25\mathsf{q}^2e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}}{\left(e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}+1\right)^2}+\frac{0.75\mathsf{q}^2e^{2\sqrt{2\mathsf{q}^2+0.0625}\varkappa}}{\left(e^{\sqrt{2\mathsf{q}^2+0.0625}\varkappa}+1\right)^2}\Bigg)\frac{\mathsf{t}^{2\wp+1}}{\Gamma(2\wp+2)}\Bigg].
\end{aligned}
\tag{4.27}
$$

If we truncate at the term $\mathsf{v}_2(\varkappa,\mathsf{t})$, the approximate series solution takes the form:

$$
\mathsf{v}(\varkappa,\mathsf{t})=\mathsf{v}_0(\varkappa,\mathsf{t})+\mathsf{v}_1(\varkappa,\mathsf{t})+\mathsf{v}_2(\varkappa,\mathsf{t}). \tag{4.28}
$$

Table 3 represents the results in [6] compared with our obtained result from solving the q-deformed SGE under initial conditions defined in case III using HAM at $\wp = 1$ and different values of q for $\mathsf{h} = -0.01$.

From Table 3, we observe that as the value of q changes, the results varies, but the accuracy remains acceptable as long as the value of q lies within the specified range $\mathsf{q} \in (0,1]$. When the value of q is increased, the quantum deformation effect becomes more prominent in the equation, this can lead to low accuracy in numerical solutions.

Table 3. A comparison between the results in [6] and our obtained results when solving the q-deformed SGE under initial conditions defined in Case III using HAM at $\wp = 1$ and different values of q for $\mathsf{h} = -0.01$.

		$\mathsf{q} = 0.2$			$\mathsf{q} = 0.7$			$\mathsf{q} = 3$
t	$\varkappa$	Exact sol. from [6]	Approx. sol. HAM	Abs. error	Exact sol. from [6]	Approx. solution HAM	Abs. error	Abs. error
0.1	-10	0.46920	0.46873	4.7291×10^{-4}	0.16821	0.16805	1.6699×10^{-4}	9.15×10^{-3}
	-7	0.49170	0.49127	5.0156×10^{-4}	0.16785	0.16768	1.6651×10^{-4}	9.15×10^{-3}
	-4	0.55570	0.55512	5.7879×10^{-4}	0.16009	0.15993	1.5638×10^{-4}	9.15×10^{-3}
	2	1.01981	1.01875	1.0530×10^{-3}	0.90274	0.90183	9.1791×10^{-4}	3.30×10^{-2}
	5	1.46155	1.46007	1.4781×10^{-3}	2.37494	2.37256	2.3735×10^{-3}	9.61×10^{-2}
	8	1.98003	1.98804	1.9850×10^{-3}	3.90352	3.89963	3.8996×10^{-3}	1.59×10^{-1}
0.2	-10	0.46893	0.46845	4.8505×10^{-4}	0.16821	0.16805	1.6382×10^{-4}	1.00×10^{-2}
	-7	0.49090	0.49037	5.3154×10^{-4}	0.16786	0.16769	1.6301×10^{-4}	1.00×10^{-2}
	-4	0.55351	0.55286	6.4703×10^{-4}	0.16029	0.16014	1.4600×10^{-4}	1.00×10^{-2}
	2	1.01140	1.01026	1.1478×10^{-3}	0.89173	0.89077	9.5553×10^{-4}	3.29×10^{-2}
	5	1.44075	1.44923	1.5218×10^{-3}	2.36251	2.36015	2.3639×10^{-3}	9.60×10^{-2}
	8	1.96813	1.96613	1.9943×10^{-3}	3.89103	3.88714	3.8872×10^{-3}	1.59×10^{-1}

5. Visual representations

Graphical representations, whether in two and three dimensions, provide a pioneering visualization of the behavior of the model under investigation. These charts facilitate a straightforward contrast between the precise solution and the approximated solution, allowing researchers to evaluate the precision of the numerical technique used to produce the estimated solution. In this study,several graphs were presented in the sake of solving the q-deformed SGE based on the initial conditions imposed on the model. Figure 1 represents the 2D representation of the solution of the proposed model under initial conditions presented in Case I. Figure 1(a) is the two-dimensions visualization once at various fractional order parameter values $\wp$ with fixed time $\mathsf{t} = 5$, and the other at $\wp = 1$ for various steps of time. Figure 1(b) represents the three-dimensions approximate and exact profile for the solution. Figure 2 shows the two and three dimensions of the solution of the model under investigation using initial conditions in Case II, in this case, the solution depends on the deformation parameter q, so all shapes in Figure 2 are drawn at $\mathsf{q} = 0.4$. As long as the solution depends on the value of q, Figure 3 represents the solution at different values of q with fixed t, h and $\wp$. It is clear from Figure 3 that, as q increased, the q-deformed SGE exhibits a more pronounced quantum deformation effect, this can lead to changes in the shape of the wave solutions, often resulting in sharper and narrower solitons. Figure 4 represents the solution of the proposed model under initial guess in Case II but at small value of q, the soliton waves are very smooth which confirms the conclusion in Figure 3. Figure 5 represents graphs of two and three dimensions for the q-deformed SGE under initial conditions presented in Case III, all graphs are presented at $\mathsf{q} = 0.2$. To notice again the effect of the deformed parameter on the solution, we present Figure 6 at distinct values of q and fixed values of the other parameters t, h and $\wp$.

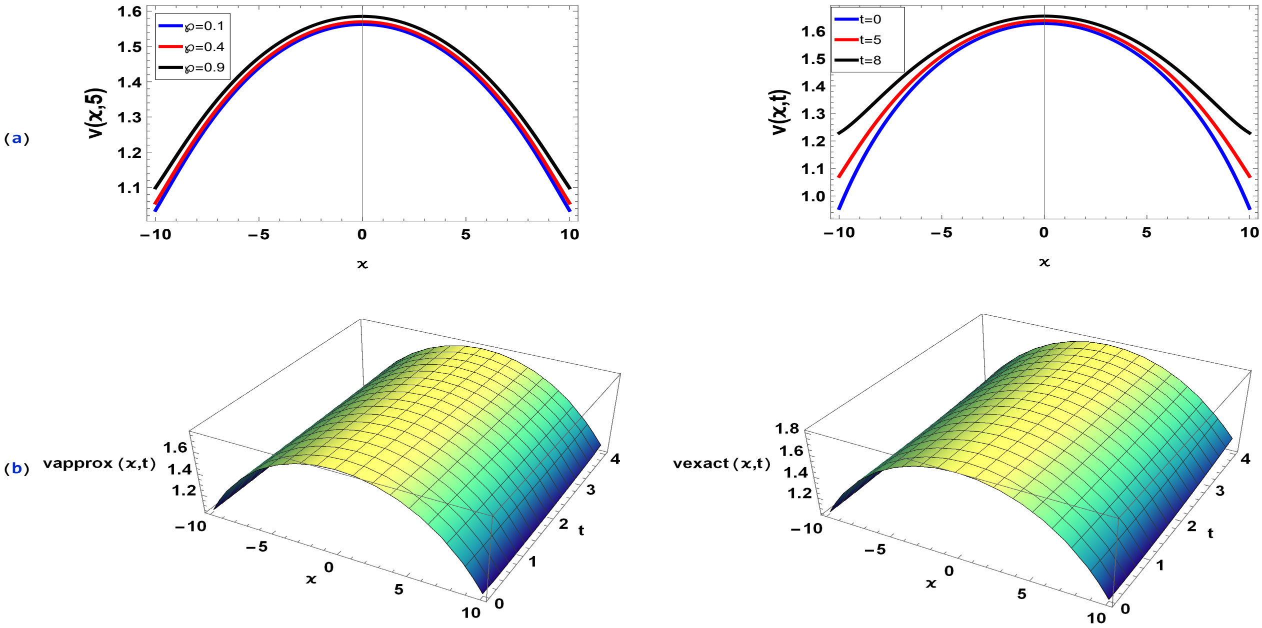

Figure 1. The approximate solution of time fractional q-deformed Sinh-Gordon equation presented in Eq (4.12) at $\mathsf{h} = -0.1$.

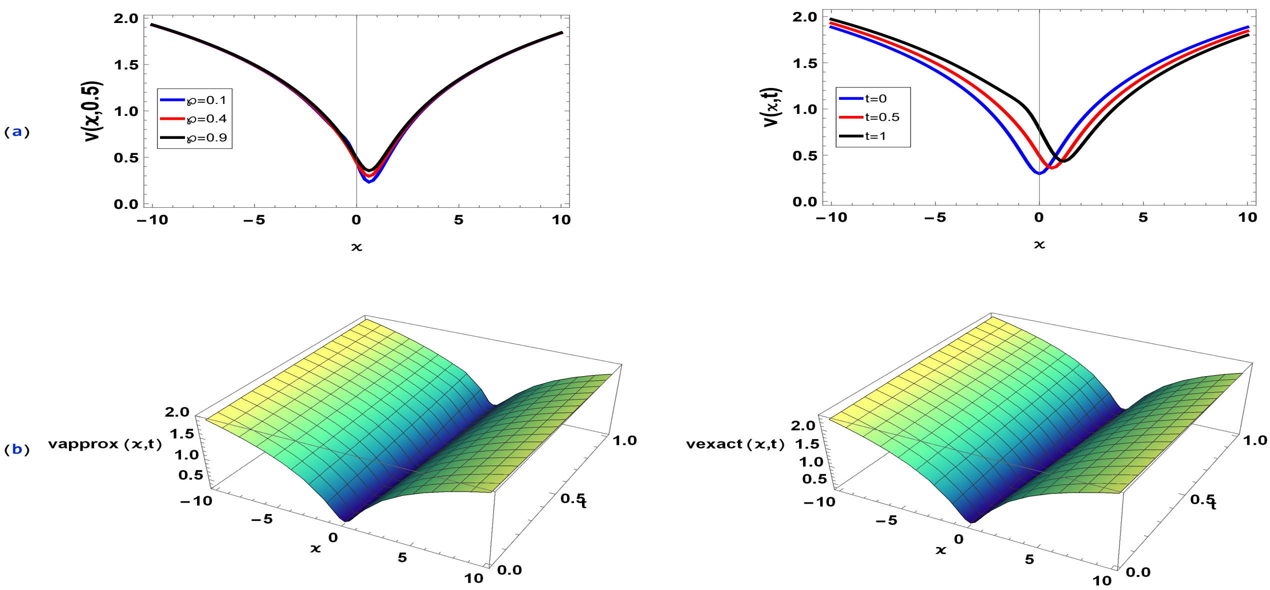

Figure 2. The estimated series solution of time fractional q-deformed SGE presented in Eq (4.21) at $\mathsf{h} = -0.2$ and $\mathsf{q} = 0.4$.

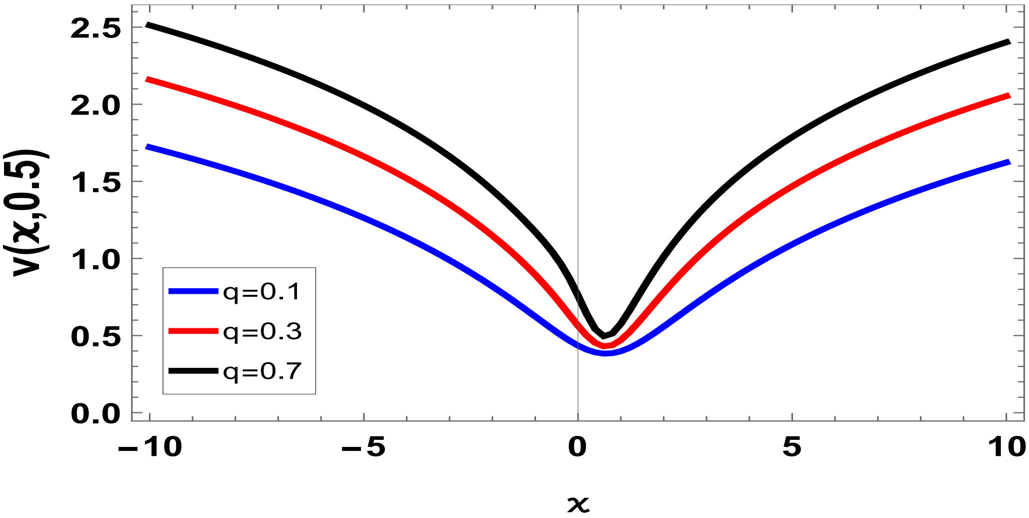

Figure 3. The estimated series solution of time fractional q-deformed SGE presented in Eq (4.21) at different values of q, for $\wp = 1$, $h = -0.01$ and $t = 0.5$.

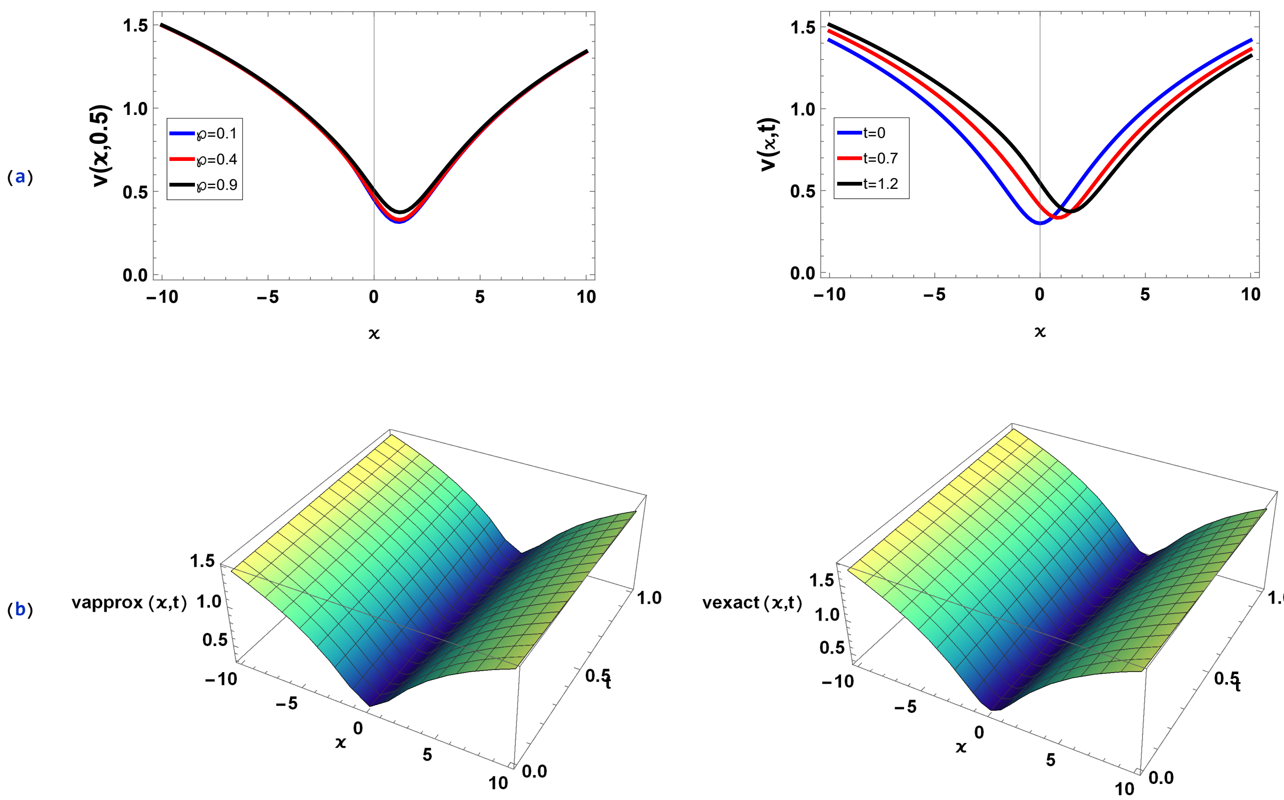

Figure 4. The estimated series solution of time fractional q-deformed SGE presented in Eq (4.21) at $h = -0.2$ and $q = 0.1$.

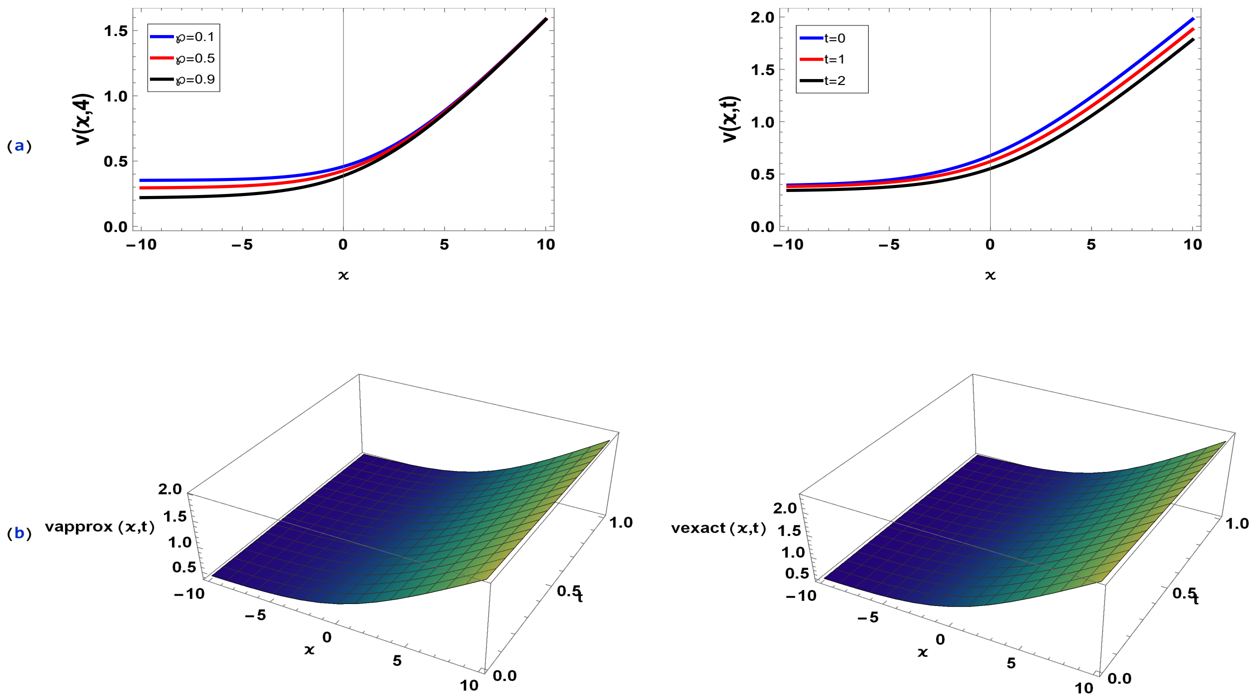

Figure 5. The estimated series solution of time fractional q-deformed SGE presented in Eq (4.28) at $\mathsf{h} = -0.2$ and $\mathsf{q} = 0.2$.

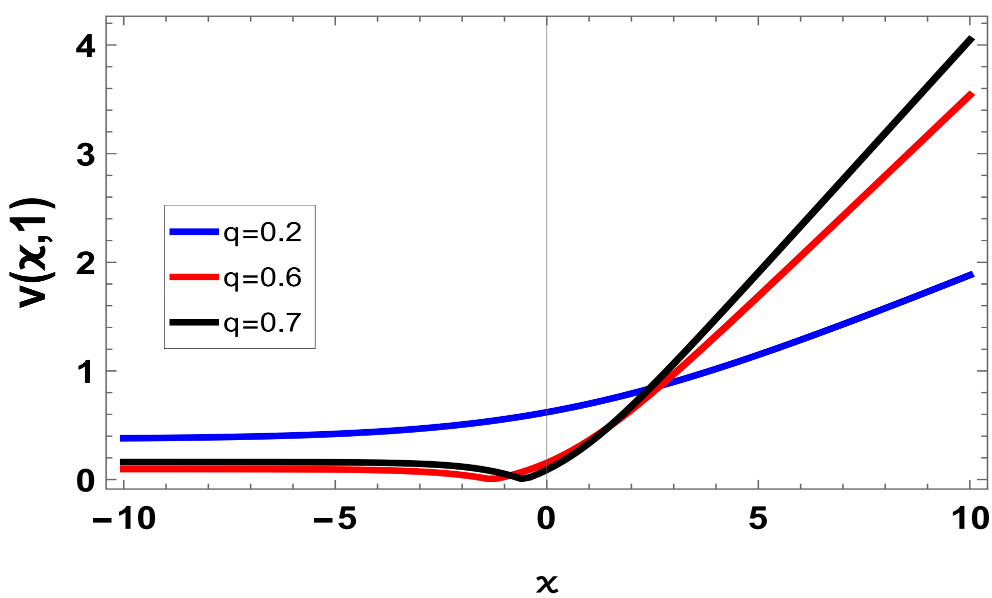

Figure 6. The estimated series solution of time fractional q-deformed SGE presented in Eq (4.28) at different values of q, for $\wp = 1$, $\mathsf{h} = -0.2$ and $\mathsf{t} = 1$.

6. Conclusions

Our study presents a significant step towards understanding the solutions of the time fractional q-deformed SGE using the HAM method. The HAM provided a robust and efficient approach for obtaining approximate solutions, which were validated through numerical simulations and graphical illustrations. We successfully derived approximate solutions for the proposed equation, the accuracy and effectiveness of the HAM were assessed through error estimations presented in Tables 1–3. Also, we examined the effects of time and the fractional order derivative, shedding light on their roles in shaping the dynamics of the system. Through our analysis, we observed that the deformation parameter q plays a pivotal role in shaping the wave solutions. Varying q led to distinct changes in the behavior of the model, influencing the amplitude, width, and localization of the solitons. Moreover, the initial conditions significantly impacted the dynamics of the solutions. To demonstrate the efficiency of the proposed solution method, a comparison was made between the results obtained through our approach and the results obtained analytically in one of the research studies presented in the literature, the comparison results illustrated the convergence of the solutions we obtained with the exact solutions under the same parameters.

As for future work, there are several promising directions to explore. Firstly, investigating other numerical techniques to validate the obtained solutions would enhance the robustness and accuracy of the results. Additionally, exploring different forms of the q-deformed SGE, such as higher-order fractional derivatives or coupling with other nonlinear equations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Authors' Contribution

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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