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Research article

Improved results for testing the oscillation of functional differential equations with multiple delays

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Abstract: In this article, we test whether solutions of second-order delay functional differential equations oscillate. The considered equation is a general case of several important equations, such as the linear, half-linear, and Emden-Fowler equations. We can construct strict criteria by inferring new qualities from the positive solutions to the problem under study. Furthermore, we can incrementally enhance these characteristics. We can use the criteria more than once if they are unsuccessful the first time thanks to their iterative nature. Sharp criteria were obtained with only one condition that guarantees the oscillation of the equation in the canonical and noncanonical forms. Our oscillation results effectively extend, complete, and simplify several related ones in the literature. An example was given to show the significance of the main results.

Keywords: functional differential equations; delay; second-order; oscillation; differential equations with multiple delays
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1. Introduction

Currently, the oscillation theory of differential equations with delay arguments, which is known as delay differential equations (DDEs), is a very active research area. This is because DDEs cover a wider field of applications than ordinary differential equations. For example, we find that the importance of this type of differential equation is evident when interpreting most of the mathematical models used to predict and analyze many scientific phenomena in life, such as dynamic systems, neural network models, electrical engineering, and epidemiology [1–3]. In epidemiology, we find that the DDEs are used to determine the time required for cell infection and the production of new viruses, as well as the period of infection, and the stages of the virus life cycle (see [4]). On the other hand, second-order DDEs are the most prevalent and visible, as they can be used to explain many phenomena in biology, physics, and engineering by mathematically modeling these phenomena. One of these models is the voltage control model of oscillating neurons in neuroengineering, see [5, 6]. We also refer the reader to the works [7, 8] for models from biological mathematics in which oscillation and/or delay actions can be expressed using cross-diffusion terms.

As it is widely known to most researchers in this field, the credit for the emergence of the theory of oscillation of differential equations is due to Sturm [9], in 1836, when he invented his famous method for deducing the oscillatory properties of solutions of a particular differential equation from those known for another equation. Then, Kneser [10] completed the work in this field in 1893 and deduced the types of solutions known by his name until now. In 1921, Fite [11] provided the first results that included the oscillation of differential equations with deviating arguments. Since then, a significant amount of research has been done to improve the field of knowledge. We suggest the monographs written by Agarwal et al. [12–14], Dosly and Rehak [15], and Gyori and Ladas [16] for an overview of the most important contributions.

In this work, we study the oscillation of solutions of the second-order quasi-linear DDE with several delays

$$\left(\varrho(\ell)\left(y'(\ell)\right)^{\alpha}\right)' + \sum_{i=1}^{m} \rho_i(\ell)\, y^{\beta}\left(\varphi_i(\ell)\right) = 0, \tag{1.1}$$

where $\ell \geq \ell_0$, and we also assume the following:

(H$_1$) α and β are quotients of odd positive integers;

(H$_2$) $\varrho \in C^1([\ell_0, \infty), (0, \infty))$ and

$$\pi(\ell) := \int_{\ell}^{\infty} \varrho^{-1/\alpha}(s)\, \mathrm{d}s,$$

with $\pi(\ell_0) < \infty$;

(H$_3$) $\rho_i \in C([\ell_0, \infty), (0, \infty))$, $\varphi_i \in C^1([\ell_0, \infty), (0, \infty))$, $\varphi_i(\ell) \leq \ell$, $\varphi_i'(\ell) \geq 0$, and $\lim_{\ell \to \infty} \varphi_i(\ell) = \infty$, for all $i = 1, 2, ..., m$.

For the solution of (1.1), we consider a real-valued function $y \in C([\ell_*, \infty), \mathbb{R})$, $\ell_* \geq \ell_0$ with the properties that $\varrho \cdot (y')^{\alpha}$ is differentiable and satisfies (1.1) on $[\ell_*, \infty)$. We consider only those solutions y of (1.1) which satisfy $\sup\{|y(\ell)| : \ell \geq \ell_*\} > 0$ for all $\ell \geq \ell_*$. A solution y of (1.1) is said to be oscillatory if it has arbitrarily large zeros. Otherwise, it is called nonoscillatory.

In this paper, we first review some studies that contributed to the development of the oscillation theory of second-order differential equations. Then, we provide new monotonic properties for the positive solutions and use them to obtain the new sharpest oscillatory criteria for (1.1). Finally, we state basic oscillation theorems that achieve the objective of the paper and an example to illustrate the importance of the study results.

2. Literature review

Among the recent contributions that had an impact on the improvement of the oscillation criteria of second-order non-canonical delay differential equations are those of Dzurina and Jadlovska [17–19]. This improvement is reflected in the neutral differential equations, which we see in the works [20–24]. On the other hand, for canonical neutral equations, Jadlovska [25], Moaaz et al. [26], and Li and Rogovchenko [27, 28] developed improved criteria to ensure the oscillation of neutral differential equations. When the rate of development depends on both the present and the future of a phenomenon, we can model it using advanced differential equations. Agarwal et al. [29], Chatzarakis et al. [30, 31], and Hassan [32] studied the oscillatory behavior of solutions to chapters of advanced second-order differential equations.

The development of approaches, techniques or criteria for studying the oscillation of second-order DDEs influences the study of the oscillation of equations of higher order, especially even-order, see for example, Li and Rogovchenko [33, 34] and Moaaz et al. [35–37].

In 2000, Koplataze et al. [38] studied the oscillation of the second-order linear DDE

$$y''(\ell) + \rho(\ell) y(\varphi(\ell)) = 0, \tag{2.1}$$

and proved that one of the following conditions is sufficient to ensure the oscillation of the solutions of (2.1):

$$\liminf_{\ell\to\infty} \int_{\varphi(\ell)}^{\ell} \left(\varphi(s) + \int_{\ell_0}^{\varphi(s)} \xi\varphi(\xi)\rho(\xi)\,\mathrm{d}\xi \right) \rho(s)\,\mathrm{d}s > \frac{1}{e} \tag{2.2}$$

or

$$\limsup_{\ell\to\infty} \int_{\varphi(\ell)}^{\ell} \left(\varphi(s) + \int_{\ell_0}^{\varphi(s)} \xi\varphi(\xi)\rho(\xi)\,\mathrm{d}\xi \right) \rho(s)\,\mathrm{d}s > 1.$$

These results are considered improvements on the results of Koplatadze [39] and Wei [40].

Chatzarakis and Jadlovska [41] considered a more general equation, the second-order half-linear DDE

$$\left(\varrho(\ell) (y'(\ell))^{\alpha} \right)' + \rho(\ell) y^{\alpha}(\varphi(\ell)) = 0, \tag{2.3}$$

in the canonical form, where $R(\ell) = \int_{\ell_0}^{\ell} \varrho^{-1/\alpha}(s)\,\mathrm{d}s \to \infty$ as $\ell \to \infty$. They extended and improved the results of Koplatadze et al. [38] by introducing a new sequence of constants $\{\gamma_k\}$ as follows: $\gamma_n \in (0, 1/e]$,

$$\gamma_1 = \liminf_{\ell\to\infty} \int_{\varphi(\ell)}^{\ell} \rho(s) R^{\alpha}(\varphi(s))\,\mathrm{d}s,$$

and

$$\gamma_{n+1} = \liminf_{\ell\to\infty} \int_{\varphi(\ell)}^{\ell} \rho(s) R_n^{\alpha}(\varphi(s))\,\mathrm{d}s,\ n \geq 1$$

where

$$R_n(\ell) = R(\ell) + \frac{\varsigma(\gamma_n)}{\alpha} \int_{\varphi(\ell)}^{\ell} R(s) R^{\alpha}(\varphi(s)) \rho(s)\,\mathrm{d}s$$

and $\varsigma(\theta)$ is the smallest positive root of the transcendental equation $\varsigma = e^{\theta\varsigma}$, $0 < \theta < 1/e$. And obtained that equation (2.3) is oscillatory if

$$\gamma_n > 1, \tag{2.4}$$

for some $n \in \mathbb{N}$. Note that, in the linear case at $\alpha = 1$, the previous criterion reduces to (2.2) at $n = 2$. Dzurina and Jadlovska [17] studied the same equation but with another approach in an attempt to improve the previous results. The criterion in [17] is considered as a simplified version of the works of Marik [42].

In 2019, Dzurina et al. [18] improved condition (2.4) by presenting new criteria for oscillation of (2.3). They proved that if

$$\liminf_{\ell\to\infty} \frac{1}{\pi(\ell)} \left(\int_{\ell_0}^{\ell} \pi^{\alpha+1}(s) \rho(s) \, \mathrm{d}s \right)^{1/\alpha} > \alpha,$$

then (2.3) is oscillatory.

Very recently, there was a research area about how to get sharper results for the oscillation of (2.3). Accordingly, Dzurina [19] established the following sharpest oscillation result for (2.3):

$$\liminf_{\ell\to\infty} \varrho^{1/\alpha}(\ell) \pi^{\alpha+1}(\ell) \rho(\ell) > \max\{c(\nu) : 0 < \nu < 1\}, \tag{2.5}$$

where $c(\nu) = \alpha\nu^{\alpha}(1-\nu)\lambda_*^{-\alpha\nu}$ and

$$\lambda_* = \liminf_{\ell\to\infty} \frac{\pi(\varphi(\ell))}{\pi(\ell)} < \infty.$$

Chatzarakis et al. [43] generalized the works of Dzurina [19] and studied the canonical form for the Euler-type half-linear differential equation with several delays

$$\left(\varrho(\ell)\left(y'(\ell)\right)^{\alpha}\right)' + \sum_{i=1}^{m} \rho_i(\ell) y^{\alpha}(\varphi_i(\ell)) = 0. \tag{2.6}$$

According to the results in [43], the oscillation of (2.6) is guaranteed according to the condition

$$\frac{1}{\alpha} \liminf_{\ell\to\infty} \varrho^{1/\alpha}(\ell) R(\varphi_i(\ell)) R(\ell) \rho_i(\ell) > 0$$

for $i = 1, 2, ..., m$,

$$R(\ell) = \int_{\ell_0}^{\ell} \frac{1}{\varrho^{1/\alpha}(s)} \mathrm{d}s,$$

and

$$\lambda_{*i} = \liminf_{\ell\to\infty} \frac{R(\ell)}{R(\varphi_i(\ell))} = \infty.$$

We note that criteria (2.2) and (2.4) need to ensure that the function $\varphi(\ell)$ is nondecreasing.

By comparing these results obtained in the previous literature with our results, we find that what distinguishes this paper is:

1. Different exponents of the first and second terms of the studied differential equation α and β affect our results and give them a wider field of application.
2. Our results work in the case of multiple delay arguments ($\varphi_i(\ell)$, $i = 1, 2, ..., m$), which do not require bounded conditions for these arguments.
3. The generality of our results allows us to apply them to a variety of differential equations, including ordinary ($\varphi_i(\ell) = \ell$), linear ($\alpha = \beta = 1$), half-linear ($\alpha = \beta$), and Emden-Fowler equations.

3. Main results

This section presents and proves some introductory lemmas that are required to conclude the main results that achieve the objectives of this paper. First of all, let us define the following notations for convenience: The class $\mathfrak{S}$ stands for all positive decreasing solutions of (1.1) for sufficiently large ℓ,

$$\varphi(\ell) = \max\{\varphi_i(\ell),\ i = 1, 2, ..., m\}, \tag{3.1}$$

and

$$\Omega(\ell) = \begin{cases} k_1 & \text{if } \beta < \alpha; \\ 1 & \text{if } \beta = \alpha; \\ k_2 \pi^{\beta-\alpha}(\ell) & \text{if } \beta > \alpha, \end{cases} \tag{3.2}$$

where k_1 and k_2 are any positive constants. The approach taken in this study is based on the assumption that there are μ_* and λ_* which are defined by

$$\mu_* = \frac{1}{\alpha} \liminf_{\ell\to\infty} \varrho^{1/\alpha}(\ell)\ \pi^{\alpha+1}(\ell)\,\Omega(\ell) \sum_{i=1}^{m} \rho_i(\ell) \tag{3.3}$$

and

$$\lambda_* = \liminf_{\ell\to\infty} \frac{\pi(\varphi(\ell))}{\pi(\ell)}.$$

Furthermore, for arbitrary fixed μ and λ, there exists $\ell_1 \geq \ell_0$, such that

$$\mu \leq \frac{1}{\alpha} \varrho^{1/\alpha}(\ell)\,\pi^{\alpha+1}(\ell)\,\Omega(\ell) \sum_{i=1}^{m} \rho_i(\ell) \tag{3.4}$$

for $0 < \mu < \mu_*$ and

$$\lambda \leq \frac{\pi(\varphi(\ell))}{\pi(\ell)}, \tag{3.5}$$

for $1 \leq \lambda \leq \lambda_*$, eventually.

Lemma 3.1. *Assume that $y \in \mathfrak{S}$. Then*

$$y^{\beta-\alpha}(\ell) \geq \Omega(\ell).$$

Proof. Let $y \in \mathfrak{S}$. Then there are three possibilities:
(1) $\beta < \alpha$: from the nonincreasing monotonicity of $y(\ell)$, it is easy to see that there exists a positive constant C_1, such that

$$y(\ell) \leq C_1,$$

which implies that

$$y^{\beta-\alpha}(\ell) \geq C_1^{\beta-\alpha} = k_1;$$

(2) $\beta = \alpha$: it is obvious that

$$y^{\beta}(\ell) = y^{\alpha}(\ell),$$

then

$$y^{\beta-\alpha}(\ell) = 1;$$

(3) $\beta > \alpha$: it is clear from the decreasing monotonicity of $\varrho\,(y')^{\alpha}$ that there is a positive constant C_2, such that

$$\varrho\,(\ell)\,(y'\,(\ell))^{\alpha} \leq -C_2 < 0,$$

then

$$y'\,(\ell) \leq \left(\frac{-C_2}{\varrho\,(\ell)}\right)^{1/\alpha}.$$

By integrating the above inequality from ℓ to ∞, we get

$$-y\,(\ell) \leq -C_2^{1/\alpha}\pi\,(\ell)\,.$$

Hence, we conclude that

$$y^{\beta-\alpha}\,(\ell) \geq C_2^{\frac{\beta}{\alpha}-1}\pi^{\beta-\alpha}\,(\ell) = k_2\pi^{\beta-\alpha}\,(\ell)\,.$$

As a result, we get $y^{\beta-\alpha}\,(\ell) \geq \Omega\,(\ell)$. This completes the proof. □

Lemma 3.2. *Assume that*

$$\int_{\ell_1}^{\infty} \frac{1}{\varrho^{1/\alpha}\,(u)} \left[\int_{\ell_1}^{u} \Omega\,(s) \sum_{i=1}^{m} \rho_i\,(s)\,\mathrm{d}s\right]^{1/\alpha} \mathrm{d}u = \infty, \tag{3.6}$$

holds. Then, for (1.1), each solution y oscillates or tends to zero.

Proof. Contrarily, assume that y is a positive solution of (1.1) for sufficiently large ℓ. From (1.1), we obtain

$$(\varrho\,(\ell)\,(y'\,(\ell))^{\alpha})' = -\sum_{i=1}^{m} \rho_i\,(\ell)\,y^{\beta}\,(\varphi_i\,(\ell)) \leq 0.$$

So, the function y' has a fixed sign, which means that it is eventually either greater than or less than zero. Firstly, let $y'\,(\ell) < 0$. Then, the decreasing monotonicity of $y\,(\ell)$ implies that there exists a non negative constant $C_3 \geq 0$ where $\lim_{\ell\to\infty} y\,(\ell) = C_3$. Assume on contrary that $y(\ell) \geq C_3 > 0$. But since

$$\begin{aligned}
(\varrho\,(\ell)\,(y'\,(\ell))^{\alpha})' &\leq -y^{\beta}\,(\varphi\,(\ell)) \sum_{i=1}^{m} \rho_i\,(\ell)\,,\\
&\leq -y^{\beta}\,(\ell) \sum_{i=1}^{m} \rho_i\,(\ell)\,,
\end{aligned}$$

since $\varphi\,(\ell)$ defined as in (3.1). Now the monotonicity of $y(\ell)$ implies that

$$\begin{aligned}
(\varrho\,(\ell)\,(y'\,(\ell))^{\alpha})' &\leq -y^{\alpha}\,(\ell)\,\Omega\,(\ell) \sum_{i=1}^{m} \rho_i\,(\ell)\\
&\leq -C_3\Omega\,(\ell) \sum_{i=1}^{m} \rho_i\,(\ell)\,.
\end{aligned}$$

Taking lim for both sides as $\ell \to \infty$ yields a contradiction, implying that $C_3 = 0$. Now, assume that $y'\,(\ell) > 0$. Let us define

$$Z\,(\ell) = \varrho\,(\ell)\,y^{-\beta}\,(\varphi_i\,(\ell))\,(y'\,(\ell))^{\alpha}\,. \tag{3.7}$$

Then, $Z(\ell) > 0$. Differentiating (3.7), we obtain

$$\begin{aligned} Z'(\ell) &= \frac{(\varrho(\ell)(y'(\ell))^{\alpha})'}{y^{\beta}(\varphi_i(\ell))} - \beta\varrho(\ell)\varphi_i'(\ell)\frac{(y'(\ell))^{\alpha}y'(\varphi_i(\ell))}{(y(\varphi_i(\ell)))^{\beta+1}} \\ &= -\sum_{i=1}^{m}\rho_i(\ell) - \beta\varphi_i'(\ell)Z(\ell)\frac{y'(\varphi_i(\ell))}{y(\varphi_i(\ell))} \le -\sum_{i=1}^{m}\rho_i(\ell). \end{aligned} \tag{3.8}$$

Integrating (3.8) from ℓ_1 to ℓ, we have

$$Z(\ell) \le Z(\ell_1) - \int_{\ell_1}^{\ell}\sum_{i=1}^{m}\rho_i(s)\,\mathrm{d}s.$$

However, (3.6) implies that $\int_{\ell_1}^{\ell}\sum_{i=1}^{m}\rho_i(\ell)\,\mathrm{d}s \to \infty$, which contradicts with our assumption that $Z(\ell) > 0$. As a result, the probability that $y'(\ell) > 0$ is impossible, and this completes the proof. □

Lemma 3.3. *Let $\mu_* > 0$. Assume that $y(\ell)$ is a positive solution of (1.1) for sufficiently large ℓ. Then*
(I) $y \in \mathfrak{S}$ *tends to zero as* $\ell \to \infty$;
(II) $(y/\pi)' \ge 0$, *eventually.*

Proof. From Lemma 3.2, we can deduce the proof of (I)-part if

$$\int_{\ell_1}^{\infty}\frac{1}{\varrho^{1/\alpha}(u)}\left[\int_{\ell_1}^{u}\Omega(s)\sum_{i=1}^{m}\rho_i(s)\,\mathrm{d}s\right]^{1/\alpha}\mathrm{d}u = \infty.$$

So, by using the condition (3.4), it is obvious that

$$\alpha\mu\varrho^{-1/\alpha}(\ell)\pi^{-(\alpha+1)}(\ell) \le \Omega(\ell)\sum_{i=1}^{m}\rho_i(\ell). \tag{3.9}$$

Integrating (3.9) from ℓ_1 to u, we get

$$\int_{\ell_1}^{u}\Omega(s)\sum_{i=1}^{m}\rho_i(s)\,\mathrm{d}s \ge \mu\int_{\ell_1}^{u}\frac{\alpha}{\varrho^{1/\alpha}(s)\pi^{\alpha+1}(s)}\mathrm{d}s,$$

Integrating once more from ℓ_1 to ℓ, yields to

$$\begin{aligned} &\int_{\ell_1}^{\ell}\frac{1}{\varrho^{1/\alpha}(u)}\left[\int_{\ell_1}^{u}\Omega(s)\sum_{i=1}^{m}\rho_i(s)\,\mathrm{d}s\right]^{1/\alpha}\mathrm{d}u \\ &\qquad \ge \sqrt[\alpha]{\mu}\int_{\ell_1}^{\ell}\varrho^{-1/\alpha}(u)\left[\alpha\int_{\ell_1}^{u}\frac{1}{\varrho^{1/\alpha}(s)\pi^{(\alpha+1)}(s)}\mathrm{d}s\right]^{1/\alpha}\mathrm{d}u \\ &\qquad = \sqrt[\alpha]{\mu}\int_{\ell_1}^{\ell}\varrho^{-1/\alpha}(u)\left[\frac{1}{\pi^{\alpha}(u)} - \frac{1}{\pi^{\alpha}(\ell_1)}\right]^{1/\alpha}\mathrm{d}u. \end{aligned}$$

From (H_2), we can conclude that the function $\pi^{-\alpha}(\ell)$ is infinite, i.e., $\lim_{\ell\to\infty}\pi^{-\alpha}(\ell) = \infty$. So, for any constant $C_4 \in (0,1)$, there is

$$\pi^{-\alpha}(\ell) - \pi^{\alpha}(\ell_1) \ge C_4^{\alpha}\pi^{-\alpha}(\ell)$$

for sufficiently large ℓ. As a result

$$\begin{aligned}\int_{\ell_1}^{\ell}\frac{1}{\varrho^{1/\alpha}(u)}\left[\int_{\ell_1}^{u}\Omega(s)\sum_{i=1}^{m}\rho_i(s)\,\mathrm{d}s\right]^{1/\alpha}\mathrm{d}u &\geq C_4\sqrt[\alpha]{\mu}\int_{\ell_1}^{\ell}\varrho^{-1/\alpha}(u)\,\pi(u)\mathrm{d}u \\ &= \ln\left(\frac{\pi(\ell_1)}{\pi(\ell)}\right)^{C_4\sqrt[\alpha]{\mu}}\end{aligned}$$

which tends to ∞ as $\ell \to \infty$. Then, we obtain

$$\int_{\ell_1}^{\infty}\frac{1}{\varrho^{1/\alpha}(u)}\left[\int_{\ell_1}^{u}\Omega(s)\sum_{i=1}^{m}\rho_i(s)\,\mathrm{d}s\right]^{1/\alpha}\mathrm{d}u=\infty, \tag{3.10}$$

and this completes the proof of this part.
(II)-part is verified as follows, by using the monotonicity of $\varrho(\ell)(y'(\ell))^{\alpha}$, we have

$$\begin{aligned}y(\ell) &\geq -\int_{\ell}^{\infty}\varrho^{-1/\alpha}(s)\varrho^{1/\alpha}(s)\,y'(s)\,\mathrm{d}s \\ &\geq -\varrho^{1/\alpha}(\ell)\,y'(\ell)\int_{\ell}^{\infty}\varrho^{-1/\alpha}(s)\mathrm{d}s \\ &= -\varrho^{1/\alpha}(\ell)\,\pi(\ell)\,y'(\ell),\end{aligned}$$

i.e.,

$$\left(\frac{y(\ell)}{\pi(\ell)}\right)'=\frac{y'(\ell)\,\pi(\ell)+\varrho^{-1/\alpha}(\ell)\,y(\ell)}{\pi^2(\ell)}\geq 0.$$

The proof is complete. □

The result illustrated in (I)-part of Lemma 3.3 can be improved by defining a sequence $\{\mu_n\}$ as

$$\begin{aligned}\mu_0 &= \sqrt[\alpha]{\mu_*}, \\ \mu_n &= \frac{\mu_0\lambda_*^{\mu_{n-1}}}{\sqrt[\alpha]{1-\mu_{n-1}}},\end{aligned} \tag{3.11}$$

for any $n \in \mathbb{N}$. It is simple to conclude through induction that for every value of $n \in \mathbb{N}$, $\mu_j < 1$, and $j = 0, 1, 2, ..., n$, then there exists μ_{n+1} satisfies that

$$\mu_{n+1}=l_n\mu_n>\mu_n, \tag{3.12}$$

where l_n is defined by

$$l_0=\frac{\lambda_*^{\mu_0}}{\sqrt[\alpha]{1-\mu_0}},$$

and

$$l_{n+1}=\lambda_*^{\mu_0(l_n-1)}\sqrt[\alpha]{\frac{1-\mu_n}{1-l_n\mu_n}},$$

for any $n \in \mathbb{N}_0$.

Remark 3.4. *Since the definition of* λ_* *and* (H_2) *states that* $\lambda_* \geq 1$, *it is obvious that* $l_0 > 1$, *which implies that* $l_n > 1$ *too for any* $n \in \mathbb{N}_0$.

Theorem 3.5. *Let* (H_1)*-*(H_3), $\mu_* > 0$, *and* $\lambda_* < \infty$ *hold . If* $y \in \mathfrak{S}$, *then* $y(\ell)/\pi^{\mu_n}(\ell)$ *is eventually decreasing for any* $n \in \mathbb{N}_0$.

Proof. Let y be a positive solution of (1.1) and (3.4) holds for every $\ell \geq \ell_1$. By integrating (1.1) under the area (ℓ_1, ℓ), we obtain

$$-\varrho(\ell)(y'(\ell))^{\alpha} + \varrho(\ell_1)(y'(\ell_1))^{\alpha} = \int_{\ell_1}^{\ell} \sum_{i=1}^{m} \rho_i(s) y^{\beta}(\varphi_i(s))\, \mathrm{d}s. \tag{3.13}$$

The (I)-part of Lemma 3.3 and condition (3.1) imply that $y(\ell)$ is a positive decreasing function and hence $y(\ell) \leq y(\varphi(\ell))$. Consequently,

$$\begin{aligned} -\varrho(\ell)(y'(\ell))^{\alpha} &\geq -\varrho(\ell_1)(y'(\ell_1))^{\alpha} + \int_{\ell_1}^{\ell} y^{\beta}(\varphi(s)) \sum_{i=1}^{m} \rho_i(s)\, \mathrm{d}s \\ &\geq -\varrho(\ell_1)(y'(\ell_1))^{\alpha} + \int_{\ell_1}^{\ell} y^{\beta}(s) \sum_{i=1}^{m} \rho_i(s)\, \mathrm{d}s, \end{aligned} \tag{3.14}$$

via Lemma 3.1, we have

$$y^{\beta}(\ell) = y^{\beta-\alpha}(\ell) y^{\alpha}(\ell) \geq \Omega(\ell) y^{\alpha}(\ell). \tag{3.15}$$

Substituting from the previous inequality into (3.14), we obtain

$$\begin{aligned} -\varrho(\ell)(y'(\ell))^{\alpha} &\geq -\varrho(\ell_1)(y'(\ell_1))^{\alpha} + \int_{\ell_1}^{\ell} \Omega(s) y^{\alpha}(s) \sum_{i=1}^{m} \rho_i(s)\, \mathrm{d}s \\ &\geq -\varrho(\ell_1)(y'(\ell_1))^{\alpha} + y^{\alpha}(\ell) \int_{\ell_1}^{\ell} \Omega(s) \sum_{i=1}^{m} \rho_i(s)\, \mathrm{d}s. \end{aligned} \tag{3.16}$$

Using (3.4) in the above inequality, yields

$$-\varrho(\ell)(y'(\ell))^{\alpha} + \varrho(\ell_1)(y'(\ell_1))^{\alpha} \geq \mu\alpha y^{\alpha}(\ell) \int_{\ell_1}^{\ell} \frac{1}{\varrho^{1/\alpha}(s)\pi^{\alpha+1}(s)} \mathrm{d}s.$$

Now, by completing the integration, we obtain

$$\begin{aligned} -\varrho(\ell)(y'(\ell))^{\alpha} &\geq -\varrho(\ell_1)(y'(\ell_1))^{\alpha} + \mu y^{\alpha}(\ell)\left[\pi^{-\alpha}(\ell) - \pi^{-\alpha}(\ell_1)\right] \\ &\geq -\varrho(\ell_1)(y'(\ell_1))^{\alpha} + \mu\left(\frac{y(\ell)}{\pi(\ell)}\right)^{\alpha} - \mu\frac{y^{\alpha}(\ell)}{\pi^{\alpha}(\ell_1)}. \end{aligned}$$

Once more, (I)-part of Lemma 3.3 implies that $y(\ell)$ tends to zero as $\ell \to \infty$. Consequently, there exists $\ell_2 \in [\ell_1, \infty)$ where

$$-\varrho(\ell_1)(y'(\ell_1))^{\alpha} > \mu\frac{y^{\alpha}(\ell)}{\pi^{\alpha}(\ell_1)}.$$

And so,

$$-\varrho(\ell)(y'(\ell))^{\alpha} > \mu\left(\frac{y(\ell)}{\pi(\ell)}\right)^{\alpha}. \tag{3.17}$$

Then

$$-\varrho^{1/\alpha}(\ell)\, y'(\ell)\, \pi(\ell) > \sqrt[\alpha]{\mu} y(\ell) = \sigma_0 \mu_0 y(\ell),$$

where $\sigma_0 = \sqrt[\alpha]{\mu}/\mu_0$ stands for any constant in $(0, 1)$. Furthermore,

$$\begin{aligned}\left(\frac{y(\ell)}{\pi^{\sqrt[\alpha]{\mu}}(\ell)}\right)' &= \frac{y'(\ell)\, \pi^{\sqrt[\alpha]{\mu}}(\ell) + \sqrt[\alpha]{\mu}\varrho^{-1/\alpha}(\ell)\, \pi^{\sqrt[\alpha]{\mu}-1}(\ell)\, y(\ell)}{\pi^{2\sqrt[\alpha]{\mu}}(\ell)} \\ &= \frac{\pi^{\sqrt[\alpha]{\mu}-1}(\ell)}{\pi^{2\sqrt[\alpha]{\mu}}(\ell)}\left(y'(\ell)\, \pi(\ell) + \sqrt[\alpha]{\mu}\varrho^{-1/\alpha}(\ell)\, y(\ell)\right) \le 0,\end{aligned} \tag{3.18}$$

for any $\ell \ge \ell_2$. Using that $y/\pi^{\sqrt[\alpha]{\mu}}$ is decreasing and integrating (1.1) from ℓ_2 to ℓ, we get

$$-\varrho(\ell)\,(y'(\ell))^{\alpha} + \varrho(\ell_2)\,(y'(\ell_2))^{\alpha} = \int_{\ell_2}^{\ell} y^{\beta}(\varphi(s)) \sum_{i=1}^{m} \rho_i(s)\, \mathrm{d}s.$$

But

$$\begin{aligned}\int_{\ell_2}^{\ell} y^{\beta}(\varphi(s))\, \mathrm{d}s &\ge \left(\frac{y(\ell)}{\pi^{\sqrt[\alpha]{\mu}}(\ell)}\right)^{\alpha} \int_{\ell_2}^{\ell} \pi^{\alpha\sqrt[\alpha]{\mu}}(\varphi(s))\, \Omega(s)\, \mathrm{d}s \\ &= \left(\frac{y(\ell)}{\pi^{\sqrt[\alpha]{\mu}}(\ell)}\right)^{\alpha} \int_{\ell_2}^{\ell} \left(\frac{\pi(\varphi(s))}{\pi(s)}\right)^{\alpha\sqrt[\alpha]{\mu}} \pi^{\alpha\sqrt[\alpha]{\mu}}(s)\, \Omega(s)\, \mathrm{d}s.\end{aligned}$$

Then

$$\begin{aligned}-\varrho(\ell)\,(y'(\ell))^{\alpha} \ge\; & -\varrho(\ell_2)\,(y'(\ell_2))^{\alpha} \\ & + \left(\frac{y(\ell)}{\pi^{\sqrt[\alpha]{\mu}}(\ell)}\right)^{\alpha} \int_{\ell_2}^{\ell} \left(\frac{\pi(\varphi(s))}{\pi(s)}\right)^{\alpha\sqrt[\alpha]{\mu}} \pi^{\alpha\sqrt[\alpha]{\mu}}(s)\, \Omega(s) \sum_{i=1}^{m} \rho_i(s)\, \mathrm{d}s.\end{aligned}$$

As a result of (3.4), we get

$$\begin{aligned}-\varrho(\ell)\,(y'(\ell))^{\alpha} &\ge -\varrho(\ell_2)\,(y'(\ell_2))^{\alpha} \\ &\quad + \mu\left(\frac{y(\ell)}{\pi^{\sqrt[\alpha]{\mu}}(\ell)}\right)^{\alpha} \int_{\ell_2}^{\ell} \frac{\alpha\left(\frac{\pi(\varphi(s))}{\pi(s)}\right)^{\alpha\sqrt[\alpha]{\mu}}}{\varrho^{1/\alpha}(s)\, \pi^{\alpha+1-\alpha\sqrt[\alpha]{\mu}}(s)} \mathrm{d}s \\ &= -\varrho(\ell_2)\,(y'(\ell_2))^{\alpha} \\ &\quad + \frac{\mu\lambda^{\alpha\sqrt[\alpha]{\mu}}}{1 - \sqrt[\alpha]{\mu}}\left(\frac{y(\ell)}{\pi^{\sqrt[\alpha]{\mu}}(\ell)}\right)^{\alpha}\left[\frac{1}{\pi^{\alpha\left(1-\sqrt[\alpha]{\mu}\right)}(\ell)} - \frac{1}{\pi^{\alpha\left(1-\sqrt[\alpha]{\mu}\right)}(\ell_2)}\right].\end{aligned} \tag{3.19}$$

Next, we are going to prove that $y/\pi^{\sqrt[\alpha]{\mu}+\sigma}$ for any $\sigma > 0$. By using the fact that $\pi^{\sqrt[\alpha]{\mu}}(\ell)$ tends to zero as ℓ approaches infinity, which implies that there exists a positive constant

$$C_5 \in \left(\frac{\sqrt[\alpha]{1 - \sqrt[\alpha]{\mu}}}{\lambda^{\sqrt[\alpha]{\mu}}}, 1\right)$$

for $\ell_3 \geq \ell_2$ such that

$$\frac{1}{\pi^{\alpha\left(1-\sqrt[\alpha]{\mu}\right)}(\ell)} - \frac{1}{\pi^{\alpha\left(1-\sqrt[\alpha]{\mu}\right)}(\ell_2)} > \frac{C_5^{\alpha}}{\pi^{\alpha\left(1-\sqrt[\alpha]{\mu}\right)}(\ell)}.$$

Using the previous inequality in (3.19), provides

$$-\varrho(\ell)\left(y'(\ell)\right)^{\alpha} \geq \left(\frac{C_5\sqrt[\alpha]{\mu}\lambda^{\sqrt[\alpha]{\mu}}}{\sqrt[\alpha]{1-\sqrt[\alpha]{\mu}}}\frac{y(\ell)}{\pi(\ell)}\right)^{\alpha}$$

and so,

$$-\varrho^{1/\alpha}(\ell)\,\pi(\ell)y'(\ell) \geq \left(\sqrt[\alpha]{\mu}+\sigma\right)y(\ell), \tag{3.20}$$

where

$$\sigma = \sqrt[\alpha]{\mu}\left(\frac{C_5\,\lambda^{\sqrt[\alpha]{\mu}}}{\sqrt[\alpha]{1-\sqrt[\alpha]{\mu}}} - 1\right) > 0.$$

Therefore, from (3.20), we conclude that

$$\left(\frac{y(\ell)}{\pi^{\sqrt[\alpha]{\mu}+\sigma}(\ell)}\right)' \leq 0, \quad \ell \geq \ell_3,$$

i.e., $y(\ell)/\pi^{\sqrt[\alpha]{\mu}+\sigma}(\ell)$ is eventually decreasing. Thus, for $\ell \geq \ell_4$, $\ell_4 \in [\ell_3,\infty)$,

$$-\varrho(\ell_2)\left(y'(\ell_2)\right)^{\alpha} - \frac{\mu\lambda^{\alpha\sqrt[\alpha]{\mu}}}{1-\sqrt[\alpha]{\mu}}\left(\frac{y(\ell)}{\pi^{\sqrt[\alpha]{\mu}}(\ell)}\right)^{\alpha}\frac{1}{\pi^{\alpha\left(1-\sqrt[\alpha]{\mu}\right)}(\ell_2)} > 0.$$

Using the above inequality and returning to (3.19), we get

$$\begin{aligned}
-\varrho(\ell)\left(y'(\ell)\right)^{\alpha} &\geq -\varrho(\ell_2)\left(y'(\ell_2)\right)^{\alpha} + \left(\frac{\sqrt[\alpha]{\mu}\lambda^{\sqrt[\alpha]{\mu}}}{\sqrt[\alpha]{1-\sqrt[\alpha]{\mu}}}\frac{y(\ell)}{\pi(\ell)}\right)^{\alpha} \\
&\quad -\frac{\mu}{1-\sqrt[\alpha]{\mu}}\lambda^{\alpha\sqrt[\alpha]{\mu}}\left(\frac{y(\ell)}{\pi^{\sqrt[\alpha]{\mu}}(\ell)}\right)^{\alpha}\frac{1}{\pi^{\alpha\left(1-\sqrt[\alpha]{\mu}\right)}(\ell_2)} \\
&\geq \frac{\mu}{1-\sqrt[\alpha]{\mu}}\lambda^{\alpha\sqrt[\alpha]{\mu}}\left(\frac{y(\ell)}{\pi(\ell)}\right)^{\alpha} > \frac{\mu}{1-\sqrt[\alpha]{\mu}}\lambda^{\alpha\sqrt[\alpha]{\mu}}y^{\alpha},
\end{aligned}$$

further,

$$-\varrho^{1/\alpha}(\ell)\,y'(\ell) > \frac{\sqrt[\alpha]{\mu}}{\sqrt[\alpha]{1-\sqrt[\alpha]{\mu}}}\lambda^{\sqrt[\alpha]{\mu}}\frac{y(\ell)}{\pi(\ell)} = \sigma_1\mu_1\frac{y(\ell)}{\pi(\ell)}, \quad \ell \geq \ell_4,$$

where σ_1 is an arbitrary constant from $(0,1)$ approaching 1 if $\mu \to \mu_*$, $\lambda \to \lambda_*$, and

$$\sigma_1 = \sqrt[\alpha]{\frac{\mu\left(1-\sqrt[\alpha]{\mu_*}\right)}{\mu_*\left(1-\sqrt[\alpha]{\mu}\right)}\frac{\lambda^{\sqrt[\alpha]{\mu}}}{\lambda_*^{\sqrt[\alpha]{\mu}}}}.$$

Hence,

$$\left(\frac{y}{\pi^{\sigma_1\mu_1}}\right)' < 0.$$

Moreover, we can use mathematical induction to prove that

$$\left(\frac{y}{\pi^{\sigma_n\mu_n}}\right)' < 0,$$

eventually for each $n \in \mathbb{N}_0$, σ_n stands for any constant in $(0, 1)$ tending to 1 if $\mu \to \mu_*$ and $\lambda \to \lambda_*$, and defined as

$$\sigma_0 = \sqrt[\alpha]{\frac{\mu}{\mu_*}}$$

and

$$\sigma_{n+1} = \sigma_0 \sqrt[\alpha]{\frac{1-\mu_n}{1-\sigma_n\mu_n}\frac{\lambda^{\sigma_n\mu_n}}{\lambda_*^{\mu_n}}}, \qquad n \in \mathbb{N}_0.$$

Finally, we claim that y/π^{μ_n} is decreasing by showing that $y/\pi^{\sigma_{n+1}\mu_{n+1}}$ is decreasing as well for any $n \in \mathbb{N}_0$. So, by using that σ_{n+1} is an arbitrary constant tending to 1 and (3.12), we get

$$\sigma_{n+1}\mu_{n+1} > \mu_n.$$

As a result,

$$-\varrho^{1/\alpha}(\ell)\, y'(\ell)\, \pi(\ell) > \sigma_{n+1}\mu_{n+1} y(\ell) > \mu_n y(\ell),$$

eventually for any $n \in \mathbb{N}_0$. And so,

$$\left(\frac{y}{\pi^{\mu_n}}\right)' < 0.$$

The proof is complete. □

4. Applications in oscillation theory

Now, we will present some oscillation theorems for (1.1).

Theorem 4.1. *Assume that $\alpha = \beta$ and*

$$\lambda_* = \liminf_{\ell\to\infty} \frac{\pi(\varphi(\ell))}{\pi(\ell)} < \infty. \tag{4.1}$$

If

$$\liminf_{\ell\to\infty} \varrho^{1/\alpha}(\ell)\, \pi^{\alpha+1}(\ell) \sum_{i=1}^{m} \rho_i(\ell) > \max\left\{c(\nu) = \alpha\nu^{\alpha}(1-\nu)\lambda_*^{-\alpha\nu} : 0 < \nu < 1\right\}, \tag{4.2}$$

then, every solution of equation (1.1) is oscillatory.

Proof. Assume that y is a positive solution of (1.1). From (4.2), we obtain that $\mu_* > 0$, which guarantees the fulfillment of (3.6), and this, in turn, excludes the existence of increasing positive solutions of (1.1). Now, from (II)-part of Lemma 3.3 and Theorem 3.5, we obtain that y/π is nondecreasing and y/π^{μ_n} is decreasing, eventually for any $n \in \mathbb{N}_0$ and

$$\mu_n < 1.$$

Thus, we can conclude that the sequence $\{\mu_n\}$ defined as in (3.11) is bounded from above and increasing for

$$\mu_0 < \mu_1 < \mu_2 < ... < \mu_n < \mu_{n+1},$$

which means that the sequence $\{\mu_n\}$ is convergent and has a finite limit

$$0 < \lim_{n\to\infty} \mu_n = \nu < 1,$$

where the positive constant ν stands for the smaller root of the characteristic equation

$$c(\nu) = \alpha\nu^\alpha (1-\nu)\lambda_*^{-\alpha\nu} = \liminf_{\ell\to\infty} \varrho^{1/\alpha}(\ell)\,\pi^{\alpha+1}(\ell)\sum_{i=1}^{m}\rho_i(\ell), \tag{4.3}$$

see [12]. But, from (4.2), it is obvious that the previous equation has no positive solutions. This is a contradiction and completes the proof. □

Corollary 4.2. *For $c(\nu)$ defined as in (4.2), let us define*

$$c(\nu_{\max}) = \max\{c(\nu)\}, \tag{4.4}$$

for any $\nu \in (0,1)$. So, according to some calculations, we obtain

$$\nu_{\max} = \begin{cases} \frac{\alpha}{\alpha+1} & \text{for } \lambda_* = 1, \\ \frac{-\sqrt{(\eta+\alpha+1)^2-4\alpha\eta}+\eta+\alpha+1}{2\eta} & \text{for } \lambda_* \neq 1 \text{ and } \eta = \ln\lambda_*. \end{cases}$$

Theorem 4.3. *Assume that*

$$\liminf_{\ell\to\infty}\frac{\pi(\varphi(\ell))}{\pi(\ell)} = \infty. \tag{4.5}$$

If

$$\liminf_{\ell\to\infty} \varrho^{1/\alpha}(\ell)\,\pi^{\alpha+1}(\ell)\,\Omega(\ell)\sum_{i=1}^{m}\rho_i(\ell) > 0,$$

then, every solution of equation (1.1) is oscillatory.

Proof. Contrarily, assume that y is a positive solution of (1.1) on $[\ell_1,\infty)$ with $y(\varphi(\ell)) > 0$ for all $\ell \geq \ell_1$. Since $\mu_* > 0$, which guarantees the fulfillment of (3.6), and this, in turn, excludes the existence of increasing positive solutions of (1.1), as proven in the (I)-part of Lemma 3.3. However, (4.5) implies that there exists a sufficiently large ℓ for any $C_6 > 0$ satisfying

$$\frac{\pi(\varphi(\ell))}{\pi(\ell)} \geq C_6^{\mu^{-1/\alpha}}.$$

Integrating (1.1) from ℓ_2 to ℓ, yields

$$-\varrho(\ell)(y'(\ell))^\alpha = -\varrho(\ell_2)(y'(\ell_2))^\alpha + \int_{\ell_2}^{\ell} y^\beta(\varphi_i(s))\sum_{i=1}^{m}\rho_i(s)\,\mathrm{d}s$$

$$\begin{aligned} &\geq -\varrho(\ell_2)(y'(\ell_2))^\alpha + \int_{\ell_2}^{\ell} y^\beta(\varphi(s)) \sum_{i=1}^{m} \rho_i(s)\,\mathrm{d}s \\ &\geq -\varrho(\ell_2)(y'(\ell_2))^\alpha + \int_{\ell_2}^{\ell} y^\alpha(\varphi(s))\,\Omega(\varphi(s)) \sum_{i=1}^{m} \rho_i(s)\,\mathrm{d}s, \end{aligned}$$

and so,

$$-\varrho(\ell)(y'(\ell))^\alpha \geq -\varrho(\ell_2)(y'(\ell_2))^\alpha + \int_{\ell_2}^{\ell} y^\alpha(\varphi(s))\,\Omega(s) \sum_{i=1}^{m} \rho_i(s)\,\mathrm{d}s.$$

Now, exactly as in the proof of Theorem 3.5, it is easy to get that $\left(y/\pi^{\sqrt[\alpha]{\mu}}\right)' < 0$ for any ℓ large enough. By using this monotonicity in the previous inequality, we obtain

$$-\varrho(\ell)(y'(\ell))^\alpha \geq -\varrho(\ell_2)(y'(\ell_2))^\alpha + C_6^\alpha y^\alpha(\ell) \int_{\ell_2}^{\ell} \Omega(s) \sum_{i=1}^{m} \rho_i(s)\,\mathrm{d}s.$$

Combining with (3.19), we arrive at

$$\begin{aligned} -\varrho(\ell)(y'(\ell))^\alpha &\geq -\varrho(\ell_2)(y'(\ell_2))^\alpha + \mu C_6^\alpha y^\alpha(\ell) \int_{\ell_2}^{\ell} \frac{\alpha\varrho^{-1/\alpha}(s)}{\pi^{\alpha+1}(s)}\mathrm{d}s \\ &= -\varrho(\ell_2)(y'(\ell_2))^\alpha + \mu C_6^\alpha y^\alpha(\ell)\left[\pi^{-\alpha}(\ell) - \pi^{-\alpha}(\ell_2)\right]. \end{aligned}$$

But since $\lim_{\ell\to\infty} y(\ell) = 0$ as in (I)-part of Lemma 3.3, there is $\ell_3 \geq \ell_2$ such that

$$-\varrho(\ell_2)(y'(\ell_2))^\alpha - \frac{\mu C_6^\alpha}{\pi^\alpha(\ell_2)} y^\alpha(\ell) > 0.$$

Implies that

$$-\varrho(\ell)(y'(\ell))^\alpha > \mu\left(\frac{C_6 y(\ell)}{\pi(\ell)}\right)^\alpha,$$

i.e.,

$$-\varrho(\ell)(y'(\ell))^\alpha > \left(\frac{C_7 y(\ell)}{\pi(\ell)}\right)^\alpha$$

for $C_7 > 0$, $C_7 = \mu^{\frac{1}{\alpha}} C_6$. Thus

$$\left(\frac{y}{\pi^{C_7}}\right)' < 0.$$

Given that C_6 stands for any arbitrary constant and therefore C_7 does too, this leads to a contradiction with the (II)-part of Lemma 3.3. This completes the proof. □

Corollary 4.4. *For the linear case, where $\alpha = \beta = 1$, we can obtain the same previous oscillation property of (1.1) to the following canonical equation:*

$$\left(\widetilde{\varrho}(\ell)\, x'(\ell)\right)' + \sum_{i=1}^{m} \widetilde{\rho}_i(\ell)\, x(\varphi_i(\ell)) = 0,\ \ell \geq \ell_0 > 0, \tag{4.6}$$

where $\widetilde{\varrho}$ and $\widetilde{\rho}_i$ are continuous positive functions, and

$$\widehat{\varrho}(\ell) = \int_{\ell_0}^{\ell} \frac{1}{\widetilde{\varrho}(s)} \mathrm{d}s \to \infty \ \text{ as } \ell \to \infty.$$

Based on this, we can deduce the following results:

Theorem 4.5. *Assume that*

$$\delta_* = \liminf_{\ell\to\infty} \frac{\widehat{\varrho}(\ell)}{\widehat{\varrho}(\varphi(\ell))} < \infty.$$

If

$$\liminf_{\ell\to\infty} \widetilde{\varrho}(\ell)\widehat{\varrho}(\ell)\widehat{\varrho}(\varphi(\ell)) \sum_{i=1}^{m} \widetilde{\rho}_i(\ell) > \max\left\{\nu(1-\nu)\,\delta_*^{-\nu} : 0 < \nu < 1\right\},$$

then , every solution of equation (4.6) is oscillatory.

Proof. As in the proof of [36, Theorem 4], we can simply show that the noncanonical equation (1.1) with $\alpha = \beta = 1$ and the canonical equation (4.6) are equivalent, where

$$y(\ell) = \frac{x(\ell)}{\widehat{\varrho}(\ell)},$$

$$\pi(\ell) = \int_{\ell}^{\infty} \frac{\mathrm{d}s}{\widetilde{\varrho}(\ell)\widehat{\varrho}^2(\ell)} = \frac{1}{\widehat{\varrho}(\ell)},$$
$$\varrho(\ell) = \widetilde{\varrho}(\ell)\widehat{\varrho}^2(\ell),$$

and

$$\sum_{i=1}^{m} \rho_i(\ell) = \sum_{i=1}^{m} \widetilde{\rho}_i(\ell)\widehat{\varrho}(\ell)\widehat{\varrho}(\varphi(\ell)).$$

The rest of the proof is exactly same as the proof of Theorem 4.1. So, for

$$\liminf_{\ell\to\infty} \frac{\pi(\varphi(\ell))}{\pi(\ell)} = \liminf_{\ell\to\infty} \frac{\widehat{\varrho}(\ell)}{\widehat{\varrho}(\varphi(\ell))} < \infty,$$

the condition (4.2) becomes

$$\begin{aligned}
\liminf_{\ell\to\infty} \varrho(\ell)\pi^2(\ell) \sum_{i=1}^{m} \rho_i(\ell) &= \liminf_{\ell\to\infty} \widetilde{\varrho}(\ell)\widehat{\varrho}^2(\ell) \frac{1}{\widehat{\varrho}^2(\ell)} \sum_{i=1}^{m} \widetilde{\rho}_i(\ell)\widehat{\varrho}(\ell)\widehat{\varrho}(\varphi(\ell)) \\
&= \liminf_{\ell\to\infty} \widetilde{\varrho}(\ell)\widehat{\varrho}(\ell)\widehat{\varrho}(\varphi(\ell)) \sum_{i=1}^{m} \widetilde{\rho}_i(\ell) \\
&> \max\left\{\nu(1-\nu)\,\delta_*^{-\nu} : 0 < \nu < 1\right\}.
\end{aligned}$$

The proof is complete. $\square$

Theorem 4.6. *Let*

$$\liminf_{\ell\to\infty}\frac{\widehat{\varrho}(\ell)}{\widehat{\varrho}(\varphi(\ell))}=\infty.$$

If

$$\liminf_{\ell\to\infty}\widetilde{\varrho}(\ell)\widehat{\varrho}(\ell)\widehat{\varrho}(\varphi(\ell))\sum_{i=1}^{m}\widetilde{\rho}_i(\ell)>0,$$

then (4.6) is oscillatory.

Proof. Proceeding exactly as in the proof of Theorem 4.3 with the equivalent noncanonical representation of (4.6), we can verify the proof and so we omit it. □

Example 4.1. *Consider the DDE with several delays*

$$\left(\ell^{\alpha+1}\left(x'(\ell)\right)^{\alpha}\right)'+\rho_0\sum_{i=1}^{m}x^{\alpha}(a_i\ell)=0, \tag{4.7}$$

where $\rho_0\in(1,\infty)$ and $a_i\in(0,1]$ for $i=1,2,...,m$. Since $\varphi_i(\ell)=a_i\ell$, we let

$$a=\max\{a_i : i=1,2,...,m\},$$

and $\varrho(\ell)=\ell^{\alpha+1}$. Thus, we get $\pi(\ell)=\alpha\ell^{-1/\alpha}$. By applying Theorem 4.1, we obtain

$$\lambda_*=\liminf_{\ell\to\infty}\left(\frac{\pi(a\ell)}{\pi(\ell)}\right)^{\alpha}=\frac{1}{a}<\infty$$

and

$$\begin{aligned}\liminf_{\ell\to\infty}\varrho^{1/\alpha}(\ell)\pi^{\alpha+1}(\ell)\sum_{i=1}^{m}\rho_i(\ell) &= \rho_0\liminf_{\ell\to\infty}\ell^{1+1/\alpha}\left(\frac{\alpha}{\ell^{1/\alpha}}\right)^{\alpha+1}\\ &= \rho_0\alpha^{\alpha+1}.\end{aligned}$$

Therefore, (4.7) is oscillatory if

$$\rho_0\alpha^{\alpha+1}>c(\nu_{\max}),$$

where $c(\nu_{\max})$ is defined as in (4.4). In contrast, conditions (2.2) and (2.5) in [19,38] cannot be applied to (4.7). Now, for the linear case where $\alpha=1$. Equation (4.7) reduced to

$$\left(\ell^2x'(\ell)\right)'+\rho_0\sum_{i=1}^{m}x(a_i\ell)=0, \tag{4.8}$$

which is equivalent to the canonical equation

$$x''(\ell)+\frac{\rho_0}{a\ell^2}\sum_{i=1}^{m}x(a_i\ell)=0,$$

where $\widetilde{\varrho}(\ell)=1$ and $\widehat{\varrho}(\ell)=\ell\to\infty$ as $\ell\to\infty$. By applying Theorem 4.5, we get

$$\delta_*=\liminf_{\ell\to\infty}\frac{\widehat{\varrho}(\ell)}{\widehat{\varrho}(\varphi(\ell))}=\frac{1}{a}<\infty$$

and

$$\liminf_{\ell\to\infty} \widetilde{\varrho}(\ell)\widehat{\varrho}(\ell)\widehat{\varrho}(\varphi(\ell))\sum_{i=1}^{m}\widetilde{\rho_i}(\ell) = \rho_0,$$

then equation (4.8) is oscillatory if

$$\rho_0 > c(\nu_{\max}).$$

Example 4.2. *Consider the following linear DDE*

$$\left(\ell^2 y'(\ell)\right)' + \rho_0 y(\ell^\gamma) = 0, \tag{4.9}$$

where $\ell \geq \ell_0 > 0$, $\alpha = \beta = 1$, $\gamma \in (0,1)$, *and* $\rho_0 \in (0,\infty)$. (H_1)*-*(H_3) *are easily satisfied, and with some calculations, we can get that*

$$\sum_{i=1}^{m}\rho_i(\ell) = \rho_0, \quad \varphi(\ell) = \ell^\gamma, \quad \pi(\ell) = \frac{1}{\ell}, \quad \Omega(\ell) = 1,$$

$$\lambda_* = \liminf_{\ell\to\infty} \ell^{1-\gamma} = \infty,$$

and

$$\mu_* = \rho_0 > 0.$$

Hence, Theorem 4.3 ensures that every solution of (4.9) oscillates.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

This work does not have any conflicts of interest.

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