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Research article

Well-posedness and stabilization of a type three layer beam system with Gurtin-Pipkin's thermal law

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Abstract: The goal of this work is to study the well-posedness and the asymptotic behavior of solutions of a triple beam system commonly known as the Rao-Nakra beam model. We consider the effect of Gurtin-Pipkin's thermal law on the outer layers of the beam system. Using standard semi-group theory for linear operators and the multiplier method, we establish the existence and uniqueness of weak global solution, as well as a stability result.

Keywords: Rao-Nakra; triple-layer beam beam; Gurtin-Pipkin conduction; well-posedness; stability analysis
Mathematics Subject Classification: 35B35, 35B40, 35D30, 35D35, 93D20

1. Introduction

In the present work, we consider the Rao-Nakra (three layer) beam system, where the top and the bottom layers of the beam are subjected to Gurtin-Pipkin's thermal law, namely

$$\begin{cases} \rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - k(-u+v+\alpha w_x) + \delta_1 \theta_x = 0, & \text{in } (0,\pi)\times\mathbb{R}_+,\\ \rho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + k(-u+v+\alpha w_x) - \delta_1\theta + \delta_2 \vartheta_x = 0, & \text{in } (0,\pi)\times\mathbb{R}_+,\\ \rho h w_{tt} + EI w_{xxxx} - \alpha k(-u+v+\alpha w_x)_x + \delta_3 w_t = 0, & \text{in } (0,\pi)\times\mathbb{R}_+,\\ \rho_4\theta_t - \beta_1 \displaystyle\int_0^{+\infty} g_1(s)\theta_{xx}(x,t-s)ds + \delta_1(u_{xt}+v_t) = 0, & \text{in } (0,\pi)\times\mathbb{R}_+,\\ \rho_5\vartheta_t - \beta_2 \displaystyle\int_0^{+\infty} g_2(s)\vartheta_{xx}(x,t-s)ds + \delta_2 v_{xt} = 0, & \text{in } (0,\pi)\times\mathbb{R}_+ \end{cases} \tag{1.1}$$

with the following boundary conditions:

$$\begin{cases} u_x(0,t) = v_x(0,t) = w(0,t) = w_{xx}(0,t) = \theta(0,t) = \vartheta(0,t) = 0, & t\geq 0,\\ u(\pi,t) = v(\pi,t) = w(\pi,t) = w_{xx}(\pi,t) = \theta_x(\pi,t) = \vartheta_x(\pi,t) = 0, & t\geq 0, \end{cases} \tag{1.2}$$

and the initial data

$$\begin{cases} u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x), & x \in (0,\pi), \\ u_t(x,0) = u_1(x),\ v_t(x,0) = v_1(x),\ w_t(x,0) = w_1(x), & x \in (0,\pi), \\ \theta(x,-t) = \theta_0(x,t), \vartheta(x,-t) = \vartheta_0(x,t), & x \in (0,\pi),\ t > 0. \end{cases} \tag{1.3}$$

The relaxation functions g_1 and g_2 are positive non-increasing functions to be specified later. The stabilization of Rao-Nakra beam systems has gathered much interest from researchers recently, and a great number of results have been established. The Rao-Nakra beam model is a beam system that takes into account the motion of two outer face plates (assumed to be relatively stiff) and a sandwiched compliant inner core layer, see [1–5] for Rao-Nakra, Mead-Markus and multilayer plates or sandwich models. The basic equations of motion of the Rao-Nakra model are derived thanks to the Euler-Bernoulli beam assumptions for the outer face plate layers, the Timoshenko beam assumptions for the sandwich layer and a "no slip" assumption for the motion along the interface. Suppose $h(j)$, $j = 1, 2, 3$ is the thickness of each layer in the beam of length π, see Figure 1 and $h = h(1) + h(2) + h(3)$ the total thickness of the beam.

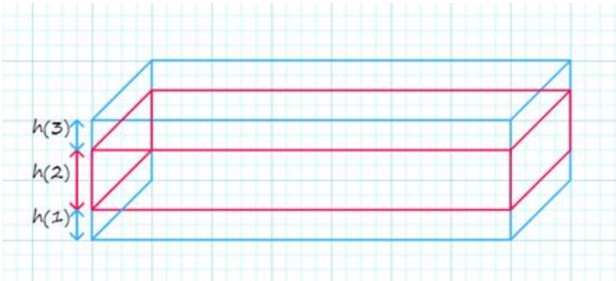

Figure 1. Triple layer beam.

Assuming the Kirchhoff hypothesis is imposed on the outer layers of beam and in addition, there is a continuous, piecewise linear displacements through the cross-sections, Liu et al. [6] gave a detailed derivation of following laminated beam system:

$$\begin{cases} \rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - \tau = 0, \\ \rho_1 I_1 y^1_{tt} - E_1 I_1 y^1_{xx} - \dfrac{h_1}{2}\tau + G_1 h_1 (w_x + y^1) = 0, \\ \rho h w_{tt} + EI w_{xxxx} - G_1 h_1 k (w_x + y^1)_x - G_3 h_3 (w_x + y^3)_x - h_2 \tau_x = 0, \\ \rho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + \tau = 0, \\ \rho_3 I_3 y^3_{tt} - E_3 I_3 y^3_{xx} - \dfrac{h_3}{2}\tau + G_3 h_3 (w_x + y^3) = 0, \end{cases} \tag{1.4}$$

where $x \in (0,\pi), t > 0$, $(u, y^1), (v, y^3)$ represent longitudinal displacement and shear angle of the bottom and top layers plates. The transverse displacement of the beam is represented by w, and τ is the shear stress of the core layer. Also, for $j = 1, 2, 3$ (from bottom to top layer), $E_j, G_j, I_j, \rho_j > 0$ are Young's modulus, shear modulus, moments of inertia and density respectively for each layer. Moreover, in $(1.4)_3$, we have that $\rho h = \rho_1 h_1 + \rho_2 h_2 + \rho_3 h_3$ and $EI = E_1 I_1 + E_3 I_3$. By neglecting the rotary inertia in top and bottom layers of the beam, we obtain $\rho_1 I_1 = \rho_3 I_3 = 0$ in $(1.4)_4$

and $(1.4)_5$. Furthermore, if we neglect the transverse shear, this leads to the Euler-Bernoulli hypothesis $w_x + y^1 = w_x + y^3 = 0$. Assuming that the core layer consists of a material that is linearly elastic with the stress-strain relationship $\tau = 2G_2\varepsilon$, where the shear strain ε is defined by

$$\varepsilon = \frac{1}{2h_2}(-u + v + \alpha w_x) \text{ where } \alpha = h_2 + \frac{h_1 + h_2}{2}.$$

Thus, we arrive at the following Rao-Nakra beam model [1], given by

$$\begin{cases} \rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - k(-u + v + \alpha w_x) = 0, \\ \rho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + k(-u + v + \alpha w_x) = 0, \\ \rho h w_{tt} + EI w_{xxxx} - \alpha k(-u + v + \alpha w_x)_x = 0, \end{cases} \tag{1.5}$$

where $k = \dfrac{G_2}{h_2}$, $G_2 = \dfrac{E_2}{2(1+\nu)}$ and $-1 < \nu < \dfrac{1}{2}$ is the Poisson ratio. Furthermore, when the extensional motion of the outer layers is neglected, system (1.4) takes the form of the two-layer laminated beam system derived by Hansen and Spies [7]. Li et al. [8] showed that system (1.5) is unstable if only one of the equations is damped. When two of the three equations in (1.5) were damped, the authors in [8] proved a polynomial stability. For recent results in literature, Méndez et al. [9] considered (1.5) with with Kelvin-Voigt damping and studied the well-posedness, lack of exponential decay and polynomial decay. Feng and Özer [10] looked at a nonlinearly damped Rao-Nakra beam system and established the global attractor with finite fractal dimension. Feng et al. [11] studied the stability of Rao-Nakra sandwich beam with time-varying weight and time-varying delay. Mukiawa et al. [12] considered (1.5) with viscoelastic damping on the first equation and heat conduction govern by Fourier's law and proved the well-posedness and a general decay result. Also, Raposo et al. [13] coupled (1.5) with Maxwell-Cattaneo heat conduction established the well-posedness. For more results related Rao-Nakra beam system with frictional, delay or thermal damping, see [14–20] and the references therein.

An interesting tool used by Mathematician in stabilizing beam models such as the Laminated and Timoshenko beam systems is the Gurtin-Pipkin's thermal law, see [21], with constitutive equation

$$\beta q(t) + \int_0^\infty g(s)\theta_x(x, t-s)ds = 0, \tag{1.6}$$

where $\theta = \theta(x,t)$ is the temperature difference, $q = q(x,t)$ is the heat flux, β is a coupling constant coefficient and the relaxation g is a summable convex $L^1([0,+\infty))$ function with unit mass. For results related to (1.6), Dell'Oro and Pata [22] studied

$$\begin{cases} \rho_1 u_{tt} - k(u_x + v)_x = 0, & \text{in } (0,\pi)\times\mathbb{R}_+, \\ \rho_2 v_{tt} - b v_{xx} + k(u_x + v) + \delta\theta_x = 0, & \text{in } (0,\pi)\times\mathbb{R}_+, \\ \rho_3\theta_t - \dfrac{1}{\beta}\displaystyle\int_0^\infty h(s)\theta_{xx}(x,t-s)ds + \delta v_{xt} = 0, & \text{in } (0,\pi)\times\mathbb{R}_+ \end{cases} \tag{1.7}$$

and proved an exponential stability result if and only if $\chi_h = 0$, where

$$\chi_h = \left(\frac{\rho_1}{k\rho_3} - \frac{\beta}{h(0)}\right)\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right) - \frac{\beta}{h(0)}\frac{\rho_1\delta^2}{kb\rho_3}.$$

For similar results with Gurtin-Pipkin's thermal law, see [23–28] and references therein. As clearly elaborated in [22], the Fourier's and Cattaneo's (second sound) thermal law can be recovered from (1.6) by defining the memory function g in (1.6) as

$$g_\delta(s) = \frac{1}{\delta} h\left(\frac{s}{\delta}\right), \ \ \delta > 0 \tag{1.8}$$

and

$$g_\tau(s) = \frac{\beta}{\tau} e^{-s\frac{\beta}{\tau}}, \ \tau > 0 \tag{1.9}$$

respectively. A closely related thermal law to the Gurtin-Pipkin's thermal law is the Coleman-Gurtin's heat conduction law, see [29], with constitutive equation given by

$$\beta q(t) + (1-\eta)\theta_x + \eta \int_0^\infty \mu(s)\theta_x(x, t-s)ds = 0, \ \ \eta \in (0,1), \tag{1.10}$$

where $\eta = 1$ and $\eta = 0$ correspond to the Gurtin-Pipkin's and Fourier thermal laws, respectively. This entails replacing $(1.7)_3$ with

$$\rho_3\theta_t - \frac{(1-\eta)}{\beta}\theta_{xx} - \frac{\eta}{\beta}\int_0^\infty \mu(s)\theta_{xx}(x, t-s)ds + \delta v_{xt} = 0, \text{ in } (0,\pi)\times\mathbb{R}_+. \tag{1.11}$$

We should note here that systems govern by Coleman-Gurtin's thermal lawa (1.10) gain additional dissipation from the term $-\frac{(1-\eta)}{\beta}\theta_{xx}$ and thus less difficult to handle compare to systems with Gurtin-Pipkin's thermal law (1.6).

Our main focus of this paper is to investigate the well-posedness and the asymptotic behavior of solutions of system (1.1)–(1.3). We mote here that, the rotational inertia term w_{xxtt} which should be in $(1.1)_3$ of the original models is neglected in the present model. However, the result in this paper is not affected by the absent of this term. Also, since the thermal coupling in system (1.1)–(1.3) is not strong enough to achieve exponential stability, a viscous damping term w_t is added to $(1.1)_3$. The rest of work is organized as follows: In Section 2, we state some assumptions and set up our problem (1.1)–(1.3) in appropriate spaces. In Section 3, we prove the existence and uniqueness result for the system (1.1)–(1.3). In Section 4, we study the asymptotic behavior of solution of system (1.1)–(1.3).

2. Assumptions, problem transformation and functional setting

2.1. Assumptions on the kernels

For the relaxation functions g_1 and g_2, we assume the following:

Assumption (A_0)**:**

(a_0) $g_1, g_2 : [0,+\infty) \longrightarrow (0,+\infty)$ are non-increasing $C^2([0,+\infty))$ and convex summable functions satisfying

$$\lim_{s\to+\infty} g_i(s) = 0 \ \text{ and } \int_0^{+\infty} g_i(s)ds = 1, \ i = 1,2. \tag{2.1}$$

(b_0) There exists $\xi_i > 0, \ i = 1,2$ such that

$$-g_i''(s) \le \xi_i\left(g_i'(s)\right), \ \forall s \ge 0, \ i = 1,2. \tag{2.2}$$

By setting

$$\mu_1(s) = -g_1'(s) \text{ and } \mu_2(s) = -g_2'(s), \tag{2.3}$$

assumption (A_0) ensues the following:
Assumption (A_1)**:**

(a_1) $\mu_1, \mu_2 : [0, +\infty) \longrightarrow (0, +\infty)$ are non-increasing $C^1([0, +\infty))$ and convex summable functions satisfying

$$\mu_{0i} = \int_0^{+\infty} \mu_i(s)ds = g_i(0) > 0, \text{ and } \int_0^{+\infty} s\mu_i(s)ds = 1, \ i = 1, 2. \tag{2.4}$$

(b_1) There exists $\xi_i > 0, \ i = 1, 2$ such that

$$\mu_i'(s) \leq -\xi_i \mu_i(s), \ \forall s \geq 0, \ i = 1, 2. \tag{2.5}$$

2.2. Problem transformation

Due to the work of Dafermos [30], we define new functions for the relative past history of θ and ϑ as follows:

$$\sigma, \zeta : (0, \pi) \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+,$$

define by

$$\sigma = \sigma(x, t, s) := \int_{t-s}^{t} \theta(x, r)dr \text{ and } \zeta = \zeta(x, t, s) := \int_{t-s}^{t} \vartheta(x, r)dr. \tag{2.6}$$

On account of the boundary conditions (1.2), we have

$$\sigma(0, t, s) = \sigma_x(\pi, t, s) = \zeta(0, t, s) = \zeta_x(\pi, t, s) = 0,$$

and routine calculation gives

$$\begin{cases} \sigma_t + \sigma_s - \theta = 0, & \text{in } (0, \pi) \times (\mathbb{R}_+ \times \mathbb{R}_+, \\ \zeta_t + \zeta_s - \vartheta = 0, & \text{in } (0, \pi) \times \mathbb{R}_+ \times \mathbb{R}_+, \\ \sigma(x, t, 0) = \zeta(x, t, 0) = 0, & \text{in } (0, \pi) \times \mathbb{R}_+, \\ \sigma(x, 0, s) = \displaystyle\int_0^s \theta_0(x, r)dr := \sigma_0(x, s), & \text{in } (0, \pi) \times \mathbb{R}_+, \\ \zeta(x, 0, s) = \displaystyle\int_0^s \vartheta_0(x, r)dr := \zeta_0(x, s), & \text{in } (0, \pi) \times \mathbb{R}_+, \end{cases} \tag{2.7}$$

where σ_0 and ζ_0 represent the history of θ and ϑ respectively. Also, using direct computations, we have

$$\begin{aligned} & \int_0^{+\infty} g_1(s)\theta_{xx}(x, t-s)ds \\ = & \lim_{a \to +\infty} g_1(s) \int_{t-s}^{t} \theta_{xx}(x, r)dr \Big|_{s=0}^{s=a} - \int_0^{+\infty} g_1'(s) \int_{t-s}^{t} \theta_{xx}(x, r)drds \\ = & \int_0^{+\infty} \mu_1(s)\sigma_{xx}(x, t, s)ds. \end{aligned} \tag{2.8}$$

Similarly, we get

$$\int_0^{+\infty} g_2(s)\vartheta_{xx}(x,t-s)ds = \int_0^{+\infty} \mu_2(s)\zeta_{xx}(x,t,s)ds. \tag{2.9}$$

On account of (2.6)–(2.9), system (1.1)–(1.3) takes the form

$$\begin{cases} \rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - k(-u + v + \alpha w_x) + \delta_1 \theta_x = 0, & \text{in } (0,\pi)\times\mathbb{R}_+, \\ \rho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + k(-u + v + \alpha w_x) - \delta_1 \theta + \delta_2 \vartheta_x = 0, & \text{in } (0,\pi)\times\mathbb{R}_+, \\ \rho h w_{tt} + EI w_{xxxx} - \alpha k(-u + v + \alpha w_x)_x + \delta_3 w_t = 0, & \text{in } (0,\pi)\times\mathbb{R}_+, \\ \rho_4 \theta_t - \beta_1 \int_0^{+\infty} \mu_1(s)\sigma_{xx}(x,t,s)ds + \delta_1(u_{xt} + v_t) = 0, & \text{in } (0,\pi)\times\mathbb{R}_+, \\ \sigma_t + \sigma_s - \theta = 0, & \text{in } (0,\pi)\times\mathbb{R}_+\times\mathbb{R}_+, \\ \rho_5 \vartheta_t - \beta_2 \int_0^{+\infty} \mu_2(s)\zeta_{xx}(x,t,s)ds + \delta_2 v_{xt} = 0, & \text{in } (0,\pi)\times\mathbb{R}_+, \\ \zeta_t + \zeta_s - \vartheta = 0, & \text{in } (0,\pi)\times\mathbb{R}_+\times\mathbb{R}_+ \end{cases} \tag{2.10}$$

with the boundary conditions

$$\begin{cases} u_x(0,t) = v_x(0,t) = w(0,t) = w_{xx}(0,t) = \theta(0,t) = \vartheta(0,t), & t \geq 0, \\ u(\pi,t) = v(\pi,t) = w(\pi,t) = w_{xx}(\pi,t) = \theta_x(\pi,t) = \vartheta_x(\pi,t) = 0, & t \geq 0, \\ \sigma(0,t,s) = \sigma_x(\pi,t,s) = \zeta(0,t,s) = \zeta_x(\pi,t,s) = 0, & s,t \in \mathbb{R}_+, \\ \sigma(x,t,0) = \zeta(x,t,0) = 0, & x \in (0,\pi), t \in \mathbb{R}_+ \end{cases} \tag{2.11}$$

and the initial data

$$\begin{cases} u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x), & x \in (0,\pi), \\ u_t(x,0) = u_1(x),\ v_t(x,0) = v_1(x),\ w_t(x,0) = w_1(x), & x \in (0,\pi), \\ \theta(x,-t) = \theta_0(x,t), \vartheta(x,-t) = \vartheta_0(x,t) & x \in (0,\pi),\ t > 0, \\ \sigma(x,0,s) = \sigma_0(x,s),\ \zeta(x,0,s) = \zeta_0(x,s), & x \in (0,\pi), s > 0. \end{cases} \tag{2.12}$$

Setting $\Psi = (u,\varphi,v,\psi,w,\phi,\theta,\sigma,\vartheta,\zeta)^T$, with $\varphi = u_t$, $\psi = v_t$ and $\phi = w_t$. Then, the semi-group formulation of system (2.10)–(2.12) is given by the Cauchy problem

$$(P)\begin{cases} \Psi_t + \mathcal{A}\Psi = 0, \\ \Psi(0) = \Psi_0, \end{cases} \tag{2.13}$$

where $\Psi_0 = (u_0, u_1, v_0, v_1, w_0, w_1, \theta_0, \sigma_0, \vartheta_0, \zeta_0)^T$ and the linear operator $\mathcal{A}$ is defined by

$$\mathcal{A}\Psi=\begin{pmatrix} -\varphi \\ -\dfrac{E_1}{\rho_1}u_{xx}-\dfrac{k}{\rho_1 h_1}(-u+v+\alpha w_x)+\dfrac{\delta_1}{\rho_1 h_1}\theta_x \\ -\psi \\ -\dfrac{E_3}{\rho_3}v_{xx}+\dfrac{k}{\rho_3 h_3}(-u+v+\alpha w_x)-\frac{\delta_1}{\rho_3 h_3}\theta+\frac{\delta_2}{\rho_3 h_3}\vartheta_x \\ -\phi \\ \dfrac{EI}{\rho h}w_{xxxx}-\dfrac{\alpha k}{\rho h}(-u+v+\alpha w_x)_x+\dfrac{\delta_3}{\rho h}\phi \\ -\frac{\beta_1}{\rho_4}\displaystyle\int_0^{+\infty}\mu_1(s)\sigma_{xx}(x,s)ds+\dfrac{\delta_1}{\rho_4}(\varphi_x+\psi) \\ \sigma_s-\theta \\ -\frac{\beta_2}{\rho_5}\displaystyle\int_0^{+\infty}\mu_2(s)\zeta_{xx}(x,s)ds+\dfrac{\delta_2}{\rho_5}\psi_x \\ \zeta_s-\vartheta \end{pmatrix}.$$

2.3. Functional spaces

Let $\langle , \rangle$ and $||.||$ denote the inner product and the norm in $L^2(0,\pi)$ respectively and we define following Sobolev spaces:

$$\begin{aligned} H_a^1 :=&\{\varpi\in H^1(0,\pi)/\varpi(0)=0\},\ H_b^1:=\{\varpi\in H^1(0,\pi)/\varpi(\pi)=0\},\\ H_a^2 :=&\{\varpi\in H^2(0,\pi)/\varpi_x\in H_a^1\},\ H_b^2:=\{\varpi\in H^2(0,\pi)/\varpi_x\in H_b^1\},\\ H_*^2 :=&H^2(0,\pi)\cap H_0^1(0,\pi), \end{aligned}$$

where H_*^2 is equip with the inner product

$$\langle\varpi,\hat{\varpi}\rangle_{H_*^2}=\langle\varpi_{xx},\hat{\varpi}_{xx}\rangle$$

and norm

$$||\varpi||_{H_*^2}^2=||\varpi_{xx}||^2.$$

It is easy to check that $(H_*^2,||.||_{H_*^2}^2)$ is a Banach space and the norm $||.||_{H_*^2}^2$ is equivalent to the usual norm in $H^2(0,\pi)$. Next, we introduce the weighted-Hilbert space of $H_a^1(0,\pi)$-real valued functions on $(0,+\infty)$ by

$$L_\mu^2:=L_\mu^2\left(\mathbb{R}_+;H_a^1(0,\pi)\right),$$

where

$$L_\mu^2\left(\mathbb{R}_+;H_a^1(0,\pi)\right)=\left\{\varpi:\mathbb{R}_+\longrightarrow H_a^1(0,\pi)/\int_0^{+\infty}\mu(s)\|\varpi_x(s)\|^2ds<\infty\right\},$$

and equip them with the inner product

$$(\varpi,\hat{\varpi})_{L_\mu^2}:=\int_0^{+\infty}\mu(s)\langle\varpi_x(s),\hat{\varpi}_x(s)\rangle ds,$$

and norm

$$\|\varpi\|_{L_\mu^2}^2=\int_0^{+\infty}\mu(s)\|\varpi_x(s)\|^2ds.$$

Also, we define

$$\mathcal{D}\left(L_\mu^2\right):=\left\{\varpi\in L_\mu^2/\varpi_s\in L_\mu^2\ \text{ and }\lim_{s\to 0}\|\varpi_x(s)\|=0\right\}.$$

Now, we introduce the phase space of our problem given by

$$\mathcal{H}:=H_b^1\times L^2\times H_b^1\times L^2\times H_*^2\times L^2\times L^2\times L_{\mu_1}^2\times L^2\times L_{\mu_2}^2$$

and equipped it with the inner product

$$\begin{aligned}&\langle(u,\varphi,v,\psi,w,\phi,\theta,\sigma,\vartheta,\zeta),(\hat{u},\hat{\varphi},\hat{v},\hat{\psi},\hat{w},\hat{\phi},\hat{\theta},\hat{\sigma},\hat{\vartheta},\hat{\zeta})\rangle_{\mathcal{H}}\\&:=E_1h_1\langle u_x,\hat{u}_x\rangle+\rho_1h_1\langle\varphi,\hat{\varphi}\rangle+k\langle(-u+v+\alpha w_x),(-\hat{u}+\hat{v}+\alpha\hat{w}_x)\rangle\\&\quad+E_3h_3\langle v_x,\hat{v}_x\rangle+\rho_3h_3\langle\psi,\hat{\psi}\rangle+EI\langle w_{xx},\hat{w}_{xx}\rangle+\rho h\langle\phi,\hat{\phi}\rangle+\rho_4\langle\theta,\hat{\theta}\rangle\\&\quad+\beta_1\langle\sigma,\hat{\sigma}\rangle_{L_{\mu_1}^2}+\rho_5\langle\vartheta,\hat{\vartheta}\rangle+\beta_2\langle\zeta,\hat{\zeta}\rangle_{L_{\mu_2}^2}\end{aligned}$$

and norm

$$\begin{aligned}\|\Psi\|_{\mathcal{H}}^2=&\|(u,\varphi,v,\psi,w,\phi,\theta,\sigma,\vartheta,\zeta)\|_{\mathcal{H}}^2\\:=&E_1h_1\|u_x\|^2+\rho_1h_1\|\varphi\|^2+k\|(-u+v+\alpha w_x)\|^2\\&+E_3h_3\|v_x\|^2+\rho_3h_3\|\psi\|^2+EI\|w_{xx}\|^2+\rho h\|\phi\|^2\\&+\rho_4\|\theta\|^2+\beta_1\|\sigma\|_{L_{\mu_1}^2}^2+\rho_5\|\vartheta\|^2+\beta_2\|\zeta\|_{L_{\mu_2}^2}^2,\end{aligned}$$

for any $\Phi=(w,\varphi,v,\psi,u,\phi,\theta,\sigma,\vartheta,\zeta)^T\in\mathcal{H}$.

The domain of the linear operator $\mathcal{A}$ in (2.13) is defined as follows:

$$\mathcal{D}(\mathcal{A}):=\left\{(u,\varphi,v,\psi,w,\phi,\theta,\sigma,\vartheta,\zeta)\in\mathcal{H}\left|\begin{array}{l}u,v\in H_b^2\cap H_b^1,\ \varphi,\psi\in H_b^1,\\w\in H^4\cap H_*^2,\ \phi\in H_*^2,\\\sigma\in\mathcal{D}\left(L_{\mu_1}^2\right),\ \theta\in H_a^1,\\\zeta\in\mathcal{D}\left(L_{\mu_2}^2\right),\ \vartheta\in H_a^1,\\(-u+v+\alpha w_x)\in H_a^1\cap H_b^1,\\\int_0^{+\infty}\mu_1(s)\sigma(s)ds\in H^2\cap H_a^1,\\\int_0^{+\infty}\mu_2(s)\zeta(s)ds\in H^2\cap H_a^1,\\w_{xx}(0)=w_{xx}(\pi)=0.\end{array}\right.\right\}.$$

Remark 2.1. *(1) Due to* (2.5)*, we can deduce that*

$$\langle -\varpi_s, \varpi \rangle_{L^2_{\mu_i}} \leq -\frac{\xi_i}{2}\|\varpi\|^2_{L^2_{\mu_i}}, \quad \forall\, \varpi \in \mathcal{D}\left(L^2_{\mu_i}\right),\ i = 1, 2. \tag{2.14}$$

(2) Using Hölder's and Young's inequalities, we have that

$$\int_0^{+\infty} \mu_i(s)\|\varpi_x(s)\|ds \leq \sqrt{g_i(0)}\|\varpi\|_{L^2_{\mu_i}},\ i = 1, 2. \tag{2.15}$$

3. Well-posedness

In this section, we establish the existence and uniqueness of global weak solution to the system (2.10)–(2.12).

3.1. Needed lemmas for well-posedness

Lemma 3.1. *The linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ defined in* (2.13) *is monotone.*

Proof. Let $\Psi = (u, \varphi, v, \psi, w, \phi, \theta, \sigma, \vartheta, \zeta) \in \mathcal{D}(\mathcal{A})$, then using integration by parts and the boundary conditions (2.11), we have

$$\begin{aligned}
\langle \mathcal{A}\Psi, \Psi\rangle_{\mathcal{H}} =& \delta_3\|\phi\|^2 + \beta_1 \int_0^{+\infty} \mu_1(s)\langle \sigma_{xs}(s), \sigma_x(s)\rangle ds + \beta_2 \int_0^{+\infty} \mu_2(s)\langle \zeta_{xs}(s), \zeta_x(s)\rangle ds \\
=& \delta_3\|\phi\|^2 + \frac{\beta_1}{2}\int_0^{+\infty} \mu_1(s)\frac{d}{ds}\left(\|\sigma_x(s)\|^2\right) ds + \frac{\beta_2}{2}\int_0^{+\infty} \mu_2(s)\frac{d}{ds}\left(\|\zeta_x(s)\|^2\right) ds \\
=& \delta_3\|\phi\|^2 - \frac{\beta_1}{2}\int_0^{+\infty} \mu_1'(s)\|\sigma_x(s)\|^2 ds + \frac{\beta_1}{2}\lim_{a\to+\infty} \mu_1(s)\|\sigma_x(s)\|^2\Big|_{s=0}^{s=a} \\
&- \frac{\beta_2}{2}\int_0^{+\infty} \mu_2'(s)\|\zeta_x(s)\|^2 ds + \frac{\beta_2}{2}\lim_{a\to+\infty} \mu_2(s)\|\zeta_x(s)\|^2\Big|_{s=0}^{s=a}.
\end{aligned}$$

From (2.5) and (2.6), we obtain

$$\lim_{a\to+\infty} \mu_1(s)\|\sigma_x(s)\|^2\Big|_{s=0}^{s=a} = \lim_{a\to+\infty} \mu_2(s)\|\zeta_x(s)\|^2\Big|_{s=0}^{s=a} = 0.$$

Therefore,

$$\langle \mathcal{A}\Psi, \Psi\rangle_{\mathcal{H}} = \delta_3\|\phi\|^2 - \frac{\beta_1}{2}\int_0^{+\infty} \mu_1'(s)\|\sigma_x(s)\|^2 ds - \frac{\beta_2}{2}\int_0^{+\infty} \mu_2'(s)\|\zeta_x(s)\|^2 ds \geq 0.$$

Therefore, $\mathcal{A}$ is monotone. □

Lemma 3.2. The linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ defined in (2.13) maximal, that is $\Re(I + \mathcal{A}) = \mathcal{H}$.

Proof. Given $F = (k^1, k^2, k^3, k^4, k^5, k^6, k^7, k^8, k^9, k^{10}) \in \mathcal{H}$, we look for a unique solution

$$\Psi = (u, \varphi, v, \psi, w, \phi, \theta, \sigma, \vartheta, \zeta) \in \mathcal{D}(\mathcal{A})$$

such that Ψ solves the stationary problem

$$\Psi + \mathcal{A}\Psi = F. \tag{3.1}$$

System (3.1) is equivalent to

$$\begin{cases} u - \varphi = k^1, & \text{in } H_b^1, \\ \rho_1 h_1 \varphi - E_1 h_1 u_{xx} - k(-u + v + \alpha w_x) + \delta_1 \theta_x = \rho_1 h_1 k^2, & \text{in } L^2, \\ v - \psi = k^3, & \text{in } H_b^1, \\ \rho_3 h_3 \psi - E_3 h_3 v_{xx} + k(-u + v + \alpha w_x) - \delta_1 \theta + \delta_2 \vartheta_x = \rho_3 h_3 k^4, & \text{in } L^2, \\ w - \phi = k^5, & \text{in } H_*^2, \\ (\rho h + \delta_3)\phi + EI w_{xxxx} - \alpha k(-u + v + \alpha w_x)_x = \rho h k^6, & \text{in } L^2, \\ \rho_4 \theta - \beta_1 \int_0^{+\infty} \mu_1(s) \sigma_{xx}(x, s) ds + \delta_1 (\varphi_x + \psi) = \rho_4 k^7, & \text{in } L^2, \\ \sigma + \sigma_s - \theta = k^8, & \text{in } L^2_{\mu_1} \\ \rho_5 \vartheta - \beta_2 \int_0^{+\infty} \mu_2(s) \zeta_{xx}(x, s) ds + \delta_2 \psi_x = \rho_5 k^9, & \text{in } L^2, \\ \zeta + \zeta_s - \vartheta = k^{10}, & \text{in } L^2_{\mu_2}. \end{cases} \tag{3.2}$$

By multiplying $(3.2)_8$ and $(3.2)_{10}$ by e^r and integrating the results over $(0, s)$, we arrive at

$$\begin{aligned} \sigma(s) &= (1 - e^{-s})\, \theta + \int_0^s e^{r-s} k^8(r) dr, \\ \zeta(s) &= (1 - e^{-s})\, \vartheta + \int_0^s e^{r-s} k^{10}(r) dr. \end{aligned} \tag{3.3}$$

From $(3.2)_1$, $(3.2)_3$ and $(3.2)_5$, we get

$$u - k^1 = \varphi, \quad v - k^3 = \psi \text{ and } \; w - k^5 = \phi, \tag{3.4}$$

respectively. Substituting (3.4) and (3.3) into $(3.2)_2$, $(3.2)_4$, $(3.2)_6$, $(3.2)_7$ and $(3.2)_9$ leads to

$$\begin{cases} \rho_1 h_1 u - E_1 h_1 u_{xx} - k(-u + v + \alpha w_x) + \delta_1 \theta_x = \underbrace{\rho_1 h_1 (k^1 + k^2)}_{f^1}, & \text{in } L^2, \\ \rho_3 h_3 v - E_3 h_3 v_{xx} + k(-u + v + \alpha w_x) - \delta_1 \theta + \delta_2 \vartheta_x = \underbrace{\rho_3 h_3 (k^3 + k^4)}_{f^2}, & \text{in } L^2, \\ \rho h w + EI w_{xxxx} - \alpha k(-u + v + \alpha w_x)_x = \underbrace{\delta_3 k^5 + \rho h (k^5 + k^6)}_{f^3}, & \text{in } L^2, \\ \rho_4 \theta - C_{\beta_1, \mu_1} \theta_{xx} + \delta_1 (u_x + v) & \\ = \underbrace{\delta_1 \left(k_x^1 + k^3\right) + \rho_4 k^7 + \beta_1 \int_0^{+\infty} \mu_1(s) \left(\int_0^s e^{r-s} k_{xx}^8(r) dr \right) ds}_{f^4}, & \text{in } H^{-1}, \\ \rho_5 \vartheta - C_{\beta_2, \mu_2} \vartheta_{xx} + \delta_2 v_x & \\ = \underbrace{\delta_2 k_x^3 + \rho_5 k^9 + \beta_2 \int_0^{+\infty} \mu_2(s) \left(\int_0^s e^{r-s} k_{xx}^{10}(r) dr \right) ds}_{f^5}, & \text{in } H^{-1}, \end{cases} \tag{3.5}$$

where

$$C_{\beta_i,\mu_i} = \beta_i \int_0^{+\infty} \mu_i(s)\,(1 - e^{-s})\,ds > 0,\ i = 1, 2.$$

Now, we observe that last terms in f^4 and f^5 are in $H^{-1}(0,\pi)$. Indeed, since $k^8 \in L^2_{\mu_1}$, we have for any

$$\varpi \in H^1_a(0,\pi),\ \text{with } \|\varpi_x\| \le 1,$$

that

$$\begin{aligned}
\left|\left\langle \int_0^{+\infty} \mu_1(s)\left(\int_0^s e^{r-s} k^8_{xx}(r)dr\right) ds, \varpi\right\rangle\right| &= \left|\left\langle \int_0^{+\infty} \mu_1(s)\left(\int_0^s e^{r-s} k^8_{x}(r)dr\right) ds, \varpi_x\right\rangle\right| \\
&\le \int_0^{+\infty} \mu_1(s)e^{-s}\left(\int_0^s e^r\|k^8_x(r)\|dr\right) ds \\
&= \int_0^{+\infty} e^r\|k^8_x(r)\|\left(\int_r^{+\infty} e^{-s}\mu_1(s)ds\right) dr \\
&= \le \int_0^{+\infty} \mu_1(r)e^r\|k^8_x(r)\| \int_r^{+\infty} e^{-s}dsdr \\
&= \int_0^{+\infty} \mu_1(r)\|k^8_x(r)\|dr < \infty.
\end{aligned}$$

In the same way, we get that

$$\int_0^{+\infty} \mu_2(s)\left(\int_0^s e^{r-s} k^{10}_{xx}(r)dr\right) ds \in H^{-1}(0,\pi).$$

Next, we consider the Banach space $\mathbb{H} := H^1_b \times H^1_b \times H^2_* \times L^2 \times L^2$ and equip it with the norm

$$\begin{aligned}
\|(u, v, w, \theta, \vartheta)\|^2_{\mathbb{H}} =&\rho_1 h_1\|u\|^2 + E_1 h_1\|u_x\|^2 + k\|(-u + v + \alpha w_x)\|^2 + \rho_3 h_3\|v\|^2 + E_3 h_3\|v_x\|^2 \\
&+ \rho h\|w\|^2 + EI\|w_{xx}\|^2 + \rho_4\|\theta\|^2 + \rho_5\|\vartheta\|^2.
\end{aligned}$$

On the account of the weak formulation of (3.5), we consider the bilinear form $\mathcal{B}$ on $\mathbb{H} \times \mathbb{H}$ and linear form $\mathcal{L}$ on $\mathbb{H}$, define as follows:

$$\begin{aligned}
&\mathcal{B}((u, v, w, \theta, \vartheta), (u^*, v^*, w^*, \theta^*, \vartheta^*)) \\
:=&\rho_1 h_1\langle u, u^*\rangle + E_1 h_1\langle u_x, u^*_x\rangle + k\langle(-u + v + \alpha w_x), (-u^* + v^* + \alpha w^*_x)\rangle \\
&+ \rho_3 h_3\langle v, v^*\rangle + E_3 h_3\langle v_x, v^*_x\rangle + \rho h\langle w, w^*\rangle + EI\langle w_{xx}, w^*_{xx}\rangle \\
&+ \rho_4\langle\theta, \theta^*\rangle + C_{\eta,\beta_1,\mu_1}\langle\theta_x, \theta^*_x\rangle + \rho_5\langle\vartheta, \vartheta^*\rangle + C_{\eta,\beta_2,\mu_2}\langle\vartheta_x, \vartheta^*_x\rangle,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}((u^*, v^*, w^*, \theta^*, \vartheta^*)) :=&\langle\rho_1 h_1(k^1 + k^2), u^*\rangle + \langle\rho_3 h_3(k^3 + k^4), v^*\rangle + \langle\delta_3 k^5 + \rho h(k^5 + k^6), u^*\rangle \\
&+ \langle\delta_1\left(k^1_x + k^3\right) + \rho_4 k^7, \theta^*\rangle + \langle\beta_1 \int_0^{+\infty} \mu_1(s)\left(\int_0^s e^{r-s} k^8_x(r)dr\right) ds, \theta^*_x\rangle \\
&+ \langle\delta_2 k^3_x + \rho_5 k^9, \vartheta^*\rangle + \langle\beta_2 \int_0^{+\infty} \mu_2(s)\left(\int_0^s e^{r-s} k^{10}_x(r)dr\right) ds, \vartheta^*_x\rangle,
\end{aligned}$$

for every $(u, v, w, \theta, \vartheta)$, $(u^*, v^*, w^*, \theta^*, \vartheta^*) \in \mathbb{H}$. Routine computations, using Cauchy-Schwarz, Young's and Poincaré's inequalities shows that $\mathcal{B}$ is a bounded and coercive bilinear form on $\mathbb{H} \times \mathbb{H}$, and $\mathcal{L}$ is a bounded linear form on $\mathbb{H}$. Therefore, using Lax-Milgram theorem, there exists a unique $(u, v, w, \theta, \vartheta) \in \mathbb{H}$ such that

$$\mathcal{B}((u, v, w, \theta, \vartheta), (u^*, v^*, w^*, \theta^*, \vartheta^*)) = \mathcal{L}((u^*, v^*, w^*, \theta^*, \vartheta^*)),\ \forall\ (u^*, v^*, w^*, \theta^*, \vartheta^*) \in \mathbb{H}.$$

From (3.4), it follows that

$$\varphi \in H_b^1,\ \psi \in H_b^1 \ and\ \phi \in H_*^2.$$

Then, using standard regularity theory, it follows from (3.5), that

$$u, v \in H_b^2 \cap H_b^1,\ w \in H^4 \cap H_*^2,\ \ \theta, \vartheta \in H^2 \cap H_a^1.$$

Since $u, v \in H_b^1, w, k^6 \in H_*^2$ and $k^6 \in L^2$, it easy to see from $(3.5)_3$ that w satisfy

$$w_{xx}(0) = w_{xx}(\pi) = 0.$$

Also, from (3.3), substituting θ and ϑ, we see that

$$\sigma \in \mathcal{D}(L^2_{\mu_1}),\ \ \zeta \in \mathcal{D}(L^2_{\mu_2}).$$

Finally, from $(3.2)_7$ and $(3.2)_9$, using regularity theory, we get that

$$\int_0^{+\infty} \mu_1(s)\sigma(s)ds, \int_0^{+\infty} \mu_2(s)\zeta(s)ds \in H^2 \cap H_a^1.$$

Thus, $\Psi = (u, \varphi, v, \psi, w, \phi, \theta, \sigma, \vartheta, \zeta) \in \mathcal{D}(\mathcal{A})$ and satisfies (3.1). That is, the operator $\mathcal{A}$ is maximal. □

3.2. Well-posedness Result

Theorem 3.1. Suppose $\Psi_0 = (u_0, u_1, v_0, v_1, w_0, w_1, \theta_0, \sigma_0, \vartheta_0, \zeta_0) \in \mathcal{H}$ is given and condition (A_1) holds, then the Cauchy problem (2.13) has a unique weak global solution

$$\Psi \in C([0, +\infty), \mathcal{H}).$$

Furthermore, if $\Psi_0 = (u_0, u_1, v_0, v_1, w_0, w_1, \theta_0, \sigma_0, \vartheta_0, \zeta_0) \in \mathcal{D}(\mathcal{A})$, then the solution is in the class

$$\Psi \in C([0, \infty), \mathcal{D}(\mathcal{A})) \cap C^1([0, \infty), \mathcal{H}).$$

Proof. On account of Lemmas 3.1 and 3.2 applying the Hille-Yosida theorem, we have that $\mathcal{A}$ is a generator of a C_0-semigroup of contractions $\mathcal{S}(t) = e^{\mathcal{A}t},\ t \geq 0$, on $\mathcal{H}$. By the semigroup theory for linear operators (Pazy [31]), we get that

$$\Psi(t) = \mathcal{S}(t)\Psi_0,\ t \geq 0,$$

on $\mathcal{H}$ is a unique solution satisfying problem (2.13). □

4. Stability result

In this section, we study the stability of solution of (2.10)–(2.12). The energy functional associated to the solution $\Psi = (u, u_t, v, v_t, w, w_t, \theta, \sigma, \vartheta, \zeta)$ of system (2.10)–(2.12) is defined by

$$\begin{aligned}\mathcal{E}(t) =& \frac{1}{2}\left[\rho_1 h_1\|u_t\|^2 + \rho_3 h_3\|v_t\|^2 + \rho h\|w_t\|^2 + E_1 h_1\|u_x\|^2 + E_3 h_3\|v_x\|^2 + EI\|w_{xx}\|^2\right]\\ &+ \frac{1}{2}\left[k\|(-u + v + \alpha w_x)\|^2 + \rho_4\|\theta\|^2 + \beta_1\|\sigma\|^2_{L^2_{\mu_1}} + \rho_5\|\vartheta\|^2 + \beta_2\|\zeta\|^2_{L^2_{\mu_2}}\right], \quad \forall\, t \geq 0.\end{aligned} \tag{4.1}$$

4.1. Needed lemmas for stability

Lemma 4.1. *Under the conditions of Theorem 3.1, the energy functional* (4.1) *satisfies*

$$\mathcal{E}'(t) = -\delta_3\|w_t\|^2 + \frac{\beta_1}{2}\int_0^{+\infty}\mu_1'(s)\|\sigma_x(s)\|^2 ds + \frac{\beta_2}{2}\int_0^{+\infty}\mu_2'(s)\|\zeta_x(s)\|^2 ds \leq 0, \forall t \geq 0. \tag{4.2}$$

Proof. Multiplication in $L^2(0,\pi)$ the Eq $(2.10)_1, (2.10)_2, (2.10)_3$, $(2.10)_4$ and $(2.10)_6$ by u_t, v_t, w_t, θ and ϑ respectively, follow by multiplying $(2.10)_5$ and $(2.10)_7$ by σ and ζ in $L^2_{\mu_1}$ and $L^2_{\mu_2}$ respectively, then using integration by parts and the boundary conditions (2.11), we have

$$\frac{1}{2}\frac{d}{dt}\left[\rho_1 h_1\|u_t\|^2 + E_1 h_1\|u_x\|^2\right] - \langle k(-u + v + \alpha w_x), u_t\rangle - \delta_1\langle\theta, u_{xt}\rangle = 0, \tag{4.3}$$

$$\frac{1}{2}\frac{d}{dt}\left[\rho_3 h_3\|v_t\|^2 + E_3 h_3\|v_x\|^2\right] + \langle k(-u + v + \alpha w_x), v_t\rangle - \delta_1\langle\theta, v_t\rangle - \delta_2\langle\vartheta, v_{xt}\rangle = 0, \tag{4.4}$$

$$\frac{1}{2}\frac{d}{dt}\left[\rho h\|w_t\|^2 + EI\|w_{xx}\|^2\right] + \langle k(-u + v + \alpha w_x), \alpha w_{xt}\rangle + \delta_3\|w_t\|^2 = 0, \tag{4.5}$$

$$\frac{1}{2}\frac{d}{dt}\left[\rho_4\|\theta\|^2\right] + \beta_1\int_0^{+\infty}\mu_1(s)\langle\sigma_x(s), \theta_x(t)\rangle ds + \delta_1\langle\theta, (u_{xt} + v_t)\rangle = 0, \tag{4.6}$$

$$\frac{1}{2}\frac{d}{dt}\left[\beta_1\|\sigma\|^2_{L^2_{\mu_1}}\right] - \frac{\beta_1}{2}\int_0^{+\infty}\mu_1'(s)\|\sigma_x(s)\|^2 ds - \beta_1\int_0^{+\infty}\mu_1(s)\langle\sigma_x(s), \theta_x(t)\rangle ds = 0, \tag{4.7}$$

$$\frac{1}{2}\frac{d}{dt}\left[\rho_5\|\vartheta\|^2\right] + \beta_2\int_0^{+\infty}\mu_2(s)\langle\zeta_x(s), \vartheta_x(t)\rangle ds + \delta_2\langle\vartheta, v_{xt}\rangle = 0, \tag{4.8}$$

and

$$\frac{1}{2}\frac{d}{dt}\left[\beta_2\|\zeta\|^2_{L^2_{\mu_2}}\right] - \frac{\beta_2}{2}\int_0^{+\infty}\mu_2'(s)\|\zeta_x(s)\|^2 ds - \beta_1\int_0^{+\infty}\mu_2(s)\langle\zeta_x(s), \vartheta_x(t)\rangle ds = 0. \tag{4.9}$$

Addition of (4.3)–(4.9) leads to

$$\mathcal{E}'(t) = -\delta_3\|w_t\|^2 + \frac{\beta_1}{2}\int_0^{+\infty}\mu_1'(s)\|\sigma_x(s)\|^2 ds + \frac{\beta_2}{2}\int_0^{+\infty}\mu_2'(s)\|\zeta_x(s)\|^2 ds \leq 0. \tag{4.10}$$

Therefore, the energy $\mathcal{E}$ is non-increasing and bounded above by $\mathcal{E}(0)$. Also, the computations here are done for regular solution. However, the result remains true for weak solution by density argument. □

Lemma 4.2. *Let $\Psi = (u, u_t, v, v_t, w, w_t, \theta, \sigma, \vartheta, \zeta) \in \mathcal{H}$ be the solution of system* (2.10)–(2.12) *given by Theorem 3.1, then the functional G_1 defined by*

$$G_1(t) = \rho_1 h_1 \langle u_t, u\rangle + \rho_3 h_3 \langle v_t, v\rangle + \rho h\langle w_t, w\rangle + \frac{\delta_3}{2}\|w\|^2$$

satisfies the estimate

$$\begin{aligned} G_1'(t) \leq & -\frac{E_1 h_1}{2}\|u_x\|^2 - \frac{E_3 h_3}{2}\|v_x\|^2 - EI\|w_{xx}\|^2 - k\|(-u + v + \alpha w_x)\|^2 \\ & + \rho_1 h_1 \|u_t\|^2 + \rho_3 h_3 \|v_t\|^2 + \rho h\|w_t\|^2 + C\|\theta\|^2 + C\|\vartheta\|^2, \ \forall\, t \geq 0. \end{aligned} \tag{4.11}$$

Proof. Differentiation of G_1 gives

$$\begin{aligned} G_1'(t) =& \rho_1 h_1 \langle u_{tt}, u\rangle + \rho_3 h_3 \langle v_{tt}, v\rangle + \rho h\langle w_{tt}, w\rangle + \delta_3 \langle w_t, w\rangle \\ & + \rho_1 h_1 \|u_t\|^2 + \rho_3 h_3 \|v_t\|^2 + \rho h\|w_t\|^2. \end{aligned}$$

Using Eq $(2.10)_1$, $(2.10)_2$ and $(2.10)_3$, then applying integration by parts over $(0, \pi)$ and making use of the boundary conditions (2.11) leads to

$$\begin{aligned} G_1'(t) =& -E_1 h_1 \|u_x\|^2 - E_3 h_3 \|v_x\|^2 - EI\|w_{xx}\|^2 - k\|(-u + v + \alpha w_x)\|^2 \\ & + \delta_1 \langle u_x, \theta\rangle + \delta_1 \langle v, \theta\rangle + \delta_2 \langle v_x, \vartheta\rangle + \rho_1 h_1 \|u_t\|^2 + \rho_3 h_3 \|v_t\|^2 + \rho h\|w_t\|^2. \end{aligned}$$

Applying Young's and Poincaré's inequalities, we obtain

$$\begin{aligned} G_1'(t) \leq & -\frac{E_1 h_1}{2}\|u_x\|^2 - \frac{E_3 h_3}{2}\|v_x\|^2 - EI\|w_{xx}\|^2 - k\|(-u + v + \alpha w_x)\|^2 \\ & + \rho_1 h_1 \|u_t\|^2 + \rho_3 h_3 \|v_t\|^2 + \rho h\|w_t\|^2 + C\|\theta\|^2 + C\|\vartheta\|^2. \end{aligned}$$

□

Lemma 4.3. *Let $\Psi = (u, u_t, v, v_t, w, w_t, \theta, \sigma, \vartheta, \zeta) \in \mathcal{H}$ be the solution of system* (2.10)–(2.12) *given by Theorem 3.1, then the functional G_2 defined by*

$$G_2(t) = -\rho_1 h_1 \rho_4 \langle \theta, \widehat{u_t}(t)\rangle, \text{ where } \widehat{u_t}(t) = \int_0^x u_t(y, t)dydx$$

satisfies, for any $\epsilon_1 > 0$ and $\epsilon_2 > 0$, the the estimate

$$\begin{aligned} G_2'(t) \leq & -\frac{\rho_1 h_1 \delta_1}{2}\|u_t\|^2 + \epsilon_1 \|u_x\|^2 + \epsilon_2 \|(-u + v + \alpha w_x)\|^2 \\ & + C\|v_t\|^2 + C\|\sigma\|^2_{L^2_{\mu_1}} + C\left(1 + \frac{1}{\epsilon_1} + \frac{1}{\epsilon_2}\right)\|\theta\|^2, \ \forall\, t \geq 0. \end{aligned} \tag{4.12}$$

Proof. Differentiation of G_2, using $(2.10)_1$ and $(2.10)_4$, integration by parts and boundary conditions (2.11), we arrive at

$$\begin{aligned} G_2'(t) =& -\rho_1 h_1 \rho_4 \langle \theta, \widehat{u_{tt}}(t)\rangle - \rho_1 h_1 \rho_4 \langle \theta_t, \widehat{u_t}(t)\rangle \\ =& -\rho_1 h_1 \delta_1 \|u_t\|^2 - \rho_4 E_1 h_1 \langle \theta, u_x\rangle + \rho_1 h_1 \delta_1 \langle v_t, \widehat{u_t}(t)\rangle \end{aligned}$$

$$- \rho_4 k\langle \theta, (\widehat{-u + v + \alpha w_x})\rangle + \rho_3\delta_1\|\theta\|^2 + \rho_1 h_1 \beta_1 \langle u_t, \int_0^{+\infty} \mu_1(s)\sigma_x(., t, s)ds\rangle.$$

Using Cauchy-Schwarz, Young's and Poincaré's inequalities yields

$$\begin{aligned} G_2'(t) \leq & - \rho_1 h_1 \delta_1 \|u_t\|^2 + \epsilon_1 \|u_x\|^2 + \frac{(\rho_4 E_1 h_1)^2}{4\epsilon_1}\|\theta\|^2 + \frac{3\rho_1 h_1 \delta_1}{4}\|v_t\|^2 \\ & + \frac{\rho_1 h_1 \delta_1}{4}\|u_t\|^2 + \epsilon_2\|(-u + v + \alpha w_x)\|^2 + \frac{(\rho_4 k)^2}{4\epsilon_2}\|\theta\|^2 \\ & + \rho_3\delta_1\|\theta\|^2 + \frac{\rho_1 h_1 \delta_1}{4}\|u_t\|^2 + \frac{3\rho_1 h_1 \beta_1^2}{4\delta_1}\|\sigma\|^2_{L^2_{\mu_1}}. \end{aligned}$$

Thus, we obtain (4.12). □

Lemma 4.4. *Let* $\Psi = (u, u_t, v, v_t, w, w_t, \theta, \sigma, \vartheta, \zeta) \in \mathcal{H}$ *be the solution of system* (2.10)–(2.12) *given by Theorem 3.1, then the functional* G_3 *defined by*

$$G_3(t) = -\rho_3 h_3 \rho_5 \langle \vartheta, \widehat{v_t}(t)\rangle, \text{ where } \widehat{v_t}(t) = \int_0^x v_t(y, t)dy$$

satisfies, for any $\epsilon_3 > 0$ *and* $\epsilon_4 > 0$, *the estimate*

$$\begin{aligned} G_3'(t) \leq & - \frac{\rho_3 h_3 \delta_2}{2}\|v_t\|^2 + \epsilon_3\|v_x\|^2 + \epsilon_4\|(-u + v + \alpha w_x)\|^2 \\ & + C\|\theta\|^2 + C\|\zeta\|^2_{L^2_{\mu_2}} + C\left(1 + \frac{1}{\epsilon_3} + \frac{1}{\epsilon_4}\right)\|\vartheta\|^2, \ \forall\, t \geq 0. \end{aligned} \tag{4.13}$$

Proof. Differentiation of G_3, using $(2.10)_2$ and $(2.10)_5$, integration by parts and boundary conditions (2.11), we arrive at

$$\begin{aligned} G_3'(t) = & - \rho_3 h_3 \rho_5 \langle \vartheta, \widehat{v_{tt}}(t)\rangle - \rho_3 h_3 \rho_5 \langle \vartheta_t, \widehat{v_t}(t)\rangle \\ = & - \rho_3 h_3 \delta_2 \|v_t\|^2 - \rho_5 E_3 h_3 \langle \vartheta, v_x\rangle - \rho_5 \delta_1 \langle \vartheta, \widehat{\theta}(t)\rangle + \rho_5 k \langle \vartheta, (\widehat{-u + v + \alpha w_x})\rangle \\ & + \rho_5 \delta_2 \|\vartheta\|^2 + \rho_3 h_3 \beta_2 \langle v_t, \int_0^{+\infty} \mu_2(s)\zeta_x(., t, s)ds\rangle. \end{aligned}$$

Applying Cauchy-Schwarz, Young's and Poincaré's inequalities, we have

$$\begin{aligned} G_3'(t) \leq & - \rho_3 h_3 \delta_2 \|v_t\|^2 + \epsilon_3\|v_x\|^2 + \frac{(\rho_5 E_3 h_3)^2}{4\epsilon_3}\|\vartheta\|^2 + \frac{\rho_5 \delta_1}{2}\|\theta\|^2 \\ & + \frac{\rho_5 \delta_1}{2}\|\vartheta\|^2 + \epsilon_4\|(-u + v + \alpha w_x)\|^2 + \frac{(\rho_5 k)^2}{4\epsilon_4}\|\vartheta\|^2 \\ & + \rho_5 \delta_2 \|\vartheta\|^2 + \frac{\rho_3 h_3 \delta_2}{4}\|v_t\|^2 + \frac{3\rho_3 h_3 \beta_2^2}{4\delta_2}\|\zeta\|^2_{L^2_{\mu_2}}. \end{aligned}$$

Hence, we get (4.13). □

Lemma 4.5. *Let* $\Psi = (u, u_t, v, v_t, w, w_t, \theta, \sigma, \vartheta, \zeta) \in \mathcal{H}$ *be the solution of system* (2.10)–(2.12) *given by Theorem 3.1, then the functional* G_4 *defined by*

$$G_4(t) = -\rho_4\langle\theta, \int_0^{+\infty} \mu_1(s)\sigma(., t, s)ds\rangle,$$

satisfies, for any $\epsilon_5 > 0$ *and* $\epsilon_6 > 0$, *the estimate*

$$G_4'(t) \leq -\frac{\rho_4 g_1(0)}{2}\|\theta\|^2 + \epsilon_5\|u_t\|^2 + \epsilon_6\|v_t\|^2 - C\int_0^{+\infty} \mu_1'(s)\|\sigma_x(s)\|^2 ds \tag{4.14}$$

$$+ C\left(1 + \frac{1}{\epsilon_5} + \frac{1}{\epsilon_6}\right)\|\sigma\|^2_{L^2_{\mu_1}}, \ \forall\, t \geq 0. \tag{4.15}$$

Proof. Differentiating G_4 with respect to t, using $(2.10)_4$ and $(2.10)_5$, integration by parts and the boundary conditions (2.11) and recalling (2.4), we get

$$\begin{aligned} G_4'(t) =& -\rho_4\langle\theta_t, \int_0^{+\infty} \mu_1(s)\sigma(., t, s)ds\rangle - \rho_4\langle\theta, \int_0^{+\infty} \mu_1(s)\sigma_t(., t, s)ds\rangle \\ =& -\rho_4 g_1(0)\|\theta\|^2 + \beta_1\left\|\int_0^{+\infty} \mu_1(s)\sigma_x(., t, s)ds\right\|^2 \\ & - \delta_1\langle u_t, \int_0^{+\infty} \mu_1(s)\sigma_x(., t, s)ds\rangle + \delta_1\langle v_t, \int_0^{+\infty} \mu_1(s)\sigma_x(., t, s)ds\rangle \\ & + \rho_4\langle\theta, \int_0^{+\infty} \mu_1(s)\sigma_s(., t, s)ds\rangle. \end{aligned}$$

Making use of Cauchy-Schwarz and Young's inequalities, we have

$$\beta_1\left\|\int_0^{+\infty} \mu_1(s)\sigma_x(., t, s)ds\right\|^2 \leq C\|\sigma\|^2_{L^2_{\mu_1}}, \tag{4.16}$$

$$\left|-\delta_1\langle u_t, \int_0^{+\infty} \mu_1(s)\sigma_x(., t, s)ds\rangle\right| \leq \epsilon_5\|u_t\|^2 + \frac{C}{\epsilon_5}\|\sigma\|^2_{L^2_{\mu_1}}, \text{ for any } \epsilon_5 > 0, \tag{4.17}$$

$$\left|\delta_1\langle v_t, \int_0^{+\infty} \mu_1(s)\sigma_x(., t, s)ds\rangle\right| \leq \epsilon_6\|v_t\|^2 + \frac{C}{\epsilon_6}\|\sigma\|^2_{L^2_{\mu_1}}, \text{ for any } \epsilon_6 > 0. \tag{4.18}$$

Also, using integration by parts with respect to s, we get

$$\begin{aligned} &\left|\rho_4\langle\theta, \int_0^{+\infty} \mu_1(s)\sigma_s(., t, s)ds\rangle\right| \\ =&\left|-\rho_4\langle\theta, \int_0^{+\infty} \mu_1(s)\sigma'(., t, s)ds\rangle\right| \\ \leq& C\|\theta\|\left(-\int_0^{+\infty} \mu_1'(s)\|\sigma_x\|^2 ds\right)^{\frac{1}{2}} \\ \leq& \frac{\rho_4 g_1(0)}{2}\|\theta\|^2 - C\int_0^{+\infty} \mu_1'(s)\|\sigma_x(s)\|^2 ds. \end{aligned} \tag{4.19}$$

On account of (4.16)–(4.19), we obtain

$$G_4'(t) \leq -\frac{\rho_4 g_1(0)}{2}\|\theta\|^2 + \epsilon_5\|u_t\|^2 + \epsilon_6\|v_t\|^2 - C\int_0^{+\infty}\mu_1'(s)\|\sigma_x(s)\|^2 ds$$
$$+ C\left(1 + \frac{1}{\epsilon_5} + \frac{1}{\epsilon_6}\right)\|\sigma\|^2_{L^2_{\mu_1}}.$$

□

Lemma 4.6. *Let* $\Psi = (u, u_t, v, v_t, w, w_t, \theta, \sigma, \vartheta, \zeta) \in \mathcal{H}$ *be the solution of system* (2.10)–(2.12) *given by Theorem 3.1, then the functional* G_5 *defined by*

$$G_5(t) = -\rho_5\langle\vartheta, \int_0^{+\infty}\mu_2(s)\zeta(., t, s)ds\rangle,$$

satisfies for any $\epsilon_7 > 0$*, the estimate*

$$G_5'(t)(t) \leq -\frac{\rho_5 g_2(0)}{2}\|\vartheta\|^2 + \epsilon_7\|v_t\|^2 - C\int_0^{+\infty}\mu_2'(s)\|\zeta_x(s)\|^2 ds + C\left(1 + \frac{1}{\epsilon_7}\right)\|\zeta\|^2_{L^2_{\mu_2}}, \ \forall\, t \geq 0. \tag{4.20}$$

Proof. Differentiation of G_5 with respect to t, using $(2.10)_6$ and $(2.10)_7$, integration by parts and the boundary conditions (2.11), and recalling (2.4), we get

$$\begin{aligned} G_5' =& -\rho_5\langle\vartheta_t, \int_0^{+\infty}\mu_2(s)\zeta(., t, s)ds\rangle - \rho_5\langle\vartheta, \int_0^{+\infty}\mu_2(s)\zeta_t(., t, s)ds\rangle \\ =& -\rho_5 g_1(0)\|\vartheta\|^2 + \beta_2\left\|\int_0^{+\infty}\mu_2(s)\zeta_x(., t, s)ds\right\|^2 \\ & - \delta_2\langle v_t, \int_0^{+\infty}\mu_2(s)\zeta_x(., t, s)ds\rangle + \rho_5\langle\vartheta, \int_0^{+\infty}\mu_2(s)\zeta_s(., t, s)ds\rangle. \end{aligned}$$

Using similar estimations as in (4.16)–(4.19) leads to (4.20). □

4.2. Main stability result

The main stability result of this work is the following:

Theorem 4.1. *Let* $\Psi_0 = (u_0, u_1, v_0, v_1, w_0, w_1, \theta_0, \sigma_0, \vartheta_0, \zeta_0) \in \mathcal{D}(\mathcal{A})$ *be given. Suppose condition* (A_1) *holds, then the energy functional* $\mathcal{E}(t)$ *defined in* (4.1) *decays exponentially. That is, there exists positive constants* M *and* λ *such that*

$$\mathcal{E}(t) \leq Me^{-\lambda t}, \ \forall t \geq 0. \tag{4.21}$$

Proof. We set

$$L(t) := N\mathcal{E}(t) + N_1G_1(t) + N_2G_2(t) + N_3G_3(t) + N_4G_4(t) + N_5G_5(t), \quad t \geq 0, \tag{4.22}$$

for some $N, N_1, N_2, N_3, N_4, N_5 > 0$ to be specified later. Direct computations, applying Young's, Cauchy-Schwarz and Poincaré's inequalities gives

$$\tilde{b}_1\mathcal{E}(t) \leq L(t) \leq \tilde{b}_2\mathcal{E}(t), \quad t \geq 0, \tag{4.23}$$

for some positive constants $\tilde{b}_1$ and $\tilde{b}_2$. Now, using Lemmas 4.1 and 4.2–4.6, we get

$$
\begin{aligned}
L'(t) \leq & -\left[\frac{\rho_1 h_1 \delta_1}{2} N_2 - \rho_1 h_1 N_1 - \epsilon_5 N_4\right] \|u_t\|^2 - [\delta_3 N - \rho h N_1] \|w_t\|^2 \\
& - \left[\frac{\rho_3 h_3 \delta_2}{2} N_3 - \rho_3 h_3 N_1 - C N_2 - \epsilon_6 N_4 - \epsilon_7 N_5\right] \|v_t\|^2 \\
& - \left[\frac{E_1 h_1}{2} N_1 - \epsilon_1 N_2\right] \|u_x\|^2 - \left[\frac{E_3 h_3}{2} N_1 - \epsilon_3 N_3\right] \|v_x\|^2 - EIN_1 \|w_{xx}\|^2 \\
& - [k N_1 - \epsilon_2 N_2 - \epsilon_4 N_3] \|(-u + v + \alpha w_x)\|^2 \\
& - \left[\frac{\rho_4 g_1(0)}{2} N_4 - C N_1 - C N_2 \left(1 + \frac{1}{\epsilon_1} + \frac{1}{\epsilon_2}\right) - C N_3\right] \|\theta\|^2 \\
& + \left[C N_2 + C N_4 \left(1 + \frac{1}{\epsilon_5} + \frac{1}{\epsilon_6}\right)\right] \|\sigma\|^2_{L^2_{\mu_1}} - \left[\frac{\beta_1}{2} N - C N_4\right] \int_0^{+\infty} \mu_1'(s) \|\sigma_x(s)\|^2 ds \\
& - \left[\frac{\rho_5 g_2(0)}{2} N_5 - C N_1 - C N_3 \left(1 + \frac{1}{\epsilon_3} + \frac{1}{\epsilon_4}\right)\right] \|\vartheta\|^2 \\
& + \left[C N_3 + C N_5 \left(1 + \frac{1}{\epsilon_7}\right)\right] \|\zeta\|^2_{L^2_{\mu_2}} - \left[\frac{\beta_2}{2} N - C N_5\right] \int_0^{+\infty} \mu_2'(s) \|\zeta_x(s)\|^2 ds.
\end{aligned} \tag{4.24}
$$

From (2.5), we have that

$$
\mu_i(s) \leq -\frac{1}{\xi_i} \mu_i'(s), \quad i = 1, 2.
$$

Also, by choosing

$$
N_1 = 1, \epsilon_1 = \frac{E_1 h_1}{4 N_2}, \ \epsilon_2 = \frac{k}{4 N_2}, \ \epsilon_3 = \frac{E_3 h_3}{4 N_3}, \ \epsilon_4 = \frac{k}{4 N_3},
$$
$$
\epsilon_5 = \frac{\rho_1 h_1 \delta_1}{4 N_4}, \ \ \epsilon_6 = \frac{\rho_3 h_3 \delta_2}{8 N_4}, \ \ \epsilon_7 = \frac{\rho_3 h_3 \delta_2}{8 N_5},
$$

then (4.24) takes the form

$$
\begin{aligned}
L'(t) \leq & -\left[\frac{\rho_1 h_1 \delta_1}{4} N_2 - \rho_1 h_1\right] \|u_t\|^2 - \left[\frac{\rho_3 h_3 \delta_2}{4} N_3 - C N_2 - \rho_3 h_3\right] \|v_t\|^2 \\
& - [\delta_3 N - \rho h] \|w_t\|^2 - \frac{E_1 h_1}{4} \|u_x\|^2 - \frac{E_3 h_3}{4} \|v_x\|^2 \\
& - EI \|w_{xx}\|^2 - \frac{k}{2} \|(-u + v + \alpha w_x)\|^2 \\
& - \left[\frac{\rho_4 g_1(0)}{2} N_4 - C N_2 \left(1 + \frac{4 N_2}{E_1 h_1} + \frac{4 N_2}{k}\right) - C N_3 - C\right] \|\theta\|^2 \\
& - \left[\frac{\beta_1 \xi_1}{2} N - C \xi_1 N_4 - \left(C N_2 + C N_4 \left(1 + \frac{4 N_4}{\rho_1 h_1 \delta_1} + \frac{8 N_4}{\rho_3 h_3 \delta_2}\right)\right)\right] \|\sigma\|^2_{L^2_{\mu_1}} \\
& - \left[\frac{\rho_5 g_2(0)}{2} N_5 - C N_3 \left(1 + \frac{4 N_3}{E_3 h_3} + \frac{4 N_3}{k}\right) - C\right] \|\vartheta\|^2 \\
& - \left[\frac{\beta_2 \xi_2}{2} N - C \xi_2 N_5 - \left(C N_3 + C N_5 \left(1 + \frac{8 N_5}{\rho_3 h_3 \delta_2}\right)\right)\right] \|\zeta\|^2_{L^2_{\mu_2}}.
\end{aligned} \tag{4.25}
$$

Next, we specified the rest of the parameters. First, we choose N_2 large such that

$$\frac{\rho_1 h_1 \delta_1}{4} N_2 - \rho_1 h_1 > 0.$$

Second, we select N_3 large enough such that

$$\frac{\rho_3 h_3 \delta_2}{4} N_3 - CN_2 - \rho_3 h_3 > 0.$$

Thirdly, we choose N_4 and N_5 large enough such that

$$\frac{\rho_4 g_1(0)}{2} N_4 - CN_2 \left(1 + \frac{4N_2}{E_1 h_1} + \frac{4N_2}{k}\right) - CN_3 - C > 0,$$

and

$$\frac{\rho_4 h_2(0)}{2} N_5 - CN_3 \left(1 + \frac{8N_3}{k} + \frac{4N_3}{b}\right) - C > 0.$$

Finally, we choose N very large so that (4.23) remain valid and

$$\delta_3 N - \rho h > 0, \ \frac{\beta_1 \xi_1}{2} N - C\xi_1 N_4 - \left(CN_2 + CN_4 \left(1 + \frac{4N_4}{\rho_1 h_1 \delta_1} + \frac{8N_4}{\rho_3 h_3 \delta_2}\right)\right) > 0,$$

$$\frac{\beta_2 \xi_2}{2} N - C\xi_2 N_5 - \left(CN_3 + CN_5 \left(1 + \frac{8N_5}{\rho_3 h_3 \delta_2}\right)\right) > 0.$$

Thus, we obtain

$$\begin{aligned} L'(t) \leq & - \gamma_0 \left[\|u_t\|^2 + \|v_t\|^2 + \|w_t\|^2 + \|u_x\|^2 + \|v_x\|^2 + \|w_{xx}\|^2\right] \\ & - \gamma_0 \left[\|(-u + v + \alpha w_x)\|^2 + |\theta\|^2 + \|\sigma\|^2_{L^2_{\mu_1}} + \|\vartheta\|^2 + \|\zeta\|^2_{L^2_{\mu_2}}\right] \end{aligned} \tag{4.26}$$

for some $\gamma_0 > 0$. Recalling (4.1), it follows from (4.26) that

$$L'(t) \leq -\gamma_1 \mathcal{E}(t), \ \forall \ t \geq 0, \tag{4.27}$$

for some $\gamma_1 > 0$. Using (4.23), we obtain

$$L'(t) \leq -\gamma_2 L(t), \ \forall \ t \geq 0, \tag{4.28}$$

for some $\gamma_2 > 0$. Integrating (4.28) over $(0, t)$ yields for some $\gamma_3 > 0$

$$L(t) \leq L(0) e^{-\gamma_3 t}, \ \forall \ t \geq 0. \tag{4.29}$$

Hence, the exponential estimate of the energy functional $\mathcal{E}(t)$ in (4.21) follows from (4.29) by using (4.23). This completes the proof. □

5. Conclusions

In this work, we investigated the the effect of Gurtin-Pipkin's thermal law on the outer layers of the Rao-Nakra beam model. Using standard semi-group theory for linear operators and the multiplier method, the well-posedness and a stability result of solutions of the triple beam system have been established.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no potential conflict of interest.

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