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Abstract: Graph literacy, the ability to interpret and create graphical representations, is an important skill for students to learn mathematics and to succeed in STEM coursework and careers. Additionally, with the rapid development of technological devices and media, students are encountering increasingly more situations in which graph literacy is needed to make sense of and respond to information. In this paper, I present a conceptual analysis of what I consider the three layers constituting a graphical representation: frames of reference, coordinate system, and graph. Relatedly, I synthesize relevant literature and propose an analytical framework that could be used to make sense of students' representations of spatial phenomena. The analytical framework was designed to answer the following question: How do students make sense of a spatial situation, establish frames of reference, and create re-presentations of the spatial situation via coordinated measurements? After I elaborate on the analytical framework, I draw data from teaching experiments to illustrate how the framework could be used to model students' representations of spatial phenomena with attention to frames of reference and coordinate systems. Finally, I discuss how this framework could be used by researchers and educators for better attending to students' graph thinking and making sense of students' graph literacy.

Keywords: Conceptual analysis, graphical representation, frames of reference, coordinate systems

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Introduction

Graphical representations are commonly used in STEM fields to model, communicate, and analyze real-world phenomena. Relatedly, the ability to interpret and create graphical representations is important for students to learn mathematics and to succeed in STEM coursework and careers (Costa, 2020). More broadly, with the rapid development of technological devices and media, students are encountering increasingly more situations in which graph literacy is needed to make sense of and respond to information (Yore et al., 2007). Undoubtedly, understanding and supporting students' development of graph literacy is important.

As the National Council of Teachers of Mathematics (2000) stated, "[t]he fact that representations are such effective tools may obscure how difficult it was to develop them and, more important, how much work it takes to understand them" (p. 68). A plethora of studies on students' graph literacy have documented how difficult graphical representations can be to understand. Students' difficulties identified in the literature include treating graphs as literal representations of a situation (Clement, 1989; Lai et al., 2016); focusing on one quantity when prompted to interpret relationships between two quantities (Leinhardt et al., 1990; Oehrtman et al., 2008); struggling with establishing axes/scales, transitioning from discrete to continuous graphs (Herscovics, 1989; Mevarech & Kramarsky, 1997); and adhering to familiar forms of regularity like linearity (Leinhardt et al., 1990).

From these studies, we know that understanding graphs could be challenging and that students across K-12+ grade levels struggle with graphical representations. However, students' difficulties with graphical representations are not so surprising considering the way they are introduced in curricular materials and how students are assumed to develop proficiency in using them in relatively unproblematic ways. That is, often times, students are taught the conventions of graphical representations but less frequently given the opportunity to consider why such conventions work (e.g., why two number lines are placed perpendicularly intersecting at zero and how doing such a thing defines points in a plane when creating a Cartesian plane). Moreover, some researchers have voiced concerns about research focusing on what students cannot do or struggle with, and called for more studies investigating the demands or abilities that could be involved in solving graphing tasks (e.g., Åberg-Bengtsson & Ottosson, 2006; Ring et al., 2019).

Some researchers have identified reasoning skills or content-specific knowledge that could support productive graph understandings. For example, studies have shown that students' conceptions of number lines constituting Cartesian axes (spatial ordered-ness and positioning of numerical values) or table-reading abilities impact students' graphing activities (Earnest, 2015; Mevarech & Kramarsky, 1997; Sarama et al., 2003; Sharma, 2006). Others have identified students' covariational reasoning (Carlson et al., 2002; Rodriguez et al., 2019; Saldanha & Thompson, 1998; Thompson & Carlson, 2017; Thompson et al., 2017), emergent shape thinking (Moore & Thompson, 2015), and operative thinking (Lee et al., 2019; Moore, 2016) as critical reasoning skills that support students' graph understandings.

These research efforts have contributed to our field's understanding of students' graph thinking. However, there has been a heavy focus on students' interpretations of pre-constructed graphs (Hattikudur et al., 2012). While interpreting graphical representations is one important component of graph literacy, equally critical is the ability to create graphical representations (Fry, 1981). In fact, NCTM (2000) emphasized, "[i]t is important that students have opportunities not only to learn conventional forms of representation but also to construct, refine, and use their own representations as tools to support learning and doing mathematics" (p. 68).

In sum, more needs to be learned about students' creations of graphical representations and the relevant graph thinking students can build upon. As an exemplar, diSessa et al. (1991) and Sherin (2000) documented students' generative activities and intuitions when creating graphs in a collaborative classroom context. These constructive resources included making drawings, intuitions on temporal sequencing, and using various features of a line segment. In line with this focus on students' constructive resources, my work, some of which I present in this paper, focuses on better understanding how students re-present[1] phenomena through their creations of graphical representations. Taking a different approach from diSessa et al. and Sherin, I distinguish three layers constituting a graphical representation and investigate students' reasoning within and across these layers.

In this paper, I present a conceptual analysis (Thompson, 2008) of what I consider the three layers constituting a graphical representation, synthesize relevant literature, and propose an

[1] I use re-presentation to emphasize the active role of the cognizing subject. That is, a re-presentation should be recognized by the user, associated with an abstracted experience, and bring forth an associated meaning to the user (von Glasersfeld, 1991).

analytical framework that could be used to make sense of students' representations of spatial phenomena. The analytical framework was designed to answer the following question: How do students make sense of a spatial situation, establish frames of reference, and create representations of the spatial situation via coordinated measurements? After elaborating on the analytical framework, I draw cases from teaching experiment (Steffe & Thompson, 2000) data to illustrate how the framework could be used to model students' representations of spatial phenomena with attention to frames of reference and coordinate systems. Finally, I discuss how this framework could be used by researchers, educators, and curriculum developers for better attending to students' graph thinking and making sense of students' graph literacy.

A Conceptual Analysis: Three Layers Constituting Graphical Representations

Thompson (2008) presented four ways in which conceptual analysis can be used:

(1) in building models of what students actually know at some specific time and what they comprehend in specific situations,

(2) in describing ways of knowing that might be propitious for students' mathematical learning, and

(3) in describing ways of knowing that might be deleterious to students' understanding of important ideas and in describing ways of knowing that might be problematic in specific situations,

(4) in analyzing the coherence, or fit, of various ways of understanding a body of ideas. Each is described in terms of their meanings, and their meanings can then be inspected in regard to their mutual compatibility and mutual support. (p. 60)

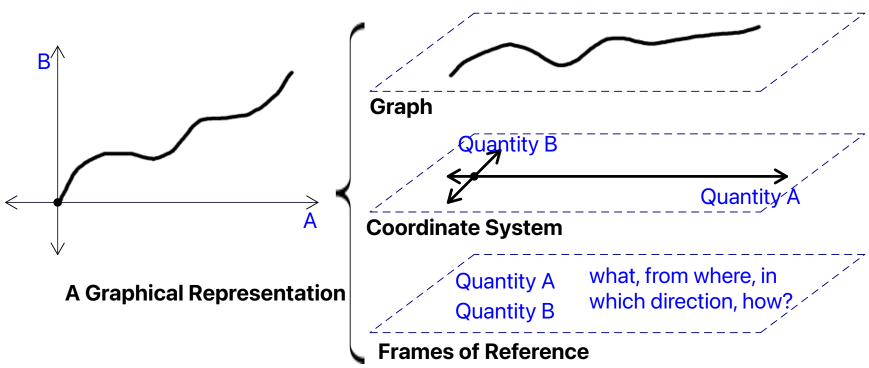
In this section, I first present a conceptual analysis (Thompson, 2008) of what it means to have a graphical representation of some phenomena to address (2) and (4) above. Second, I propose an analytical framework based on the conceptual analysis for making sense of students' graphical representations with attention to students' frames of reference and coordinate systems. In the following section, I exemplify the use of this framework to provide an example of (1) above with student data. Note that although presented linearly, my engagement in conceptual analysis, building an analytical framework, and analysis of student thinking occurred in tandem and each activity reciprocally interplayed with others.

Graphical Representations: Three Layers

By graphical representation, I mean a spatial depiction of quantities (Thompson, 2011) used to mathematize some phenomenon. As shown in Figure 1, I distinguish three interconnected layers constituting a graphical representation that build from the bottom up: frames of reference, a coordinate system (e.g., Cartesian plane), and a graph. What I consider the final layer laying atop the other two is the graph. By graph, I refer to a collection of points, depicted upon an underlying coordinate system, through which an individual can hold in mind multiple quantities' (potentially varying) magnitudes simultaneously (Thompson et al., 2017). For example, the graph in Figure 1 depicts some quantitative relationship between two quantities *A* and *B*.

Figure 1

Three Layers of a Graphical Representation



Most attention in extant research on students' graphing understandings has been given to this third layer, graphs themselves. Often, the first two layers, frames of reference and coordinate systems, are taken for granted (Joshua et al., 2015; Lee et al., 2020). There needs to be more attention paid to students' frames of reference and coordinate systems because the nature of graphs and hence, ways of thinking about a graph, fundamentally depend on the frames of reference and coordinate systems upon which they are created. Differently put, students are likely to use different strategies for making sense of graphs based on their frames of reference and coordinate systems they read or write. The analytical framework I present was designed to attend to how students re-presented locations of objects in space through these first two layers.

Frames of Reference

In order to interpret or create a graph, one will need to situate the various quantities involved by thinking about: where do the quantities in a graph come from, in which direction do they change/move, and how could the extents of these quantities be gauged? Frames of reference refer to mental structures that are used to gauge the relative extents of various attributes in the phenomenon being depicted (Lee, 2017; Levinson, 2003). Thinking within frames of reference entails establishing reference points, defining directionality, and having an idea of what and how to measure the quantities being depicted (Joshua et al., 2015; Lee et al., 2019). Before elaborating on each of these elements, I first synthesize how researchers across several fields have described frames of reference and how my notion of frame of reference was informed by and differs from those in the literature.

Frames of Reference in Literature

The central questions in spatial cognition research have involved how people perceive, organize, and represent space. An essential construct in the spatial cognition literature is the notion of frame of reference, which Levinson (2003) defined as "a unit or organization of units that collectively serve to identify a coordinate system with respect to which certain properties of objects, including the phenomenal self, are gauged" (p. 404). My basic definition of frame of reference stems from Levinson's (2003) definition above and later I elaborate on some of the relevant constructs I borrow from this area of research.

Spatial cognitionists have conducted various experimental studies to provide accounts for different types of frames of reference people use for characterizing spatial relationships among objects in space (e.g., Levinson, 2003; Taylor & Tversky, 1996). While these studies afford useful frameworks for characterizing different types of frames of reference individuals use when representing space, many of these researchers tended to assume an ontological space, which participants were expected to discover through embodied senses (e.g., through vision, sound, touch, and gestures). For example[2], in their experiments, Taylor and Tversky (1996) used pre-constructed maps and assumed "spatial environments have an objective reality" (p. 388).

In contrast, and in line with my perspective, Piaget and colleagues (Piaget, Inhelder, and Szeminska, 1960; Piaget & Inhelder, 1967) emphasized the perceiver's active role in re-presenting space and assumed "perception of space involves a gradual construction and certainly does not exist ready made at the outset of mental development" (Piaget & Inhelder, 1967, p. 6). Piaget et al. (1960) defined a reference system as a "general system embracing moving objects and stationary sites, as determined by reference points" (p. 164) and investigated how children re-presented space through coordinated systems of measurements.

[2] See Lee et al. (2019) for another elaborated example.

In their report on children's development of reference systems, Piaget and Inhelder (1967) used reference systems, co-ordinate systems, and frames of reference interchangeably. These terms referred to the basis of three-dimensional Euclidean space, the coordination of three (or less) orthogonal axes, which results in "an organizing whole forming an all-embracing system" (p. 418). The term "all-embracing system" refers to a reference frame having an orienting function accounting for potential positions for any but no particular object within the space. In this system, "each object is linked simultaneously with the rest in three directions; left-right, above-below and before-behind, along straight lines parallel to each other along one dimension and intersecting those belonging to the other two dimensions at right-angles" (p. 375). This organizing system entails a topological organization of order and dimensionality and a metric organization of measurements, such as distances between objects. Piaget et al. (1960) distinguished between these two organizing structures as qualitative and quantitative reference systems, the former comparable to my notion of frame of reference and the latter compatible with my notion of coordinate system, which I elaborate on later.

In contrast to scholarship focused on individuals' perception and organization of space, other researchers have examined how individuals construct and coordinate quantities. Building on Thompson's (2011) quantitative reasoning framework, Joshua et al. (2015) defined a frame of reference as "a set of mental actions through which an individual might organize processes and products of quantitative reasoning" (p. 2). The mental actions involved in conceptualizing measurable attributes within a frame of reference involve committing to a unit, a reference point, and directionality of measure comparison. Joshua et al. (2015) made a distinction between conceptualizing frames of reference as mental activity and a coordinate system as the product of the mental activity involved in conceptualizing multiple frames of reference and coordinating those frames of reference. This product "allows us (mathematicians, teachers, and students) to represent the measures of different quantities simultaneously when those measures stem from potentially different frames of reference" (Joshua et al., 2015, p. 35). Like Joshua et al. (2015), I consider a frame of reference to involve committing to a reference point and directionality; however, I do not require that a frame of reference involve a unit of measure[3]. Instead, all that is required is an idea of what and how to measure some attribute involved.

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[3] When frames of reference are coordinated and a unit of measure has also been adopted for each frame of reference, I consider an individual to have established a coordinate system.

Establishing Reference Points/Objects

Sadalla, Burroughs, and Staplin (1980) defined spatial reference points as places within a region whose locations serve to define the location of adjacent places when building cognitive representations of large-scale space. Drawing from their experiments with undergraduate students in identifying locations on a university campus, Sadalla et al. suggested that “spatial information is organized into conceptual units, with a number of locations cognitively located in relation to reference points” (p. 527). Further, they claimed that spatial reference points “provide an organizational structure that facilitates the location of adjacent points in space” (p. 526). Indeed, reference points or objects are crucial bases for organizing spatial phenomena. However, to gauge the relative locations of objects in space, a frame of reference should entail more than a reference point/object. Therefore, solely attending to reference points does not account for different frames of reference that one might take in locating objects (Levinson, 2003).

Defining Directionality and Having an Idea of What to Measure

In addition to establishing a reference point, in order to locate a point in space, one will need to have a means to describe/interpret the location of the point in relation to the reference point. This involves defining a direction of change/movement from the reference point and an idea for what and how to measure some relevant attributes such that the point of interest could be gauged in relation to the reference point.

For example, in one-dimensional space, with a fixed reference point (A), the location of any other point in this space could be gauged by determining which direction to move—either to one side or the other of point A along the line—and coordinating that with how far away it is from A. In the one-dimensional case, you can only move in one direction or in the opposite. However, directionality in two- and three-dimensional space is not as simple as in one-dimensional space as there are infinitely many directions in which one could move. Depending on how directionality is operationalized, the spatial attributes one will need to gauge may differ. So, there could be multiple ways people can think about the location of one point in reference to another depending on the frames of reference they construct. In other words, a frame of reference must entail establishing directionality.

Here, the perspective a person takes plays an important role when establishing directionality. Consider one theory of perspective taking in spatial cognition literature. Drawing from multiple experiments of participants describing spatial environments they learned from maps, Taylor and Tversky (1996) distinguished different perspectives people take in generating descriptions of space: *gaze, route, and survey.* In a *gaze perspective*, speakers adopt locations of objects taking an "outside viewpoint, as if their eyes were moving around the scene" (Taylor & Tversky, 1996, p. 375). According to Taylor and Tversky, because this type of description does not account for the listener's perspective nor the listener's motion within the environment but only for the speaker from a fixed outside viewpoint, the perspective is also termed "ego-oriented" (p. 375). In a *route* perspective, the speaker describes the objects within the environment in relation to the listener and accounts for the changing viewpoint the listener might take from within the environment, as if the listener is following a route in the space. Finally, the *survey perspective* refers to the viewpoint the speaker takes from viewing the environment from above the space using cardinal directions, like a map. Taylor and Tversky explained that speakers can change perspectives, "for example, to take an addressee on a mental tour but describe locations of landmarks using the cardinal direction terms" (p. 377) and that "the choice of description perspective depends on characteristics of the environment themselves" (p. 384).

Although inspired by their work that perspective plays an important role when establishing frames of reference, different from Taylor and Tversky, I distinguish two perspectives, one embedded within the space (corresponding to Taylor and Tversky's route perspective) and one taken from outside the space (corresponding to Taylor and Tversky's gaze or survey perspective). Additionally, I consider all types of spatial descriptions to be centered at the speaker/perceiver but entail different levels of *translation* of viewpoint to others, i.e., different levels of *decentration* involved. Here, I use Piaget and Inhelder's (1967) use of decentration, "The passage from one centration to another" (p. 24). For example, in the route perspective, although the speaker describes the objects in relation to the changing viewpoint of the listener in the environment, this perspective still requires the speaker to imagine the listener's viewpoint (through *decentering*) as if the speaker is the listener.

In summary, the objects an individual establishes as references, the way the individual operationalizes directionality, and the perspective the individual takes all work together when gauging relative locations of objects in space. These activities also interplay with the spatial attributes the individual can consider measuring in order to quantify the extent of various relations. At this point, the individual need not actually carry out the measurement, but have some idea of what and how they could measure in order to locate objects. Therefore, I consider the following questions critical to attend to when analyzing how students might be thinking within spatial frames of reference:

- References: What do students select as reference objects (e.g., points, segments, lines, etc.) and how do they use those objects in their locating activity?
- Directionality: How do students define directionality from one point to another in space? What perspective do students take (embedded within the space, from outside the space, or both) and how do they orient themselves in relation to the space? How is this directionality used to describe the relative position of a point in space?
- Attention to some attribute: What kind of attributes do students consider in order to locate objects within the space?

Coordinate Systems

By coordinate systems, I refer to the geometric embodiment of the frames of reference (e.g., axes). A coordinate system allows an individual to systematically express and coordinate frames of reference. For example, in Figure 1, each quantity is overlaid onto a number line and each number line is arranged perpendicularly with their reference points coinciding to form a Cartesian coordinate system. This geometric object and the space it creates allows the individual to coordinate both quantities.

I have distinguished between two types of coordinate systems depending on the goal they serve: spatial and quantitative (Lee, 2017; Lee et al., 2020). Spatial coordinate systems, relevant to this paper, are used to quantitatively organize a space in which a phenomenon is situated. Constructing a spatial coordinate system involves an individual (mentally) overlaying a coordinate system onto some physical or imagined space being re-presented where objects within that space are tagged with coordinates.

In spatial coordinate systems, frames of reference are established in the space being depicted, such that objects' locations within the space can be quantitatively described in reference to reference objects. Therefore, prior to constructing a spatial coordinate system, an individual must establish frames of reference imposed onto the spatial phenomenon to gauge extents of various spatial attributes of objects (e.g., relative location of an object) within the space. Once such frames of reference are established, an individual can enact or imagine measurements of spatial attributes of the objects and tag those quantities' values *onto* the space in which the phenomenon occurs. Through coordinating these measurements, an individual constructs a spatial coordinate system. In this sense, the coordinate system layer in a graphical representation builds upon the frames of reference layer.

In summary, coordinate systems can be created to serve an individual's goal of systematically locating objects in space when re-presenting spatial phenomena via coordinated measurements. Relatedly, I consider the following questions critical to attend to when analyzing how students might be thinking about coordinate systems:

- Measurements measured: What kind of attributes did students measure? How did their frames of reference guide their measuring activities? What unit of measure did students select?
- What measurements are coordinated and how?
- Relevance to conventional coordinate systems: Is their system of coordinated measurements similar to any of the conventional coordinate systems, including Cartesian, polar, bipolar, etc.? Could students' coordinate systems be leveraged to introduce conventional coordinate systems?

An Analytical Framework

Based on the conceptual analysis I presented above, I propose an analytical framework that could be used for making sense of students' representations of spatial phenomena, with a focus on students' frames of reference and coordinate systems (Table 1).

Table 1

An Analytical Framework for Making Sense of Students' Representations of Spatial Phenomena with Attention to Frames of Reference and Coordinate Systems

Related to...	Category	Questions to consider
Frames of Reference	References	• What do students select as reference objects (e.g., points, segments, lines, etc.) and how do they use those objects in their locating activity?
	Directionality	• How do students define directionality from one point to another in space? • What perspective do students take (embedded within the space, from outside the space, or both) and how do they orient themselves in relation to the space? • How is this directionality used to describe the relative position of a point in space?
	Attention to some attribute	• What kind of attributes do students consider in order to locate objects within the space?

Coordinate Systems	Measurements measured	• What kind of attributes did students measure? • How did their frames of reference guide their measuring activities? • What unit of measure did students select?
	Coordinated measurements	• What measurements are coordinated and how?
	Relevance to conventional coordinate systems?	• Cartesian? • Polar? • Bipolar? • Other?

Making Sense of Students' Representations of Space Using the Framework

In this section, I present four illustrative cases of how the framework (Table 1) could be used. Using the analytical framework, I will model students' thinking about frames of reference and coordinate systems as they engaged in re-presenting objects' locations in space.

Information about the Teaching Experiments

In this paper, I draw data from a single task that I created and used during three teaching experiments (Steffe & Thompson, 2000), all guided by a broader goal of investigating students' constructions and reasoning about frames of reference and coordinate systems. The first two year-long teaching experiments were conducted at a large public high school in the southern U. S. During these teaching experiments, I worked with two pairs of ninth-graders, Kaylee & Morgan, and Craig & Dan. The third teaching experiment was conducted at a public university in the southern U. S. over eight weeks. In this teaching experiment, I worked with two pairs of undergraduate, elementary pre-service teachers (PSTs), Hermione & Ginny, and Leann & Bella[4]. In this paper, I present Ginny, Kaylee, Dan, and Hermione's activities in the North Pole Rescue Task as representative cases.

The North Pole Rescue Task

In the North Pole Rescue (NPR) Task, I asked students to imagine being in a helicopter hovering over the North Pole region holding a map. Given an irregular-shaped map (see Figure 2) including a single road to the North Pole (NP), the North Pole point (P), and a missing person

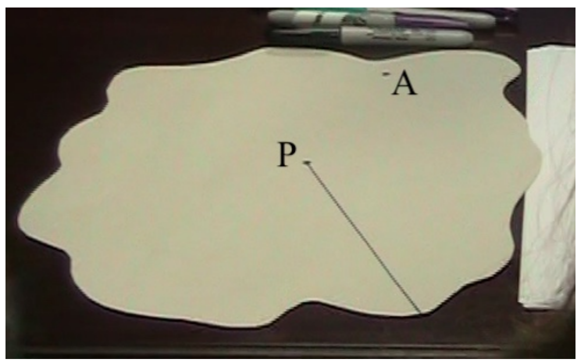
[4] The high school students were selected through teacher recommendations of students with various mathematical backgrounds and were paired based on their interactions with units coordinating tasks (Steffe & Olive, 2010) during their selection interviews. The PSTs were selected based on interest and availability during the summer and also paired similarly as the high school students. Since my focus in this paper is to provide illustrations of how the framework could be used for each individual student's activities, I do not discuss in detail the pairing process. For more information about the students, see Lee (2017; under review).

(point A) in the region, students were asked to provide instructions to a rescuer on the ground holding another identical map (without point A) so she/he could find the missing person. For detailed instructions of the setup of activity, the story context read to the students, and materials to prepare for the task, see Lee (2020).

This task was inspired by and modified from Piaget et al.'s (1960) task involving children copying a point from one rectangular sheet of paper to another blank, identical rectangular sheet of paper[5]. I made the maps irregularly shaped to eliminate any suggestive cues, such as orthogonal edges of a rectangular sheet of paper. My overarching goal in this task was to provide students with a context that could motivate students to establish frames of reference and create coordinate systems to fulfill their goal of re-presenting objects' locations in space.

Figure 2

North Pole Rescue Map Sample



Data and Analysis

For each teaching episode, I collected video recordings of participants' actions and digitized students' written work. I conducted both on-going and retrospective analyses and modeled students' constructive activities (Steffe & Thompson, 2000). On-going analyses involved testing and formulating hypotheses during the teaching experiment based on ways students engaged in each teaching episode. After the completion of the teaching experiment, I re-visited the data corpus to do an in-depth retrospective analysis. The retrospective analysis involved refining the initial explanatory models developed during the teaching experiment. Specific to the NPR task, the questions outlined in Table 2 guided the retrospective analysis.

[5] See Lee (in press) for a detailed summary of Piaget et al.'s task and how children engaged in copying the point. Other detailed principles behind the design of the NPR task including how the original task was modified to develop the NPR task could be found in Lee (under review).

Table 2

Analytical Framework Used for the NPR Task

Related to...	Category	Questions to consider
Frames of Reference	References	• What do students select as reference objects (e.g., points, segments, lines, etc.) and how do they use those objects in their locating activity?
	Directionality	• How do students define directionality from one point to another in space? Specifically, how do the students describe the direction in which the rescuer should move? • When developing instructions for the rescuer, which perspective did students take? For example, did they take on the rescuer's perspective (on the ground in the North Pole region) or their own perspective (from the helicopter looking down onto the rescuer)? Or did they coordinate both? • How did the student orient the map? Did the orientation of the map seem to matter to the student? Did they orient the map in a specific way? If so, how? • How is this directionality used to describe the relative position of the missing person for the rescuer in the North Pole region?
	Attention to some attribute	• What kind of attributes do students consider in order to locate objects within the space?
Coordinate Systems	Attributes measured	• What kind of attributes did students measure? • How did their frames of reference guide their measuring activities? • What unit of measure did students select?
	Coordinated measurements	• How did students put their measurements together in order to locate points/objects in space? • What were the instructions the student relayed to the rescuer such that the rescuer could find the missing person?
	Relevance to conventional coordinate systems?	• Cartesian? • Polar? • Bipolar? • Other?

In the following section, I describe each students' activities when locating the missing person in the NPR task. After describing what each student did to solve the task, I present my analysis of their frames of reference and coordinate systems following the questions in the analytical framework (Table 2).

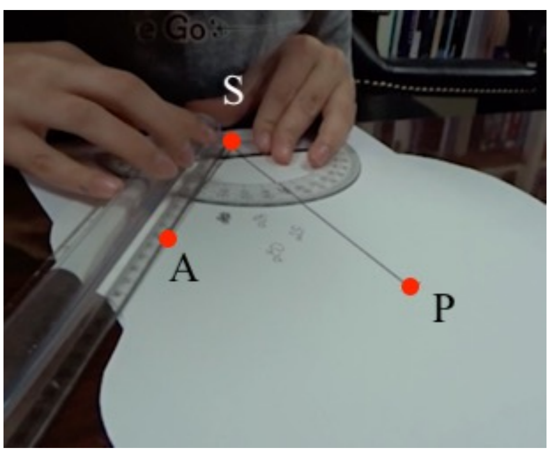
The Case of Ginny

Ginny's Activities

Ginny placed her map in front of her such that the road faced herself. She then connected the starting point of the road at the edge of her map, say point *S*, with the missing person's location, *A*, with a line segment. Next, she used a protractor to measure the angle formed by the road and the line segment connecting *S* and *A* (see Figure 3a). When it was her turn, Ginny relayed the instructions to Hermione (the rescuer) as summarized in the top part of Figure 3b. During that process, Hermione asked if the rescuer would be able to see the missing person, which led Ginny to revise her instructions to include a distance. So, Ginny measured the line segment *SA* and relayed the remaining instructions (Figure 3b, bottom) to Hermione. In summary, her instructions told the rescuer to start at the beginning of the path, facing P on the road, turn to the right about 43 degrees, and then go 12 cm along that degree line.

Figure 3

(a) Ginny's Measurement Activity to Locate A and (b) Ginny's Instructions for the Rescuer



(a)

"Turn your map so that you're looking straight at P from the road. So, you're at the beginning of the path and you're about to walk to P... And then take a protractor and put the path on the ninety. Then, to your right, go about 43 degrees, it should be around where it says 50/130. Then go straight until you find her."

...

"Put your ruler along that line and go... It'll be 12cm along that degree line. You're at the edge of your map, you're turning a bit to the right and then going straight."

(b)

Ginny's Frames of Reference

Based on Ginny's measurement activities and her communication with Hermione about the missing person's location, I infer Ginny selected point *S* as a reference point and the road as an initial/orienting ray in order to orient the map in a specific way. She deliberately selected these objects so she could define the direction in which the rescuer would have to move and from where. By the way she described the rescuer's steps as "you're at the edge of your map, you're turning a bit to the right and then going straight," she was able to embed her perspective into the rescuer's as if she were on the ground. Doing so, she established frames of reference that would allow her to describe directionality using angle as how much turn the rescuer would need to make at the reference point. Initially, she only attended to angularity as an attribute in order to locate the missing person for the rescuer; however, she later also attended to the distance traveled from the reference point to the desired location of a point.

Ginny's Coordinate System

Guided by the reference point, initial/orienting ray, and the line segment connecting *S* and *A*, Ginny measured angularity and distance using a protractor and ruler, respectively. She coordinated the angle measure, 43 degrees to the right (clockwise), and (radial) distance, 12 cm along the degree line, to locate *A* in relation to *S*. Her coordinated measurements resembled those one would see in a polar coordinate system.

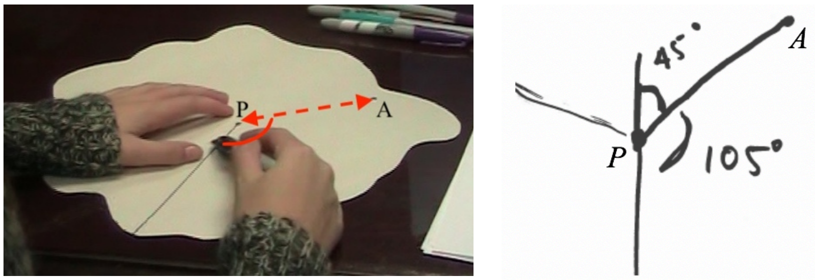
The Case of Kaylee

Kaylee's Activities

Given her map, Kaylee also placed her map such that the road faced herself. She made an invisible segment connecting points *P* and *A* with the cap of her pen but commented that telling the rescuer to go "just go straight" from *P* to *A* was not going to work. Kaylee considered using cardinal directions but rejected this idea because that would leave the ambiguity of how much northeast the rescuer would have to go. Instead, Kaylee decided to form an angle at *P* with the road and line segment PA (see Figure 4a) and estimate the measure of that angle, 105-degrees. However, Kaylee was concerned that "if we just told him one-hundred-and-five degrees, what if he drew this way and started to go this way?" while moving her finger along the alternative path (see Figure 4b). Finally, Kaylee extended the road segment farther, and estimated the angle formed by that road extension and *PA* (see Figure 4b). She came up with an instruction as summarized in Figure 4c. In summary, she told the rescuer to start at the North Pole facing in the direction he walked down the road, then turn 45 degrees to the right and then go straight until he runs into the missing person.

Figure 4

(a) Kaylee Establishing an Angular Frame of Reference, (b) Kaylee's Sketch of Three Angles to Consider, and (c) Instructions for the Rescuer



"So, it could be like, okay, when you get to the North Pole, turn forty-five degrees to the right and then go straight and you'll run into her."

(a) (b) (c)

Kaylee's Frames of Reference

Based on Kaylee's measurement activities, I infer she selected point *P* as a reference point and the road as an initial/orienting ray in order to orient the map in a specific way. She deliberately selected these objects so she could define the direction in which the rescuer would have to move and from where. Similar to Ginny, Kaylee was able to embed her perspective into the rescuer's as if she was the rescuer on the ground. Kaylee used the road to the North Pole as an orienting line segment to situate the rescuer's line of sight before moving. By doing so, Kaylee described directionality using angle as how much turn the rescuer would need to make at the reference point. Doing so, she established frames of reference that would allow her to describe directionality using angle as how much turn the rescuer would need to make at the North Pole. After deciding the amount of turn, Kaylee knew that as far as the instructor followed that path, then he will eventually see her. So, while she did not explicitly measure the distance from *P* to *A*, she had an awareness that the rescuer would need to travel along the terminal ray for a certain amount of length.

Kaylee's Coordinate System

Guided by the reference point, initial/orienting road, and the line segment connecting *P* and *A*, Kaylee estimated an angle measure[6]. Although Kaylee did not explicitly measure the distance from *PA*, in later discussions during the teaching episode, it was apparent she was aware of a

[6] Kaylee's estimation of 45 degrees was inaccurate if it were supplementary to the 105-degree angle she originally drew. However, because her estimating of an angle measure was not my focus of interest, I did not question her estimation.

distance that needed to be traveled from the reference point to the desired location of a point. Her coordinated measurements resembled those one would see as early developments of a polar coordinate system.

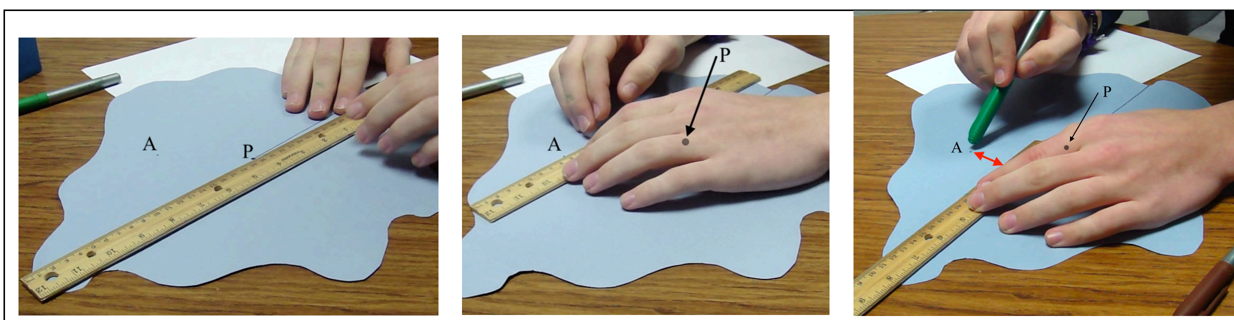
The Case of Dan

Dan's Activities

Consistent with Ginny and Kaylee, Dan also first oriented the map so that the road to the North Pole was facing him. Similar to Kaylee, Dan swept his pen from P to A in a straight motion and started giving his instructions by saying, "Alright, at the North Pole, you're going to... You want to go... Let's see..." and paused. It seemed as if Dan, similar to Kaylee, realized that telling the rescuer to go straight from *P* to *A* was not going to be sufficient. Instead, Dan moved his ruler in various positions. First, he placed his ruler aligned with the road (see Figure 5a) and translated it horizontally (from his perspective) from *P* to *A* (Figure 5b) then moved it back to the road, this time such that the ruler's end was aligned at *P* (Figure 5c). Next, Dan moved his pen left to right, from point A to the ruler, as if he was envisioning a horizontal movement from the ruler towards *A* (Figure 5c). Finally, Dan measured two orthogonal lengths the rescuer would need to trace out and provided the instructions, "Go one and a half inches straight, and then... To the right, you go two and a half inches."

Figure 5

(a-b) Dan Translates Ruler (c) Dan Moves Pen from A to Ruler



(a) (b) (c)

Dan's Frames of Reference

Based on Dan's measurement activities, I infer he selected point *P* as a reference point, like Kaylee, and the road as an orienting segment in order to orient his measuring activity. He deliberately selected these objects so he could define the direction in which the rescuer would have to move, forward and to the right, from *P*. Note, both Kaylee and Dan used point *P* as a reference point and all three students, Ginny, Kaylee, and Dan used the road to the North Pole as an orienting line of reference. However, both Ginny and Kaylee explicitly oriented the rescuer's perspective; Ginny said, "you're at the beginning of the path and you're about to walk to *P*" and Kaylee said, "when you get to the North Pole." On the other hand, Dan's instructions did not start with such an orienting comment, from which I inferred that Dan assumed the rescuer was taking the same perspective as he was. In summary, Dan described directionality along two directional axes, straight/backward and left/right, from his reference point *P*. In order to consider the rescuer's movement along these two directional axes, he attended to the distance traveled.

Dan's Coordinate System

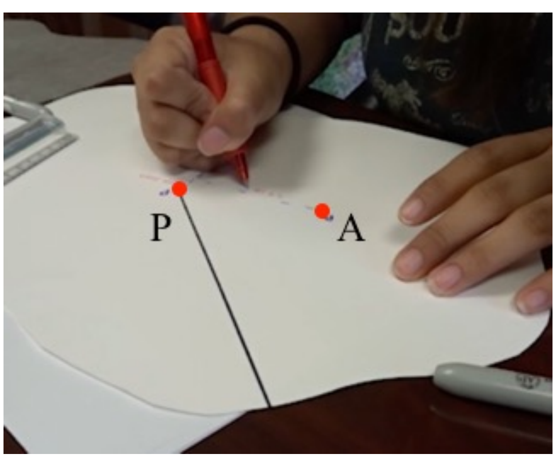
Guided by this frame of reference, Dan measured two lengths, in inches, each perpendicular to the other. Together these lengths indicated how far straight and then to the right from the reference point (North Pole) the missing person would be. Dan's coordinated measurements resembled those one would see in a Cartesian coordinate system.

The Case of Hermione

Hermione's Activities

Figure 6

(a) Hermione's Measurement Activity to Locate A and (b) Hermione's Instructions for the Rescuer



"Your pole line (referring to the road to the North Pole) is going to be on your left side up, so on the upper left side, that's where the line is going to end. From the pole you go 2cm straight down. And then you go 7.5cm to your left. And where you ended on that 7.5 is where the dot (A) should be."

(a) (b)

Given her map, Hermione left the map as it was originally placed on the table in front of her without re-orienting it in any specific way like the other three students did. She used a ruler to measure two orthogonal distances from *P*, one vertical, going down, and the other horizontal, going left, from her perspective, tracing out a path from *P* to *A* (see Figure 6a). When it was her turn, Hermione relayed the instructions as summarized in Figure 6b. In summary, she told the rescuer to make her pole line on the upper left side of the map, and from the pole to go straight down 2cm and to the left 7.5cm. Her partner, Ginny, who was acting as the rescuer, followed her instructions on the wax paper copy of Hermione's map and marked where she thought *A* would have to be. When Ginny was finished marking point *A*, the two students overlaid their maps and noticed that their two maps were pretty closely matched.

Hermione's Frames of Reference

Based on Hermione's measurement activities and through her communication with Ginny about the missing person's location, I infer Hermione selected point *P* as a reference point and the road as an initial/orienting ray in order to orient the map in a specific way, similar to the other students. However, different from the other three students who re-oriented their maps such that the road was facing them, Hermione left the road as it was originally oriented when the map was placed in front of her. Instead, later when relaying her instructions for Ginny, she attended to how the pole line should be oriented: on the upper left side of the map.

After establishing the orientation of the map, to define the direction in which the rescuer should move, Hermione broke down the rescuer's movement along vertical and horizontal directional axes, like Dan. These orthogonal movements were defined from her perspective looking down at the map, as going down and to the left[7]. In order to consider the rescuer's movement along these two directional axes, she attended to the distance traveled.

Hermione's Coordinate System

Guided by the reference point, initial/orienting ray, and lines vertical and horizontal from her perspective, Hermione measured two orthogonal distances using a ruler. She coordinated 2cm down, and 7.5cm to the left from the reference point, to locate *A*. Similar to Dan, Hermione's coordinated measurements of orthogonal distances along two directional axes, down/up and left/right, from her reference point *P* resembled a Cartesian coordination.

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[7] Had her instructions have been embedded within the NP space and aligned with that of the rescuer on the ground, the rescuer should be turning right.

Looking Across the Students' Frames of Reference and Coordinate Systems

The four students' representation activity as presented above illustrate the variety of ways students made sense of a spatial situation, established frames of reference, and coordinated measurements. Interestingly, across the eight students I worked with in the NPR task, four students, Ginny, Kaylee, Leann, and Craig coordinated angle measure and distance from a reference point, hence creating a polar-like coordination. On the other hand, the other four students, Hermione, Dan, Morgan, and Bella coordinated two orthogonal distances from a reference point, hence creating a Cartesian-like coordination. In the following, I summarize students' frames of reference and coordinate systems following the analytical framework in Table 2 across the four students' cases I presented in this paper.

Frames of Reference

References. In terms of references, although students attended to similar features or objects of the map, they used them differently, in a way that made sense to them. For example, while some students used the road segment to the North Pole as part of their frame of reference to measure certain attributes, some only used it to orient their map. Some students chose the North Pole point (*P*) as a reference point (e.g., Kaylee, Dan, Hermione) and some chose the start of the road at the edge of the map (e.g., point *S* is Ginny's case) as their reference point. In other words, students selected a point of reference that worked for them. Also, students used the road to the North Pole (or its extension line) as a spatial reference. The road was used in two ways: as an orienting object (e.g., Hermione) or as a frame of reference used to gauge some attribute (e.g., angle measure; Kaylee).

Directionality. Some students started with their map oriented in a specific way (such that they were facing the road; e.g., Ginny, Kaylee, and Dan). Some students left their map oriented in the way it was originally placed in front of them (e.g., Hermione). Either way, attending to the orientation of their map seemed important to them when defining directionality. When developing instructions for the rescuer, some students explicitly embedded their perspective into the rescuer's perspective on the ground (e.g., Kaylee and Ginny) while some students' explanations seemed to assume the rescuer was also looking down at the map or already aligned with their perspective (e.g., Dan and Hermione). After establishing some orientation and viewpoint, students described direction in terms of angular turns (e.g., Ginny, Kaylee) or orthogonal movements (e.g., forward/backward, up/down, left/right; Dan, Hermione).

Attention to Attribute. Students attended to spatial attributes such as angularity and distance traveled in order to describe the relative location of A to the reference point of their choice. The attributes students considered interplayed with the frame of reference they established.

Coordinate Systems

Attributes Measured. Students measured angularity to determine the amount of turn, and/or measured distances between two points to determine the amount of travel. Angularity was measured in degrees and length was measured in cm or inches. Angularity was gauged via initial and terminating rays through a reference point and distance between two points was gauged by horizontal/vertical directional axes.

Coordinated Measurements. Students coordinated angle measure and (radial) distance (e.g., Ginny, Kaylee) or two orthogonal distances from a reference point (e.g., Dan, Hermione). Related to perspective, students who tended to explicitly embed their perspective into that of the rescuer on the ground (e.g., Kaylee and Ginny) established an angular frame of reference and defined directionality in terms of an angular turn. Relatedly, they coordinated angle measure with (radial) distance. On the other hand, students who seemed to have assumed the rescuer was also looking down onto the map like they were (e.g., Dan and Hermione) established a Cartesian frame of reference and defined directionality in terms of orthogonal movements (e.g., front/back, up/down, left/right). Relatedly, they coordinated two lengths. I emphasize that although the students' coordinated measurements had similarities, each student came up with instructions for the rescuer in a variety of ways.

Relevance to Conventional Coordinate Systems. Dan and Hermione coordinated their measurements in a Cartesian sense while Ginny and Kaylee coordinated measurements like a polar coordinate system when locating the missing person in the NPR task[8]. Surprisingly, although these students have received formal instruction of the Cartesian coordinate system in

[8] I acknowledge that students were asked to locate only one single point in the NPR task. The students created coordinate systems in the sense that a particular system of coordinated measurements worked for at least one point in the space. However, I also note that in more recent work with middle school students with an upgraded version of the NPR task, students tended to use the same system of coordinated measurements for several points and for describing paths (graphs) in the North Pole region.

school, using it for locating the missing person in the NPR task was not as immediate as one might expect, nor did all students create a Cartesian coordinate system to achieve their goal in the task. It is also worth pointing out that the majority of these students did not receive formal instruction on the polar coordinate system, yet some of them created one.

As demonstrated above in Kaylee and Dan's case, some students started out by simply connecting the North Pole point (P) and the missing person (A) in a line segment but soon realized that telling the rescuer to go straight was going to be insufficient. Even after they made such a realization, they had to think carefully about how to break down the movement of the rescuer to find the missing person. Across all four students, it was evident that the process of having to relay instructions for the rescuer, who they know is unable to see the missing person seemed to have perturbed the students to carefully consider orientation, reference objects, measurable attributes, how to measure such attributes, and how to use those measurements to precisely mark a location. When Ginny and Hermione took turns to relay their instructions to the other person acting as the rescuer, often times the rescuer asked follow-up clarification questions (e.g., "Wait, where am I placing my protractor?) and/or looked puzzled, which led them to add more detail to their instructions for the rescuer. The rescuer not being able to see the missing person seemed to perturb them to consider orientation, reference objects, attributes, how to measure them, and how to use those measurements to precisely mark a location. An example of this occurred, after Hermione followed Ginny's instructions on the wax paper copy of Ginny's map and marked where she thought *A* would have to be. When Hermione was finished marking point *A*, the two students overlaid their maps and were impressed that their locations of A on their maps actually matched. They agreed that relaying instructions and following the instructions required a clear communication of where and how to place the protractor and ruler, emphasizing their need for establishing a reference point through which objects could be gauged systematically.

Conclusion and Discussion

In this paper, I first presented a conceptual analysis (Thompson, 2008) of what it means to conceptualize (either by creating or interpreting) a graphical representation of some phenomena by unpacking its three constituting layers. Second, I proposed an analytical framework that could be used for making sense of students' representations of spatial phenomena with attention to their frames of reference and coordinate systems. Finally, I exemplified how one might use this framework to model students' re-presentations of spatial locations.

Recall, the work presented in this paper was motivated by the call for studies that could help the field shift focus from looking into what students are not capable of doing to better understanding the constructive resources they can leverage in their graphing activity (Åberg-Bengtsson & Ottosson, 2006; Ring et al., 2019). As Thompson (2008) explained, conceptual analysis can provide researchers the tool to "provide imagistically-grounded descriptions of mathematical cognition that capture the dynamic aspects of knowing and comprehending" and teachers the tool to engage in "mathematically substantive, conceptually-grounded conversations" with students (p. 60).

Relatedly, using the proposed conceptual analysis and relevant analytical framework, I was able to model each student's unique ways of organizing the spatial situation via coordinated measurements and also find patterns across students' organization of space. From a researcher's perspective, the analytical framework helped me attend to the constructive resources—their frames of reference and coordinate systems—students can build upon to create re-presentations of a spatial situation. From a teacher standpoint, attending to students' frames of reference and coordinate systems allowed me to design the NPR task and pose questions during the task that provided opportunities for students to engage in conceptually grounded discussions. As I articulated above, having a specific goal of relaying instructions for the rescuer when locating a point in space seemed to have supported students to carefully consider elements that are foundational for constructing a coordinate system. Such a design supported students to shift from making visual estimations (e.g., marking locations where they "seem to be") or describing a location based on how they see it (e.g., just go straight) to systematically locating objects using measurements. And, in order to use and coordinate measurements, students were encouraged to establish frames of reference. In other words, students were motivated to think carefully about measurable attributes and reflect on spatial references they used. For example, students had to carefully consider their placement of the ruler/protractor. To do so, they found a common reference point for consistency and communication and made sure that they provided enough information to define a single location for the rescuer.

In closing, I propose how the analytical framework could be used for better attending to students' graph thinking and making sense of students' graph literacy more broadly. Although the framework was designed specifically for a spatial context and with attention to frames of reference and coordinate systems, the framework can be extended beyond spatial contexts and connected to the third layer of graphical representations, graphs. In Table 3 below, I provide examples of questions one can pose when attending to students' graphing activity. I invite researchers to test the affordances and limitations of this framework.

Table 3

An Illustration of Analytical Questions to Consider when Making Sense of Students' Graphing Activity

Focus on	Category	Example of relevant questions to consider
Frames of Reference	References	• When interpreting a graph, what do students attend to as reference objects (e.g., points, segments, lines, etc. in the graphical representation) and how do they use those objects in their graphing activity? • When creating a graph, what do students establish as reference objects (e.g., points, segments, lines) in the phenomena they are graphing and how do they use those objects to make sense of the quantities they are graphing?
	Directionality	• Given a graph, how do students make sense of directionality from one point to another? • What perspective do students take (embedded within the graph, from outside the graph, or both) and how do they orient themselves in relation to the graph? For example, do they envision walking along the graph or looking down onto an object tracing out the graph? • How is their sense of directionality used to describe the relative position of a point in the graph?
	Attention to some attribute	• What kind of attributes of the phenomena do students attend to in order to make sense of a given graph? • What kind of attributes in the phenomena do students attend to when creating a graph?
Coordinate Systems	Measurements measured	• What kind of attributes did students measure or imagine having been measured when creating or interpreting a graph, respectively? • How did their frames of reference guide their measuring activities or interpretation of measured quantities when creating or interpreting a graph, respectively? • How do students attend to unit of measure of a given graph or when creating a graph?
	Coordinated measurements	• How do students make sense of the coordinate system of a given graph?

		• When creating a graph, what measurements are coordinated and how, and how does that relate to their coordinate system?

Clearly, all students in the teaching experiments presented in this paper were able to construct, refine, and use their own re-presentations of the spatial context at hand in the NPR task. As I mentioned earlier, studies on students' graph literacy have heavily focused on students' interpretations of pre-constructed graphs (Hattikudur et al., 2012). I hope that the conceptual analysis, analytical framework, and illustrations of students' frames of reference and coordinate systems in this paper can motivate teachers and researchers to provide more opportunities for students to construct, refine, and use their own representations. Relatedly, future studies investigating effective ways to help students connect their coordinate systems with conventional forms of coordinate systems are needed.

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