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Deborah Moore-Russo

Courtney Nagle

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Deborah Moore-Russo[*]

University of Oklahoma

Courtney Nagle

Penn State Erie, The Behrend College; Larson Texts, Inc. and Big Ideas Learning

Abstract: While slope is a topic in the algebra curriculum, having a robust understanding of slope is needed for students to truly understand several single and multivariable calculus topics with any depth. We begin with a review of the topic of slope and present what is known from its existing corpus of literature. We then outline the tenets of APOS theory. Building from there, we suggest what a robust, flexible understanding of slope involves, as well as how slope is used, with the APOS-slope framework acting as a theoretical lens. This is followed by the cases of two hypothetical students built from amalgamations of research and experience to emphasize why moving easily between different ways of thinking about and the various uses of slope is vital to successfully transition into calculus. We offer suggestions as to how university instructors might consider slope understanding when teaching calculus, then conclude with suggestions for future research on slope.

Keywords: Slope, APOS Theory, Calculus

[*]dmr@ou.edu

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Introduction

Mathematics courses typically follow a path of progression where topics rely on students' understanding of previous content (Treisman, 1992). Many calculus concepts build on topics that have first been introduced to students in precalculus and even algebra (Habre & Abboud, 2006). Students' lack of prerequisite knowledge can lead to challenges in understanding later topics that appear in calculus (Pyzdrowski et al., 2013).

Although typically introduced in the middle grades in the U.S. (Stanton & Moore-Russo, 2012; Nagle & Moore-Russo, 2014b; Nagle et al., 2022), slope serves as a foundational idea for topics in more advanced mathematics courses. Slope helps contrast covariational relationships in linear and nonlinear functions in algebra (Lobato & Thanheiser, 2002; Teuscher & Reys, 2010). Slope as a measure of steepness ties into the tangent of an angle in trigonometry (Nagle & Moore-Russo, 2013a). Slope plays a key role in statistics in describing the nature of a data set when a regression line is used (Nagle et al., 2017). Slope also has an important role in the initial development of derivatives in calculus (Zandieh, 2000; Zandieh & Knapp, 2006), as noted in Asiala and colleagues' (1997) extensive genetic decomposition of derivatives, and in directional derivatives in multivariable calculus (McGee & Moore-Russo, 2015; McGee et al., 2015). In this paper, we will focus on how students with different understandings of slope might be impacted as they proceed to calculus. More specifically, we report on an amalgamation of cases to create two fictional accounts of students who are representative of the very different understandings of slope that students can have in introductory university calculus courses.

Slope

Without an ability to comfortably move between representations and to think about slope as a parameter for linear relationships, students will have a hard time navigating the extension of slope to non-linear functions in calculus. Calculus requires students to extend the linear notion of slope to recognize that nonlinear functions have variable slope, and to define the slope of a curve at a point as the slope of the tangent line to the curve at the point. Calculus also requires students to interpret the tangent line's slope as the instantaneous rate of change of outputs with respect to inputs, and to define the derivative function as a new function which produces the instantaneous rate of change of f for any input.

To develop a deep, flexible understanding of foundational concepts in calculus, such as instantaneous rate of change and derivative, students must first understand average rates of change and the difference between linear and nonlinear functions (Nagle & Moore-Russo, 2013b). Yet, it has been reported that students can have a limited understanding of linear functions, even when they are able to transition between their different representations (Adu-Gyamfi & Bossé, 2014). This is the case even for STEM majors at the university level (Newton, 2018). Students often over rely on symbolic calculations or shape thinking, where a graph is treated as a static object (Moore & Thompson, 2015) and may not make critical connections between the graphical and symbolic representations used for linear functions (Örnekçi & Çetin, 2021). When dealing with linear equations, students may reply with recalled responses to common symbolic manipulations rather than engaging in meaningful covariational reasoning (Thompson & Carlson, 2017) about the linear relation in the task. This has been noted in research of students' explanations of graphical relationships when students reported a slope of 4 when they had actually calculated the task's solution to be $x = 4$ (Huntley et al., 2007). Students who rely solely on static shape thinking may be challenged when interpreting slope in non-standard setting, such as non-homogeneous coordinate systems (Zaslavsky et al., 2002) or when using dynagraphs (Nagle & Moore-Russo, in press). They may also struggle to determine the derivative for a relatively common function, such as a parabola, as reported by Aspinwall and colleagues (1997), due to thinking that the far-right behavior of the parabola would result in tangent lines whose slope is undefined.

Even though secondary teachers express concern for students' understanding of the slope of linear functions, they may reduce slope to rote computations based on memorized procedures or focus on buzz words (i.e., "rate of change") without assessing, or providing adequate support for developing, a student's robust conceptual understanding of slope (Stump, 1999; Styers et al., 2020). Stump's (2001a) research findings suggest that teachers rely primarily on ratios as the dominant representations of slope. As a result, slope frequently is introduced as a mnemonic (e.g., rise over run, change in y over change in x) that can hinder meaningful understanding of slope as a constant rate of change of two variables (Walter & Gerson, 2007). This might be a result of the limited understanding of slope held by some pre-service and in-service teachers (Avcu & Türker Biber, 2022; Coe, 2007; Moore-Russo et al., 2011; Stump, 1999, 2001a). It might also be related to the variations found in the ways state standards and other curricular materials present slope (Frank & Thompson, 2021; Kim, 2012; Bateman et al., 2021; Dolores Flores et al., 2020; Nagle & Moore-Russo, 2014a; Moore-Russo, 2012; Tuluk, 2020). The result is that many students can apply slope only in particular problem contexts (Byerley & Thompson, 2017), and they are not always able to work with slope in a conceptual way on application tasks (Christensen & Thompson, 2012; Lingefjärd & Farahani, 2017).

How students think of slope often depends on the learning context including the task at hand, what it involves, and the representations that are used (Byerley & Thompson, 2017; De Bock et al., 2015; Tall & Vinner, 1981) as well as prior knowledge and experiences (Vinner, 1992). Slope can be thought of in many ways, but previous research suggests that both students and teachers often fail to make connections between the various interpretations of slope (Coe, 2007; Frank & Thompson, 2021; Hattikudur et al., 2011; Hoban, 2021; Lobato & Siebert, 2002; Mudaly & Moore-Russo, 2011, Planinic et al., 2012). As a result, the notions that students entering post-secondary calculus have of slope may be quite shallow (Dolores-Flores et al., 2019; Mielicki & Wiley, 2016) and different from the ways professors think of, represent, and communicate slope (Nagle et al., 2013). When students have a fragmented or disjoint understanding without connections between various ways of thinking about slope, they are not able to interpret different representations of slope (Glen, 2017; Tanışlı & Bike Kalkan, 2018). So, students may enter calculus with isolated notions of slope and not be able to connect slope as a ratio to other ways of conceptualizing slope, such as slope as a measure of steepness (Nagle & Moore-Russo, 2013a; Stump, 2001b). Students' fragile understanding of slope can persist as they advance in mathematics. This was noted when 15% of university students tested were unable to correctly rank the slope magnitudes of tangent lines at marked points on a single-variable function curve at the end of a multivariable calculus course (Christensen & Thompson, 2012).

Slope can be linked to developing student understanding of multivariable calculus, such as has been done with directional derivatives (see Martínez-Planell & Trigueros, 2021; McGee & Moore-Russo, 2015) and with tasks that consider the tilt of planes (Bos et al., 2022). However, for this paper we focus on how different notions of slope impact student understanding of concepts typically addressed in an introductory, single-variable calculus course. We do this by building on previous research that has considered how slope is conceptualized by students (Moore-Russo et al., 2011; Nagle et al., 2016; Stump 1999, 2001b) and that has identified common challenges, such as confusion between slope and the output value of a function at a point (Beichner, 1994; Planinic et al., 2012). Such misunderstandings can continue into calculus. For example, Orton (1983) reported that almost 20% of the students in his study confused the derivative at a point on a function with the point's *y*-coordinate. Moreover, it has been reported that even "good" calculus students who perform well on routine calculus tasks struggle with nonroutine problems that cover basic calculus concepts (Selden et al., 1994). By working with what is known, we consider how different understandings of slope might impact how students come to understand common introductory calculus concepts.

APOS Theory

APOS theory was proposed by Dubinsky (1984) as an adaptation of Piaget's reflective abstraction (Piaget, 1971; Piaget & Campbell, 2001). APOS theory provides a means to study an individual's development of mathematical knowledge through four stages: Action, Process, Object, and Schema (Dubinsky, 2014). According to APOS theory, individuals' abilities to respond to diverse mathematical tasks involving a particular mathematical topic vary depending on the stage of their understandings of the topic. In the following paragraphs, we describe each of the four stages.

An *Action* is a step-by-step transformation of a mathematical notion that an individual perceives as external since it is not connected to the person's other mathematical knowledge. It may involve the rigid application of a procedure or a memorized fact, and it is often associated with a specific mathematical representation. As an individual repeats and reflects on an Action, it may be interiorized into a Process. An individual with a *Process* stage of understanding can perform the same transformation as the Action, but it is internal since it no longer requires external stimuli and occurs in the individual's mind. A Process is linked by some meaningful connections to other mathematical knowledge. These connections allow the individual to imagine the transformation in a less rigid way, quite often omitting some of the steps that were required in the Action stage. These connections also enable the individual to be able to anticipate results even before performing the transformation, and they allow the individual to work with different mathematical representations of the transformation.

As a person becomes aware of the total Process as its own entity and develops the ability to extend the Process beyond its original context to deal with new situations, then the Process has been encapsulated by the person into a mental *Object*. This stage involves realizing that Actions can act on the Process. These Actions may be applied, or the application of such Actions might just be imagined. At the Object stage, individuals can extend across, and even beyond, the different representations of slope to consider how it may apply in contexts that are novel. While an individual is developing an understanding of some mathematical topic, many Actions, Processes, and Objects may be constructed. Once a mental framework is constructed so that these Actions, Processes, and Objects form an organized, coherent collection, the individual has constructed a *Schema* for the topic. The Schema allows the individual to determine if and how this topic might be applied when encountering different situations, even situations where the topic is not explicitly mentioned.

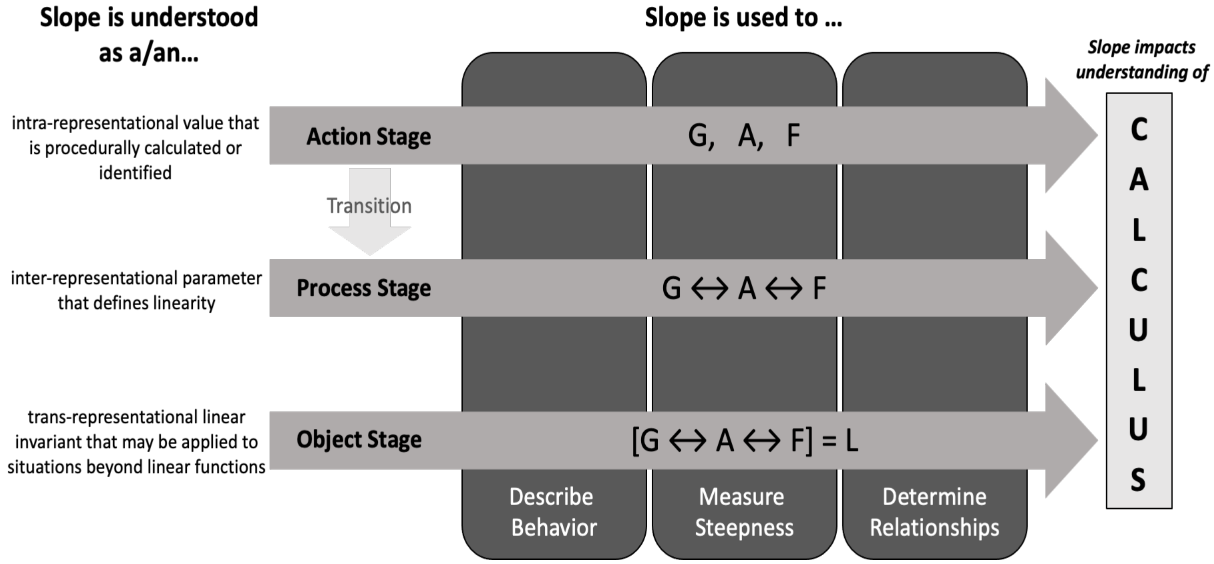
In APOS theory, Actions, Processes, Objects, and Schemas are stages, but it is also possible to consider the transition "levels" between these stages (Arnon et al., 2014). What the transition levels involve, and how many there are, depend on the mathematical topic. Rather than thinking of APOS theory as a strict linear progression from Actions to Processes to Objects, all of which are finally organized in Schemas, it is important to realize that there can be partial developments, passages to, and returns from the stages (Arnon et al., 2014). When encountering a mathematical task, individuals may try to address the task at hand through existing Actions, Processes, Objects, or Schemas. Individuals may complete an Action on a partially constructed Process. In this case it may seem that the individuals have constructed a Process on which they can do Actions. However, on another task that is related to the same topic but that involves a slightly different scenario, the individuals' transition level structures may not suffice to solve the problem. The individuals would need to review their ways of thinking to accommodate sufficient mental structures to handle the problem situation. That is, they will need to reconstruct or reorganize their existing mental structures, perhaps advancing towards the construction of a Process, to assimilate the new problem situation. To an observer, these individuals seem to be moving back and forth between Action and Process stages. We will refer to individuals doing this to be at a *Transition* level. Note that a level of development between the Process and Object stages has been suggested (Arnon et al., 2014). While one can imagine what this level (referred to as *Totality*) might look like as students encapsulate an idea and are starting to perform Actions on it, we leave it for another paper.

APOS-Slope Framework

The APOS-slope framework, in Figure 1, was first introduced by Nagle and colleagues in 2016 and then more thoroughly vetted (Nagle et al., 2019). In that time period, Deniz and Kabael (2017) also used APOS theory to study 8th graders' understanding of slope.

Figure 1

APOS-Slope Framework (adapted from Nagle et al., 2019).



The APOS-slope framework, in Figure 1, evolved from earlier work by Sheryl Stump (1999; 2001a; 2001b) and then by Moore-Russo and colleagues (Moore-Russo, Conner, & Rugg, 2011; Nagle et al., 2013; Stanton & Moore-Russo, 2012). These earlier works described 11 conceptualizations of slope evidenced by students, teachers, and standards documents. For this paper, we have slightly modified and extended Nagle and colleagues' APOS-slope framework to consider how different understandings of slope impact students when they are learning introductory calculus.

The APOS-Slope framework builds on decades of research on slope while distinguishing between how slope is understood and the purposes for which it is used. It can extend to uses of slope in trigonometry, but, in this paper, we consider only slope extensions to calculus, especially topics where students build on the slope of a tangent line at a point as the first derivative for a specific input in nonlinear functions, as outlined in Table 1.

Uses of Slope

The APOS-slope framework distinguishes between the three uses of slope, as shown in the vertical columns in Figure 1. Each use is emphasized in algebra (Bateman et al., 2021) but is also foundational for different topics in calculus, a select few of which are described in Table 1.

Table 1

Common Uses of Slope in Introductory Calculus

Slope used in	To Describe Behavior	To Measure Steepness	To Determine Relationships
Related Rates	Express a rate of change mathematically (e.g., dy/dt) by interpreting the direction of change in the quantity (y) over a change in time (e.g., if y is the distance between two cars driving away from each other, then dy/dt is positive); Interpret the sign of dy/dt by considering the signs of both dy/dx and dx/dt, where each can be interpreted as slopes	Recognize the role of slope (dy/dx) when determining the rate of change of y with respect to time (dy/dt); for instance, in examples of a point moving along a curve at a constant rate of change dx/dt, recognize that how quickly y changes depends both on dx/dt and the steepness of the tangent line to f at a given point	
Mean Value Theorem			Visually approximate points where a tangent line is parallel to a given secant line; use the derivative to identify input values where the instantaneous rate of change is equal to the average rate of change over an interval
Differentials & Linearization	Extend the relationship between sign of slope and (increasing or decreasing) behavior of a line to reason about whether dx and dy on a linear approximation for a function have the same or opposite signs; use $dy = \mathrm{m}dx$ to recognize direction of change in dy is dependent on both the sign of m and the value of dx	Use slope as a tool to measure steepness of a linear approximation to determine whether a given dx will result in a large or small corresponding value of dy; compare $f'(a)$ for different values of a to describe how the same value of dx can result in different values of dy, as determined by the steepness of the tangent lines at various inputs on the function f	
Function Behavior	Coordinate behavior of a tangent line to $f(x)$ at $x = a$ with the sign of $f'(a)$; coordinate intervals on which f is increasing, decreasing, horizontal with intervals on which f' is positive, negative, zero; recognize that horizontal tangent lines to points can signal extrema or inflection points	Visually compare severity of tilts of tangent lines at different points along a function's domain to determine if slopes are increasing or decreasing; looking at the change in $f'(a)$ for a sequential series of inputs over a limited domain of a function to determine concavity of f over an interval	
Indefinite Integrals			Visualize that tangent lines at fixed values $x = c$ are parallel for functions with vertical translations; recognize that functions $f(x) = g(x) + C$ will have the same derivative and therefore $f(x)$ and $g(x)$ will have equal slopes for tangent lines at corresponding inputs

To *describe behavior* is to use slope to determine if a line's graph is (or the value for outputs in a linear function are) increasing, decreasing, or horizontal (i.e., constant output values) as input values increase. To *measure steepness* is to use slope to determine the angle of inclination of a graph that impacts the severity of tilt of a line's graph or the rate of increase in the outputs in a table with equal increments between inputs. To *determine relationships* is to use slope to decide if linear graphs intersect or systems of linear functions have a solution. This involves recognizing slopes of parallel and perpendicular lines as being equal or negative reciprocals, respectively. It also entails students recognizing that a system of two linear equations with the same slope will either have no solutions or infinitely many.

Stages and Levels of Slope Understanding

We now concentrate on the three ways of thinking of slope, namely as a geometric ratio (G), algebraic ratio (A), or functional property (F), when at the Action or Process stage (or the Transition level between the two), until the three merge into a linear constant (L) conceptualization, which would occur at the Object stage. The ways of thinking of slope are shown in Figure 1 and summarized in Table 2; each is now described in more detail.

Action Stage

At the Action stage, students are constrained by isolated notions of slope where they use memorized procedures to do plug-and-chug calculations to produce a slope value. A student can exhibit more than one notion of slope but does not exhibit behaviors that suggest any ability to make connections between these notions.

The geometric ratio at the Action stage (A_G) represents a student who thinks of slope as the rise-over-run calculation without considering the *x*- and *y*-coordinates of points in the calculation; this stage is restricted to numeric and geometric representations of slope. They may only think of up and over, concentrating on the two distances, which they may (or may not) envision on a single, static slope triangle.

Students operating with A_G might occasionally misremember whether it is the horizontal or vertical displacement that goes in the numerator of the slope fraction since they are substituting in numbers without considering what the resulting values represent. These students might also fail to consider any negative displacement values that would be associated with the slope of a decreasing line, or they might ignore axes that are non- homogeneous when comparing the slopes of the graphs of two linear functions (Zaslavsky, Sela, & Leron, 2002).

Table 2

Descriptions of Possible Stages and Levels for Slope Understanding

Stage/Level	Description
Action Stage	
Geometric Ratio (A_G)	Students calculate slope using the formula ΔV/ΔH, which is rise over run or the vertical change over horizontal change using two corresponding distances on a line's graph.
Algebraic Ratio (A_A)	Students calculate slope using the formula $(y_2 - y_1)/(x_2 - x_1)$, which is the difference in y-values of two given points from a line divided by the difference in the point's corresponding x-values.
Functional Property (A_F)	Students identify slope as the coefficient of the x term in a linear function; slope is verbalized as "rate of change."
Action to Process Transition Level	
Geometric Ratio (T_G)	Students are starting to use ΔV/ΔH to calculate slope and recognize that all ΔV/ΔH ratios calculated for a particular line are equivalent using mental imagery to see calculations involve similar triangles.
Algebraic Ratio (T_A)	Students are starting to use the formula $(y_2 - y_1)/(x_2 - x_1)$ to calculate slope and to recognize that the difference ratios calculated using any two given points from a particular line yield the same slope.
Functional Property (T_F)	Students are starting to recognize that a single linear function can be represented with equations in different forms (e.g., $y = \mathrm{m}x + \mathrm{b}$ versus $\mathrm{a}x + \mathrm{b}y = \mathrm{c}$), are beginning to convert between those forms, and are equating slope as it is indicated in both formats (e.g., m = -a/b).
Connecting Alg. & Geom. Ratio ($T_{G\text{-}A}$)	Students can reason at both the T_A and T_G levels described above and are starting to understand the connections between the geometric and algebraic representations (e.g., ΔV is $y_2 - y_1$) in the formulas.
Connecting Geom. Ratio & Funct. Property ($T_{G\text{-}F}$)	Students can reason at both the T_G and T_F levels described above and are starting to make connections between the geometric ratio and the steepness of the line such that larger values of |m| yield steeper lines for $y = \mathrm{m}x + \mathrm{b}$.
Connecting Alg. Ratio & Funct. Property ($T_{A\text{-}F}$)	Students can reason at both the T_A and T_F levels described above and are starting to understand the connections for slope in the algebraic ratio formula from the point-slope equation of a line.
Connecting Geom. Ratio, Alg. Ratio & Funct. Property ($T_{G\text{-}A\text{-}F}$)	Students can reason at the T_G, T_A, T_F, $T_{G\text{-}A}$, $T_{G\text{-}F}$, and $T_{A\text{-}F}$ levels described above and are starting to reason between representation pairs; they are also starting to realize that each unit increase in the input corresponds to a fixed change in the output, m.
Process Stage (G ↔ A ↔ F)	Students understand that slope is a constant property of a linear relationship which is independent of representation and move fluidly between geometric, algebraic, numeric, and verbal situations involving slope realizing that certain ways of thinking about slope might be more efficient.
Object Stage ([G ↔ A ↔ F] = L)	Students understand slope is a linear invariant that describes an equivalence class of ratios; they can extend across contexts and representations, including dealing with situations in which they have not previously encountered slope, such as applying slope in three dimensions to determine the slope of a plane in relation to a given line (or an axis).

The algebraic ratio at the Action stage (A_A) represents a student who is limited to using the formula $(y_2 - y_1)/(x_2 - x_1)$ as a memorized fact or to using an explicitly available copy of the formula without any geometric inference connecting or justifying the formula. This is what

Reiken (2008) called a procedural understanding of slope, seeing it as a “number from a formula.” Since using rote procedures, students operating with A_A might confuse whether the *x*- or *y*-values are “on the top of the fraction” in the algebraic formula for slope. A student operating with A_A might perform multiple calculations when given more than two points on a line, not yet recognizing that the slope will be independent of the pair of points chosen.

The functional property at the Action stage (A_F) represents students who work with and are able to interpret some symbolic and verbal representations of slope that are related to linear functions. They may identify the coefficient of the *x*-term in a linear function to be the slope, but they may not differentiate between linear functions in different forms, even when both forms are equations. For example, they might identify 2 as the slope for the equation $y = 2x + 7$ as well as for $2x - 3y = 1$. They are simply following the procedure of identifying the number in front of *x* in the equation and not thinking about how the 2 or any of the other coefficients interact in either equation. These students may parrot back the phrase “slope is the rate of change of a function” in an empty verbal identification without understanding what holding covariation constant means.

While students operating with A_F might memorize familiar contextual interpretations of slope as a rate of change (e.g., recalling that a time, in hours, versus distance, in miles, function has slope in terms of velocity in mph units) they would not be able to explain the rate of change and what it represents contextually. They would be reciting words from memory without any geometric imagery of what this notion means or any connection to the difference quotient that could be calculated for two ordered pairs that satisfy a linear equation.

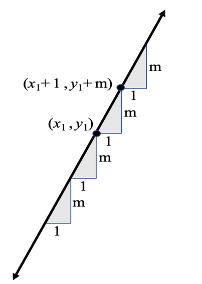
Transition from Action to Process Stage

Students transitioning from the Action to the Process stage exhibit a deeper understanding of slope that extends past simply following a procedure through mindless plugging-and-chugging. Note that whether transitioning to T_G, T_A, or T_F, the student is extending beyond a single action and starting to be able to repeat the action in order to describe linearity (e.g., justifying that a set of points represents a line by verifying the ratio or rate of change is constant). See Table 3 for reasoning that is representative of the dynamic nature of each of the three transitions that we now describe.

Students transitioning from the Action to the Process stage for geometric ratio (T_G) are beginning to realize that the slope of a line does not depend on which “rise over run triangle” is used to determine the vertical displacement (ΔV) and horizontal displacement (ΔH). They might use mental images where they visualize a slope triangle moving up and down a line or changing size to understand that the ΔV/ΔH ratios for a line have equal values, since the ratios are formed from the corresponding sides of similar triangles.

Table 3

Reasoning Involved in Transitioning from the Action to Process Stage

<table>
<tr><th>Student is starting to understand the dynamic covariation in …</th><th>Representative Reasoning</th></tr>
<tr><td>Slope triangles consistent with T_G</td><td>
</td></tr>
<tr><td>Input-output pairs consistent with T_A</td><td>$$\begin{array}{ccc|ccc} & & x & y & & \\ \hline & & -2 & -2m+b & & \\ & & -1 & -1m+b & & \\ & +1 & 0 & b & +m & \\ & +1 & 1 & m+b & +m & \\ +4 & +1 & 2 & 2m+b & +m & +4m \\ & +1 & 3 & 3m+b & +m & \\ & & 4 & 4m+b & & \\ & & 5 & 5m+b & & \end{array}$$</td></tr>
<tr><td>Functional relationships consistent with T_F</td><td>$$f(x) = mx + b$$ $$f(x+1) = m(x+1) + b$$ $$f(x+1) = mx + m1 + b$$ $$f(x+1) = (mx+b) + m$$ $$f(x+1) = f(x) + m$$</td></tr>
</table>

Students transitioning from the Action to the Process stage for algebraic ratio (T_A) are beginning to realize that the slope of a line does not depend on which two points are used to determine the difference in outputs (Δy) and the difference in inputs (Δx). They are starting to look at an input-output table and see patterns in the output changes for constant, incremental increases in the input values. They are beginning to realize that the slope formula does not need two given points but may be used with any general point (x, y) and a specific, fixed point from a line to obtain a calculation for slope that involves symbols. They might be able to determine the pay rate for a part-time hourly worker given how much the worker earns when working 12 hours per week and when working 20 hours per week.

Students transitioning from the Action to the Process stage for functional property (T_F) are realizing why linear functions must be in a certain format (e.g., the slope-intercept form rather than the standard form) for the coefficient of the x term to act as a parameter since all coefficients in a function's equation interact. Students are recognizing that x and y are variables and as such can take on different values, where y is a function of and dependent on x, while the numeric coefficients are held constant. They are seeing that m in the slope-intercept form of a line denoted by $y = \mathrm{m}x + \mathrm{b}$ (or the -a/b ratio in the $\mathrm{a}x + \mathrm{b}y = \mathrm{c}$ standard form of a line) is constant, acting as a parameter, and does not depend on the input and output values used in the function. They are also realizing that the equations $\mathrm{a}x + \mathrm{b}y = \mathrm{c}$ and $\mathrm{ka}x + \mathrm{kb}y = \mathrm{kc}$ (for some nonzero constant k) have the same slope since -kb/ka = b/a, indicating either represents the same function when rewritten in a simplified point-slope form. This reasoning is facilitated by understanding how the parameter (either m or -a/b) in the function's equation indicates the amount of change in y related to the change in x. In short, this reasoning is facilitated by understanding how covariation works in linear functions.

Students transitioning from the Action to the Process stage are also starting to make connections between different ways of thinking about slope. Students transitioning from the Action to the Process stage, who are connecting the geometric and algebraic ratios ($T_{G\text{-}A}$), are able to reason at both the T_G and T_A levels. They are connecting the geometric and algebraic formulas and understand that vertical displacement can be represented by $(y_2 - y_1)$ while horizontal displacement can be denoted as $(x_2 - x_1)$ for two ordered pairs (x_1, y_1) and (x_2, y_2). They may also be realizing that if the slope triangles for two different lines have the same run (i.e., horizontal displacement), then the severity of the angle of inclination between the horizontal and the line increases, or decreases, in a manner that corresponds to the rise or fall of the line (i.e., the vertical displacement). They are beginning to internalize steps so they could apply this same reasoning to conclude that the line through (x_1, y_1) and $(x_2, y_1 + \mathrm{c})$ is steeper than the line through (x_1, y_1) and $(x_2, y_1 + \mathrm{k})$, for constants c and k where $|\mathrm{c}| > |\mathrm{k}|$, without having to actually calculate the slope as either an algebraic or geometric ratio.

Students transitioning from the Action to the Process stage who are connecting the geometric ratio and functional property ($T_{G\text{-}F}$) are able to reason at both the T_G and T_F levels. They are making connections for slope between the geometric ratio and the graphical referents (i.e., the horizontal and vertical displacement) to the m coefficient found in the linear function $f(x) = \mathrm{m}x + \mathrm{b}$. They are starting to realize that larger values of |m| yield steeper lines regardless of the *y*-intercept when graphed on the same axes, and they may be beginning to justify their observations by connecting the magnitude of the rate of change to the rise per common run for the two linear relationships. They may also be noticing that the slope for one line may appear to be steeper than another when comparing two graphs even when this is not actually the case (i.e., when the scale factors for the graphs' axes differ). This reasoning when encountering nonhomogeneous coordinate systems may be supported by verbal descriptions of the changes represented by vertical displacement and horizontal displacement that move beyond counting to describing those physical changes in the graph with descriptions of change in quantity and units. Furthermore, they recognize how the rate of change is determined by the parameter m given a function $f(x) = \mathrm{m}x + \mathrm{b}$, since m will determine how much the outputs change for a given change in inputs. Such a student might be able to create a graph showing the height of a classroom set of textbooks stacked on a teacher's desk using the linear function where the height, *H*, of the stack from the ground is given by the equation $H(t) = \mathrm{m}t + d$, and interpret the slope, m, as the thickness of each textbook while *t* represents number of textbooks and *d* is the height of the teacher's desk.

Students transitioning from the Action to the Process stage who are connecting the algebraic ratio and functional property ($T_{A\text{-}F}$) are able to reason at both the T_A and T_F levels. They may be realizing how to connect the algebraic formula for slope and m in the point-slope form of a line, moving from $m = \frac{y_2 - y_1}{x_2 - x_1}$ to $y_2 - y_1 = \mathrm{m}(x_2 - x_1)$ to $y_2 = \mathrm{m}(x_2 - x_1) + y_1$. In the special case where the *y*-intercept (0, b) is substituted for the point (x_1, y_1), this then leads to the equation $y = \mathrm{m}x + b$. These associations allow the student to understand how slope, in association with the *y*-intercept, acts as a defining parameter for linear functions. Such a student might also be able to interpret a given contextual scenario to interpret the constant rate of change and initial value to yield additional points using the algebraic ratio of slope. For instance, given a scenario where a 16-gallon tank empties at a rate of 3 gallons per hour, the student might apply algebraic ratio reasoning to determine that, if the tank was initially full, then after 3.5 hours there would be $-3 = \frac{y_2 - 16}{3.5 - 0}$ or $y_2 = 5.5$ gallons left in the tank.

Students transitioning from the Action to the Process stage who are connecting the geometric ratio, algebraic ratio, and functional property ($T_{G\text{-}A\text{-}F}$), can reason at the T_G, T_A, T_F, $T_{A\text{-}G}$, $T_{G\text{-}F}$, and $T_{A\text{-}F}$ levels. Students are realizing the connections between the point-slope and the slope-intercept forms of the line for a horizontal displacement of 1. So, they are realizing that holding the horizontal displacement in a slope triangle to 1, relates to selecting points where $x_2 - x_1 = 1$. This, in turn, results in m = ΔV/ΔH = ΔV/1 = ΔV, which is equivalent to having a slope of m = $(y_2 - y_1)/(x_2 - x_1) = (y_2 - y_1)/1 = (y_2 - y_1)$ or ΔV. They also see how the slope triangles and the difference quotient are based on inputs and outputs that relate to a more functional reasoning that $f(x + 1) - f(x)$ = m. They may be extending their reasoning to understand not only how a unit increase in the input corresponds to a fixed change in the output, m, but also how an increase in k units, for any k, corresponds to a km change in the output for equations in the slope-intercept form. While the students can move between levels, this might take significant thought to move between pairs of levels rather than the fluid reasoning that easily moves to the way of thinking and working with slope that is most efficient for the scenario at hand that is now described for students at the Process stage. A student at this level might be able to find the function for Total Cost (C) in terms of distance (d) in miles corresponding to the scenario of a cab charging $12 for the first 2 miles and then 60 cents for each fifth of a mile. However, this process might not be very efficient. The student might first determine that slope is a rate of change that is ($0.60)/(1/5 mile) or $3 per mile. With this information the student might then create a table of ordered pairs starting with (2, 12) then moving to (3, 15) then adding additional ordered pairs using only integer inputs. Next, the student might then verify the slope using an algebraic ratio and confirm the geometric ratio using a graph of the scenario on a Cartesian coordinate system. From the graph the student would then finally report that C(d) = 3d + 6, for d $\geq$ 2 miles.

Process Stage of Slope

Students at the Process stage have an understanding that is not grounded in one specific form of representation. Instead, students at this stage have developed the ability to attribute various components of a generalized meaning for slope through different representations. They can move fluidly and easily between geometric, algebraic, numeric, and verbal situations involving the slope of a linear function realizing that in certain cases one way of thinking about slope might be more efficient. They have covariational reasoning (Carlson et al., 2002) abilities that allow the mental actions for coordinating the average rate of change of a function with continuous changes in its input variable. In other words, these students understand and are able to visualize that a horizontal displacement (or change in input values) of 4 for a line would result in a vertical displacement (or change in outputs) of 4 times the slope value for the line. They would understand how this would be reflected by corresponding points on the line. They would be able to mentally picture moving over 1 unit and up m units using 4 slope triangles, and they would be able to mentally picture this as changes in the input of a table that correspond to changes in the outputs of a table. The student would also have the simultaneous awareness that the equation could be written as $y = \mathrm{m}x + \mathrm{b}$ or some other form, where b is vertical displacement of the function's output for a given input of 0 (or the function's initial height) and that mx is the vertical change given as the rate of change m, times the change in inputs x. (Some of the different possible mental images for the process stage are displayed in Table 3.) The student would also be able to interpret this rate of change for a given context. For instance, if x represents time and y represents the volume of water in an emptying tank, the student not only interprets m as the rate at which the volume is decreasing but can apply the above reasoning to think about how a horizontal displacement of 2 represents a time increment of two seconds and the corresponding vertical displacement represents the decrease in volume over that time. Students at the Process stage recognize that a function that fails to meet any of the criteria previously listed would not be linear.

Students at the Process stage can view slope as a linear parameter that involves understanding that slope is the defining constant property that serves as a unique parameter in a linear equation that determines the "straightness" associated with linear graphs. Students at the Process stage view slope as a constant property or a constant rate of change between two covarying quantities. At the Process stage, students are aware of slope as a parameter that allows lines to be parallel. They understand that any pair of points on a single line can be used to determine what the defining slope parameter is. So, students at the Process stage also understand that this constant property is also captured by thinking of slope as a constant rate of change between covarying quantities that can move between representations. So, not only would students with a Process stage of slope be able to connect and move fluidly between all three representations displayed in Table 3, but they would also be able to select the representation that is most efficient to the task at hand. In the previous contextual example of water emptying from a tank, viewing slope as a linear parameter enables a student to extend that reasoning to conclude that the volume decreases by the same amount between t = 1 and t = 3 hours as it does for the same duration of time with different starting and stopping points (e.g., $t = 2$ and $t = 4$ hours). The student may also reason about how the starting volume of water in the tank would impact the remaining volume at any given time. However, this student would note the rate at which the water exits the tub and could reason about how adjusting the rate of change for water to empty from the tank would influence the change in volume over any period of time. Furthermore, the student could reason that a different tub emptying at the same rate might have a different starting height, but the same constant rate of change, hence yielding a parallel linear relationship if comparing the volumes of the two tanks.

Object Stage of Slope

Students at the Object stage understand all the connections within the Process stage, and they are also able to view slope as a linear invariant. At the Object stage students are able to do everything in the previous paragraph; however, they are also able to do more. Viewing slope as an Object extends beyond seeing slope as a constant to recognizing that slope is an invariant that transcends contexts and representations. A student at the Object stage interprets a single slope, m, as a representative of the set including all equivalent ratios. As such, the student can view slope as a new object, an equivalence class of ratios, which can be applied in different contexts. For instance, provided the linear relationship $y = (2/5)x,$ where x represents the amount of water and y represents the amount of concentrate in an orange juice mixture, the student would view slope not just as 2/5, but as the set of all ratios y/x that would preserve the strength of the orange juice concentrate mixture, independent of the amount of orange juice made. This understanding moves beyond a procedural understanding of equivalent fractions (e.g., 6/15 = 2/5) to understanding that the solution set of ordered pairs generate a graph that maintains a constant concentration ratio no matter the quantity of orange juice. The individual could also switch contexts to interpret 2/5 as the same equivalence class of ratios in different real-world scenarios or using different representations (e.g., parallel lines rather than equivalent ratios).

Thus, the individual now sees slope as an invariant within an equivalence class of ratios that can be described through various forms of representation. They can extend past the Process stage to make applications beyond the contexts in which they have previously encountered slope.

They are able to apply Actions on a Process to deal with the new scenarios, even those involving unfamiliar units or contexts. For example, situations where students must extend their linear notions of slope to interpret slope fields or apply directional slope of a plane in three-dimensional space extend into this stage since students would need to perform Actions on slope as a Process to work in these new scenarios.

Methodology

We now consider how different stages might impact students' understanding of calculus and how they are able to use slope in calculus. To do this, we apply the methods adopted by Liljedahl and colleagues (2015) following the lead of others (Leron & Hazzan, 1997; Zazkis & Koichu, 2014).

We created the amalgamations of two cases to create an account that is a "fictionalized aggregate" based on our decades of "collective experiences" (Liljedahl et al., 2015, p. 195) both as post-secondary calculus instructors and mathematics education researchers who have significant work in the study of slope. We report on two amalgams, each with a different understanding of slope. We then share experiences for each amalgam, whose initials correspond to a stage or level of slope understanding, using different topics that arise in an introductory calculus course. In this manner we explore how students who lack a deep, flexible understanding of slope might struggle to learn calculus.

Amalgamated Cases

Due to space limitations, we only present two amalgams. They are fictionalized aggregates based on the literature on slope reviewed at the beginning of the paper as well as on the researchers' over 50 years of combined experience teaching both algebra and calculus courses. These cases were selected to represent students who are not yet at the Process stage while representing all three types of understanding (A, G, F). The first case, Albert Algar, demonstrates understanding of slope as an algebraic ratio at the Action stage (A_A). The second case, Tracy Graff-Fund, demonstrates a student transitioning from the Action to the Process stage who is connecting the geometric ratio and functional property ($T_{G\text{-}F}$) and, hence, who can reason at both the T_G and T_F levels.

The Case of Albert Algar (A_A)

Consider a hypothetical student named Albert who entered a first-semester calculus class confident that he understood the topic of slope. He explained that slope was the change in *y* over the change in x. If you gave him two points or a table of input and output values, he was able to quickly calculate the value of the slope of the segment connecting them using an algebraic ratio. His calculus instructor had spent the first few classes reviewing concepts he had seen in high school. Albert understood them well enough, although he did not follow when the instructor encouraged the class to look at graphical or functional representations of otherwise familiar ideas. This included understanding slope as what would be classified as A_G and A_F. He hoped the tests would only require him to apply the formula he had memorized to calculate slope using two points.

In class one day, Albert watched his instructor show an animation of a secant line where one endpoint remained in place and the second endpoint moved ever closer to the first endpoint to define a derivative. He tried to follow the visual demonstration and worried less when he heard the instructor mention that slope would be used. When completing problems that involved finding the slope of secant lines, Albert was able to calculate slope values using coordinates of two given points. He felt this often involved some tedious calculations, such as 10.1 then 10.01 then 10.001 for inputs, but he was able to complete the tasks using $(y_2 - y_1)/(x_2 - x_1)$. Albert did relatively well on the related homework, even though he was not sure how the animation the professor had been so excited about tied to the limit definition of the derivative. He remembered that the professor implied you could find the slope at a point on a function (or as the professor put it, "you can find the slope of the tangent line"). However, it did not seem logical to Albert that slope could be calculated using just one point. He felt a person must have two sets of ordered pairs to plug in all the necessary information for the slope formula. Albert did not worry about this since he received passing grades on the homework assignments for the limit definition of a derivative that followed. He was able to perform the algebraic calculations to answer most of the problems using multiplication by the conjugate to rationalize either the numerator or the denominator or another "algebraic trick" that his professor had shown the class. Even though Albert was able to complete the algebraic simplifications and then plug in $h = 0$ to arrive at answers, he became increasingly worried that he did not understand what his final answer meant or how it related to slope since he was never plugging in coordinates. Without understanding the geometric ratio, he could tell from the instructor's occasional inclusion of graphs that the derivative was supposed to represent slope, but Albert was unsure how. His confusion was even more pronounced when confronted with application problems that required using slope as a functional property when asking for an interpretation of the derivative in the context of real-world variables (e.g., time, distance, cost).

Albert performed better when he was given shortcut formulas for derivatives, especially after a friend taught him the "low-d high" mnemonic for the quotient rule. One day, the professor provided a graph of two unfamiliar functions, *f* and *g*, and asked the class to find $(fg)'$ and $(\frac{f}{g})'$ for a particular input value, see Figure 2 for the task assigned. Albert became frustrated because although he knew finding the value of each derivative required him to calculate slope, he could not use the algebraic ratio since he was only given an *x* coordinate. Without being able to interpret slope as a geometric ratio, Albert could not calculate the values to plug into the product and quotient rules nor could he understand the professor's solution and how the values of f' and g' needed for the formulas were determined when only the graphs of *f* and *g* were provided.

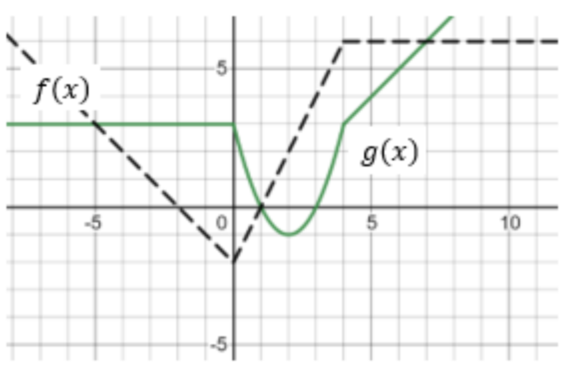
Without an understanding of the geometric ratio, Albert was not able to visualize the slope of a tangent line to a function. Thus, things became worse for Albert when implicit differentiation was introduced. The derivative was now a function of the ordered pair (*x*, *y*) and the first example presented by the professor required at least A_G to visualize the slopes of two different tangent lines to an ellipse at a given *x* input with more than one *y* output.

Figure 2

Derivative Problem Emphasizing Visual Interpretation of Slope

Using the graphs of *f* and *g* provided below, determine the following:

a. $(fg)'(-3)$
b. $(\frac{f}{g})'(2)$



Note: Understanding slope at the Action stage as an algebraic ratio, Albert struggles to calculate the derivative of *fg* given only graphical representations of *f* and *g*.

Without an understanding of slope as a functional property, Albert did not see slope as a rate of change constant and was completely overwhelmed when the professor discussed related rates. Every example involved interpreting rates of change in real-world scenarios. These problems required calculating and interpreting both the rate of change of y with respect to x and the rate of change of x with respect to t. Albert had tried his best to pay attention in lectures and reread his notes before class, but this did not seem to be enough; so, he dropped the course.

The Case of Tracy Graff-Fund ($T_{G\text{-}F}$)

Consider a second hypothetical first-semester calculus student named Tracy. Like Albert, Tracy was confident that she understood slope. When asked about slope, she would report that slope was the rate of change noted as m when a linear equation is solved in terms of the y-variable, thinking of slope as a functional property. She also thought of slope as a geometric ratio and might describe what can be seen on a linear graph as the ratio of the vertical change to the horizontal change. She could see that this ratio did not depend on which rise-over-run triangle was used on the line's graph and recognized the rise-over-run ratio, which had the same value as m, as determining the tilt of the line's graph and if it increased or decreased. She loved when her calculus professor drew on the board and gave a visual interpretation of secant lines approaching a tangent line to show the slope of a curve at a point as the slope, or the instantaneous rate of change of y with respect to x. Tracy noted that the professor was drawing a line through a point on the graph that just barely touched the graph and tried to follow as the limit of the secant lines became a tangent line. Tracy felt she understood this visual relationship. However, without understanding slope as an algebraic ratio, the formula that was given for the limit definition of a derivative with the professor's explanation that it was "simply the limit of the difference quotient" seemed to come out of nowhere.

On the assigned homework problems, she had difficulty remembering the "limit formula for the derivative" (as the professor called it) and occasionally used $f(x) + h$ instead of $f(x + h)$ when applying the definition to a particular function since she was not connecting slope as an algebraic ratio to slope as a functional property. Without the algebraic ratio, she struggled when given tables with points and tedious decimal numbers for inputs using only the geometric ratio. Tracy plotted the points on a graph to find the slope of the secant line that would be closest to the tangent line for the points given, which was laborious due to the very small horizontal and vertical changes from one point to the next. Even though Tracy could not always perform the procedures, she did well on conceptual and visual questions that asked her to interpret what the derivative meant, even in applied situations.

Tracy could plug into the shortcut formulas for calculating derivatives, even though she felt lost when the professor tried to show how the shortcuts related to the limit definition of derivative. She was relieved to see the limit definition used less and less frequently as shortcuts were learned. Since she was comfortable with the geometric ratio being used to describe behavior and determine relationships, Tracy particularly liked the problems that required her to identify where a function had a horizontal tangent line or to identify points on a function where a graph would have a tangent line parallel to a given line.

With her functional property understanding of slope, Tracy did well on the chain rule section and felt she understood the "rate of change with respect to what" question that the professor would ask. She also did well on implicit differentiation, especially when problems provided a graph of the function and asked students to find all solutions for the slopes of the tangent lines for a given value of x since they required her to use slope as a geometric ratio to describe behavior and measure steepness.

Tracy memorized all the derivative rules and did well on the next exam that contained mostly conceptual problems and related rates that required interpreting the slope as a geometric ratio or a functional property. Tracy followed the professor's drawings and explanations of the Mean Value Theorem using parallel secant and tangent lines, since she understood how slope is used to determine relationships both as a geometric ratio and a functional property. However, without an understanding of the algebraic ratio for slope, she struggled with most of the problems in the section that relied heavily on what the professor called the difference quotient.

Tracy followed most of the professor's lecture on differentials, which included drawings that relied on slope as a geometric ratio, especially when the professor referenced vertical displacement and went between the actual function graph and the tangent line using dy and Δy markings on his drawing. However, she struggled to understand the problems in the assignment on linearization since the professor's explanation of this only involved symbols. Without understanding slope as an algebraic ratio, Tracy had trouble making the leap first to $\Delta y = y_2 - y_1$ (shown in the graphical explanation that Δy was approximately equivalent to dy) and then to using the calculation $\Delta y = f(c + \Delta x) - f(c)$.

Tracy did well visualizing increasing/decreasing intervals, which corresponded to the first derivative test, where she often sketched a picture while solving problems that involved interpreting whether a critical value produced a relative maximum or minimum (or neither). These allowed her to leverage her understanding of slope being used to describe behavior both as a geometric ratio and as a functional property. She also did well when the professor provided a graph of a function and asked the class to determine intervals on which the function was concave upward or downward. She would often apply her knowledge of slope as a geometric ratio used to measure steepness to sketch several tangent lines and visually inspect the steepness or "tilt" as she thought to herself to determine whether the slope values increased (concave upward) or decreased (concave downward). She was able to move between her understanding of slope as both a functional property and a geometric ratio to describe behavior and measure steepness in order to sketch the graphs of f' based on the graphs of f.

The course ended with antiderivatives and an introduction to integration. Tracy felt confident in the class discussion related to how a family of antiderivatives involved an infinite number of curves, each of which had the same slope at every x-value on their graphs, since the tangent lines at these points were all parallel. Her understanding of slope as a geometric ratio and as a functional property used to determine relationships allowed her to see why the solution to an indefinite integral was not unique.

Tracy was concerned when the Fundamental Theorem of Calculus was introduced with a proof which relied heavily on the difference quotient from the Mean Value Theorem. Without a strong understanding of slope as an algebraic ratio, Tracy had struggled with these ideas earlier in the semester and was discouraged when they reemerged. She was relieved when the follow-up problems did not require the difference quotient and instead only involved finding an antiderivative $F(x)$ or evaluating $F(b) - F(a)$.

At the end of the semester, Tracy was able to explain that the slope of a curve at a point is the slope of the tangent line to the curve at that point. As in algebra, slope could be used to describe the behavior or measure the steepness of a graph. What was new was extending these ideas to describe concavity. Tracy was also able to explain that slope could be interpreted as the instantaneous rate of change between the outputs and inputs of a function and could interpret this in real-world contexts or for families of functions that differed by a constant. However, she was unaware of how the many rules and shortcuts she learned for finding derivatives were generated, explaining that these are "just the steps we need to memorize to get the right slope."

Discussion and Conclusion

The accounts of Albert and Tracy and their experiences in introductory calculus provide two cases that are based on research findings which informed the APOS-Slope framework, including those cited in the literature review, and the authors' teaching experiences. They exemplify challenges faced by calculus students who lack a robust understanding of slope. The stages and levels of slope understanding (in Table 1), indicate that Albert and Tracy are just two examples of many possible amalgams that might be used to consider the challenges for students encountering calculus topics that involve slope.

The most immediate implication of this work is to bring to light the diversity of student background knowledge related to slope that can be expected in an introductory calculus course. This may present instructional challenges to a professor, especially since it has been documented that instructors and students often have different ways of thinking about slope (Nagle et al., 2013). Students who are unable to make the connections between the ways of thinking about slope and its uses may have difficulty passing calculus or may pass with some gaps in their knowledge.

Past research provides evidence that students often enter calculus with limited slope understanding (Arcavi, 2003; Orton, 1984; Schoenfeld et al., 1993). Precalculus curriculum and instruction may not provide opportunities for students to connect slope as a visual property of a line with the idea of a constant rate of change (Frank & Thompson, 2021). So, calculus instructors should consider how to help students transition from thinking of slope as a value that can be calculated or identified to understanding it as an inter-representational parameter that defines linearity. Slope merits, at minimum, a review by calculus instructors in which direct connections between slope as an algebraic ratio, a geometric ratio, and a functional property are made explicit using a contextual example, like one of the ones presented in this paper. This might also involve engaging students in reasoning about linear tasks using dynagraphs, which move students away from more familiar calculations and representations to thinking about slope as a constant that describes a covariational relationship (Nagle & Moore-Russo, *in press*). Building such covariational perspective is more likely to facilitate students operating at the

Process stage, which is necessary to fully understand certain topics that arise in introductory calculus.

The cases also suggest that it may not be enough to only consider what stage or level of slope understanding calculus students bring from algebra. They also highlight that slope can be used to *describe behavior*, *measure steepness*, and *determine relationships* in calculus just as in algebra. Instructors should ensure that students can fluently apply all three uses of slope in constant rate of change settings before extending these uses to the variable rate of change settings encountered in calculus. For instance, previous research found that some university students were unable to use slope to measure steepness by comparing the magnitudes of tangent lines to a curve (Christensen & Thompson, 2012). So, students should not be expected to describe concavity without first addressing their ability to use slope to measure steepness in a linear context. Explicit review of the uses of slope prior to or during the teaching of certain calculus topics is suggested. For example, reviewing that slope is used to determine relationships should occur when teaching the Mean Value Theorem.

From the perspective of research, the amalgams provided offer an example of how the APOS-Slope framework, which is built on numerous research studies as delineated by Nagle and colleagues (2019), can be combined with professional experience to make sense of and connect findings in a way that should resonate with both researchers and instructors. The APOS-Slope framework helps move research on slope away from a deficit-grounded model focusing solely on what students are unable to do or conceptualize. Future researchers can use the APOS-Slope framework to take an asset-based approach to describe how students think of and are able to use slope and how this might, in turn, impact their understanding of calculus.

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