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Expressing Distance in Graphs of Functions in the Cartesian Plane: Obstacles and Interventions

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Abstract: Connecting algebraic and graphical representations encompasses a large portion of mathematical activity for students in grades 8-14. In Calculus, the ability to represent distances on graphs using algebraic expressions is foundational for a wide range of results. However, research has shown that students may struggle to make such connections. In this article, we seek to answer the following questions: (1) What are the underlying conceptions critical to expressing distances on graphs of functions algebraically? and (2) What types of tasks may support students in developing this skill? We first offer a conceptual analysis of the connections between algebraic expressions and distances in graphs of functions. Next, we describe a hypothetical learning trajectory of tasks and associated learning goals we developed to support students in expressing distances. We then report the outcome of implementing the tasks with groups of Calculus students. While the results include some success in supporting students to express distances, they also pointed to some persistent obstacles as well as limitations of the tasks. We focus on two such obstacles that emerged: the role of conceptualizing symbols as variables in the algebraic register and the role of conceptualizing distances within the graphical register. We discuss directions for future research and ways to support students in connecting algebraic expressions and graphs.

Keywords: distances in graphs, difference expressions, Calculus students, conceptual analysis, hypothetical learning trajectory

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Introduction

Connecting mathematical representations, especially algebraic and graphical ones, encompasses a large portion of mathematical activity for students in grades 8-14 (NGA & CCSSO, 2010; NCTM, 1989). These activities may include sketching a graph to display a functional relationship between variables or expressions, interpreting a graph as displaying a functional relationship to describe how the values of two quantities vary together, or classifying graphs based on certain characteristics. The ability to make connections across algebraic and graphical representations becomes increasingly important as students prepare for the study of Calculus. Visual representations of significant results in Calculus rely on a particular connection between algebraic expressions and graphs: distances on graphs may be represented with algebraic expressions (e.g., the horizontal distance shown in Figure 1 may be represented as $2-x$). Indeed, the notion of distance underlies the foundation of the Cartesian coordinate system as a way of representing values. In Calculus, conceptualizing and expressing distances are key to graphically representing expressions such as the difference quotient, or for understanding why integrals afford calculating areas and volumes of irregular regions. The applications of this use of Integral Calculus extend broadly into later mathematics and the natural sciences. Although connections between algebraic expressions and graphs are fundamental to the study of mathematics, especially at the early undergraduate level, research has shown that students may struggle to make such connections (e.g., Glen & Zazkis, 2021; Knuth, 2000; Moon et al., 2013). For instance, one Calculus III student who was interpreting a statement with a difference quotient was unable to describe relevant distances within a graph. Instead, the student counted along axes and the graph to explain what various difference expressions (such as a change in x) represented (Parr, 2021).

Our recent research has been motivated by this documented need to support students in expressing distances within graphs algebraically, as a fundamental component to conceptualizing central ideas of secondary and tertiary mathematics. In order to better understand why students may struggle with this and offer support accordingly, we seek to explore the following questions: (1) *What are the underlying conceptions critical to expressing distances on graphs of functions algebraically?* and (2) *What types of tasks may support students in developing this skill*?

Prior research has looked at how students move between representations in mathematics, in terms of representational fluency (e.g., Fonger, 2019), to develop coherent mathematical ideas for students beginning in the early years (e.g., Fennell & Rowan, 2001). Increasing attention has been paid to students of Calculus, as the content is rich with connections, especially among representations in the algebraic and graphical register (Duval, 1999). These studies include reports on students' fluency among representations in finding volumes of solids of revolution (Gulkilik, 2022) and the Fundamental Theorem of Calculus (García-García & Dolores-Flores, 2018). To support students in developing strong connections between representations in the algebraic and graphical register, researchers have begun developing tasks with this aim. For instance, Moon (2020) developed tasks to support students in conceptualizing solutions to algebraic inequalities within graphs in the Cartesian plane via the Cartesian connection. Still, research that describes how students may learn to develop rich connections among representations in different registers is limited, especially in later years. This study seeks to help fill this gap.

In this article, we first offer a conceptual analysis (Thompson, 2008) of the connections between algebraic expressions and distances in graphs of functions along with empirical findings of obstacles that exist for students in making these connections. Next, we describe a hypothetical learning trajectory (Simon & Tzur, 2004) we developed to support students, rooted in our conceptual analysis and prior research. We then report the outcomes of two iterations of implementing the tasks with groups of Calculus students. These findings include some success in using these tasks as an intervention to support students in expressing distances from graphs. The results also revealed some persistent obstacles in students' developing conceptions of distances. We focus on two such obstacles that emerged: the role of conceptualizing symbols as variables in the algebraic register and the role of conceptualizing distances within the graphical register. We close with directions for future research and recommendations for supporting students in building connections between algebraic expressions and graphs of functions that can be used at the secondary level, prior to the study of Calculus.

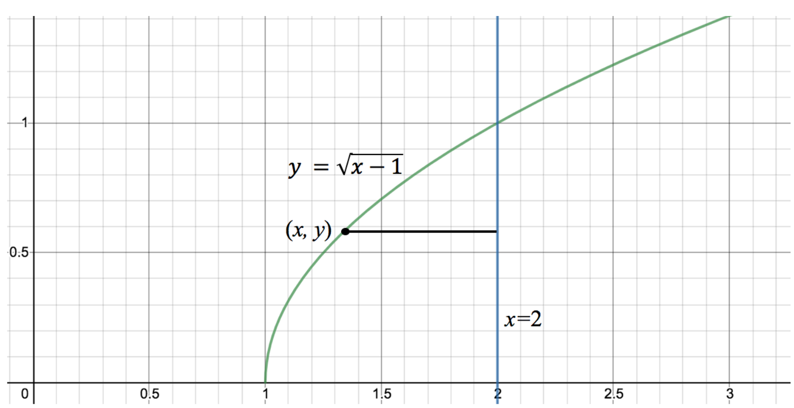
Conceptual Analysis & Related Literature

The central goal of this article focuses on the particular connection of representing distances between functions graphed in the Cartesian plane using algebraic expressions. In this section, we will share our conceptual analysis (Thompson, 2008) of the underlying conceptions critical to expressing distances on graphs of functions algebraically. We frame our discussion around the following task and ask: what are the conceptions involved in algebraically expressing the horizontal distance between any point on $\boldsymbol{y = \sqrt{x - 1}}$ and a point on $\boldsymbol{x = 2}$ (for all *x* such that $\boldsymbol{1 < x < 2}$) in terms of *x* and in terms of *y*? (See Figure 1). We note that the length of this segment may be expressed as 2–*x* or equivalently as $\mathbf{2 - (y^2 + 1)}$.

We engage in the process of conceptual analysis consistent with Thompson (2008) as describing "what students might understand when they know a particular idea in various ways" (p. 42) to detail the mental steps involved for students to conceptualize an algebraic difference expression as representing a horizontal (or vertical) distance within a graph in the Cartesian plane. Based on previous research and work with students on such tasks (e.g., Parr et al., 2021), we theorize that there are three main connections between the graphical register and algebraic register that underly this conception: (1) differences express distances between two positions, (2) points are ordered pairs of distances from the axes in the Cartesian plane and (3) equations give the relation between *x* and *y* for every ordered pair represented at a point on a graph. These three connections build upon each other to comprise the connection among *distances between functions in graphs* and *(algebraic) difference expressions*. We describe each of these connections between the graphical and algebraic register, as well as the underlying mechanism for how the connection is forged. We see potential obstacles for students within each component of the graphical register, algebraic register, and the mechanism of the connection. We will describe each of the three steps and previous research highlighting potential obstacles for students within each.

Figure 1

Graph with a Horizontal Segment from a Point (x, y) on $\boldsymbol{y = \sqrt{x - 1}}$ *to* $\boldsymbol{x = 2}$.

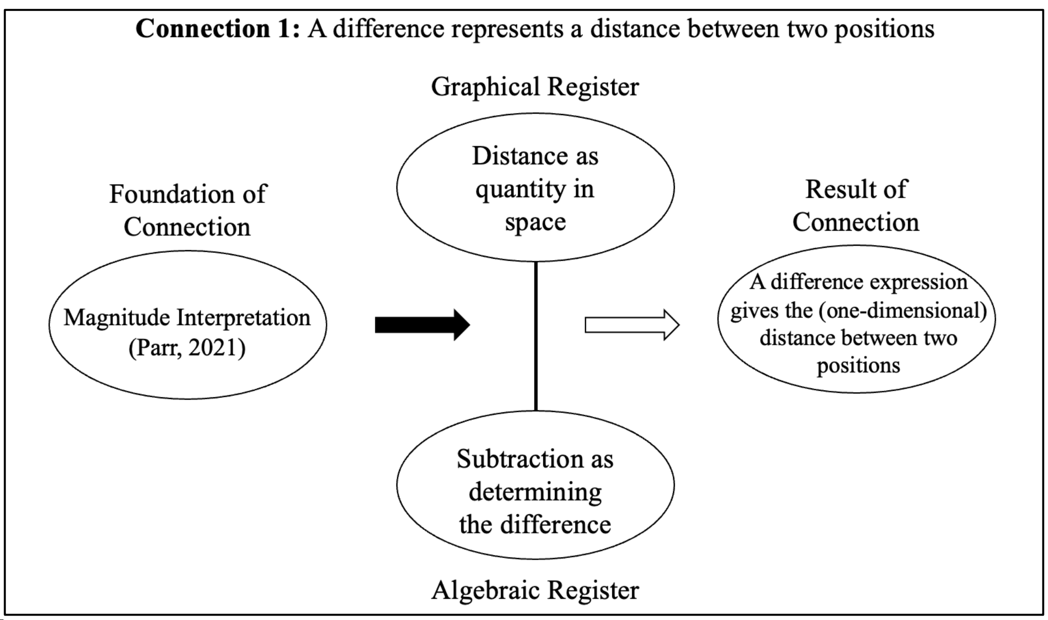


Connection 1: A Difference Represents a Distance Between Two Positions

From our perspective, the first main cognitive step to expressing distances on graphs of functions involves using a difference expression to represent a distance between two positions in a single dimension. Within the graphical register, this connection involves conceptualizing distance as a quantity, that is, a measurable attribute (Thompson, 1990, 2011) of the spatial arrangement of two positions within a number line or graph. Within the algebraic register, this connection involves operating from a "*determine the difference*," (van den Heuvel-Panhuizen & Treffers, 2009; Selter et al., 2012) also referred to as a "*comparison*" (Usiskin, 2008) model of subtraction, rather than a *takeaway* model. Connecting these conceptualized distances with difference operations relies on a magnitude interpretation of symbols (Parr, 2021). We summarize these components that result in this connection between differences and distances between two positions in Figure 2.

Figure 2

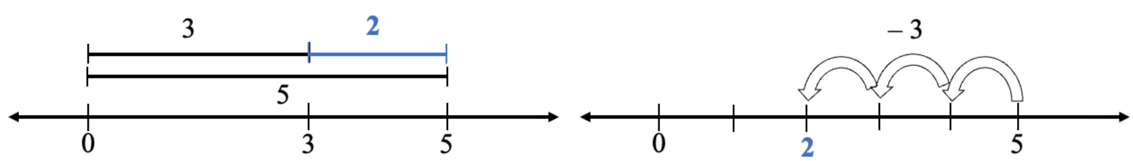
The Components of Connection 1: Connecting Distances in Graphs with Difference Expressions



Previous research suggests that students may not use these ways of reasoning underlying the connection involved in representing distances with differences. Within the graphical register, conceptualizing a distance as a quantity in space is a non-trivial component of this connection. The Cartesian coordinate system relies on distances to relate points, yet students may not conceive of distance between points as relevant when viewing positions on a number line or graph. Alternatively, they may conceive of positions as labeled at arbitrary locations, or use some other comparison, such as relative location, without explicitly attending to or conceptualizing measurable distance between positions (Parr, 2021). At the elementary level, students may not necessarily place non-consecutive numerical values at appropriate distances apart from one another on a linear-scaled number line (Saxe et al., 2013). Instead of using a number line as a measurement model, some students may use it as a counting model (Diezmann & Lowrie, 2006), counting the number of tick marks or intervals (Mitchell & Horne, 2008). Within the algebraic register, conceiving of subtraction as an operation that determines the difference between two amounts that are compared is not a given for students. Previous research has found that the comparison operations are those that pose the most difficulty for students among types of subtraction problems in early grades (Stern, 1993). When asked to interpret an expression involving a difference, students may think exclusively of a counting down or takeaway model, rather than a comparison one (Figure 3).

Figure 3

A Determine the Difference (Comparison) Model (left) vs. A Takeaway Model of 5–3 (right)

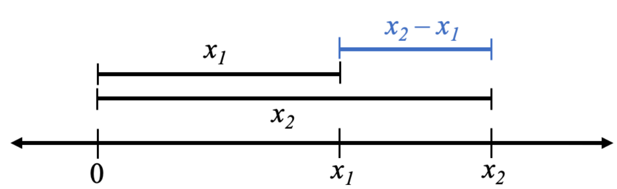


Students may also default to using the takeaway model for subtraction on a number line and may not as readily use a determine the difference model, perhaps because the former is more commonly encountered in school mathematics and everyday situations (e.g., Selter et al., 2012). Even at the undergraduate level, students may default to using a takeaway model of subtraction. This was the case with a student, Peter who was at first unable to reconcile a difference expression in $|x–1| < \delta$ with a graph of a function showing a shaded vertical strip centered at $x = 1$ to represent all values of x within a given distance (δ) of 1. (David, 2018).

We view the magnitude interpretation (Parr, 2021) of a symbol (either a number or variable) or expression as the foundation of the connection between the graphical representation of distance and the algebraic operation of subtraction to determine the difference. A magnitude interpretation of a symbol recognizes that a symbol refers to both a position as well as a measure of a distance from 0 (Parr, 2021). To represent the "determining the difference" model of subtraction on a number line, one employs what Parr et al. (2021) refer to as a *composed* magnitude interpretation. For example, to understand conceptually why the difference expression $x_2 - x_1$ yields the distance between two positions, x_1 and x_2, in one-dimensional Cartesian space, one must first understand the positions x_1 and x_2 as distances from the origin themselves. Then, one can apply the operation of subtraction to find the difference between the length of x_2 and the length of x_1 to express the distance between position x_1 and x_2 as $x_2 - x_1$, as shown in Figure 4.

Figure 4

Composed Magnitude Interpretation of the Expression $x_2 - x_1$



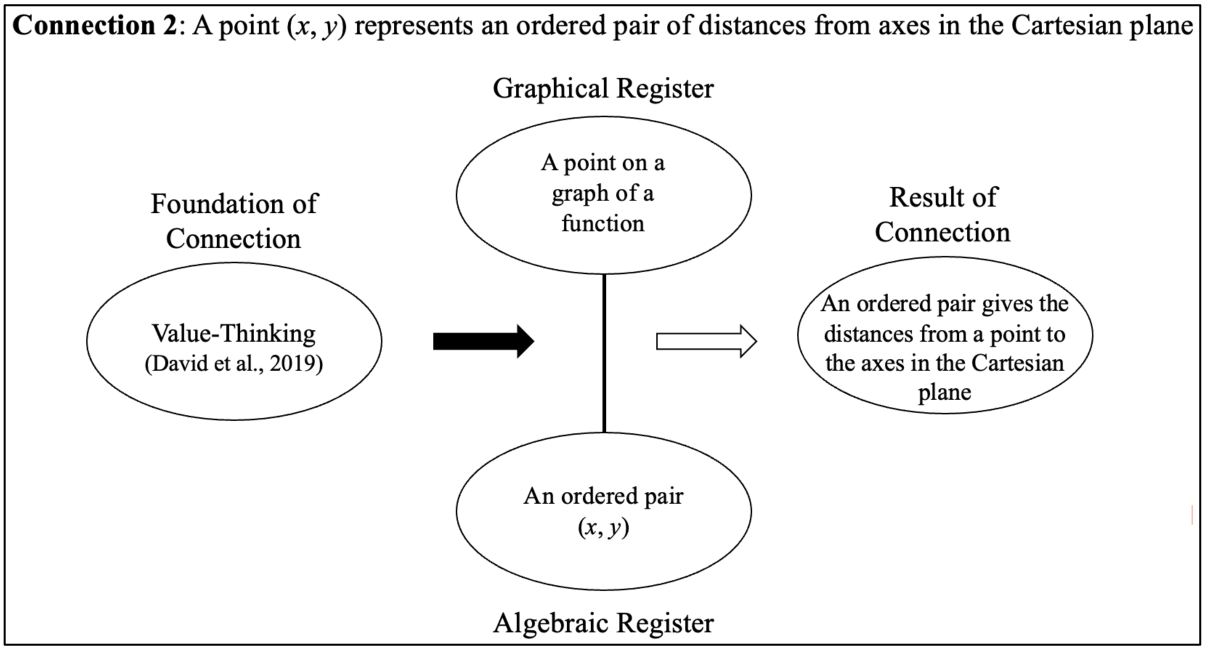
Students may not readily use a composed magnitude interpretation of difference expressions, even in situations where it would support further mathematical activity. At the undergraduate level, students may still use a cardinal interpretation, counting tick marks or spaces to interpret a difference expression on an axis, rather than as a distance between two points, as in the case of Annie and Kate reported in Parr (2021).

Connection 2: A Point (x, y) Represents an Ordered Pair of Distances from Axes in the Cartesian Plane

The next connection involved in expressing distances in the Cartesian plane is the connection that an ordered pair of values, (x, y), that locate a point in the Cartesian plane represents an ordered pair of distances measured from the point to each axis. This connection builds on the prior one, such that x is a magnitude from the origin horizontally and y is a magnitude from the origin vertically. What is new in this step is the combination of the two magnitudes into two-dimensional space, so that a single entity given by an ordered pair is connected to the pair of distances. The foundation of this connection is value-thinking (David et al., 2019), in which one envisions a point as a multiplicative object (Saldanha & Thompson, 1998); that is, a point is a single entity that is comprised of two components conceived of simultaneously. When combined with a magnitude interpretation from the previous connection, one envisions a point as a multiplicative object of a pair of distances to the axes (see Figure 5).

Figure 5

The Components of Connection 2: Connecting Points on Graphs with Ordered Pairs of Variables



Prior research suggests that students, both at the secondary and undergraduate level, may not connect a point on a graph of a function with an ordered pair of input and output values of the function via value-thinking. Students may instead associate a point solely with a single value-often the output of a function, rather than a pair of values, as in the case of location-thinking (David et al., 2019). When students do connect a point with an ordered pair, they may conceive of the ordered pair purely as a directive for how to locate a point (Thompson et al., 2017)- right (or left) some amount x and then up (or down) some other amount y. Students who view ordered pairs this way may not be able to unite pairs of magnitudes represented on axes in a single point, even after appropriately plotting points themselves using the over and up technique (Frank, 2016; Goldenberg, 1988). Thus, we view building the connection between ordered pairs and pairs of distances from axes as an essential part of supporting students in expressing distances between points on functions of graphs.

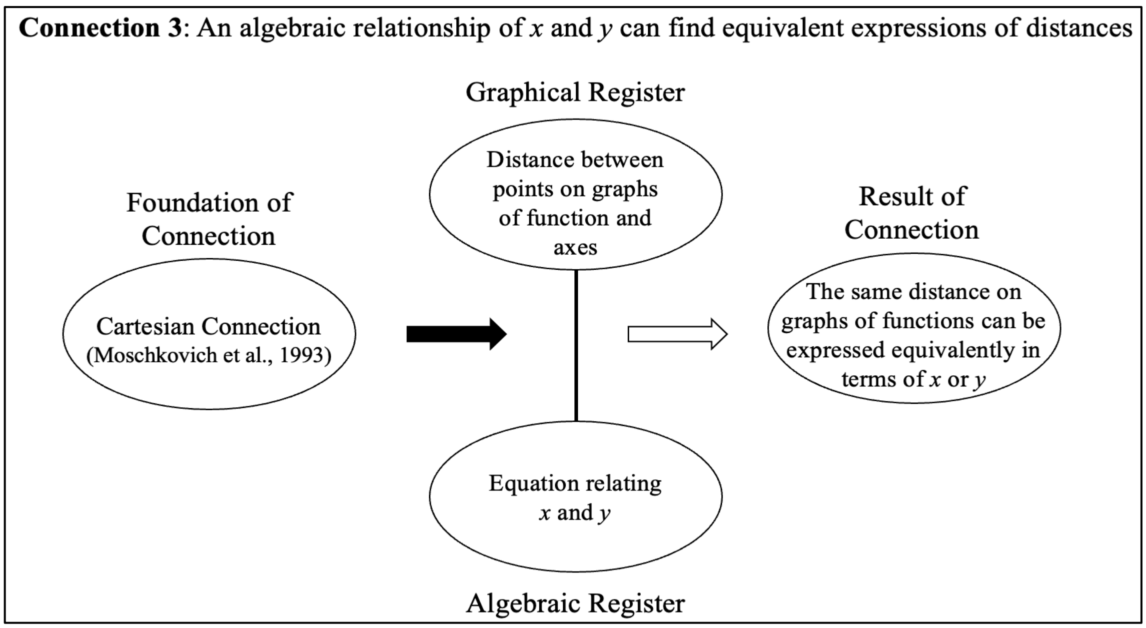
Connection 3: An Algebraic Relationship of x and y Can Be Used to Find Equivalent Expressions of Distances

The third connection between the algebraic and graphical register that we view as essential to expressing distances is the Cartesian connection (Moschkovich et al., 1993). The

Cartesian connection states that the set of all ordered pairs of points on a graph of an equation correspond to the complete set of pairs of values satisfying the equation. Using this connection with the prior two, in which ordered pairs for points give pairs of distances to the axes, allows one to conceive of the algebraic equation relating x and y as a way to express distances in the graph flexibly in terms of x or y as needed (see Figure 6).

Figure 6

The Components of Connection 3: Connecting Distances on Graphs with Equations of x and y



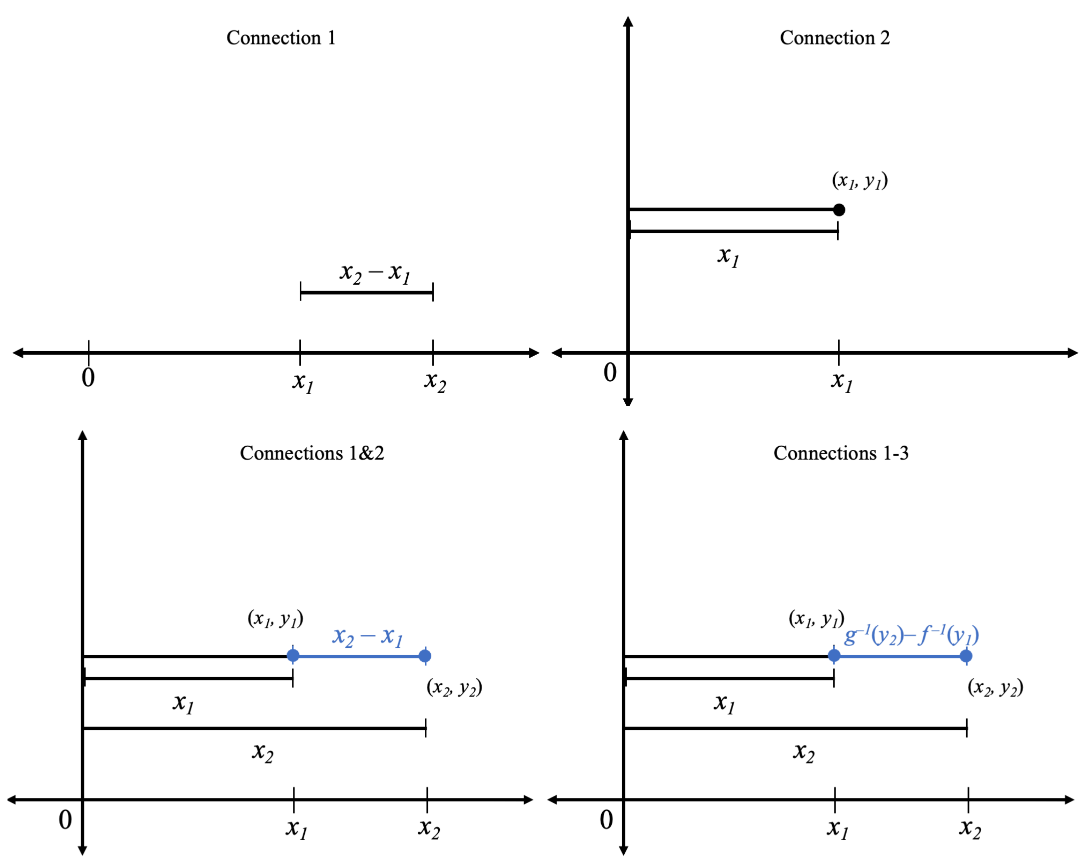
Previous studies have shown that students may not recognize the Cartesian connection when working with graphs of functions and their algebraic relationships (Dufour-Janvier et al., 1987; Glen & Zazkis, 2021; Knuth, 2000; Moon, et al., 2013; Moon, 2020). Instead of viewing an equation for a linear function as comprising the set of all ordered pairs of values satisfying the equation, which can be plotted as points comprising the graph of the line, students may see a linear equation as a directive for how to draw a graph of a line using slope and intercept information (Parr et al., 2021). In the words of Todd, a Calculus student, "$y = 2x+1$… tells you how it's [the graph is] going to look, so the slope would be 2, the y-intercept would be 1" (Parr et al., 2021, p. 219).

Representing distances on two-dimensional graphs of functions using algebraic expressions is a complex cognitive activity. In order to conceptualize why expressions such as $2–x$ and $2–(y^2+1)$ in Figure 1 represent the distance depicted, students must connect the graphical and algebraic register in the three ways described in this section: 1) a distance between can be modeled using a difference, 2) an ordered pair of values gives the distances the associated point is located from the axes in the Cartesian plane, and 3) an algebraic relationship between x and y can be used to flexibly express distances within the graph of the relationship in terms of x or y.

In Figure 7, we show how these connections combine to connect difference expressions with distances between two points on functions in the Cartesian plane. Connection 1 supports students in expressing a straight-line distance between two positions x_1 and x_2 as $x_2–x_1$. Connection 2 supports students in expressing a straight-line distance from an axis in the Cartesian plane using the coordinates of the point, in this case, the horizontal distances as x_1. Combining these two connections supports students in describing the horizontal distance between two points in the Cartesian plane as $x_2–x_1$. Assuming the two points shown are located on functions f and g, respectively, Connection 3, the Cartesian connection, would support a student in expressing this same horizontal distance in terms of y, as $g^{-1}(y_2) – f^{-1}(y_1)$.

Figure 7

The Combination of Connections 1–3 to Express Horizontal Distance Between Two Points in the Cartesian Plane



Based on prior research, we anticipate potential issues in students' reasoning to make these connections in each of the three steps. These issues may arise in either conceptualizing the objects in the graphical register, the objects in the algebraic register, or the way in which these objects are connected. The tasks that follow are designed to support each of these connections, which we describe in their associated learning goals. When we implemented these tasks in a Calculus class, we found that the tasks showed some moderate success in supporting students in expressing distances algebraically (see Table 2). Our results also indicated that students faced some persistent obstacles in coming to make these connections, which underscore the complex nature of the cognitive steps involved in these connections.

Description of Tasks & Learning Goals

In this section, we describe a series of tasks, which we refer to as the *Interpreting Graphs for Calculus Activity*, and their associated learning goals that comprise a hypothetical learning trajectory (Simon & Tzur, 2004) for algebraically expressing distances within graphs of functions. The first author developed a first iteration of these tasks in collaboration with other researchers based on experience in previous research and practice, as briefly described in Parr et al. (2021). The tasks shared in this article are a second iteration with minor revisions made for clarity as well as an additional task added to the beginning of the activity. The tasks that comprise this activity correspond with the three main connections described earlier: First, Tasks 1–4 are intended to support students with Connection 1 by evoking students' conception of distance within a number line (either a horizontal or vertical axis) and supporting them in using a difference expression to represent a distance between two points on an axis. Second, Task 5 is designed to support students with Connection 2, in conceiving of a point in the two-dimensional Cartesian plane as an ordered pair of distances from the axes. Third, Tasks 6–7 are intended to support students in making and using the combination of Connection 1 and 2 to express distances on points of functions to axes and to make Connection 3 to flexibly movie between expressing these distances in terms of x or y. Finally, Task 8 asks students to combine all three connections to express a horizontal distance between two relations in terms of x and in terms of y. We provide an overview of these tasks, the corresponding connections used in each, and associated learning goals in Table 1.

Table 1

Tasks and Learning Goals in Interpreting Graphs for Calculus Activity

Task		Learning Goals
1	Conn. 1	• Conceptualize distance as a measurable attribute of positions within a horizontal number line • Represent the distance between two horizontal positions using a difference expression
2	Conn. 1	• Represent a difference expression as a distance between two positions on a horizontal axis.
3	Conn. 1	• Represent a difference expression as a distance between two positions on a vertical axis.
4	Conn. 1	• Represent a distance between two points on a horizontal or vertical axis with a difference expression
5	Conn. 2	• Conceptualize a point in the two-dimensional Cartesian plane as an ordered pair of distances from the axes
6	Conn. 2 & Conn. 3	• Represent horizontal and vertical distances on points of functions to axes • Express these distances both in terms of x or in terms of y on the graph of a linear function
7	Conn. 2 & Conn. 3	• Represent horizontal and vertical distances on points of functions to axes • Express these distances both in terms of x or in terms of y on the graph of a non-linear relationship
8	Conn. 1-3	• Represent a horizontal distance between two relations in terms of x and in terms of y

In the remainder of this section, we describe the tasks and their intended learning goals and provide portions of the tasks as examples to the reader. For the full *Interpreting Graphs for Calculus Activity* as given to the students, see the Appendix.

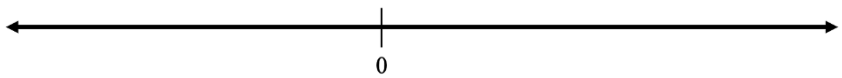
Task 1: Distances as Measurable Quantities & Distances Between as Differences

As part of supporting Connection 1, Task 1 is designed to engage students in conceptualizing distance as a measurable attribute of positions within a number line (one-dimensional Cartesian space), and to represent the distance between two positions using a difference expression. The task provides students with a number line that has 0 labeled at its center. Students are instructed to use rulers to measure and label an exact distance either to the left or to the right of 0. Then, students are asked to draw a segment between pairs of points they had previously marked and to measure the distance between these pairs of points. Through the experience of physically measuring distances, students are encouraged to conceptualize distance as a measurable attribute of the "space" between two numbers within a number line, which is the graphical register component of Connection 1. Further, this measuring and labeling activity encourages the use of a magnitude interpretation of these numbers on the number line, the foundation for Connection 1. The goal of visually comparing segments of total distances from zero and distances between positions is to promote the use of the difference model of subtraction (rather than a takeaway model) to represent the distance between two positions, the algebraic component of Connection 1. A portion of this task is shown in Figure 8.

Figure 8

Portion of Task 1 to Measure and Label Distances from 0 and Between Positions

1. Use the number line below and your ruler to answer the following questions:



a. On the number line above, mark and label a distance of 2 inches to the right of 0.
b. On the number line above, mark and label a distance of 0.5 inches to the right of 0.
c. On the number line above, mark and label a distance of 1 inch to the left of 0.
d. On the number line above, mark and label a distance of 1.2 inches to the right of 0.
e. Draw a segment to show the distance between 2 and 0.5. Measure this distance. What is it? Label this segment.

Tasks 2 & 3: Differences Give Distances between Two Points on an Axis

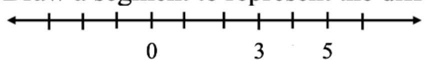
Continuing to support Connection 1, the goal of Task 2 is to orient students to the idea of representing difference expressions of the form $b–a$ as a distance from a to b on a number line (i.e., using the difference model of subtraction). Task 2 provides students with horizontal number lines (a single-dimensional Cartesian system of representation) labeled with 0 and two other values, which are either numbers or the variable x. Students are given a difference expression involving the two values and asked to draw a segment on the number line to represent the expression. After students draw a segment for the first difference, they are asked "what is the length of the segment you drew?" The purpose in asking this question is two-fold: (1) for students to reflect on the quantity of distance as the quantity used to represent an expression and (2) to orient students to representing a difference expression of the form $b–a$ as a distance from a to b, rather than using a takeaway model to represent the result of $b–a$ as a singular point on the number line.

The five differences presented to students are sequenced to build from natural numbers to integers, and finally to involve the variable x. The last difference of this task, which asks students to find distance between the variable x and an integer, is followed with a prompt asking students to draw another segment representing the difference between x and the same integer. This is done so that students are encouraged to think of variables as representing varying rather than fixed values. Again, the goal is to build toward generalizing the idea that the expression $b–a$ can be represented in a Cartesian system as a one-dimensional distance between any value a and any other value b, the outcome of Connection 1. Task 3 is similar to Task 2, but instead asks students to represent differences on vertical number lines, referred to as y-axes. A portion of Task 2 is shown in Figure 9.

Figure 9

Portion of Task 2 to Draw Segments whose Lengths Represent Given Differences

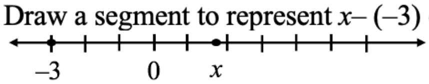
2. Draw the segments described using the x-axes provided.
   a. Draw a segment with a length of 5 on the x-axis below, beginning at 0.
   b. Draw a segment with a length of 3 on the x-axis below, beginning at 0.
   c. Draw a segment to represent the difference 5 – 3 on the x-axis below and label it "5 – 3."



   d. What is the length of the segment you drew in part c.?

   …

   h. Draw a segment to represent x– (–3) on the x-axis below.



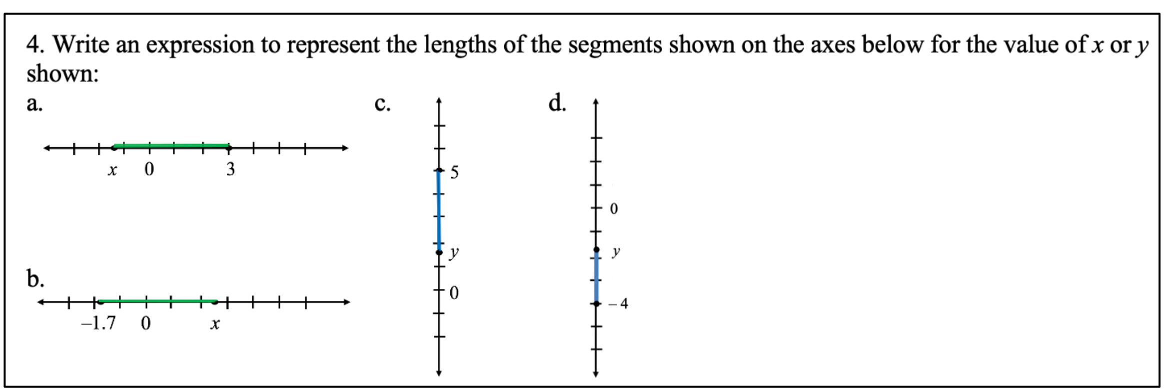
   i. Place another value of x on the number line above and draw another segment representing x– (–3).

Task 4: Distances Can Be Represented by Differences (variables used)

Task 4 is the final task provided for students to practice making Connection 1 on its own. In Task 4, students are asked to reverse the process of Tasks 2–3 by representing distances between positions with differences. Task 4 provides students with segments highlighted on horizontal and vertical number lines, with values labeled at the end points. For each segment, one value is numerical and the other is a variable. The goal of this task is for students to write difference expressions to represent the lengths of the segments shown. This will prepare them for moving to the two-dimensional Cartesian coordinate system in the subsequent tasks. Task 4 is shown in Figure 10.

Figure 10

Task 4 to Express Given Segment Lengths between Two Positions using Differences



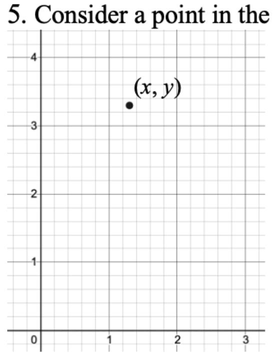
Task 5: A Coordinate Pair in the Cartesian Plane Gives Two Distances

The goal of Task 5 is to support Connection 2 by focusing students' attention to the two distances that a point in the Cartesian plane, as represented by an ordered pair, provides: the horizontal distance between the point and the y-axis and the vertical distance between the point and the x-axis. Task 5 shows a point plotted and labeled generically as "(x, y)" in the first quadrant of a Cartesian plane. This task asks students to represent and label the horizontal distance between the point and the y-axis and the vertical distance between the point and the x-axis. Task 5 is shown in Figure 11.

Figure 11

Task 5 to Express Distance from a Point in the Cartesian Plane to the Axes

5. Consider a point in the coordinate plane (x, y).



a. Use a segment to show the distance between the point and the x-axis.
b. Label this distance that you represented in part a. with a variable given.
c. Use a segment to show the distance between the point and the y-axis.
d. Label this distance that you represented in part c. with a variable given.

Task 6 & 7: Distances from Points on Graphs of Functions to Axes in Terms of *x* and in Terms of *y*

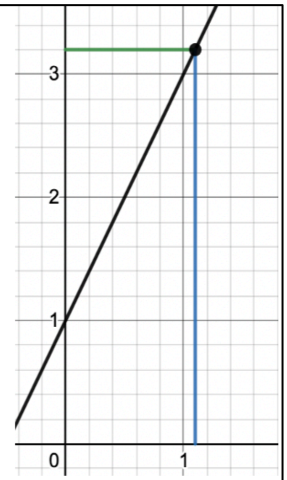
The purpose of Task 6 is to further support students in using Connection 2 and to introduce Connection 3, which allows students to give the same distance in terms of either x or y using the algebraic relationship of the function depicted in the graph. Task 6 provides a linear function and its graph and asks students to represent lengths of segments between a linear function and each of the axes in terms of both x and y. Further, 6c and 6f ask students to check their expressions to confirm that they represent the same distance at a fixed point, which is intended to support students in reflecting on Connection 3, the Cartesian connection. The goal of these questions is to bring to students' awareness the idea that two distinct expressions (one involving x and one involving y) can be used to represent the same distance within the graph.

Task 7 is similar to Task 6 but contains a graph of a function that is not linear. We provide Task 6 in Figure 12.

Figure 12

Task 6 to Represent Horiztonal and Vertical Distances from a Point on a Linear Function to the Axes

6. The graph of $y = 2x + 1$ is shown to the right. Let (x, y) represent a generic point on this function.
   a. Represent the length of the horizontal segment (green) in terms of x. (Your expression should have "x" in it).
   b. Represent the length of the horizontal segment (green) in terms of y. (Your expression should have "y" in it).
   c. Evaluate your expressions at the point where x equals 1, and confirm that you get the same result from a. and b.
   d. Represent the length of the vertical segment (blue) in terms of y.
   e. Represent the length of the vertical segment (blue) in terms of x.
   f. Evaluate your expressions at the point where y equals 3, and confirm that you get the same result from d. and e.



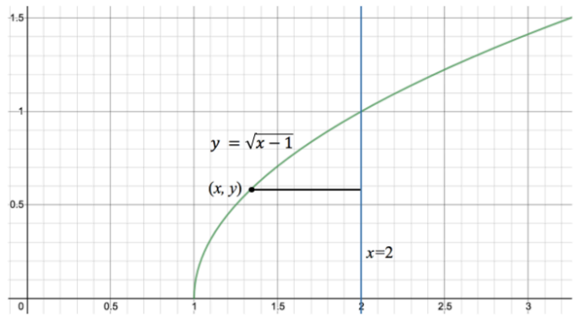
Task 8: Distances between Two Relations Represented as Differences in Terms of *x* and in Terms of *y*

Task 8 brings together Connections 1-3 together, combining the ideas that were promoted in the preceding tasks. The goal of Task 8 is to support students in expressing the distance between two relations, similar to those needed to define an integral to find the area between two curves or the volume of a solid of revolution. Task 8 asks students to represent a horizontal segment's length between two relations. Students will use the idea of expressing distance between two points as a difference (Connection 1) that they developed and practiced in Tasks 1–4. Parts 8a and 8b support students in recalling the magnitude interpretation which will allow them to conceive of distance between using a difference. Then, students will use the notion of representing distances with variables related to the coordinates of an ordered pair from Tasks 5–7 (Connections 2 & 3). We also note that Task 8c-f served as a pre-test item for our data collection. Figure 13 shows Task 8.

Figure 13

Task 8 to Express a Distance between Two Relations in Terms of x and in Terms of y

8.



a. Use a segment to show the distance between the point (x, y) and the y-axis. Label this segment.
b. Use a segment to show the distance between the line $x = 2$ and the y-axis. Label this segment.
c. Represent the horizontal segment's length in terms of x (Your expression should have "x" in it).
d. In a sentence or two, explain how you came up with your expression.
e. Represent the horizontal segment's length in terms of y (Your expression should have "y" in it).
f. In a sentence or two, explain how you came up with your expression.

Implementation of Tasks

This activity was first piloted with a small group of students in a Calculus course at a public university in 2020 (Parr et al., 2021). The current version of the tasks was implemented in 2022 at a small, private college in two sections of a Calculus course taught by the first author. Students first completed a pre-test item (Task 8c-f, see Figure 13) individually the class period before the activity was administered. The following class period, students worked through the activity in groups of two or three, with the instructor-researcher circulating and offering clarification as needed. Students recorded video and audio of their conversations as they worked through the activity in class and submitted their individual work on the activity. Additionally, follow-up items related to the tasks were given on the subsequent midterm exam (Exam 3) and final exam in the course. In the next section, we will share some of the initial results of implementing the activity.

Results

We found evidence that the activity, in some instances, seemed to support students in their ability to express distances in graphs of functions. Table 2 compares the number of correct responses on the pre-test (completed individually) and the number of correct responses at the end of the activity (completed in groups of 2-3).

Table 2

Number of Correct Responses on the Pre-Test and Post-Test Item

Represent segment's length	*Pre-Test (individual)*	*End of Activity (group)*
in terms of x: $2–x$	12/31 (38.7%)	24/31 (77.4%)
in terms of y: $2–(y^2+1)$	10/31 (32.3%)	20/31 (64.5%)
Both x and y correct	7/31 (22.5%)	20/31 (64.5%)

We first find that the majority of the 31 students in this Calculus course were not able to correctly express this distance in the graph algebraically. Of the 31, 12 correctly expressed the distance in terms of x as $2–x$, and only seven of these students also correctly expressed this distance in terms of y. These findings alone suggest that students need support with building this connection. Comparing the number of correct responses on the pre-test item and the same item at the end of the activity (Task 8c & 8e) indicates some improvement in students' ability to appropriately express distances algebraically. We note that students completed the activity in groups of two to three, so that these results on the final task do not necessarily reflect individual student's responses. However, some obstacles still appeared to persist for students in their process of learning to express these distances, especially related to Connection 1. When we looked more closely at the conversations students were having in groups, these obstacles became more evident. Even when students could provide correct responses to the tasks, the reasoning to support these responses did not always include the rich connections we were targeting, such as a magnitude interpretation of symbols and expressions as a foundation for Connection 1. Here, we describe two broad themes in the results of implementing the activity with two groups of Calculus students especially related to Connection 1: 1) issues within the algebraic register – ambiguity around symbols intended to represent *varying* quantities and 2) issues within the graphical register- a lack of necessity to conceptualize distances. We frame these issues as shortcomings of the tasks in their current design, as well as potential sources of challenge in the teaching and learning of expressing distances on graphs.

In the remainder of this section, we draw on data from two pairs of students, Pair 1: Kevin & Amir and Pair 2: Tom & Alan. We first provide their pre-test responses and responses to the end of the activity in Table 3. We chose to look at these pairs because all four students initially provided incorrect responses to the pre-test items and ended the activity with responses we considered appropriate. For these two pairs of students, then, the activity appeared to support their ability to express distances with difference expressions. However, for both pairs, we found certain aspects of their reasoning did not reflect the ways of reasoning we were intending to promote in the tasks, especially related to Connection 1.

Table 3

Two Pairs of Students Pre-Test and End of Activity responses to Task 8c and 8d

	Pre-Test Response (Individual)	*End of Activity Response (Pair)*
Amir	$2 - \sqrt{x-1}$ I identified the end points of the segment and subtracted their respective *x*-coordinates.	$z = 2 - x$ We defined the endpoints of the horizontal segment and subtract the values.
Kevin	$L = 2 - \sqrt{x-1}$ The height of the segments is just *f*(*x*) at the *x* location of (*x*, *y*). That is the starting point of the horizontal segment & the final point is *x*=2. Evaluating the distance between these points gives you the length.	$z = 2 - x$ We defined the endpoints of the segment as a value and then subtracted the values.
Tom	$x = 1.65$ The length of the segment starts at 2 and goes to the 1.65 mark. 2 is 0 in this case so it is 1.65 units long.	$b - x$ *b* is the total distance between *x*=2 and the *y*-axis. The horizontal segment is the difference between *b* and the distance from (*x*, *y*) to *y*-axis.
Alan	$= 2 - \sqrt{x-1}$ Not sure about the reason.	$b - x$ *b* is the total distance between *x* =2 and the *y*-axis. The horizontal segment is the difference between *x*,*y* to the *y*-axis.

We note that Tom and Alan represented the horizontal distance at the end of the activity as *b*–*x*. They had labeled the segment of length 2, from *x*=2 to the y-axis as *b*, because the item letter that asked them to label the segment was item b within Task 8. We coded this as an equivalent response to 2–*x* since they demonstrated the appropriate reasoning and were perhaps misguided by the directions to label the segment.

Ambiguity of Symbols in the Algebraic Register

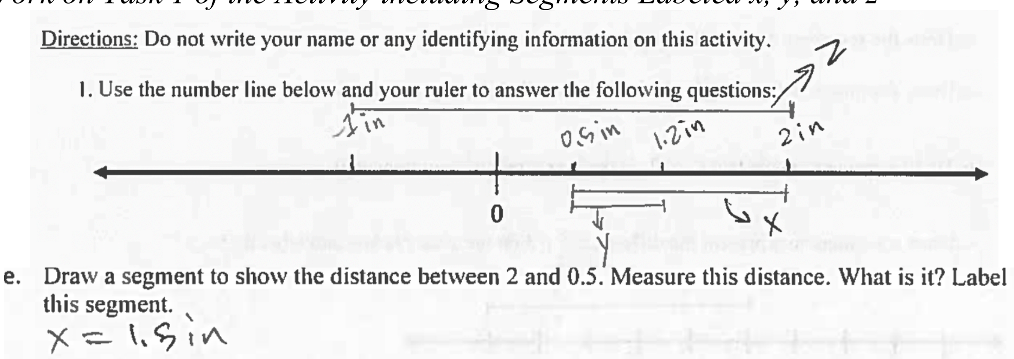
We found that the way in which students interpreted and used symbols in the algebraic register emerged as a salient feature of their reasoning related to Connection 1. At times, students' interpretations of symbols as indicating fixed, rather than varying values, contributed to issues with connecting difference expressions with distances. We describe two such instances and discuss how the ambiguous nature of the symbols used in the tasks may have contributed to issues in making Connection 1.

Kevin & Amir – Interpreted Symbols as Labels and Parameters

Both sets of responses from Kevin and Amir indicate that they tended to use symbols (such as *x*, *y*, and *z*) as either labels or parameters, rather than as variables to represent varying values. We found evidence of this tendency beginning with Amir's work on Task 1 in which he labeled segments with symbols *x*, *y*, and *z* that he introduced himself.

Figure 14

Amir's Work on Task 1 of the Activity including Segments Labeled x, y, and z



On Task 1, Amir measured and drew the indicated segments on the given number line. To label these segments, Amir introduced three symbols, *x*, *y*, and *z*. He wrote that "$x = 1.5$ in" and then labeled the corresponding segment *x*, and continued in this way with *y* and *z* (see Figure 13). Amir's use of the symbols *x*, *y*, and *z* which he set equal to the lengths he measured, and to label the segments, first indicated to us that he may be treating these symbols as labels to refer to distinct segment lengths.

We found other evidence of Amir's tendency to use symbols as labels for fixed segments or values when it created some hesitation for him on Task 2i. The discussion between Kevin and Amir around this task reveals some limitations of Amir's interpretation and use of *x*.

Amir: (*reading 2i*) Oh, so like place a value for *x* and just draw it, *x* equals 2…

Kevin: What is it asking, again?

Amir: Place another value of *x*, like 1, 2, 0, why did I put 0?... It doesn't make sense to place another value of *x*. I mean even if you put *x* over here, so x_0 equals x_1, it's the same thing.

Kevin: Ah (*leaning back*)

Amir: I mean, I like the zero thing.

Kevin: So, wait, what are you doing? (*looking at Amir's work*)

Amir: I'm saying *x* equals 0, I'm setting

Kevin: Okay

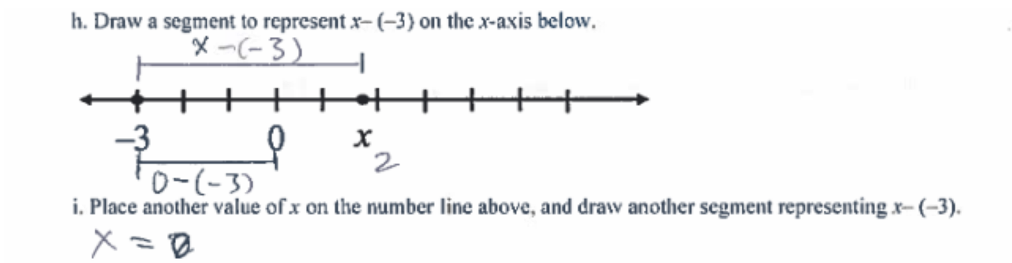
Amir: the segment from 0

Kevin: Yeah

Amir: I don't know.

Figure 15

Amir's Work on Task 2h and 2i



When Amir read the directions to 2i to "place a value," he remarked that "*x* equals 2" and labeled 2 near the *x* labeled on the number line. We note that in the task, *x* was placed between 1 and 2, closer to 2, but purposely not exactly on the tick mark at 2. When Kevin asked Amir what the task is asking, Amir gave some examples of other values of *x*, like 1, 2, 0, but then added that "it doesn't make sense to place another value of *x*." He then distinguished other values of *x* saying "x_0" and "x_1" but claimed, "it's the same thing." Amir decided to set *x* equal to 0, and wrote "*x*=0" over the "*x*=2" he had previously written below 2i. Amir labeled "the segment from 0" with the expressions, "0–(–3)" but indicated his uncertainty with "I don't know" before moving on to the next task.

We infer from these comments that Amir may have interpreted *x* as a parameter, in which *x* may take only take on different values in different contexts, rather than as a variable taking on varying values in the same context (Thompson & Carlson, 2017). Amir appeared comfortable using *x* to label a different distance in Task 1. Within the context of Task 2h and 2i, however, Amir did not think it made "sense" to have *x* take on another value. Amir's interpretation of *x* as a parameter is also supported by his written work on this task, in which he sets *x* to 0, but then he labels the new segment as "0–(–3)" rather than "*x*–(–3)" as directed. We take his choice to use 0 rather than label another *x* on the number line as evidence that Amir was not comfortable with using *x* to take on two different values within the same number line.

Amir's claim that "it doesn't make sense to place another value of *x*" is a valid deduction in the task as designed. In Task 2, *x* is introduced without a domain and in the number line for Task 2h and 2i, *x* is placed statically at a single value on the number line. Further, the tasks up to this point do not indicate that *x* is intended as a variable. We take Amir's hesitancy to label a different segment length with the same expression *x*–(–3) (which was our intended response to the task) to be in part a result of the ambiguity of the use of *x* in the task design. Amir's response highlights the nuance in the use of symbols as variables to represent varying positions within Connection 1, which was not an emphasis in our task design. When students interpret symbols and the positions and lengths they represent as fixed in contexts in which we intended for them to vary, it may contribute to issues with Connection 1. We present such a case next, from Tom and Alan.

Tom & Alan – Interpreted an Expression to Represent a Fixed Length

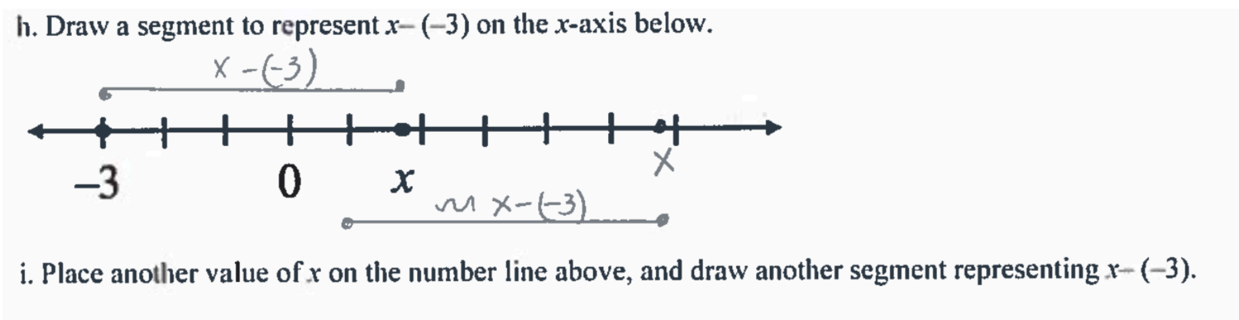
We found related reasoning from Tom and Alan on this task. We note that their reasoning led to what we would consider an incorrect response on this item. When working on Task 2i, Tom asks Alan a question about where to place *x* and Alan responds as follows:

Tom: "We can put our *x* anywhere?"

Alan: "Yeah, but we want it to be the same length as the other segment"

Figure 16

Tom's Work on Task 2h and 2i



Following Alan's response, Tom did not question Alan's statement that the new segment needed to have the "same length as the other segment." Instead, both drew another point on the number line, just to the left of the sixth tick mark to the right of 0, and drew a segment of the same length as the first segment they had labeled as "*x*– (–3)." They both labeled this segment with the same expression, and use a "~" symbol, which we infer that they intended that this new segment have approximately the same length as the previous one.

Tom and Alan's brief conversation and response to Task 2i indicate that they viewed the expression "*x*–(–3)" as a fixed amount of a length on a number line. Alan claimed that he wanted the new segment to "be the same length as the other segment" and both students drew segments and labeled them accordingly. Notably, their new segment did not include –3 at either endpoint. This did not appear to bother either student as they then proceeded to the next task.

Tom and Alan's response indicates an issue in making Connection 1 between the expression $x-(-3)$ and the distance between –3 and x, as the new segment they drew did not extend to –3. Although Tom and Alan did not appear to make Connection 1 in their response, the ambiguity of the use of symbols in the instructions again may have contributed to not making this connection. Tom and Alan may have interpreted the instructions to draw another segment representing $x-(-3)$ to refer to a similar segment to the previous segment drawn representing $x-(-3)$. They indicated that they were comfortable with x varying, and represented x in a new position, but they did not connect the distance between the new position of x and the position of –3 to the expression $x-(-3)$.

From the examples above from both Kevin and Amir and from Alan and Tom, we conjecture that interpreting and conceptualizing symbols as taking on varying values, and how these varying values are represented in the graphical register to be a significant component to Connection 1. Had Kevin and Amir been instructed to consider x as a variable and been shown dynamic imagery, they may have been comfortable with labeling x at a new position on the horizontal axis. In the case of Tom and Alan, had they been instructed that $x-(-3)$ could represent varying distances, they may have had the opportunity to confront Connection 1 more directly in the task.

Motivating a Magnitude Interpretation in the Graphical Register

Related to Connection 1, we also noticed a theme emerge of students using a pattern or formula they found to work in order to express distances using difference expressions. In using such a formula, they circumvented using a composed magnitude interpretation, in which the distance between two positions they were expressing can be thought of as composed of two other distances. The students who correctly completed Task 8 after the activity may have used a magnitude interpretation to connect difference expressions with distances between positions on graphs of functions. However, we found evidence that students may have correctly connected expressions with distances without this foundation of a magnitude interpretation, instead using a pattern or formula for their expressions. We found the formulas or patterns that students used to have one of two common bases: a numerical comparison or a spatial comparison. We describe two such instances of "formulas" and evidence that indicated that the students were using interpretations other than a composed magnitude interpretation.

Tom & Alan – Larger Minus Smaller (Numeric)

Although Tom and Alan gave correct responses to the final tasks of the activity, we found evidence that they may have followed a pattern in order to express the indicated distances. Tom and Alan briefly described their solutions to task 4d. Tom asked Alan about his expression in the following exchange:

Tom: I had 4 – (–*y*), should I change it?

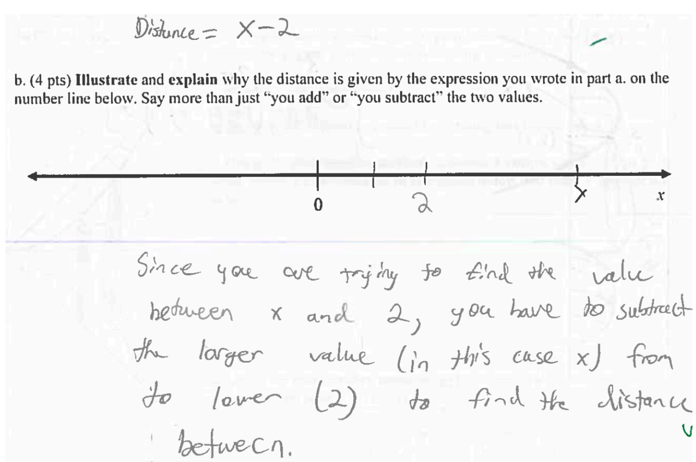
Alan: I don't know this always has the bigger number, or the number closer… first (*Alan had this segment labeled y–(–4)*)

Tom: I believe you (*changes segment label to y–(–4)*)

When Tom asked Alan if he should change his expression, Alan appealed to a pattern of "always… bigger number… first." Tom accepted this justification based on a pattern, rather than one rooted in how to measure distances on the number line. Further, he verbalized his satisfaction with Alan's reasoning when he responded "I believe you." Interestingly, this pattern of larger number minus smaller number persisted in Tom's reasoning later in the semester. On the item on the third exam in the course, Tom writes that "*x*–2" gives the distance between *x* and 2 (see Figure 12). However, when asked to explain why the distance is given by the expression, Tom writes "you have to subtract the larger value (*x*) from [the] lower (2) to find the distance between." For Tom, this response was sufficient to justify why the expression represented the distance between *x* and 2, as he did not describe a composed magnitude interpretation. Regardless of whether Tom is able to conceptualize distances in the graphs, he appears to be able to successfully use this pattern to correctly represent distances.

Figure 17

Tom's Explanation of x–2 as a Distance on Exam 3



Kevin & Amir – Higher Minus Lower (Spatial)

Although Kevin and Amir gave correct responses to the final tasks of the activity, we found evidence that Kevin and Amir were also following a pattern in order to express the indicated distances. Their reliance on this pattern, rather than a magnitude interpretation, became most apparent when Amir and Kevin were working through task 4c and 4d together and Kevin proposed an alternate expression to describe the length. Their conversation was as follows:

Amir: See directions matter. So now like, this above, 5 minus *y* and this is *y* minus

Kevin: Yeah. (*pause*) I guess you could say that it's also *y* plus 5.

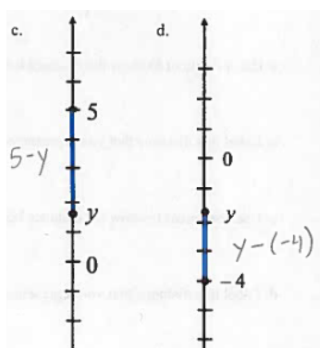
Amir: *y* plus 5 (*slowly, thinking*). Well, yeah, you can say that, too. But because it's differences. Here it doesn't say differences (*the instructions for Task 4*), here it does (*the instructions for task 3 on the same page of the activity*).

Kevin: I'm just going to use minus for consistency.

Amir: Yeah

Figure 18

Kevin's Work on Task 4c and 4d



We infer from Amir's comment that "directions matter" that he was referring to the positions on the number line corresponding with the order of the terms in the expression. Following this comment, he contrasts the two examples where 5 is "above" and where y is, and notes the difference in the order of the terms. At first, it is unclear whether Amir was conceptualizing distances when thinking of the position of 5 and *y* on the vertical number lines. However, when Kevin proposed that the length could also be described with the expression "*y* plus 5," and Amir accepted it, it becomes clearer that the order of these expressions was a pattern to follow, rather than rooted in a meaning for distance for Amir. Had Amir been thinking of distances, he would consider the length of the segment *y* from 0 to *y*, 5 to give the length from 0 to 5, and rejected y+5 to describe the length of the given segment. Amir's acceptance of Kevin's proposal indicates that Amir was following a pattern without a conceptual basis in a magnitude interpretation. Both Amir and Kevin displayed evidence that they followed a pattern when they discussed whether to use "differences" by referring to the directions and in their ultimate decision to use a "minus for consistency."

From the examples above, we conjecture that conceptualizing distances as quantities in graphs is a significant step in learning to describe distances in Connection 1. Further, the tasks did not necessitate that students conceive of distances throughout the activity. In fact, the tasks may have supported students in learning to follow a pattern to correctly represent the distances. We discuss this issue as another limitation of the tasks and area for future improvement and research.

Discussion

In this article, we describe a conceptual analysis (Thompson, 2008) of the connection between the algebraic and graphical registers in order to express distances within graphs of functions. We also describe a series of tasks and associated learning goals comprising a hypothetical learning trajectory (Simon & Tzur, 2004) based on these components as an attempt to support students in developing this skill. We illustrated through the implementation of these tasks some outstanding obstacles in the process of building the connections necessary to express distances with strong conceptual underpinnings, namely the conception of variable and the motivation of conceptualizing distance in a graphical representation. Our findings underscore the need for developing robust conceptions of these fundamental mathematical objects within each register as a prerequisite for meaningful connections among these representations.

The conceptual analysis we present offers a detailed account of the three cognitive connections we theorize that students need to meaningfully use an algebraic difference expression to represent distances between functions in graphs. These connections are: 1) differences express distances between positions, which relies on a magnitude interpretation 2) the coordinates of a point in the Cartesian plane are an ordered pair of distances to the axes, which relies on value-thinking and 3) algebraic relationships of x and y can find equivalent distance expressions in terms of either x or y, which uses the Cartesian connection. Making each of these connections includes conceptualizing objects within each respective register, as well as using the interpretation necessary to connect the relevant components. The three connections we detail between the algebraic and graphical register uncover the complex nature of the cognitive steps involved in expressing distances algebraically.

More broadly, the structure of the conceptual analysis we offer here may serve as a model for others related to connections among registers. Many prior examples of conceptual analysis in the literature attend to students' understanding of mathematical ideas, but may not explicitly attend to representational structures involved in conceiving of or communicating these ideas. We add to the example of Lee et al.'s (2018) conceptual analysis of coordinate systems by attending to the representational structures involved in connecting graphical distances with algebraic expressions. We decompose the larger connection in question into sub-connections, which each include components in the algebraic register, graphical register, and an underlying interpretation. This structure may serve as a model for future conceptual analyses on other connections between the graphical and algebraic registers, or any other combinations among mathematical registers.

The tasks and learning goals that we present, which are rooted in our conceptual analysis, are an example of a hypothetical learning trajectory to support students in connecting algebraic expressions and distances in graphs using the three connections we described. The first four tasks focus on connecting difference expressions with distances between two positions horizontally and vertically. The fifth task emphasizes the second connection, combining the use of value-thinking and a magnitude interpretation to interpret the coordinates of a point as an ordered pair of distances from the axes. The final three tasks add in the Cartesian connection to use an algebraic relationship between x and y to describe the same distances in the graphs both in terms of x and in terms of y.

The initial results of implementing the tasks show some promise, in that many students who were previously unable to correctly represent distances algebraically were able to do so after working through the tasks. However, the results also revealed that students did not always show evidence of using the meanings we aimed to support, including a composed magnitude interpretation. These findings indicate that the tasks may be improved and also point to some persistent obstacles students may face in connecting graphical and algebraic representations. The two issues we observed were related to interpreting symbols as representing varying values within a graph and motivating a conception of distance within a graph. In our next stage of research, we plan to revise the tasks to better accommodate these issues.

In completing the tasks, some students indicated that they interpreted symbols such as x and y as representing fixed, rather than varying values. This may have in part been due to some ambiguity of the tasks. In the next iteration of the tasks, we plan to frame x and y as variables more clearly, to support students in interpreting x and y as taking on varying values within a context. We will do this in the directions when introducing these symbols. The ambiguity in representing a varying or "generic" value x by placing it at one position on an axis is also a limitation of static representations on paper. Accordingly, we plan to provide some tasks in an interactive graphical software with sliders for students to be able to vary the value of a variable themselves and see the corresponding adjustments in the graph. The use of a slider may also support students in conceiving of varying distances, and that the same algebraic expression may represent varying distances between two relations. In general, the divergence in the student responses around symbols from what we anticipated points to a broader issue of implied meaning for variables which may be an obstacle to making Connection 1 for students.

Additionally, we recognize that the tendency for students to use a "formula" to express distances, rather than conceptualizing distances within the graphs, may have been a result of the task design. In fact, the tasks may have supported students in developing these formulas to express distances, and reinforced the use of formulas, rather than reflecting on a magnitude interpretation of symbols, as students progressed through the tasks. To support students in conceptualizing distances rather than relying on a pattern, we may consider introducing tasks with a physical context that may be familiar to students and use distance units. Further, we may consider the types of reasoning the tasks promoted for the students who completed them, and reframe our task revision in terms of the intellectual needs that were (or were not) created for students (Harel, 2013; Weinberg et al., 2023). Again, conceptualizing distance (and necessitating it) may be a broader obstacle to consider in teaching students to make Connection 1.

Further research is needed to understand how a student develops the components for making the connections between the graphical and algebraic register specified here. For instance, teaching experiments may be designed and conducted to uncover how students develop value-thinking or a magnitude interpretation in order to connect a symbol to a component of a point in the graph or to a distance within the graph. The insights from such studies may offer clearer direction for subsequent task design and curricular materials.

We close with several recommendations for mathematics instructors and curriculum designers across mathematics. At the elementary level, we recommend that students are given many opportunities to use a *determine the difference* model for subtraction in various contexts, rather than solely focus on a *takeaway* model, and to use number lines as measurement models in addition to counting models. When the Cartesian plane is introduced, students should be given repeated opportunities to decompose points as magnitudes on the axes, and plotting points should be introduced as uniting two magnitudes on the axes, rather the over and up formula to locate a point that does not attend to distances. In algebra, students need to develop a robust meaning for variables as taking on the set of all varying values in a given domain, beyond an unknown, a label, or parameter. We also recommend the use of tasks that necessitate the use of the Cartesian connection early and often, so that students develop a conception of an equation beyond a cue for how to draw the graph of a function. Finally, in later grades, we recommend continuing to return to these ideas and a general attitude of not assuming fluency among representations. Attention to students' thinking, especially around conceptions of variable and distance, and use of tasks that evoke such thinking, are likely to promote strong connections among algebraic and graphical representations that will lay a solid foundation for students' future mathematical studies in Calculus and beyond.

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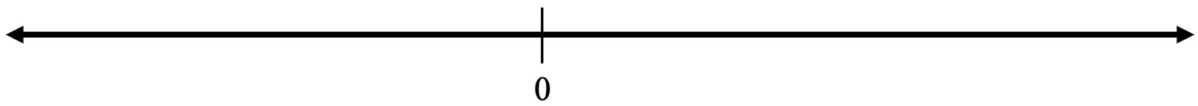
Appendix- Interpreting Graphs for Calculus Activity

1. Use the number line below and your ruler to answer the following questions:



a. On the number line above, mark and label a distance of 2 inches to the right of 0.

b. On the number line above, mark and label a distance of 0.5 inches to the right of 0.

c. On the number line above, mark and label a distance of 1 inch to the left of 0.

d. On the number line above, mark and label a distance of 1.2 inches to the right of 0.

e. Draw a segment to show the distance between 2 and 0.5. Measure this distance. What is it? Label this segment.

f. What expression (not a single number) can you write to represent this distance?

g. Draw a segment to show the distance between 1.2 and 0.5. Measure this distance. What is it? Label this segment.

h. What expression (not a single number) can you write to represent this distance?

i. Draw a segment to show the distance between 2 and 1 (to the left of 0). Measure this distance. What is it? Label this segment.

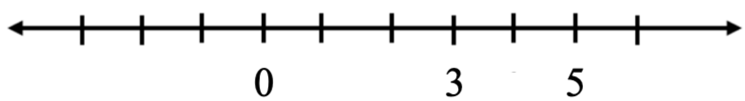
j. What expression (not a single number) can you write to represent this distance?

2. Draw the segments described using the *x*-axes provided.

a. Draw a segment with a length of 5 on the x-axis below, beginning at 0.

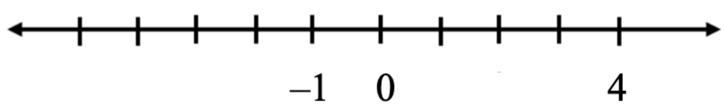
b. Draw a segment with a length of 3 on the x-axis below, beginning at 0.

c. Draw a segment to represent the difference 5 – 3 on the x-axis below and label it “5 – 3.”

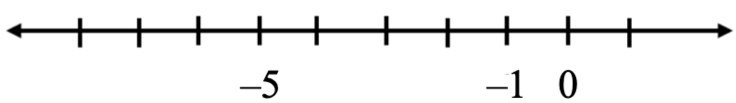


d. What is the length of the segment you drew in part c.?

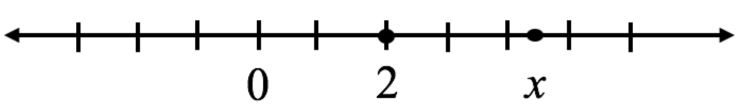
e. Draw a segment to represent 4 – (–1) on the x-axis below.



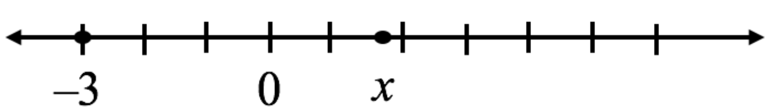
f. Draw a segment to represent –1 – (–5) on the x-axis below.



g. Draw a segment to represent x – 2 on the x-axis below.



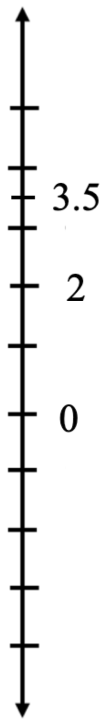
h. Draw a segment to represent x– (–3) on the x-axis below.



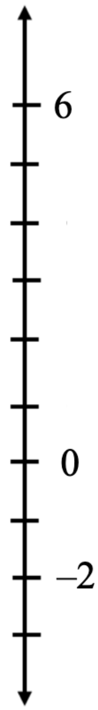
i. Place another value of x on the number line above, and draw another segment representing x– (–3).

3. Draw a segment to represent the differences on the y-axes below

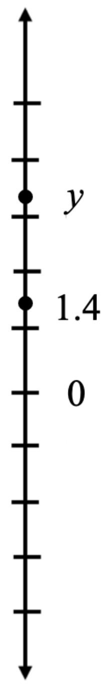
a. Represent 3.5 – 2:



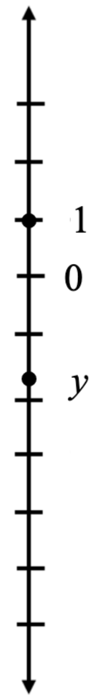
b. Represent 6– (–2):



c. Represent y –1.4:

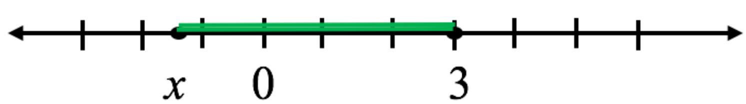


d. Represent 1 – y:

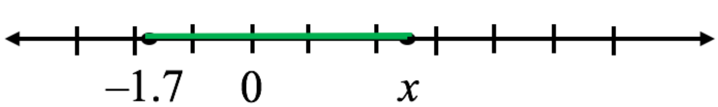


4. Write an expression to represent the lengths of the segments shown on the axes below for the value of x or y shown:

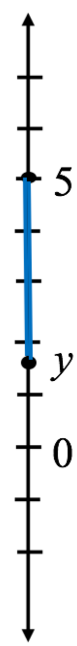
a.



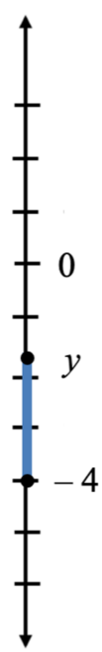
b.



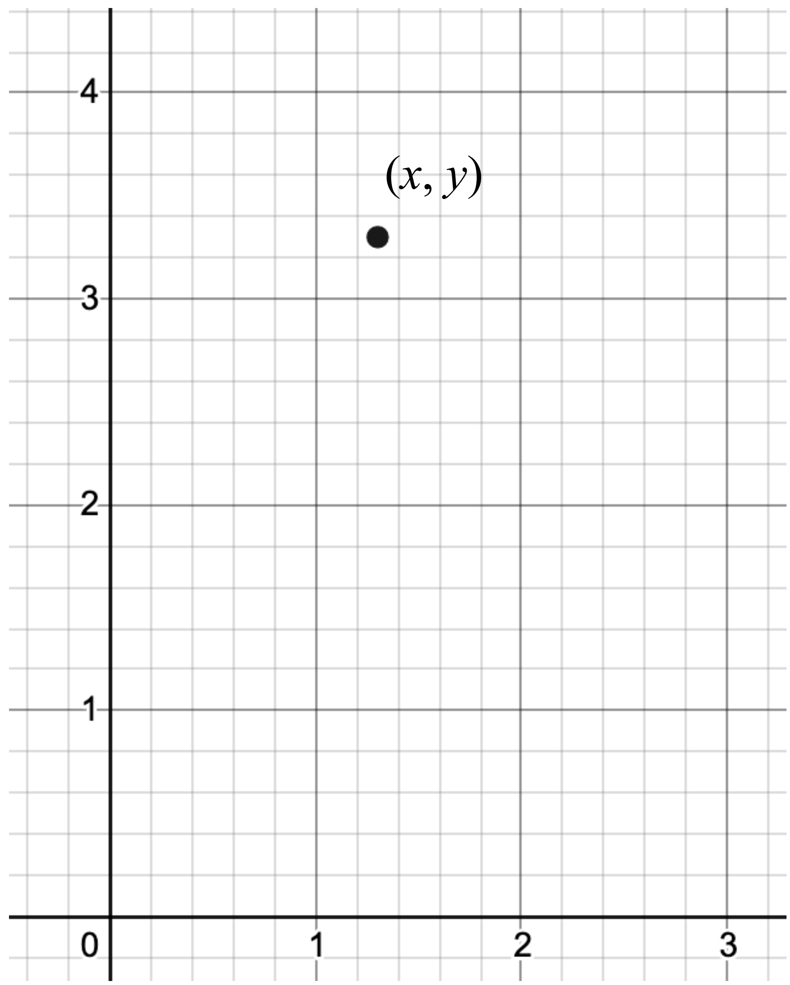
c.



d.



5. Consider a point in the coordinate plane (x, y).



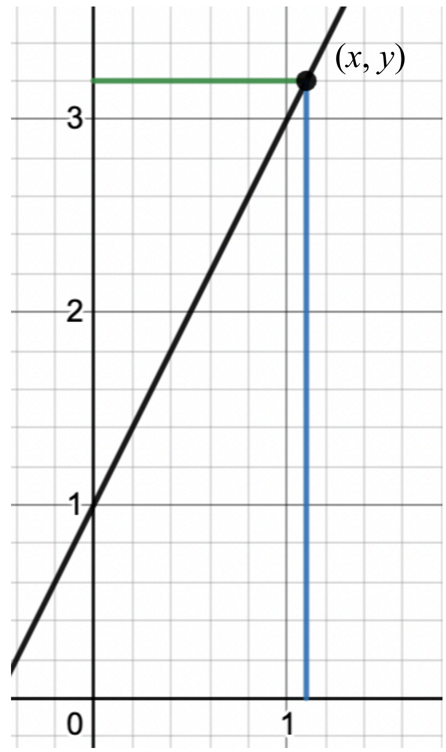
a. Use a segment to show the distance between the point and the x-axis.

b. Label this distance that you represented in part a. with a variable given.

c. Use a segment to show the distance between the point and the y-axis.

d. Label this distance that you represented in part c. with a variable given.

6. The graph of $y = 2x + 1$ is shown to the right. Let (x, y) represent a generic point on this function.



a. Represent the length of the horizontal segment (green) in terms of x. (Your expression should have "x" in it).

b. Represent the length of the horizontal segment (green) in terms of y. (Your expression should have "y" in it).

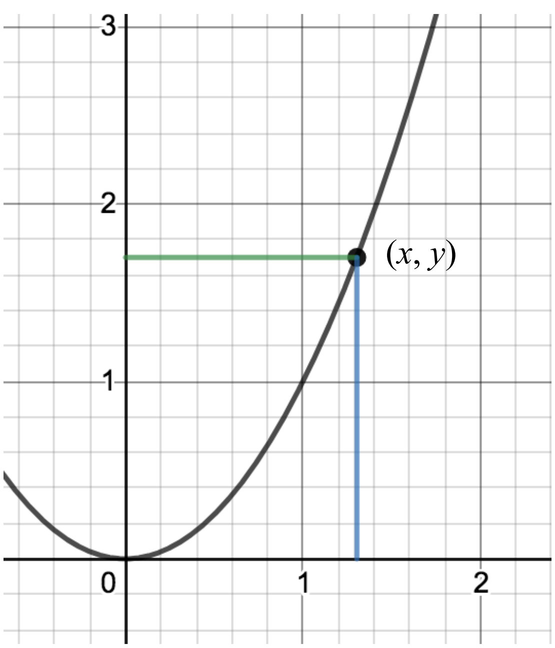
c. Evaluate your expressions at the point where x equals 1, and confirm that you get the same result from a. and b.

d. Represent the length of the vertical segment (blue) in terms of y.

e. Represent the length of the vertical segment (blue) in terms of x.

f. Evaluate your expressions at the point where y equals 3, and confirm that you get the same result from d. and e.

7. The graph of $y = x^2$ is shown below. Let (x, y) represent a generic point on this function.



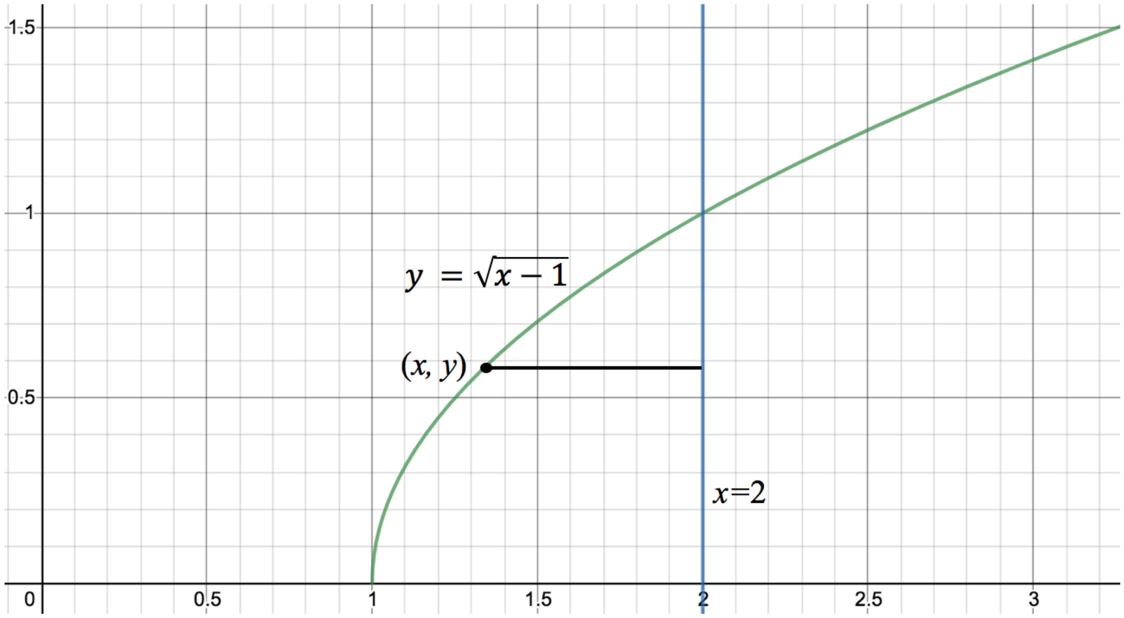
a. Represent the length of the horizontal segment (green) in terms of x.

b. Represent the length of the horizontal segment (green) in terms of y.

c. Represent the length of the vertical segment (blue) in terms of y.

d. Represent the length of the vertical segment (blue) in terms of x.

8.



a. Use a segment to show the distance between the point (x, y) and the y-axis. Label this segment.

b. Use a segment to show the distance between the line $x = 2$ and the y-axis. Label this segment.

c. Represent the horizontal segment's length in terms of x (Your expression should have "x" in it).

d. In a sentence or two, explain how you came up with your expression.

e. Represent the horizontal segment's length in terms of y (Your expression should have "y" in it).

f. In a sentence or two, explain how you came up with your expression.