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Nonexistence of solutions to fractional parabolic problem with general nonlinearities

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Abstract

In this content, we investigate a class of fractional parabolic equation with general nonlinearities

$$\frac{\partial z(x,t)}{\partial t} - (\Delta + \lambda)^{\frac{\beta}{2}} z(x,t) = a(x_1) f(z),$$

where a and f are nondecreasing functions. We first prove that the monotone increasing property of the positive solutions in x_1 direction. Based on this, nonexistence of the solutions are obtained by using a contradiction argument. We believe these new ideas we introduced will be applied to solve more fractional parabolic problems.

Keywords Fractional parabolic equation · General nonlinearity · Tempered fractional Laplacian · Monotonicity

Mathematics Subject Classification 35R11 · 35K91

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1 Introduction

As we all know, the nonexistence of solutions to indefinite elliptic and parabolic problems have been studied extensively. In [1–3], for the following elliptic problem with nonlinearities and the regular Laplacian

$$-\Delta u(x) = a(x_1)u^p(x),\ x \in \mathbb{R}^n,\ 1 < p < \infty.$$

the Liouville theorems were studied. In [4], Chen and Zhu applied the extension method [5] to transform the problem to a local one. They derived the nonexistence of positive solutions to the following equation:

$$(-\Delta)^s u(x) = x_1 u^p(x),$$

where $\frac{1}{2} < s < 1$ and $1 < p < \infty$. In [6], Poláčik and Quittner introduced indefinite parabolic problem with the regular Laplacian as follows,

$$\frac{\partial z(x,t)}{\partial t} - \Delta z(x,t) = a(x_1)z^p(x,t), \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

where $x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^n$, $t \in \mathbb{R}$, a is nondecreasing continuous function. They received the nonexistence of bounded positive solutions of the above equation. In [7], Chen, Wu and Wang studied the indefinite fractional parabolic equation

$$\frac{\partial u(x,t)}{\partial t} + (-\Delta)^s u(x,t) = x_1 u^p(x,t),\ (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

where $0 < s < 1$ and $1 < p < \infty$ and they obtained the monotone increasing and nonexistence of positive bounded solutions. More details can be seen in [8–11].

In 2023, with the aid of the direct method of moving planes, we [12] studied a tempered fractional Laplacian parabolic equation with logarithmic nonlinearity, asymptotic symmetry and monotonicity of radial solution of the parabolic equation were obtained. As a supplement and continuation of our above research results, in this work, we will study the nonexistence of solutions for a class of tempered fractional Laplacian parabolic problem with general nonlinearity, which will further enrich the theory of tempered fractional Laplacian parabolic problem.

To our knowledge, the nonexistence of solutions to parabolic equation with general nonlinearity is rarely studied. Here, we mainly focus on the following equation:

$$\frac{\partial z(x,t)}{\partial t} - (\Delta + \lambda)^{\frac{\beta}{2}} z(x,t) = a(x_1) f(z), \tag{1.1}$$

where a and f are nondecreasing functions and the tempered fractional Laplacian operator is defined as

$$(\Delta + \lambda)^{\frac{\beta}{2}} z(x,t) = -C_{n,\beta,\lambda} P.V. \int_{R^n} \frac{z(x,t) - z(y,t)}{e^{\lambda|x-y|}|x-y|^{n+\beta}} dy,$$

where $\beta \in (0,2)$, λ is a sufficient small positive constant and $C_{n,\beta,\lambda} = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}|\Gamma(-\beta)|}$. $P.V.$ presents the cauchy principle value and $\Gamma(t) = \int_0^\infty s^{t-1}e^{-s}ds$ is the Gamma function. Obviously, when $z \in C^{1,1}_{loc} \bigcap \mathcal{L}_\beta$, $(\Delta + \lambda)^{\frac{\beta}{2}} z(x,t)$ is well defined, where $\mathcal{L}_\beta = \{z(\cdot,t) \in L^1_{loc}(R^n) | \int_{R^n} \frac{|z(x,t)|}{1+|x|^{n+\beta}} dx < +\infty\}$.

The fractional Laplacian $\Delta^{\frac{2}{\beta}}$ is the generator of the β-stable Lévy process, in which the second and all higher order moments diverge. It sometimes is referred to as a shortcoming when applied to physical processes. So a parameter λ is introduced to temper the L$\acute{e}$vy process. Tempered L$\acute{e}$vy process is the scaling limit of the tempered L$\acute{e}$vy flight, which makes the L$\acute{e}$vy flight a more suitable physical model. Moreover, the tempered fractional Laplacian equation governs the probability distribution function of the position of the particles and some works on the tempered fractional Laplacian have been done by scholars. For example, in [13], Zhang, Deng and Fan developed the finite difference schemes for the tempered fractional Laplacian equation with the generalized Dirichlet type boundary condition. In [14], Zhang, Deng and Karniadakis established numerical methods in the Riesz basis Galerkin framework with respect to the tempered fractional Laplacian. In [15], Zhang, Hou, Ahmad and Wang studied the Choquard equation involving a generalized nonlinear tempered fractional p-Laplacian operator. In addition, more results on tempered fractional Laplacian operator can be found in [16–19].

The nonlocal property of the fractional Laplacian operator creates some difficulties to study it. To overcome this difficulty, an extension method was introduced by Caffarelli and Slivestre [20], which converts the nonlocal problem into a high dimensional local one. In addition, the method of moving planes in integral forms also has been widely used to study the nonlocal problems, please see [21, 22]. However, some nonlocal operators cannot be solved by the above method. In [23], Chen, Li and Li put forward a novel approach: a direct method of moving planes method, which is a new idea to solve the fractional Laplacian problems. By using direct method of moving planes, in [24], Wang and Ren devoted to a nonlinear Schr$\ddot{o}$dinger equation with the fractional Laplacian and Hardy potential and in [25], Zhang and Nie studied two nonlinear equations concerning Logarithmic Laplacian. Recently, in [26], in view of nonlocal parabolic problems, Chen, Wang and Niu developed the asymptotic method of moving planes and applied it on bounded or unbounded domains. Numerous results can be seen in [27–29].

In this article, we study parabolic equation involving the general nonlinearity by the direct method of moving planes. A mass of elliptic equations involving general nonlinearity have been studied by many authors. Here, *we make a new attempt to study parabolic equation with the general nonlinearity to obtain monotonicity and nonexistence of its solution.*

2 Preliminaries

in order for the lemma to work, we introduce the following notations. We define

$$T_\alpha = \{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n |\, x_1 = \alpha,\ \text{for}\ \in \mathbb{R}\}$$

being the moving planes and

$$\Sigma_\alpha = \{x \in \mathbb{R}^n |\, x_1 < \alpha\}$$

being the region to the left of T_α.

Also,

$$x^\alpha = (2\alpha - x_1, x_2, \cdots, x_n)$$

is the reflection of x about T_α. Meanwhile, we denote

$$z_\alpha(x,t) = z(x^\alpha, t),\ Z_\alpha(x,t) = z_\alpha(x,t) - z(x,t).$$

In order to continue the proof, we show the following lemma.

Lemma 2.1 *[9] Given any $N(t) > 0$, there exists a positive constant k_0 such that if $N(t) \leq -N(0)$, then*

$$\frac{C}{|x_1(t) - \alpha|^\beta} > k_0 > 0, \tag{2.1}$$

where $x(t) = (x_1(t), \cdots, x_n(t))$ is a minimum point of $\bar{Z}_\alpha(x,t)$ in Σ_α for each fixed t.

3 Main results

For this part, our main results are given. The monotonicity of solutions in x_1 direction and the nonexistence of positive solutions are established by **Theorem.**3.1 and **Theorem.**3.2 respectively. The main content of Theorems are as follows.

Theorem 3.1 *Let $z(x,t) \in (C^{1,1}_{loc}(\mathbb{R}^n) \cap \mathcal{L}_\beta) \times \mathrm{C}^1(\mathbb{R})$ be a positive bounded classical solution of (1.1), assume that (1.1) satisfies the following conditions:*

(H_1) *$a(p) \leq 0$, for $p \leq 0$;*
(H_2) *$a(p) > 0$ somewhere for $p > 0$;*
(H_2) *f is positive and locally Lipschitz continuous.*

Then $z(x,t)$ is monotone increasing in x_1 direction.

Proof From the equation (1.1), we deduce that

$$\begin{aligned}
&\frac{\partial Z_\alpha(x,t)}{\partial t} - (\Delta + \lambda)^{\frac{\beta}{2}} Z_\alpha(x,t) \\
&\quad = a(x_1^\alpha) f(z_\alpha) - a(x_1) f(z) \\
&\quad = (a(x_1^\alpha) - a(x_1)) f(z_\alpha) + a(x_1)(f(z_\alpha) - f(z)) \\
&\quad \geq a(x_1)(f(z_\alpha) - f(z)) \\
&\quad = a(x_1) N(\alpha, x) Z_\alpha(x,t)
\end{aligned}$$

where $N(\alpha, x) = \frac{f(z_\alpha) - f(z)}{z_\alpha - z}$. Meanwhile, we impose the condition that $N(\alpha, x)$ is nonnegative.

Next, we consider the following problem

$$\begin{cases} \frac{\partial Z_\alpha(x,t)}{\partial t} - (\Delta + \lambda)^{\frac{\beta}{2}} Z_\alpha(x,t) \geq a(x_1) N(\alpha, x) Z_\alpha(x,t), & (x,t) \in \Sigma_\alpha \times \mathbb{R}, \\ Z_\alpha(x,t) = -Z_\alpha(x^\alpha, t), & (x,t) \in \Sigma_\alpha \times \mathbb{R}, \end{cases} \tag{3.1}$$

Step 1. As usual, we want to show that

$$Z_\alpha(x,t) \geq 0, \quad (x,t) \in \Sigma_\alpha \times \mathbb{R}, \text{ for is sufficiently negative.} \tag{3.2}$$

The assumption that z is bounded, which cannot guarantee the minimum of Z_α can be obtained. To overcome this difficulty, we introduce an auxiliary function

$$\bar{Z}_\alpha(x,t) = \frac{Z_\alpha(x,t)}{h(x)},$$

where $h(x) = |x - (\alpha + 1)e_1|^o$ with $e_1 = (1, 0, \cdots, 0)$, o is a small positive constant. Based on above, we know that the sign of $\bar{Z}_\alpha(x,t)$ is same as $Z_\alpha(x,t)$.

Letting $|x| \to +\infty$, we have

$$\lim_{|x|\to+\infty} \bar{Z}_\alpha(x,t) \to 0. \tag{3.3}$$

In the following processes, we will consider $\bar{Z}_\alpha(x,t)$.

According to (3.3), we deduce that there is $x(t)$, then

$$\bar{Z}_\alpha(x(t),t) = \inf_{x\in\Sigma_\alpha} \bar{Z}_\alpha(x,t), \text{ for arbitrary fixed } t \in \mathbb{R}.$$

Next, we infer that

$$\text{if } \bar{Z}_\alpha(x(t),t) < 0, \ \frac{\partial \bar{Z}_\alpha}{\partial t}(x(t),t) \geq \frac{-C}{|x_1(t)-\alpha|^\beta}\bar{Z}_\alpha(x(t),t). \tag{3.4}$$

In reality, on the basis of a similar calculation as (22) in [9], we have

$$\text{if } Z_\alpha(x(t),t) < 0, \ -(\Delta+\lambda)^{\frac{\beta}{2}} Z_\alpha(x(t),t) \leq \frac{C}{|x_1(t)-\alpha|^\beta} Z_\alpha(x(t),t).$$

Combined with (3.1), then

$$\frac{\partial Z_\alpha}{\partial t}(x(t),t) \geq \frac{-C}{|x_1(t)-\alpha|^\beta} Z_\alpha(x(t),t).$$

According to the definition of $\bar{Z}_\alpha(x,t)$, we deduce (3.4).

For arbitrary fixed $t \in \mathbb{R}$, let

$$N(t) := \bar{Z}_\alpha(x(t),t) = \inf_{x\in\Sigma_\alpha} \bar{Z}_\alpha(x,t).$$

Proving (3.2) is equivalent to prove the following (3.5)

$$N(t) \geq 0, \ \forall t \in \mathbb{R}. \tag{3.5}$$

Now we proceed with the proof of (3.5).

Suppose that (3.5) is invalid, there is a $t^{'} \in \mathbb{R}$, then

$$-N(0) := N(t^{'}) = \bar{Z}_\alpha(x(t^{'}),t^{'}) < 0. \tag{3.6}$$

For arbitrary $\bar{t} < t^{'}$, we set up a subsolution

$$m(t) = -\bar{R}e^{-k_0(t-\bar{t})},$$

here k_0 is defined in (2.1) and

$$-\bar{R} = \inf_{\Sigma_\alpha\times\mathbb{R}} \bar{Z}_\alpha(x,t).$$

We show that

$$\bar{Z}_\alpha(x,t) \geq m(t), \ \ (x,t) \in \overline{\Sigma_\alpha} \times [\bar{t}, t^{'}]. \tag{3.7}$$

Think about the function

$$V(x,t) = \bar{Z}_\alpha(x,t) - m(t), \ \ (x,t) \in \overline{\Sigma_\alpha} \times [\bar{t}, t^{'}].$$

From the construction of $m(t)$, we have

$$V(x,t) = \bar{Z}_\alpha(x,t) - m(t) = \bar{Z}_\alpha(x,t) - (-\bar{R}) \geq 0, (x,t) \in \Sigma_\alpha \times \{\bar{t}\};$$

and

$$V(x,t)=\bar{Z}_\alpha(x,t)-m(t)=-m(t)\geq 0,\,(x,t)\in T_\alpha\times[\bar{t},t^{'}].$$

Assume that (3.7) is not true, there is $(x(\hat{t}),\hat{t})\in\Sigma_\alpha\times(\bar{t},t^{'}]$, then

$$V(x(\hat{t}),\hat{t})=\inf_{\Sigma_\alpha\times(\bar{t},t^{'}]}V(x,t)<0, \tag{3.8}$$

$$\frac{\partial V}{\partial t}(x(\hat{t}),\hat{t})\leq 0. \tag{3.9}$$

On one hand, in view of the definition of $V(x,t)$, then

$$\bar{Z}_\alpha(x(\hat{t}),\hat{t})=\inf_{\Sigma_\alpha}\bar{Z}_\alpha(x,\hat{t})<m(\hat{t})<0.$$

Hence, from (3.4), one has

$$\frac{\partial\bar{Z}_\alpha}{\partial t}(x(\hat{t}),\hat{t})\geq\frac{-C}{|x_1(\hat{t})-\alpha|^\beta}\bar{Z}_\alpha(x(\hat{t}),\hat{t}). \tag{3.10}$$

But on the other, by (3.9), we derive that

$$V(x(\hat{t}),\hat{t})\leq V(x(t^{'}),t^{'}),$$

that is

$$\bar{Z}_\alpha(x(\hat{t}),\hat{t})-\bar{Z}_\alpha(x(t^{'}),t^{'})\leq m(\hat{t})-m(t^{'})\leq 0$$

due to monotone increasing property of $m(t)$. As a result,

$$N(\hat{t})=\bar{Z}_\alpha(x(\hat{t}),\hat{t})\leq\bar{Z}_\alpha(x(t^{'}),t^{'})=N(t^{'})=-N(0). \tag{3.11}$$

Taking account of Lemma 2.1, by (3.11), then

$$\frac{C}{|x_1(\hat{t})-\alpha|^\beta}>k_0>0.$$

Combined with (3.10), we derive that

$$\frac{\partial\bar{Z}_\alpha}{\partial t}(x(\hat{t}),\hat{t})\geq -k_0\bar{Z}_\alpha(x(\hat{t}),\hat{t}). \tag{3.12}$$

Then from (3.9), one has

$$-k_0m(\hat{t})=\frac{\partial m}{\partial t}(\hat{t})\geq\frac{\partial\bar{Z}_\alpha}{\partial t}(x(\hat{t}),\hat{t})\geq -k_0\bar{Z}_\alpha(x(\hat{t}),\hat{t}),$$

which concludes that

$$V(x(\hat{t}),\hat{t})=\bar{Z}_\alpha(x(\hat{t}),\hat{t})-m(\hat{t})\geq 0,$$

which yields a contradiction to

$$V(x(\hat{t}),\hat{t})<0.$$

Therefore, we derive that (3.7) holds. It means that

$$\bar{Z}_\alpha(x,t)\geq m(t),\;\;(x,t)\in\overline{\Sigma_\alpha}\times[\bar{t},t^{'}].$$

For any $\bar{t}$, the above formula is true. Letting $\bar{t}\to-\infty$, we have $m(t)\to 0$. Consequently, $\bar{Z}_\alpha(x,t)\geq 0$, $(x,t)\in\overline{\Sigma_\alpha}\times(-\infty,t^{'}]$, which contradicts to (3.6). Therefore, (3.5) is true and so does (3.2).

Remark 3.1 Using a proof similar to *Step* 1, we deduce that for arbitrary $\alpha > 0$, $\bar{Z}_\alpha(x,t)$ is a solution to (3.1) and

$$\bar{Z}_\alpha(x(t),t) = \inf_{x\in\Sigma_\alpha} \bar{Z}_\alpha(x,t) < 0,$$

it follows that $x_1(t) > 0$. The above will be employed in *Step* 2.

Step 2. On account of (3.2), we move the T_α as long as the inequality holds. Let

$$\alpha_0 = sup\{\alpha | Z_\nu(x,t) \geq 0,\ \forall (x,t) \in \Sigma_\nu \times \mathbb{R}, \nu \leq \alpha\}.$$

Now we verify that

$$\alpha_0 = +\infty. \tag{3.13}$$

We use a contradiction argument. Assume that $0 < \alpha_0 < +\infty$, then in view of the definition of α_0, there is a sequence $\alpha_k \searrow \alpha_0$ such that

$$\inf_{\Sigma_{\alpha_k}\times\mathbb{R}} Z_{\alpha_k}(x,t) < 0.$$

Let

$$\bar{Z}_{\alpha_k}(x,t) = \frac{Z_{\alpha_k}(x,t)}{h(x)},$$

where $h(x)$ is mentioned earlier. Then obviously,

$$-N_k := \inf_{\Sigma_{\alpha_k}\times\mathbb{R}} \bar{Z}_{\alpha_k}(x,t) < 0. \tag{3.14}$$

Since $t \in \mathbb{R}$, the minimum point of $\bar{Z}_{\alpha_k}$ might not be obtained for finite value t. For more information about $\frac{\partial \bar{Z}_{\alpha_k}(x,t)}{\partial t}$, we pick a sequence t_k, and $x(t_k)$ and $\tau_k \searrow 0$, then

$$\bar{Z}_{\alpha_k}(x(t_k),t_k) = \inf_{\Sigma_{\alpha_k}} \bar{Z}_{\alpha_k}(\cdot,t) = -N_k + \tau_k N_k. \tag{3.15}$$

We construct an auxiliary function

$$\widetilde{Z}_{\alpha_k}(x,t) = \bar{Z}_{\alpha_k}(x,t) - \tau_k N_k \vartheta_k(t),$$

here $\vartheta_k(t) = \vartheta(t - t_k)$, $\vartheta(t) \in C_0^\infty(\mathbb{R})$, $|\vartheta'(t)| \leq 1$ and

$$\vartheta(t) = \begin{cases} 1 & |t| \leq \frac{1}{2}, \\ 0 & |t| \geq 2. \end{cases}$$

Next we study the value of $\widetilde{Z}_{\alpha_k}(x,t)$ in $\Sigma_{\alpha_k} \times (t_k - 2, t_k + 2)$. In view of the definition of $\widetilde{Z}_{\alpha_k}(x,t)$, we have

$$\widetilde{Z}_{\alpha_k}(x(t_k),t_k) = -N_k.$$

Otherwise, when $|t - t_k| \geq 2$,

$$\widetilde{Z}_{\alpha_k}(x,t) = \bar{Z}_{\alpha_k}(x,t) \geq -N_k.$$

To sum up, the minimum point of $\widetilde{Z}_{\alpha_k}(x,t)$ is obtained in $\Sigma_{\alpha_k} \times (t_k - 2, t_k + 2)$. We denote it as $(x(\hat{t}_k), \hat{t}_k)$. That is

$$\widetilde{Z}_{\alpha_k}(x(\hat{t}_k),\hat{t}_k) = \inf_{\Sigma_{\alpha_k}\times\mathbb{R}} (x,t) < 0.$$

Hence,

$$\frac{\partial \widetilde{Z}_{\alpha_k}}{\partial t}(x(\hat{t}_k), \hat{t}_k) = 0,$$

which means that

$$|\frac{\partial \bar{Z}_{\alpha_k}}{\partial t}(x(\hat{t}_k), \hat{t}_k)| = |\tau_k N_k \frac{\partial \vartheta_k}{\partial t}| \le \tau_k N_k. \tag{3.16}$$

Combined the definition of N_k in (3.14) and

$$\widetilde{Z}_{\alpha_k}(x(\hat{t}_k), \hat{t}_k) \le \widetilde{Z}_{\alpha_k}(x(t_k), t_k),$$

we have

$$-N_k \le \bar{Z}_{\alpha_k}(x(\hat{t}_k), \hat{t}_k) \le \bar{Z}_{\alpha_k}(x(t_k), t_k) = -N_k + \tau_k N_k. \tag{3.17}$$

From the definition of $\widetilde{Z}_{\alpha_k}(x, t)$, we have

$$\bar{Z}_{\alpha_k}(x(\hat{t}_k), \hat{t}_k) = \inf_{\Sigma_{\alpha_k}} (x, \hat{t}_k) < 0.$$

According to Remark 3.1, we know that $x_1(\hat{t}_k) > 0$. Therefore, we could assume $0 < x_1(\hat{t}_k) < \alpha_0 + 1$. Then by a similar process as (22) in [9], we deduce

$$-(\Delta + \lambda)^{\frac{\beta}{2}} Z_{\alpha_k}(x(\hat{t}_k), \hat{t}_k) \le \frac{C}{|x_1(\hat{t}_k) - \alpha_k|^\beta} Z_{\alpha_k}(x(\hat{t}_k), \hat{t}_k). \tag{3.18}$$

Obverse that there is a positive number m_1, then

$$a(x_1(\hat{t}_k))N(\alpha, x) \le m_1.$$

Together with (3.1), (3.18) and $Z_{\alpha_k}(x(\hat{t}_k), \hat{t}_k) < 0$, we arrive at

$$\begin{aligned} &\frac{\partial Z_{\alpha_k}}{\partial t}(x(\hat{t}_k), \hat{t}_k) + \frac{C}{|x_1(\hat{t}_k) - \alpha_k|^\beta} Z_{\alpha_k}(x(\hat{t}_k), \hat{t}_k) \\ \ge & a(x_1(\hat{t}_k))N(\alpha, x) Z_{\alpha_k}(x(\hat{t}_k), \hat{t}_k) \\ \ge & m_1 Z_{\alpha_k}(x(\hat{t}_k), \hat{t}_k). \end{aligned} \tag{3.19}$$

For above inequality, we divide $h(x(\hat{t}_k), \hat{t}_k)$ on both sides. Then we obtain

$$\frac{\partial \bar{Z}_{\alpha_k}}{\partial t}(x(\hat{t}_k), \hat{t}_k) + \frac{C}{|x_1(\hat{t}_k) - \alpha_k|^\beta} \bar{Z}_{\alpha_k}(x(\hat{t}_k), \hat{t}_k) \ge m_1 \bar{Z}_{\alpha_k}(x(\hat{t}_k), \hat{t}_k). \tag{3.20}$$

Together with (3.16),(3.17) and (3.20), we divide $-N_k$ on both sides, one can arrive at

$$\frac{C}{|x_1(\hat{t}_k) - \alpha_k|^\beta} \le \frac{m_1}{2}, \text{ for } \o_k \text{ is small,} \tag{3.21}$$

which concludes that

$$|x_1(\hat{t}_k) - \alpha_k| \ge m_2 > 0$$

and

$$|x_1(\hat{t}_k) - \alpha_0| \ge \frac{m_2}{2} > 0. \tag{3.22}$$

When k is sufficiently large, one has

$$\begin{aligned}&\frac{\partial Z_{\alpha_k}}{\partial t}(x(\hat{t}_k),\hat{t}_k)+\frac{C}{|x_1(\hat{t}_k)-\alpha_k|^{\beta}}Z_{\alpha_k}(x(\hat{t}_k),\hat{t}_k)\\ \geq&(a(x_1^{\alpha}(\hat{t}_k))-x_1(\hat{t}_k))f(Z_{\alpha_k})N(\alpha,x)Z_{\alpha_k}(x(\hat{t}_k),\hat{t}_k)\\ \geq&m_3>0,\end{aligned} \tag{3.23}$$

where we use the fact that f is locally Lipschitz continuous and

$$Z_{\alpha_k}(x,t)\rightrightarrows Z_{\alpha_0}(x,t)\geq 0.$$

On account of $Z_{\alpha_k}(x(\hat{t}_k),\hat{t}_k)$, from (3.23), we derive that

$$\frac{\partial Z_{\alpha_k}}{\partial t}(x(\hat{t}_k),\hat{t}_k)\geq m_3>0. \tag{3.24}$$

Next we let

$$\hat{Z}_{\alpha_k}(x,t)=Z_{\alpha_k}(x+x(\hat{t}_k),t+\hat{t}_k),$$

from (3.24), we arrive at

$$\frac{\partial \hat{Z}_{\alpha_k}}{\partial t}(0,0)\geq m_3>0. \tag{3.25}$$

In view of [30], we have

$$\parallel \hat{Z}_{\alpha_k}\parallel_{t,x}^{1+\varepsilon,\beta(1+\varepsilon)}\leq m_4,\ \forall\,(x,t)\in\Omega\times(-T,T)\subset\subset\mathbb{R}^n\times\mathbb{R},$$

it concludes that there is a subsequence of $(x(\hat{t}_k),\hat{t}_k)$ and when $k\to+\infty$,

$$\hat{Z}_{\alpha_k}(x,t)\to\hat{Z}_{\alpha_0}(x,t),\ \frac{\partial\hat{Z}_{\alpha_k}}{\partial t}(x,t)\to\frac{\partial\hat{Z}_{\alpha_0}}{\partial t}(x,t).$$

On account of

$$0<x_1(\hat{t}_k)\leq\alpha_k,$$

and

$$\alpha_k\to\alpha_0,\ \ k\to+\infty.$$

Hence, there is a subsequence of $x_1(\hat{t}_k)$ and $0\leq x_1^0\leq\alpha_0$, then

$$x_1(\hat{t}_k)\to x_1^0.$$

Now we think about $\hat{Z}_{\alpha_0}(x,t)$. Obviously, one has

$$\hat{Z}_{\alpha_0}(x,t)\geq 0,\ \ (x,t)\in\Sigma_{\alpha_0-x_1^0}\times\mathbb{R}.$$

Taking account of

$$Z_{\alpha_k}(x(\hat{t}_k),\hat{t}_k)<0,$$

we arrive at

$$\hat{Z}_{\alpha_0}(0,0)=0=\inf_{\Sigma_{\alpha_0-x_1^0}\times\mathbb{R}}\hat{Z}_{\alpha_0}(x,t).$$

Hence

$$\frac{\partial \hat{Z}_{\alpha_0}}{\partial t}(0,0) = 0.$$

It contradicts to (3.25). As a result, we have $\alpha_0 = +\infty$.

Therefore, $z(x,t)$ is monotone increasing in x_1 direction. □

Theorem 3.2 *Let $z(x,t) \in (C^{1,1}_{loc}(\mathbb{R}^n) \cap \mathcal{L}_\beta) \times \mathrm{C}^1(\mathbb{R})$ and satisfy the conditions of Theorem 3.1, then (1.1) has no positive bounded classical solutions.*

Proof We will use a contradiction to verify it. Suppose in the contrary, there exists a positive bounded solution of (1.1), we will obtain a contradiction.

We define λ_1 being the first eigenvalue of the following problem

$$\begin{cases} -(\Delta+\lambda)^{\frac{\beta}{2}}\varphi(x) = \lambda_1\varphi(x), & x \in B_1(b+2, 0^{'}), \\ \varphi(x) = 0, & x \in B_1^c(b+2, 0^{'}), \end{cases}$$

with $0 \le b \in \mathbb{R}$ is sufficiently large.

For the sake of integration by parts, we mollify $\varphi(x)$ to

$$\varphi_1(x) = \varrho * \varphi(x) \in C_0^\infty(\mathbb{R}^n).$$

Consequently,

$$-(\Delta+\lambda)^{\frac{\beta}{2}}\varphi_1(x) \le \lambda_1\varphi_1(x), \ x \in \mathbb{R}^n, \tag{3.26}$$

here $*$ stands for convolution, $\varrho(x) \in C_0^\infty(B_1(b+2, 0^{'}))$ is a mollifier which satisifies $\int_{\mathbb{R}^n}\varrho(x)dx = 1$.

The above (3.26) will be verified by adopting the idea of ([7] Lemma A.1).

Suppose that

$$\int_{\mathbb{R}^n}\varphi_1(x)dx = 1.$$

Owing to the mollification, the support of φ_1 is included in $B_2(b+2, 0^{'})$. Let

$$\Phi_b(t) := \int_{\mathbb{R}^n} z(x,t)\varphi_1(x)dx = \int_{B_2(b+2,0^{'})} z(x,t)\varphi_1(x)dx.$$

Together with Jensen inequality and (3.26), we derive that

$$\begin{aligned} \frac{d}{dt}\Phi_b(t) &= -\int_{\mathbb{R}^n} -(\Delta+\lambda)^{\frac{\beta}{2}} z(x,t)\varphi_1(x)dx + \int_{\mathbb{R}^n} a(x_1)f(z)\varphi_1(x)dx \\ &= -\int_{\mathbb{R}^n} z(x,t) - (\Delta+\lambda)^{\frac{\beta}{2}}\varphi_1(x)dx + \int_{\mathbb{R}^n} a(x_1)f(z)\varphi_1(x)dx \\ &\ge -\lambda_1\int_{\mathbb{R}^n} z(x,t)\varphi_1(x)dx + b\int_{\mathbb{R}^n} f(z)\varphi_1(x)dx \\ &\ge -\lambda_1\Phi_b(t) + bf(\int_{\mathbb{R}^n} f(z)\varphi_1(x)dx) \\ &\ge -\lambda_1\Phi_b(t). \end{aligned} \tag{3.27}$$

In view of the monotone increasing property in x_1 by Theorem 3.1, we conclude that for arbitrary fixed $t \in \mathbb{R}$, $\Phi_b(t)$ is monotone increasing about b. As a result,

$$\Phi_b(0) \ge c_0 := \Phi_0(0), \tag{3.28}$$

where we choose c_0 to be a number so that $-\lambda_1 c_0$ is positive. From (3.27), we have

$$\frac{d}{dt}\Phi_b(t) \geq -\lambda_1 \Phi_b(t).$$

Combined with (3.28), we get

$$\Phi_b(t) \geq -\lambda_1 c_0 e^t.$$

Hence, $\Phi_b(t)$ is monotone increasing about t. Letting $t \to +\infty$, $\Phi_b(t) \to +\infty$, which yields a contradiction to the boundedness of $z(x, t)$. □

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