Mathematical Methods of Operations Research (2023) 98:57–73
https://doi.org/10.1007/s00186-023-00824-1

ORIGINAL ARTICLE

On egalitarian values for cooperative games with level structures

J. M. Alonso-Meijide[1] · J. Costa[2] · I. García-Jurado[3] · J. C. Gonçalves-Dosantos[4]

Received: 28 November 2022 / Revised: 23 May 2023 / Accepted: 13 June 2023 /
Published online: 1 July 2023

Abstract
In this paper we extend the equal division and the equal surplus division values for transferable utility cooperative games to the more general setup of transferable utility cooperative games with level structures. In the case of the equal surplus division value we propose three possible extensions, one of which has already been described in the literature. We provide axiomatic characterizations of the values considered, apply them to a particular cost sharing problem and compare them in the framework of such an application.

Keywords Cooperative games · Level structures · Equal division value · Equal surplus division value

1 Introduction

In many practical situations, when a group of agents are faced with sharing the costs of a project they are jointly developing, they use an egalitarian approach to cost sharing. Egalitarian sharing is a clear criterion, computationally simple and easily acceptable

✉ J. C. Gonçalves-Dosantos
juan.carlos.goncalves@udc.es

1 Grupo MODESTYA, CITMAga and Departamento de Estatística, Análise Matemática e Optimización, Facultade de Ciencias, Universidade de Santiago de Compostela, Campus de Lugo, 27002 Lugo, Spain

2 Grupo MODES, Departamento de Matemáticas, Universidade da Coruña, Campus de Elviña, 15071 A Coruña, Spain

3 Grupo MODES, CITMAga and Departamento de Matemáticas, Universidade da Coruña, Campus de Elviña, 15071 A Coruña, Spain

4 Grupo MODES, CITIC and Departamento de Matemáticas, Universidade da Coruña, Campus de Elviña, 15071 A Coruña, Spain

to all as a non-conflict generating procedure. Selten (1972) indicates that egalitarian considerations explain in a successful way observed outcomes in experimental cooperative games.

Cooperative game theory, which is concerned, among other things, with the study of fair distributions of the outcomes of cooperation, has analysed various distribution rules based on egalitarian criteria. For instance, van den Brink (2007) provides a comparison of the equal division value and the Shapley value, and Casajus and Hüttner (2014) compare those two solutions with the equal surplus division value (studied first in Driessen and Funaki 1991).

More recently, egalitarian solutions have been studied in the context of coalition-structured cooperative games. More specifically, Alonso-Meijide et al. (2020) extend the equal division and the equal surplus division values for cooperative games with a priori unions; in the case of the equal surplus division value Alonso-Meijide et al. (2020) propose three possible extensions. Hu and Li (2021) extend the equal surplus division value for cooperative games with level structures. Level structures were introduced in Owen (1977) and further studied in Winter (1989). A level structure is a collection of nested partitions of the set of players that conditions their negotiation. When instead of a collection of nested partitions we have a single partition, the level structure is called an a priori union structure. The Hu and Li's equal surplus division value for cooperative games with level structures turns out to be an extension of one of the three equal surplus division values for cooperative games with a priori unions introduced in Alonso-Meijide et al. (2020).

In this paper we extend to the case with levels the other two equal surplus division values for cooperative games with a priori unions introduced in Alonso-Meijide et al. (2020), as well as the equal division value. Moreover we provide new insights and axiomatic characterizations of all the values considered here, including the Hu and Li value, and illustrate their interest with a motivating example.

The structure of the paper is as follows. In Sect. 2 we describe the model of cooperative games with level structures and define four egalitarian values in this context; furthermore, we elaborate a particular cost sharing problem to motivate the presentation, and discussion, of our egalitarian approach. In Sect. 3 we provide axiomatic characterizations of the previous values. Finally, Sect. 4 is devoted to the concluding remarks.

2 Four egalitarian values for cooperative games with level structures

We begin this section with a cost sharing problem to motivate the definition of the egalitarian values we are about to introduce. After the definitions, we will return to the problem for a comparison of the different solutions.

Housing legislation in most democratic states includes regulations on how to share the costs of improving common elements in homeowners' associations. In some cases, such regulations recommend the use of equal sharing criteria. For example, in the Netherlands, each of the owners of the dwellings involved in an improvement of the common elements of a building must share equally in the debts and costs

Table 1 Owners' individual fees

Owner	Individual fee
1	$50 + 50 \cdot 1 + 4 \cdot 1 \cdot 1 + 10 \cdot 1 = 114$
2	$50 + 50 \cdot 1 + 4 \cdot 2 \cdot 1 + 10 \cdot 1 = 118$
3	$50 + 50 \cdot 1 + 4 \cdot 1 \cdot 1 + 10 \cdot 1 = 114$
4	$50 + 50 \cdot 1 + 4 \cdot 2 \cdot 1 + 10 \cdot 1 = 118$
5	$50 + 50 \cdot 1 + 4 \cdot 2 \cdot 1 + 10 \cdot 1 = 118$

involved, unless the internal community agreements provide for a different proportion of participation.

In this section we deal with an example that arises in the sharing of the ordinary maintenance costs of a parking area serving two residential buildings. The specific characteristics of the example are as follows. We consider a small residential complex consisting of two buildings that share an underground parking area. The first building is a two-storey building with one dwelling on each floor. The second building is also two-storey with one dwelling on the first floor and two dwellings on the second floor. From the parking area there is a lift to the first building and another lift to the second building. The owners of the dwellings in the residential complex have entrusted the maintenance of the common parking area (where each owner has one parking place) to a company which is responsible for the cleaning and security of the parking area, as well as for the maintenance of the lifts.

The company in charge of maintenance has a standard monthly fee for each community of owners which has two parts: a fixed part of 50 euros, and a variable part consisting of 50 euros for each lift, 4 euros times the highest floor the lift should reach (for each lift) and 10 euros for each parking place. Accordingly, the community in this example should pay the following monthly amount in euros (broken down as the sum of the fee components):

$$50 + 50 \cdot 2 + 4 \cdot 2 \cdot 2 + 10 \cdot 5 = 216.$$

Now the question is how to distribute this fee in an equitable way among the five owners involved. Below are two proposals.

- One proposal is to divide the total fee equally between the five owners. This distribution is known as the equal division value (ED-value). Accordingly, each of them would have to pay 43.2 euros.
- Another proposal is to take into account that not all owners would have the same individual fee; when we refer to the owners, by individual fee we mean the fee that each owner would have to pay if only he had access to the parking area. According to this proposal each owner should pay his individual fee plus the remainder of the total fee divided equally among the owners; by the remainder of the total fee we mean the difference between the total fee and the sum of the individual fees (notice that this remainder may be a negative amount). This distribution is known as the equal surplus division value (ESD-value). Table 1 shows the individual fee of

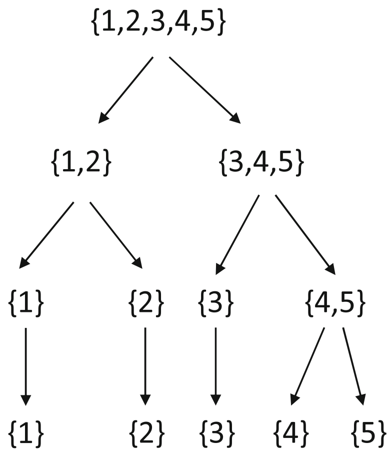

Fig. 1 The level structure of the property owners

each owner. Therefore, each owner must pay his individual fee plus $\frac{1}{5}(216-582)$, resulting in the following distribution vector: (40.8, 44.8, 40.8, 44.8, 44.8).

The two proposals above do not reflect that, in setting the monthly fee, it would be appropriate to take into account that the owners participate in the residential complex according to a nested structure; hereafter, we call it a level structure. Firstly, the owners are grouped according to the lifts that serve them; secondly, they are grouped according to the floor on which their property is located. The level structure of the owners is shown in Fig. 1.

Below, we describe the model of cooperative games with level structures. Transferable utility games (abbreviated, TU-games) are probably the most studied model of cooperative game theory. A TU-game is defined by a finite set of players $N = \{1, 2, \dots, n\}$ and a real-valued function $v : 2^N \to \mathbb{R}$, called a characteristic function, where $v(S)$ is the worth of $S \subseteq N$, that is, the benefits (or costs) that coalition S is able to generate. By convention, $v(\emptyset) = 0$. A TU-game with a priori unions is a triplet $(N, v, \mathcal{C})$, where $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$ is a partition of the player set N into classes, which in cooperative game theory are known as unions or a priori unions.

A level structure $\mathcal{L}$ is a sequence of nested partitions of the set of players N, $\mathcal{L} = \{\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_{k+1}\}$. Level zero comprises each player $i \in N$ as an a priori union, $\mathcal{C}_0 = \{\{i\} : i \in N\}$. At each level $l \geq 1$, the partition is obtained by aggregation of a priori unions from level $l-1$; if $C \in \mathcal{C}_l$ then $C = \bigcup_{C' \in \hat{\mathcal{C}}_{l-1}} C'$, where $\hat{\mathcal{C}}_{l-1} \subseteq \mathcal{C}_{l-1}$. Finally, $\mathcal{C}_{k+1}$ is the trivial partition with the grand coalition N as the sole a priori union, $\mathcal{C}_{k+1} = \{N\}$. (See Fig. 1 for an example of a level structure, with $n = 5$ and $k = 2$.) A level game is a triplet $(N, v, \mathcal{L})$, where (N, v) is a TU-game and $\mathcal{L}$ is a level structure. A TU-game with a priori unions can be viewed as a particular case of a level game with $k = 1$.

For every player $i \in N$ and level $l \in \{0, 1, \dots, k+1\}$, we denote by $C_l(i)$ the a priori union in $\mathcal{C}_l \in \mathcal{L}$ containing i, that is, $i \in C_l(i) \in \mathcal{C}_l$. Also, for every level $l \in \{1, 2, \dots, k+1\}$ and a priori union $C \in \mathcal{C}_l \in \mathcal{L}$, we denote by $\lfloor C \rfloor$ the set of all a priori unions in $\mathcal{C}_{l-1}$ contained in C. We can refer to these a priori unions as the direct subordinates of C.

$$\lfloor C \rfloor = \{C' \in \mathcal{C}_{l-1} \in \mathcal{L} : C' \subseteq C\}.$$

By a value, we mean a map g that assigns to every level game $(N, v, \mathcal{L})$ a vector $g(N, v, \mathcal{L}) \in \mathbb{R}^N$ with components $g_i(N, v, \mathcal{L})$ for all $i \in N$. Each component represents the player's payoff according to g. Examples of values that share an egalitarian approach but do not take into account the level structure are the equal division value (ED-value)

$$ED_i = \frac{v(N)}{n}$$

and the equal surplus division value (ESD-value)

$$ESD_i = v(i) + \frac{v(N) - \sum_{i \in N} v(i)}{n},$$

which have been studied for example in van den Brink (2007) and in Casajus and Hüttner (2014), respectively. Alonso-Meijide et al. (2020) define and study values for games with a priori unions using an egalitarian approach. Next, we propose four values that extend the previous ones to the context of level games. The third value has already been introduced in Hu and Li (2021).

The first proposal, the level equal division value (LED-value), divides the worth of the grand coalition, $v(N)$, equally among the players at each level of the level structure, that is, for a player $i \in N$, the value divides $v(N) = v(C_{k+1}(i))$ by the cardinals of the direct subordinates of all a priori unions containing i. Hereafter, we will call this an equitable allocation among the players, but taking into account the level structure.

Definition 2.1 The LED-value is defined for every $(N, v, \mathcal{L})$ and $i \in N$ by

$$LED_i = \frac{v(C_{k+1}(i))}{\Pi_{l=1}^{k+1} |\lfloor C_l(i) \rfloor|}.$$

The second proposal is to calculate the worth of each a priori union at level k (the last non-trivial partition) and share it equally among its players, but taking into account the level structure. The remainder of the total worth is then divided equally among the players taking again into account the level structure. We call it the level equal surplus division value 1 (LESD^1-value).

Definition 2.2 The LESD^1-value is defined for every $(N, v, \mathcal{L})$ and $i \in N$ by

$$LESD_i^1 = \frac{v(C_k(i))}{\Pi_{l=1}^{k} |\lfloor C_l(i) \rfloor|} + \frac{v(C_{k+1}(i)) - \sum_{C \in \lfloor C_{k+1}(i) \rfloor} v(C)}{\Pi_{l=1}^{k+1} |\lfloor C_l(i) \rfloor|}.$$

The third proposal is to allocate to each player its individual worth, plus the remainder of the worth of the a priori union to which it belongs on level 1 divided equally among the members of the a priori union, but taking into account the level structure, and so on.

Definition 2.3 The LESD^2-value is defined for every $(N, v, \mathcal{L})$ and $i \in N$ by

$$LESD_i^2 = v(i) + \sum_{l=1}^{k+1} \frac{v(C_l(i)) - \sum_{C \in \lfloor C_l(i) \rfloor} v(C)}{\Pi_{l'=1}^{l} |\lfloor C_{l'}(i) \rfloor|}.$$

The fourth and last proposal is to allocate to each player its individual worth, plus the remainder of the worth of the grand coalition divided equally among the players, but taking into account the level structure.

Definition 2.4 The LESD^3-value is defined for every $(N, v, \mathcal{L})$ and $i \in N$ by

$$LESD_i^3 = v(i) + \frac{v(C_{k+1}(i)) - \sum_{j \in N} v(j)}{\Pi_{l=1}^{k+1} |\lfloor C_l(i) \rfloor|}.$$

We now go back to the example at the start of the section to compute the new values in that setting. The four proposals for distributing the total fee take into account the level structure of the problem.

- The first of these four proposals, the LED-value, is an equal division at each level of the level structure; i.e. first we divide the 216 euros equally between the two lifts, then we divide the amount allocated to each lift equally between the floors it serves; finally, we divide the amount allocated to each lift and floor equally between the corresponding owners. As we stated before, we say that the total fee is divided equally between the owners, but taking into account their level structure. This proposal results in the following distribution vector: (54, 54, 54, 27, 27).

The following three proposals take into account the level structure of the owners, but also the individual fees that the owners, floors and lifts would have to pay if being alone. The latter can be done in various ways, resulting in the three different proposals.

- One proposal, the LESD^1-value, is to calculate the individual fee for each lift, i.e. the fee that the owners who use each lift would have to pay if only they had access to the parking area, and share it equally among the owners it serves taking into account the level structure. The remainder of the total fee is then divided equally among the owners taking again into account the level structure. The individual fees for the lifts would be 128 and 138, respectively. The remainder of the total fee is −50. Therefore, for instance, the forth component of the corresponding distribution vector is

$$\frac{138}{2 \cdot 2} - \frac{50}{2 \cdot 2 \cdot 2} = 28.25$$

  and the complete distribution vector is (51.5, 51.5, 56.5, 28.25, 28.25).

Table 2 Floor's and lifts' individual fees

	Individual fee
Floor 1, Lift 1	$50 + 50 \cdot 1 + 4 \cdot 1 \cdot 1 + 10 \cdot 1 = 114$
Floor 2, Lift 1	$50 + 50 \cdot 1 + 4 \cdot 2 \cdot 1 + 10 \cdot 1 = 118$
Floor 1, Lift 2	$50 + 50 \cdot 1 + 4 \cdot 1 \cdot 1 + 10 \cdot 1 = 114$
Floor 2, Lift 2	$50 + 50 \cdot 1 + 4 \cdot 2 \cdot 1 + 10 \cdot 2 = 128$
Lift 1	$50 + 50 \cdot 1 + 4 \cdot 2 \cdot 1 + 10 \cdot 2 = 128$
Lift 2	$50 + 50 \cdot 1 + 4 \cdot 2 \cdot 1 + 10 \cdot 3 = 138$

Table 3 The six proposals

	1	2	3	4	5
ED	43.2	43.2	43.2	43.2	43.2
ESD	40.8	44.8	40.8	44.8	44.8
LED	54	54	54	27	27
LESD^1	51.5	51.5	56.5	28.25	28.25
LESD^2	49.5	53.5	49.5	31.75	31.75
LESD^3	22.5	26.5	22.5	72.25	72.25

- Another proposal, the LESD^2-value, is to calculate the individual fee for each owner, for each floor and for each lift and then allocate to each owner his individual fee, plus the remainder of the individual fee for his floor divided equally among the owners of his floor (taking into account the level structure), plus the remainder of the individual fee for his lift divided equally among the owners of his lift (taking into account the level structure), plus the remainder of the total fee divided equally among the owners (taking into account the level structure). Table 2 shows the individual fees of floors and lifts. Therefore, for instance, the forth component of the corresponding distribution vector is

$$118 + \frac{128 - 2 \cdot 118}{2} + \frac{138 - 114 - 128}{2 \cdot 2} + \frac{216 - 128 - 138}{2 \cdot 2 \cdot 2} = 31.75$$

  and the complete distribution vector is (49.5, 53.5, 49.5, 31.75, 31.75).
- The last proposal we put forward, using the LESD^3-value, is to allocate to each owner his individual fee plus the remainder of the total fee divided equally among the owners taking into account the level structure. Therefore, for instance, the forth component of the corresponding distribution vector is

$$118 + \frac{216 - 114 - 118 - 114 - 118 - 118}{2 \cdot 2 \cdot 2} = 72.25$$

  and the complete distribution vector is (22.5, 26.5, 22.5, 72.25, 72.25).

Table 3 shows the six proposed fee allocations among the owners that we have calculated. All of them are adaptations of the equal share criterion and all seem reasonable, at least theoretically. However, in practice, they are not all equally reasonable;

for instance, in this example, the LESD^3-value seems difficult for the fourth and fifth owners to accept.

In order to decide in which cases it is more appropriate to use each of the six proposals it is necessary to analyse them mathematically. The ED-value and the ESD-value have been extensively studied in the literature. The other four will be analysed in the following section. The LESD^2-value has been introduced and studied in Xu and Li (2021), but in this article we obtain a new axiomatic characterization for it.

3 Axiomatic characterizations

First, we introduce a few new concepts that we need in this section, in which we provide axiomatic characterizations of the four values defined in Sect. 2. Given a level game $(N, v, \mathcal{L})$ and level $l \in \{0, 1, \ldots, k\}$, we define v^l, a new characteristic function on $\mathcal{C}_l$, as

$$v^l(S) = v\left(\bigcup_{C \in S} C\right) \text{ for all } S \subseteq \mathcal{C}_l.$$

The l-th quotient game $(\mathcal{C}_l, v^l)$ is induced from $(N, v, \mathcal{L})$ by treating a priori unions of $\mathcal{C}_l \in \mathcal{L}$ as players.

Given a level structure $\mathcal{L} = \{\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{k+1}\}$ and level $l \in \{0, 1, \ldots, k\}$, we define the l-th truncation of $\mathcal{L}$, denoted by $\mathcal{L}^l$, as a new level structure in which the set of players is $\mathcal{C}_l \in \mathcal{L}$ and $\mathcal{L}^l = \{\mathcal{C}_0^l, \mathcal{C}_1^l, \ldots, \mathcal{C}_{k-l+1}^l\}$, with

- $\mathcal{C}_0^l = \{\{S\} : S \in \mathcal{C}_l\}$.
- $\mathcal{C}_j^l = \{\{T \in \mathcal{C}_0^l : T \subseteq Q\} : Q \in \mathcal{C}_{l+j}\}$, $j = 1, 2, \ldots, k - l + 1$.

We call $(\mathcal{C}_l, v^l, \mathcal{L}^l)$ the l-th truncated game.

Given a TU-game (N, v) and players $i, j \in N$, we say that i, j are indistinguishable in (N, v) if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \backslash \{i, j\}$. We say that i is a nullifying player in (N, v) if $v(S \cup i) = 0$ for all $S \subseteq N$. We say that i is a dummifying player in (N, v) if $v(S \cup i) = \sum_{j \in S \cup i} v(j)$ for all $S \subseteq N$. The above definitions extend to level games in a natural way; thus, for example, we say that i, j are indistinguishable in $(N, v, \mathcal{L})$ if they are indistinguishable in (N, v). Finally, we denote the restriction of (N, v) on $S \subset N$ as (S, v).

We show below a set of properties that characterizes the first proposal. The first two properties are standard in the literature. The property of symmetry among a priori unions on each level says that two indistinguishable coalitions at a particular level, both included in the same coalition at the next level, receive the same. The nullifying player property states that a nullifying player receives zero.

Efficiency A value g for level games satisfies efficiency if, for all $(N, v, \mathcal{L})$, it holds that

$$\sum_{i \in N} g_i(N, v, \mathcal{L}) = v(N).$$

Additivity A value g for level games satisfies additivity if, for all $(N, v, \mathcal{L})$ and $(N, w, \mathcal{L})$, it holds that

$$g(N, v, \mathcal{L}) + g(N, w, \mathcal{L}) = g(N, v + w, \mathcal{L}).$$

Symmetry among a priori unions on each level A value g for level games satisfies symmetry among a priori unions on each level if, for all $(N, v, \mathcal{L})$, with $\mathcal{L} = \{\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{k+1}\}$, $l \in \{0, 1, \ldots, k\}$ and $C, C' \in \mathcal{C}_l$, with $C \cup C' \subseteq C'' \in \mathcal{C}_{l+1}$, indistinguishable in $(\mathcal{C}_l, v^l, \mathcal{L}^l)$, it holds that

$$\sum_{i \in C} g_i(N, v, \mathcal{L}) = \sum_{j \in C'} g_j(N, v, \mathcal{L}).$$

Nullifying player property A value g for level games satisfies the nullifying player property if, for all $(N, v, \mathcal{L})$ and all $i \in N$ nullifying player in $(N, v, \mathcal{L})$, it holds that

$$g_i(N, v, \mathcal{L}) = 0.$$

Theorem 3.1 *The LED-value is the unique value for level games that satisfies efficiency, additivity, symmetry among a priori unions on each level and nullifying player property.*

Proof It is immediate to check that the LED-value satisfies efficiency, additivity, symmetry among a priori unions on each level and nullifying player property. To prove the unicity, consider a value g for level games satisfying efficiency, additivity, symmetry among a priori unions on each level and nullifying player property. Fix N and define, for all $\alpha \in \mathbb{R}$ and all non-empty $T \subseteq N$, the TU-game (N, e_T^α) given by $e_T^\alpha(S) = \alpha$ if $S = T$ and $e_T^\alpha(S) = 0$ if $S \neq T$. If $T = N$, since g satisfies efficiency and symmetry among a priori unions on each level, it is clear that

$$g_i(N, e_N^\alpha, \mathcal{L}) = \frac{\alpha}{\Pi_{l=1}^{k+1} |\lfloor C_l(i) \rfloor|}$$

for any $i \in N$ and $\mathcal{L} = \{\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{k+1}\}$, because all a priori unions in $\mathcal{C}_l$ are indistinguishable in $(\mathcal{C}_l, (e_N^\alpha)^l, \mathcal{L}^l)$, with $l \in \{0, 1, \ldots, k\}$. If $T \subset N$, notice that all players in $N \setminus T$ are nullifying players in (N, e_T^α) and therefore, since g satisfies efficiency and nullifying player property,

$$\sum_{i \in T} g_i(N, e_T^\alpha, \mathcal{L}) = \sum_{i \in N} g_i(N, e_T^\alpha, \mathcal{L}) = e_T^\alpha(N) = 0.$$

Since g satisfies symmetry among a priori unions on each level it is not difficult to check that $g_i(N, e_T^\alpha, \mathcal{L}) = 0$ for all $i \in N$. Finally, the additivity of g and the fact that $v = \sum_{T \subseteq N} e_T^{v(T)}$ imply that

$$g_i(N, v, \mathcal{L}) = \sum_{T \subseteq N} g_i(N, e_T^{v(T)}, \mathcal{L}) = g_i(N, e_N^{v(N)}, \mathcal{L}) = \frac{v(N)}{\Pi_{l=1}^{k+1} |\lfloor C_l(i) \rfloor|}$$

and thus

$$g(N, v, \mathcal{L}) = LED(N, v, \mathcal{L}).$$

□

The property of dummifying level/nullifying player property says that if a player of a dummifying a priori union in the last non trivial partition is a nullifying player in the game restricted to this a priori union, receives zero. This property replaces the nullifying player property in the characterization of the first proposal.

Dummifying level/nullifying player property A value g for level games satisfies the dummifying level/nullifying player property if, for all $(N, v, \mathcal{L})$, with $\mathcal{L} = \{\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{k+1}\}$, and all $i \in N$ nullifying player in $(C_k(i), v)$ such that $C_k(i)$ is a dummifying player in $(\mathcal{C}_k, v^k)$, then it holds that

$$g_i(N, v, \mathcal{L}) = 0.$$

Theorem 3.2 *The LESD*[1]*-value is the unique value for level games that satisfies efficiency, additivity, symmetry among a priori unions on each level and dummifying level/nullifying player property.*

Proof It is immediate to check that the LESD[1]-value satisfies efficiency, additivity, symmetry among a priori unions on each level and dummifying level/nullifying player property. To prove the unicity, consider a value g for level games satisfying efficiency, additivity, symmetry among a priori unions on each level and dummifying level/nullifying player property. Take $(N, v, \mathcal{L})$ with $\mathcal{L} = \{\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{k+1}\}$ and $\mathcal{C}_k = \{C_1, C_2, \ldots, C_m\}$. Now, define the TU-game (N, v^*) given by

$$v^*(S) = \sum_{C \in \mathcal{C}_k : C \subseteq S} v(C) = \sum_{j=1}^{m} v^{C_j}(S)$$

for all $S \subseteq N$, where $v^{C_j}(S) = v(C_j)$ if $C_j \subseteq S$ and $v^{C_j}(S) = 0$ otherwise. Take $C_r \in \mathcal{C}_k$. Since g satisfies efficiency,

$$\sum_{i \in N} g_i(N, v^{C_r}, \mathcal{L}) = v^{C_r}(N) = v(C_r).$$

All a priori unions $C_j \in \mathcal{C}_k$ are dummifying players in $(\mathcal{C}_k, (v^{C_r})^k)$ and all players $i \in C_j$, with $j \neq r$, are nullifying players in $(C_k(i), v^{C_r})$. By dummifying level/nullifying player property, $g_i(N, v^{C_r}, \mathcal{L}) = 0$ for all $i \notin C_r$. For each level $l \in \{0, 1, \ldots, k-1\}$, a priori unions $C, C' \in \mathcal{C}_l$, with $C \cup C' \subseteq C'' \in \mathcal{C}_{l+1}$ and $C \cup C' \subseteq C_r$, are indistinguishable in $(\mathcal{C}_l, (v^{C_r})^l, \mathcal{L}^l)$, therefore symmetry among a priori unions on each level implies that, for all $i \in C_r$,

$$g_i(N, v^{C_r}, \mathcal{L}) = \frac{v(C_r)}{\Pi_{l=1}^{k} |\lfloor C_l(i) \rfloor|} = \frac{v(C_k(i))}{\Pi_{l=1}^{k} |\lfloor C_l(i) \rfloor|}.$$

Using the additivity of g, for all $i \in N$,

$$g_i(N, v^*, \mathcal{L}) = \frac{v(C_k(i))}{\Pi_{l=1}^{k}|\lfloor C_l(i)\rfloor|}. \tag{1}$$

Define now the characteristic functions $v^{**} = v - v^*$ and, for all $\alpha \in \mathbb{R}$ and all non-empty $T \subseteq N$, e_T^α given by $e_T^\alpha(S) = \alpha$ if $S = T$ and $e_T^\alpha(S) = 0$ if $S \neq T$. It is clear that $v^{**} = \sum_{T \subseteq N} e_T^{v^{**}(T)}$. If $T = N$, since g satisfies efficiency and symmetry among a priori unions on each level, it is clear that, for all $i \in N$, $g_i(N, e_N^{v^{**}(N)}, \mathcal{L})$ is given by

$$\frac{v^{**}(N)}{\Pi_{l=1}^{k+1}|\lfloor C_l(i)\rfloor|} = \frac{v(N) - \sum_{C \in \mathcal{C}_k} v(C)}{\Pi_{l=1}^{k+1}|\lfloor C_l(i)\rfloor|} = \frac{v(C_{k+1}(i)) - \sum_{C \in \lfloor C_{k+1}(i)\rfloor} v(C)}{\Pi_{l=1}^{k+1}|\lfloor C_l(i)\rfloor|}.$$

If $T \subset N$, consider two cases:

- Take $T = \cup_{C \in R} C$, with $\emptyset \subset R \subset \mathcal{C}_k$. For all $C_j \in \mathcal{C}_k$, if $T \neq C_j$ then $e_T^{v^{**}(T)}(C_j) = 0$ and if $T = C_j$ then $e_T^{v^{**}(T)}(C_j) = v^{**}(C_j) = 0$. Hence, it is easy to see that all a priori unions in $\mathcal{C}_k \setminus R$ are dummifying players in $(\mathcal{C}_k, (e_T^{v^{**}(T)})^k)$. Also, since all players $i \in N \setminus T$ are nullifying players in $(C_k(i), e_T^{v^{**}(T)})$, dummifying level/nullifying player property implies that $g_i(N, e_T^{v^{**}(T)}, \mathcal{L}) = 0$ for all $i \in N \setminus T$. Notice that all a priori unions in R are indistinguishable in $(\mathcal{C}_k, (e_T^{v^{**}(T)})^k, \mathcal{L}^k)$, therefore by symmetry among a priori unions on each level, $\sum_{i \in C} g_i(N, e_T^{v^{**}(T)}, \mathcal{L}) = \sum_{i \in C'} g_i(N, e_T^{v^{**}(T)}, \mathcal{L})$ for all $C, C' \in R$; notice also that

$$\sum_{i \in T} g_i(N, e_T^{v^{**}(T)}, \mathcal{L}) = \sum_{i \in N} g_i(N, e_T^{v^{**}(T)}, \mathcal{L}) = e_T^{v^{**}(T)}(N) = 0,$$

  therefore $\sum_{i \in C} g_i(N, e_T^{v^{**}(T)}, \mathcal{L}) = 0$ for all $C \in R$. To conclude, symmetry among a priori unions on each level implies that $g_i(N, e_T^{v^{**}(T)}, \mathcal{L}) = 0$ for all $i \in C \in R$, and therefore for all $i \in T$.
- For any other $T \subset N$ that is not in the previous case, the game $(\mathcal{C}_k, (e_T^{v^{**}(T)})^k)$ satisfies that $(e_T^{v^{**}(T)})^k(R) = 0$ for all $R \subseteq \mathcal{C}_k$ and, thus, all a priori unions in $\mathcal{C}_k$ are indistinguishable and dummifying players in $(\mathcal{C}_k, (e_T^{v^{**}(T)})^k)$. If $i \in N \setminus T$, then i is a nullifying player in $(C_k(i), e_T^{v^{**}(T)})$ and dummifying level/nullifying player property implies that $g_i(N, e_T^{v^{**}(T)}, \mathcal{L}) = 0$. Analogously as in the previous case, symmetry among a priori unions on each level implies that $g_i(N, e_T^{v^{**}(T)}, \mathcal{L}) = 0$ for all $i \in T$.

Now additivity implies that, for all $i \in N$,

$$g_i(N, v^{**}, \mathcal{L}) = \sum_{T \subseteq N} g_i(N, e_T^{v^{**}(T)}, \mathcal{L}) = \frac{v^{**}(N)}{\Pi_{l=1}^{k+1}|\lfloor C_l(i)\rfloor|}. \tag{2}$$

Finally, from (1), (2), additivity and $v = v^* + v^{**}$ it is clear that

$$g(N, v, \mathcal{L}) = LESD^1(N, v, \mathcal{L}).$$

□

To characterize the third proposal, we use the same set of properties included in the previous theorems to characterize the first and second proposal, with a unique modification. The property of dummifying a priori unions for a player replaces the nullifying player property of the first proposal or the dummifying level/nullifying player property of the second. The dummifying a priori unions for a player property states that if the a priori union containing a particular player is a dummifying a priori union at every level, then this player receives his/her individual worth.

Dummifying a priori unions for a player property A value g for level games satisfies the dummifying a priori unions for a player property if, for all $(N, v, \mathcal{L})$ with $\mathcal{L} = \{\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{k+1}\}$, and all $i \in N$ such that $C_l(i) \in \mathcal{C}_l$ is a dummifying player in $(\mathcal{C}_l, v^l)$ for all $l \in \{0, 1, \ldots, k\}$, then it holds that

$$g_i(N, v, \mathcal{L}) = v(i).$$

Theorem 3.3 *The LESD^2-value is the unique value for level games that satisfies efficiency, additivity, symmetry among a priori unions on each level and dummifying a priori unions for a player property.*

Proof It is immediate to check that the LESD^2-value satisfies efficiency, additivity, symmetry among a priori unions on each level and dummifying a priori unions for a player property. To prove the unicity, consider a value g for level games that satisfies efficiency, additivity, symmetry among a priori unions on each level and dummifying a priori unions for a player property. Take $(N, v, \mathcal{L})$ with $\mathcal{L} = \{\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{k+1}\}$ and define v^a, for all $S \subseteq N$, by

$$v^a(S) = \sum_{i \in S} v(i).$$

Define $\tilde{v}$, for all $S \subseteq N$, by

$$\tilde{v}(S) = (v - v^a)(S) - \sum_{C \in \mathcal{C}_k : C \subseteq S} (v - v^a)(C).$$

Now define v_l^*, for all $l \in \{1, 2, \ldots, k\}$ with $\mathcal{C}_l = \{C_{l,1}, C_{l,2}, \ldots, C_{l,|\mathcal{C}_l|}\}$, by

$$v_l^*(S) = \sum_{C \in \mathcal{C}_l : C \subseteq S} \left(v(C) - \sum_{C' \in \mathcal{C}_{l-1} : C' \subseteq C} v(C') \right) = \sum_{j=1}^{|\mathcal{C}_l|} v_l^{C_{l,j}}(S)$$

for all $S \subseteq N$, where $v_l^{C_{l,j}}(S) = v(C_{l,j}) - \sum_{C' \in \mathcal{C}_{l-1} : C' \subseteq C_{l,j}} v(C')$ if $C_{l,j} \subseteq S$ and $v_l^{C_{l,j}}(S) = 0$ otherwise. For all $l \in \{0, 1, \ldots, k\}$, $C_l(i) \in \mathcal{C}_l$ is a dummifying player in $(\mathcal{C}_l, (v^a)^l)$; hence, dummifying a priori unions for a player property implies that, for all $i \in N$,

$$g_i(N, v^a, \mathcal{L}) = v^a(i) = v(i). \tag{3}$$

Take now $l \in \{1, 2, \ldots, k\}$ and $C_r \in \mathcal{C}_l$. Since g satisfies efficiency,

$$\sum_{i \in N} g_i(N, v_l^{C_r}, \mathcal{L}) = v_l^{C_r}(N) = v(C_r) - \sum_{C' \in \mathcal{C}_{l-1}: C' \subseteq C_r} v(C').$$

For all $l' \in \{0, 1, \ldots, l\}$, all a priori unions $C \in \mathcal{C}_{l'}$ such that $C \nsubseteq C_r$ are dummifying players in $(\mathcal{C}_{l'}, (v_l^{C_r})^{l'})$. By dummifying a priori unions for a player property, $g_i(N, v_l^{C_r}, \mathcal{L}) = v_l^{C_r}(i) = 0$ for all $i \notin C_r$. And since all players in C_r are indistinguishable in $(N, v_l^{C_r}, \mathcal{L})$, symmetry among a priori unions on each level implies that, for all $i \in C_r$,

$$g_i(N, v_l^{C_r}, \mathcal{L}) = \frac{v_l^{C_r}(C_r)}{\Pi_{l'=1}^{l+1} |\lfloor C_{l'}(i) \rfloor|}.$$

Using additivity, for all $i \in N$,

$$g_i(N, v_l^*, \mathcal{L}) = \frac{v_l^{C_l(i)}(C_l(i))}{\Pi_{l'=1}^{l+1} |\lfloor C_{l'}(i) \rfloor|}. \tag{4}$$

Define, for all $\alpha \in \mathbb{R}$ and all non-empty $T \subseteq N$, the TU-game (N, e_T^α) given by $e_T^\alpha(S) = \alpha$ if $S = T$ and $e_T^\alpha(S) = 0$ if $S \neq T$. Take now into account that $\tilde{v} = \sum_{T \subseteq N} e_T^{\tilde{v}(T)}$. If $T = N$, since g satisfies efficiency and symmetry among a priori unions on each level, it is clear that for all $i \in N$,

$$g_i(N, e_N^{\tilde{v}(N)}, \mathcal{L}) = \frac{\tilde{v}(N)}{\Pi_{l=1}^{k+1} |\lfloor C_l(i) \rfloor|}.$$

If $T \subset N$, consider two cases:

- Take $T = \cup_{C \in R} C$, with $\emptyset \subset R \subset \mathcal{C}_k$. For all $C_j \in \mathcal{C}_k$, if $T \neq C_j$ then $e_T^{\tilde{v}(T)}(C_j) = 0$ and if $T = C_j$ then $e_T^{\tilde{v}(T)}(C_j) = \tilde{v}(C_j) = 0$. Also, if $Q \subseteq \mathcal{C}_k$ with $(\mathcal{C}_k \backslash R) \cap Q \neq \emptyset$, then $e_T^{\tilde{v}(T)}(\cup_{C \in Q} C) = 0$. For all $i \in N \setminus T$ and $l \in \{0, 1, \ldots, k\}$, $C_l(i) \in \mathcal{C}_l$ is a dummifying player in $(\mathcal{C}_l, (e_T^{\tilde{v}(T)})^l, \mathcal{L}^l)$, therefore dummifying a priori unions for a player property implies that $g_i(N, e_T^{\tilde{v}(T)}, \mathcal{L}) = e_T^{\tilde{v}(T)}(i) = 0$. Notice that all a priori unions in R are indistinguishable in $(\mathcal{C}_k, (e_T^{\tilde{v}(T)})^k, \mathcal{L}^k)$, therefore by symmetry among a priori unions on each level, $\sum_{i \in C} g_i(N, e_T^{\tilde{v}(T)}, \mathcal{L}) =$

$\sum_{i \in C'} g_i(N, e_T^{\tilde{v}(T)}, \mathcal{L})$ for all $C, C' \in R$; and notice also that

$$\sum_{i \in T} g_i(N, e_T^{\tilde{v}(T)}, \mathcal{L}) = \sum_{i \in N} g_i(N, e_T^{\tilde{v}(T)}, \mathcal{L}) = e_T^{\tilde{v}(T)}(N) = 0,$$

therefore $\sum_{i \in C} g_i(N, e_T^{\tilde{v}(T)}, \mathcal{L}) = 0$ for all $C \in R$. Hence, symmetry among a priori unions on each level implies that $g_i(N, e_T^{\tilde{v}(T)}, \mathcal{L}) = 0$ for all $i \in T$.

- For any other $T \subset N$ that is not in the previous case, notice that $e_T^{\tilde{v}(T)}(\cup_{C \in Q} C) = 0$ for all $Q \subseteq \mathcal{C}_k$. If $i \in N \setminus T$, then a priori union $C_l(i) \in \mathcal{C}_l$ is a dummifying player in $(\mathcal{C}_l, (e_T^{\tilde{v}(T)})^l, \mathcal{L}^l)$ for all $l \in \{0, 1, \ldots, k\}$, therefore by dummifying a priori unions for a player property, $g_i(N, e_T^{\tilde{v}(T)}, \mathcal{L}) = e_T^{\tilde{v}(T)}(i) = 0$. Analogously as in the previous case, symmetry among a priori unions on each level implies that $g_i(N, e_T^{\tilde{v}(T)}, \mathcal{L}) = 0$ for all $i \in T$.

Now additivity implies that, for all $i \in N$,

$$g_i(N, \tilde{v}, \mathcal{L}) = \sum_{T \subseteq N} g_i(N, e_T^{\tilde{v}(T)}, \mathcal{L}) = \frac{\tilde{v}(N)}{\Pi_{l=1}^{k+1} |\lfloor C_l(i) \rfloor|}. \tag{5}$$

Finally, from (3), (4), (5), additivity and $v = v^a + \tilde{v} + \sum_{l=1}^{k} v_l^*$ it is clear that

$$g(N, v, \mathcal{L}) = LESD^2(N, v, \mathcal{L}).$$

□

Finally, we present a set of properties that characterize the last proposal. Efficiency and additivity are common to the previous characterizations. The two new properties are: weak symmetry among a priori unions on each level and dummifying player property. The dummifying player property states that a dummifying player receives his/her individual worth. Similar to the nullifying player of the first proposal, this property only takes into account the characteristic function of the game, and does not depend on the level structure. The weak symmetry among a priori unions property on each level is a weaker version of the symmetry among a priori unions property on each level (used to characterize the first proposal) because only indistinguishable coalitions formed by players with individual worth equal to zero receive the same. It is clear that if a value satisfies the property of symmetry among a priori unions on each level, it also satisfies the weak property of symmetry among a priori unions on each level. Therefore, there does not exist a value that satisfies efficiency, additivity, symmetry among a priori unions on each level and dummifying player property.

Dummifying player property A value g for level games satisfies the dummifying player property if, for all $(N, v, \mathcal{L})$ and all $i \in N$ dummifying player in $(N, v, \mathcal{L})$, it holds that

$$g_i(N, v, \mathcal{L}) = v(i).$$

Weak symmetry among a priori unions on each level. A value g for level games satisfies weak symmetry among a priori unions on each level if, for all $(N, v, \mathcal{L})$, with $\mathcal{L} = \{\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{k+1}\}$ and $v(i) = 0$ for all $i \in N$, $l \in \{0, 1, \ldots, k\}$ and $C, C' \in \mathcal{C}_l$, with $C \cup C' \subseteq C'' \in \mathcal{C}_{l+1}$, indistinguishable in $(\mathcal{C}_l, v^l, \mathcal{L}^l)$, it holds that

$$\sum_{i \in C} g_i(N, v, \mathcal{L}) = \sum_{j \in C'} g_j(N, v, \mathcal{L}).$$

Theorem 3.4 *The LESD^3-value is the unique value for level games that satisfies efficiency, additivity, weak symmetry among a priori unions on each level and dummifying player property.*

Proof It is immediate to check that the LESD^3-value satisfies efficiency, additivity, weak symmetry among a priori unions on each level and dummifying player property. To prove the unicity, consider a value g for level games satisfying efficiency, additivity, weak symmetry among a priori unions on each level and dummifying player property. Take $(N, v, \mathcal{L})$ and define v^a, for all $S \subseteq N$, by

$$v^a(S) = \sum_{i \in S} v(i).$$

Define $v^0 = v - v^a$. Additivity implies that

$$g(N, v, \mathcal{L}) = g(N, v^a, \mathcal{L}) + g(N, v^0, \mathcal{L}). \tag{6}$$

Since i is a dummifying player in (N, v^a), for all $i \in N$, dummifying player property implies that

$$g_i(N, v^a, \mathcal{L}) = v^a(i) = v(i). \tag{7}$$

Now, using for $(N, v^0, \mathcal{L})$ analogous arguments as those used in the proof of Theorem 3.1, it is easy to see that efficiency, additivity, weak symmetry among a priori unions on each level and dummifying player property imply that

$$g(N, v^0, \mathcal{L}) = LED(N, v^0, \mathcal{L}). \tag{8}$$

Finally, from (6), (7) and (8) it is clear that

$$g(N, v, \mathcal{L}) = LESD^3(N, v, \mathcal{L}).$$

□

4 Concluding remarks

In this article we have extended the equal division value and the equal surplus division value to level games. In the latter case, we have relied on three extensions of such a value for cooperative games with a priori unions described in Alonso-Meijide et al.

(2020). In Gonçalves-Dosantos and Alonso-Meijide (2023) two new variants of the equal surplus value for cooperative games with a priori unions are proposed. A topic for future work is the extension of these two variants to games with levels.

One of the advantages of the values based on the equal sharing criterion is that they are generally calculable in moderate times even for games with many players, because they do not make use of the full characteristic function of the game. In any case, in order for managers to be able to make use of the values considered here, it would be convenient to make available to the community a computer tool developed in free software to calculate them. In the near future it would be interesting to generate such a tool.

Acknowledgements This work is part of the R+D+I project grants MTM2017-87197-C3-1-P, MTM2017-87197- C3-3-P, PID2021-124030NB-C31 and PID2021-124030NB-C32, funded by MCIN/AEI/10.13039/501100011033/ and by "ERDF A way of making Europe"/EU. This research was also funded by Grupos de Referencia Competitiva ED431G-2019/01, ED431C-2020/14 and ED431C-2021/24 from the Consellería de Cultura, Educación e Universidades, Xunta de Galicia. During the completion of this work, J. C. Gonçalves-Dosantos is a "Margarita Salas" postdoctoral researcher, carrying out a research stay at the CIO, *Centro de Investigación Operativa* of the Miguel Hernández University of Elche (Spain), to which he is grateful for the welcome. Funding for open access charge: Universidade da Coruña/CISUG

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Declarations

Conflicts of interest The authors state that there is no conflict of interest.

References

Alonso-Meijide JM, Costa J, García-Jurado I, Gonçalves-Dosantos JC (2020) On egalitarian values for cooperative games with a priori unions. TOP 28:672–688

Casajus A, Hüttner F (2014) Null, nullifying, or dummifying players: the difference between the Shapley value, the equal division value, and the equal surplus division value. Econ Lett 122:167–169

Driessen TSH, Funaki Y (1991) Coincidence of and collinearity between game theoretic solutions. OR Spectr 13:15–30

Gonçalves-Dosantos JC, Alonso-Meijide JM (2023) New results on egalitarian values for games with a priori unions. Optimization 72:861–881

Hu XF, Li DF (2021) The equal surplus division value for cooperative games with a level structure. Group Decis Negot 30:1315–1341

Owen G (1977) Values of games with a priori unions. In: Henn R, Moeschlin O (eds) Mathematical economics and game theory. Springer, Berlin, pp 76–88

Selten R (1972) Equal share analysis of characteristic function experiments. In: Sauermann H (ed) Contributions to experimental economics III. Mohr, Siebeck, pp 130–165 van den Brink R (2007) Null or nullifying players: the difference between the Shapley value and equal division solutions. J Econ Theory 136:767–775

Winter E (1989) A value for cooperative games with levels structure of cooperation. Int J Game Theory 18:227–240