Universidade de Aveiro Departamento de Matemática
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CATARINA MARIA NETO DA CRUZ

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THE NON-EXISTENCE OF PERFECT 2-ERROR CORRECTING LEE CODES OF WORD LENGTH 7 OVER Z

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Tese apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, realizada sob a orientação científica da Professora Doutora Ana Maria Reis D'Azevedo Breda, Professora Associada com Agregação do Departamento de Matemática da Universidade de Aveiro, do Professor Doutor Peter Horak, Professor do Departamento de Artes e Ciências Interdisciplinares da Universidade de Washington, Tacoma, EUA e da Professora Doutora Maria Raquel Rocha Pinto, Professora Auxiliar do Departamento de Matemática da Universidade de Aveiro

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Aos meus pais

o júri

presidente

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palavras-chave Códigos de Lee perfeitos, Conjetura de Golomb-Welch, Pavimentações, Métrica de Lee.

resumo

A conjetura de Golomb-Welch estabelece que não existem códigos de Lee perfeitos, corretores de *r*-erros, de palavras de comprimento *n* sobre Z para $n \geq 3$ e $r \geq 2$. Este problema tem recebido particular atenção devido à sua importância em aplicações em várias áreas que não apenas a da matemática e das ciências da computação. Apesar de terem sido obtidos muitos resultados no sentido de provar a conjetura, esta tem resistido estando estabelecida apenas para alguns valores particulares de *n* e *r*, nomeadamente: $3 \leq n \leq 5$ e $r \geq 2$; $n = 6$ e $r = 2$.

Nesta tese é dada uma contribuição que reforça a conjetura, sendo provada a não existência de códigos de Lee perfeitos, corretores de 2-erros, de palavras de comprimento 7 sobre Z.

keywords

Perfect Lee codes, Golomb-Welch conjecture, Tilings, Lee metric.

abstract

The Golomb-Welch conjecture states that there is no perfect *r*-error correcting Lee code of word length *n* over Z for $n \geq 3$ and $r \geq 2$. This problem has received great attention due to its importance in applications in several areas beyond mathematics and computer sciences. Many results on this subject have been achieved, however the conjecture has resisted, although its validity has been proved for some particular values of *n* and *r*, namely: $3 \leq n \leq 5$ and $r \geq 2$; $n = 6$ and $r = 2$.

Here we give a contribution for the proof of the Golomb-Welch conjecture which reinforces it, proving the non-existence of perfect 2-error correcting Lee codes of word length 7 over Z.

Contents

Introduction

Tiling problems have literally been popular for thousands of years. Although some of them belong to recreational mathematics, nowadays many of these problems are motivated by real-life applications.

Problems involving space tilings are common in coding theory. In fact, special types of tilings can be regarded as error correcting codes which are essential on correct transmission of information over a noisy channel, see [9] and [14]. For example, tilings of $\mathbb{R}^n$ by crosses and semicrosses constitute different types of error correcting codes, see [9] and [21]. Another application of these tilings can be found in [18], where tilings by crosses are related to both a disturb and a retention error in flash memories. The existence of such tilings has been researched by various authors for special cases, in [6] we completely solve the problem for the two-dimensional Euclidean space.

Here, we are interested in dealing with tilings of spaces by Lee spheres. The Lee metric is frequently used in coding theory. Since its first applications, related with signal transmission over noisy channels, see [14] and [23], many studies involving the Lee metric have appeared, in particular, studies of different types of codes in the Lee metric. There exists an extensive literature on codes in the Lee metric. See, for instance, [2], [5] and [17]. The interest in Lee codes has been increasing due to their several applications. Some examples can be seen in [3], [4], [7], [16] and [22].

The study of tiling spaces by Lee spheres was introduced by Golomb and Welch ([8] and [9]) which related these tilings with error correcting codes considering the center of a Lee sphere as a codeword and the other elements of the sphere as words which are decoded by the central codeword. When a Lee sphere of radius r tiles the n-dimensional space, the set of all centers of the Lee spheres, that is, the set of all codewords, produces a perfect r-error correcting Lee code of word length n.

In the study of Lee codes particular attention is given to the perfect r-error correcting Lee codes of word length n over $\mathcal{S}$, with $\mathcal{S} = \mathbb{Z}$ or $\mathcal{S} = \mathbb{Z}_q$, where r, n and q are positive integer numbers. In fact, tilings of $\mathbb{Z}^n$ or $\mathbb{Z}_q^n$ by Lee spheres of radius r, whose set of all codewords is denoted by $\mathrm{PL}(n,r)$ or $\mathrm{PL}(n,r,q)$ code, respectively, have been central subjects in the area of the Lee codes. We can relate these two types of codes since, if n, r and q are positive integer numbers, with $q \geq 2r+1$, so that there exists a $\mathrm{PL}(n,r,q)$ code, then the periodic repetition of this code results in a $\mathrm{PL}(n,r)$ code. As an immediate consequence, if n and r are positive integer numbers so that there is no $\mathrm{PL}(n,r)$ code, then no $\mathrm{PL}(n,r,q)$ code exists for $q \geq 2r+1$.

This research work is focused on the study of $\mathrm{PL}(n,r)$ codes, that is, in the study of tilings of $\mathbb{Z}^n$ by Lee spheres of radius r.

By a Lee sphere of radius r in $\mathbb{Z}^n$ centered at $Z = (z_1, \ldots, z_n) \in \mathbb{Z}^n$ we understand the set

$$\left\{ X = (x_1, \ldots, x_n) \in \mathbb{Z}^n : \sum_{i=1}^{n} |x_i - z_i| \leq r \right\}.$$

Considering the n-dimensional space $\mathbb{R}^n$, a small step is needed to establish a relation between tilings of $\mathbb{Z}^n$ and tilings of $\mathbb{R}^n$ by Lee spheres. In $\mathbb{R}^n$ the unit cube centered at $X = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is the set

$$\left\{ Y = (y_1, \ldots, y_n) \in \mathbb{R}^n : |y_i - x_i| \leq \frac{1}{2},\ 1 \leq i \leq n \right\}.$$

Considering $Z \in \mathbb{Z}^n$, a Lee sphere of radius r in $\mathbb{R}^n$ centered at Z is the union of the unit cubes centered at $X \in \mathbb{Z}^n$ satisfying $\sum_{i=1}^{n} |x_i - z_i| \leq r$. In Figure 1 are depicted Lee spheres of radius 2 in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. A $\mathrm{PL}(n,r)$ code exists if an only if there exists a tiling of $\mathbb{R}^n$ by Lee spheres of radius r.

The question "*for what values of n and r does the n-dimensional Lee sphere of radius r tile a n-dimensional space?*" was formulated by Golomb and Welch in [9], where they proved:

i) n-dimensional Lee sphere of radius 1 tiles the n-dimensional space for any positive integer n;

ii) for each $r \geq 1$, there exists a tiling of the n-dimensional space by Lee spheres of radius r for $n = 1, 2$.

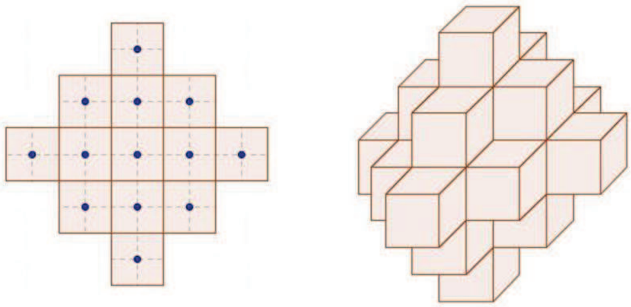

Figure 1: Lee spheres of radius 2 in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

In other words, there exist PL$(n, 1)$, PL$(1, r)$ and PL$(2, r)$ codes for any positive integer numbers n and r, respectively.

Based in these results Golomb and Welch have conjectured:

Conjecture (Golomb-Welch). *There is no PL(n, r) code for $n \geq 3$ and $r \geq 2$.*

Having in view PL(n, r, q) codes, the Golomb-Welch conjecture implies that there are no PL(n, r, q) codes for $n \geq 3$, $r \geq 2$ and $q \geq 2r + 1$.

Motivated by the Golomb-Welch conjecture, the existence and enumeration of PL(n, r) and PL(n, r, q) codes have captured the attention of many mathematicians. These codes have been extensively studied by several authors, however the conjecture is still to be solved.

In [9] Golomb and Welch have proved that:

i) there is no PL$(3, 2)$ code;

ii) there is no PL(n, r) code for $n > 4$ and $r > \rho_n$, not being specified the value of ρ_n.

There are other results supporting this conjecture. In fact, in [10] the conjecture is stated for $n = 3$ and $r \geq 2$. Špacapan [19] has showed the non-existence of a PL(n, r) code for $n = 4$ and $r \geq 2$. Horak has proved in [12] that there are no PL(n, r) codes for $3 \leq n \leq 5$ and $r \geq 2$. In [11] Horak has also stated the conjecture for the parameters $n = 6$ and $r = 2$. These are the only values of the parameters for which the Golomb-Welch conjecture is known to be true.

It seems that an immediate generalization of the proofs of the referred cases of the conjecture cannot be easily done to the unproved cases. The difficulty in proving the Golomb-Welch conjecture in its generality has led some authors to prove the non-existence of $\mathrm{PL}(n, r, q)$ codes. Next we present some of the known results.

Golomb and Welch [8] constructed a $\mathrm{PL}(n, r, q)$ code for the parameters:

i) $(1, r, 2r + 1)$;

ii) $(2, r, r^2 + (r + 1)^2)$;

iii) $(n, 1, 2n + 1)$.

In [15] Post has proved that $\mathrm{PL}(n, r, q)$ codes, with $q \geq 2r + 1$, do not exist for:

i) $3 \leq n \leq 5$ and $r \geq n - 1$;

ii) $n \geq 6$ and $r \geq \frac{\sqrt{2}}{2}n - \frac{1}{4}(3\sqrt{2} - 2)$.

These results were improved by Špacapan in [20], where it is shown the non-existence of $\mathrm{PL}(n, r, q)$ codes for $q \geq 2r + 1$ and $r \geq n \geq 3$.

Astola [1] proved the non-existence of $\mathrm{PL}(n, 2, q)$ codes for:

i) $q = 13$;

ii) q not divisible by a prime of the form $4m + 1$;

iii) $q = p^k$, p is a prime, $p \neq 13$, $p < \sqrt{2n^2 + 2n + 1}$.

Some authors have studied the non-existence of special codes imposing additional conditions. This is the case of codes in which the set of all codewords forms a group with respect to the vector addition, the so called linear codes. See Horak and Grošek [13].

As stated previously, a Lee sphere of radius 1 tiles the n-dimensional space for any positive integer n. Following an intuitive and geometric reasoning, it seems that the bigger the radius of the Lee sphere is more difficult is to tile the space with the sphere. Then, it seems that the most difficult cases of the Golomb-Welch conjecture to deal with are those in which $r = 2$.

Since the non-existence of $\mathrm{PL}(n, 2)$ codes for $3 \leq n \leq 6$ has already been proved, our goal is to give a contribution to the establishment of the Golomb-Welch conjecture proving that there is no PL$(7,2)$ code. It should be pointed out that Horak and Grošek, in [13], have proved, using a new approach, the non-existence of linear PL$(n,2)$ codes for $7 \leq n \leq 11$.

Our strategy to prove the non-existence of PL$(7,2)$ codes is based on the assumption of their existence, being focused on the cardinality restrictions of some codewords sets. We assume the existence of a PL$(7,2)$ code $\mathcal{M}$ and, without loss of generality, we suppose that $O \in \mathcal{M}$, where $O = (0,\ldots,0)$. Since we are dealing with Lee spheres of radius 2, all words $W \in \mathbb{Z}^7$, with $W = (w_1,\ldots,w_7)$, satisfying $\sum_{i=1}^{7} |w_i| \leq 2$ are covered by the codeword O. Having in view the definition of PL$(7,2)$ code, each word $W \in \mathbb{Z}^7$ which is distant three units from O must be covered by a unique codeword of $\mathcal{M}$. These words have to be covered by codewords which dist five units from O, being these codewords of the types $[\pm 5]$, $[\pm 4, \pm 1]$, $[\pm 3, \pm 2]$, $[\pm 3, \pm 1^2]$, $[\pm 2^2, \pm 1]$, $[\pm 2, \pm 1^3]$ and $[\pm 1^5]$. Denoting, respectively, by $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ the sets containing the codewords of these types, we prove the non-existence of PL$(7,2)$ codes showing that it is not possible to cover all the referred words without superposing Lee spheres centered at codewords of $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E} \cup \mathcal{F} \cup \mathcal{G}$.

Next we present a brief outline of the contents of each chapter of the thesis.

In Chapter 1, basic notions and notations used throughout the document are given. Here, some necessary conditions for the existence of PL$(n,2)$ codes, when $n \geq 7$, are presented.

In Chapter 2, our study is concentrated in PL$(7,2)$ codes, being presented necessary conditions for their existence based on restrictions on the cardinality of index subsets of $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E} \cup \mathcal{F} \cup \mathcal{G}$. Particular attention is given to the sets $\mathcal{G}_i$ being proved that $3 \leq |\mathcal{G}_i| \leq 8$ for any $i \in \mathcal{I} = \{+1,+2,\ldots,+7,-1,-2,\ldots,-7\}$. In this chapter are also established relations between the cardinality of index subsets when $|\mathcal{G}_i|$ assume different admissible values.

The following chapters are dedicated to the analysis of $|\mathcal{G}_i|$, $i \in \mathcal{I}$. In Chapters 3, 4, 5 and 6 we refine the variation of $|\mathcal{G}_i|$, proving that $|\mathcal{G}_i| \neq 3,4,5,8$ for any $i \in \mathcal{I}$.

In Chapter 7, under the assumption $6 \leq |\mathcal{G}_i| \leq 7$ for any $i \in \mathcal{I}$, we conclude the proof of the main result:

Theorem. *There is no PL$(7,2)$ code.*

The last section of the document is devoted to conclusions, being also presented our intention about some future work.

Chapter 1

Perfect error correcting Lee codes

In this chapter we introduce the notion of a perfect error correcting Lee code and some basic results involving this notion. The notation that will be used throughout the document is mostly based on Horak work [11]. Particular attention will be given to perfect 2-error correcting Lee codes, where necessary conditions for their existence will be presented.

1.1 Definitions

Let $(\mathcal{S}, \mu)$ be a metric space, where $\mathcal{S}$ is a nonempty set and μ a metric on $\mathcal{S}$. Any subset $\mathcal{M}$ of $\mathcal{S}$ satisfying $|\mathcal{M}| \geq 2$ is a **code**. The elements of $\mathcal{S}$ are called **words** and, in particular, the elements of a code $\mathcal{M}$ are called **codewords**.

A sphere centered at $W \in \mathcal{S}$ with radius r, denoted by $S(W, r)$, is defined as follows

$$S(W, r) = \{V \in \mathcal{S} : \mu(V, W) \leq r\}.$$

If $W \in \mathcal{M}$ and $V \in S(W, r)$, with $V \neq W$, then we say that **the codeword** W **covers the word** V.

Definition 1.1 *A code* $\mathcal{M}$ *is a* ***perfect*** r***-error correcting code*** *if:*

i) $S(W, r) \cap S(V, r) = \varnothing$ *for any two distinct codewords* W *and* V *in* $\mathcal{M}$*;*

ii) $\bigcup_{W \in \mathcal{M}} S(W, r) = \mathcal{S}$.

In other words, $\mathcal{M}$ is a perfect r-error correcting code if the spheres of radius r centered at codewords of $\mathcal{M}$ form a partition of $\mathcal{S}$. Equivalently, $\mathcal{M}$ is a perfect r-error correcting code if the spheres of radius r centered at codewords of $\mathcal{M}$ tile $\mathcal{S}$.

When a code $\mathcal{M}$ satisfies the condition $i)$ in Definition 1.1, we say that $\mathcal{M}$ is a **r-error correcting code**.

We are interested in dealing with metric spaces $(\mathbb{Z}^n, \mu_L)$, where $\mathbb{Z}^n$ is the n-fold Cartesian product of the set of the integer numbers, with n a positive integer number, and μ_L is the **Lee metric**, that is, for any $W, V \in \mathbb{Z}^n$, with $W = (w_1, \ldots, w_n)$ and $V = (v_1, \ldots, v_n)$, the Lee distance between W and V, shortly $\mu_L(W, V)$, is given by

$$\mu_L(W, V) = \sum_{i=1}^{n} |w_i - v_i|.$$

If $\mathcal{M} \subset \mathbb{Z}^n$ is a perfect r-error correcting code of $(\mathbb{Z}^n, \mu_L)$, then $\mathcal{M}$ is called a **perfect r-error correcting Lee code of word length n over** $\mathbb{Z}$, shortly a **PL(n,r) code**.

The following result gives us a necessary and sufficient condition on the Lee distance between two words to avoid superposition of spheres centered at them.

Lemma 1.1 *Given $W, V \in \mathbb{Z}^n$, with $W \neq V$, and r a positive integer number, $S(W, r) \cap S(V, r) = \varnothing$ if and only if $\mu_L(W, V) \geq 2r + 1$.*

Proof. Let W and V be distinct elements in $\mathbb{Z}^n$, with $W = (w_1, \ldots, w_n)$ and $V = (v_1, \ldots, v_n)$. Consider r a positive integer number.

We begin by showing the necessary condition. Suppose, by contradiction, that $\mu_L(W, V) \geq 2r + 1$ and $S(W, r) \cap S(V, r) \neq \varnothing$. In these conditions, there exists $X = (x_1, \ldots, x_n) \in \mathbb{Z}^n$, such that $X \in S(W, r) \cap S(V, r)$, that is, $\mu_L(W, X) \leq r$ and $\mu_L(V, X) \leq r$. Thus,

$$\sum_{i=1}^{n} |w_i - x_i| + \sum_{i=1}^{n} |v_i - x_i| = |w_1 - x_1| + |v_1 - x_1| + \cdots + |w_n - x_n| + |v_n - x_n| \leq 2r. \quad (1.1)$$

Since $|v_i - x_i| = |x_i - v_i|$ and $|w_i - x_i| + |x_i - v_i| \geq |w_i - x_i + x_i - v_i| = |w_i - v_i|$ for all $i \in \{1, \ldots, n\}$, from (1.1) it follows that

$$\mu_L(W, V) = |w_1 - v_1| + \cdots + |w_n - v_n| \leq 2r,$$

contradicting our assumption.

The sufficient condition will be proved supposing, by contradiction, that $S(W,r) \cap S(V,r) = \varnothing$ and $\mu_L(W,V) \leq 2r$. If $\mu_L(W,V) \leq r$, then $S(W,r) \cap S(V,r) \neq \varnothing$, contradicting the hypothesis. Therefore, let us consider $r < \mu_L(W,V) \leq 2r$.

Assume, without loss of generality, that $V = (0,\ldots,0)$. Then, $r < \sum_{i=1}^{n} |w_i| \leq 2r$. The word W can be rewritten as follows

$$W = (x_1 + y_1, \ldots, x_n + y_n),$$

where, for each $i = 1,\ldots,n$, x_i and y_i satisfy $x_i y_i \geq 0$ and $\sum_{i=1}^{n} |x_i| = r$. Thus,

$$\sum_{i=1}^{n} |w_i| = |x_1| + |y_1| + \cdots + |x_n| + |y_n| = \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i|.$$

Since,

$$r < \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i| \leq 2r \text{ and } \sum_{i=1}^{n} |x_i| = r,$$

then,

$$0 < \sum_{i=1}^{n} |y_i| \leq r.$$

Now, $X = (x_1,\ldots,x_n)$ is such that $\mu_L(X,V) = \sum_{i=1}^{n} |x_i| = r$, that is, $X \in S(V,r)$. On the other hand, $\mu_L(X,W) = \sum_{i=1}^{n} |x_i - x_i - y_i| = \sum_{i=1}^{n} |y_i| \leq r$, and so $X \in S(W,r) \cap S(V,r)$, which is a contradiction. □

Next lemma presents three equivalent conditions to define a perfect error correcting Lee code.

Lemma 1.2 *Let $\mathcal{M} \subset \mathbb{Z}^n$ and r a positive integer number. The following statements are equivalent:*

i) $(\forall\ W,V \in \mathcal{M},\ S(W,r) \cap S(V,r) = \varnothing) \ \wedge\ \bigcup_{W \in \mathcal{M}} S(W,r) = \mathbb{Z}^n$;

ii) $\forall\ V \in \mathbb{Z}^n,\ \exists^1\ W \in \mathcal{M} : \mu_L(V,W) \leq r$;

iii) $(\forall\ W,V \in \mathcal{M},\ \mu_L(W,V) \geq 2r+1) \ \wedge\ \bigcup_{W \in \mathcal{M}} S(W,r) = \mathbb{Z}^n$.

Proof. *i)* $\Rightarrow$ *ii)*

By hypothesis $\bigcup_{W\in\mathcal{M}} S(W,r) = \mathbb{Z}^n$, consequently, for all $V \in \mathbb{Z}^n$ there exists $W \in \mathcal{M}$ such that $V \in S(W,r)$, that is, $\mu_L(V,W) \leq r$. Suppose, by contradiction, that there are two distinct elements $W,U \in \mathcal{M}$ satisfying $\mu_L(V,W) \leq r$ and $\mu_L(V,U) \leq r$. In these conditions, $V \in S(W,r) \cap S(U,r)$, contradicting the hypothesis.

ii) $\Rightarrow$ *iii)*

Since $\bigcup_{W\in\mathcal{M}} S(W,r) \subset \mathbb{Z}^n$, to show that $\bigcup_{W\in\mathcal{M}} S(W,r) = \mathbb{Z}^n$ it is enough to prove that $\mathbb{Z}^n \subset \bigcup_{W\in\mathcal{M}} S(W,r)$.

Let $V \in \mathbb{Z}^n$. By hypothesis there exists $W \in \mathcal{M}$ such that $\mu_L(V,W) \leq r$, that is, $V \in S(W,r)$.

By contradiction, assume that there are two distinct elements $W,V \in \mathcal{M}$ so that $\mu_L(W,V) \leq 2r$. By Lemma 1.1, $S(W,r) \cap S(V,r) \neq \varnothing$. Thus, there exists $U \in \mathbb{Z}^n$ such that $\mu_L(U,W) \leq r$ and $\mu_L(U,V) \leq r$, contradicting the assumption.

iii) $\Rightarrow$ *i)*

Follows immediately from Lemma 1.1. □

1.2 Necessary conditions for the existence of PL$(n,2)$ codes when $n \geq 7$

The Golomb-Welch conjecture states that: *there is no PL(n,r) code for $n \geq 3$ and $r \geq 2$.* Our contribution for the proof of this conjecture is focused on the analysis of the non-existence of PL$(n,2)$ codes. Golomb and Welch have proved in [9] the existence of PL$(1,r)$ and PL$(2,r)$ codes for any positive integer number r, in particular, for $r = 2$. On the other hand, Horak [11] has proved the non-existence of PL$(n,2)$ codes for $3 \leq n \leq 6$, establishing the Golomb-Welch conjecture for $r = 2$ and some lower values of n. Since, so far, the non-existence of PL$(n,2)$ codes is proved only for these values of n, we are interested into reinforcing this conjecture proving that there are no PL$(7,2)$ codes.

The non-existence of PL$(7,2)$ codes will be proved by contradiction, that is, assuming the existence of such codes. We begin by deducing some necessary conditions for the existence of PL$(n,2)$ codes when $n \geq 7$, centering, later, our attention on PL$(7,2)$ codes.

Let us assume the existence of a PL$(n,2)$ code $\mathcal{M} \subset \mathbb{Z}^n$, $n \geq 7$, and suppose, without loss of generality, that $O \in \mathcal{M}$, with $O = (0,\ldots,0)$. Thus, all words $W \in \mathbb{Z}^n$ such that $\mu_L(W,O) \leq 2$ are covered by the codeword O. Taking into account Lemma 1.2, for each word $W \in \mathbb{Z}^n$ satisfying $\mu_L(W,O) = 3$ there exists an unique codeword $V \in \mathcal{M}$ such that $\mu_L(W,V) \leq 2$. The conditions for the existence of PL$(n,2)$ codes derive essentially from the analysis of the codewords which cover all words $W \in \mathbb{Z}^n$ which are distant three units from O.

Let $W \in \mathbb{Z}^n$ such that $\mu_L(W,O) = 3$. Then, $W = (w_1,\ldots,w_n)$ is of one and only one of the types:

- $[\pm\mathbf{3}]$, if there exists $i \in \{1,\ldots,n\}$ so that $|w_i| = 3$ and $w_j = 0$ for all $j \in \{1,\ldots,n\} \setminus \{i\}$;
- $[\pm\mathbf{2},\pm\mathbf{1}]$, if $|w_i| = 2$ and $|w_j| = 1$ for some $i,j \in \{1,\ldots,n\}$, and $w_k = 0$ for all $k \in \{1,\ldots,n\} \setminus \{i,j\}$;
- $[\pm\mathbf{1}^\mathbf{3}]$, if $|w_i| = |w_j| = |w_k| = 1$ for some $i,j,k \in \{1,\ldots,n\}$, and $w_l = 0$ for all $l \in \{1,\ldots,n\} \setminus \{i,j,k\}$.

Let $\mathcal{T} \subset \mathcal{M}$ be the set of codewords which cover all words $W \in \mathbb{Z}^n$ satisfying $\mu_L(W,O) = 3$. Any codeword $V \in \mathcal{T}$ is such that $\mu_L(V,O) = 5$. In fact, since O and V are codewords in $\mathcal{M}$, by Lemma 1.1, $\mu_L(V,O) \geq 5$. On the other hand, if we suppose $\mu_L(V,O) \geq 6$ then, for W so that $\mu_L(W,O) = 3$, we get

$$\mu_L(V,W) = |v_1-w_1|+\cdots+|v_n-w_n| \geq |v_1|-|w_1|+\cdots+|v_n|-|w_n| = \sum_{i=1}^{n}|v_i|-\sum_{i=1}^{n}|w_i| \geq 3.$$

That is, the codeword $V \in \mathcal{T}$ does not cover any word whose distance from O is three units.

Following the same idea used in the characterization of the words which are distant three units from O, we conclude that $V \in \mathcal{T}$ is of one and only one of the types: $[\pm\mathbf{5}]$,

$[\pm\mathbf{4}, \pm\mathbf{1}]$, $[\pm\mathbf{3}, \pm\mathbf{2}]$, $[\pm\mathbf{3}, \pm\mathbf{1^2}]$, $[\pm\mathbf{2^2}, \pm\mathbf{1}]$, $[\pm\mathbf{2}, \pm\mathbf{1^3}]$ and $[\pm\mathbf{1^5}]$. We will denote the subsets of $\mathcal{T}$ containing codewords of each one of these types by, respectively, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$. Furthermore, we set $a = |\mathcal{A}|$, $b = |\mathcal{B}|$, $c = |\mathcal{C}|$, $d = |\mathcal{D}|$, $e = |\mathcal{E}|$, $f = |\mathcal{F}|$ and $g = |\mathcal{G}|$, where $|\mathcal{A}|$ denotes the cardinality of the set $\mathcal{A}$ and so on.

Consider

$$\mathcal{I} = \{+1, +2, \ldots, +n, -1, -2, \ldots, -n\}$$

the **set of signed coordinates**. Let $W, V \in \mathbb{Z}^n$, with $W = (w_1, \ldots, w_n)$ and $V = (v_1, \ldots, v_n)$. If $iw_{|i|} > 0$ for $i \in \mathcal{I}$, then i and $w_{|i|}$ have the same sign. If $iw_{|i|} > 0$ and $iv_{|i|} > 0$, with $i \in \mathcal{I}$, then the $|i| - th$ coordinates of W and V have the same sign and we say that W and V are sign equivalent in the $|i| - th$ coordinate.

Let $\mathcal{H} \subset \mathbb{Z}^n$. For $i, j \in \mathcal{I}$, with $|i| \neq |j|$, and k a positive integer number, $\mathcal{H}_i$, $\mathcal{H}_{ij}$ and $\mathcal{H}_i^{(k)}$ will denote, respectively, the sets:

- $\mathcal{H}_i = \{W \in \mathcal{H} \ : \ iw_{|i|} > 0\}$;
- $\mathcal{H}_{ij} = \{W \in \mathcal{H} \ : \ iw_{|i|} > 0 \ \wedge \ jw_{|j|} > 0\}$;
- $\mathcal{H}_i^{(k)} = \{W \in \mathcal{H} \ : \ iw_{|i|} > 0 \ \wedge \ |w_{|i|}| = k\}$.

These sets are called **index subsets** of $\mathcal{H}$. We note that, it makes no sense to consider $\mathcal{H}_{ij}$ for $i = j$ or $i = -j$, so, in the rest of the document, when we write $\mathcal{H}_{ij}$, with $\mathcal{H} \subset \mathbb{Z}^n$ and $i, j \in \mathcal{I}$, we assume $|i| \neq |j|$.

Consider, for instance, $W \in \mathcal{G}$. Since the codewords of $\mathcal{G}$ are of type $[\pm 1^5]$, then there are $i, j, k, l, m \in \mathcal{I}$ such that $W \in \mathcal{G}_{ijklm}$. In this case i, j, k, l and m characterize the index distribution of $W \in \mathcal{G}$. If we consider $W \in \mathcal{F}$, since the codewords of $\mathcal{F}$ are of type $[\pm 2, \pm 1^3]$, there exist $i, j, k, l \in \mathcal{I}$ so that $W \in \mathcal{F}_{ijkl}$, more precisely, $W \in \mathcal{F}_i^{(2)} \cap \mathcal{F}_j^{(1)} \cap \mathcal{F}_k^{(1)} \cap \mathcal{F}_l^{(1)}$, being characterized the index value distribution of W.

Let $W \in \mathbb{Z}^n$ such that $\mu_L(W, O) = 3$. By definition of PL$(n,2)$ code, there exists a unique codeword $V \in \mathcal{T}$, with $\mathcal{T} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E} \cup \mathcal{F} \cup \mathcal{G}$, so that $\mu_L(W, V) \leq 2$. Having in mind that $\mu_L(W, V) = \sum_{i=1}^{n} |w_i - v_i|$, if W is of type:

- $[\pm 3]$, then $V \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$;

- $[\pm 2, \pm 1]$, then $V \in \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E} \cup \mathcal{F}$;

- $[\pm 1^3]$, then $V \in \mathcal{D} \cup \mathcal{E} \cup \mathcal{F} \cup \mathcal{G}$.

The following three lemmas impose restrictions to the cardinality of index subsets of $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E} \cup \mathcal{F} \cup \mathcal{G}$ warranting, respectively, that any word of the types $[\pm 3]$, $[\pm 2, \pm 1]$ or $[\pm 1^3]$ is covered by a unique codeword of $\mathcal{T}$.

Lemma 1.3 *For each $i \in \mathcal{I}$, $|\mathcal{A}_i \cup \mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(3)}| = 1$.*

Proof. For each $i \in \mathcal{I}$ there exists a word $W \in \mathbb{Z}^n$ of type $[\pm 3]$, with $W = (w_1,...,w_n)$, satisfying $iw_{|i|} > 0$ and $|w_{|i|}| = 3$. This word W must be covered by a codeword $V \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$, in particular, $V \in \mathcal{A}_i \cup \mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(3)}$. Thus, we conclude that $|\mathcal{A}_i \cup \mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(3)}| \geq 1$. If, by contradiction, we assume $|\mathcal{A}_i \cup \mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(3)}| \geq 2$, then there are two distinct codewords V and V' in $\mathcal{A}_i \cup \mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(3)}$ satisfying $\mu_L(V, W) \leq 2$ and $\mu_L(V', W) \leq 2$, which contradicts the definition of PL$(n, 2)$ code.

□

Lemma 1.4 *For each $i, j \in \mathcal{I}$, with $|i| \neq |j|$,*

$$|\mathcal{B}_i^{(4)} \cap \mathcal{B}_j^{(1)}| + |\mathcal{C}_i \cap \mathcal{C}_j| + |\mathcal{D}_i^{(3)} \cap \mathcal{D}_j^{(1)}| + |\mathcal{E}_i^{(2)} \cap \mathcal{E}_j| + |\mathcal{F}_i^{(2)} \cap \mathcal{F}_j^{(1)}| = 1.$$

Proof. For each $i, j \in \mathcal{I}$, with $|i| \neq |j|$, there exists a word $W \in \mathbb{Z}^n$ of type $[\pm 2, \pm 1]$, with $W = (w_1, \ldots, w_n)$, satisfying $iw_{|i|}, jw_{|j|} > 0$, $|w_{|i|}| = 2$ and $|w_{|j|}| = 1$. This word must be covered by a codeword $V \in \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E} \cup \mathcal{F}$ satisfying one of the following conditions: $V \in \mathcal{B}_i^{(4)} \cap \mathcal{B}_j^{(1)}$; $V \in \mathcal{C}_i \cap \mathcal{C}_j$; $V \in \mathcal{D}_i^{(3)} \cap \mathcal{D}_j^{(1)}$; $V \in \mathcal{E}_i^{(2)} \cap \mathcal{E}_j$; $V \in \mathcal{F}_i^{(2)} \cap \mathcal{F}_j^{(1)}$. Consequently, since $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ are disjoint sets,

$$|\mathcal{B}_i^{(4)} \cap \mathcal{B}_j^{(1)}| + |\mathcal{C}_i \cap \mathcal{C}_j| + |\mathcal{D}_i^{(3)} \cap \mathcal{D}_j^{(1)}| + |\mathcal{E}_i^{(2)} \cap \mathcal{E}_j| + |\mathcal{F}_i^{(2)} \cap \mathcal{F}_j^{(1)}| \geq 1.$$

If, by contradiction, we suppose

$$|\mathcal{B}_i^{(4)} \cap \mathcal{B}_j^{(1)}| + |\mathcal{C}_i \cap \mathcal{C}_j| + |\mathcal{D}_i^{(3)} \cap \mathcal{D}_j^{(1)}| + |\mathcal{E}_i^{(2)} \cap \mathcal{E}_j| + |\mathcal{F}_i^{(2)} \cap \mathcal{F}_j^{(1)}| \geq 2,$$

then, there are distinct codewords V and V' satisfying

$$V, V' \in (\mathcal{B}_i^{(4)} \cap \mathcal{B}_j^{(1)}) \cup (\mathcal{C}_i \cap \mathcal{C}_j) \cup (\mathcal{D}_i^{(3)} \cap \mathcal{D}_j^{(1)}) \cup (\mathcal{E}_i^{(2)} \cap \mathcal{E}_j) \cup (\mathcal{F}_i^{(2)} \cap \mathcal{F}_j^{(1)}).$$

Consequently, $\mu_L(V, W) \leq 2$ and $\mu_L(V', W) \leq 2$, which contradicts the definition of perfect 2-error correcting Lee code. □

Lemma 1.5 *For each $i, j, k \in \mathcal{I}$, with $|i|$, $|j|$ and $|k|$ pairwise distinct,*

$$|\mathcal{D}_{ijk} \cup \mathcal{E}_{ijk} \cup \mathcal{F}_{ijk} \cup \mathcal{G}_{ijk}| = 1.$$

Proof. For each $i, j, k \in \mathcal{I}$, with $|i|$, $|j|$ and $|k|$ distinct between them, there exists a word $W \in \mathbb{Z}^n$ of type $[\pm 1^3]$, with $W = (w_1,\ldots,w_n)$, so that, $iw_{|i|}, jw_{|j|}, kw_{|k|} > 0$ and $|w_{|i|}| = |w_{|j|}| = |w_{|k|}| = 1$. This word must be covered by a codeword $V \in \mathcal{D}_{ijk} \cup \mathcal{E}_{ijk} \cup \mathcal{F}_{ijk} \cup \mathcal{G}_{ijk}$, therefore $|\mathcal{D}_{ijk} \cup \mathcal{E}_{ijk} \cup \mathcal{F}_{ijk} \cup \mathcal{G}_{ijk}| \geq 1$. If, by contradiction, we suppose that $|\mathcal{D}_{ijk} \cup \mathcal{E}_{ijk} \cup \mathcal{F}_{ijk} \cup \mathcal{G}_{ijk}| \geq 2$, then there are distinct codewords $V, V' \in \mathcal{D}_{ijk} \cup \mathcal{E}_{ijk} \cup \mathcal{F}_{ijk} \cup \mathcal{G}_{ijk}$ and, consequently, $\mu_L(V, W) \leq 2$ and $\mu_L(V', W) \leq 2$, contradicting the definition of PL(n, 2) code. □

Taking into account the number of words of each one of the types $[\pm 3]$, $[\pm 2, \pm 1]$ and $[\pm 1^3]$, and considering the type of codewords which cover them, Horak has deduced in [11] the following proposition involving the parameters $a = |\mathcal{A}|$, $b = |\mathcal{B}|$, $c = |\mathcal{C}|$, $d = |\mathcal{D}|$, $e = |\mathcal{E}|$, $f = |\mathcal{F}|$ and $g = |\mathcal{G}|$.

Proposition 1.1 *The parameters a, b, c, d, e, f and g satisfy the system of equations*

$$\begin{cases} a + b + c + d = 2n \\ b + 2c + 2d + 4e + 3f = 8\binom{n}{2} \\ d + e + 4f + 10g = 8\binom{n}{3}. \end{cases}$$

There exist many nonnegative integer solutions for this system of equations. However, we are only interested in determining "good" solutions, that is, solutions which do not contradict the definition of a perfect 2-error correcting Lee code. For that, we will focus our attention on the cardinality of the index subsets of $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E} \cup \mathcal{F} \cup \mathcal{G}$.

We can find a relation between the cardinality of each set of codewords $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ and the cardinality of their index subsets. Considering, for instance, the set

$\mathcal{G}$, since the codewords of $\mathcal{G}$ are of type $[\pm 1^5]$, we get

$$g = |\mathcal{G}| = \frac{1}{5} \sum_{i \in \mathcal{I}} |\mathcal{G}_i|.$$

Besides, for $i \in \mathcal{I}$,

$$|\mathcal{G}_i| = \frac{1}{4} \sum_{j \in \mathcal{I} \setminus \{i,-i\}} |\mathcal{G}_{ij}|.$$

Similar equalities for the other subsets of $\mathcal{T}$ can be derived.

Looking at the words of type $[\pm 1^3]$, Horak proved in [11] the following two lemmas in which a relation between the cardinality of index subsets of $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ is given.

Lemma 1.6 *For each* $i \in \mathcal{I}$, $|\mathcal{D}_i \cup \mathcal{E}_i| + 3|\mathcal{F}_i| + 6|\mathcal{G}_i| = 4\binom{n-1}{2}$. *Consequently, if* $n \not\equiv 0\,(mod\ 3)$ *then* $|\mathcal{D}_i \cup \mathcal{E}_i| \equiv 0\,(mod\ 3)$, *and if* $n \equiv 0\,(mod\ 3)$, *then* $|\mathcal{D}_i \cup \mathcal{E}_i|$ *and* $|\mathcal{F}_i|$ *have the same parity, and* $|\mathcal{D}_i \cup \mathcal{E}_i| \equiv 1(mod\ 3)$.

Lemma 1.7 *For each* $i, j \in \mathcal{I}$, $|i| \neq |j|$,

$$|\mathcal{D}_{ij} \cup \mathcal{E}_{ij}| + 2|\mathcal{F}_{ij}| + 3|\mathcal{G}_{ij}| = 2(n-2).$$

Consequently, $|\mathcal{D}_{ij} \cup \mathcal{E}_{ij}|$ *and* $|\mathcal{G}_{ij}|$ *have the same parity.*

Lemma 1.6 restricts the variation of the cardinality of $\mathcal{D}_i, \mathcal{E}_i, \mathcal{F}_i$ and $\mathcal{G}_i$ for any $i \in \mathcal{I}$, consequently, reduces the possible values for the parameters d, e, f and g. In next result we refine the variation of $|\mathcal{D}_i \cup \mathcal{E}_i|$, establishing a smaller upper bound for the cardinality of $\mathcal{D}_i \cup \mathcal{E}_i$ than the one derived by Horak in Lemma 1.6.

Lemma 1.8 *For each* $i \in \mathcal{I}$, $|\mathcal{D}_i \cup \mathcal{E}_i| \leq 2n - 1$.

Proof. Suppose, by contradiction, that $|\mathcal{D}_i \cup \mathcal{E}_i| \geq 2n$ for some $i \in \mathcal{I}$.

Recall that the codewords of $\mathcal{D}$ and $\mathcal{E}$ are, respectively, of types $[\pm 3, \pm 1^2]$ and $[\pm 2^2, \pm 1]$. Then, by assumption, $|\mathcal{D}_i^{(3)} \cup \mathcal{D}_i^{(1)} \cup \mathcal{E}_i^{(2)} \cup \mathcal{E}_i^{(1)}| \geq 2n$.

By Lemma 1.3 we get $|\mathcal{D}_i^{(3)}| \leq 1$. Thus, $|\mathcal{D}_i^{(1)} \cup \mathcal{E}_i^{(2)} \cup \mathcal{E}_i^{(1)}| \geq 2n-1$ and, consequently,

$$\sum_{j \in \mathcal{I} \setminus \{i,-i\}} |(\mathcal{D}_i^{(1)} \cap \mathcal{D}_j^{(3)}) \cup (\mathcal{E}_i \cap \mathcal{E}_j^{(2)})| \geq 2n - 1. \tag{1.2}$$

Since $|\mathcal{I}\backslash\{i,-i\}| = 2n-2$, from (1.2) we conclude that there exists $j \in \mathcal{I}\backslash\{i,-i\}$ such that $|(\mathcal{D}_i^{(1)} \cap \mathcal{D}_j^{(3)}) \cup (\mathcal{E}_i \cap \mathcal{E}_j^{(2)})| \geq 2$, contradicting Lemma 1.4. □

This lemma gives us a range of variation for the parameters d and e. In fact, since

$$d = \frac{1}{3}\sum_{i\in\mathcal{I}} |\mathcal{D}_i| \text{ and } e = \frac{1}{3}\sum_{i\in\mathcal{I}} |\mathcal{E}_i|,$$

by Lemma 1.8 we get

$$d + e = \frac{1}{3}\sum_{i\in\mathcal{I}} |\mathcal{D}_i \cup \mathcal{E}_i| \leq \frac{2n(2n-1)}{3}.$$

Having in view the words of type $[\pm 2, \pm 1]$, Horak [11] has established the next two results.

Lemma 1.9 *For each $i \in \mathcal{I}$, $|\mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)}| + 2|\mathcal{D}_i^{(3)} \cup \mathcal{E}_i^{(2)}| + 3|\mathcal{F}_i^{(2)}| = 2(n-1)$ and $|\mathcal{B}_i^{(1)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(1)} \cup \mathcal{E}_i^{(2)} \cup \mathcal{F}_i^{(1)}| + 2|\mathcal{E}_i^{(1)}| = 2(n-1)$.*

As an immediate consequence, it follows that:

Lemma 1.10 *Let $i \in \mathcal{I}$. Then, $|\mathcal{F}_i^{(1)}| \leq 2(n-1) - (|\mathcal{D}_i^{(1)}| + |\mathcal{E}_i| + |\mathcal{E}_i^{(1)}|)$ and $|\mathcal{F}_i^{(2)}| \leq \left\lfloor \frac{2(n-1)-2(|\mathcal{D}_i^{(3)}|+|\mathcal{E}_i^{(2)}|)}{3} \right\rfloor$. In accordance,*

$$|\mathcal{F}_i| \leq 2(n-1) - (|\mathcal{D}_i^{(1)}| + |\mathcal{E}_i| + |\mathcal{E}_i^{(1)}|) + \left\lfloor \frac{2(n-1) - 2(|\mathcal{D}_i^{(3)}| + |\mathcal{E}_i^{(2)}|)}{3} \right\rfloor.$$

We note that $\lfloor x \rfloor$ denotes the highest integer number less or equal to x.

As we have mentioned before, we are looking for the "good" solutions of the system of equations in Proposition 1.1, that is, the solutions satisfying the definition of perfect error correcting Lee code. In this sense, a particular attention will be given to the parameters f and g, since $\mathcal{F}$ and $\mathcal{G}$ are the subsets of $\mathcal{T}$ in which the codewords have more nonzero coordinates. Having in mind these sets, we establish the following new results which restrict the variation of $|\mathcal{F}_i|$ and $|\mathcal{G}_i|$ for any $i \in \mathcal{I}$ and, consequently, restrict the range of variation for the parameters f and g.

The following result follows from Lemma 1.10.

Lemma 1.11 *For each $i \in \mathcal{I}$, $|\mathcal{F}_i| \leq \frac{8(n-1)+1}{3} - |\mathcal{D}_i \cup \mathcal{E}_i| - \frac{2}{3}|\mathcal{E}_i|$.*

Proof. Let $i \in \mathcal{I}$. By Lemma 1.10 it follows that

$$|\mathcal{F}_i| \leq 2(n-1) - (|\mathcal{D}_i^{(1)}| + |\mathcal{E}_i| + |\mathcal{E}_i^{(1)}|) + \left\lfloor \frac{2(n-1) - 2(|\mathcal{D}_i^{(3)}| + |\mathcal{E}_i^{(2)}|)}{3} \right\rfloor.$$

Then,

$$|\mathcal{F}_i| \leq 2(n-1) - (|\mathcal{D}_i^{(1)}| + |\mathcal{E}_i| + |\mathcal{E}_i^{(1)}|) + \frac{2(n-1) - 2(|\mathcal{D}_i^{(3)}| + |\mathcal{E}_i^{(2)}|)}{3}$$

and, equivalently,

$$|\mathcal{F}_i| \leq \frac{8(n-1)}{3} - \left(|\mathcal{D}_i^{(1)}| + \frac{2}{3}|\mathcal{D}_i^{(3)}|\right) - \left(|\mathcal{E}_i| + |\mathcal{E}_i^{(1)}| + \frac{2}{3}|\mathcal{E}_i^{(2)}|\right). \tag{1.3}$$

The codewords of $\mathcal{D}$ and $\mathcal{E}$ are, respectively, of types $[\pm 3, \pm 1^2]$ and $[\pm 2^2, \pm 1]$. Since $\mathcal{D}_i = \mathcal{D}_i^{(3)} \cup \mathcal{D}_i^{(1)}$ and $\mathcal{D}_i^{(3)} \cap \mathcal{D}_i^{(1)} = \varnothing$, then $|\mathcal{D}_i| = |\mathcal{D}_i^{(3)}| + |\mathcal{D}_i^{(1)}|$. By a similar reasoning, $|\mathcal{E}_i| = |\mathcal{E}_i^{(2)}| + |\mathcal{E}_i^{(1)}|$. In these conditions, (1.3) can be rewritten in the form

$$|\mathcal{F}_i| \leq \frac{8(n-1)}{3} - \left(|\mathcal{D}_i| - \frac{1}{3}|\mathcal{D}_i^{(3)}|\right) - \left(2|\mathcal{E}_i| - \frac{1}{3}|\mathcal{E}_i^{(2)}|\right).$$

As $|\mathcal{E}_i| \geq |\mathcal{E}_i^{(2)}|$ and, by Lemma 1.3, $|\mathcal{D}_i^{(3)}| \leq 1$, we get

$$|\mathcal{F}_i| \leq \frac{8(n-1)}{3} - |\mathcal{D}_i| + \frac{1}{3} - 2|\mathcal{E}_i| + \frac{1}{3}|\mathcal{E}_i| = \frac{8(n-1)+1}{3} - |\mathcal{D}_i \cup \mathcal{E}_i| - \frac{2}{3}|\mathcal{E}_i|.$$

□

Next two lemmas establish, respectively, an upper and lower bound for the cardinality of $|\mathcal{G}_i|$.

Lemma 1.12 *For each $i \in \mathcal{I}$, $|\mathcal{G}_i| \leq \frac{(n-1)(n-2)}{3}$. In particular, if $n \equiv 0 \,(mod\ 3)$, then $|\mathcal{G}_i| \leq \frac{(n-1)(n-3)}{3}$. If $n \equiv 1 \,(mod\ 3)$, then $|\mathcal{G}_i| \leq \frac{(n-1)(2n-5)}{6}$.*

Proof. From Lemma 1.6 we get

$$6|\mathcal{G}_i| \leq 4\binom{n-1}{2}$$

for all $i \in \mathcal{I}$. Equivalently,

$$|\mathcal{G}_i| \leq \frac{(n-1)(n-2)}{3}$$

for each $i \in \mathcal{I}$.

Lemma 1.7 leads to

$$|\mathcal{G}_{ij}| \leq \frac{2(n-2)}{3} \tag{1.4}$$

for each $i, j \in \mathcal{I}$, with $|i| \neq |j|$.

If $n \equiv 0 \pmod 3$, then there is a positive integer number k so that $n = 3k$. Thus, (1.4) assumes the form

$$|\mathcal{G}_{ij}| \leq \frac{2(3k-2)}{3} = 2k - 1 - \frac{1}{3}.$$

Since $|\mathcal{G}_{ij}|$ is a nonnegative integer number, it follows that

$$|\mathcal{G}_{ij}| \leq 2k - 2.$$

Taking into account that $k = \frac{n}{3}$, we get

$$|\mathcal{G}_{ij}| \leq 2\left(\frac{n}{3} - 1\right). \tag{1.5}$$

The codewords of $\mathcal{G}$ are of type $[\pm 1^5]$. Therefore,

$$|\mathcal{G}_i| = \frac{1}{4} \sum_{j \in \mathcal{I} \setminus \{i, -i\}} |\mathcal{G}_{ij}| \tag{1.6}$$

for $i \in \mathcal{I}$. As $|\mathcal{I} \setminus \{i, -i\}| = 2(n-1)$, from (1.5) and (1.6) it follows that

$$|\mathcal{G}_i| \leq \frac{1}{4} \times 2(n-1) \times 2\left(\frac{n}{3} - 1\right) = \frac{(n-1)(n-3)}{3}.$$

If $n \equiv 1 \pmod 3$, then $n = 3k + 1$, where k is a positive integer number. In these conditions (1.4) can be written in the form

$$|\mathcal{G}_{ij}| \leq 2k - \frac{2}{3}.$$

As $|\mathcal{G}_{ij}|$ is a nonnegative integer number, $|\mathcal{G}_{ij}| \leq 2k - 1$. Taking into account that $k = \frac{n-1}{3}$, it follows that

$$|\mathcal{G}_{ij}| \leq 2\left(\frac{n-1}{3}\right) - 1 = \frac{2n-5}{3}. \tag{1.7}$$

Thus, by (1.6) and (1.7) we conclude that

$$|\mathcal{G}_i| \leq \frac{1}{4} \times 2(n-1) \times \frac{2n-5}{3} = \frac{(n-1)(2n-5)}{6}.$$

□

Lemma 1.13 *For each* $i \in \mathcal{I}$, $|\mathcal{G}_i| \geq \frac{|\mathcal{D}_i \cup \mathcal{E}_i| + (n-1)(n-6)}{3} - \frac{1}{6}$.

Proof. From Lemma 1.6 we get

$$|\mathcal{D}_i \cup \mathcal{E}_i| + 3|\mathcal{F}_i| + 6|\mathcal{G}_i| = 4\binom{n-1}{2}$$

for all $i \in \mathcal{I}$. Therefore,

$$|\mathcal{F}_i| = \frac{2(n-1)(n-2) - 6|\mathcal{G}_i| - |\mathcal{D}_i \cup \mathcal{E}_i|}{3},$$

for each $i \in \mathcal{I}$. Considering Lemma 1.11, it follows that

$$\frac{2(n-1)(n-2) - 6|\mathcal{G}_i| - |\mathcal{D}_i \cup \mathcal{E}_i|}{3} \leq \frac{8(n-1)+1}{3} - |\mathcal{D}_i| - \frac{5}{3}|\mathcal{E}_i| \leq \frac{8(n-1)+1}{3} - |\mathcal{D}_i \cup \mathcal{E}_i|.$$

Thus,

$$|\mathcal{G}_i| \geq \frac{|\mathcal{D}_i \cup \mathcal{E}_i| + (n-1)(n-6)}{3} - \frac{1}{6}.$$

□

Lemmas 1.12 and 1.13 reduce the number of the required solutions for the system of equations given in Proposition 1.1. In fact, these results restrict the variation of $|\mathcal{G}_i|$ for any $i \in \mathcal{I}$. Since it is possible to relate the cardinality of the index subsets $\mathcal{G}_i$ with the cardinality of $\mathcal{G}$, from these lemmas we get an interval of variation for g, as we will see in the next corollary.

Corollary 1.1 *The parameter* g *satisfies* $\frac{n[2(n-1)(n-6)-1]}{15} \leq g \leq \frac{2n(n-1)(n-2)}{15}$. *In particular, if* $n \equiv 0$ *(mod 3), then* $g \leq \frac{2n(n-1)(n-3)}{15}$. *If* $n \equiv 1$ *(mod 3), then* $g \leq \frac{n(n-1)(2n-5)}{15}$.

Proof. The codewords of $\mathcal{G}$ are the codewords of type $[\pm 1^5]$, and so

$$g = \frac{1}{5}\sum_{i \in \mathcal{I}} |\mathcal{G}_i|. \tag{1.8}$$

By Lemma 1.13, for all $i \in \mathcal{I}$,

$$|\mathcal{G}_i| \geq \frac{(n-1)(n-6)}{3} - \frac{1}{6}. \tag{1.9}$$

Taking into account that $|\mathcal{I}| = 2n$, from (1.8) and (1.9) it follows that

$$g \geq \frac{1}{5} \times 2n \times \left[\frac{(n-1)(n-6)}{3} - \frac{1}{6}\right],$$

equivalently,

$$g \geq \frac{n[2(n-1)(n-6)-1]}{15}.$$

By Lemma 1.12, for any $i \in \mathcal{I}$,

$$|\mathcal{G}_i| \leq \frac{(n-1)(n-2)}{3}. \tag{1.10}$$

Thus, from (1.8) and (1.10), we get

$$g \leq \frac{1}{5} \times 2n \times \frac{(n-1)(n-2)}{3} = \frac{2n(n-1)(n-2)}{15}.$$

If $n \equiv 0 \pmod 3$, considering Lemma 1.12 we have, for each $i \in \mathcal{I}$,

$$|\mathcal{G}_i| \leq \frac{(n-1)(n-3)}{3},$$

consequently,

$$g \leq \frac{2n(n-1)(n-3)}{15}.$$

If $n \equiv 1 \pmod 3$, by Lemma 1.12 we get, for all $i \in \mathcal{I}$,

$$|\mathcal{G}_i| \leq \frac{(n-1)(2n-5)}{6}.$$

Therefore,

$$g \leq \frac{n(n-1)(2n-5)}{15}.$$

□

In this section we have presented some results which must be satisfied by PL$(n,2)$ codes, when $n \geq 7$, assuming their existence. From now, our research will be focused in the study of PL$(7,2)$ codes.

Chapter 2

PL$(7,2)$ codes: necessary conditions for their existence

This chapter is devoted to the study of the conditions that a perfect 2-error correcting Lee code of word length 7 over $\mathbb{Z}$ must obey, assuming its existence.

Firstly, we present some results, having into account the results given in the previous chapter for PL$(n,2)$ codes, when $n \geq 7$, considering now $n = 7$. In Section 2.2, the range of variation for the cardinality of $\mathcal{G}_i$, for each $i \in \mathcal{I}$, is improved. In the last section, some conditions on the cardinality of the index subsets of $\mathcal{T}$ are achieved, considering particular values for $|\mathcal{G}_i|$.

Assuming the existence of a perfect 2-error correcting Lee code $\mathcal{M}$ in $(\mathbb{Z}^7, \mu_L)$, let us assume, without loss of generality, that $O \in \mathcal{M}$, with $O = (0, \ldots, 0)$.

All words $W \in \mathbb{Z}^7$ satisfying $\mu_L(W,O) \leq 2$ are covered by the codeword O. On the other hand, considering Lemma 1.2, for each $W \in \mathbb{Z}^7$ so that $\mu_L(W,O) = 3$ there exists a unique codeword $V \in \mathcal{M}$ such that $\mu_L(W,V) \leq 2$.

Using the notation presented in the previous chapter, let $\mathcal{T} \subset \mathcal{M}$ be the set of codewords which cover all words $W \in \mathbb{Z}^7$ satisfying $\mu_L(W,O) = 3$ and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ be the subsets of $\mathcal{T}$ defined in Section 1.2. The proof of the non-existence of PL$(7,2)$ codes is centered on the analysis of the codewords which cover all words W that are distant three units from O, that is, is focused on the study of the codewords of $\mathcal{T} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E} \cup \mathcal{F} \cup \mathcal{G}$. We will prove that there is no PL$(7,2)$ code showing that it is not possible to cover all these words without contradicting the definition of perfect Lee code.

2.1 Previous results

Assuming the existence of a PL$(7,2)$ code $\mathcal{M}$, it follows immediately from Proposition 1.1 that $a = |\mathcal{A}|$, $b = |\mathcal{B}|$, $c = |\mathcal{C}|$, $d = |\mathcal{D}|$, $e = |\mathcal{E}|$, $f = |\mathcal{F}|$ and $g = |\mathcal{G}|$ must satisfy the system of equations given bellow.

Proposition 2.1 *The parameters a, b, c, d, e, f and g satisfy the system of equations*

$$\begin{cases} a+b+c+d = 14 \\ b+2c+2d+4e+3f = 168 \\ d+e+4f+10g = 280. \end{cases}$$

Our aim is to prove that any nonnegative integer solution of this system of equations contradicts the definition of perfect 2-error correcting Lee code leading, consequently, to the non-existence of PL$(7,2)$ codes.

The four following results derive directly from Lemmas 1.6, 1.7, 1.8 and 1.9, respectively, when considered $n = 7$.

Lemma 2.1 *For each $i \in \mathcal{I}$,*

$$|\mathcal{D}_i \cup \mathcal{E}_i| + 3|\mathcal{F}_i| + 6|\mathcal{G}_i| = 60.$$

Consequently, $|\mathcal{D}_i \cup \mathcal{E}_i| \equiv 0 \,(mod\, 3)$.

Lemma 2.2 *For each $i, j \in \mathcal{I}$, $|i| \neq |j|$,*

$$|\mathcal{D}_{ij} \cup \mathcal{E}_{ij}| + 2|\mathcal{F}_{ij}| + 3|\mathcal{G}_{ij}| = 10.$$

Consequently, $|\mathcal{D}_{ij} \cup \mathcal{E}_{ij}|$ and $|\mathcal{G}_{ij}|$ have the same parity.

Lemma 2.3 *For each $i \in \mathcal{I}$, $|\mathcal{D}_i \cup \mathcal{E}_i| \leq 13$.*

Lemma 2.4 *For each $i \in \mathcal{I}$,*

$$|\mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)}| + 2|\mathcal{D}_i^{(3)} \cup \mathcal{E}_i^{(2)}| + 3|\mathcal{F}_i^{(2)}| = 12$$

and

$$|\mathcal{B}_i^{(1)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(1)} \cup \mathcal{E}_i^{(2)} \cup \mathcal{F}_i^{(1)}| + 2|\mathcal{E}_i^{(1)}| = 12.$$

Having in view Lemma 2.2 we detach the following condition on $\mathcal{F}_{ij}$, for $i, j \in \mathcal{I}$ and $|i| \neq |j|$, when $|\mathcal{F}_{ij}|$ assumes the highest possible value.

Lemma 2.5 *For any $i, j \in \mathcal{I}$, with $|i| \neq |j|$, $|\mathcal{F}_{ij}| \leq 5$. Furthermore, if $|\mathcal{F}_{ij}| = 5$, then $|\mathcal{F}_{ijk}| = 1$ for all $k \in \mathcal{I}\backslash\{i, -i, j, -j\}$.*

Proof. Let $i, j \in \mathcal{I}$ with $|i| \neq |j|$. From Lemma 2.2 it follows that $|\mathcal{F}_{ij}| \leq 5$. Suppose that $|\mathcal{F}_{ij}| = 5$ and W_1,...,$W_5 \in \mathcal{F}_{ij}$, with $W_1 \in \mathcal{F}_{ijw_1w_2}$, $W_2 \in \mathcal{F}_{ijw_3w_4}$,...,$W_5 \in \mathcal{F}_{ijw_9w_{10}}$. Note that, w_1,...,$w_{10} \in \mathcal{I}\backslash\{i, -i, j, -j\}$. By Lemma 1.5 we must impose w_1,...,w_{10} pairwise distinct. Consequently, since $|\mathcal{I}\backslash\{i, -i, j, -j\}| = 10$, we get $|\mathcal{F}_{ijk}| = 1$ for all $k \in \mathcal{I}\backslash\{i, -i, j, -j\}$. □

Next statements are obtained immediately from Lemmas 1.11, 1.12 and 1.13, respectively, for $n = 7$, and restrict the variation of the cardinality of $\mathcal{F}_i$ and $\mathcal{G}_i$ for each $i \in \mathcal{I}$.

Lemma 2.6 *For each $i \in \mathcal{I}$, $|\mathcal{F}_i| \leq \frac{49}{3} - |\mathcal{D}_i \cup \mathcal{E}_i| - \frac{2}{3}|\mathcal{E}_i|$.*

Lemma 2.7 *For each $i \in \mathcal{I}$, $|\mathcal{G}_i| \leq 9$.*

Lemma 2.8 *For each $i \in \mathcal{I}$, $|\mathcal{G}_i| \geq \frac{|\mathcal{D}_i \cup \mathcal{E}_i| + 6}{3} - \frac{1}{6}$.*

Lemmas 2.7 and 2.8 lead to the following corollary.

Corollary 2.1 *For each $i \in \mathcal{I}$, $2 \leq |\mathcal{G}_i| \leq 9$.*

Using Corollary 2.1 and taking into account that, $g = \frac{1}{5}\sum_{i\in\mathcal{I}} |\mathcal{G}_i|$ and $|\mathcal{I}| = 14$, it follows that $6 \leq g \leq 25$. It is evident that the smaller is the range of the variation of $|\mathcal{G}_i|$, the smaller is the number of solutions we are looking for.

The proof of the non-existence of PL$(7,2)$ codes is based on the analysis of the possible cardinalities for $\mathcal{G}_i$. In next section we improve the result given in Corollary 2.1 restricting further the possible values for $|\mathcal{G}_i|$, $i \in \mathcal{I}$.

2.2 Refining the variation of the cardinality of $\mathcal{G}_i$

By Corollary 2.1, $2 \leq |\mathcal{G}_i| \leq 9$ for all $i \in \mathcal{I}$. Our intention is to reduce more and more the range of the variation of $|\mathcal{G}_i|$. Our first move is to prove that $3 \leq |\mathcal{G}_i| \leq 8$ for any $i \in \mathcal{I}$.

Proposition 2.2 *For any $i \in \mathcal{I}$, $|\mathcal{G}_i| \neq 2$.*

Proof. Suppose, by contradiction, that there exists $i \in \mathcal{I}$ such that $|\mathcal{G}_i| = 2$. By Lemma 2.8 it follows that

$$2 \geq \frac{|\mathcal{D}_i \cup \mathcal{E}_i| + 6}{3} - \frac{1}{6},$$

and, consequently, $|\mathcal{D}_i \cup \mathcal{E}_i| = 0$.

Since we are assuming $|\mathcal{G}_i| = 2$, let $\mathcal{G}_i = \{W, V\}$. As the codewords of $\mathcal{G}$ are of type $[\pm 1^5]$, there are $j,k,l,m,n,o,p,q \in \mathcal{I}\backslash\{i,-i\}$ such that $W \in \mathcal{G}_{ijklm}$ and $V \in \mathcal{G}_{inopq}$, with $|j|$, $|k|$, $|l|$, $|m|$ pairwise distinct as well as $|n|$, $|o|$, $|p|$ and $|q|$.

Now, $|\{j,k,l,m\} \cap \{n,o,p,q\}| \leq 1$, otherwise, there are two distinct elements $\alpha, \beta \in \{j,k,l,m,n,o,p,q\}$ such that $|\mathcal{G}_{i\alpha\beta}| = 2$, contradicting Lemma 1.5. Since $\mathcal{G}_i = \{W, V\}$, there are, at least, six elements $\alpha \in \{j,k,l,m,n,o,p,q\}$ such that $|\mathcal{G}_{i\alpha}| = 1$. By Lemma 2.2, $|\mathcal{G}_{i\alpha}|$ and $|\mathcal{D}_{i\alpha} \cup \mathcal{E}_{i\alpha}|$ both have the same parity. Then, $|\mathcal{D}_{i\alpha} \cup \mathcal{E}_{i\alpha}|$ is odd, and, consequently, $|\mathcal{D}_i \cup \mathcal{E}_i| > 0$, which is a contradiction. □

Proposition 2.3 *For any $i \in \mathcal{I}$, $|\mathcal{G}_i| \neq 9$.*

Proof. By contradiction, let us assume the existence of an $i \in \mathcal{I}$ such that $|\mathcal{G}_i| = 9$.

Since

$$|\mathcal{G}_i| = \frac{1}{4} \sum_{j \in \mathcal{I} \backslash \{i,-i\}} |\mathcal{G}_{ij}|,$$

it follows that

$$\sum_{j \in \mathcal{I} \backslash \{i,-i\}} |\mathcal{G}_{ij}| = 36. \tag{2.1}$$

By Lemma 2.2, $|\mathcal{G}_{ij}| \leq 3$ for all $j \in \mathcal{I}\backslash\{i,-i\}$. As $|\mathcal{I}\backslash\{i,-i\}| = 12$, taking into account (2.1) we get $|\mathcal{G}_{ij}| = 3$ for each $j \in \mathcal{I}\backslash\{i,-i\}$.

Let $W_1 \in \mathcal{G}_i$ such that $W_1 \in \mathcal{G}_{i\alpha\beta\gamma\delta}$, where $\alpha, \beta, \gamma, \delta \in \mathcal{I}\backslash\{i,-i\}$ and $|\alpha|$, $|\beta|$, $|\gamma|$, $|\delta|$ are pairwise distinct. Since $|\mathcal{G}_{ij}| = 3$ for all $j \in \mathcal{I}\backslash\{i,-i\}$, then $|\mathcal{G}_{i\alpha}| = |\mathcal{G}_{i\beta}| = |\mathcal{G}_{i\gamma}| = |\mathcal{G}_{i\delta}| = 3$. Considering Lemma 1.5, for $W \in \mathcal{G}_i\backslash\{W_1\}$ there exists, at most, one element $\varepsilon \in \{\alpha, \beta, \gamma, \delta\}$ so that $W \in \mathcal{G}_{i\varepsilon}$. Thus, the index distribution of the codewords $W_1, \ldots, W_9 \in \mathcal{G}_i$ must satisfy the conditions presented in the next table.

W_1	i	α	β	γ	δ
W_2	i	α			
W_3	i	α			
W_4	i	β			
W_5	i	β			
W_6	i	γ			
W_7	i	γ			
W_8	i	δ			
W_9	i	δ			

Table 2.1: Partial index distribution of the codewords of $\mathcal{G}_i$.

Let $\mathcal{J} = \{\alpha, \beta, \gamma, \delta\}$ and $\mathcal{J}^- = \{-\alpha, -\beta, -\gamma, -\delta\}$. Denoting by $\mathcal{K}$ the set $\mathcal{K} = \mathcal{I}\backslash(\{i,-i\} \cup \mathcal{J} \cup \mathcal{J}^-)$, $\mathcal{K} = \{x, -x, y, -y\}$.

Let $W, W' \in \mathcal{G}_{i\varepsilon}\backslash\{W_1\}$, with $\varepsilon \in \mathcal{J}$, such that $W \in \mathcal{G}_{i\varepsilon w_1 w_2 w_3}$ and $W' \in \mathcal{G}_{i\varepsilon w_4 w_5 w_6}$, where $w_1, \ldots, w_6 \in \mathcal{J}^-\backslash\{-\varepsilon\} \cup \mathcal{K}$. Taking into account Lemma 1.5, $w_1, \ldots, w_6$ are pairwise distinct and, consequently, $|\{w_1, \ldots, w_6\} \cap (\mathcal{J}^-\backslash\{-\varepsilon\} \cup \mathcal{K})| = 6$. Since $|\{w_1, \ldots, w_6\} \cap \mathcal{J}^-\backslash\{-\varepsilon\}| \leq 3$, then $|\{w_1, \ldots, w_6\} \cap \mathcal{K})| \geq 3$. On the other hand, $|\{w_1, w_2, w_3\} \cap \mathcal{K}| \leq 2$. In fact, $|w_1|$, $|w_2|$ and $|w_3|$ must be pairwise distinct and

$\{|k| : k \in \mathcal{K}\} = \{|x|, |y|\}$. Similarly, $|\{w_4, w_5, w_6\} \cap \mathcal{K}| \leq 2$. Thus, for each $\varepsilon \in \mathcal{J}$ and $W, W' \in \mathcal{G}_{i\varepsilon}\backslash\{W_1\}$, there are, at least, three distinct elements $k, k', k'' \in \mathcal{K}$ so that $W \in \mathcal{G}_{i\varepsilon kk'}$ and $W' \in \mathcal{G}_{i\varepsilon k''}$. Therefore, the index distribution of the codewords of $\mathcal{G}_i$ satisfies the conditions presented in Table 2.2, where $k_1, \ldots, k_{12} \in \mathcal{K}$.

W_1	i	α	β	γ	δ
W_2	i	α	k_1	k_2	
W_3	i	α	k_3		
W_4	i	β	k_4	k_5	
W_5	i	β	k_6		
W_6	i	γ	k_7	k_8	
W_7	i	γ	k_9		
W_8	i	δ	k_{10}	k_{11}	
W_9	i	δ	k_{12}		

Table 2.2: Partial index distribution of the codewords of $\mathcal{G}_i$.

Since $\mathcal{K} \subset \mathcal{I}\backslash\{i, -i\}$, then for all $k \in \mathcal{K}$ we get $|\mathcal{G}_{ik}| = 3$. By the analysis of the Table 2.2, for each $\varepsilon \in \mathcal{J}$ and $W, W' \in \mathcal{G}_{i\varepsilon}\backslash\{W_1\}$, with $W \in \mathcal{G}_{i\varepsilon w_1 w_2 w_3}$ and $W' \in \mathcal{G}_{i\varepsilon w_4 w_5 w_6}$, we must impose $|\{w_1, \ldots, w_6\} \cap \mathcal{K}| = 3$ and, consequently, $|\{w_1, \ldots, w_6\} \cap \mathcal{J}^-\backslash\{-\varepsilon\}| = 3$.

Considering the codewords W_2, W_4, W_6 and W_8, see Table 2.2, as $\mathcal{K} = \{x, -x, y, -y\}$, all possible combinations between the elements of $\mathcal{K}$ are exhausted in the characterization of these codewords. We may assume, without loss of generality, that $k_1 = x$, $k_2 = y$, $k_3 = -x$, $k_4 = -x$ and $k_5 = y$. The partial index distribution takes now the form given in the Table 2.3.

By Lemma 1.5, the elements $j_1, j_2, j_3 \in \mathcal{J}^-$ must be pairwise distinct with $j_4 \neq j_1, j_2, j_3$, otherwise, $|\mathcal{G}_{iyj_4}| \geq 2$ or $|\mathcal{G}_{i,-x,j_4}| \geq 2$. Thus, $\{j_1, \ldots, j_4\} = \mathcal{J}^-$. As $|\mathcal{G}_{iy}| = 3$ and $W_2, W_4 \in \mathcal{G}_{iy}$, there exists a unique $k \in \{k_6, \ldots, k_{12}\}$ such that $k = y$. If $y = k_6$, then $|\mathcal{G}_{i\beta y}| \geq 2$, contradicting Lemma 1.5. If $y \in \{k_7, k_8, k_{10}, k_{11}\}$, since $k_7, k_8, k_{10}, k_{11} \in \{x, -x, y, -y\} = \mathcal{K}$, we get $W_6 \in \mathcal{G}_{iyx} \cup \mathcal{G}_{i,y,-x}$ or $W_8 \in \mathcal{G}_{iyx} \cup \mathcal{G}_{i,y,-x}$. Considering $W_2 \in \mathcal{G}_{ixy}$ and $W_4 \in \mathcal{G}_{i,-x,y}$, it follows that $|\mathcal{G}_{ixy}| \geq 2$ or $|\mathcal{G}_{i,-x,y}| \geq 2$, a contradiction. Therefore, $y = k_9$ or $y = k_{12}$.

W_1	i	α	β	γ	δ
W_2	i	α	x	y	j_1
W_3	i	α	$-x$	j_2	j_3
W_4	i	β	$-x$	y	j_4
W_5	i	β	k_6		
W_6	i	γ	k_7	k_8	
W_7	i	γ	k_9		
W_8	i	δ	k_{10}	k_{11}	
W_9	i	δ	k_{12}		

Table 2.3: Partial index distribution of the codewords of $\mathcal{G}_i$.

Suppose that $y = k_9$. Then $W_7 \in \mathcal{G}_{i\gamma y j_5 j_6}$, with $j_5, j_6 \in \mathcal{J}^-$ distinct between them. As $\mathcal{J}^- = \{j_1, ..., j_4\}$, to avoid the contradiction of Lemma 1.5 we must impose $j_5, j_6 \neq j_1, j_4$, that is, $j_5 = j_2$ and $j_6 = j_3$. But, even so, we get an absurdity since $W_3, W_7 \in \mathcal{G}_{ij_2j_3}$.

Considering the assumption $y = k_{12}$ and using a similar reasoning we would end up once again with a contradiction. □

Next corollary it follows immediately from Corollary 2.1 and Propositions 2.2 and 2.3.

Corollary 2.2 *For each* $i \in \mathcal{I}$, $3 \leq |\mathcal{G}_i| \leq 8$.

Corollary 2.2 allows us further constrain in the range of variation of the parameter g. In fact, in these conditions we get $9 \leq g \leq 22$.

Our intention is to prove that any one of the admissible values for $|\mathcal{G}_i|$ leads to a contradiction. The proof of the impossibility of $|\mathcal{G}_i| = \alpha$ for $\alpha = 3, 4, 5, 8$ is much more complex and laborious than the previous ones, as we will see in the following chapters.

Next section is devoted to the establishment of new relations involving the cardinality of the index subsets of $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E} \cup \mathcal{F} \cup \mathcal{G}$. These results will be crucial on the analysis of the hypothesis $|\mathcal{G}_i| = \alpha$ for some $3 \leq \alpha \leq 8$.

2.3 Establishment of relations between the cardinality of index subsets of $\mathcal{T}$

Here, we present conditions which must be satisfied by index subsets of $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E} \cup \mathcal{F}$ when $|\mathcal{G}_i|$, $i \in \mathcal{I}$, assumes a specific value.

Lemma 2.9 *If* $|\mathcal{G}_i| = 3$*, for some* $i \in \mathcal{I}$*, then* $|\mathcal{A}_i| = 1$, $|\mathcal{B}_i \cup \mathcal{C}_i \cup \mathcal{E}_i| = 0$, $|\mathcal{D}_i| = 3$ *and* $|\mathcal{F}_i| = 13$*. More precisely,* $|\mathcal{D}_i^{(3)}| = 0$, $|\mathcal{D}_i^{(1)}| = 3$, $|\mathcal{F}_i^{(2)}| = 4$ *and* $|\mathcal{F}_i^{(1)}| = 9$*.*

Proof. Let $i \in \mathcal{I}$ such that $|\mathcal{G}_i| = 3$. By Lemma 2.8 we get

$$3 \geq \frac{|\mathcal{D}_i \cup \mathcal{E}_i| + 6}{3} - \frac{1}{6}$$

and, consequently, $|\mathcal{D}_i \cup \mathcal{E}_i| \leq 3$. From Lemma 2.1 it follows that $|\mathcal{D}_i \cup \mathcal{E}_i| \equiv 0 \pmod 3$, thus $|\mathcal{D}_i \cup \mathcal{E}_i| = 0$ or $|\mathcal{D}_i \cup \mathcal{E}_i| = 3$.

Consider $\mathcal{G}_i = \{W_1, W_2, W_3\}$. Let $W_1 \in \mathcal{G}_{ijklm}$, with $j, k, l, m \in \mathcal{I} \backslash \{i, -i\}$ and $|j|$, $|k|$, $|l|$, $|m|$ pairwise distinct. By Lemma 1.5 there exists, at most, one element $w \in \{j, k, l, m\}$ and, at most, one element $w' \in \{j, k, l, m\}$ such that $W_2 \in \mathcal{G}_{iw}$ and $W_3 \in \mathcal{G}_{iw'}$. Then, there are, at least, two distinct elements $x, y \in \{j, k, l, m\}$ satisfying $|\mathcal{G}_{ix}| = |\mathcal{G}_{iy}| = 1$. Taking into account Lemma 2.2, $|\mathcal{D}_{ix} \cup \mathcal{E}_{ix}|$ and $|\mathcal{G}_{ix}|$, as well as, $|\mathcal{D}_{iy} \cup \mathcal{E}_{iy}|$ and $|\mathcal{G}_{iy}|$ have the same parity. Thus, $|\mathcal{D}_{ix} \cup \mathcal{E}_{ix}|$ and $|\mathcal{D}_{iy} \cup \mathcal{E}_{iy}|$ are odd and, consequently, $|\mathcal{D}_i \cup \mathcal{E}_i| > 0$. Therefore, $|\mathcal{D}_i \cup \mathcal{E}_i| = 3$.

Since $|\mathcal{D}_i \cup \mathcal{E}_i| = 3$ and, by hypothesis, $|\mathcal{G}_i| = 3$, from Lemma 2.1 it follows that $|\mathcal{F}_i| = 13$. Consequently, by Lemma 2.6, we get

$$13 \leq \frac{49}{3} - 3 - \frac{2}{3}|\mathcal{E}_i|,$$

which implies $|\mathcal{E}_i| = 0$ and so $|\mathcal{D}_i| = 3$.

The codewords of $\mathcal{D}$ are of type $[\pm 3, \pm 1^2]$, thus $\mathcal{D}_i = \mathcal{D}_i^{(3)} \cup \mathcal{D}_i^{(1)}$. As $\mathcal{D}_i^{(3)} \cap \mathcal{D}_i^{(1)} = \varnothing$, we have $|\mathcal{D}_i| = |\mathcal{D}_i^{(3)}| + |\mathcal{D}_i^{(1)}|$. From Lemma 1.3 it follows that $|\mathcal{D}_i^{(3)}| \leq 1$ and so $|\mathcal{D}_i^{(1)}| \geq 2$.

The codewords of $\mathcal{F}$ are of type $[\pm 2, \pm 1^3]$. Thus, $\mathcal{F}_i = \mathcal{F}_i^{(2)} \cup \mathcal{F}_i^{(1)}$. Since $\mathcal{F}_i^{(2)} \cap \mathcal{F}_i^{(1)} = \varnothing$, then $|\mathcal{F}_i| = |\mathcal{F}_i^{(2)}| + |\mathcal{F}_i^{(1)}|$. By Lemma 2.4 we get $|\mathcal{F}_i^{(2)}| \leq 4$. As seen before $|\mathcal{F}_i| = 13$ and so $|\mathcal{F}_i^{(1)}| \geq 9$.

Taking into account that $|\mathcal{E}_i| = 0$, from Lemma 2.4 it follows that

$$|\mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)}| + 2|\mathcal{D}_i^{(3)}| + 3|\mathcal{F}_i^{(2)}| = 12 \tag{2.2}$$

and

$$|\mathcal{B}_i^{(1)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(1)} \cup \mathcal{F}_i^{(1)}| = 12. \tag{2.3}$$

Assuming $|\mathcal{F}_i^{(2)}| \leq 2$, then $|\mathcal{F}_i^{(1)}| \geq 11$. By (2.3) we get $|\mathcal{B}_i^{(1)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(1)}| \leq 1$, which is a contradiction since, as we have just seen, $|\mathcal{D}_i^{(1)}| \geq 2$. Then, $3 \leq |\mathcal{F}_i^{(2)}| \leq 4$.

Now suppose that $|\mathcal{F}_i^{(2)}| = 3$. Thus, $|\mathcal{F}_i^{(1)}| = 10$. As $|\mathcal{D}_i^{(1)}| \geq 2$, considering (2.3) we must have $|\mathcal{D}_i^{(1)}| = 2$ and $|\mathcal{B}_i^{(1)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)}| = 0$ implying $|\mathcal{D}_i^{(3)}| = 1$. Since $|\mathcal{F}_i^{(2)}| = 3$ and $|\mathcal{D}_i^{(3)}| = 1$, by (2.2) we get $|\mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)}| = 1$. But $|\mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)}| = 0$ and so $|\mathcal{B}_i^{(4)}| = 1$, contradicting Lemma 1.3 since $|\mathcal{B}_i^{(4)}| = |\mathcal{D}_i^{(3)}| = 1$.

Therefore, $|\mathcal{F}_i^{(2)}| = 4$ and $|\mathcal{F}_i^{(1)}| = 9$. Accordingly, from (2.2) it follows that $|\mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)}| = |\mathcal{D}_i^{(3)}| = 0$. As $\mathcal{C}_i = \mathcal{C}_i^{(3)} \cup \mathcal{C}_i^{(2)}$, then $|\mathcal{C}_i| = 0$.

Now, $|\mathcal{D}_i^{(1)}| = 3$ and $|\mathcal{D}_i^{(1)} \cup \mathcal{F}_i^{(1)}| = 12$. Considering (2.3) we get $|\mathcal{B}_i^{(1)}| = 0$. As $|\mathcal{B}_i^{(4)}| = |\mathcal{B}_i^{(1)}| = 0$ and $\mathcal{B}_i = \mathcal{B}_i^{(4)} \cup \mathcal{B}_i^{(1)}$, then $|\mathcal{B}_i| = 0$.

Using the fact that $|\mathcal{B}_i^{(4)}| = |\mathcal{C}_i^{(3)}| = |\mathcal{D}_i^{(3)}| = 0$ and Lemma 1.3, we get $|\mathcal{A}_i| = 1$. □

Lemma 2.10 *If $|\mathcal{G}_i| = 4$, for some $i \in \mathcal{I}$, then one and only one of the following conditions must occur:*

i) $|\mathcal{D}_i \cup \mathcal{E}_i| = 3$ *and* $|\mathcal{F}_i| = 11$*;*

ii) $|\mathcal{D}_i| = 6$, $|\mathcal{E}_i| = 0$ *and* $|\mathcal{F}_i| = 10$.

Besides, if ii) is satisfied, then $|\mathcal{A}_i| = 1$, $|\mathcal{B}_i \cup \mathcal{C}_i| = 0$, $|\mathcal{D}_i^{(1)}| = 6$, $|\mathcal{F}_i^{(2)}| = 4$ *and* $|\mathcal{F}_i^{(1)}| = 6$.

Proof. Let $i \in \mathcal{I}$ such that $|\mathcal{G}_i| = 4$. From Lemma 2.8 it follows that

$$4 \geq \frac{|\mathcal{D}_i \cup \mathcal{E}_i| + 6}{3} - \frac{1}{6}$$

and, consequently, $|\mathcal{D}_i \cup \mathcal{E}_i| \leq 6$.

Assume $\mathcal{G}_i = \{W_1, \ldots, W_4\}$. Let $W_1 \in \mathcal{G}_{ijklm}$, with $j, k, l, m \in \mathcal{I} \backslash \{i, -i\}$ and $|j|$, $|k|$, $|l|$, $|m|$ pairwise distinct. Taking into account Lemma 1.5, for each $W \in \mathcal{G}_i \backslash \{W_1\}$ there exists, at most, one element $w \in \{j, k, l, m\}$ so that $W \in \mathcal{G}_{iw}$. Accordingly, there exists, at least, one element $v \in \{j, k, l, m\}$ satisfying $|\mathcal{G}_{iv}| = 1$. By Lemma 2.2, $|\mathcal{G}_{iv}|$ and $|\mathcal{D}_{iv} \cup \mathcal{E}_{iv}|$ have the same parity. Thus, $|\mathcal{D}_{iv} \cup \mathcal{E}_{iv}|$ is odd and $0 < |\mathcal{D}_i \cup \mathcal{E}_i| \leq 6$.

From Lemma 2.1 it follows that

$$|\mathcal{D}_i \cup \mathcal{E}_i| + 3|\mathcal{F}_i| = 36. \tag{2.4}$$

Consequently, $|\mathcal{D}_i \cup \mathcal{E}_i| \equiv 0 \pmod 3$, that is, $|\mathcal{D}_i \cup \mathcal{E}_i| = 3$ or $|\mathcal{D}_i \cup \mathcal{E}_i| = 6$.

By (2.4), if $|\mathcal{D}_i \cup \mathcal{E}_i| = 3$ or $|\mathcal{D}_i \cup \mathcal{E}_i| = 6$ then $|\mathcal{F}_i| = 11$ or $|\mathcal{F}_i| = 10$, respectively.

Let us suppose that $|\mathcal{D}_i \cup \mathcal{E}_i| = 6$ and $|\mathcal{F}_i| = 10$.

Using Lemma 2.6 we get

$$10 \leq \frac{31}{3} - \frac{2}{3}|\mathcal{E}_i|,$$

and so $|\mathcal{E}_i| = 0$.

Since $|\mathcal{D}_i \cup \mathcal{E}_i| = 6$, $|\mathcal{D}_i| = 6$.

Noting that, $|\mathcal{D}_i| = |\mathcal{D}_i^{(3)}| + |\mathcal{D}_i^{(1)}|$, by Lemma 1.3 it follows that $|\mathcal{D}_i^{(3)}| \leq 1$. Since $|\mathcal{D}_i| = 6$, we get $|\mathcal{D}_i^{(1)}| \geq 5$.

Considering Lemma 2.4 and taking into account that $|\mathcal{E}_i| = 0$ we obtain the following equalities

$$|\mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)}| + 2|\mathcal{D}_i^{(3)}| + 3|\mathcal{F}_i^{(2)}| = 12 \tag{2.5}$$

and

$$|\mathcal{B}_i^{(1)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(1)} \cup \mathcal{F}_i^{(1)}| = 12. \tag{2.6}$$

From (2.5) we conclude that $|\mathcal{F}_i^{(2)}| \leq 4$. As $|\mathcal{F}_i| = |\mathcal{F}_i^{(2)}| + |\mathcal{F}_i^{(1)}|$ and $|\mathcal{F}_i| = 10$, then $|\mathcal{F}_i^{(1)}| \geq 6$.

As seen before $|\mathcal{D}_i^{(1)}| \geq 5$, then, by (2.6), $|\mathcal{F}_i^{(1)}| \leq 7$. Accordingly, $6 \leq |\mathcal{F}_i^{(1)}| \leq 7$ and so $3 \leq |\mathcal{F}_i^{(2)}| \leq 4$.

Suppose that $|\mathcal{F}_i^{(2)}| = 3$ and $|\mathcal{F}_i^{(1)}| = 7$. Taking into account (2.6) we must have $|\mathcal{D}_i^{(1)}| = 5$ and, consequently, $|\mathcal{B}_i^{(1)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)}| = 0$. Thus, $|\mathcal{D}_i^{(3)}| = 1$ and, by (2.5), $|\mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)}| = 1$. Since $|\mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)}| = 0$, it follows that $|\mathcal{B}_i^{(4)}| = 1$ leading to the contradiction of Lemma 1.3 ($|\mathcal{B}_i^{(4)}| = |\mathcal{D}_i^{(3)}| = 1$).

Let us now assume that $|\mathcal{F}_i^{(2)}| = 4$ and $|\mathcal{F}_i^{(1)}| = 6$. Considering (2.5) we get $|\mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(3)}| = 0$. Consequently, $|\mathcal{D}_i^{(1)}| = 6$ and from (2.6) it follows that $|\mathcal{B}_i^{(1)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)}| = 0$. As $\mathcal{B}_i = \mathcal{B}_i^{(4)} \cup \mathcal{B}_i^{(1)}$ and $\mathcal{C}_i = \mathcal{C}_i^{(3)} \cup \mathcal{C}_i^{(2)}$, then $|\mathcal{B}_i| = |\mathcal{C}_i| = 0$. Since $|\mathcal{B}_i^{(4)}| = |\mathcal{C}_i^{(3)}| = |\mathcal{D}_i^{(3)}| = 0$, from Lemma 1.3 it follows that $|\mathcal{A}_i| = 1$ □

Lemma 2.11 *If $|\mathcal{G}_i| = 5$, for some $i \in \mathcal{I}$, then one and only one of the following conditions must occur:*

i) $|\mathcal{D}_i \cup \mathcal{E}_i| = 0$ *and* $|\mathcal{F}_i| = 10$*;*

ii) $|\mathcal{D}_i \cup \mathcal{E}_i| = 3$ *and* $|\mathcal{F}_i| = 9$*;*

iii) $|\mathcal{D}_i \cup \mathcal{E}_i| = 6$, $|\mathcal{F}_i| = 8$ *and* $|\mathcal{D}_i| \geq 3$*;*

iv) $|\mathcal{D}_i| = 9$, $|\mathcal{E}_i| = 0$ *and* $|\mathcal{F}_i| = 7$*.*

Besides, if iv) is satisfied, then $|\mathcal{A}_i| = 1$, $|\mathcal{B}_i \cup \mathcal{C}_i| = 0$, $|\mathcal{D}_i^{(1)}| = 9$, $|\mathcal{F}_i^{(2)}| = 4$ *and* $|\mathcal{F}_i^{(1)}| = 3$*.*

Proof. Suppose that $|\mathcal{G}_i| = 5$ for some $i \in \mathcal{I}$. By Lemma 2.8 we get

$$5 \geq \frac{|\mathcal{D}_i \cup \mathcal{E}_i| + 6}{3} - \frac{1}{6}$$

which implies $|\mathcal{D}_i \cup \mathcal{E}_i| \leq 9$.

From Lemma 2.1 it follows that

$$|\mathcal{D}_i \cup \mathcal{E}_i| + 3|\mathcal{F}_i| = 30 \tag{2.7}$$

and, as an immediate consequence, $|\mathcal{D}_i \cup \mathcal{E}_i| \equiv 0 \pmod 3$.

Taking into account (2.7),

- if $|\mathcal{D}_i \cup \mathcal{E}_i| = 0$, then $|\mathcal{F}_i| = 10$;

- if $|\mathcal{D}_i \cup \mathcal{E}_i| = 3$, then $|\mathcal{F}_i| = 9$;

- if $|\mathcal{D}_i \cup \mathcal{E}_i| = 6$, then $|\mathcal{F}_i| = 8$;

- if $|\mathcal{D}_i \cup \mathcal{E}_i| = 9$, then $|\mathcal{F}_i| = 7$.

Assuming that $|\mathcal{D}_i \cup \mathcal{E}_i| = 6$ and $|\mathcal{F}_i| = 8$, considering Lemma 2.6 we get

$$8 \leq \frac{31}{3} - \frac{2}{3}|\mathcal{E}_i|.$$

Consequently, $|\mathcal{E}_i| \leq 3$ and as we are assuming $|\mathcal{D}_i \cup \mathcal{E}_i| = 6$, then $|\mathcal{D}_i| \geq 3$.

Assuming that $|\mathcal{D}_i \cup \mathcal{E}_i| = 9$ and $|\mathcal{F}_i| = 7$, from Lemma 2.6 it follows that

$$7 \leq \frac{22}{3} - \frac{2}{3}|\mathcal{E}_i|,$$

which implies $|\mathcal{E}_i| = 0$, and as an immediate consequence we obtain $|\mathcal{D}_i| = 9$.

Since $|\mathcal{D}_i| = |\mathcal{D}_i^{(3)}| + |\mathcal{D}_i^{(1)}|$, by Lemma 1.3, $|\mathcal{D}_i^{(3)}| \leq 1$ and so $|\mathcal{D}_i^{(1)}| \geq 8$.

By Lemma 2.4 and taking into account that $|\mathcal{E}_i| = 0$, we get

$$|\mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)}| + 2|\mathcal{D}_i^{(3)}| + 3|\mathcal{F}_i^{(2)}| = 12 \tag{2.8}$$

and

$$|\mathcal{B}_i^{(1)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(1)} \cup \mathcal{F}_i^{(1)}| = 12. \tag{2.9}$$

From (2.8) it follows that $|\mathcal{F}_i^{(2)}| \leq 4$. Thus, as $|\mathcal{F}_i| = 7$, we get $|\mathcal{F}_i^{(1)}| \geq 3$. On the other hand, since $|\mathcal{D}_i^{(1)}| \geq 8$, we conclude, by (2.9), that $|\mathcal{F}_i^{(1)}| \leq 4$. Therefore, $3 \leq |\mathcal{F}_i^{(1)}| \leq 4$ and, consequently, $3 \leq |\mathcal{F}_i^{(2)}| \leq 4$.

Suppose that $|\mathcal{F}_i^{(2)}| = 3$ which implies $|\mathcal{F}_i^{(1)}| = 4$. Considering (2.9), we must have $|\mathcal{D}_i^{(1)}| = 8$ and $|\mathcal{B}_i^{(1)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)}| = 0$. Since $|\mathcal{D}_i| = 9$, we get $|\mathcal{D}_i^{(3)}| = 1$. From (2.8) it follows that $|\mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)}| = 1$. As $|\mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)}| = 0$, then $|\mathcal{B}_i^{(4)}| = 1$, contradicting Lemma 1.3 ($|\mathcal{B}_i^{(4)}| = |\mathcal{D}_i^{(3)}| = 1$).

Let us now assume that $|\mathcal{F}_i^{(2)}| = 4$ and $|\mathcal{F}_i^{(1)}| = 3$. By (2.8) we obtain $|\mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(3)}| = 0$. Consequently, $|\mathcal{D}_i^{(1)}| = 9$. Considering (2.9) we get $|\mathcal{B}_i^{(1)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)}| = 0$. Accordingly, $|\mathcal{B}_i \cup \mathcal{C}_i| = 0$. From Lemma 1.3 it follows that $|\mathcal{A}_i \cup \mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(3)}| = 1$. Since $|\mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(3)}| = 0$, then $|\mathcal{A}_i| = 1$. □

Lemma 2.12 *If $|\mathcal{G}_i| = 6$, for some $i \in \mathcal{I}$, then one and only one of the following conditions must occur:*

i) $|\mathcal{D}_i \cup \mathcal{E}_i| = 0$ *and* $|\mathcal{F}_i| = 8$*;*

ii) $|\mathcal{D}_i \cup \mathcal{E}_i| = 3$ *and* $|\mathcal{F}_i| = 7$*;*

iii) $|\mathcal{D}_i \cup \mathcal{E}_i| = 6$ *and* $|\mathcal{F}_i| = 6$*;*

iv) $|\mathcal{D}_i \cup \mathcal{E}_i| = 9$*,* $|\mathcal{F}_i| = 5$ *and* $|\mathcal{D}_i| \geq 6$*;*

v) $|\mathcal{D}_i| = 12$*,* $|\mathcal{E}_i| = 0$ *and* $|\mathcal{F}_i| = 4$*.*

Besides, if v) is satisfied, then $|\mathcal{A}_i| = 1$*,* $|\mathcal{B}_i \cup \mathcal{C}_i| = 0$*,* $|\mathcal{D}_i^{(1)}| = 12$ *and* $|\mathcal{F}_i^{(2)}| = 4$*.*

Proof. Let us assume $|\mathcal{G}_i| = 6$ for $i \in \mathcal{I}$. By Lemmas 2.3 and 2.1 one has:

$$|\mathcal{D}_i \cup \mathcal{E}_i| \leq 13;$$

$$|\mathcal{D}_i \cup \mathcal{E}_i| + 3|\mathcal{F}_i| = 24 \tag{2.10}$$

with $|\mathcal{D}_i \cup \mathcal{E}_i| \equiv 0 \pmod 3$.

Considering (2.10), and analyzing all the possibilities for $|\mathcal{D}_i \cup \mathcal{E}_i|$ and $|\mathcal{F}_i|$ we get,

- if $|\mathcal{D}_i \cup \mathcal{E}_i| = 0$, then $|\mathcal{F}_i| = 8$;
- if $|\mathcal{D}_i \cup \mathcal{E}_i| = 3$, then $|\mathcal{F}_i| = 7$;
- if $|\mathcal{D}_i \cup \mathcal{E}_i| = 6$, then $|\mathcal{F}_i| = 6$;
- if $|\mathcal{D}_i \cup \mathcal{E}_i| = 9$, then $|\mathcal{F}_i| = 5$;
- if $|\mathcal{D}_i \cup \mathcal{E}_i| = 12$, then $|\mathcal{F}_i| = 4$.

Assuming that $|\mathcal{D}_i \cup \mathcal{E}_i| = 9$ and $|\mathcal{F}_i| = 5$, by Lemma 2.6 we get

$$5 \leq \frac{22}{3} - \frac{2}{3}|\mathcal{E}_i|.$$

Consequently, $|\mathcal{E}_i| \leq 3$ and so $|\mathcal{D}_i| \geq 6$.

Assuming that $|\mathcal{D}_i \cup \mathcal{E}_i| = 12$ and $|\mathcal{F}_i| = 4$, from Lemma 2.6 it follows that

$$4 \leq \frac{13}{3} - \frac{2}{3}|\mathcal{E}_i|$$

which implies $|\mathcal{E}_i| = 0$. Consequently, $|\mathcal{D}_i| = 12$. Besides, taking into account Lemma 1.3, $|\mathcal{D}_i^{(3)}| \leq 1$, and so, $|\mathcal{D}_i^{(1)}| \geq 11$.

Considering Lemma 2.4 and having in mind that $|\mathcal{E}_i| = 0$, we get

$$|\mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)}| + 2|\mathcal{D}_i^{(3)}| + 3|\mathcal{F}_i^{(2)}| = 12 \tag{2.11}$$

and

$$|\mathcal{B}_i^{(1)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(1)} \cup \mathcal{F}_i^{(1)}| = 12. \tag{2.12}$$

Since $|\mathcal{D}_i^{(1)}| \geq 11$, by (2.12) we conclude that $|\mathcal{F}_i^{(1)}| \leq 1$. As we are supposing $|\mathcal{F}_i| = 4$, we get $3 \leq |\mathcal{F}_i^{(2)}| \leq 4$.

Suppose that $|\mathcal{F}_i^{(2)}| = 3$ and $|\mathcal{F}_i^{(1)}| = 1$. Accordingly, from (2.12), $|\mathcal{D}_i^{(1)}| = 11$ and $|\mathcal{B}_i^{(1)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)}| = 0$. In these conditions, $|\mathcal{D}_i^{(3)}| = 1$. Considering (2.11) we get $|\mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)}| = 1$. However, as $|\mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)}| = 0$, we conclude that $|\mathcal{B}_i^{(4)}| = 1$, which is a contradiction since, by Lemma 1.3, $|\mathcal{B}_i^{(4)} \cup \mathcal{D}_i^{(3)}| \leq 1$.

Now assume that $|\mathcal{F}_i^{(2)}| = 4$. By (2.11) it follows that $|\mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(3)}| = 0$. Thus, $|\mathcal{D}_i^{(1)}| = 12$ and, by (2.12), we conclude that $|\mathcal{B}_i^{(1)} \cup \mathcal{C}_i^{(2)} \cup \mathcal{C}_i^{(3)}| = 0$. Therefore, $|\mathcal{B}_i \cup \mathcal{C}_i| = 0$. As, by Lemma 1.3, $|\mathcal{A}_i \cup \mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(3)}| = 1$, we must have $|\mathcal{A}_i| = 1$. □

Lemma 2.13 *If $|\mathcal{G}_i| = 7$, for some $i \in \mathcal{I}$, then one and only one of the following conditions must occur:*

i) $|\mathcal{D}_i \cup \mathcal{E}_i| = 3$ *and* $|\mathcal{F}_i| = 5$*;*

ii) $|\mathcal{D}_i \cup \mathcal{E}_i| = 6$ *and* $|\mathcal{F}_i| = 4$*;*

iii) $|\mathcal{D}_i \cup \mathcal{E}_i| = 9$, $|\mathcal{F}_i| = 3$ *and* $|\mathcal{D}_i| \geq 3$*;*

iv) $|\mathcal{D}_i \cup \mathcal{E}_i| = 12$, $|\mathcal{F}_i| = 2$ *and* $|\mathcal{D}_i| \geq 9$.

Proof. Let $i \in \mathcal{I}$ such that $|\mathcal{G}_i| = 7$. By Lemmas 1.8 and 2.1, we get $|\mathcal{D}_i \cup \mathcal{E}_i| \leq 13$ and $|\mathcal{D}_i \cup \mathcal{E}_i| \equiv 0 \pmod 3$. Therefore, $0 \leq |\mathcal{D}_i \cup \mathcal{E}_i| \leq 12$.

The cardinality of $\mathcal{G}_i$ can be related with the cardinality of its index subsets as follows

$$|\mathcal{G}_i| = \frac{1}{4} \sum_{j \in \mathcal{I} \setminus \{i,-i\}} |\mathcal{G}_{ij}|.$$

As, by hypothesis, $|\mathcal{G}_i| = 7$, then

$$\sum_{j \in \mathcal{I} \setminus \{i,-i\}} |\mathcal{G}_{ij}| = 28. \tag{2.13}$$

From Lemma 2.2 it follows that $|\mathcal{G}_{ij}| \leq 3$ for all $j \in \mathcal{I} \setminus \{i, -i\}$. Since $|\mathcal{I} \setminus \{i, -i\}| = 12$, taking into account (2.13), there are, at least, four distinct elements $j \in \mathcal{I} \setminus \{i, -i\}$ such that $|\mathcal{G}_{ij}| = 3$. Applying Lemma 2.2 for these indices j the cardinality of $\mathcal{D}_{ij} \cup \mathcal{E}_{ij}$ is odd. Thus, $|\mathcal{D}_i \cup \mathcal{E}_i| > 0$ and, consequently, $3 \leq |\mathcal{D}_i \cup \mathcal{E}_i| \leq 12$.

By Lemma 2.1 we get

$$|\mathcal{D}_i \cup \mathcal{E}_i| + 3|\mathcal{F}_i| = 18. \tag{2.14}$$

Considering (2.14), for each admissible value of $|\mathcal{D}_i \cup \mathcal{E}_i|$ we obtain the correspondent value for $|\mathcal{F}_i|$. Namely,

- if $|\mathcal{D}_i \cup \mathcal{E}_i| = 3$, then $|\mathcal{F}_i| = 5$;
- if $|\mathcal{D}_i \cup \mathcal{E}_i| = 6$, then $|\mathcal{F}_i| = 4$;
- if $|\mathcal{D}_i \cup \mathcal{E}_i| = 9$, then $|\mathcal{F}_i| = 3$;
- if $|\mathcal{D}_i \cup \mathcal{E}_i| = 12$, then $|\mathcal{F}_i| = 2$.

Suppose that $|\mathcal{D}_i \cup \mathcal{E}_i| = 9$ and $|\mathcal{F}_i| = 3$. From Lemma 2.6 it follows that

$$3 \leq \frac{22}{3} - \frac{2}{3}|\mathcal{E}_i|.$$

Consequently, $|\mathcal{E}_i| \leq 6$. As we are supposing $|\mathcal{D}_i \cup \mathcal{E}_i| = 9$, in these conditions we get $|\mathcal{D}_i| \geq 3$.

Now assume that $|\mathcal{D}_i \cup \mathcal{E}_i| = 12$ and $|\mathcal{F}_i| = 2$. Considering Lemma 2.6 we obtain

$$2 \leq \frac{13}{3} - \frac{2}{3}|\mathcal{E}_i|$$

which implies $|\mathcal{E}_i| \leq 3$ and, consequently, $|\mathcal{D}_i| \geq 9$. □

In the previous lemmas we have found, for each admissible value of $|\mathcal{G}_i|$, $i \in \mathcal{I}$, a range for the variation of $|\mathcal{F}_i|$. In the next chapters we will prove that $|\mathcal{G}_i| = \alpha$ for $\alpha = 3, 4, 5, 8$ lead us to an absurdity. To achieve the contradiction we will focus our attention mostly in the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$, since these are the codewords that have more nonzero coordinates. Our interest is to characterize all codewords of $\mathcal{G}_i \cup \mathcal{F}_i$, $i \in \mathcal{I}$, assuming a certain admissible value for $|\mathcal{G}_i|$ and having in mind the preceding lemmas.

In the characterization of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$, $i \in \mathcal{I}$, it is natural to think that the most difficult cases are those for which $|\mathcal{F}_i|$ is the lower possible value when assumed a certain value for $|\mathcal{G}_i|$. So, it is convenient to get some conditions to overcome this difficulty.

Observing Lemmas 2.10, 2.11 and 2.12, we verify that there exists a common condition when $|\mathcal{F}_i|$ is minimal. In fact, in all cases we get $|\mathcal{F}_i^{(2)}| = 4$.

We would like to pointed out that, by Lemma 2.9, if $|\mathcal{G}_i| = 3$, then we must also have $|\mathcal{F}_i^{(2)}| = 4$. The condition $|\mathcal{F}_i^{(2)}| = 4$ is quite strong. As we will see in the next result, the characterization of the four codewords of $\mathcal{F}_i^{(2)}$ involves all index subsets of $\mathcal{F} \backslash \mathcal{F}_{-i}$.

Lemma 2.14 *For each $i \in \mathcal{I}$, $|\mathcal{F}_i^{(2)}| \leq 4$. If $|\mathcal{F}_i^{(2)}| = 4$, then $|\mathcal{F}_i^{(2)} \cap \mathcal{F}_j| = 1$ for all $j \in \mathcal{I} \backslash \{i, -i\}$*

Proof. From Lemma 2.4 it follows right away that $|\mathcal{F}_i^{(2)}| \leq 4$ for all $i \in \mathcal{I}$.

Suppose that $|\mathcal{F}_i^{(2)}| = 4$ for some $i \in \mathcal{I}$. If there exists $j \in \mathcal{I} \backslash \{i, -i\}$ such that $|\mathcal{F}_i^{(2)} \cap \mathcal{F}_j| \geq 2$, then Lemma 1.4 is contradicted. Therefore, for each $j \in \mathcal{I} \backslash \{i, -i\}$ we get $|\mathcal{F}_i^{(2)} \cap \mathcal{F}_j| \leq 1$.

Let W_1,...,$W_4 \in \mathcal{F}_i^{(2)}$ such that $W_1 \in \mathcal{F}_{iw_1w_2w_3}$, $W_2 \in \mathcal{F}_{iw_4w_5w_6}$, $W_3 \in \mathcal{F}_{iw_7w_8w_9}$ and $W_4 \in \mathcal{F}_{iw_{10}w_{11}w_{12}}$, with $w_1, \ldots, w_{12} \in \mathcal{I} \backslash \{i, -i\}$. Since $|\mathcal{F}_i^{(2)} \cap \mathcal{F}_j| \leq 1$ for each $j \in \mathcal{I} \backslash \{i, -i\}$, then $w_1, \ldots, w_{12}$ must be pairwise distinct. As $|\mathcal{I} \backslash \{i, -i\}| = 12$, we conclude that $\{w_1,...,w_{12}\} = \mathcal{I} \backslash \{i, -i\}$. Thus, $|\mathcal{F}_i^{(2)} \cap \mathcal{F}_j| = 1$ for all $j \in \mathcal{I} \backslash \{i, -i\}$. $\square$

The next lemma establishes the variation of $|\mathcal{F}_i|$, $i \in \mathcal{I}$, when $4 \leq |\mathcal{G}_i| \leq 6$ under certain conditions.

Lemma 2.15 *Let $\mathcal{G}_i$ for $i \in \mathcal{I}$. For all $j \in \mathcal{I}\backslash\{i, -i\}$, $|\mathcal{G}_{ij}| \leq 3$. If $|\mathcal{G}_{ij}| = 3$ for some $j \in \mathcal{I}\backslash\{i, -i\}$, then $|\mathcal{F}_i^{(2)}| \leq 3$. Besides,*

i) $|\mathcal{G}_i| \neq 3$;

ii) *if* $|\mathcal{G}_i| = 4$, *then* $|\mathcal{F}_i| = 11$;

iii) *if* $|\mathcal{G}_i| = 5$, *then* $8 \leq |\mathcal{F}_i| \leq 10$;

iv) *if* $|\mathcal{G}_i| = 6$, *then* $5 \leq |\mathcal{F}_i| \leq 8$.

Proof. Let us consider $\mathcal{G}_i$ for $i \in \mathcal{I}$. By Lemma 2.2, $|\mathcal{G}_{ij}| \leq 3$ for all $j \in \mathcal{I}\backslash\{i, -i\}$.

Suppose that there exists $k \in \mathcal{I}\backslash\{i, -i\}$ so that $|\mathcal{G}_{ik}| = 3$. From Lemma 2.2 it follows that $|\mathcal{F}_{ik}| = 0$. Looking at Lemma 1.9, one has $|\mathcal{F}_i^{(2)}| \leq 4$.

Assume, by contradiction, that $|\mathcal{F}_i^{(2)}| = 4$. Then, by Lemma 1.9, $|\mathcal{F}_i^{(2)} \cap \mathcal{F}_j| = 1$ for all $j \in \mathcal{I}\backslash\{i, -i\}$, and so $|\mathcal{F}_{ik}| \geq 1$ contradicting $|\mathcal{F}_{ik}| = 0$. Therefore, for $|\mathcal{G}_{ik}| = 3$, one has $|\mathcal{F}_i^{(2)}| \leq 3$.

Lemmas 2.9, 2.10, 2.11 and 2.12, lead straightaway to $|\mathcal{G}_i| \neq 3$ and:

- if $|\mathcal{G}_i| = 4$, then $|\mathcal{F}_i| = 11$;

- if $|\mathcal{G}_i| = 5$, then $8 \leq |\mathcal{F}_i| \leq 10$;

- if $|\mathcal{G}_i| = 6$, then $5 \leq |\mathcal{F}_i| \leq 8$.

□

Chapter 3

Proof of $|\mathcal{G}_i| \neq 8$ for any $i \in \mathcal{I}$

In this chapter we intend to restrict the range of variation of $|\mathcal{G}_i|$ proving that $|\mathcal{G}_i| \neq 8$ for any $i \in \mathcal{I}$. The referred proof will be achieved by contradiction assuming that there exists an element $i \in \mathcal{I}$ so that $|\mathcal{G}_i| = 8$.

Initially some conditions that subsets of $\mathcal{T}$ must satisfy are derived. In last section we present the proof of the main result of this chapter, that is, $|\mathcal{G}_i| \neq 8$ for any $i \in \mathcal{I}$.

Let us suppose that there exists $i \in \mathcal{I}$ such that $|\mathcal{G}_i| = 8$. Thus, since

$$8 = \frac{1}{4} \sum_{j \in \mathcal{I} \backslash \{i,-i\}} |\mathcal{G}_{ij}|,$$

we get

$$\sum_{j \in \mathcal{I} \backslash \{i,-i\}} |\mathcal{G}_{ij}| = 32.$$

From Lemma 2.2 it follows that $|\mathcal{G}_{ij}| \leq 3$ for all $j \in \mathcal{I} \backslash \{i, -i\}$. Particular attention will be given to the elements $j \in \mathcal{I} \backslash \{i, -i\}$ so that $|\mathcal{G}_{ij}| = 3$ or $|\mathcal{G}_{ij}| = 2$.

Throughout this chapter $\mathcal{J}$ and $\mathcal{K}$ will denote the following sets:

$$\mathcal{J} = \{j \in \mathcal{I} \backslash \{i, -i\} : |\mathcal{G}_{ij}| = 3\}$$

and

$$\mathcal{K} = \{k \in \mathcal{I} \backslash \{i, -i\} : |\mathcal{G}_{ik}| = 2\}.$$

3.1 Preliminary results

We are interested in proving that the existence of an element $i \in \mathcal{I}$ for which $|\mathcal{G}_i| = 8$ will bring out contradictions on the definition of being a PL(7, 2) code. We begin by characterizing partially the index distribution of the codewords $W_1, \ldots, W_8 \in \mathcal{G}_i$.

Proposition 3.1 *If* $|\mathcal{G}_i| = 8$, $i \in \mathcal{I}$, *then* $\mathcal{I}\backslash\{i, -i\} = \mathcal{J} \cup \mathcal{K}$, *with* $|\mathcal{J}| = 8$ *and* $|\mathcal{K}| = 4$. *The partial index distribution of the codewords* $W_1, \ldots, W_8 \in \mathcal{G}_i$ *satisfies:*

W_1	i	k_1	x	y
W_2	i	k_2	x	$-y$
W_3	i	k_3	x	
W_4	i	k_4	$-x$	y
W_5	i	k_5	$-x$	$-y$
W_6	i	k_6	$-x$	
W_7	i	k_7	y	
W_8	i	k_8	$-y$	

where $x, -x, y, -y \in \mathcal{J}$ *and* $k_1, \ldots, k_8 \in \mathcal{K}$. *Consequently, for all* $W \in \mathcal{G}_i$ *there exists a unique element* $k \in \mathcal{K}$ *such that* $W \in \mathcal{G}_{ik}$.

Proof. Let $i \in \mathcal{I}$ such that $|\mathcal{G}_i| = 8$. In these conditions,

$$8 = \frac{1}{4} \sum_{j \in \mathcal{I}\backslash\{i,-i\}} |\mathcal{G}_{ij}|.$$

Equivalently,

$$\sum_{j \in \mathcal{I}\backslash\{i,-i\}} |\mathcal{G}_{ij}| = 32. \tag{3.1}$$

By Lemma 2.2, for any $j \in \mathcal{I}\backslash\{i, -i\}$ we get $|\mathcal{G}_{ij}| \leq 3$. Since $|\mathcal{I}\backslash\{i, -i\}| = 12$, taking into account (3.1) we conclude that there are, at least, eight elements $j \in \mathcal{I}\backslash\{i, -i\}$ satisfying $|\mathcal{G}_{ij}| = 3$. We have just concluded that $|\mathcal{J}| \geq 8$.

Let us consider

$$\mathcal{L} = \{l \in \mathcal{I}\backslash\{i, -i\} : |\mathcal{G}_{il}| \leq 2\}.$$

Observing that, $\mathcal{J} \cup \mathcal{L} = \mathcal{I}\backslash\{i, -i\}$, $\mathcal{J} \cap \mathcal{L} = \varnothing$, $|\mathcal{I}\backslash\{i, -i\}| = 12$ and $|\mathcal{J}| \geq 8$, then $|\mathcal{L}| \leq 4$. Thus, there are, at most, four distinct elements $j \in \mathcal{J}$ such that $-j \in \mathcal{L}$.

Since $|\mathcal{J}| \geq 8$, there exist $x, y \in \mathcal{J}$, distinct between them, so that $-x, -y \in \mathcal{J}$. Then, let us consider $x, -x, y, -y \in \mathcal{J}$.

By definition of $\mathcal{J}$, $|\mathcal{G}_{ix}| = |\mathcal{G}_{i,-x}| = |\mathcal{G}_{iy}| = |\mathcal{G}_{i,-y}| = 3$. Taking into account Lemma 1.5, the partial index distribution of the codewords $W_1, \ldots, W_8 \in \mathcal{G}_i$ must satisfy the conditions presented in the following table.

W_1	i	x	y
W_2	i	x	$-y$
W_3	i	x	
W_4	i	$-x$	y
W_5	i	$-x$	$-y$
W_6	i	$-x$	
W_7	i	y	
W_8	i	$-y$	

Table 3.1: Partial index distribution of the codewords of $\mathcal{G}_i$.

That is, $W_1 \in \mathcal{G}_{ixy}$, $W_2 \in \mathcal{G}_{i,x,-y}$ and so on.

Looking at $W_1 \in \mathcal{G}_{ixy}$, there are $\alpha, \beta \in \mathcal{I}\backslash\{i, -i, x, -x, y, -y\}$ so that $W_1 \in \mathcal{G}_{ixy\alpha\beta}$. Suppose that $\alpha, \beta \in \mathcal{J}$, that is, $|\mathcal{G}_{i\alpha}| = |\mathcal{G}_{i\beta}| = 3$. Considering Lemma 1.5, $|\mathcal{G}_{ix\alpha}| = |\mathcal{G}_{iy\alpha}| = |\mathcal{G}_{ix\beta}| = |\mathcal{G}_{iy\beta}| = 1$. Besides, $\mathcal{G}_{ix\alpha} = \mathcal{G}_{iy\alpha} = \mathcal{G}_{ix\beta} = \mathcal{G}_{iy\beta} = \{W_1\}$. Since $|\mathcal{G}_{i\alpha}| = 3$, taking into account Table 3.1 and Lemma 1.5, $\mathcal{G}_{i\alpha}\backslash\{W_1\} \subset \{W_5, W_6, W_8\}$ and $\mathcal{G}_{i\beta}\backslash\{W_1\} \subset \{W_5, W_6, W_8\}$. As $|\mathcal{G}_{i\alpha}\backslash\{W_1\}| = |\mathcal{G}_{i\beta}\backslash\{W_1\}| = 2$, there exists $W \in \{W_5, W_6, W_8\}$ such that $W \in \mathcal{G}_{i\alpha\beta}$, which contradicts Lemma 1.5 since $W, W_1 \in \mathcal{G}_{i\alpha\beta}$. Therefore, there exists $l_1 \in \mathcal{L}$ so that $W_1 \in \mathcal{G}_{ixyl_1}$. Similarly, there are $l_2, l_4, l_5 \in \mathcal{L}$ such that $W_2 \in \mathcal{G}_{i,x,-y,l_2}$, $W_4 \in \mathcal{G}_{i,-x,y,l_4}$ and $W_5 \in \mathcal{G}_{i,-x,-y,l_5}$.

Let us consider $W_3 \in \mathcal{G}_{ix}$. Having in view $W_1, W_2 \in \mathcal{G}_{ix}$ and Lemma 1.5, there are $\alpha, \beta, \gamma \in \mathcal{I}\backslash\{i, -i, x, -x, y, -y\}$ so that $W_3 \in \mathcal{G}_{ix\alpha\beta\gamma}$. Assume that $\{\alpha, \beta, \gamma\} \subset \mathcal{J}$. Then, $|\mathcal{G}_{i\alpha}| = |\mathcal{G}_{i\beta}| = |\mathcal{G}_{i\gamma}| = 3$. Accordingly, by Lemma 1.5, $|\mathcal{G}_{ix\alpha}| = |\mathcal{G}_{ix\beta}| = |\mathcal{G}_{ix\gamma}| = 1$ and, consequently, $\mathcal{G}_{ix\alpha} = \mathcal{G}_{ix\beta} = \mathcal{G}_{ix\gamma} = \{W_3\}$. Taking into account Table 3.1 and Lemma 1.5, we get: $\mathcal{G}_{i\alpha}\backslash\{W_3\} \subset \{W_4, \ldots, W_8\}$; $\mathcal{G}_{i\beta}\backslash\{W_3\} \subset \{W_4, \ldots, W_8\}$; $\mathcal{G}_{i\gamma}\backslash\{W_3\} \subset \{W_4, \ldots, W_8\}$. As $|\mathcal{G}_{i\alpha}\backslash\{W_3\}| = |\mathcal{G}_{i\beta}\backslash\{W_3\}| = |\mathcal{G}_{i\gamma}\backslash\{W_3\}| = 2$ and

$|\{W_4,\ldots,W_8\}| = 5$, there exists $W \in \{W_4,\ldots,W_8\}$ such that $W \in \mathcal{G}_{i\varepsilon\theta}$ for $\varepsilon,\theta \in \{\alpha,\beta,\gamma\}$, which contradicts Lemma 1.5 since $W, W_3 \in \mathcal{G}_{i\varepsilon\theta}$. Thus, there exists $l_3 \in \mathcal{L}$ so that $W_3 \in \mathcal{G}_{ixl_3}$. Likewise, there are $l_6, l_7, l_8 \in \mathcal{L}$ so that $W_6 \in \mathcal{G}_{i,-x,l_6}$, $W_7 \in \mathcal{G}_{iyl_7}$ and $W_8 \in \mathcal{G}_{i,-y,l_8}$.

Therefore, for all $W \in \mathcal{G}_i$ there exists $l \in \mathcal{L}$ such that $W \in \mathcal{G}_{il}$.

By definition of $\mathcal{L}$, $|\mathcal{G}_{il}| \leq 2$ for all $l \in \mathcal{L}$. We have concluded before that $|\mathcal{L}| \leq 4$. Since for any $W \in \mathcal{G}_i$ there exists $l \in \mathcal{L}$ such that $W \in \mathcal{G}_{il}$ and $|\mathcal{G}_i| = 8$, we must impose $|\mathcal{L}| = 4$ and $|\mathcal{G}_{il}| = 2$ for any $l \in \mathcal{L}$. That is, $\mathcal{K} = \{k \in \mathcal{I}\backslash\{i,-i\} : |\mathcal{G}_{ik}| = 2\}$ is such that $|\mathcal{K}| = 4$. Consequently, for each $W \in \mathcal{G}_i$ there exists a unique element $k \in \mathcal{K}$ such that $W \in \mathcal{G}_{ik}$. Furthermore, $|\mathcal{J}| = 8$ and $\mathcal{I}\backslash\{i,-i\} = \mathcal{J} \cup \mathcal{K}$.

Thus, the partial index distribution of the codewords of $\mathcal{G}_i$ satisfies:

W_1	i	k_1	x	y
W_2	i	k_2	x	$-y$
W_3	i	k_3	x	
W_4	i	k_4	$-x$	y
W_5	i	k_5	$-x$	$-y$
W_6	i	k_6	$-x$	
W_7	i	k_7	y	
W_8	i	k_8	$-y$	

Table 3.2: Partial index distribution of the codewords of $\mathcal{G}_i$.

where $x, -x, y, -y \in \mathcal{J}$ and $k_1, \ldots, k_8 \in \mathcal{K}$. □

The following result characterizes with more detail the set $\mathcal{K}$ and, consequently, the set $\mathcal{J}$.

Proposition 3.2 *If $k \in \mathcal{K}$, then $-k \in \mathcal{K}$.*

Proof. We are assuming $|\mathcal{G}_i| = 8$ for $i \in \mathcal{I}$. The partial index distribution of the codewords $W_1, \ldots, W_8 \in \mathcal{G}_i$ satisfies the conditions enunciated in Proposition 3.1. We recall that, from this proposition it follows that $\mathcal{I}\backslash\{i,-i\} = \mathcal{J} \cup \mathcal{K}$, with $|\mathcal{J}| = 8$ and $|\mathcal{K}| = 4$. Furthermore, $\{x,-x,y,-y\} \subset \mathcal{J}$ and $\{k_1,\ldots,k_8\} = \mathcal{K}$.

Let us consider $\mathcal{N} = \mathcal{J}\backslash\{x,-x,y,-y\} = \{\alpha,\beta,\gamma,\delta\}$. We note that,

$$\mathcal{I}\backslash\{i,-i\} = \{k_1,\ldots,k_8\} \cup \{x,-x,y,-y\} \cup \{\alpha,\beta,\gamma,\delta\}.$$

By Proposition 3.1, for each $W \in \mathcal{G}_i$ there exists a unique element $k \in \mathcal{K}$ so that $W \in \mathcal{G}_{ik}$. On the other hand, since $|\mathcal{G}_{ij}| = 3$ for all $j \in \mathcal{J}$, we have identified all codewords of $\mathcal{G}_{ix}$, $\mathcal{G}_{i,-x}$, $\mathcal{G}_{iy}$ and $\mathcal{G}_{i,-y}$. Thus, to characterize completely the index distribution of all codewords of $\mathcal{G}_i$ we must fill in with elements of $\mathcal{N}$ the empty entries of the table presented in Proposition 3.1.

Consider $W_1, W_2, W_3 \in \mathcal{G}_{ix}$, see table in Proposition 3.1. Taking into account Lemma 1.5, the index distribution of the codewords of $\mathcal{G}_{ix}$ must satisfy:

W_1	i	k_1	x	y	α
W_2	i	k_2	x	$-y$	β
W_3	i	k_3	x	γ	δ
W_4	i	k_4	$-x$	y	
W_5	i	k_5	$-x$	$-y$	
W_6	i	k_6	$-x$		
W_7	i	k_7	y		
W_8	i	k_8	$-y$		

Table 3.3: Partial index distribution of the codewords of $\mathcal{G}_i$.

Let us now consider the codeword $W_4 \in \mathcal{G}_{i,k_4,-x,y}$. Having in mind Lemma 1.5 we conclude that $W_4 \notin \mathcal{G}_\alpha$, otherwise we would get $W_1, W_4 \in \mathcal{G}_{iy\alpha}$. Suppose that $W_4 \in \mathcal{G}_\beta$. In these conditions, $W_4, W_2 \in \mathcal{G}_{i\beta}$, with $W_4 \in \mathcal{G}_{i,k_4,-x,y,\beta}$ and $W_2 \in \mathcal{G}_{i,k_2,x,-y,\beta}$. Since $|\mathcal{G}_{i\beta}| = 3$ ($\beta \in \mathcal{J}$), there exists $W \in \mathcal{G}_i\backslash\{W_1,W_2,W_3,W_4\}$ such that $W \in \mathcal{G}_{i\beta}$. Analyzing Table 3.3 we verify that $W \in \mathcal{G}_{i,\beta,-x} \cup \mathcal{G}_{i\beta y} \cup \mathcal{G}_{i,\beta,-y}$. Consequently, taking into account W_2 and W_4, $|\mathcal{G}_{i\beta z}| \geq 2$ for some $z \in \{-x,y,-y\}$, contradicting Lemma 1.5.

Therefore, $W_4 \in \mathcal{G}_\gamma \cup \mathcal{G}_\delta$. By a similar reasoning, we are led to the conclusion that $W_5 \in \mathcal{G}_\gamma \cup \mathcal{G}_\delta$.

We are assuming $W_3 \in \mathcal{G}_{ik_3x\gamma\delta}$. As $k_3 \in \mathcal{K}$, by definition of $\mathcal{K}$ we get $|\mathcal{G}_{ik_3}| = 2$. Thus, there exists $k \in \{k_1,...,k_8\}\backslash\{k_3\}$ such that $k = k_3$. We note that, $k_3 \neq k_1, k_2$, otherwise Lemma 1.5 is contradicted. Since $W_4, W_5 \in \mathcal{G}_\gamma \cup \mathcal{G}_\delta$, taking into account Lemma 1.5 we conclude that $k_3 \neq k_4, k_5$. Therefore, $k \in \{k_6, k_7, k_8\}$. If $k_3 = k_7$, then Lemma 1.5 forces $W_7 \in \mathcal{G}_{ik_7y\alpha\beta}$, which is a contradiction, since $W_1, W_7 \in \mathcal{G}_{iy\alpha}$. Then, $k_3 \neq k_7$. By a similar reasoning we may conclude that $k_3 \neq k_8$. Consequently, $k_3 = k_6$ and, applying once again Lemma 1.5, we must impose $W_6 \in \mathcal{G}_{i,k_3,-x,\alpha,\beta}$.

Note that $|\mathcal{G}_{i\alpha}| = |\mathcal{G}_{i\beta}| = 3$. Since $W_4, W_5 \in \mathcal{G}_\gamma \cup \mathcal{G}_\delta$, we must obligate $W_7, W_8 \in \mathcal{G}_\alpha \cup \mathcal{G}_\beta$. Considering W_1 and W_2, Lemma 1.5 leads us to conclude that $W_7 \in \mathcal{G}_\beta$ and $W_8 \in \mathcal{G}_\alpha$.

Accordingly, the partial index distribution of the codewords of $\mathcal{G}_i$ satisfies:

W_1	i	k_1	x	y	α
W_2	i	k_2	x	$-y$	β
W_3	i	k_3	x	γ	δ
W_4	i	k_4	$-x$	y	
W_5	i	k_5	$-x$	$-y$	
W_6	i	k_3	$-x$	α	β
W_7	i	k_7	y	β	
W_8	i	k_8	$-y$	α	

Table 3.4: Partial index distribution of the codewords of $\mathcal{G}_i$.

Note that, as $|\mathcal{G}_{i\gamma}| = |\mathcal{G}_{i\delta}| = 3$, the four empty entries of this table must be filled in with γ and δ. Thus, $W_4, W_5, W_7, W_8 \in \mathcal{G}_\gamma \cup \mathcal{G}_\delta$.

Consider the elements of $\mathcal{K}$. By the analysis of the entries of the previous table, to avoid the contradiction of Lemma 1.5, one should have $k_1 = k_5$, $k_2 = k_4$ and $k_7 = k_8$. That is, $\mathcal{K} = \{k_1, k_2, k_3, k_7\}$ and the codewords of $\mathcal{G}_i$ are characterize as it is presented in Table 3.5.

We intend to show that if $k \in \mathcal{K}$, then $-k \in \mathcal{K}$. Let us focus our attention on $k_3 \in \mathcal{K}$. We have concluded before that $W_3, W_6 \in \mathcal{G}_{ik_3}$, with $W_3 \in \mathcal{G}_{ik_3x\gamma\delta}$ and $W_6 \in \mathcal{G}_{i,k_3,-x,\alpha,\beta}$. In these conditions, $-k_3 \in \mathcal{I}\backslash(\{i, -i, x, -x, y, -y\} \cup \mathcal{N})$. That is, $-k_3 \in \mathcal{I}\backslash(\{i, -i\} \cup \mathcal{J})$. Since $\mathcal{I} = \{i, -i\} \cup \mathcal{J} \cup \mathcal{K}$, then $-k_3 \in \mathcal{K}$.

W_1	i	k_1	x	y	α
W_2	i	k_2	x	$-y$	β
W_3	i	k_3	x	γ	δ
W_4	i	k_2	$-x$	y	
W_5	i	k_1	$-x$	$-y$	
W_6	i	k_3	$-x$	α	β
W_7	i	k_7	y	β	
W_8	i	k_7	$-y$	α	

Table 3.5: Partial index distribution of the codewords of $\mathcal{G}_i$.

Looking at the codewords $W_7, W_8 \in \mathcal{G}_{ik_7}$, we get $W_7 \in \mathcal{G}_\gamma$ and $W_8 \in \mathcal{G}_\delta$, or, $W_7 \in \mathcal{G}_\delta$ and $W_8 \in \mathcal{G}_\gamma$. In both cases $-k_7 \in \mathcal{I}\backslash(\{i,-i\} \cup \mathcal{J})$, accordingly $-k_7 \in \mathcal{K}$.

Now, $\mathcal{K} = \{k_1, k_2, k_3, k_7\}$ and $-k_3, -k_7 \in \mathcal{K}$. Either $k_3 \neq -k_7$ or $k_3 = -k_7$.

If $k_3 \neq -k_7$, then $-k \in \mathcal{K}$ for all $k \in \mathcal{K}$.

If $k_3 = -k_7$ and $k_1 = -k_2$, then $-k \in \mathcal{K}$ for all $k \in \mathcal{K}$.

Assume that $k_3 = -k_7$ and $k_1 \neq -k_2$. By this assumption it follows that $-k_1, -k_2 \in \mathcal{N} = \{\alpha, \beta, \gamma, \delta\}$. Thus, there are $\varepsilon_1, \varepsilon_2 \in \mathcal{N}$ so that $-k_1 = \varepsilon_1$, $-k_2 = \varepsilon_2$ and the remaining elements of $\mathcal{N}$, ε_3 and ε_4, satisfy $\varepsilon_3 = -\varepsilon_4$. As $W_1 \in \mathcal{G}_{ik_1xy\alpha}$, then $-k_1 \in \{\beta, \gamma, \delta\}$. On the other hand, since $W_2 \in \mathcal{G}_{i,k_2,x,-y,\beta}$, then $-k_2 \in \{\alpha, \gamma, \delta\}$. We note that, as $k_1 \neq k_2$, then $-k_1 \neq -k_2$.

If $-k_1 = \beta$ and $-k_2 = \alpha$, then $\gamma = -\delta$, which is a contradiction since $W_3 \in \mathcal{G}_{ik_3x\gamma\delta}$.

If $-k_1 = \beta$ and $-k_2 = \gamma$, then $\alpha = -\delta$. Analyzing Table 3.5 and taking into account that $W_4 \in \mathcal{G}_\gamma \cup \mathcal{G}_\delta$, we conclude that $W_4 \in \mathcal{G}_{i,k_2,-x,y,\delta}$. Consequently, having in mind Lemma 1.5, $W_5 \in \mathcal{G}_{i,k_1,-x,-y,\gamma}$, $W_7 \in \mathcal{G}_{ik_7y\beta\gamma}$ and $W_8 \in \mathcal{G}_{i,k_7,-y,\alpha,\delta}$, which is not possible since we are supposing $\alpha = -\delta$.

If $-k_1 = \beta$ and $-k_2 = \delta$, then $\alpha = -\gamma$. Consequently, $W_8 \in \mathcal{G}_{i,k_7,-y,\alpha,\delta}$, $W_7 \in \mathcal{G}_{ik_7y\beta\gamma}$ and $W_4 \in \mathcal{G}_{i,k_2,-x,y,\delta}$. We get a contradiction since, by hypothesis, $-k_2 = \delta$.

Combining all possibilities for $-k_1 \in \{\beta, \gamma, \delta\}$ and $-k_2 \in \{\alpha, \gamma, \delta\}$, by a similar reasoning we get always a contradiction. Therefore, $-k \in \mathcal{K}$ for all $k \in \mathcal{K}$. □

From Proposition 3.1 we get $\mathcal{I}\backslash\{i,-i\} = \mathcal{J} \cup \mathcal{K}$. We have just seen that, if $k \in \mathcal{K}$ then $-k \in \mathcal{K}$. So, if $j \in \mathcal{J}$ then $-j \in \mathcal{J}$.

Until this moment we have centered our attention on the characterization of the codewords of $\mathcal{G}_i$. The two following propositions arise from the analysis of other type of codewords, in particular, codewords of $\mathcal{D} \cup \mathcal{E} \cup \mathcal{F}$.

Proposition 3.3 *If $|\mathcal{G}_i| = 8$, $i \in \mathcal{I}$, then $|\mathcal{F}_i| = 0$.*

Proof. Let $|\mathcal{G}_i| = 8$ for $i \in \mathcal{I}$. Suppose, by contradiction, that $|\mathcal{F}_i| > 0$. Let $U \in \mathcal{F}_i$. Since the codewords of $\mathcal{F}$ are of type $[\pm 2, \pm 1^3]$, there are $u_1, u_2, u_3 \in \mathcal{I}\backslash\{i,-i\}$, with $|u_1|, |u_2|$ and $|u_3|$ distinct between them, such that $U \in \mathcal{F}_{iu_1u_2u_3}$.

By Proposition 3.1, $\mathcal{I}\backslash\{i,-i\} = \mathcal{J} \cup \mathcal{K}$, therefore $u_1, u_2, u_3 \in \mathcal{J} \cup \mathcal{K}$. Recall that $|\mathcal{G}_{ij}| = 3$ for any $j \in \mathcal{J}$. Then, by Lemma 2.2 one has $|\mathcal{F}_{ij}| = 0$ for all $j \in \mathcal{J}$. Consequently, $u_1, u_2, u_3 \in \mathcal{K}$. From Proposition 3.1 it follows that $|\mathcal{K}| = 4$ and, taking into account Proposition 3.2, $-k \in \mathcal{K}$ for all $k \in \mathcal{K}$. Thus, is not possible to have $u_1, u_2, u_3 \in \mathcal{K}$ satisfying $|u_1|$, $|u_2|$ and $|u_3|$ pairwise distinct, contradicting our assumption. □

Proposition 3.4 *For all $j \in \mathcal{J}$, $|\mathcal{D}_{ij} \cup \mathcal{E}_{ij}| = 1$. For all $k \in \mathcal{K}$, $|\mathcal{D}_{ik} \cup \mathcal{E}_{ik}| = 4$. Furthermore, if $k \in \mathcal{K}$, the codewords $U_1, U_2, U_3, U_4 \in \mathcal{D}_{ik} \cup \mathcal{E}_{ik}$ are such that $U_1 \in \mathcal{D}_{iku_1} \cup \mathcal{E}_{iku_1}$, $U_2 \in \mathcal{D}_{iku_2} \cup \mathcal{E}_{iku_2}$, $U_3 \in \mathcal{D}_{iku_3} \cup \mathcal{E}_{iku_3}$ and $U_4 \in \mathcal{D}_{iku_4} \cup \mathcal{E}_{iku_4}$, with $u_1, u_2 \in \mathcal{J}$, $u_1 \neq u_2$, and $u_3, u_4 \in \mathcal{K}\backslash\{k,-k\}$, with $u_3 = -u_4$.*

Proof. From Lemma 2.2 we get

$$|\mathcal{D}_{il} \cup \mathcal{E}_{il}| + 2|\mathcal{F}_{il}| + 3|\mathcal{G}_{il}| = 10 \tag{3.2}$$

for all $l \in \mathcal{I}\backslash\{i,-i\}$. By Proposition 3.3 we know that $|\mathcal{F}_i| = 0$ and, consequently, $|\mathcal{F}_{il}| = 0$ for all $l \in \mathcal{I}\backslash\{i,-i\}$. As $|\mathcal{G}_{ij}| = 3$ for any $j \in \mathcal{J}$, from (3.2) we obtain $|\mathcal{D}_{ij} \cup \mathcal{E}_{ij}| = 1$ for all $j \in \mathcal{J}$. Considering again (3.2), we conclude that $|\mathcal{D}_{ik} \cup \mathcal{E}_{ik}| = 4$ for each $k \in \mathcal{K}$, since $|\mathcal{G}_{ik}| = 2$ for all $k \in \mathcal{K}$.

Let $k \in \mathcal{K}$. Then, there exist codewords $V_1, V_2 \in \mathcal{G}_{ik}$ and $U_1, \ldots, U_4 \in \mathcal{D}_{ik} \cup \mathcal{E}_{ik}$. We note that, the codewords of $\mathcal{D}$ are of type $[\pm 3, \pm 1^2]$ and the codewords of $\mathcal{E}$ are of type $[\pm 2^2, \pm 1]$. Thus, there are $v_1, \ldots, v_6, u_1, \ldots, u_4$ in $\mathcal{I} \backslash \{i, -i, k, -k\}$ such that:

V_1	i	k	v_1	v_2	v_3
V_2	i	k	v_4	v_5	v_6

U_1	i	k	u_1
U_2	i	k	u_2
U_3	i	k	u_3
U_4	i	k	u_4

Table 3.6: Index distribution of the codewords of $\mathcal{G}_{ik} \cup \mathcal{D}_{ik} \cup \mathcal{E}_{ik}$.

It should be pointed out that, by Lemma 1.5, $v_1, \ldots, v_6, u_1, \ldots, u_4$ must be pairwise distinct. Therefore, $\{v_1, \ldots, v_6, u_1, \ldots, u_4\} = \mathcal{I} \backslash \{i, -i, k, -k\}$.

By Proposition 3.1, $\mathcal{I} \backslash \{i, -i\} = \mathcal{J} \cup \mathcal{K}$, with $|\mathcal{J}| = 8$ and $|\mathcal{K}| = 4$. Furthermore, from Proposition 3.2, $-k \in \mathcal{K}$. Then, $\{v_1, \ldots, v_6, u_1, \ldots, u_4\} = \mathcal{J} \cup \mathcal{K} \backslash \{k, -k\}$.

Since $V_1, V_2 \in \mathcal{G}_{ik}$, with $k \in \mathcal{K}$, taking into account Proposition 3.1 we must impose $\{v_1, \ldots, v_6\} \subset \mathcal{J}$. Consequently, without loss of generality, $u_1, u_2 \in \mathcal{J}$ and $u_3, u_4 \in \mathcal{K} \backslash \{k, -k\}$. Considering Proposition 3.2 we conclude that $u_3 = -u_4$. □

3.2 $|\mathcal{G}_i| \neq 8$ for any $i \in \mathcal{I}$

We are now in conditions to establish the main result of this chapter.

Theorem 3.1 *For any $i \in \mathcal{I}$, $|\mathcal{G}_i| \neq 8$.*

Proof. By contradiction, consider $i \in \mathcal{I}$ such that $|\mathcal{G}_i| = 8$.

From Proposition 3.1 we have $|\mathcal{K}| = 4$, so let k be an element of $\mathcal{K}$. By Proposition 3.4, there exist $U_1, \ldots, U_4 \in \mathcal{D}_{ik} \cup \mathcal{E}_{ik}$ whose index distribution satisfies the conditions presented in Table 3.7, where $u, -u \in \mathcal{K} \backslash \{k, -k\}$ and $j_1, j_2 \in \mathcal{J}$, with $j_1 \neq j_2$. We note that, in these conditions, $\mathcal{K} = \{k, -k, u, -u\}$.

U_1	i	k	u
U_2	i	k	$-u$
U_3	i	k	j_1
U_4	i	k	j_2

Table 3.7: Index distribution of the codewords of $\mathcal{D}_{ik} \cup \mathcal{E}_{ik}$.

Let us denote by $\mathcal{H}$ the set of words of type $[\pm 2, \pm 1]$. Consider the words $P_1, P_2 \in \mathcal{H}$ such that $P_1 \in \mathcal{H}_i^{(2)} \cap \mathcal{H}_{j_1}^{(1)}$ and $P_2 \in \mathcal{H}_i^{(2)} \cap \mathcal{H}_{j_2}^{(1)}$. The index distribution of the codewords of $\mathcal{D}_{ik} \cup \mathcal{E}_{ik}$ and the index value distribution of the words P_1 and P_2 are represented in the following table:

	i	k	u	$-u$	j_1	j_2
U_1	x	x	x			
U_2	x	x		x		
U_3	x	x			x	
U_4	x	x				x
P_1	± 2				± 1	
P_2	± 2					± 1

Table 3.8: Index distribution of $U_1, \ldots, U_4 \in \mathcal{D}_{ik} \cup \mathcal{E}_{ik}$ and index value distribution of $P_1, P_2 \in \mathcal{H}_i$.

By definition of perfect 2-error correcting Lee code, for each $P \in \{P_1, P_2\}$ there exists a unique codeword $V \in \mathcal{T}$ such that $\mu_L(P, V) \leq 2$. Since $P_1, P_2 \in \mathcal{H}_i^{(2)}$, with $\mathcal{H}$ the set of words of type $[\pm 2, \pm 1]$, then each one of these words must be covered by $V \in \mathcal{B}_i^{(4)} \cup \mathcal{C}_i \cup \mathcal{D}_i^{(3)} \cup \mathcal{E}_i^{(2)} \cup \mathcal{F}_i^{(2)}$. As, by Proposition 3.3, $|\mathcal{F}_i| = 0$, then $V \in \mathcal{B}_i^{(4)} \cup \mathcal{C}_i \cup \mathcal{D}_i^{(3)} \cup \mathcal{E}_i^{(2)}$.

More concretely, P_1 is covered by $V \in (\mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_1}^{(1)}) \cup \mathcal{C}_{ij_1} \cup (\mathcal{D}_i^{(3)} \cap \mathcal{D}_{j_1}^{(1)}) \cup (\mathcal{E}_i^{(2)} \cap \mathcal{E}_{j_1})$. Likewise, P_2 is covered by $V' \in (\mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_2}^{(1)}) \cup \mathcal{C}_{ij_2} \cup (\mathcal{D}_i^{(3)} \cap \mathcal{D}_{j_2}^{(1)}) \cup (\mathcal{E}_i^{(2)} \cap \mathcal{E}_{j_2})$. Thus, we may consider U_3 and U_4 as possible codewords to cover P_1 and P_2, respectively.

Suppose that P_1 is covered by U_3 and P_2 is covered by U_4. Then, we must impose

$$U_3 \in (\mathcal{D}_i^{(3)} \cap \mathcal{D}_k^{(1)} \cap \mathcal{D}_{j_1}^{(1)}) \cup (\mathcal{E}_i^{(2)} \cap \mathcal{E}_k \cap \mathcal{E}_{j_1})$$

and

$$U_4 \in (\mathcal{D}_i^{(3)} \cap \mathcal{D}_k^{(1)} \cap \mathcal{D}_{j_2}^{(1)}) \cup (\mathcal{E}_i^{(2)} \cap \mathcal{E}_k \cap \mathcal{E}_{j_2}),$$

which contradicts Lemma 1.4, since $U_3, U_4 \in (\mathcal{D}_i^{(3)} \cap \mathcal{D}_k^{(1)}) \cup (\mathcal{E}_i^{(2)} \cap \mathcal{E}_k)$. Therefore, either P_1 is not covered by U_3 or P_2 is not covered by U_4.

Without loss of generality, let us assume that P_1 is not covered by U_3. Note that, $U_3 \in \mathcal{D}_{ikj_1} \cup \mathcal{E}_{ikj_1}$. As $j_1 \in \mathcal{J}$, by Proposition 3.4 we get $|\mathcal{D}_{ij_1} \cup \mathcal{E}_{ij_1}| = 1$. Consequently, $\mathcal{D}_{ij_1} \cup \mathcal{E}_{ij_1} = \{U_3\}$. Since we are assuming that U_3 does not cover P_1, then P_1 is covered by a codeword V satisfying $V \in (\mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_1}^{(1)}) \cup \mathcal{C}_{ij_1}$.

Next, we will analyze, separately, the hypotheses:

1) $V \in \mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_1}^{(1)}$;

2) $V \in \mathcal{C}_{ij_1}$.

1) **Assume that P_1 is covered by $V \in \mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_1}^{(1)}$.**

If P_1 is covered by $V \in \mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_1}^{(1)}$, then, by Lemma 1.3, $|\mathcal{B}_i^{(4)} \backslash \{V\} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(3)}| = 0$. Consequently, if $U \in \{U_1, \ldots, U_4\}$ is such that $U \in \mathcal{D}$, then $U \in \mathcal{D}_i^{(1)}$. Furthermore, under the assumption, P_2 must be covered by

$$V' \in (\mathcal{C}_i^{(2)} \cap \mathcal{C}_{j_2}^{(3)}) \cup (\mathcal{E}_i^{(2)} \cap \mathcal{E}_{j_2}).$$

If $V' \in \mathcal{E}_i^{(2)} \cap \mathcal{E}_{j_2}$, since $j_2 \in \mathcal{J}$ we conclude, by Proposition 3.4, that $V' = U_4$. Having in mind U_1, U_2 and U_3, see Table 3.8, if $U \in \{U_1, U_2, U_3\}$ is such that $U \in \mathcal{E}$, then $U \in \mathcal{E}_i^{(1)}$, otherwise, $U, U_4 \in \mathcal{E}_i^{(2)} \cap \mathcal{E}_k$, contradicting Lemma 1.4. Therefore, since we have concluded before that $\{U_1, U_2, U_3\} \cap \mathcal{D}_i^{(3)} = \varnothing$, we get $U_1, U_2, U_3 \in \mathcal{D}_i^{(1)} \cup \mathcal{E}_i^{(1)}$. Taking into account the index distribution of U_1 and U_2, we must have $U_1 \in \mathcal{D}_u^{(3)}$ or $U_2 \in \mathcal{D}_{-u}^{(3)}$, otherwise, $U_1, U_2 \in (\mathcal{D}_i^{(1)} \cap \mathcal{D}_k^{(3)}) \cup (\mathcal{E}_i^{(1)} \cap \mathcal{E}_k^{(2)})$, contradicting, once again, Lemma 1.4.

If $V' \in \mathcal{C}_i^{(2)} \cap \mathcal{C}_{j_2}^{(3)}$, to avoid the contradiction of Lemma 1.4 we must impose $U_4 \in \mathcal{D}_k^{(3)}$. Consequently, considering again Lemma 1.4, $U_1, U_2, U_3 \in \mathcal{D}_k^{(1)} \cup \mathcal{E}_k^{(1)}$. We recall that, we have seen before that $\{U_1, U_2, U_3\} \cap \mathcal{D}_i^{(3)} = \varnothing$. Thus, in these conditions, $U_1 \in \mathcal{D}_u^{(3)}$ or $U_2 \in \mathcal{D}_{-u}^{(3)}$, otherwise, $U_1, U_2 \in \mathcal{E}_i^{(2)} \cap \mathcal{E}_k^{(1)}$, contradicting again Lemma 1.4.

Therefore, in both cases, supposing $V' \in \mathcal{E}_i^{(2)} \cap \mathcal{E}_{j_2}$ or $V' \in \mathcal{C}_i^{(2)} \cap \mathcal{C}_{j_2}^{(3)}$, we conclude that $U_1 \in \mathcal{D}_u^{(3)}$ or $U_2 \in \mathcal{D}_{-u}^{(3)}$.

Suppose, without loss of generality, that $U_1 \in \mathcal{D}_u^{(3)}$. As $u \in \mathcal{K}$, by Proposition 3.4 there are $U_5, U_6 \in \mathcal{D}_{iu} \cup \mathcal{E}_{iu}$ satisfying $U_5 \in \mathcal{D}_{iuj_3} \cup \mathcal{E}_{iuj_3}$ and $U_6 \in \mathcal{D}_{iuj_4} \cup \mathcal{E}_{iuj_4}$, with $j_3, j_4 \in \mathcal{J}$ distinct. Note that, $j_1, \ldots, j_4 \in \mathcal{J}$ are pairwise distinct, since, by Proposition 3.4, $|\mathcal{D}_{ij} \cup \mathcal{E}_{ij}| = 1$ for all $j \in \mathcal{J}$.

Let us consider $P_3 \in \mathcal{H}_i^{(2)} \cap \mathcal{H}_{j_3}^{(1)}$ and $P_4 \in \mathcal{H}_i^{(2)} \cap \mathcal{H}_{j_4}^{(1)}$. Table bellow summarizes the conditions that the index distribution, and, in some cases, the index value distribution, of the codewords and words described until now, must satisfy:

	i	k	u	$-u$	j_1	j_2	j_3	j_4
U_1	±1	±1	±3					
U_2	x	x		x				
U_3	x	x			x			
U_4	x	x				x		
P_1	±2				±1			
P_2	±2					±1		
V	±4				±1			
U_5	x		x				x	
U_6	x		x					x
P_3	±2						±1	
P_4	±2							±1

Table 3.9: Index conditions on $\mathcal{B}_i \cup \mathcal{D}_i \cup \mathcal{E}_i$ and on 4 words of type $[\pm2, \pm1]$.

Taking into account the words P_3 and P_4 we may conclude, as we have concluded before for P_1 and P_2, that either P_3 is not covered by U_5 or P_4 is not covered by U_6. In fact, if U_5 covers P_3 and U_6 covers P_4, then $U_5, U_6 \in (\mathcal{D}_i^{(3)} \cap \mathcal{D}_u^{(1)}) \cup (\mathcal{E}_i^{(2)} \cap \mathcal{E}_u)$, contradicting Lemma 1.4. Let us assume, without loss of generality, that P_3 is not covered by U_5. By Proposition 3.4 it follows that $|\mathcal{D}_{ij_3} \cup \mathcal{E}_{ij_3}| = 1$. Consequently, $\mathcal{D}_{ij_3} \cup \mathcal{E}_{ij_3} = \{U_5\}$. As a consequence of the assumption $V \in \mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_1}^{(1)}$ we get $|\mathcal{B}_i^{(4)} \backslash \{V\} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(3)}| = 0$. Furthermore, from Proposition 3.3, $|\mathcal{F}_i| = 0$. Thus, under these conditions, P_3 must be covered by a codeword R satisfying $R \in \mathcal{C}_i^{(2)} \cap \mathcal{C}_{j_3}^{(3)}$. Consequently, $U_5 \in \mathcal{D}_u^{(3)}$, otherwise, $U_5 \in (\mathcal{D}_i^{(1)} \cap \mathcal{D}_{j_3}^{(3)}) \cup (\mathcal{E}_i^{(2)} \cap \mathcal{E}_{j_3}) \cup (\mathcal{E}_i \cap \mathcal{E}_{j_3}^{(2)})$ and contradicts with the codeword R Lemma 1.4. However, $U_1, U_5 \in \mathcal{D}_u^{(3)}$, contradicting Lemma 1.3.

Accordingly, P_1 can not be covered by the codeword $V \in \mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_1}^{(1)}$.

2) **Assume that P_1 is covered by $V \in \mathcal{C}_{ij_1}$.**

Since $V \in \mathcal{C}$, then V is a codeword of type $[\pm 3, \pm 2]$. According with what is being supposed, $V \in \mathcal{C}_i^{(3)} \cap \mathcal{C}_{j_1}^{(2)}$ or $V \in \mathcal{C}_i^{(2)} \cap \mathcal{C}_{j_1}^{(3)}$. Consider $U_3 \in \mathcal{D}_{ikj_1} \cup \mathcal{E}_{ikj_1}$. In order to have Lemma 1.4 fulfilled we must force $U_3 \in \mathcal{D}_i^{(1)} \cap \mathcal{D}_k^{(3)} \cap \mathcal{D}_{j_1}^{(1)}$. Schematically:

	i	k	u	$-u$	j_1	j_2
U_1	x	x	x			
U_2	x	x		x		
U_3	± 1	± 3			± 1	
U_4	x	x				x
P_1	± 2				± 1	
P_2	± 2					± 1
V	x				x	

Table 3.10: Index distribution on $\mathcal{C}_i \cup \mathcal{D}_i \cup \mathcal{E}_i$ and on 2 words of type $[\pm 2, \pm 1]$.

Taking into account U_3, by Lemma 1.4 we must have $U_1, U_2, U_4 \in \mathcal{D}_k^{(1)} \cup \mathcal{E}_k^{(1)}$. Besides, $U_1 \in \mathcal{D}_u^{(3)}$ or $U_2 \in \mathcal{D}_{-u}^{(3)}$, otherwise, $U_1, U_2 \in (\mathcal{D}_i^{(3)} \cap \mathcal{D}_k^{(1)}) \cup (\mathcal{E}_i^{(2)} \cap \mathcal{E}_k^{(1)})$, contradicting Lemma 1.4.

Let us assume, without loss of generality, that $U_1 \in \mathcal{D}_u^{(3)}$.

Proceeding as in the previous case, we will consider $U_5 \in \mathcal{D}_{iuj_3} \cup \mathcal{E}_{iuj_3}$ and $U_6 \in \mathcal{D}_{iuj_4} \cup \mathcal{E}_{iuj_4}$, with $j_3, j_4 \in \mathcal{J}$ and distinct. We will consider also $P_3 \in \mathcal{H}_i^{(2)} \cap \mathcal{H}_{j_3}^{(1)}$ and $P_4 \in \mathcal{H}_i^{(2)} \cap \mathcal{H}_{j_4}^{(1)}$. Gathering the information obtained so far, one has the index distribution presented in Table 3.11.

	i	k	u	$-u$	j_1	j_2	j_3	j_4
U_1	±1	±1	±3					
U_2	x	x		x				
U_3	±1	±3			±1			
U_4	x	x				x		
P_1	±2				±1			
P_2	±2					±1		
V	x				x			
U_5	x		x				x	
U_6	x		x					x
P_3	±2						±1	
P_4	±2							±1

Table 3.11: Index distribution on $\mathcal{C}_i \cup \mathcal{D}_i \cup \mathcal{E}_i$ and on 4 words of type $[\pm2, \pm1]$.

As seen in the previous case, either U_5 does not cover P_3 or U_6 does not cover P_4. Assume, without loss of generality, that P_3 is not covered by U_5. By Propositions 3.3 and 3.4 we get, respectively, $|\mathcal{F}_i| = 0$ and $\mathcal{D}_{ij_3} \cup \mathcal{E}_{ij_3} = \{U_5\}$. Therefore, P_3 must be covered by a codeword $R \in (\mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_3}^{(1)}) \cup \mathcal{C}_{ij_3}$. If $R \in \mathcal{C}_{ij_3}$, then, by Lemma 1.4, we must impose $U_5 \in \mathcal{D}_u^{(3)}$ and, consequently, $|\mathcal{D}_u^{(3)}| \geq 2$, contradicting Lemma 1.3. Accordingly, $R \in \mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_3}^{(1)}$.

Taking into account Lemma 1.3, $|\mathcal{B}_i^{(4)} \backslash \{R\} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(3)}| = 0$. Thus, since, by Proposition 3.3, $|\mathcal{F}_i| = 0$, we may conclude that P_4 must be covered by a codeword

$$S \in (\mathcal{C}_i^{(2)} \cap \mathcal{C}_{j_4}^{(3)}) \cup (\mathcal{E}_i^{(2)} \cap \mathcal{E}_{j_4}).$$

Note that, if $S \in \mathcal{C}_i^{(2)} \cap \mathcal{C}_{j_4}^{(3)}$, then, by Lemma 1.4, $U_6 \in \mathcal{D}_u^{(3)}$ implying $|\mathcal{D}_u^{(3)}| \geq 2$ and contradicting Lemma 1.3. Thus, $S \in \mathcal{E}_i^{(2)} \cap \mathcal{E}_{j_4}$. By Proposition 3.4, $|\mathcal{D}_{ij_4} \cup \mathcal{E}_{ij_4}| = 1$ leading to $\mathcal{D}_{ij_4} \cup \mathcal{E}_{ij_4} = \{U_6\}$ and, consequently, $S = U_6$. Since $U_1 \in \mathcal{D}_i^{(1)} \cap \mathcal{D}_k^{(1)} \cap \mathcal{D}_u^{(3)}$, taking into account Lemma 1.4, we must force $U_6 \in \mathcal{E}_i^{(2)} \cap \mathcal{E}_u^{(1)} \cap \mathcal{E}_{j_4}^{(2)}$. The index distribution, and, in some cases the index value distribution, of the codewords and words which we are dealing with are presented in Table 3.12.

	i	k	u	$-u$	j_1	j_2	j_3	j_4
U_1	±1	±1	±3					
U_2	x	x		x				
U_3	±1	±3			±1			
U_4	x	x				x		
P_1	±2				±1			
P_2	±2					±1		
V	x				x			
U_5	x		x				x	
U_6	±2		±1					±2
P_3	±2						±1	
P_4	±2							±1
R	±4						±1	

Table 3.12: Index distribution on $\mathcal{B}_i \cup \mathcal{C}_i \cup \mathcal{D}_i \cup \mathcal{E}_i$ and on 4 words of type $[\pm2, \pm1]$.

Let us now focus our attention on $-u \in \mathcal{K}$. By Proposition 3.4, there are codewords $U_7, U_8 \in \mathcal{D}_{i,-u} \cup \mathcal{E}_{i,-u}$, so that, $U_7 \in \mathcal{D}_{i,-u,j_5} \cup \mathcal{E}_{i,-u,j_5}$ and $U_8 \in \mathcal{D}_{i,-u,j_6} \cup \mathcal{E}_{i,-u,j_6}$, with $j_5, j_6 \in \mathcal{J}$ distinct. Note that, by Proposition 3.4, $|\mathcal{D}_{ij} \cup \mathcal{E}_{ij}| = 1$ for all $j \in \mathcal{J}$, and so $j_1, \ldots, j_6$ are pairwise distinct. Taking into account the existence of the words $P_5 \in \mathcal{H}_i^{(2)} \cap \mathcal{H}_{j_5}^{(1)}$ and $P_6 \in \mathcal{H}_i^{(2)} \cap \mathcal{H}_{j_6}^{(1)}$, we obtain the index distribution presented schematically in Table 3.13.

By a similar reasoning to the one done with the words $P_1, P_2 \in \mathcal{H}_i^{(2)}$ and $P_3, P_4 \in \mathcal{H}_i^{(2)}$, we conclude that either P_5 is not covered by U_7 or P_6 is not covered by U_8. Let us assume, without loss of generality, that U_7 does not cover P_5. Then, Propositions 3.3 and 3.4 lead us to conclude that P_5 must be covered by a codeword

$$Q \in (\mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_5}^{(1)}) \cup (\mathcal{C}_{ij_5}).$$

As $R \in \mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_3}^{(1)}$, by Lemma 1.3, $Q \in \mathcal{C}_i^{(2)} \cap \mathcal{C}_{j_5}^{(3)}$. Consequently, taking into account Lemma 1.4, we must force $U_7 \in \mathcal{D}_{-u}^{(3)}$.

Focus our attention on the codeword $U_2 \in \mathcal{D}_{i,k,-u} \cup \mathcal{E}_{i,k,-u}$. Having in mind the index value distribution of the codewords R, U_3 and U_7 and considering Lemma 1.3, we conclude that $U_2 \in \mathcal{E}_i$. Consequently, either $U_2 \in \mathcal{E}_i \cap \mathcal{E}_k^{(2)}$ or $U_2 \in \mathcal{E}_i \cap \mathcal{E}_{-u}^{(2)}$. If $U_2 \in \mathcal{E}_i \cap \mathcal{E}_k^{(2)}$, then the index value distribution of U_2 and U_3 contradicts Lemma 1.4. If $U_2 \in \mathcal{E}_i \cap \mathcal{E}_{-u}^{(2)}$, the index value distribution of U_2 and U_7 contradicts also Lemma 1.4.

	i	k	u	$-u$	j_1	j_2	j_3	j_4	j_5	j_6
U_1	±1	±1	±3							
U_2	x	x		x						
U_3	±1	±3			±1					
U_4	x	x				x				
P_1	±2				±1					
P_2	±2					±1				
V	x				x					
U_5	x		x				x			
U_6	±2		±1					±2		
P_3	±2						±1			
P_4	±2							±1		
R	±4						±1			
U_7	x			x					x	
U_8	x			x						x
P_5	±2								±1	
P_6	±2									±1

Table 3.13: Index distribution on $\mathcal{B}_i \cup \mathcal{C}_i \cup \mathcal{D}_i \cup \mathcal{E}_i$ and on 6 words of type $[\pm2, \pm1]$.

In both hypotheses, P_1 covered by $V \in \mathcal{B}_i^{(4)} \cap \mathcal{B}_{j_1}^{(1)}$ or P_1 covered by $V \in \mathcal{C}_{ij_1}$, we get a contradiction. □

From Corollary 2.2 and Theorem 3.1 it follows immediately:

Corollary 3.1 *For any $i \in \mathcal{I}$, $3 \leq |\mathcal{G}_i| \leq 7$.*

Consequently, the required solutions for the system of equations presented in Proposition 2.1 must satisfy $9 \leq g \leq 19$. As we have been saying, our strategy to prove the non-existence of PL$(7,2)$ codes consists in getting a minimum range for the variation of $|\mathcal{G}_i|$, with $i \in \mathcal{I}$. In next chapter we get another result which reinforces this aim proving that $|\mathcal{G}_i| \neq 3$ for any $i \in \mathcal{I}$.

Until now we have been denoting by i a general element of the set $\mathcal{I}$, since it is a natural and intuitive choice. However, we call attention to the fact that in the following chapters we will consider $\mathcal{I} = \{i, -i, j, -j, k, -k, l, -l, m, -m, n, -n, o, -o\}$, being i representing a certain element of $\mathcal{I}$ although not specified.

Chapter 4

Proof of $|\mathcal{G}_i| \neq 3$ for any $i \in \mathcal{I}$

In the previous chapters we have proved that $3 \leq |\mathcal{G}_i| \leq 7$ for any $i \in \mathcal{I}$. Here, we prove that $|\mathcal{G}_i| \neq 3$ for any $i \in \mathcal{I}$, restricting even more the range of variation of $|\mathcal{G}_i|$, that is, $4 \leq |\mathcal{G}_i| \leq 7$ for any $i \in \mathcal{I}$.

This chapter is organized as follows. Under the assumption $|\mathcal{G}_i| = 3$, for some $i \in \mathcal{I}$, we present some results from which we get conditions that necessarily must be satisfied by the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$. In the second section we show how we apply these results in the characterization of the index distribution of such codewords. At the end of this chapter we present the methodology used to show that any one of the obtained index distribution for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ contradicts the definition of PL$(7, 2)$ code.

4.1 Necessary conditions for the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$

Let us suppose $|\mathcal{G}_i| = 3$ for some $i \in \mathcal{I}$. Under this condition, by Lemma 2.9, we have $|\mathcal{F}_i| = 13$ and, in particular, $|\mathcal{F}_i^{(2)}| = 4$.

Our intention is to show that the hypothesis $|\mathcal{G}_i| = 3$, for some $i \in \mathcal{I}$, lead us to contradictions on the definition of PL$(7, 2)$ code. In this sense, we will characterize the possible index distributions for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$, taking into account that $|\mathcal{G}_i| = 3$ and $|\mathcal{F}_i| = 13$, having in mind to prove that such codewords do not satisfy the definition of perfect 2-error correcting Lee code over $\mathbb{Z}^7$.

The following results impose conditions in the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$.

Proposition 4.1 *If $|\mathcal{G}_i| = 3$, for some $i \in \mathcal{I}$, then there are $\alpha, \beta, \gamma \in \mathcal{I} \setminus \{i, -i\}$, with α, β and γ pairwise distinct, such that, $|\mathcal{F}_{i\alpha}| = |\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = 5$. Furthermore, $|\mathcal{F}_{i\omega}| \leq 3$ for all $\omega \in \mathcal{I}\backslash\{i, -i, \alpha, \beta, \gamma\}$.*

Proof. Let $i \in \mathcal{I}$ be such that $|\mathcal{G}_i| = 3$. The three codewords W_1, W_2, W_3 of $\mathcal{G}_i$ satisfy $W_1 \in \mathcal{G}_{iw_1w_2w_3w_4}$, $W_2 \in \mathcal{G}_{iw_5w_6w_7w_8}$ and $W_3 \in \mathcal{G}_{iw_9w_{10}w_{11}w_{12}}$, with $w_1, \ldots, w_{12} \in \mathcal{I}\backslash\{i, -i\}$ and not necessarily pairwise distinct.

As $|\mathcal{F}_i| = \frac{1}{3}\sum_{\omega\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{F}_{i\omega}|$ and $|\mathcal{F}_i| = 13$ one has,

$$\sum_{\omega\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{F}_{i\omega}| = 39. \tag{4.1}$$

Since $|\mathcal{I}\backslash\{i, -i\}| = 12$ and, by Lemma 2.2, $|\mathcal{F}_{i\omega}| \leq 5$ for all $\omega \in \mathcal{I}\backslash\{i, -i\}$, the equation (4.1) implies the existence of, at least, two elements $\alpha, \beta \in \mathcal{I}\backslash\{i, -i\}$, with $\alpha \neq \beta$, such that, $|\mathcal{F}_{i\alpha}|, |\mathcal{F}_{i\beta}| \geq 4$.

Let us show, now, that there are, at most, three elements $\alpha, \beta, \gamma \in \mathcal{I}\backslash\{i, -i\}$, distinct between them, such that, $|\mathcal{F}_{i\alpha}|, |\mathcal{F}_{i\beta}|, |\mathcal{F}_{i\gamma}| \geq 4$. Suppose, by contradiction, that there exist $\alpha, \beta, \gamma, \delta \in \mathcal{I}\backslash\{i, -i\}$, distinct between them, such that, $|\mathcal{F}_{i\alpha}|, |\mathcal{F}_{i\beta}|, |\mathcal{F}_{i\gamma}|, |\mathcal{F}_{i\delta}| \geq 4$. By Lemma 2.2, $|\mathcal{G}_{i\alpha}| = |\mathcal{G}_{i\beta}| = |\mathcal{G}_{i\gamma}| = |\mathcal{G}_{i\delta}| = 0$ and having in account the index distribution of $W_1, W_2, W_3 \in \mathcal{G}_i$, we may conclude that $w_1, \ldots, w_{12} \in \mathcal{I}\backslash\{i, -i, \alpha, \beta, \gamma, \delta\}$. As $|\mathcal{I}\backslash\{i, -i, \alpha, \beta, \gamma, \delta\}| = 8$, there are $\omega, \theta \in \mathcal{I}\backslash\{i, -i, \alpha, \beta, \gamma, \delta\}$ such that $|\mathcal{G}_{i\omega\theta}| \geq 2$, contradicting Lemma 1.5. Thus, there are, at most, three distinct elements $\alpha, \beta, \gamma \in \mathcal{I}\backslash\{i, -i\}$ satisfying $|\mathcal{F}_{i\alpha}|, |\mathcal{F}_{i\beta}|, |\mathcal{F}_{i\gamma}| \geq 4$.

Next, we prove that there is no $\omega \in \mathcal{I}\backslash\{i, -i\}$ satisfying $|\mathcal{F}_{i\omega}| = 4$. By contradiction, assume that $\alpha \in \mathcal{I}\backslash\{i, -i\}$ is such that $|\mathcal{F}_{i\alpha}| = 4$.

In view of (4.1) and in spite of the conditions established until now, one and only one of the following conditions is verified:

i) there is $\beta \in \mathcal{I}\backslash\{i,-i,\alpha\}$ such that $|\mathcal{F}_{i\beta}| = 5$ and $|\mathcal{F}_{i\omega}| = 3$ for any $\omega \in \mathcal{I}\backslash\{i,-i,\alpha,\beta\}$;

ii) there are $\beta, \gamma \in \mathcal{I}\backslash\{i,-i,\alpha\}$, with $\beta \neq \gamma$, such that $|\mathcal{F}_{i\beta}|, |\mathcal{F}_{i\gamma}| \geq 4$ and $|\mathcal{F}_{i\omega}| \leq 3$ for all $\omega \in \mathcal{I}\backslash\{i,-i,\alpha,\beta,\gamma\}$.

As $|\mathcal{G}_i| = 3$ and $|\mathcal{G}_i| = \frac{1}{4}\sum_{\omega\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{G}_{i\omega}|$, then $\sum_{\omega\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{G}_{i\omega}| = 12$.

Let us analyze the hypothesis *i*). By Lemma 2.2, $|\mathcal{G}_{i\alpha}| = |\mathcal{G}_{i\beta}| = 0$ and $|\mathcal{G}_{i\omega}| \leq 1$ for all $\omega \in \mathcal{I}\backslash\{i,-i,\alpha,\beta\}$. As $|\mathcal{I}\backslash\{i,-i,\alpha,\beta\}| = 10$, it follows that $\sum_{\omega\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{G}_{i\omega}| \leq 10$, which is a contradiction.

Now assume that the conditions stated in *ii*) are fulfilled. In these conditions, $|\mathcal{F}_{i\alpha}| + |\mathcal{F}_{i\beta}| + |\mathcal{F}_{i\gamma}| \leq 14$, then having in consideration (4.1) we get $\sum_{\omega\in\mathcal{I}\backslash\{i,-i,\alpha,\beta,\gamma\}} |\mathcal{F}_{i\omega}| \geq 25$. Since $|\mathcal{I}\backslash\{i,-i,\alpha,\beta,\gamma\}| = 9$ and $|\mathcal{F}_{i\omega}| \leq 3$ for all $\omega \in \mathcal{I}\backslash\{i,-i,\alpha,\beta,\gamma\}$, then $|\mathcal{F}_{i\omega}| \geq 1$ for all $\omega \in \mathcal{I}\backslash\{i,-i,\alpha,\beta,\gamma\}$, furthermore, there are, at most, two distinct elements $\theta, \theta' \in \mathcal{I}\backslash\{i,-i,\alpha,\beta,\gamma\}$ so that $1 \leq |\mathcal{F}_{i\theta}|, |\mathcal{F}_{i\theta'}| \leq 2$. Thus, by Lemma 2.2, $\sum_{\omega\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{G}_{i\omega}| \leq 11$, contradicting our assumption.

Accordingly:

- there are exactly two distinct elements $\alpha, \beta \in \mathcal{I}\backslash\{i,-i\}$ so that $|\mathcal{F}_{i\alpha}| = |\mathcal{F}_{i\beta}| = 5$ and $|\mathcal{F}_{i\omega}| \leq 3$ for all $\omega \in \mathcal{I}\backslash\{i,-i,\alpha,\beta\}$;

- there are exactly three distinct elements $\alpha, \beta, \gamma \in \mathcal{I}\backslash\{i,-i\}$ such that $|\mathcal{F}_{i\alpha}| = |\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = 5$ and $|\mathcal{F}_{i\omega}| \leq 3$ for all $\omega \in \mathcal{I}\backslash\{i,-i,\alpha,\beta,\gamma\}$.

Let us assume first that there are only two distinct elements $\alpha, \beta \in \mathcal{I}\backslash\{i,-i\}$ such that $|\mathcal{F}_{i\alpha}| = |\mathcal{F}_{i\beta}| = 5$. By (4.1), there exists a unique element $\theta \in \mathcal{I}\backslash\{i,-i,\alpha,\beta\}$ such that $|\mathcal{F}_{i\theta}| = 2$ and $|\mathcal{F}_{i\omega}| = 3$ for all $\omega \in \mathcal{I}\backslash\{i,-i,\alpha,\beta,\theta\}$. Consequently, by Lemma 2.2, we conclude that $\sum_{\omega\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{G}_{i\omega}| \leq 11$, which is a contradiction.

Summarizing, if $|\mathcal{G}_i| = 3$, there are exactly three distinct elements $\alpha, \beta, \gamma \in \mathcal{I}\backslash\{i,-i\}$, such that, $|\mathcal{F}_{i\alpha}| = |\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = 5$ and $|\mathcal{F}_{i\omega}| \leq 3$ for all $\omega \in \mathcal{I}\backslash\{i,-i,\alpha,\beta,\gamma\}$. □

Proposition 4.2 *Let $\alpha, \beta, \gamma \in \mathcal{I}\backslash\{i, -i\}$ such that $|\mathcal{F}_{i\alpha}| = |\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = 5$. Then, $|\alpha|$, $|\beta|$ and $|\gamma|$ are pairwise distinct and there exist $U_1, U_2, U_3, U_4 \in \mathcal{F}_i$ whose index distributions satisfy:*

U_1	i	α	β	x_1
U_2	i	α	γ	x_2
U_3	i	β	γ	x_3
U_4	i	y_1	y_2	y_3

where $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathcal{I}\backslash\{i, -i, \alpha, \beta, \gamma\}$.

Proof. Let $\alpha, \beta, \gamma \in \mathcal{I}\backslash\{i, -i\}$ so that $|\mathcal{F}_{i\alpha}| = |\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = 5$.

Let us assume, by contradiction, that $|\alpha|$, $|\beta|$ and $|\gamma|$ are not pairwise distinct. Without loss of generality we may assume that $\alpha = -\beta$. Thus, $\mathcal{F}_{i\alpha} \cap \mathcal{F}_{i\beta} = \varnothing$ and, consequently, $|\mathcal{F}_{i\alpha} \cup \mathcal{F}_{i\beta}| = 10$. As $|\mathcal{F}_i| = 13$, then $|\mathcal{F}_{i\alpha\gamma}| = |\mathcal{F}_{i\beta\gamma}| = 1$ and so $\mathcal{F}_i = \mathcal{F}_{i\alpha} \cup \mathcal{F}_{i\beta} \cup \mathcal{F}_{i\gamma}$.

From Lemma 2.9, $|\mathcal{F}_i^{(2)}| = 4$. That is, $|\mathcal{F}_i^{(2)} \cap (\mathcal{F}_{i\alpha} \cup \mathcal{F}_{i\beta} \cup \mathcal{F}_{i\gamma})| = 4$. Consequently, there exists $\omega \in \{\alpha, \beta, \gamma\}$ such that $|\mathcal{F}_i^{(2)} \cap \mathcal{F}_{i\omega}| \geq 2$, contradicting Lemma 2.14.

Therefore, $|\alpha|$, $|\beta|$ and $|\gamma|$ are pairwise distinct.

We have just seen that if $\mathcal{F}_i = \mathcal{F}_{i\alpha} \cup \mathcal{F}_{i\beta} \cup \mathcal{F}_{i\gamma}$, then Lemma 2.14 is contradicted. Thus, $\mathcal{F}_i \supset \mathcal{F}_{i\alpha} \cup \mathcal{F}_{i\beta} \cup \mathcal{F}_{i\gamma}$ which implies $|\mathcal{F}_{i\alpha\beta\gamma}| = 0$. As $|\mathcal{F}_{i\omega}| = 5$ for all $\omega \in \{\alpha, \beta, \gamma\}$, by Lemma 2.5 we get $|\mathcal{F}_{i\omega u}| = 1$ for all $u \in \mathcal{I}\backslash\{i, -i, \omega, -\omega\}$. As a consequence, $|\mathcal{F}_{i\alpha\beta}| = |\mathcal{F}_{i\alpha\gamma}| = |\mathcal{F}_{i\beta\gamma}| = 1$. That is, there are $U_1, U_2, U_3 \in \mathcal{F}_i$ satisfying:

U_1	i	α	β	x_1
U_2	i	α	γ	x_2
U_3	i	β	γ	x_3

Table 4.1: Partial index distribution of $U_1, U_2, U_3 \in \mathcal{F}_i$.

where $x_1, x_2, x_3 \in \mathcal{I}\backslash\{i, -i, \alpha, \beta, \gamma\}$.

As $|\mathcal{F}_{i\alpha} \cup \mathcal{F}_{i\beta} \cup \mathcal{F}_{i\gamma}| = 12$ and $|\mathcal{F}_i| = 13$, there exists $U_4 \notin \mathcal{F}_{i\alpha} \cup \mathcal{F}_{i\beta} \cup \mathcal{F}_{i\gamma}$, that is, $U_4 \in \mathcal{F}_{iy_1y_2y_3}$ where $y_1, y_2, y_3 \in \mathcal{I}\backslash\{i, -i, \alpha, \beta, \gamma\}$. □

The previous proposition gives us a complete picture of the index distribution of four codewords $U_1, U_2, U_3, U_4 \in \mathcal{F}_i$. Namely,

U_1	i	α	β	x_1
U_2	i	α	γ	x_2
U_3	i	β	γ	x_3
U_4	i	y_1	y_2	y_3

Table 4.2: Partial index distribution of $U_1, \ldots, U_4 \in \mathcal{F}_i$.

where $\alpha, \beta, \gamma \in \mathcal{I} \backslash \{i, -i\}$, with $|\alpha|$, $|\beta|$ and $|\gamma|$ pairwise distinct and such that $|\mathcal{F}_{i\alpha}| = |\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = 5$ and $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathcal{I} \backslash \{i, -i, \alpha, \beta, \gamma\}$.

Moreover,

$$\mathcal{F}_i = \mathcal{F}_{i\alpha} \cup \mathcal{F}_{i\beta} \cup \mathcal{F}_{i\gamma} \cup \{U_4\}.$$

Corollary 4.1 *In the considered conditions* $\mathcal{F}_i^{(2)} = \{U_4, U', U'', U'''\}$, *where* $U' \in \mathcal{F}_{i\alpha} \backslash (\mathcal{F}_\beta \cup \mathcal{F}_\gamma)$, $U'' \in \mathcal{F}_{i\beta} \backslash (\mathcal{F}_\alpha \cup \mathcal{F}_\gamma)$ *and* $U''' \in \mathcal{F}_{i\gamma} \backslash (\mathcal{F}_\alpha \cup \mathcal{F}_\beta)$.

Proof. In view of Lemma 2.9, $|\mathcal{F}_i^{(2)}| = 4$. By Lemma 2.14 it follows that $|\mathcal{F}_i^{(2)} \cap \mathcal{F}_u| = 1$ for each $u \in \mathcal{I} \backslash \{i, -i\}$. Since $\mathcal{F}_i = \mathcal{F}_{i\alpha} \cup \mathcal{F}_{i\beta} \cup \mathcal{F}_{i\gamma} \cup \{U_4\}$, then $U_4 \in \mathcal{F}_i^{(2)}$, otherwise $|\mathcal{F}_i^{(2)} \cap \mathcal{F}_\omega| \geq 2$ for some $\omega \in \{\alpha, \beta, \gamma\}$. On the other hand, $|\mathcal{F}_i^{(2)} \cap (\mathcal{F}_{i\alpha} \cup \mathcal{F}_{i\beta} \cup \mathcal{F}_{i\gamma})| = 3$ and so there are $U', U'', U''' \in \mathcal{F}_i^{(2)} \cap (\mathcal{F}_{i\alpha} \cup \mathcal{F}_{i\beta} \cup \mathcal{F}_{i\gamma})$ such that $U' \in \mathcal{F}_{i\alpha} \backslash (\mathcal{F}_\beta \cup \mathcal{F}_\gamma)$, $U'' \in \mathcal{F}_{i\beta} \backslash (\mathcal{F}_\alpha \cup \mathcal{F}_\gamma)$ and $U''' \in \mathcal{F}_\gamma \backslash (\mathcal{F}_\alpha \cup \mathcal{F}_\beta)$, otherwise there exists again $\omega \in \{\alpha, \beta, \gamma\}$ such that $|\mathcal{F}_i^{(2)} \cap \mathcal{F}_{i\omega}| \geq 2$. □

Proposition 4.3 *If $|\mathcal{F}_{i\alpha}| = |\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = 5$, then $|\mathcal{G}_{i\alpha}| = |\mathcal{G}_{i\beta}| = |\mathcal{G}_{i\gamma}| = 0$. Furthermore, there exist $\delta, \varepsilon, \theta \in \mathcal{I}\backslash\{i, -i, \alpha, \beta, \gamma\}$ such that $|\mathcal{G}_{i\delta}| = |\mathcal{G}_{i\varepsilon}| = |\mathcal{G}_{i\theta}| = 2$ and $|\mathcal{G}_{i\omega}| = 1$ for all $\omega \in \mathcal{I}\backslash\{i, -i, \alpha, \beta, \gamma, \delta, \varepsilon, \theta\}$. The index distributions of the three codewords $W_1, W_2, W_3 \in \mathcal{G}_i$ satisfy:*

W_1	i	δ	ε	w_1	w_2
W_2	i	δ	θ	w_3	w_4
W_3	i	ε	θ	w_5	w_6

where $\delta, \varepsilon, \theta, w_1, \ldots, w_6 \in \mathcal{I}\backslash\{i, -i, \alpha, \beta, \gamma\}$ are pairwise distinct.

Proof. Let $\alpha, \beta, \gamma \in \mathcal{I}\backslash\{i, -i\}$ be such that $|\mathcal{F}_{i\alpha}| = |\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = 5$ which means that $|\mathcal{G}_{i\alpha}| = |\mathcal{G}_{i\beta}| = |\mathcal{G}_{i\gamma}| = 0$, in view of Lemma 2.2. Let W_1, W_2, W_3 be the three codewords of $\mathcal{G}_i$. Then,

W_1	i	w_1	w_2	w_3	w_4
W_2	i	w_5	w_6	w_7	w_8
W_3	i	w_9	w_{10}	w_{11}	w_{12}

Table 4.3: Partial index distribution of $W_1, W_2, W_3 \in \mathcal{G}_i$.

where $w_1, \ldots, w_{12} \in \mathcal{I}\backslash\{i, -i, \alpha, \beta, \gamma\}$.

Regarding the cardinality of $\mathcal{G}_i$, we may conclude that $|\mathcal{G}_{i\omega}| \leq 2$ for all $\omega \in \mathcal{I}\backslash\{i, -i, \alpha, \beta, \gamma\}$ since $|\mathcal{G}_{i\omega}| = 3$, for some ω, implies, by Lemma 1.5, the existence of nine distinct elements in $\mathcal{I}\backslash\{i, -i, \alpha, \beta, \gamma, \omega\}$, which is a contradiction because $|\mathcal{I}\backslash\{i, -i, \alpha, \beta, \gamma, \omega\}| = 8$.

As $|\mathcal{G}_i| = 3$ and, consequently, $\sum_{\omega\in\mathcal{I}\backslash\{i,-i,\alpha,\beta,\gamma\}} |\mathcal{G}_{i\omega}| = 12$, there exist, at least, three distinct elements $\omega \in \mathcal{I}\backslash\{i, -i, \alpha, \beta, \gamma\}$ satisfying $|\mathcal{G}_{i\omega}| = 2$. Taking into account the partial index distribution of the codewords of $\mathcal{G}_i$ and Lemma 1.5, there exist exactly three elements $\delta, \varepsilon, \theta \in \mathcal{I}\backslash\{i, -i, \alpha, \beta, \gamma\}$ in these conditions satisfying,

W_1	i	δ	ε	w_1	w_2
W_2	i	δ	θ	w_3	w_4
W_3	i	ε	θ	w_5	w_6

Table 4.4: Partial index distribution of $W_1, W_2, W_3 \in \mathcal{G}_i$.

where $\delta, \varepsilon, \theta, w_1, \ldots, w_6 \in \mathcal{I}\backslash\{i, -i, \alpha, \beta, \gamma\}$ are pairwise distinct. □

Let us consider

$$\mathcal{I} = \{i, -i, j, -j, k, -k, l, -l, m, -m, n, -n, o, -o\}.$$

Since the index distribution of $U_1, U_2, U_3, U_4 \in \mathcal{F}_i$ is the one illustrated in Table 4.2, we may assume, without loss of generality, that $\alpha = j$, $\beta = k$ and $U_1 \in \mathcal{F}_{ijkl}$, that is:

U_1	i	j	k	l
U_2	i	j	γ	x_1
U_3	i	k	γ	x_2
U_4	i	y_1	y_2	y_3

Table 4.5: Partial index distribution of $U_1, \ldots, U_4 \in \mathcal{F}_i$.

where $x_1, x_2 \in \mathcal{I}\backslash\{i, -i, j, k, \gamma, l\}$ and $y_1, y_2, y_3 \in \mathcal{I}\backslash\{i, -i, j, k, \gamma\}$.

In what follows, the index distribution of the codewords of $\mathcal{G}_i$ and $U_1, U_2, U_3, U_4 \in \mathcal{F}_i$ are, respectively, the ones given in the Tables 4.4 and 4.5, respectively. Next, we will analyze how the codewords of $\mathcal{G}_i$ and $\mathcal{F}_i$ fit together.

Proposition 4.4 *If $l \neq \delta, \varepsilon, \theta$, then, without loss of generality, $W_1 \in \mathcal{G}_{i\delta\varepsilon l}$, and either $\theta = -l$ or $\theta = -j$ or $\theta = -k$.*

Proof. Let us assume that $l \neq \delta, \varepsilon, \theta$. Since $l \notin \{i, -i, j, k, \gamma\}$, by Proposition 4.3, $|\mathcal{G}_{il}| = 1$. Without loss of generality, we set $W_1 \in \mathcal{G}_{i\delta\varepsilon l}$.

Observe that, with the relabeling of the indices, $U_1 \in \mathcal{F}_{ijkl}$.

Suppose, by contradiction, that $\theta \neq -l, -j, -k$. As $|\mathcal{F}_{ij}| = |\mathcal{F}_{ik}| = 5$ then, by Lemma 2.5, $|\mathcal{F}_{ij\omega}| = 1$ for all $\omega \in \mathcal{I}\backslash\{i, -i, j, -j\}$, and $|\mathcal{F}_{ik\omega}| = 1$ for each $\omega \in \mathcal{I}\backslash\{i, -i, k, -k\}$. Thus, there exist $U \in \mathcal{F}_{ij\theta u_1}$ and $U' \in \mathcal{F}_{ik\theta u_2}$, with $U \neq U'$, otherwise Lemma 1.5 is contradicted.

As we have seen, two of the three codewords of $\mathcal{G}_i$, namely W_2 and W_3, belong, respectively, to $\mathcal{G}_{i\delta\theta w_3 w_4}$ and $\mathcal{G}_{i\varepsilon\theta w_5 w_6}$. Since $U, U' \in \mathcal{F}_{i\theta}$ and $W_2, W_3 \in \mathcal{G}_{i\theta}$, by Lemma 1.5, $\delta, \varepsilon, u_1, u_2, w_3, w_4, w_5, w_6 \in \mathcal{I}\backslash\{i, -i, j, k, \theta, -\theta\}$ are pairwise distinct. As $|\mathcal{I}\backslash\{i, -i, j, k, \theta, -\theta\}| = 8$ and we are assuming $\theta \neq l, -l$, then we have $l \in \{\delta, \varepsilon, u_1, u_2, w_3, w_4, w_5, w_6\}$. But $\delta, \varepsilon \neq l$ and so $l \in \{u_1, u_2, w_3, w_4, w_5, w_6\}$. Consequently either $|\mathcal{F}_{ijl}| \geq 2$ or $|\mathcal{F}_{ikl}| \geq 2$ or $|\mathcal{G}_{i\delta l}| \geq 2$ or $|\mathcal{G}_{i\varepsilon l}| \geq 2$, which implies, in any case, the contradiction of Lemma 1.5. □

Bringing to the scene the index characterization of the codeword $U_4 \in \mathcal{F}_i$, in particular, $U_4 \in \mathcal{F}_i\backslash(\mathcal{F}_j \cup \mathcal{F}_k \cup \mathcal{F}_\gamma)$, we already know that $U_4 \in \mathcal{F}_{iy_1y_2y_3}$ where $y_1, y_2, y_3 \in \mathcal{I}\backslash\{i, -i, j, k, \gamma\}$. Next result bounds the range of y_1, y_2, y_3 even more.

Proposition 4.5 *$U_4 \in \mathcal{F}_{iy_1y_2y_3}$ for $y_1, y_2, y_3 \in \{-j, -k, -\gamma, l, x_1, x_2\}$, where x_1 and x_2 are such that $U_2 \in \mathcal{F}_{ij\gamma x_1}$ and $U_3 \in \mathcal{F}_{ik\gamma x_2}$ (see Table 4.5).*

Proof. As seen before $U_4 \in \mathcal{F}_{iy_1y_2y_3}$, where $y_1, y_2, y_3 \in \mathcal{I}\backslash\{i, -i, j, k, \gamma\}$.

Suppose, by contradiction, that there exists $y \in \{y_1, y_2, y_3\}$ such that $y \notin \{-j, -k, -\gamma, l, x_1, x_2\}$, that is, $y \in \mathcal{I}\backslash\{i, -i, j, -j, k, -k, \gamma, -\gamma, l, x_1, x_2\}$. Since $|\mathcal{F}_{ij}| = |\mathcal{F}_{ik}| = |\mathcal{F}_{i\gamma}| = 5$, for each $\omega \in \{j, k, \gamma\}$ we have $|\mathcal{F}_{i\omega u}| = 1$ for all $u \in \mathcal{I}\backslash\{i, -i, \omega, -\omega\}$. In accordance, there are three codewords, U', U'', U''' in $\mathcal{F}_i$ satisfying $U' \in \mathcal{F}_{ijy}$, $U'' \in \mathcal{F}_{iky}$ and $U''' \in \mathcal{F}_{i\gamma y}$. As $y \neq l, x_1, x_2$ the codewords U', U'', U''' are pairwise distinct. Thus, U', U'', U''', U_4 are four distinct codewords in $\mathcal{F}_{iy}$. In spite of Proposition 4.3, $|\mathcal{G}_{iy}| \geq 1$. Therefore, $|\mathcal{F}_{iy}| \geq 4$ and $|\mathcal{G}_{iy}| \geq 1$, contradicting Lemma 2.2. Consequently, $\{y_1, y_2, y_3\} \subset \{-j, -k, -\gamma, l, x_1, x_2\}$. □

Now, we know that $U_1 \in \mathcal{F}_{ijkl}$, $U_2 \in \mathcal{F}_{ij\gamma x_1}$ and $U_3 \in \mathcal{F}_{ik\gamma x_2}$. Next result establishes connections between the index distributions of these codewords and the codewords of $\mathcal{G}_i$. Namely,

Proposition 4.6 *If* $|\mathcal{G}_{i\omega}| = 2$ *for some* $\omega \in \mathcal{I}\backslash\{i, -i, j, -j, k, -k, \gamma, -\gamma, l\}$*, then either* $\omega = x_1$ *or* $\omega = x_2$*.*

Proof. Let $\omega \in \mathcal{I}\backslash\{i, -i, j, -j, k, -k, \gamma, -\gamma, l\}$ be such that $|\mathcal{G}_{i\omega}| = 2$. Suppose, by contradiction, that $\omega \neq x_1, x_2$. As $|\mathcal{F}_{ij}| = |\mathcal{F}_{ik}| = |\mathcal{F}_{i\gamma}| = 5$ and, by assumption, $\omega \neq -j, -k, -\gamma$, taking into account Lemma 2.5, there are $U', U'', U''' \in \mathcal{F}_i$ so that $U' \in \mathcal{F}_{ij\omega}$, $U'' \in \mathcal{F}_{ik\omega}$ and $U''' \in \mathcal{F}_{i\gamma\omega}$. By hypothesis $\omega \notin \{l, x_1, x_2\}$, then U',U'' and U''' are pairwise distinct, which is not possible since we would get $|\mathcal{F}_{i\omega}| \geq 3$ and $|\mathcal{G}_{i\omega}| = 2$, contradicting Lemma 2.2. □

Under the conditions and notation of the previous results one has:

Proposition 4.7 *The indices* $\delta, \varepsilon, \theta \in \{-j, -k, -\gamma, l, x_1, x_2\}\backslash\{y_1, y_2, y_3\}$ *furthermore* $|\{-j, -k, -\gamma, l, x_1, x_2\}| = 6$.

Proof. As seen in Proposition 4.3, the three codewords of $\mathcal{G}_i$, W_1, W_2 and W_3, satisfy $W_1 \in \mathcal{G}_{i\delta\varepsilon}$, $W_2 \in \mathcal{G}_{i\delta\theta}$ and $W_3 \in \mathcal{G}_{i\varepsilon\theta}$, with $\delta, \varepsilon, \theta \in \mathcal{I}\backslash\{i, -i, j, k, \gamma\}$. Suppose that $\omega \in \{\delta, \varepsilon, \theta\}$ is such that $\omega \neq -j, -k, -\gamma, l$. From Proposition 4.6, we conclude that $\omega = x_1$ or $\omega = x_2$. Thus, if $\omega \in \{\delta, \varepsilon, \theta\}$, then $\omega \in \{-j, -k, -\gamma, l, x_1, x_2\}$.

By Proposition 4.5, $\{y_1, y_2, y_3\} \subset \{-j, -k, -\gamma, l, x_1, x_2\}$. Suppose, by contradiction, that $\{\delta, \varepsilon, \theta\} \cap \{y_1, y_2, y_3\} \neq \varnothing$. Without loss of generality, suppose that $\delta = y_1$. Thus, $\{\varepsilon, \theta\} \cap \{y_2, y_3\} = \varnothing$, otherwise U_4 and W_1, or, U_4 and W_2 contradict Lemma 1.5. Then, $y_2, y_3 \in \{-j, -k, -\gamma, l, x_1, x_2\}\backslash\{\delta, \varepsilon, \theta\}$. By Proposition 4.3 there are $W' \in \mathcal{G}_{iy_2}$ and $W'' \in \mathcal{G}_{iy_3}$. To avoid superposition between U_4 and the codewords $W_1, W_2 \in \mathcal{G}_{i\delta}$, with $\delta = y_1$, we must impose $W_3 \in \mathcal{G}_{i\varepsilon\theta y_2 y_3}$, which is not possible since U_4 and W_3 contradict Lemma 1.5. Therefore, $\delta, \varepsilon, \theta \in \{-j, -k, -\gamma, l, x_1, x_2\}\backslash\{y_1, y_2, y_3\}$.

As y_1, y_2, y_3 are pairwise distinct and $\delta, \varepsilon, \theta$ are also pairwise distinct, we conclude that $|\{-j, -k, -\gamma, l, x_1, x_2\}| = 6$. □

4.2 Index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$

In the previous section we have concluded that there are $U_1, U_2, U_3, U_4 \in \mathcal{F}_i$ satisfying the following conditions

U_1	i	j	k	l
U_2	i	j	γ	x_1
U_3	i	k	γ	x_2
U_4	i	y_1	y_2	y_3

Table 4.6: Partial index distribution of $U_1, \dots, U_4 \in \mathcal{F}_i$.

where:

- $|\mathcal{F}_{ij}| = |\mathcal{F}_{ik}| = |\mathcal{F}_{i\gamma}| = 5$ and $\gamma \in \mathcal{I} \backslash \{i, -i, j, -j, k, -k, l\}$;
- $x_1, x_2 \in \mathcal{I} \backslash \{i, -i, j, k, \gamma, -\gamma, l\}$ are distinct;
- $y_1, y_2, y_3 \in \{-j, -k, -\gamma, l, x_1, x_2\}$.

We have verified also that the codewords $W_1, W_2, W_3 \in \mathcal{G}_i$ satisfy the conditions

W_1	i	δ	ε	w_1	w_2
W_2	i	δ	θ	w_3	w_4
W_3	i	ε	θ	w_5	w_6

Table 4.7: Partial index distribution of $W_1, W_2, W_3 \in \mathcal{G}_i$.

with

- $\delta, \varepsilon, \theta \in \{-j, -k, -\gamma, l, x_1, x_2\} \backslash \{y_1, y_2, y_3\}$ pairwise distinct;
- $w_1, \dots, w_6 \in \mathcal{I} \backslash \{i, -i, j, k, \gamma, \delta, \varepsilon, \theta\}$ pairwise distinct.

Next, we characterize in more detail the index distributions of the codewords of $\mathcal{G}_i$.

4.2.1 Partial index distribution of the codewords of $\mathcal{G}_i$

In this section we present the different possible partial index distributions for the codewords of $\mathcal{G}_i$. In this sense, we will consider Propositions 4.4 and 4.3 from which follows, respectively, the statements:

- if $l \neq \delta, \varepsilon, \theta$, then $W_1 \in \mathcal{G}_{i\delta\varepsilon l}$ and either $\theta = -l$ or $\theta = -j$ or $\theta = -k$;
- for all $\omega \in \mathcal{I} \backslash \{i, -i, j, k, \gamma, \delta, \varepsilon, \theta\}$, $|\mathcal{G}_{i\omega}| = 1$.

Taking into account Proposition 4.4, we are going to distinguish the cases:

i) $l \neq \delta, \varepsilon, \theta$;

ii) $\exists^1\, \omega \in \{\delta, \varepsilon, \theta\}$ such that $\omega = l$.

i) **Suppose** $l \neq \delta, \varepsilon, \theta$

If $l \neq \delta, \varepsilon, \theta$, then, by Proposition 4.4, $W_1 \in \mathcal{G}_{i\delta\varepsilon l}$, and, $\theta = -l$ or $\theta = -j$ or $\theta = -k$. Considering $U_1, \ldots, U_4 \in \mathcal{F}_i$ and $W_1, W_2, W_3 \in \mathcal{G}_i$, see, respectively, Tables 4.6 and 4.7, it is indifferent to consider $\theta = -j$ or $\theta = -k$, thus, we only consider the two following hypotheses:

W_1	i	δ	ε	l	w_1
W_2	i	δ	$-j$	w_2	w_3
W_3	i	ε	$-j$	w_4	w_5

W_1	i	δ	ε	l	w_1
W_2	i	δ	$-l$	w_2	w_3
W_3	i	ε	$-l$	w_4	w_5

Table 4.8: Partial index distribution of the codewords of $\mathcal{G}_i$.

where the distinct elements w_1,...,w_5 are so that w_1,...,$w_5 \in \mathcal{I} \backslash \{i, -i, j, -j, k, \gamma, \delta, \varepsilon, l\}$ if $W_1, W_2, W_3 \in \mathcal{G}_i$ satisfy the conditions in Table 4.8 on the left, on the other hand, $w_1, \ldots, w_5 \in \mathcal{I} \backslash \{i, -i, j, k, \gamma, \delta, \varepsilon, l, -l\}$ if $W_1, W_2, W_3 \in \mathcal{G}_i$ satisfy the conditions in Table 4.8 on the right.

Next, we will analyze each one of these possible partial index distributions.

• Assume $\theta = -j$. From Proposition 4.3, $1 \leq |\mathcal{G}_{i,-k}| \leq 2$, since $-k \in \mathcal{I}\backslash\{i, -i, j, k, \gamma\}$. If $|\mathcal{G}_{i,-k}| = 2$, then $\delta = -k$ or $\varepsilon = -k$. Suppose, without loss of generality, $\delta = -k$. Accordingly, concretizing other indices, schematically we have:

W_1	i	$-k$	m	l	n
W_2	i	$-k$	$-j$	w_1	w_2
W_3	i	m	$-j$	w_3	w_4

Table 4.9: Partial index distribution of the codewords of $\mathcal{G}_i$.

If $|\mathcal{G}_{i,-k}| = 1$, we distinguish between the cases $W_1 \in \mathcal{G}_{i,-k}$ or $W_1 \notin \mathcal{G}_{i,-k}$. Accordingly, concretizing other indices, we get the following possibilities for the index distribution of the codewords of $\mathcal{G}_i$:

W_1	i	m	n	l	$-k$
W_2	i	m	$-j$	w_1	w_2
W_3	i	n	$-j$	w_3	w_4

Table 4.10: $W_1 \in \mathcal{G}_{i,-k}$.

W_1	i	m	n	l	o
W_2	i	m	$-j$	$-k$	w_1
W_3	i	n	$-j$	w_2	w_3

Table 4.11: $W_1 \notin \mathcal{G}_{i,-k}$.

• Assume now $\theta = -l$. Since $-j, -k \in \mathcal{I}\backslash\{i, -i, j, k, \gamma\}$, by Proposition 4.3, $1 \leq |\mathcal{G}_{i,-j}| \leq 2$ and $1 \leq |\mathcal{G}_{i,-k}| \leq 2$. So, we will consider the cases:

1. $\delta = -j$ and $\varepsilon = -k$;
2. $\exists^1\ \omega \in \{\delta, \varepsilon\}$ such that $\omega \in \{-j, -k\}$ (without loss of generality, we will assume $\omega = -j$);
3. $\{\delta, \varepsilon\} \cap \{-j, -k\} = \varnothing$.

In the following schemes are presented the partial index distributions of the codewords of $\mathcal{G}_i$ satisfying, respectively, the conditions in 1, 2 and 3. In each case we concretize other indices.

W_1	i	$-j$	$-k$	l	m
W_2	i	$-j$	$-l$	w_1	w_2
W_3	i	$-k$	$-l$	w_3	w_4

Table 4.12: $\delta = -j$ and $\varepsilon = -k$.

W_1	i	$-j$	m	l	w_1
W_2	i	$-j$	$-l$	w_2	w_3
W_3	i	m	$-l$	w_4	w_5

Table 4.13: $\delta = -j$ and $\varepsilon \neq -k$.

W_1	i	m	n	l	w_1
W_2	i	m	$-l$	w_2	w_3
W_3	i	n	$-l$	w_4	w_5

Table 4.14: $\{\delta, \varepsilon\} \cap \{-j, -k\} = \varnothing$.

$ii)$ **Suppose that there is a unique** $\omega \in \{\delta, \varepsilon, \theta\}$ **such that** $\omega = l$

Without loss of generality, consider $\delta = l$. Such as in the previous case, one of the following conditions is satisfied:

1. $\varepsilon = -j$ and $\theta = -k$;

2. $\exists^1\ \omega \in \{\varepsilon, \theta\}$ such that $\omega \in \{-j, -k\}$ (without loss of generality, we will assume $\varepsilon = -j$);

3. $\{\varepsilon, \theta\} \cap \{-j, -k\} = \varnothing$.

The partial index distribution of the codewords of $\mathcal{G}_i$ satisfying, respectively, the conditions in 1, 2 and 3 are presented in the following schemes:

W_1	i	l	$-j$	m	n
W_2	i	l	$-k$	w_1	w_2
W_3	i	$-j$	$-k$	w_3	w_4

Table 4.15: $\varepsilon = -j$ and $\theta = -k$.

W_1	i	l	$-j$	w_1	w_2
W_2	i	l	m	w_3	w_4
W_3	i	$-j$	m	w_5	w_6

Table 4.16: $\varepsilon = -j$ and $\theta \neq -k$.

W_1	i	l	m	w_1	w_2
W_2	i	l	n	w_3	w_4
W_3	i	m	n	w_5	w_6

Table 4.17: $\{\varepsilon, \theta\} \cap \{-j, -k\} = \varnothing$.

4.2.2 Partial index distribution of the codewords of $\mathcal{F}_i$

In the previous subsection we have characterized the possible partial index distributions of the codewords $W_1, W_2, W_3 \in \mathcal{G}_i$. Here, for each one of these index distributions we describe, in more detail, the index distribution of $U_2, U_3, U_4 \in \mathcal{F}_i$.

Let us consider $W_1, W_2, W_3 \in \mathcal{G}_i$ satisfying one of the obtained partial index distributions, in particular,

W_1	i	$-k$	m	l	n
W_2	i	$-k$	$-j$	w_1	w_2
W_3	i	m	$-j$	w_3	w_4

Table 4.18: Partial index distribution of the codewords of $\mathcal{G}_i$.

where $\delta = -k$, $\varepsilon = m$, $\theta = -j$ and $w_1, \ldots, w_4 \in \mathcal{I} \backslash \{i, -i, j, -j, k, -k, l, m, n, \gamma\}$ are pairwise distinct.

Let $U_1, \ldots, U_4 \in \mathcal{F}_i$ be such that:

U_1	i	j	k	l
U_2	i	j	γ	x_1
U_3	i	k	γ	x_2
U_4	i	y_1	y_2	y_3

Table 4.19: Partial index distribution of $U_1, \ldots, U_4 \in \mathcal{F}_i$.

By Propositions 4.5 and 4.7, $\{-k, m, -j, y_1, y_2, y_3\} = \{-j, -k, -\gamma, l, x_1, x_2\}$.

From Proposition 4.3 it follows that $|\mathcal{G}_{i\gamma}| = 0$, then, taking into account the index distribution of the codewords of $\mathcal{G}_i$, and considering the elements of $\mathcal{I}$, we conclude that $\gamma \in \{-l, -m, -n, o, -o\}$. If:

- $\gamma = -l$, then $|\{-j, -k, -\gamma, l, x_1, x_2\}| \leq 5$, contradicting Proposition 4.7;
- $\gamma = -m$, then $\{y_1, y_2, y_3\} = \{l, x_1, x_2\}$, that is, $U_4 \in \mathcal{F}_{ilx_1x_2}$;
- $\gamma = -n$, then $l, n \in \{y_1, y_2, y_3\}$ and $U_4 \in \mathcal{F}_{iln}$ implying $|\mathcal{G}_{iln} \cup \mathcal{F}_{iln}| \geq 2$, which contradicts Lemma 1.5;
- $\gamma = o$, then $m \in \{x_1, x_2\}$ and $U_4 \in \mathcal{F}_{i,l,-o,x}$, with $x \in \{x_1, x_2\} \backslash \{m\}$.

Note that, it is indifferent to consider $\gamma = o$ or $\gamma = -o$.

Therefore, if $W_1, W_2, W_3 \in \mathcal{G}_i$ verify the conditions in Table 4.18, then $\gamma = -m$ or $\gamma = o$ and $U_1, \ldots, U_4 \in \mathcal{F}_i$ satisfy, respectively:

U_1	i	j	k	l
U_2	i	j	$-m$	x_1
U_3	i	k	$-m$	x_2
U_4	i	l	x_1	x_2

Table 4.20: $\gamma = -m$.

U_1	i	j	k	l
U_2	i	j	o	x_1
U_3	i	k	o	x_2
U_4	i	l	$-o$	x

Table 4.21: $\gamma = o$; $m \in \{x_1, x_2\}$; $x \in \{x_1, x_2\} \backslash \{m\}$.

By a similar reasoning we get, for each one of the partial index distributions of the codewords of $\mathcal{G}_i$ obtained in the previous subsection, the following conditions.

If $W_1, W_2, W_3 \in \mathcal{G}_i$ satisfy the partial index distribution presented in:

♢ Table 4.10, then $U_4 \in \mathcal{F}_{i,-k,l}$ and, consequently, $|\mathcal{G}_{i,-k,l} \cup \mathcal{F}_{i,-k,l}| \geq 2$, which is a contradiction;

♢ Table 4.11, then is satisfied one of the following hypotheses

- ⋄ $\gamma = -m$; $n \in \{x_1, x_2\}$; $U_4 \in \mathcal{F}_{i,-k,l,x}$, with $x \in \{x_1, x_2\} \backslash \{n\}$;
- ⋄ $\gamma = -n$; $m \in \{x_1, x_2\}$; $U_4 \in \mathcal{F}_{i,-k,l,x}$, with $x \in \{x_1, x_2\} \backslash \{m\}$;

♢ Table 4.12, then

⋄ $\gamma = n$; $-l \in \{x_1, x_2\}$; $U_4 \in \mathcal{F}_{i,l,-n,x}$, with $x \in \{x_1, x_2\} \backslash \{-l\}$;

♢ Table 4.13, then is satisfied one of the following hypotheses

⋄ $\gamma = -m$; $-l \in \{x_1, x_2\}$; $U_4 \in \mathcal{F}_{i,-k,l,x}$, with $x \in \{x_1, x_2\} \backslash \{-l\}$;

⋄ $\gamma = n$; $x_1 = m$ and $x_2 = -l$, or, $x_1 = -l$ and $x_2 = m$; $U_4 \in \mathcal{F}_{i,-k,-n,l}$;

♢ Table 4.14, then

⋄ $\gamma = -m$; $x_1 = n$ and $x_2 = -l$, or, $x_1 = -l$ and $x_2 = n$; $U_4 \in \mathcal{F}_{i,-j,-k,l}$;

♢ Table 4.15, then is satisfied one of the following hypotheses

⋄ $\gamma = -m$ and $U_4 \in \mathcal{F}_{imx_1x_2}$;

⋄ $\gamma = o$ and $U_4 \in \mathcal{F}_{i,-o,x_1,x_2}$;

♢ Table 4.16, then is satisfied one of the following hypotheses

⋄ $\gamma = -m$ and $U_4 \in \mathcal{F}_{i,-k,x_1,x_2}$;

⋄ $\gamma = n$; $m \in \{x_1, x_2\}$; $U_4 \in \mathcal{F}_{i,-k,-n,x}$, with $x \in \{x_1, x_2\} \backslash \{m\}$;

♢ Table 4.17, then is satisfied one of the following hypotheses

⋄ $\gamma = -m$, $n \in \{x_1, x_2\}$; $U_4 \in \mathcal{F}_{i,-j,-k,x}$, with $x \in \{x_1, x_2\} \backslash \{n\}$;

⋄ $\gamma = o$; $x_1 = m$ and $x_2 = n$,or, $x_1 = n$ and $x_2 = m$; $U_4 \in \mathcal{F}_{i,-j,-k,-o}$.

4.2.3 Complete characterization of the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$

Until now we have presented the possible partial index distributions of the codewords of $\mathcal{G}_i$ and respective codewords $U_1, U_2, U_3, U_4 \in \mathcal{F}_i$. However, our aim is to describe completely the index distribution of all codewords of $\mathcal{G}_i \cup \mathcal{F}_i$, taking into account that $|\mathcal{G}_i| = 3$ and $|\mathcal{F}_i| = 13$. Here, we show, throughout illustrative examples, the method applied in the characterization of the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$.

By the analysis of all partial index distributions of the codewords of $\mathcal{G}_i \cup \{U_1, \ldots, U_4\}$, identified in the previous subsections, we have verified that in the majority of the cases the complete characterization of the index distribution of all codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ implies $|\mathcal{G}_{i\omega\rho} \cup \mathcal{F}_{i\omega\rho}| \geq 2$ for some $\omega, \rho \in \mathcal{I}\backslash\{i, -i\}$, contradicting Lemma 1.5. There exist some cases in which it is possible to describe the index distribution of all codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ such that $|\mathcal{G}_{i\omega\rho} \cup \mathcal{F}_{i\omega\rho}| \leq 1$ for all $\omega, \rho \in \mathcal{I}\backslash\{i, -i\}$. However, in most of these cases there exists $\varphi \in \mathcal{I}\backslash\{i, -i\}$ such that $|\mathcal{F}_i^{(2)} \cap \mathcal{F}_\varphi| \geq 2$, which contradicts Lemma 1.4.

There exist only two possible index distributions for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ in which $|\mathcal{G}_{i\omega\rho} \cup \mathcal{F}_{i\omega\rho}| \leq 1$ and $|\mathcal{F}_i^{(2)} \cap \mathcal{F}_\varphi| \leq 1$ for all $\omega, \rho, \varphi \in \mathcal{I}\backslash\{i, -i\}$. In these cases to show that the definition of PL$(7, 2)$ code is contradicted it is necessary to analyze the complete index distribution of all codewords of $\mathcal{G}_\omega \cup \mathcal{F}_\omega$ for other index $\omega \in \mathcal{I}\backslash\{i\}$, as we will see.

Next, we show how we have gotten the conclusions referred before, presenting illustrative examples in which:

1. $|\mathcal{G}_{i\omega\rho} \cup \mathcal{F}_{i\omega\rho}| \geq 2$ for some $\omega, \rho \in \mathcal{I}\backslash\{i, -i\}$;
2. $|\mathcal{G}_{i\omega\rho} \cup \mathcal{F}_{i\omega\rho}| \leq 1$ for all $\omega, \rho \in \mathcal{I}\backslash\{i, -i\}$ and $|\mathcal{F}_i^{(2)} \cap \mathcal{F}_\varphi| \geq 2$ for some $\varphi \in \mathcal{I}\backslash\{i, -i\}$;
3. $|\mathcal{G}_{i\omega\rho} \cup \mathcal{F}_{i\omega\rho}| \leq 1$ and $|\mathcal{F}_i^{(2)} \cap \mathcal{F}_\varphi| \leq 1$ for all $\omega, \rho, \varphi \in \mathcal{I}\backslash\{i, -i\}$.

- $|\mathcal{G}_{i\omega\rho} \cup \mathcal{F}_{i\omega\rho}| \geq 2$ **for some** $\omega, \rho \in \mathcal{I}\backslash\{i, -i\}$

Consider $W_1, W_2, W_3 \in \mathcal{G}_i$ satisfying the following partial index distribution:

W_1	i	$-k$	m	l	n
W_2	i	$-k$	$-j$	w_1	w_2
W_3	i	m	$-j$	w_3	w_4

Table 4.22: Partial index distribution of the codewords of $\mathcal{G}_i$.

where $\delta = -k$, $\varepsilon = m$ and $\theta = -j$.

For $U_1, \dots, U_4 \in \mathcal{F}_i$,

U_1	i	j	k	l
U_2	i	j	γ	x_1
U_3	i	k	γ	x_2
U_4	i	y_1	y_2	y_3

Table 4.23: Partial index distribution of $U_1, \dots, U_4 \in \mathcal{F}_i$.

we have concluded in the previous subsection that one of the following conditions must be verified:

i) $\gamma = -m$ and $U_4 \in \mathcal{F}_{ilx_1x_2}$;

ii) $\gamma = o$; $m \in \{x_1, x_2\}$; $U_4 \in \mathcal{F}_{i,l,-o,x}$, with $x \in \{x_1, x_2\} \backslash \{m\}$.

Let us consider $\gamma = o$ and assume $m = x_1$. Accordingly, $U_1, \dots, U_4 \in \mathcal{F}_i$ satisfy:

U_1	i	j	k	l
U_2	i	j	o	m
U_3	i	k	o	x_2
U_4	i	l	$-o$	x_2

Table 4.24: Partial index distribution of $U_1, \dots, U_4 \in \mathcal{F}_i$.

From Proposition 4.7 it follows that $|\{-j, -k, -o, l, m, x_2\}| = 6$. Taking into account the index distribution of the codewords $W_1, W_2, W_3 \in \mathcal{G}_i$, $U_1, \dots, U_4 \in \mathcal{F}_i$ and Lemma 1.5 we conclude that $x_2 \in \{-m, -n\}$. So, we are going to analyze, separately, the cases: $x_2 = -m$ and $x_2 = -n$.

Suppose first that $x_2 = -m$. In this case $U_1, \ldots, U_4 \in \mathcal{F}_i$ are such that:

U_1	i	j	k	l
U_2	i	j	o	m
U_3	i	k	o	$-m$
U_4	i	l	$-o$	$-m$

Table 4.25: Index distribution of $U_1, \ldots, U_4 \in \mathcal{F}_i$.

Considering Proposition 4.3, by the analysis of the index distribution of the codewords known at this moment, $W_1, W_2, W_3 \in \mathcal{G}_i$, see Table 4.22, are such that $w_1, \ldots, w_4 \in \{-l, -m, -n, -o\}$ and are pairwise distinct. Taking into account the codewords $W_1, W_2, W_3 \in \mathcal{G}_i$, $U_1, \ldots, U_4 \in \mathcal{F}_i$ and Lemma 1.5, two possible index distributions for the codewords of $\mathcal{G}_i$ are obtained:

W_1	i	$-k$	m	l	n
W_2	i	$-k$	$-j$	$-m$	$-l$
W_3	i	m	$-j$	$-n$	$-o$

W_1	i	$-k$	m	l	n
W_2	i	$-k$	$-j$	$-m$	$-n$
W_3	i	m	$-j$	$-l$	$-o$

Table 4.26: Possible index distributions for the codewords of $\mathcal{G}_i$.

We recall that $|\mathcal{F}_{ik}| = 5$ and, until now, we have only characterized the index distribution of two codewords of $\mathcal{F}_{ik}$, $U_1 \in \mathcal{F}_{ijkl}$ and $U_3 \in \mathcal{F}_{i,k,o,-m}$. Then, we must describe the index distribution of $U_5, U_6, U_7 \in \mathcal{F}_{ik} \backslash \{U_1, U_3\}$. Let us consider $U_5 \in \mathcal{F}_{iku_1u_2}$, $U_6 \in \mathcal{F}_{iku_3u_4}$ and $U_7 \in \mathcal{F}_{iku_5u_6}$. By Lemma 2.5 we must impose $u_1, \ldots, u_6 \in \{-j, -l, m, n, -n, -o\}$ pairwise distinct.

Assume that $W_1, W_2, W_3 \in \mathcal{G}_i$ satisfy the index distribution presented in Table 4.26 on the left. Taking into account the index distribution of all codewords already characterized, in particular $W_3 \in \mathcal{G}_{i,m,-j,-n,-o}$, and Lemma 1.5 we must impose $U_5 \in \mathcal{F}_{i,k,-j,u_1}$, $U_6 \in \mathcal{F}_{i,k,-n,u_3}$ and $U_7 \in \mathcal{F}_{i,k,-o,u_5}$, with $\{u_1, u_3, u_5\} = \{-l, m, n\}$. Consequently, $U_5 \in \mathcal{F}_{i,k,-j,n}$ and $U_6 \in \mathcal{F}_{i,k,-n,-l}$, implying $U_7 \in \mathcal{F}_{i,k,-o,m}$, contradicting Lemma 1.5 with the codeword W_3.

If we suppose $W_1, W_2, W_3 \in \mathcal{G}_i$ satisfying the index distribution in Table 4.26 on the right, we get an analogous conclusion.

Let us now suppose that $x_2 = -n$. In this case the index distribution of the codewords $U_1, \ldots, U_4 \in \mathcal{F}_i$ is such that:

U_1	i	j	k	l
U_2	i	j	o	m
U_3	i	k	o	$-n$
U_4	i	l	$-o$	$-n$

Table 4.27: Index distribution of $U_1, \ldots, U_4 \in \mathcal{F}_i$.

Considering $W_1, W_2, W_3 \in \mathcal{G}_i$, see Table 4.22, and Proposition 4.3, we must impose $\{w_1, \ldots, w_4\} = \{-l, -m, -n, -o\}$ and, having in mind Lemma 1.5, the index distribution of the codewords of $\mathcal{G}_i$ satisfies one of the following hypotheses:

W_1	i	$-k$	m	l	n
W_2	i	$-k$	$-j$	$-m$	$-o$
W_3	i	m	$-j$	$-l$	$-n$

W_1	i	$-k$	m	l	n
W_2	i	$-k$	$-j$	$-m$	$-n$
W_3	i	m	$-j$	$-l$	$-o$

Table 4.28: Possible index distributions for the codewords of $\mathcal{G}_i$.

Under the assumption $|\mathcal{F}_{ij}| = |\mathcal{F}_{ik}| = |\mathcal{F}_{io}| = 5$, taking into account the codewords $U_1, \ldots, U_4 \in \mathcal{F}_i$ and Lemma 2.5, we must consider:

- $U_5, U_6, U_7 \in \mathcal{F}_{ik} \backslash \{U_1, U_3\}$ so that $U_5 \in \mathcal{F}_{iku_1u_2}$, $U_6 \in \mathcal{F}_{iku_3u_4}$ and $U_7 \in \mathcal{F}_{iku_5u_6}$, with $u_1, \ldots, u_6 \in \{-j, -l, m, -m, n, -o\}$ pairwise distinct;

- $U_8, U_9, U_{10} \in \mathcal{F}_{io} \backslash \{U_2, U_3\}$ so that $U_8 \in \mathcal{F}_{iou_7u_8}$, $U_9 \in \mathcal{F}_{iou_9u_{10}}$ and $U_{10} \in \mathcal{F}_{iou_{11}u_{12}}$, with $u_7, \ldots, u_{12} \in \{-j, -k, l, -l, -m, n\}$ pairwise distinct;

- $U_{11}, U_{12}, U_{13} \in \mathcal{F}_{ij} \backslash \{U_1, U_2\}$ so that $U_{11} \in \mathcal{F}_{iju_{13}u_{14}}$, $U_{12} \in \mathcal{F}_{iju_{15}u_{16}}$ and $U_{13} \in \mathcal{F}_{iju_{17}u_{18}}$, with $u_{13}, \ldots, u_{18} \in \{-k, -l, -m, n, -n, -o\}$ pairwise distinct.

Assuming that the codewords of $\mathcal{G}_i$ satisfy the index distribution presented in Table 4.28 on the left, taking into account the conditions referred before and considering all codewords characterized at this moment and Lemma 1.5, the index distribution of the remaining codewords of $\mathcal{F}_{ik} \cup \mathcal{F}_{io}$ must satisfies:

U_5	i	k	m	$-o$
U_6	i	k	$-j$	n
U_7	i	k	$-l$	$-m$

U_8	i	o	$-k$	$-l$
U_9	i	o	$-j$	l
U_{10}	i	o	$-m$	n

Table 4.29: Index distribution of codewords of $\mathcal{F}_{ik} \cup \mathcal{F}_{io}$.

However, in the characterization of the index distribution of $U_{11}, U_{12}, U_{13} \in \mathcal{F}_{ij} \backslash \{U_1, U_2\}$ we verify that $|\mathcal{G}_{i\omega\rho} \cup \mathcal{F}_{i\omega\rho}| \geq 2$ for some $\omega, \rho \in \mathcal{I} \backslash \{i, -i\}$, contradicting Lemma 1.5.

If the codewords of $\mathcal{G}_i$ satisfy the index distribution in Table 4.28 on the right, the characterization of all codewords of $\mathcal{F}_{ik}$ implies $|\mathcal{G}_{i\omega\rho} \cup \mathcal{F}_{i\omega\rho}| \geq 2$ for some $\omega, \rho \in \mathcal{I} \backslash \{i, -i\}$.

- $|\mathcal{G}_{i\omega\rho} \cup \mathcal{F}_{i\omega\rho}| \leq 1$ **for all** $\omega, \rho \in \mathcal{I} \backslash \{i, -i\}$ **and** $|\mathcal{F}_i^{(2)} \cap \mathcal{F}_{\varphi}| \geq 2$ **for some** $\varphi \in \mathcal{I} \backslash \{i, -i\}$

Consider $W_1, W_2, W_3 \in \mathcal{G}_i$ satisfying, respectively, the following partial index distribution:

W_1	i	$-j$	m	l	w_1
W_2	i	$-j$	$-l$	w_2	w_3
W_3	i	m	$-l$	w_4	w_5

Table 4.30: Partial index distribution of the codewords of $\mathcal{G}_i$.

where $\delta = -j$, $\varepsilon = m$ and $\theta = -l$.

Accordingly, we have concluded in the previous subsection that one of the following conditions must be satisfied:

i) $\gamma = -m$; $-l \in \{x_1, x_2\}$; $U_4 \in \mathcal{F}_{i,-k,l,x}$, with $x \in \{x_1, x_2\} \backslash \{-l\}$;

ii) $\gamma = n$; $x_1 = m$ and $x_2 = -l$, or, $x_1 = -l$ and $x_2 = m$; $U_4 \in \mathcal{F}_{i,-k,-n,l}$.

Let us consider $\gamma = -m$ and $-l = x_1$. In these conditions, $U_1, \ldots, U_4 \in \mathcal{F}_i$ satisfy:

U_1	i	j	k	l
U_2	i	j	$-m$	$-l$
U_3	i	k	$-m$	x_2
U_4	i	$-k$	l	x_2

Table 4.31: Partial index distribution of $U_1, \ldots, U_4 \in \mathcal{F}_i$.

By Proposition 4.7 it follows that $|\{-j, -k, m, l, -l, x_2\}| = 6$. Taking into account the index distribution of the codewords of $\mathcal{G}_i$ and $U_1, \ldots, U_4 \in \mathcal{F}_i$, and Lemma 1.5, we conclude that $x_2 \in \{n, -n, o, -o\}$.

Without loss of generality, suppose that $x_2 = n$. Then,

U_1	i	j	k	l
U_2	i	j	$-m$	$-l$
U_3	i	k	$-m$	n
U_4	i	$-k$	l	n

Table 4.32: Index distribution of $U_1, \ldots, U_4 \in \mathcal{F}_i$.

Consider the codewords $W_1, W_2, W_3 \in \mathcal{G}_i$, see Table 4.30. By Proposition 4.3 we must impose $w_1, \ldots, w_5 \in \{-k, n, -n, o, -o\}$. By the analysis of the codewords of $\mathcal{G}_i \cup \{U_1, \ldots, U_4\}$ and taking into account Lemma 1.5, four possible index distributions for the codewords of $\mathcal{G}_i$ are obtained.

W_1	i	$-j$	m	l	$-n$
W_2	i	$-j$	$-l$	o	$-k$
W_3	i	m	$-l$	$-o$	n

W_1	i	$-j$	m	l	$-n$
W_2	i	$-j$	$-l$	o	n
W_3	i	m	$-l$	$-o$	$-k$

W_1	i	$-j$	m	l	o
W_2	i	$-j$	$-l$	n	$-o$
W_3	i	m	$-l$	$-n$	$-k$

W_1	i	$-j$	m	l	o
W_2	i	$-j$	$-l$	$-n$	$-k$
W_3	i	m	$-l$	n	$-o$

Table 4.33: Possible index distributions for the codewords of $\mathcal{G}_i$.

Since, by assumption, $|\mathcal{F}_{ij}| = |\mathcal{F}_{ik}| = |\mathcal{F}_{i,-m}| = 5$, taking into account the codewords $U_1, \ldots, U_4 \in \mathcal{F}_i$ and Lemma 2.5, we must consider:

– $U_5, U_6, U_7 \in \mathcal{F}_{ij} \setminus \{U_1, U_2\}$ such that $U_5 \in \mathcal{F}_{iju_1u_2}$, $U_6 \in \mathcal{F}_{iju_3u_4}$ and $U_7 \in \mathcal{F}_{iju_5u6}$, with $u_1, \ldots, u_6 \in \{-k, m, n, -n, o, -o\}$ pairwise distinct;

– $U_8, U_9, U_{10} \in \mathcal{F}_{ik} \setminus \{U_1, U_3\}$ such that $U_8 \in \mathcal{F}_{iku_7u_8}$, $U_9 \in \mathcal{F}_{iku_9u_{10}}$ and $U_{10} \in \mathcal{F}_{iku_{11}u_{12}}$, with $u_7, \ldots, u_{12} \in \{-j, -l, m, -n, o, -o\}$ pairwise distinct;

– $U_{11}, U_{12}, U_{13} \in \mathcal{F}_{i,-m} \setminus \{U_2, U_3\}$ such that $U_{11} \in \mathcal{F}_{i,-m,u_{13}u_{14}}$, $U_{12} \in \mathcal{F}_{i,-m,u_{15},u_{16}}$ and $U_{13} \in \mathcal{F}_{i,-m,u_{17}u_{18}}$, with $u_{13}, \ldots, u_{18} \in \{-j, -k, l, -n, o, -o\}$ pairwise distinct.

Suppose that $W_1, W_2, W_3 \in \mathcal{G}_i$ are such that: $W_1 \in \mathcal{G}_{i,-j,m,l,-n}$, $W_2 \in \mathcal{G}_{i,-j,-l,o,n}$ and $W_3 \in \mathcal{G}_{i,m,-l,-o,-k}$. Considering the index distributions of the codewords already known and Lemma 1.5 the index distribution of the remaining codewords of $\mathcal{F}_{ij} \cup \mathcal{F}_{ik} \cup \mathcal{F}_{i,-m}$ must satisfy:

U_5	i	j	$-k$	o
U_6	i	j	m	n
U_7	i	j	$-n$	$-o$

U_8	i	k	$-j$	$-o$
U_9	i	k	$-l$	$-n$
U_{10}	i	k	m	o

Table 4.34: Index distribution of codewords of $\mathcal{F}_{ij} \cup \mathcal{F}_{ik}$.

U_{11}	i	$-m$	$-j$	$-k$
U_{12}	i	$-m$	l	$-o$
U_{13}	i	$-m$	$-n$	o

Table 4.35: Index distribution of codewords of $\mathcal{F}_{i,-m}$.

We know, by Lemma 2.9, that $|\mathcal{F}_i^{(2)}| = 4$. Accordingly, by Corollary 4.1, $\mathcal{F}_i^{(2)} = \{U_4, U', U'', U'''\}$ with $U' \in \mathcal{F}_{ij}\backslash(\mathcal{F}_k \cup \mathcal{F}_{-m})$, $U'' \in \mathcal{F}_{ik}\backslash(\mathcal{F}_j \cup \mathcal{F}_{-m})$ and $U''' \in \mathcal{F}_{i,-m}\backslash(\mathcal{F}_j \cup \mathcal{F}_k)$. Then, let $U', U'', U''' \in \mathcal{F}_i^{(2)}$ such that: $U' \in \mathcal{F}_{ijz_1z_2}$, $U'' \in \mathcal{F}_{ikz_3z_4}$ and $U''' \in \mathcal{F}_{i,-m,z_5,z_6}$. By Lemma 2.14, taking into account that $U_4 \in \mathcal{F}_{i,-k,l,n}$ is a codeword in $\mathcal{F}_i^{(2)}$, we must impose $z_1, \ldots, z_6 \in \{-j, -l, m, -n, o, -o\}$ pairwise distinct. By the analysis of the codewords $U_5, U_6, U_7 \in \mathcal{F}_{ij}$ we must impose $U' = U_7$. Considering $U_{11}, U_{12}, U_{13} \in \mathcal{F}_{-m}$ we verify that any one of these codewords does not verify the required conditions.

If $W_1, W_2, W_3 \in \mathcal{G}_i$ satisfy $W_1 \in \mathcal{G}_{i,-j,m,l,o}$, $W_2 \in \mathcal{G}_{i,-j,-l,n,-o}$ and $W_3 \in \mathcal{G}_{i,m,-l,-n,-k}$, such as in the previous example we can characterize the index distribution of all remaining codewords of $\mathcal{F}_{ij} \cup \mathcal{F}_{ik} \cup \mathcal{F}_{i,-m}$ with $|\mathcal{G}_{i\omega\rho} \cup \mathcal{F}_{i\omega\rho}| \leq 1$ for all $\omega, \rho \in \mathcal{I}\backslash\{i, -i\}$. However, when we identify the codewords of $\mathcal{F}_i^{(2)}$ we also get contradictions.

If the index distribution of the codewords of $\mathcal{G}_i$ satisfies any one of the remaining hypotheses presented in Table 4.33, then the characterization of all codewords of $\mathcal{F}_{ij} \cup \mathcal{F}_{ik} \cup \mathcal{F}_{i,-m}$ implies the existence of some $\omega, \rho \in \mathcal{I}\backslash\{i, -i\}$ so that $|\mathcal{G}_{i\omega\rho} \cup \mathcal{F}_{i\omega\rho}| \geq 2$, which is a contradiction.

- $|\mathcal{G}_{i\omega\rho} \cup \mathcal{F}_{i\omega\rho}| \leq 1$ **and** $|\mathcal{F}_i^{(2)} \cap \mathcal{F}_\varphi| \leq 1$ **for all** $\omega, \rho, \varphi \in \mathcal{I}\backslash\{i, -i\}$

As we have said before, for the majority of the possible partial index distributions of the codewords $W_1, W_2, W_3 \in \mathcal{G}_i$ and $U_1, \ldots, U_4 \in \mathcal{F}_i$ we get, when we characterize completely the index distribution of all codewords of $\mathcal{G}_i \cup \mathcal{F}_i$, one of the following conclusions:

– $|\mathcal{G}_{i\omega\rho} \cup \mathcal{F}_{i\omega\rho}| \geq 2$ for some $\omega, \rho \in \mathcal{I}\backslash\{i, -i\}$;

– $|\mathcal{G}_{i\omega\rho} \cup \mathcal{F}_{i\omega\rho}| \leq 1$ for all $\omega, \rho \in \mathcal{I}\backslash\{i, -i\}$ and $|\mathcal{F}_i^{(2)} \cap \mathcal{F}_\varphi| \geq 2$ for some $\varphi \in \mathcal{I}\backslash\{i, -i\}$.

Concluding that, in these cases, the definition of PL(7, 2) code is contradicted. However, there exist two cases in which it is possible to characterize the index distribution of all codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ not being verified any one of the previous statements. Next, we present the referred cases.

Consider $W_1, W_2, W_3 \in \mathcal{G}_i$ satisfying, respectively, the following index distribution:

W_1	i	$-k$	m	l	n
W_2	i	$-k$	$-j$	w_1	w_2
W_3	i	m	$-j$	w_3	w_4

Table 4.36: Partial index distribution of the codewords of $\mathcal{G}_i$.

where $\delta = -k$, $\varepsilon = m$ and $\theta = -j$.

We have concluded in the previous subsection that, in these conditions, one of the following hypotheses must occurs:

i) $\gamma = -m$ and $U_4 \in \mathcal{F}_{ilx_1x_2}$;

ii) $\gamma = o$; $m \in \{x_1, x_2\}$; $U_4 \in \mathcal{F}_{i,l,-o,x}$ with $x \in \{x_1, x_2\}\backslash\{m\}$.

Let us consider $\gamma = -m$ and $U_4 \in \mathcal{F}_{ilx_1x_2}$. Accordingly, $U_1, \ldots, U_4 \in \mathcal{F}_i$ satisfy:

U_1	i	j	k	l
U_2	i	j	$-m$	x_1
U_3	i	k	$-m$	x_2
U_4	i	l	x_1	x_2

Table 4.37: Partial index distribution of $U_1, \ldots, U_4 \in \mathcal{F}_i$.

By Proposition 4.7 we have $|\{-j, -k, m, l, x_1, x_2\}| = 6$. Then, taking into account the codewords $W_1, W_2, W_3 \in \mathcal{G}_i$ and $U_1, \ldots, U_4 \in \mathcal{F}_i$, and Lemma 1.5, it follows that

$x_1, x_2 \in \{-n, o, -o\}$. Thus, without loss of generality, we distinguish the following two cases:

◇ $x_1 = o$ and $x_2 = -n$;

◇ $x_1 = -n$ and $x_2 = o$.

Next, we analyze each one of these hypotheses.

Let us suppose first $x_1 = o$ and $x_2 = -n$. Then, the index distribution of the codewords $U_1, \ldots, U_4 \in \mathcal{F}_i$ is such that:

U_1	i	j	k	l
U_2	i	j	$-m$	o
U_3	i	k	$-m$	$-n$
U_4	i	l	o	$-n$

Table 4.38: Index distribution of $U_1, \ldots, U_4 \in \mathcal{F}_i$.

Considering the codewords $W_1, W_2, W_3 \in \mathcal{G}_i$, from Proposition 4.3 it follows that $w_1, \ldots, w_4 \in \{-l, -n, o, -o\}$. Taking into account the index distributions of all codewords known at this moment and Lemma 1.5, we conclude that there exist only two possible index distributions for the codewords of $\mathcal{G}_i$:

W_1	i	$-k$	m	l	n
W_2	i	$-k$	$-j$	o	$-l$
W_3	i	m	$-j$	$-o$	$-n$

W_1	i	$-k$	m	l	n
W_2	i	$-k$	$-j$	$-o$	$-n$
W_3	i	m	$-j$	o	$-l$

Table 4.39: Possible index distributions for the codewords of $\mathcal{G}_i$.

Since $|\mathcal{F}_{ij}| = |\mathcal{F}_{ik}| = |\mathcal{F}_{i,-m}| = 5$, considering the codewords $U_1, \ldots, U_4 \in \mathcal{F}_i$ and Lemma 2.5, the remaining codewords of $\mathcal{F}_{ij} \cup \mathcal{F}_{ik} \cup \mathcal{F}_{i,-m}$ must satisfy:

– $U_5, U_6, U_7 \in \mathcal{F}_{ij} \backslash \{U_1, U_2\}$ such that $U_5 \in \mathcal{F}_{iju_1u_2}$, $U_6 \in \mathcal{F}_{iju_3u_4}$ and $U_7 \in \mathcal{F}_{iju_5u_6}$, with $u_1, \ldots, u_6 \in \{-k, -l, m, n, -n, -o\}$ pairwise distinct;

– $U_8, U_9, U_{10} \in \mathcal{F}_{ik}\backslash\{U_1, U_3\}$ such that $U_8 \in \mathcal{F}_{iku_7u_8}$, $U_9 \in \mathcal{F}_{iku_9u_{10}}$ and $U_{10} \in \mathcal{F}_{iku_{11}u_{12}}$, with $u_7, \ldots, u_{12} \in \{-j, -l, m, n, o, -o\}$ pairwise distinct;

– $U_{11}, U_{12}, U_{13} \in \mathcal{F}_{i,-m}\backslash\{U_2, U_3\}$ such that $U_{11} \in \mathcal{F}_{i,-m,u_{13}u_{14}}$, $U_{12} \in \mathcal{F}_{i,-m,u_{15}u_{16}}$ and $U_{13} \in \mathcal{F}_{i,-m,u_{17}u_{18}}$, with $u_{13}, \ldots, u_{18} \in \{-j, -k, l, -l, n, -o\}$ pairwise distinct.

If $W_1, W_2, W_3 \in \mathcal{G}_i$ satisfy the index distribution presented in Table 4.39 on the right, the characterization of all codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ implies the existence of some $\omega, \rho \in \mathcal{I}\backslash\{i, -i\}$ so that $|\mathcal{G}_{i\omega\rho} \cup \mathcal{F}_{i\omega\rho}| \geq 2$.

Thus, let us assume that the codewords $W_1, W_2, W_3 \in \mathcal{G}_i$ satisfy the index distribution presented in Table 4.39 on the left. In these conditions, there exists a unique possible index distribution for the remaining codewords of $\mathcal{F}_{ij} \cup \mathcal{F}_{ik} \cup \mathcal{F}_{i,-m}$ satisfying $|\mathcal{G}_{i\omega\rho} \cup \mathcal{F}_{i\omega\rho}| \leq 1$ for all $\omega, \rho \in \mathcal{I}\backslash\{i, -i\}$ and $|\mathcal{F}_i^{(2)} \cap \mathcal{F}_\varphi| = 1$ for all $\varphi \in \mathcal{I}\backslash\{i, -i\}$:

U_5	i	j	m	$-l$
U_6	i	j	$-k$	$-n$
U_7	i	j	n	$-o$

U_8	i	k	m	o
U_9	i	k	$-j$	n
U_{10}	i	k	$-l$	$-o$

U_{11}	i	$-m$	$-j$	l
U_{12}	i	$-m$	$-k$	$-o$
U_{13}	i	$-m$	$-l$	n

Table 4.40: Index distribution of codewords of $\mathcal{F}_{ij} \cup \mathcal{F}_{ik} \cup \mathcal{F}_{i,-m}$.

Taking into account Corollary 4.1 and Lemma 2.14, we conclude that, in this case, $\mathcal{F}_i^{(2)} = \{U_4, U_5, U_9, U_{12}\}$.

This is one of the cases which, apparently, does not contradict necessary conditions for the existence of PL(7, 2) codes. As such, its analysis require more some work, as we will see in next section. There exists only one more index distribution for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ in these conditions, such index distribution is derived from the analysis of the hypothesis $x_1 = -n$ and $x_2 = o$.

Consider now $x_1 = -n$ and $x_2 = o$. Accordingly, the codewords $U_1, \ldots, U_4 \in \mathcal{F}_i$ satisfy:

U_1	i	j	k	l
U_2	i	j	$-m$	$-n$
U_3	i	k	$-m$	o
U_4	i	l	$-n$	o

Table 4.41: Index distribution of $U_1, \ldots, U_4 \in \mathcal{F}_i$.

Such as in the previous case, $W_1, W_2, W_3 \in \mathcal{G}_i$ must satisfy one of the index distributions presented in Table 4.39.

Under the assumption $|\mathcal{F}_{ij}| = |\mathcal{F}_{ik}| = |\mathcal{F}_{i,-m}| = 5$, from the analysis of the codewords $U_1, \ldots, U_4 \in \mathcal{F}_i$, and taking into account Lemma 2.5, we conclude that the remaining codewords of $\mathcal{F}_{ij} \cup \mathcal{F}_{ik} \cup \mathcal{F}_{i,-m}$ must satisfy the following conditions:

– $U_5, U_6, U_7 \in \mathcal{F}_{ij} \backslash \{U_1, U_2\}$ such that $U_5 \in \mathcal{F}_{iju_1u_2}$, $U_6 \in \mathcal{F}_{iju_3u_4}$ and $U_7 \in \mathcal{F}_{iju_5u_6}$, with $u_1, \ldots, u_6 \in \{-k, -l, m, n, o, -o\}$ pairwise distinct;

– $U_8, U_9, U_{10} \in \mathcal{F}_{ik} \backslash \{U_1, U_3\}$ such that $U_8 \in \mathcal{F}_{iku_7u_8}$, $U_9 \in \mathcal{F}_{iku_9u_{10}}$ and $U_{10} \in \mathcal{F}_{iku_{11}u_{12}}$, with $u_7, \ldots, u_{12} \in \{-j, -l, m, n, -n, -o\}$ pairwise distinct;

– $U_{11}, U_{12}, U_{13} \in \mathcal{F}_{i,-m} \backslash \{U_2, U_3\}$ such that $U_{11} \in \mathcal{F}_{i,-m,u_{13},u_{14}}$, $U_{12} \in \mathcal{F}_{i,-m,u_{15},u_{16}}$ and $U_{13} \in \mathcal{F}_{i,-m,u_{17},u_{18}}$, with $u_{13}, \ldots, u_{18} \in \{-j, -k, l, -l, n, -o\}$ pairwise distinct.

If the codewords of $\mathcal{G}_i$ satisfy the index distribution presented in Table 4.39 on the left, the characterization of the index distribution of all codewords of $\mathcal{F}_{ij} \cup \mathcal{F}_{ik} \cup \mathcal{F}_{i,-m}$ implies the existence of some $\omega, \rho \in \mathcal{I} \backslash \{i, -i\}$ so that $|\mathcal{G}_{i\omega\rho} \cup \mathcal{F}_{i\omega\rho}| \geq 2$, contradicting Lemma 1.5.

Thus, let us consider $W_1, W_2, W_3 \in \mathcal{G}_i$ satisfying the index distribution presented in Table 4.39 on the right. In this case, we must impose the following index distribution for the remaining codewords of $\mathcal{F}_{ij} \cup \mathcal{F}_{ik} \cup \mathcal{F}_{i,-m}$:

U_5	i	j	m	$-o$
U_6	i	j	$-k$	o
U_7	i	j	$-l$	n

U_8	i	k	m	$-n$
U_9	i	k	$-j$	n
U_{10}	i	k	$-l$	$-o$

U_{11}	i	$-m$	$-j$	l
U_{12}	i	$-m$	$-k$	$-l$
U_{13}	i	$-m$	$-o$	n

Table 4.42: Index distribution of codewords of $\mathcal{F}_{ij} \cup \mathcal{F}_{ik} \cup \mathcal{F}_{i,-m}$.

Accordingly, $|\mathcal{G}_{i\omega\rho} \cup \mathcal{F}_{i\omega\rho}| \leq 1$ for all $\omega, \rho \in \mathcal{I}\backslash\{i, -i\}$. Furthermore, $\mathcal{F}_i^{(2)} = \{U_4, U_5, U_9, U_{12}\}$.

4.3 Analysis of the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$

We have presented in the previous section the unique two index distributions for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ which verify the following conditions: $|\mathcal{G}_{i\omega\rho} \cup \mathcal{F}_{i\omega\rho}| \leq 1$ for any $\omega, \rho \in \mathcal{I}\backslash\{i, -i\}$; $|\mathcal{F}_i^{(2)} \cap \mathcal{F}_\varphi| \leq 1$ for any $\varphi \in \mathcal{I}\backslash\{i, -i\}$, more precisely, $|\mathcal{F}_i^{(2)} \cap \mathcal{F}_\varphi| = 1$ for each $\varphi \in \mathcal{I}\backslash\{i, -i\}$. Apparently, these index distributions do not contradict necessary conditions for the existence of PL$(7, 2)$ codes.

In this section, considering one of the cases as an illustrative example, we show that both hypotheses for the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ contradict the definition of PL$(7, 2)$ code. We do it considering other element $\omega \in \mathcal{I}\backslash\{i\}$ and analyzing the complete index distribution of all codewords of $\mathcal{G}_\omega \cup \mathcal{F}_\omega$.

Let us consider $W_1, W_2, W_3 \in \mathcal{G}_i$ satisfying

W_1	i	$-k$	m	l	n
W_2	i	$-k$	$-j$	o	$-l$
W_3	i	m	$-j$	$-o$	$-n$

Table 4.43: Index distribution of the codewords of $\mathcal{G}_i$.

and $U_1, \ldots, U_{13} \in \mathcal{F}_i$ such that:

U_1	i	j	k	l
U_2	i	j	$-m$	o
U_3	i	k	$-m$	$-n$
U_4	i	l	o	$-n$
U_5	i	j	m	$-l$
U_6	i	j	$-k$	$-n$
U_7	i	j	n	$-o$

U_8	i	k	m	o
U_9	i	k	$-j$	n
U_{10}	i	k	$-l$	$-o$
U_{11}	i	$-m$	$-j$	l
U_{12}	i	$-m$	$-k$	$-o$
U_{13}	i	$-m$	$-l$	n

Table 4.44: Index distribution of the codewords of $\mathcal{F}_i$.

Our aim is to analyze the index distribution of all codewords of a set $\mathcal{G}_\omega \cup \mathcal{F}_\omega$ with $\omega \in \mathcal{I}\backslash\{i\}$. We focus our attention on these sets since their codewords have more nonzero coordinates, helping our study.

The choice of the element $\omega \in \mathcal{I}\backslash\{i\}$ it will be done giving preference to the elements $\omega \in \mathcal{I}\backslash\{i\}$ for which the known codewords of $\mathcal{G}_{i\omega} \cup \mathcal{F}_{i\omega}$ generate a partition of $\mathcal{I}\backslash\{i, \omega, -\omega\}$ with less elements, since in these conditions we reduce the number of possible index distributions for the remaining codewords of $(\mathcal{G}_\omega \cup \mathcal{F}_\omega)\backslash(\mathcal{G}_i \cup \mathcal{F}_i)$, as we will see. Accordingly, we will concentrate our attention on an element $\omega \in \mathcal{I}\backslash\{i\}$ so that $|\mathcal{G}_{i\omega}| = |\mathcal{F}_{i\omega}| = 2$. The elements $-j, -k, m \in \mathcal{I}$ are in the required conditions.

We will analyze the referred index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ considering $m \in \mathcal{I}$, that is, analyzing the index distribution of all codewords of $\mathcal{G}_m \cup \mathcal{F}_m$.

The codewords $W_1, W_3 \in \mathcal{G}_{im}$ and $U_5, U_8 \in \mathcal{F}_{im}$ induce the following partition $\mathcal{P}$ of

$\mathcal{I}\backslash\{i, m, -m\}$:

$$\mathcal{P}_1 = \{-k, l, n\};\;\; \mathcal{P}_2 = \{-j, -n, -o\};\;\; \mathcal{P}_3 = \{j, -l\};\;\; \mathcal{P}_4 = \{k, o\};\;\; \mathcal{P}_5 = \{-i\}. \quad (4.2)$$

This partition will be useful in the characterization of the index distribution of the codewords of $(\mathcal{G}_m \cup \mathcal{F}_m)\backslash(\mathcal{G}_i \cup \mathcal{F}_i)$.

By Corollary 3.1 we know that $3 \leq |\mathcal{G}_m| \leq 7$. Since $|\mathcal{G}_{im}| = 2$, then $|\mathcal{G}_m\backslash\mathcal{G}_i| \geq 1$. Let $W \in \mathcal{G}_m\backslash\mathcal{G}_i$ so that $W \in \mathcal{G}_{mw_1w_2w_3w_4}$, with $w_1, w_2, w_3, w_4 \in \mathcal{I}\backslash\{i, m, -m\}$. Considering the partition $\mathcal{P}$, we conclude that $w_1 \in \mathcal{P}_p$, $w_2 \in \mathcal{P}_q$, $w_3 \in \mathcal{P}_r$ and $w_4 \in \mathcal{P}_s$, with $p, q, r, s \in \{1, \ldots, 5\}$ pairwise distinct, otherwise, $|\mathcal{G}_{m\omega\rho} \cup \mathcal{F}_{m\omega\rho}| \geq 2$ for some $\omega, \rho \in \mathcal{I}\backslash\{m, -m\}$, which contradicts Lemma 1.5.

Combining the elements of the partition $\mathcal{P}$ and taking into account the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$, presented in Tables 4.43 and 4.44, as well as Lemma 1.5, if $W \in \mathcal{G}_m\backslash\mathcal{G}_i$ then W must satisfy one of the following conditions:

1				-n	o
2				-n	-l
3			-k	-o	j
4				-o	-l
5				o	j
6				-j	k
7	m	-i		-j	o
8				-n	k
9			l	-o	k
10				-n	j
11				-o	j
12				o	j

13				-j	o
14				-o	k
15				-j	-l
16			n	-o	-l
17				k	j
18				k	-l
19	m	-i		o	j
20				o	-l
21			-j	k	-l
22				k	j
23			-n	k	-l
24				o	j
25				o	-l
26			-o	k	j

Table 4.45: Possible index distributions for $W \in \mathcal{G}_m\backslash\mathcal{G}_i$.

By the analysis of the above table we conclude that if $W \in \mathcal{G}_m\backslash\mathcal{G}_i$, then $W \in \mathcal{G}_{m,-i}$. Since, by Lemma 2.2, $|\mathcal{G}_{m,-i}| \leq 3$, then $1 \leq |\mathcal{G}_{m,-i}| \leq 3$. Accordingly, $1 \leq |\mathcal{G}_m\backslash\mathcal{G}_i| \leq 3$ and $3 \leq |\mathcal{G}_m| \leq 5$.

The cardinality of $\mathcal{F}_m$ it depends on the cardinality of $\mathcal{G}_m$. In fact, by Lemmas 2.9, 2.10 and 2.11, respectively, we know that

- if $|\mathcal{G}_m| = 3$, then $|\mathcal{F}_m| = 13$;

- if $|\mathcal{G}_m| = 4$, then $10 \leq |\mathcal{F}_m| \leq 11$;

– if $|\mathcal{G}_m| = 5$, then $7 \leq |\mathcal{F}_m| \leq 10$.

Furthermore, by the same lemmas,

– if $|\mathcal{G}_m| = 3$, then $|\mathcal{F}_m^{(2)}| = 4$;

– if $|\mathcal{G}_m| = 4$ and $|\mathcal{F}_m| = 10$, then $|\mathcal{F}_m^{(2)}| = 4$;

– if $|\mathcal{G}_m| = 5$ and $|\mathcal{F}_m| = 7$, then $|\mathcal{F}_m^{(2)}| = 4$.

That is, in any case the minimal possible value for $|\mathcal{F}_m|$ implies $|\mathcal{F}_m^{(2)}| = 4$.

Next proposition characterizes $\mathcal{F}_m^{(2)}$ when $|\mathcal{F}_m^{(2)}| = 4$.

Proposition 4.8 *If $|\mathcal{F}_m^{(2)}| = 4$, then $\mathcal{F}_m^{(2)} = \{U_8, M, M', M''\}$, where $U_8 \in \mathcal{F}_{ikmo}$ and the index distribution of M, M', M'' satisfies one of the following conditions:*

M	m, j, -k, -o
M'	m, -l, n, -j
M''	m, -i, l, -n

M	m, j, l, -n		
M'	m, -l, -k, -o	m, -l, n, -j	m, -l, n, -o
M''	m, -i, n, -j	m, -i, -k, -o	m, -i, -k, -j

M	m, j, l, -o	
M'	m, -l, -k, -n	m, -l, n, -j
M''	m, -i, n, -j	m, -i, -k, -n

Proof. If $|\mathcal{F}_m^{(2)}| = 4$, by Lemma 2.14 we get $|\mathcal{F}_m^{(2)} \cap \mathcal{F}_\omega| = 1$ for any $\omega \in \mathcal{I} \backslash \{m, -m\}$. Accordingly, $|\mathcal{F}_m^{(2)} \cap \mathcal{F}_i| = 1$. Thus, there exists $U \in \mathcal{F}_{im}$, with $\mathcal{F}_{im} = \{U_5, U_8\}$, such that $U \in \mathcal{F}_m^{(2)}$. We have concluded in the previous subsection that $|\mathcal{F}_i^{(2)}| = 4$ with $\mathcal{F}_i^{(2)} = \{U_4, U_5, U_9, U_{12}\}$. As the codewords of $\mathcal{F}$ are of type $[\pm 2, \pm 1^3]$ it follows that $U_8 \in \mathcal{F}_m^{(2)}$, with $U_8 \in \mathcal{F}_{ikmo}$.

Let $M, M', M'' \in \mathcal{F}_m^{(2)} \backslash \{U_8\}$. Taking into account $W_1 \in \mathcal{G}_{i,-k,m,l,n}$ and Lemmas 2.14 and 1.5, we must impose $M \in \mathcal{F}_{m,-k,u_1,u_2}$, $M' \in \mathcal{F}_{mlu_3u_4}$ and $M'' \in \mathcal{F}_{mnu_5u_6}$, with $u_1, \ldots, u_6 \in \{-i, j, -j-l-n, -o\}$. Considering $W_3 \in \mathcal{G}_{i,m,-j,-n,-o}$, we must impose $u_1, u_3, u_5 \in \{-j, -n, -o\}$ and, consequently, $u_2, u_4, u_6 \in \{j, -l, -i\}$. Combining all possibilities for $u_1, \ldots, u_6$ and considering the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ and Lemma 1.5, we conclude that the index distribution of the codewords $M, M', M'' \in \mathcal{F}_m^{(2)}$ satisfies one of the conditions presented in the following schemes.

M	m, j, -k, -o
M'	m, -l, n, -j
M"	m, -i, l, -n

M	m, j, l, -n		
M'	m, -l, -k, -o	m, -l, n, -j	m, -l, n, -o
M"	m, -i, n, -j	m, -i, -k, -o	m, -i, -k, -j

M	m, j, l, -o	
M'	m, -l, -k, -n	m, -l, n, -j
M"	m, -i, n, -j	m, -i, -k, -n

Table 4.46: Possible index distribution for the codewords of $\mathcal{F}_m^{(2)} \backslash \{U_8\}$.

□

Corollary 4.2 *If $|\mathcal{G}_m| = 5$, then $8 \leq |\mathcal{F}_m| \leq 10$.*

Proof. Let us suppose $|\mathcal{G}_m| = 5$. Accordingly, $|\mathcal{G}_m \backslash \mathcal{G}_i| = 3$ and, taking into account the possible index distributions for $W \in \mathcal{G}_m \backslash \mathcal{G}_i$, see Table 4.45, $|\mathcal{G}_{m,-i}| = 3$. From Lemma 2.15 it follows that $8 \leq |\mathcal{F}_m| \leq 10$. □

Next, we will analyze the index distribution of the codewords of $\mathcal{G}_m \cup \mathcal{F}_m$, when $|\mathcal{G}_m|$ assumes each one of the possible values. We will study separately the cases: $|\mathcal{G}_m| = 3$ and $4 \leq |\mathcal{G}_m| \leq 5$.

4.3.1 Analysis of $\mathcal{G}_m \cup \mathcal{F}_m$ when $|\mathcal{G}_m| = 3$

Let us suppose that $|\mathcal{G}_m| = 3$. Accordingly, $|\mathcal{G}_m \backslash \mathcal{G}_i| = 1$. By Proposition 4.1, there are $\alpha, \beta, \gamma \in \mathcal{I} \backslash \{m, -m\}$ so that $|\mathcal{F}_{m\alpha}| = |\mathcal{F}_{m\beta}| = |\mathcal{F}_{m\gamma}| = 5$. Consequently, from Proposition 4.3 it follows that $|\mathcal{G}_{m\alpha}| = |\mathcal{G}_{m\beta}| = |\mathcal{G}_{m\gamma}| = 0$.

Let $W \in \mathcal{G}_m \backslash \mathcal{G}_i$. We have seen before that if $W \in \mathcal{G}_m \backslash \mathcal{G}_i$, then its index distribution satisfies one of the conditions presented in Table 4.45. Let us consider, for instance, $W \in \mathcal{G}_{m,-i,-k,o,j}$. Then, the three codewords $W_1, W_3, W \in \mathcal{G}_m$ are such that:

W_1	m	i	$-k$	l	n
W_3	m	i	$-j$	$-n$	$-o$
W	m	$-i$	$-k$	o	j

Table 4.47: Index distribution of the codewords of $\mathcal{G}_m$.

In these conditions, k and $-l$ are the unique elements in $\mathcal{I}\backslash\{m,-m\}$ satisfying $|\mathcal{G}_{mk}| = |\mathcal{G}_{m,-l}| = 0$, which contradicts Proposition 4.3. Applying a similar reasoning to each one of the other possible index distributions for $W \in \mathcal{G}_m \backslash \mathcal{G}_i$, we get the same conclusion when W satisfies the index distributions identified in Table 4.45 by the numbers: 5; 12; 17 to 26.

Since we are assuming $|\mathcal{G}_m| = 3$, by Lemma 2.9 we have $|\mathcal{F}_m^{(2)}| = 4$. Accordingly, the index distribution of the codewords of $\mathcal{F}_m^{(2)}$ must satisfy one of the conditions presented in Proposition 4.8.

Let us suppose that $W \in \mathcal{G}_m \backslash \mathcal{G}_i$ is such that $W \in \mathcal{G}_{m,-i,-k,-n,-l}$. Observing all possible hypotheses for the index distribution of the codewords of $\mathcal{F}_m^{(2)}$, we conclude that in any case there exists $V \in \mathcal{F}_m^{(2)}$ such that $V, W \in \mathcal{G}_{m\omega\rho} \cup \mathcal{F}_{m\omega\rho}$ for some $\omega, \rho \in \mathcal{I}\backslash\{m,-m\}$, contradicting Lemma 1.5. By a similar reasoning we verify that $W \in \mathcal{G}_m \backslash \mathcal{G}_i$ can not satisfy the index distributions identified in Table 4.45 by the numbers: 2 to 4; 10; 11; 13; 15; 16.

Consider now that $W \in \mathcal{G}_{m,-i,-k,-n,o}$. Thus, $W_1, W_3, W \in \mathcal{G}_m$ are such that:

W_1	m	i	$-k$	l	n
W_3	m	i	$-j$	$-n$	$-o$
W	m	$-i$	$-k$	$-n$	o

Table 4.48: Index distribution of the codewords of $\mathcal{G}_m$.

In these conditions, j, k and $-l$ are the unique elements in $\mathcal{I}\backslash\{m,-m\}$ satisfying $|\mathcal{G}_{mj}| = |\mathcal{G}_{mk}| = |\mathcal{G}_{m,-l}| = 0$. Taking into account Propositions 4.1 and 4.3 we must impose $|\mathcal{F}_{mj}| = |\mathcal{F}_{mk}| = |\mathcal{F}_{m,-l}| = 5$.

Since $|\mathcal{F}_{m,-l}| = 5$, by Lemma 2.5 we have $|\mathcal{F}_{m,-l,\omega}| = 1$ for any $\omega \in \mathcal{I}\backslash\{m,-m,l,-l\}$. In particular, $|\mathcal{F}_{m,-l,-i}| = |\mathcal{F}_{m,-l,-k}| = |\mathcal{F}_{m,-l,-n}| = |\mathcal{F}_{m,-l,o}| = 1$. Considering $W \in \mathcal{G}_{m,-i,-k,-n,o}$ and Lemma 1.5, $\mathcal{F}_{m,-l,-i}$, $\mathcal{F}_{m,-l,-k}$, $\mathcal{F}_{m,-l,-n}$ and $\mathcal{F}_{m,-l,o}$ must be pairwise disjoint. Thus, noting that $U_5 \in \mathcal{F}_{i,j,m,-l}$, the partial index distribution of the codewords of $\mathcal{F}_{m,-l}$ must satisfy the conditions presented in the following table, where $u_1, \ldots, u_4 \in \{-j, k, n, -o\}$.

U_5	m	i	j	$-l$
U_{14}	m	$-l$	$-i$	u_1
U_{15}	m	$-l$	$-k$	u_2
U_{16}	m	$-l$	$-n$	u_3
U_{17}	m	$-l$	o	u_4

Table 4.49: Partial index distribution of the codewords of $\mathcal{F}_{m,-l}$.

Taking into account the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i \cup \{W\}$ and Lemma 1.5 we must impose:

U_5	m	i	j	$-l$
U_{14}	m	$-l$	$-i$	$-j$
U_{15}	m	$-l$	$-k$	$-o$
U_{16}	m	$-l$	$-n$	k
U_{17}	m	$-l$	o	n

Table 4.50: Index distribution of the codewords of $\mathcal{F}_{m,-l}$.

As $|\mathcal{F}_m^{(2)}| = 4$ and the codewords of $\mathcal{F}_m^{(2)}$ must satisfy one of the index distributions presented in Proposition 4.8, we verify that for each one of those index distributions there exist $V \in \mathcal{F}_m^{(2)}$ and $U \in \mathcal{F}_{m,-l}$ so that $V, U \in \mathcal{F}_{m\omega\rho}$, with V and U distinct, for some $\omega, \rho \in \mathcal{I}\backslash\{m, -m\}$, facing up to a contradiction.

If $W \in \mathcal{G}_m \backslash \mathcal{G}_i$ satisfies one of the remaining possible index distributions presented in Table 4.45, that is, one of the index distributions identified in Table 4.45, respectively, by the numbers 6, 7, 8, 9 or 14, such as in the previous example, having in view Propositions 4.1, 4.3 and 4.8, the characterization of the index distribution of all codewords of $\mathcal{F}_m^{(2)} \cup \mathcal{F}_{m\alpha} \cup \mathcal{F}_{m\beta} \cup \mathcal{F}_{m\gamma}$, with $\alpha, \beta, \gamma \in \mathcal{I}\backslash\{m, -m\}$ so that $|\mathcal{F}_{m\alpha}| = |\mathcal{F}_{m\beta}| = |\mathcal{F}_{m\gamma}| = 5$, contradicts Lemma 1.5.

Therefore, $|\mathcal{G}_m| \neq 3$. Next, we will verify what happens when we consider $|\mathcal{G}_m| = 4$ or $|\mathcal{G}_m| = 5$.

4.3.2 Analysis of $\mathcal{G}_m \cup \mathcal{F}_m$ when $4 \leq |\mathcal{G}_m| \leq 5$

Let us now assume that $|\mathcal{G}_m| = 4$ or $|\mathcal{G}_m| = 5$. Since $|\mathcal{G}_{mi}| = 2$, then $2 \leq |\mathcal{G}_m \backslash \mathcal{G}_i| \leq 3$. So, we must identify, at least, two codewords in $\mathcal{G}_m \backslash \mathcal{G}_i$. For that, we will take into account the possible index distributions for $W \in \mathcal{G}_m \backslash \mathcal{G}_i$ presented in Table 4.45.

Considering Lemma 2.10 and Corollary 4.2, one of the two following conditions must occurs:

– $|\mathcal{G}_m| = 4$ and $10 \leq |\mathcal{F}_m| \leq 11$;

– $|\mathcal{G}_m| = 5$ and $8 \leq |\mathcal{F}_m| \leq 10$.

We note that, $|\mathcal{F}_{mi}| = 2$, then, accordingly with we have just said, $|\mathcal{F}_m \backslash \mathcal{F}_i| \geq 6$.

The possible index distributions for the codewords $U \in \mathcal{F}_m \backslash \mathcal{F}_i$ can be identified following the same reasoning applied in the characterization of the possible index distributions of the codewords of $\mathcal{G}_m \backslash \mathcal{G}_i$. Considering the partition $\mathcal{P}$ of $\mathcal{I} \backslash \{i, m, -m\}$,

$$\mathcal{P}_1 = \{-k, l, n\}; \ \ \mathcal{P}_2 = \{-j, -n, -o\}; \ \ \mathcal{P}_3 = \{j, -l\}; \ \ \mathcal{P}_4 = \{k, o\}; \ \ \mathcal{P}_5 = \{-i\},$$

if $U \in \mathcal{F}_m \backslash \mathcal{F}_i$, with $U \in \mathcal{F}_{mu_1u_2u_3}$, then $u_1 \in \mathcal{P}_p$, $u_2 \in \mathcal{P}_q$ and $u_3 \in \mathcal{P}_r$, with $p, q, r \in \{1, \ldots, 5\}$ pairwise distinct. Thus, taking into account the partition $\mathcal{P}$, the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ and Lemma 1.5, if $U \in \mathcal{F}_m \backslash \mathcal{F}_i$, then the index distribution of U satisfies one of the conditions presented in Table 4.51.

Our goal is to characterize all possible index distributions for the codewords of $(\mathcal{G}_m \cup \mathcal{F}_m) \backslash (\mathcal{G}_i \cup \mathcal{F}_i)$. Taking into account Table 4.45, we will identify all possible index distributions for the codewords of $\mathcal{G}_m \backslash \mathcal{G}_i$. Furthermore, for each one of them we will characterize, considering Table 4.51, the respective possible index distributions for the codewords of $\mathcal{F}_m \backslash \mathcal{F}_i$. To show how we have proceeded we will present some illustrative examples.

m	-k	-n	o
			-l
		-o	j
			-l
		o	j
	l	-j	k
			o
		k	-n
			-o
		j	-n
			-o
			o
	n	-l	-j
			-o
			k
		k	-o
			j
		o	j
			-j
			-l

m	-j	k	-l
	-n	k	j
			-l
		o	j
			-l
	-o	k	j
	-i	-k	-j
			-n
			-o
			o
			j
			-l
		l	-j
			-n
			-o
			k
			o
			j

m	-i	n	-j
			-o
			k
			o
			j
			-l
		-j	k
			o
			-l
		-n	k
			o
			j
			-l
		-o	k
			j
			-l
		k	j
			-l
		o	j
			-l

Table 4.51: Possible index distributions for $U \in \mathcal{F}_m \backslash \mathcal{F}_i$.

We have seen before that $|\mathcal{G}_m \backslash \mathcal{G}_i| \geq 2$, so let us consider $W, W' \in \mathcal{G}_m \backslash \mathcal{G}_i$, with $W \neq W'$. Suppose that $W \in \mathcal{G}_{m,-i,-k,-n,o}$. Taking into account the possible index distributions presented in Table 4.45 and Lemma 1.5, $W' \in \mathcal{G}_m \backslash \mathcal{G}_i$ must satisfy one of the following index distributions:

m	$-i$	l	$-j$	k
m	$-i$	l	$-o$	k
m	$-i$	l	$-o$	j
m	$-i$	n	$-o$	k
m	$-i$	n	$-j$	$-l$

m	$-i$	n	$-o$	$-l$
m	$-i$	n	k	j
m	$-i$	n	k	$-l$
m	$-i$	$-j$	k	$-l$
m	$-i$	$-o$	k	j

Table 4.52: Possible index distributions for $W' \in \mathcal{F}_m \backslash \mathcal{F}_i$.

By the analysis of the tables above we verify that there are ten different possible index distributions for the codeword $W' \in \mathcal{G}_m \backslash \mathcal{G}_i$, that is, assuming $|\mathcal{G}_m| = 4$ and $W \in \mathcal{G}_{m,-i,-k,-n,o}$, there exist ten distinct index distributions for the codewords of $\mathcal{G}_m$. On the other hand, by the analysis of the same tables, and taking into account Lemma 1.5, if we suppose $|\mathcal{G}_m| = 5$ and $W \in \mathcal{G}_{m,-i,-k,-n,o}$, then $W', W'' \in \mathcal{G}_m \backslash \mathcal{G}_i$ must satisfy one of the index distributions presented in the next tables. We note that, in the following tables, W' on the left is matching to W'' presented on the right.

W'	m	$-i$	l	$-j$	k
W'	m	$-i$	l	$-o$	k
W'	m	$-i$	l	$-o$	j
W'	m	$-i$	l	$-o$	j
W'	m	$-i$	l	$-o$	j
W'	m	$-i$	n	$-j$	$-l$

W''	m	$-i$	n	$-o$	$-l$
W''	m	$-i$	n	$-j$	$-l$
W''	m	$-i$	n	$-j$	$-l$
W''	m	$-i$	n	k	$-l$
W''	m	$-i$	$-j$	k	$-l$
W''	m	$-i$	$-o$	k	j

Table 4.53: Index distributions for $W', W'' \in \mathcal{F}_m \backslash \mathcal{F}_i$.

In what follows, under the assumption $W \in \mathcal{G}_{m,-i,-k,-n,o}$, we will analyze cases in which:

- $|\mathcal{G}_m| = 4$ and $|\mathcal{F}_m| = 10$;
- $|\mathcal{G}_m| = 4$ and $|\mathcal{F}_m| = 11$;
- $|\mathcal{G}_m| = 5$ and $8 \leq |\mathcal{F}_m| \leq 10$.

- $|\mathcal{G}_m| = 4$ **and** $|\mathcal{F}_m| = 10$

Let us suppose $|\mathcal{G}_m| = 4$ and $|\mathcal{F}_m| = 10$. From Lemma 2.10 it follows that $|\mathcal{F}_m^{(2)}| = 4$. Taking into account the codeword $W \in \mathcal{G}_{m,-i,-k,-n,o}$, Proposition 4.8 and Lemma 1.5, $U_8, M, M', M'' \in \mathcal{F}_m^{(2)}$ must satisfy:

U_8	m	i	k	o
M	m	j	l	$-n$
M'	m	$-l$	$-k$	$-o$
M''	m	$-i$	n	$-j$

Table 4.54: Index distribution of the codewords of $\mathcal{F}_m^{(2)}$.

Accordingly, taking into account the possible index distributions for $W' \in \mathcal{G}_m \backslash \mathcal{G}_i$ presented in Table 4.52 and Lemma 1.5, $W' \in \mathcal{G}_m \backslash \mathcal{G}_i$ is such that: $W' \in \mathcal{G}_{m,-i,l,-o,k}$ or $W' \in \mathcal{G}_{m,-i,-o,k,j}$.

Assume first that $W' \in \mathcal{G}_{m,-i,l,-o,k}$. Until now we know the index distribution of only five codewords in $\mathcal{F}_m$, namely, $U_5, U_8 \in \mathcal{F}_{mi}$ and $M, M', M'' \in \mathcal{F}_m^{(2)} \backslash \mathcal{F}_i$. Then, we have to characterize the index distribution of the five codewords of $\mathcal{F}_m \backslash (\mathcal{F}_i \cup \mathcal{F}_m^{(2)})$. Considering the index distribution of the codewords of $\mathcal{G}_m \cup \mathcal{F}_m^{(2)}$ and Lemma 1.5, by the analysis of Table 4.51 we conclude that if $U \in \mathcal{F}_m \backslash (\mathcal{F}_i \cup \mathcal{F}_m^{(2)})$, then the index distribution of U satisfies one of the following conditions:

m	l	$-j$	o
m	n	$-l$	k
m	n	k	j
m	n	o	j

m	n	o	$-l$
m	$-j$	k	$-l$
m	$-n$	k	$-l$

Table 4.55: Possible index distributions for $U \in \mathcal{F}_m \backslash (\mathcal{F}_i \cup \mathcal{F}_m^{(2)})$.

That is, if $U \in \mathcal{F}_m \backslash (\mathcal{F}_i \cup \mathcal{F}_m^{(2)})$, then

$$U \in \mathcal{F}_{m,l,-j,o} \cup \mathcal{F}_{mnk} \cup \mathcal{F}_{mno} \cup \mathcal{F}_{m,k,-l}.$$

Consequently, taking into account Lemma 1.5, $|\mathcal{F}_m \backslash (\mathcal{F}_i \cup \mathcal{F}_m^{(2)})| \leq 4$, which is a contradiction.

If we consider $W' \in \mathcal{G}_{m,-i,-o,k,j}$, proceeding as in the previous case we conclude again that $|\mathcal{F}_m| < 10$. Therefore, if $|\mathcal{G}_m| = 4$ and $W \in \mathcal{G}_{m,-i,-k,-n,o}$, then $|\mathcal{F}_m| \neq 10$.

- $|\mathcal{G}_m| = 4$ **and** $|\mathcal{F}_m| = 11$

Suppose that $|\mathcal{G}_m| = 4$ and $|\mathcal{F}_m| = 11$. Unlike the previous case, now we do not have any information about $|\mathcal{F}_m^{(2)}|$. In this case, for each one of the possible index distributions for $W' \in \mathcal{G}_m \backslash \mathcal{G}_i$ presented in Table 4.52 we must identify, by the analysis of Table 4.51, the possible index distributions for the codewords of $\mathcal{F}_m \backslash \mathcal{F}_i$.

Let us assume $W' \in \mathcal{G}_{m,-i,l,-j,k}$. Since $|\mathcal{F}_{mi}| = 2$ and, by assumption, $|\mathcal{F}_m| = 11$, then $|\mathcal{F}_m \backslash \mathcal{F}_i| = 9$. Considering Table 4.51, taking into account the index distribution of the codewords $W, W' \in \mathcal{G}_m \backslash \mathcal{G}_i$ and Lemma 1.5, we conclude that if $U \in \mathcal{F}_m \backslash \mathcal{F}_i$, then U must satisfy one of the index distributions presented in the following table.

m	-k	-o	j
			-l
m	l	j	-n
			-o
			o
m	n	o	j
			-j

m	n	-l	-j
			-o
			k
			o
			-i
m	n	k	-o
			j

m	-n	k	j
			-l
m	-i	n	-o
			j
m	-i	-o	j
			-l
m	-o	k	j

Table 4.56: Possible index distribution for $U \in \mathcal{F}_m \backslash \mathcal{F}_i$.

By the analysis of the above tables, if $U \in \mathcal{F}_m \backslash \mathcal{F}_i$, then

$$U \in \mathcal{F}_{m,-k,-o} \cup \mathcal{F}_{mlj} \cup \mathcal{F}_{mno} \cup \mathcal{F}_{m,n,-l} \cup \mathcal{F}_{mnk} \cup \mathcal{F}_{m,-n,k} \cup \mathcal{F}_{m,-i,n} \cup \mathcal{F}_{m,-i,-o} \cup \mathcal{F}_{m,-o,k,j}.$$

Thus, considering Lemma 1.5, $U_{14} \in \mathcal{F}_{m,-o,k,j}$ and $U_{15}, \ldots, U_{22} \in \mathcal{F}_m \backslash \mathcal{F}_i$ satisfy, respectively, the partial index distribution presented bellow.

U_{15}	m	$-k$	$-o$
U_{16}	m	l	j
U_{17}	m	n	o
U_{18}	m	n	$-l$

U_{19}	m	n	k
U_{20}	m	$-n$	k
U_{21}	m	$-i$	n
U_{22}	m	$-i$	$-o$

Table 4.57: Partial index distributions for $U_{15}, \ldots, U_{22} \in \mathcal{F}_m \backslash \mathcal{F}_i$.

Let us consider $U_{19} \in \mathcal{F}_{mnk}$. Taking into account Table 4.56,

$$U_{19} \in \mathcal{F}_{m,n,-l,k} \cup \mathcal{F}_{m,n,k,-o} \cup \mathcal{F}_{mnkj}.$$

Accordingly:

– if $U_{19} \in \mathcal{F}_{m,n,-l,k}$, then $U_{19}, U_{18} \in \mathcal{F}_{m,n,-l}$;

– if $U_{19} \in \mathcal{F}_{m,n,k,-o}$, then $U_{19}, U_{14} \in \mathcal{F}_{m,k,-o}$;

– if $U_{19} \in \mathcal{F}_{mnkj}$, then $U_{19}, U_{14} \in \mathcal{F}_{mkj}$.

In any case Lemma 1.5 is contradicted.

Therefore, if $\mathcal{G}_m \backslash \mathcal{G}_i = \{W, W'\}$, with $W \in \mathcal{G}_{m,-i,-k,-n,o}$ and $W' \in \mathcal{G}_{m,-i,l,-j,k}$, then $|\mathcal{F}_m| \neq 11$.

If we consider W' satisfying any other index distribution presented in Table 4.52, applying a similar reasoning we can not describe the index distributions of all codewords of $\mathcal{F}_m \backslash \mathcal{F}_i$ without contradictions.

Thus, if $W \in \mathcal{G}_{m,-i,-k,-n,o}$, then $|\mathcal{G}_m| \neq 4$. In what follows, we study the hypothesis $|\mathcal{G}_m| = 5$ assuming that $W \in \mathcal{G}_{m,-i,-k,-n,o}$.

- $|\mathcal{G}_m| = 5$ **and** $8 \leq |\mathcal{F}_m| \leq 10$

Suppose now that $|\mathcal{G}_m| = 5$. Let us consider $\mathcal{G}_m \backslash \mathcal{G}_i = \{W, W', W''\}$, where $W \in \mathcal{G}_{m,-i,-k,-n,o}$. In these conditions, $W', W'' \in \mathcal{G}_m \backslash \mathcal{G}_i$ satisfy one of the possible index distributions presented in Table 4.53.

Since $8 \leq |\mathcal{F}_m| \leq 10$ and $|\mathcal{F}_{mi}| = 2$, then $|\mathcal{F}_m \backslash \mathcal{F}_i| \geq 6$.

Let us consider $W' \in \mathcal{G}_{m,-i,l,-j,k}$ and $W'' \in \mathcal{G}_{m,-i,n,-o,-l}$. The characterization of the possible index distributions for the codewords $U \in \mathcal{F}_m \backslash \mathcal{F}_i$ is done considering Table 4.51, the index distribution of the codewords of $\mathcal{G}_m \backslash \mathcal{G}_i$ and Lemma 1.5. Accordingly, if $U \in \mathcal{F}_m \backslash \mathcal{F}_i$, then

$$U \in \mathcal{F}_{m,-o,j} \cup \mathcal{F}_{mlj} \cup \mathcal{F}_{mkj} \cup \mathcal{F}_{mno} \cup \mathcal{F}_{m,-n,k,-l}.$$

Consequently, taking into account Lemma 1.5, $|\mathcal{F}_m \backslash \mathcal{F}_i| \leq 5$, contradicting the assumption $8 \leq |\mathcal{F}_m| \leq 10$. Thus, if $|\mathcal{G}_m| = 5$, then $\mathcal{G}_m \backslash \mathcal{G}_i \neq \{W, W', W''\}$, with $W \in \mathcal{G}_{m,-i,-k,-n,o}$, $W' \in \mathcal{G}_{m,-i,l,-j,k}$ and $W'' \in \mathcal{G}_{m,-i,n,-o,-l}$.

If we consider W' and W'' satisfying any other possible index distribution presented in Table 4.53, applying a similar reasoning we get always contradictions.

Therefore, the assumption $W \in \mathcal{G}_{m,-i,-k,-n,o}$ contradicts necessary conditions for the existence of PL$(7,2)$ codes. Then, if $W \in \mathcal{G}_m \backslash \mathcal{G}_i$, then $W \notin \mathcal{G}_{m,-i,-k,-n,o}$.

Considering each one of the remaining possible index distributions for $W \in \mathcal{G}_m \backslash \mathcal{G}_i$, see Table 4.45, proceeding as in the analysis of the hypothesis $W \in \mathcal{G}_{m,-i,-k,-n,o}$, we conclude that, in the majority of the cases, the characterization of the index distribution of all codewords of $\mathcal{G}_m \cup \mathcal{F}_m$ implies the existence of contradictions. However, there are cases, although few cases, in which it is possible to identify completely the index distribution of all codewords of $\mathcal{G}_m \cup \mathcal{F}_m$ without contradictions, being in these cases necessary to analyze other set $\mathcal{G}_\omega \cup \mathcal{F}_\omega$, with $\omega \in \mathcal{I}\backslash\{i,m\}$. Next, we present an illustrative example of one of these cases.

4.3.3 When there are no contradictions in the characterization of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i \cup \mathcal{G}_m \cup \mathcal{F}_m$

Let us consider $|\mathcal{G}_m| = 4$ with $W, W' \in \mathcal{G}_m \backslash \mathcal{G}_i$ so that $W \in \mathcal{G}_{m,-i,-k,o,j}$ and $W' \in \mathcal{G}_{m,-i,l,-o,k}$. Assume $|\mathcal{F}_m| = 10$, with $U_{14}, \ldots, U_{21} \in \mathcal{F}_m \backslash \mathcal{F}_i$ such that:

U_{14}	m	j	l	$-n$
U_{15}	m	$-l$	$-k$	$-o$
U_{16}	m	$-i$	n	$-j$
U_{17}	m	l	$-j$	o

U_{18}	m	n	k	j
U_{19}	m	n	o	$-l$
U_{20}	m	$-j$	k	$-l$
U_{21}	m	$-i$	$-n$	$-l$

Table 4.58: Index distribution of $U_{14}, \ldots, U_{21} \in \mathcal{F}_m \backslash \mathcal{F}_i$.

By Lemma 2.10, since we are supposing $|\mathcal{F}_m| = 10$, we have $|\mathcal{F}_m^{(2)}| = 4$. Consequently, from Proposition 4.8 it follows that $\mathcal{F}_m^{(2)} = \{U_8, U_{14}, U_{15}, U_{16}\}$, with $U_8 \in \mathcal{F}_{ikmo}$.

This is an example in which we have found a possible index distribution for all codewords of $\mathcal{G}_i \cup \mathcal{F}_i \cup \mathcal{G}_m \cup \mathcal{F}_m$ such that:

- $|\mathcal{G}_{\omega\rho\varphi} \cup \mathcal{F}_{\omega\rho\varphi}| \leq 1$ for any $\omega, \rho, \varphi \in \mathcal{I}$;
- it is possible to identify the codewords of $\mathcal{F}_m^{(2)}$.

To verify that, in fact, this index distribution implies contradictions, we will analyze other set $\mathcal{G}_\omega \cup \mathcal{F}_\omega$ for other element $\omega \in \mathcal{I}\backslash\{i,m\}$.

The choice of that element $\omega \in \mathcal{I}\backslash\{i,m\}$ it depends on the index distribution of the known codewords of $\mathcal{G}_i \cup \mathcal{F}_i \cup \mathcal{G}_m \cup \mathcal{F}_m$. Since, until now, we have identified only one codeword in $\mathcal{G}_j$, $W \in \mathcal{G}_{m,-i,-k,o,j}$, we will concentrate our attention on the characterization of all codewords of $\mathcal{G}_j \cup \mathcal{F}_j$.

We note that $|\mathcal{F}_{ij}| = 5$. Considering the index distribution of the codewords of $\mathcal{F}_{ij}$, see Table 4.44, the codewords $U_1, U_2, U_5, U_6, U_7 \in \mathcal{F}_{ij}$ induce a partition $\mathcal{Q}$ of

$\mathcal{I}\backslash\{i, m, j, -j\}$:

$$\mathcal{Q}_1 = \{k, l\};\ \mathcal{Q}_2 = \{-m, o\};\ \mathcal{Q}_3 = \{-l\};\ \mathcal{Q}_4 = \{-k, -n\};\ \mathcal{Q}_5 = \{n, -o\};\ \mathcal{Q}_6 = \{-i\}.$$

Such as in the previous cases, this partition will be useful in the characterization of the possible index distributions for the codewords of $(\mathcal{G}_j \cup \mathcal{F}_j)\backslash(\mathcal{G}_i \cup \mathcal{F}_i \cup \mathcal{G}_m \cup \mathcal{F}_m)$.

We note that, by Corollary 3.1, it follows $3 \leq |\mathcal{G}_j| \leq 7$, then $|\mathcal{G}_j\backslash(\mathcal{G}_i \cup \mathcal{G}_m)| \geq 2$. Taking into account the partition $\mathcal{Q}$, the known index distribution of all codewords of $\mathcal{G}_i \cup \mathcal{F}_i \cup \mathcal{G}_m \cup \mathcal{F}_m$ and Lemma 1.5, we conclude that if $V \in \mathcal{G}_j\backslash(\mathcal{G}_i \cup \mathcal{G}_m)$, then V must satisfy one of the following index distributions:

j	$-l$	$-n$	k	o
j	$-l$	$-n$	$-m$	$-o$
j	$-i$	$-m$	$-n$	$-o$

j	$-i$	$-m$	$-l$	$-o$
j	$-i$	$-m$	l	n
j	$-i$	$-m$	k	$-l$

Table 4.59: Possible index distributions for $V \in \mathcal{G}_j\backslash(\mathcal{G}_i \cup \mathcal{G}_m)$.

Analyzing the above tables, if $V \in \mathcal{G}_j\backslash(\mathcal{G}_i \cup \mathcal{G}_m)$, then $V \in \mathcal{G}_{j,-l,-n} \cup \mathcal{G}_{j,-i,-m}$. Consequently, having in view Lemma 1.5, we conclude that $|\mathcal{G}_j\backslash(\mathcal{G}_i \cup \mathcal{G}_m)| \leq 2$. However, by Corollary 3.1 we must impose $|\mathcal{G}_j\backslash(\mathcal{G}_i \cup \mathcal{G}_m)| = 2$, which implies $|\mathcal{G}_j| = 3$. Considering Lemma 1.5 and the possible index distributions for the codewords of $\mathcal{G}_j\backslash(\mathcal{G}_i \cup \mathcal{G}_m)$, presented in Table 4.59, the codewords $V, V' \in \mathcal{G}_j\backslash(\mathcal{G}_i \cup \mathcal{G}_m)$ must verify one of the conditions bellow. We note that, V on the left is matching to V' on the right.

V	j	$-l$	$-n$	k	o
V	j	$-l$	$-n$	k	o
V	j	$-l$	$-n$	k	o
V	j	$-l$	$-n$	$-m$	$-o$

V'	j	$-i$	$-m$	$-n$	$-o$
V'	j	$-i$	$-m$	$-l$	$-o$
V'	j	$-i$	$-m$	l	n
V'	j	$-i$	$-m$	l	n

Table 4.60: Possible index distributions for $V, V' \in \mathcal{G}_j\backslash(\mathcal{G}_i \cup \mathcal{G}_m)$.

Since $|\mathcal{G}_j| = 3$, by Proposition 4.1 there exist $\alpha, \beta, \gamma \in \mathcal{I}\backslash\{j, -j\}$ so that $|\mathcal{F}_{j\alpha}| = |\mathcal{F}_{j\beta}| = |\mathcal{F}_{j\gamma}| = 5$. Furthermore, by Proposition 4.3, $|\mathcal{G}_{j\alpha}| = |\mathcal{G}_{j\beta}| = |\mathcal{G}_{j\gamma}| = 0$.

Let us suppose that $V, V' \in \mathcal{G}_j\backslash(\mathcal{G}_i \cup \mathcal{G}_m)$ are such that $V \in \mathcal{G}_{j,-l,-n,k,o}$ and $V' \in \mathcal{G}_{j,-i,-m,-n,-o}$.

Accordingly, the codewords of $\mathcal{G}_j$ are such that:

W_4	j	m	$-i$	$-k$	o
V	j	$-l$	$-n$	k	o
V'	j	$-i$	$-m$	$-n$	$-o$

Table 4.61: Index distribution of the codewords of $\mathcal{G}_j$.

Since i, l and n are the unique elements in $\mathcal{I}\backslash\{j,-j\}$ satisfying $|\mathcal{G}_{ji}| = |\mathcal{G}_{jl}| = |\mathcal{G}_{jn}| = 0$, then $|\mathcal{F}_{ji}| = |\mathcal{F}_{jl}| = |\mathcal{F}_{jn}| = 5$.

Let us consider $\mathcal{F}_{jl}$. Taking into account the codewords of $\mathcal{F}_i \cup \mathcal{F}_m$, we already know the index distribution of two codewords in $\mathcal{F}_{jl}$: $U_1 \in \mathcal{F}_{ijkl}$ and $U_{14} \in \mathcal{F}_{m,j,l,-n}$. As $|\mathcal{F}_{jl}| = 5$, by Lemma 2.5 we have $|\mathcal{F}_{jl\omega}| = 1$ for any $\omega \in \mathcal{I}\backslash\{j,-j,l,-l\}$. Thus, $|\mathcal{F}_{j,l,-k}| = |\mathcal{F}_{j,l,-i}| = |\mathcal{F}_{jlo}| = 1$. Considering $W_4 \in \mathcal{G}_{j,m,-i,-k,o}$ and Lemma 1.5 we must impose $\mathcal{F}_{j,l,-k} \cap \mathcal{F}_{j,l,-i} \cap \mathcal{F}_{jlo} = \varnothing$. Then, the index distribution of $U_1, U_{14}, J_1, J_2, J_3 \in \mathcal{F}_{jl}$ must satisfy:

U_1	j	l	i	k
U_{14}	j	l	m	$-n$
J_1	j	l	$-k$	j_1
J_2	j	l	$-i$	j_2
J_3	j	l	o	j_3

Table 4.62: Index distribution of the codewords of $\mathcal{F}_{jl}$.

with $j_1, j_2, j_3 \in \{-m, n, -o\}$ pairwise distinct. Considering $V' \in \mathcal{G}_{j,-i,-m,-n,-o}$, we must impose $J_2 \in \mathcal{F}_{j,l,-i,n}$. Consequently, $J_3 \in \mathcal{F}_{j,l,o,-m}$. In these conditions we have $J_3, U_2 \in \mathcal{F}_{j,-m,o}$, which contradicts Lemma 1.5.

If we suppose $V, V' \in \mathcal{G}_j\backslash(\mathcal{G}_i \cup \mathcal{G}_m)$ satisfying any other index distribution presented in Table 4.60, such as in the presented example, we get contradictions.

Therefore, the considered index distribution for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i \cup \mathcal{G}_m \cup \mathcal{F}_m$ contradicts necessary conditions for the existence of PL(7, 2) codes.

We have analyzed all possible sets $\mathcal{G}_m \backslash \mathcal{G}_i$ whose codewords satisfy the index distributions presented in Table 4.45. For each one of them we have obtained one of the following conclusions:

– it is not possible to characterize completely the index distribution of all codewords of $\mathcal{G}_m \cup \mathcal{F}_m$ without contradictions;

– it is possible to characterize completely all codewords of $\mathcal{G}_m \cup \mathcal{F}_m$, however when we consider another element $\omega \in \mathcal{I} \backslash \{i, m\}$, the characterization of the index distribution of all codewords of $\mathcal{G}_\omega \cup \mathcal{F}_\omega$ implies contradictions.

In Subsection 4.2.3 we have identified other possible index distribution for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$. Applying a similar reasoning, we conclude also that this index distribution implies contradictions on necessary conditions for the existence of $\mathrm{PL}(7, 2)$ codes.

Thus, we are able to establish the following theorem:

Theorem 4.1 *For any* $\alpha \in \mathcal{I}$, $|\mathcal{G}_\alpha| \neq 3$.

As an immediate consequence of the previous theorem and Corollary 3.1 we get:

Corollary 4.3 *For any* $\alpha \in \mathcal{I}$, $4 \leq |\mathcal{G}_\alpha| \leq 7$.

Chapter 5

Proof of $|\mathcal{G}_i| \neq 4$ for any $i \in \mathcal{I}$

In last chapter we have proved that $4 \leq |\mathcal{G}_\alpha| \leq 7$ for any $\alpha \in \mathcal{I}$. Here, we intend to restrict even more the range of variation of $|\mathcal{G}_\alpha|$ proving that $|\mathcal{G}_\alpha| \neq 4$ for any $\alpha \in \mathcal{I}$. Such as in the previous case, we assume, without loss of generality, that there exists an $i \in \mathcal{I}$ such that $|\mathcal{G}_i| = 4$ and taking into account that the codewords of $\mathcal{G} \cup \mathcal{F}$ have more nonzero coordinates, we will focus our attention on the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$. Our aim is to show that any index distribution for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ contradicts the definition of a perfect 2-error correcting Lee code.

Firstly, we deduce some necessary conditions which must be satisfied by the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$. These conditions will help us in the identification of all possible index distributions for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$. We note that, under the assumption $|\mathcal{G}_i| = 4$, by Lemma 2.10, $10 \leq |\mathcal{F}_i| \leq 11$.

In the last section we show how we may conclude that any index distribution for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ contradicts the definition of PL$(7, 2)$ code, that is $|\mathcal{G}_\alpha| \neq 4$ for any $\alpha \in \mathcal{I}$.

5.1 Necessary conditions for the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$

Let $i \in \mathcal{I}$ be so that $|\mathcal{G}_i| = 4$. The first results presented in this section are focused on the characterization of the index distribution of the four codewords of $\mathcal{G}_i$.

Proposition 5.1 *If* $|\mathcal{G}_i| = 4$, *for* $i \in \mathcal{I}$, *then* $|\mathcal{G}_{i\alpha}| \leq 2$ *for any* $\alpha \in \mathcal{I}\backslash\{i, -i\}$.

Proof. By Lemma 2.2 we know that $|\mathcal{G}_{i\alpha}| \leq 3$ for all $\alpha \in \mathcal{I}\backslash\{i, -i\}$. Suppose, by contradiction, that $j \in \mathcal{I}\backslash\{i, -i\}$ is such that $|\mathcal{G}_{ij}| = 3$.

As $|\mathcal{G}_i| = 4$, from Lemma 2.10 it follows that $|\mathcal{F}_i| = 10$ or $|\mathcal{F}_i| = 11$. Next, we analyze, separately, these two hypotheses: $|\mathcal{F}_i| = 10$ and $|\mathcal{F}_i| = 11$.

Suppose first that $|\mathcal{F}_i| = 10$. Then, by Lemma 2.10, $|\mathcal{B}_i \cup \mathcal{C}_i \cup \mathcal{E}_i| = 0$.

Considering Lemma 2.2 and taking into account that, by hypothesis, $|\mathcal{G}_{ij}| = 3$, then $|\mathcal{D}_{ij} \cup \mathcal{E}_{ij}| + 2|\mathcal{F}_{ij}| = 1$. Since $|\mathcal{E}_i| = 0$, it follows that $|\mathcal{D}_{ij}| = 1$ and $|\mathcal{F}_{ij}| = 0$.

Let us consider two words V_1 and V_2 of type $[\pm 2, \pm 1]$ satisfying:

	i	j
V_1	± 2	± 1
V_2	± 1	± 2

Table 5.1: Index value distribution of the words V_1 and V_2.

These words must be covered by codewords of $\mathcal{B}_{ij} \cup \mathcal{C}_{ij} \cup \mathcal{D}_{ij} \cup \mathcal{E}_{ij} \cup \mathcal{F}_{ij}$. As $|\mathcal{B}_{ij}| = |\mathcal{C}_{ij}| = |\mathcal{E}_{ij}| = |\mathcal{F}_{ij}| = 0$, then V_1 and V_2 must be covered by the unique codeword in $\mathcal{D}_{ij}$, which is not possible since the codewords of $\mathcal{D}$ are of type $[\pm 3, \pm 1^2]$.

Now assume that $|\mathcal{F}_i| = 11$. Since we are under the assumption $|\mathcal{G}_{ij}| = 3$, let us consider $W_1, W_2, W_3 \in \mathcal{G}_{ij}$ such that $W_1 \in \mathcal{G}_{ijw_1w_2w_3}$, $W_2 \in \mathcal{G}_{ijw_4w_5w_6}$ and $W_3 \in \mathcal{G}_{ijw_7w_8w_9}$, with $w_1, \ldots, w_9 \in \mathcal{I}\backslash\{i, -i, j, -j\}$. We note that, by Lemma 1.5, $w_1, \ldots, w_9$ must be pairwise distinct. As $|\mathcal{G}_i| = 4$, let $W_4 \in \mathcal{G}_i\backslash\mathcal{G}_j$ so that $W_4 \in \mathcal{G}_{iw_{10}w_{11}w_{12}w_{13}}$, where $w_{10}, w_{11}, w_{12}, w_{13} \in \mathcal{I}\backslash\{i, -i, j\}$. In Table 5.2, the codewords $W_1, \ldots, W_4 \in \mathcal{G}_i$ are schematically represented.

Since $w_1, \ldots, w_9 \subset \mathcal{I}\backslash\{i, -i, j, -j\}$ with $w_1, \ldots, w_9$ pairwise distinct, taking into account that $|\mathcal{I}| = 14$, let $\{\beta\} = \mathcal{I}\backslash\{i, -i, j, -j, w_1, \ldots, w_9\}$. Note that,

$$\mathcal{I}\backslash\{i, -i\} = \{j\} \cup \{-j\} \cup \{\beta\} \cup \{w_1, \ldots, w_9\}.$$

W_1	i	j	w_1	w_2	w_3
W_2	i	j	w_4	w_5	w_6
W_3	i	j	w_7	w_8	w_9
W_4	i	w_{10}	w_{11}	w_{12}	w_{13}

Table 5.2: Partial index distribution of the codewords of $\mathcal{G}_i$.

Considering $W_4 \in \mathcal{G}_{iw_{10}w_{11}w_{12}w_{13}}$, with $w_{10}, \ldots, w_{13} \in \mathcal{I}\backslash\{i, -i, j\}$, we conclude that $|\{w_{10},...,w_{13}\} \cap \{w_1,...,w_9\}| \geq 2$. On the other hand, $|\{w_{10},...,w_{13}\} \cap \{w_1,...,w_9\}| \leq 3$, otherwise Lemma 1.5 is contradicted. We will consider separately the cases:

i) $|\{w_{10}, \ldots, w_{13}\} \cap \{w_1, \ldots, w_9\}| = 2$;

ii) $|\{w_{10}, \ldots, w_{13}\} \cap \{w_1, \ldots, w_9\}| = 3$.

Suppose that $|\{w_{10},...,w_{13}\}\cap\{w_1,...,w_9\}| = 2$, in these conditions $W_4 \in \mathcal{G}_{i,-j,\beta,w_{10},w_{11}}$, with $w_{10}, w_{11} \in \{w_1, \ldots, w_9\}$. Accordingly, we have $|\mathcal{G}_{ij}| = 3$, $|\mathcal{G}_{iw_{10}}| = |\mathcal{G}_{iw_{11}}| = 2$ and $|\mathcal{G}_{iw}| = 1$ for all $w \in \mathcal{I}\backslash\{i, -i, j, w_{10}, w_{11}\}$. Consequently, from Lemma 2.2 it follows that $|\mathcal{F}_{ij}| = 0$, $|\mathcal{F}_{iw_{10}}|, |\mathcal{F}_{iw_{11}}| \leq 2$ and $|\mathcal{F}_{iw}| \leq 3$ for all $w \in \mathcal{I}\backslash\{i, -i, j, w_{10}, w_{11}\}$. As $|\mathcal{F}_i| = \frac{1}{3}\sum_{\alpha\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{F}_{i\alpha}|$ and we are assuming $|\mathcal{F}_i| = 11$, then $\sum_{\alpha\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{F}_{i\alpha}| = 33$. However, taking into account what was been said before, $\sum_{\alpha\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{F}_{i\alpha}| \leq 31$, which is a contradiction.

Now consider that $|\{w_{10}, \ldots, w_{13}\} \cap \{w_1, \ldots, w_9\}| = 3$. Thus, $W_4 \in \mathcal{G}_{ixw_{10}w_{11}w_{12}}$, with $x \in \{-j, \beta\}$ and $w_{10}, w_{11}, w_{12} \in \{w_1, \ldots, w_9\}$. In these conditions, $|\mathcal{G}_{ij}| = 3$, $|\mathcal{G}_{iw_{10}}| = |\mathcal{G}_{iw_{11}}| = |\mathcal{G}_{iw_{12}}| = 2$, $|\mathcal{G}_{iy}| = 0$ for $\{y\} = \{-j, \beta\}\backslash\{x\}$ and $|\mathcal{G}_{iw}| = 1$ for all $w \in \mathcal{I}\backslash\{i, -i, j, y, w_{10}, w_{11}, w_{12}\}$. Consequently, by Lemma 2.2, we get $|\mathcal{F}_{ij}| = 0$, $|\mathcal{F}_{iw_{10}}|, |\mathcal{F}_{iw_{11}}|, |\mathcal{F}_{iw_{12}}| \leq 2$, $|\mathcal{F}_{iy}| \leq 5$ and $|\mathcal{F}_{iw}| \leq 3$ for all $w \in \mathcal{I}\backslash\{i, -i, j, y, w_{10}, w_{11}, w_{12}\}$. Accordingly, $\sum_{\alpha\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{F}_{i\alpha}| \leq 32$, obtaining again a contradiction. □

We have just proved that for any $\alpha \in \mathcal{I}\backslash\{i, -i\}$ we get $|\mathcal{G}_{i\alpha}| \leq 2$. Let us consider the subset $\mathcal{J} \subset \mathcal{I}\backslash\{i, -i\}$ so that:

$$\mathcal{J} = \{\alpha \in \mathcal{I}\backslash\{i, -i\} : |\mathcal{G}_{i\alpha}| = 2\}.$$

The following result restricts the variation of $|\mathcal{J}|$.

Proposition 5.2 *The cardinality of* $\mathcal{J}$ *satisfies* $4 \leq |\mathcal{J}| \leq 6$.

Proof. Since $|\mathcal{G}_i| = \frac{1}{4}\sum_{\alpha\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{G}_{i\alpha}|$ and, by assumption, $|\mathcal{G}_i| = 4$, then

$$\sum_{\alpha\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{G}_{i\alpha}| = 16. \tag{5.1}$$

By Proposition 5.1, $|\mathcal{G}_{i\alpha}| \leq 2$ for all $\alpha \in \mathcal{I}\backslash\{i,-i\}$. As $|\mathcal{I}\backslash\{i,-i\}| = 12$, having in view (5.1) we conclude that there exist, at least, four elements $\alpha \in \mathcal{I}\backslash\{i,-i\}$ satisfying $|\mathcal{G}_{i\alpha}| = 2$, that is, $|\mathcal{J}| \geq 4$.

Let $W, W' \in \mathcal{G}_i$. Taking into account Lemma 1.5, there exists, at most, one element $\alpha \in \mathcal{I}\backslash\{i,-i\}$ so that $W, W' \in \mathcal{G}_{i\alpha}$. As $|\mathcal{G}_i| = 4$, at most, there are $6 = \binom{4}{2}$ distinct elements $\alpha \in \mathcal{I}\backslash\{i,-i\}$ such that $|\mathcal{G}_{i\alpha}| = 2$, that is, $|\mathcal{J}| \leq 6$. □

Next, we establish conditions which must be verified by the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ when $|\mathcal{J}|$ assumes each one of the possible values.

Proposition 5.3 *If* $|\mathcal{J}| = 4$, *then* $|\mathcal{G}_{i\alpha}| = 1$ *for any* $\alpha \in \mathcal{I}\backslash(\{i,-i\}\cup\mathcal{J})$ *and* $|\mathcal{F}_i| = 10$.

Proof. By assumption $|\mathcal{G}_i| = 4$, consequently $\sum_{\alpha\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{G}_{i\alpha}| = 16$. That is,

$$\sum_{\alpha\in\mathcal{I}\backslash(\{i,-i\}\cup\mathcal{J})} |\mathcal{G}_{i\alpha}| + \sum_{\alpha\in\mathcal{J}} |\mathcal{G}_{i\alpha}| = 16.$$

As $|\mathcal{G}_{i\alpha}| = 2$ for all $\alpha \in \mathcal{J}$ and, by assumption, $|\mathcal{J}| = 4$, it follows that:

$$\sum_{\alpha\in\mathcal{I}\backslash(\{i,-i\}\cup\mathcal{J})} |\mathcal{G}_{i\alpha}| = 8.$$

Taking into account Proposition 5.1, $|\mathcal{G}_{i\alpha}| \leq 1$ for all $\alpha \in \mathcal{I}\backslash(\{i,-i\} \cup \mathcal{J})$. Since that $|\mathcal{I}\backslash(\{i,-i\} \cup \mathcal{J})| = 8$, we must impose $|\mathcal{G}_{i\alpha}| = 1$ for all $\alpha \in \mathcal{I}\backslash(\{i,-i\} \cup \mathcal{J})$.

From Lemma 2.10 we know that $|\mathcal{F}_i| = 10$ or $|\mathcal{F}_i| = 11$. Let us suppose that $|\mathcal{F}_i| = 11$. In these conditions, taking into account that $|\mathcal{F}_i| = \frac{1}{3}\sum_{\alpha\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{F}_{i\alpha}|$, we have $\sum_{\alpha\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{F}_{i\alpha}| = 33$. Having in mind what was proved before, from Lemma 2.2, we get $|\mathcal{F}_{i\alpha}| \leq 2$ for all $\alpha \in \mathcal{J}$ and $|\mathcal{F}_{i\alpha}| \leq 3$ for any $\alpha \in \mathcal{I}\backslash(\{i,-i\} \cup \mathcal{J})$. That is, $\sum_{\alpha\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{F}_{i\alpha}| \leq 32$, which is an absurdity. Therefore, $|\mathcal{F}_i| = 10$. □

Proposition 5.4 *If $|\mathcal{J}| = 4$, with $\mathcal{J} = \{\beta, \gamma, \delta, \varepsilon\}$, then $|\mathcal{F}_{i\alpha}| \geq 1$ for all $\alpha \in \mathcal{I}\backslash\{i, -i\}$ and there exist, at least, two elements $\alpha \in \mathcal{J}$ such that $|\mathcal{F}_{i\alpha}| = 2$. Furthermore:*

i) if $\beta, \gamma \in \mathcal{J}$ are the unique elements in $\mathcal{J}$ satisfying $|\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = 2$, then $|\mathcal{F}_{i\delta}| = |\mathcal{F}_{i\varepsilon}| = 1$ and $|\mathcal{F}_{i\alpha}| = 3$ for all $\alpha \in \mathcal{I}\backslash(\{i, -i\} \cup \mathcal{J})$;

ii) if $\beta, \gamma, \delta \in \mathcal{J}$ are the unique elements in $\mathcal{J}$ satisfying $|\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = |\mathcal{F}_{i\delta}| = 2$, then $|\mathcal{F}_{i\varepsilon}| = 1$ and there are seven elements $\alpha \in \mathcal{I}\backslash(\{i, -i\} \cup \mathcal{J})$ such that $|\mathcal{F}_{i\alpha}| = 3$;

iii) if $|\mathcal{F}_{i\alpha}| = 2$ for all $\alpha \in \mathcal{J}$, then there are, at least, six elements $\alpha \in \mathcal{I}\backslash(\{i, -i\} \cup \mathcal{J})$ satisfying $|\mathcal{F}_{i\alpha}| = 3$.

Proof. Let us suppose $|\mathcal{J}| = 4$ with $\mathcal{J} = \{\beta, \gamma, \delta, \varepsilon\}$. By Proposition 5.3 we know that $|\mathcal{F}_i| = 10$. Consequently, by Lemma 2.10, we get $|\mathcal{F}_i^{(2)}| = 4$. In these conditions, from Lemma 2.14 we conclude that $|\mathcal{F}_{i\alpha}| \geq 1$ for all $\alpha \in \mathcal{I}\backslash\{i, -i\}$.

Since $|\mathcal{F}_i| = 10$, then $\sum_{\alpha \in \mathcal{I}\backslash\{i,-i\}} |\mathcal{F}_{i\alpha}| = 30$. Taking into account Lemma 2.2, $|\mathcal{F}_{i\alpha}| \leq 2$ for all $\alpha \in \mathcal{J}$. By Proposition 5.3, $|\mathcal{G}_{i\alpha}| = 1$ for all $\alpha \in \mathcal{I}\backslash(\{i, -i\} \cup \mathcal{J})$ and considering Lemma 2.2 it follows that $|\mathcal{F}_{i\alpha}| \leq 3$ for all $\alpha \in \mathcal{I}\backslash(\{i, -i\} \cup \mathcal{J})$. As

$$\sum_{\alpha \in \mathcal{I}\backslash(\{i,-i\}\cup\mathcal{J})} |\mathcal{F}_{i\alpha}| + \sum_{\alpha \in \mathcal{J}} |\mathcal{F}_{i\alpha}| = 30 \tag{5.2}$$

and $\sum_{\alpha \in \mathcal{I}\backslash(\{i,-i\}\cup\mathcal{J})} |\mathcal{F}_{i\alpha}| \leq 24$, we must impose $\sum_{\alpha \in \mathcal{J}} |\mathcal{F}_{i\alpha}| \geq 6$, consequently, there exist, at least, two distinct elements $\alpha \in \mathcal{J}$ satisfying $|\mathcal{F}_{i\alpha}| = 2$.

Suppose that there are exactly two elements $\alpha \in \mathcal{J}$ satisfying $|\mathcal{F}_{i\alpha}| = 2$. Without loss of generality, we may suppose that $|\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = 2$. Therefore, considering what was proved before, $|\mathcal{F}_{i\delta}| = |\mathcal{F}_{i\varepsilon}| = 1$. Thus, $\sum_{\alpha \in \mathcal{J}} |\mathcal{F}_{i\alpha}| = 6$ and, by (5.2), we must impose $\sum_{\alpha \in \mathcal{I}\backslash(\{i,-i\}\cup\mathcal{J})} |\mathcal{F}_{i\alpha}| = 24$. Consequently, $|\mathcal{F}_{i\alpha}| = 3$ for all $\alpha \in \mathcal{I}\backslash(\{i, -i\} \cup \mathcal{J})$.

Consider the existence of exactly three elements $\alpha \in \mathcal{J}$ satisfying $|\mathcal{F}_{i\alpha}| = 2$. Without loss of generality, let $|\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = |\mathcal{F}_{i\delta}| = 2$. Thus, $|\mathcal{F}_{i\varepsilon}| = 1$. Since $\sum_{\alpha \in \mathcal{J}} |\mathcal{F}_{i\alpha}| = 7$, taking into account (5.2) we conclude that $\sum_{\alpha \in \mathcal{I}\backslash(\{i,-i\}\cup\mathcal{J})} |\mathcal{F}_{i\alpha}| = 23$ and, consequently, there are seven elements $\alpha \in \mathcal{I}\backslash(\{i, -i\} \cup \mathcal{J})$ satisfying $|\mathcal{F}_{i\alpha}| = 3$.

If $|\mathcal{F}_{i\alpha}| = 2$ for any $\alpha \in \mathcal{J}$, then $\sum_{\alpha \in \mathcal{J}} |\mathcal{F}_{i\alpha}| = 8$ and, by (5.2), we conclude that there are, at least, six elements $\alpha \in \mathcal{I}\backslash(\{i, -i\} \cup \mathcal{J})$ so that $|\mathcal{F}_{i\alpha}| = 3$. □

Next, we derive equivalent results considering now $|\mathcal{J}| = 5$.

Proposition 5.5 *If $|\mathcal{J}| = 5$, there exists $x \in \mathcal{I} \backslash (\{i, -i\} \cup \mathcal{J})$ such that $|\mathcal{G}_{ix}| = 0$. Furthermore, $|\mathcal{G}_{i\alpha}| = 1$ for any $\alpha \in \mathcal{I} \backslash (\{i, -i, x\} \cup \mathcal{J})$.*

Proof. Let us suppose $|\mathcal{J}| = 5$. By definition of $\mathcal{J}$ we have $|\mathcal{G}_{i\alpha}| = 2$ for any $\alpha \in \mathcal{J}$. Thus, $\sum_{\alpha \in \mathcal{J}} |\mathcal{G}_{i\alpha}| = 10$. Since, by assumption, $\sum_{\alpha \in \mathcal{I} \backslash \{i,-i\}} |\mathcal{G}_{i\alpha}| = 16$, then $\sum_{\alpha \in \mathcal{I} \backslash (\{i,-i\} \cup \mathcal{J})} |\mathcal{G}_{i\alpha}| = 6$. Taking into account Proposition 5.1, if $\alpha \notin \mathcal{J}$ then $|\mathcal{G}_{i\alpha}| \leq 1$. As $|\mathcal{I} \backslash (\{i, -i\} \cup \mathcal{J})| = 7$, we conclude that there exists a unique element $x \in \mathcal{I} \backslash (\{i, -i\} \cup \mathcal{J})$ satisfying $|\mathcal{G}_{ix}| = 0$ and $|\mathcal{G}_{i\alpha}| = 1$ for all $\alpha \in \mathcal{I} \backslash (\{i, -i, x\} \cup \mathcal{J})$. $\square$

Since by Lemma 2.10 we have $|\mathcal{F}_i| = 11$ or $|\mathcal{F}_i| = 10$, the following two propositions give us conditions for the index distribution of the codewords of $\mathcal{F}_i$ when $|\mathcal{J}| = 5$ and $|\mathcal{F}_i|$ assumes each one of these values.

Proposition 5.6 *Let $|\mathcal{J}| = 5$ and $x \in \mathcal{I} \backslash (\{i, -i\} \cup \mathcal{J})$ be such that $|\mathcal{G}_{ix}| = 0$. If $|\mathcal{F}_i| = 11$, then:*

- *$|\mathcal{F}_{i\alpha}| = 2$ for any $\alpha \in \mathcal{J}$;*

- *$|\mathcal{F}_{ix}| = 5$;*

- *$|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I} \backslash (\{i, -i, x\} \cup \mathcal{J})$.*

Proof. If, by hypothesis, $|\mathcal{F}_i| = 11$, then $\sum_{\alpha \in \mathcal{I} \backslash \{i,-i\}} |\mathcal{F}_{i\alpha}| = 33$. That is,

$$\sum_{\alpha \in \mathcal{I} \backslash (\{i,-i,x\} \cup \mathcal{J})} |\mathcal{F}_{i\alpha}| + \sum_{\alpha \in \mathcal{J}} |\mathcal{F}_{i\alpha}| + |\mathcal{F}_{ix}| = 33. \tag{5.3}$$

From Lemma 2.2 it follows that $|\mathcal{F}_{i\alpha}| \leq 2$ for all $\alpha \in \mathcal{J}$ and $|\mathcal{F}_{ix}| \leq 5$. By Proposition 5.5, $|\mathcal{G}_{i\alpha}| = 1$ for all $\alpha \in \mathcal{I} \backslash (\{i, -i, x\} \cup \mathcal{J})$ and using Lemma 2.2 we get $|\mathcal{F}_{i\alpha}| \leq 3$ for any $\alpha \in \mathcal{I} \backslash (\{i, -i, x\} \cup \mathcal{J})$.

Consequently, by (5.3), we conclude that:

- $|\mathcal{F}_{i\alpha}| = 2$ for any $\alpha \in \mathcal{J}$;

- $|\mathcal{F}_{ix}| = 5$;

- $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i, x\} \cup \mathcal{J})$.

□

Proposition 5.7 *Let $|\mathcal{J}| = 5$, with $\mathcal{J} = \{\beta, \gamma, \delta, \varepsilon, \theta\}$, and $x \in \mathcal{I}\backslash(\{i, -i\} \cup \mathcal{J})$ such that $|\mathcal{G}_{ix}| = 0$. If $|\mathcal{F}_i| = 10$, then there are, at least, two elements $\alpha \in \mathcal{J}$ such that $|\mathcal{F}_{i\alpha}| = 2$. Furthermore:*

i) if $\beta, \gamma \in \mathcal{J}$ are the unique elements in $\mathcal{J}$ satisfying $|\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = 2$, then $|\mathcal{F}_{i\delta}| = |\mathcal{F}_{i\varepsilon}| = |\mathcal{F}_{i\theta}| = 1$, $|\mathcal{F}_{ix}| = 5$ and $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i, x\} \cup \mathcal{J})$;

ii) if $\beta, \gamma, \delta \in \mathcal{J}$ are the unique elements in $\mathcal{J}$ satisfying $|\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = |\mathcal{F}_{i\delta}| = 2$, then one of the following conditions must occurs:

- *$|\mathcal{F}_{ix}| = 5$ and $|\mathcal{F}_{i\alpha}| = 3$ for, at least, five elements $\alpha \in \mathcal{I}\backslash(\{i, -i, x\} \cup \mathcal{J})$;*
- *$|\mathcal{F}_{ix}| = 4$, $|\mathcal{F}_{i\varepsilon}| = |\mathcal{F}_{i\theta}| = 1$ and $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i, x\} \cup \mathcal{J})$;*

iii) if $\beta, \gamma, \delta, \varepsilon \in \mathcal{J}$ are the unique elements in $\mathcal{J}$ satisfying $|\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = |\mathcal{F}_{i\delta}| = |\mathcal{F}_{i\varepsilon}| = 2$, then one of the following conditions must occurs:

- *$|\mathcal{F}_{ix}| = 5$ and $|\mathcal{F}_{i\alpha}| = 3$ for, at least, four elements $\alpha \in \mathcal{I}\backslash(\{i, -i, x\} \cup \mathcal{J})$;*
- *$|\mathcal{F}_{ix}| = 4$ and $|\mathcal{F}_{i\alpha}| = 3$ for, at least, five elements $\alpha \in \mathcal{I}\backslash(\{i, -i, x\} \cup \mathcal{J})$;*
- *$|\mathcal{F}_{ix}| = 3$, $|\mathcal{F}_{i\theta}| = 1$ and $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i, x\} \cup \mathcal{J})$;*

iv) if $|\mathcal{F}_{i\alpha}| = 2$ for any $\alpha \in \mathcal{J}$, then one of the following conditions must occurs:

- *$|\mathcal{F}_{ix}| = 5$ and $|\mathcal{F}_{i\alpha}| = 3$ for, at least, three elements $\alpha \in \mathcal{I}\backslash(\{i, -i, x\} \cup \mathcal{J})$;*
- *$|\mathcal{F}_{ix}| = 4$ and $|\mathcal{F}_{i\alpha}| = 3$ for, at least, four elements $\alpha \in \mathcal{I}\backslash(\{i, -i, x\} \cup \mathcal{J})$;*
- *$|\mathcal{F}_{ix}| = 3$ and $|\mathcal{F}_{i\alpha}| = 3$ for, at least, five elements $\alpha \in \mathcal{I}\backslash(\{i, -i, x\} \cup \mathcal{J})$;*
- *$|\mathcal{F}_{ix}| = 2$ and $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i, x\} \cup \mathcal{J})$.*

Proof. Since $|\mathcal{F}_i| = 10$, then $\sum_{\alpha\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{F}_{i\alpha}| = 30$. Consequently,

$$\sum_{\alpha\in\mathcal{I}\backslash(\{i,-i,x\}\cup\mathcal{J})} |\mathcal{F}_{i\alpha}| + \sum_{\alpha\in\mathcal{J}} |\mathcal{F}_{i\alpha}| + |\mathcal{F}_{ix}| = 30. \tag{5.4}$$

Note that, from Proposition 5.5 we get $|\mathcal{G}_{i\alpha}| = 1$ for all $\alpha \in \mathcal{I}\backslash(\{i,-i,x\} \cup \mathcal{J})$. Furthermore, $|\mathcal{I}\backslash(\{i,-i,x\} \cup \mathcal{J})| = 6$.

By Lemma 2.2, it follows that:

$$|\mathcal{F}_{i\alpha}| \leq 2 \text{ for all } \alpha \in \mathcal{J};\ \ |\mathcal{F}_{ix}| \leq 5;\ \ |\mathcal{F}_{i\alpha}| \leq 3 \text{ for all } \alpha \in \mathcal{I}\backslash(\{i,-i,x\} \cup \mathcal{J}). \tag{5.5}$$

Taking into account (5.5), $\sum_{\alpha\in\mathcal{I}\backslash(\{i,-i,\}\cup\mathcal{J})} |\mathcal{F}_{i\alpha}| \leq 23$. Consequently, considering (5.4), $\sum_{\alpha\in\mathcal{J}} |\mathcal{F}_{i\alpha}| \geq 7$. Thus, as $|\mathcal{J}| = 5$, there are, at least, two elements in $\mathcal{J}$ satisfying $|\mathcal{F}_{i\alpha}| = 2$.

Consider $\mathcal{J} = \{\beta, \gamma, \delta, \varepsilon, \theta\}$. Suppose that $\beta, \gamma \in \mathcal{J}$ are the unique elements in $\mathcal{J}$ satisfying $|\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = 2$. Therefore, $|\mathcal{F}_{i\delta}|$, $|\mathcal{F}_{i\varepsilon}|$, $|\mathcal{F}_{i\theta}| \leq 1$. Taking into account (5.4) and (5.5) we must impose:

- $|\mathcal{F}_{i\delta}| = |\mathcal{F}_{i\varepsilon}| = |\mathcal{F}_{i\theta}| = 1$;
- $|\mathcal{F}_{ix}| = 5$;
- $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I}\backslash(\{i,-i,x\} \cup \mathcal{J})$.

Now consider that β, γ and δ are the unique elements in $\mathcal{J}$ such that $|\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = |\mathcal{F}_{i\delta}| = 2$. Taking into account (5.4) and (5.5) we must impose $4 \leq |\mathcal{F}_{ix}| \leq 5$, otherwise, $\sum_{\alpha\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{F}_{i\alpha}| \leq 29$, which is a contradiction. So, we will distinguish the cases: $|\mathcal{F}_{ix}| = 5$ and $|\mathcal{F}_{ix}| = 4$.

If $|\mathcal{F}_{ix}| = 5$, then $\sum_{\alpha\in\mathcal{J}} |\mathcal{F}_{i\alpha}| + |\mathcal{F}_{ix}| \leq 13$ and, by (5.4), we conclude that $\sum_{\alpha\in\mathcal{I}\backslash(\{i,-i,x\}\cup\mathcal{J})} |\mathcal{F}_{i\alpha}| \geq 17$. As $|\mathcal{I}\backslash(\{i,-i,x\} \cup \mathcal{J})| = 6$ and $|\mathcal{F}_{i\alpha}| \leq 3$ for all $\alpha \in \mathcal{I}\backslash(\{i,-i,x\}\cup\mathcal{J})$, there are, at least, five elements $\alpha \in \mathcal{I}\backslash(\{i,-i,x\}\cup\mathcal{J})$ satisfying $|\mathcal{F}_{i\alpha}| = 3$.

Supposing $|\mathcal{F}_{ix}| = 4$, then $\sum_{\alpha\in\mathcal{I}\backslash(\{i,-i,x\}\cup\mathcal{J})} |\mathcal{F}_{i\alpha}| + \sum_{\alpha\in\mathcal{J}} |\mathcal{F}_{i\alpha}| = 26$. Considering (5.5), we must impose $|\mathcal{F}_{i\varepsilon}| = |\mathcal{F}_{i\theta}| = 1$ and $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I}\backslash(\{i,-i,x\} \cup \mathcal{J})$.

Let us assume that $|\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = |\mathcal{F}_{i\delta}| = |\mathcal{F}_{i\varepsilon}| = 2$. In these conditions, taking into account (5.4) and (5.5), $\sum_{\alpha \in \mathcal{I}\backslash(\{i,-i,x\}\cup\mathcal{J})} |\mathcal{F}_{i\alpha}| + \sum_{\alpha\in\mathcal{J}} |\mathcal{F}_{i\alpha}| \leq 27$. Accordingly, $3 \leq |\mathcal{F}_{ix}| \leq 5$.

If $|\mathcal{F}_{ix}| = 5$, as $\sum_{\alpha\in\mathcal{J}} |\mathcal{F}_{i\alpha}| \leq 9$, by (5.4) it follows that $\sum_{\alpha \in \mathcal{I}\backslash(\{i,-i,x\}\cup\mathcal{J})} |\mathcal{F}_{i\alpha}| \geq 16$. Consequently, there are, at least four elements $\alpha \in \mathcal{I}\backslash(\{i,-i,x\}\cup\mathcal{J})$ so that $|\mathcal{F}_{i\alpha}| = 3$.

Following a similar reasoning we get:

- if $|\mathcal{F}_{ix}| = 4$, there are, at least, five elements $\alpha \in \mathcal{I}\backslash(\{i,-i,x\}\cup\mathcal{J})$ such that $|\mathcal{F}_{i\alpha}| = 3$;

- if $|\mathcal{F}_{ix}| = 3$, then $|\mathcal{F}_{i\theta}| = 1$ and $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I}\backslash(\{i,-i,x\}\cup\mathcal{J})$.

Now consider $|\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = |\mathcal{F}_{i\delta}| = |\mathcal{F}_{i\varepsilon}| = |\mathcal{F}_{i\theta}| = 2$. By a similar reasoning to the one applied in the previous cases, we conclude in these conditions that $2 \leq |\mathcal{F}_{ix}| \leq 5$ and one of the following conditions must be satisfied:

- if $|\mathcal{F}_{ix}| = 5$, there are, at least, three elements $\alpha \in \mathcal{I}\backslash(\{i,-i,x\}\cup\mathcal{J})$ such that $|\mathcal{F}_{i\alpha}| = 3$;

- if $|\mathcal{F}_{ix}| = 4$, there are, at least, four elements $\alpha \in \mathcal{I}\backslash(\{i,-i,x\}\cup\mathcal{J})$ such that $|\mathcal{F}_{i\alpha}| = 3$;

- if $|\mathcal{F}_{ix}| = 3$, there are, at least, five elements $\alpha \in \mathcal{I}\backslash(\{i,-i,x\}\cup\mathcal{J})$ such that $|\mathcal{F}_{i\alpha}| = 3$;

- if $|\mathcal{F}_{ix}| = 2$, then $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I}\backslash(\{i,-i,x\}\cup\mathcal{J})$.

□

The last results of this section are devoted to the characterization of the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ when $|\mathcal{J}| = 6$.

Proposition 5.8 *If* $|\mathcal{J}| = 6$*, then there exist* $x, y \in \mathcal{I}\backslash(\{i,-i\}\cup\mathcal{J})$ *such that* $|\mathcal{G}_{ix}| = |\mathcal{G}_{iy}| = 0$*. Furthermore,* $|\mathcal{G}_{i\alpha}| = 1$ *for any* $\alpha \in \mathcal{I}\backslash(\{i,-i,x,y\}\cup\mathcal{J})$.

Proof. Suppose that $|\mathcal{J}| = 6$. Since $\sum_{\alpha \in \mathcal{I}\setminus\{i,-i\}} |\mathcal{G}_{i\alpha}| = 16$ and, by assumption, $\sum_{\alpha \in \mathcal{J}} |\mathcal{G}_{i\alpha}| = 12$, then $\sum_{\alpha \in \mathcal{I}\setminus(\{i,-i\}\cup\mathcal{J})} |\mathcal{G}_{i\alpha}| = 4$. By Proposition 5.1, $|\mathcal{G}_{i\alpha}| \leq 2$ for any $\alpha \in \mathcal{I}\setminus\{i,-i\}$, thus $|\mathcal{G}_{i\alpha}| \leq 1$ for any $\alpha \in \mathcal{I}\setminus(\{i,-i\}\cup\mathcal{J})$. As $|\mathcal{I}\setminus(\{i,-i\}\cup\mathcal{J})| = 6$, we conclude that there exist exactly four elements $\alpha \in \mathcal{I}\setminus(\{i,-i\}\cup\mathcal{J})$ satisfying $|\mathcal{G}_{i\alpha}| = 1$ and, on the other hand, there are $x, y \in \mathcal{I}\setminus(\{i,-i\}\cup\mathcal{J})$ so that $|\mathcal{G}_{ix}| = |\mathcal{G}_{iy}| = 0$. □

Such as it was done for the assumption $|\mathcal{J}| = 5$, the following propositions present conditions for the index distribution of the codewords of $\mathcal{F}_i$ when $|\mathcal{F}_i| = 11$ and $|\mathcal{F}_i| = 10$, respectively, assuming now that $|\mathcal{J}| = 6$.

Proposition 5.9 *Let $|\mathcal{J}| = 6$, with $\mathcal{J} = \{\beta,\gamma,\delta,\varepsilon,\theta,\mu\}$, and $x, y \in \mathcal{I}\setminus(\{i,-i\}\cup\mathcal{J})$ such that $|\mathcal{G}_{ix}| = |\mathcal{G}_{iy}| = 0$. If $|\mathcal{F}_i| = 11$, then there are, at least, five elements $\alpha \in \mathcal{J}$ satisfying $|\mathcal{F}_{i\alpha}| = 2$. Furthermore, if there exist exactly five elements in these conditions, then:*

- $|\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = |\mathcal{F}_{i\delta}| = |\mathcal{F}_{i\varepsilon}| = |\mathcal{F}_{i\theta}| = 2$;
- $|\mathcal{F}_{i\mu}| = 1$;
- $|\mathcal{F}_{ix}| = |\mathcal{F}_{iy}| = 5$;
- $|\mathcal{F}_{i\alpha}| = 3$ *for any* $\alpha \in \mathcal{I}\setminus(\{i,-i,x,y\}\cup\mathcal{J})$.

Proof. Suppose that $|\mathcal{J}| = 6$ with $\mathcal{J} = \{\beta,\gamma,\delta,\varepsilon,\theta,\mu\}$. By Proposition 5.8 we know that there are $x, y \in \mathcal{I}\setminus(\{i,-i\}\cup\mathcal{J})$ so that $|\mathcal{G}_{ix}| = |\mathcal{G}_{iy}| = 0$, furthermore, $|\mathcal{G}_{i\alpha}| = 1$ for any $\alpha \in \mathcal{I}\setminus(\{i,-i,x,y\}\cup\mathcal{J})$, with $|\mathcal{I}\setminus(\{i,-i,x,y\}\cup\mathcal{J})| = 4$. Accordingly, from Lemma 2.2 it follows that:

- $|\mathcal{F}_{i\alpha}| \leq 2$ for any $\alpha \in \mathcal{J}$;
- $|\mathcal{F}_{ix}|, |\mathcal{F}_{iy}| \leq 5$;
- $|\mathcal{F}_{i\alpha}| \leq 3$ for any $\alpha \in \mathcal{I}\setminus(\{i,-i,x,y\}\cup\mathcal{J})$.

By assumption, $|\mathcal{F}_i| = 11$, then $\sum_{\alpha \in \mathcal{I}\setminus\{i,-i\}} |\mathcal{F}_{i\alpha}| = 33$. Taking into account what was said before, $\sum_{\alpha \in \mathcal{I}\setminus(\{i,-i\}\cup\mathcal{J})} |\mathcal{F}_{i\alpha}| \leq 22$, then $\sum_{\alpha \in \mathcal{J}} |\mathcal{F}_{i\alpha}| \geq 11$. Consequently, there are, at least, five elements $\alpha \in \mathcal{J}$ satisfying $|\mathcal{F}_{i\alpha}| = 2$.

If there exist exactly five elements $\alpha \in \mathcal{J}$ so that $|\mathcal{F}_{i\alpha}| = 2$, then we must impose:

– $|\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = |\mathcal{F}_{i\delta}| = |\mathcal{F}_{i\varepsilon}| = |\mathcal{F}_{i\theta}| = 2$;

– $|\mathcal{F}_{i\mu}| = 1$;

– $|\mathcal{F}_{ix}| = |\mathcal{F}_{iy}| = 5$;

– $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i, x, y\} \cup \mathcal{J})$.

□

Proposition 5.10 *Let $|\mathcal{J}| = 6$, with $\mathcal{J} = \{\beta, \gamma, \delta, \varepsilon, \theta, \mu\}$, and $x, y \in \mathcal{I}\backslash(\{i, -i\} \cup \mathcal{J})$ be such that $|\mathcal{G}_{ix}| = |\mathcal{G}_{iy}| = 0$. If $|\mathcal{F}_i| = 10$, then there are, at least, three elements $\alpha \in \mathcal{J}$ satisfying $|\mathcal{F}_{i\alpha}| = 2$. Furthermore:*

i) if $\beta, \gamma, \delta \in \mathcal{J}$ are the unique elements in $\mathcal{J}$ satisfying $|\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = |\mathcal{F}_{i\delta}| = 2$, then $|\mathcal{F}_{i\varepsilon}| = |\mathcal{F}_{i\theta}| = |\mathcal{F}_{i\mu}| = 1$, $|\mathcal{F}_{ix}| = 5$, $|\mathcal{F}_{iy}| = 4$ and $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i, x, y\} \cup \mathcal{J})$;

ii) if $\beta, \gamma, \delta, \varepsilon \in \mathcal{J}$ are the unique elements in $\mathcal{J}$ satisfying $|\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = |\mathcal{F}_{i\delta}| = |\mathcal{F}_{i\varepsilon}| = 2$, then $|\mathcal{F}_{ix}| + |\mathcal{F}_{iy}| \geq 8$; if $|\mathcal{F}_{ix}| + |\mathcal{F}_{iy}| = 8$, then $|\mathcal{F}_{i\theta}| = |\mathcal{F}_{i\mu}| = 1$ and $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i, x, y\} \cup \mathcal{J})$;

iii) if $\beta, \gamma, \delta, \varepsilon, \theta \in \mathcal{J}$ are the unique elements in $\mathcal{J}$ satisfying $|\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = |\mathcal{F}_{i\delta}| = |\mathcal{F}_{i\varepsilon}| = |\mathcal{F}_{i\theta}| = 2$, then $|\mathcal{F}_{ix}| + |\mathcal{F}_{iy}| \geq 7$; if $|\mathcal{F}_{ix}| + |\mathcal{F}_{iy}| = 7$, then $|\mathcal{F}_{i\mu}| = 1$ and $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i, x, y\} \cup \mathcal{J})$;

iv) if $|\mathcal{F}_{i\alpha}| = 2$ for any $\alpha \in \mathcal{J}$, then $|\mathcal{F}_{ix}| + |\mathcal{F}_{iy}| \geq 6$; if $|\mathcal{F}_{ix}| + |\mathcal{F}_{iy}| = 6$, then $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i, x, y\} \cup \mathcal{J})$.

Proof. Consider $|\mathcal{J}| = 6$, with $\mathcal{J} = \{\beta, \gamma, \delta, \varepsilon, \theta, \mu\}$. Let $x, y \in \mathcal{I}\backslash(\{i, -i\} \cup \mathcal{J})$ satisfying $|\mathcal{G}_{ix}| = |\mathcal{G}_{iy}| = 0$.

By assumption, $|\mathcal{F}_i| = 10$, that is, $\sum_{\alpha \in \mathcal{I}\backslash\{i,-i\}} |\mathcal{F}_{i\alpha}| = 30$. Accordingly,

$$\sum_{\alpha \in \mathcal{I}\backslash(\{i,-i,x,y\} \cup \mathcal{J})} |\mathcal{F}_{i\alpha}| + \sum_{\alpha \in \mathcal{J}} |\mathcal{F}_{i\alpha}| + |\mathcal{F}_{ix}| + |\mathcal{F}_{iy}| = 30. \tag{5.6}$$

From Proposition 5.8 it follows that $|\mathcal{G}_{i\alpha}| = 1$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i, x, y\} \cup \mathcal{J})$. Taking into account Lemma 2.2 we get:

$$|\mathcal{F}_{i\alpha}| \leq 2 \text{ for any } \alpha \in \mathcal{J}; \ |\mathcal{F}_{ix}|, |\mathcal{F}_{iy}| \leq 5; \ |\mathcal{F}_{i\alpha}| \leq 3 \text{ for any } \alpha \in \mathcal{I}\backslash(\{i, -i, x, y\} \cup \mathcal{J}). \tag{5.7}$$

Let us begin by proving that it is not possible to have $|\mathcal{F}_{ix}| = |\mathcal{F}_{iy}| = 5$. Suppose, by contradiction, that $|\mathcal{F}_{ix}| = |\mathcal{F}_{iy}| = 5$. By Lemma 1.5, there exists, at most, one codeword $U \in \mathcal{F}_{ix} \cap \mathcal{F}_{iy}$. So, $|\mathcal{F}_{ix} \cup \mathcal{F}_{iy}| \geq 9$ and, consequently, $|\mathcal{F}_i\backslash(\mathcal{F}_{ix} \cup \mathcal{F}_{iy})| \leq 1$. That is, $\mathcal{F}_i$ satisfies one of the following conditions:

i) $\mathcal{F}_i = \mathcal{F}_{ix} \cup \mathcal{F}_{iy} \cup \mathcal{F}_{ipqr}$, with $p, q, r \in \mathcal{I}\backslash\{i, -i, x, y\}$;

ii) $\mathcal{F}_i = \mathcal{F}_{ix} \cup \mathcal{F}_{iy}$.

Since $|\mathcal{F}_i| = 10$, then by Lemma 2.10 we get $|\mathcal{F}_i^{(2)}| = 4$. However, taking into account the hypotheses *i)* and *ii)* we conclude that Lemma 2.14 is not satisfied. Therefore, there exists, at most, one element $\alpha \in \{x, y\}$ satisfying $|\mathcal{F}_{i\alpha}| = 5$. Consequently, $\sum_{\alpha \in \mathcal{I}\backslash(\{i,-i,x,y\} \cup \mathcal{J})} |\mathcal{F}_{i\alpha}| + |\mathcal{F}_{ix}| + |\mathcal{F}_{iy}| \leq 21$ and $\sum_{\alpha \in \mathcal{J}} |\mathcal{F}_{i\alpha}| \geq 9$, which implies the existence of, at least, three elements $\alpha \in \mathcal{J}$ satisfying $|\mathcal{F}_{i\alpha}| = 2$.

Next, we will verify what happens when we consider the existence of three, four, five and six elements in $\mathcal{J}$ in these conditions.

Suppose, without loss of generality, that β, γ, δ are the unique elements in $\mathcal{J}$ such that $|\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = |\mathcal{F}_{i\delta}| = 2$. In these conditions, taking into account (5.6) and (5.7) we must impose:

- $|\mathcal{F}_{i\varepsilon}| = |\mathcal{F}_{i\theta}| = |\mathcal{F}_{i\mu}| = 1$;
- $|\mathcal{F}_{ix}| = 5$ and $|\mathcal{F}_{iy}| = 4$;
- $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i, x, y\} \cup \mathcal{J})$.

Now consider that there are exactly four elements $\alpha \in \mathcal{J}$ so that $|\mathcal{F}_{i\alpha}| = 2$. That is, $|\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = |\mathcal{F}_{i\delta}| = |\mathcal{F}_{i\varepsilon}| = 2$. In these conditions, taking into account (5.6) and (5.7), we have $|\mathcal{F}_{ix}| + |\mathcal{F}_{iy}| \geq 8$. If $|\mathcal{F}_{ix}| + |\mathcal{F}_{iy}| = 8$, then $|\mathcal{F}_{i\theta}| = |\mathcal{F}_{i\mu}| = 1$ and $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i, x, y\} \cup \mathcal{J})$.

The rest of the proposition follows applying the same reasoning. □

5.2 Index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$

In the previous section we have derived necessary conditions on the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$, $i \in \mathcal{I}$, for the existence of PL(7, 2) codes. Here, we will apply such results, describing a process which allow us to get all possible index distributions for the referred codewords. We note that, were obtained many distinct possible characterizations for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ being presented here only some of them.

We begin by showing how we obtain all possible index distributions for the codewords of $\mathcal{G}_i$. In the last part of this section, considering certain index distributions for the codewords $W_1, \ldots, W_4 \in \mathcal{G}_i$, we exemplify how we get the respective possible index distributions for the codewords of $\mathcal{F}_i$.

5.2.1 Index distribution of the codewords of $\mathcal{G}_i$

Let us consider $W_1, \ldots, W_4 \in \mathcal{G}_i$. By Proposition 5.1 we know that $|\mathcal{G}_{i\alpha}| \leq 2$ for any $\alpha \in \mathcal{I}\backslash\{i, -i\}$. Furthermore, from Proposition 5.2 it follows that $4 \leq |\mathcal{J}| \leq 6$, where $\mathcal{J} = \{\alpha \in \mathcal{I}\backslash\{i, -i\} : |\mathcal{G}_{i\alpha}| = 2\}$.

We intend to describe a methodology to get all possible index distributions for $W_1, \ldots, W_4 \in \mathcal{G}_i$. For that, we will analyze, separately, the cases: $|\mathcal{J}| = 4$, $|\mathcal{J}| = 5$ and $|\mathcal{J}| = 6$.

- **Index distribution of the codewords of $\mathcal{G}_i$ considering $|\mathcal{J}| = 4$**

If $|\mathcal{J}| = 4$, this means that there are exactly four elements in $\mathcal{I}\backslash\{i, -i\}$ being, each one of them, "shared" by two codewords of $\mathcal{G}_i$. The following schemes traduce this idea and help us in the identification of possible structures for the index distributions of the codewords of $\mathcal{G}_i$.

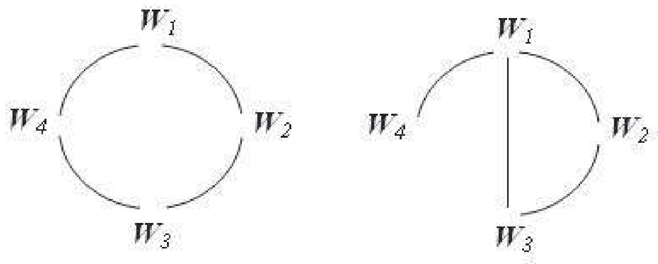

Figure 5.1: Possible structures for the index distribution of the codewords of $\mathcal{G}_i$.

That is, considering $\mathcal{J} = \{\beta, \gamma, \delta, \varepsilon\}$, there are two possible structures for the index distribution of the codewords of $\mathcal{G}_i$:

W_1	i	β	γ	w_1	w_2
W_2	i	β	δ	w_3	w_4
W_3	i	δ	ε	w_5	w_6
W_4	i	γ	ε	w_7	w_8

W_1	i	β	γ	δ	w_1
W_2	i	β	ε	w_2	w_3
W_3	i	γ	ε	w_4	w_5
W_4	i	δ	w_6	w_7	w_8

Table 5.3: Possible structures for the index distribution of the codewords of $\mathcal{G}_i$.

Considering $\mathcal{I} = \{i, -i, j, -j, k, -k, l, -l, m, -m, n, -n, o, -o\}$, our aim is to concretize the possible index distributions for $W_1, \ldots, W_4 \in \mathcal{G}_i$.

We begin by considering that the codewords $W_1, \ldots, W_4 \in \mathcal{G}_i$ satisfy:

W_1	i	β	γ	w_1	w_2
W_2	i	β	δ	w_3	w_4
W_3	i	δ	ε	w_5	w_6
W_4	i	γ	ε	w_7	w_8

Table 5.4: Index distribution structure of $W_1, \ldots, W_4 \in \mathcal{G}_i$.

where $\beta, \gamma, \delta, \varepsilon, w_1, \ldots, w_8 \in \mathcal{I} \backslash \{i, -i\}$ are pairwise distinct.

The following results will be useful in the characterization of these codewords.

Proposition 5.11 *Let $|\mathcal{J}| = 4$, with $\mathcal{J} = \{\beta, \gamma, \delta, \varepsilon\}$, and $W_1, \ldots, W_4 \in \mathcal{G}_i$ satisfying:*

W_1	i	β	γ	w_1	w_2
W_2	i	β	δ	w_3	w_4
W_3	i	δ	ε	w_5	w_6
W_4	i	γ	ε	w_7	w_8

where $\beta, \gamma, \delta, \varepsilon, w_1, \ldots, w_8 \in \mathcal{I}\backslash\{i, -i\}$ are pairwise distinct. If $x, y \in \mathcal{J}$ are such that $|\mathcal{G}_{ixy}| = 0$ and $|x| \neq |y|$, then $|\mathcal{F}_{ix}| \neq 2$ and $|\mathcal{F}_{iy}| \neq 2$.

Proof. Let $W_1, \ldots, W_4 \in \mathcal{G}_i$ satisfying the stated conditions. Without loss of generality, consider β and ε so that $|\mathcal{G}_{i\beta\varepsilon}| = 0$. By contradiction, suppose that $|\beta| \neq |\varepsilon|$ and $|\mathcal{F}_{i\beta}| = 2$.

Let us consider $U_1, U_2 \in \mathcal{F}_{i\beta}$ so that

U_1	i	β	u_1	u_2
U_2	i	β	u_3	u_4

Table 5.5: Partial index distribution of $U_1, U_2 \in \mathcal{F}_{i\beta}$.

with $u_1, \ldots, u_4 \in \mathcal{I}\backslash\{i, -i, \beta, -\beta\}$. By Lemma 1.5, $u_1, \ldots, u_4$ must be pairwise distinct. Note that, $\mathcal{I}\backslash\{i, -i\} = \{\beta, \gamma, \delta, \varepsilon, w_1, \ldots, w_8\}$. Taking into account the codewords $W_1, W_2 \in \mathcal{G}_{i\beta}$ and Lemma 1.5 we must impose $u_1, \ldots, u_4 \in \{\varepsilon, w_5, \ldots, w_8\}$. Since $-\beta \in \{\varepsilon, w_5, \ldots, w_8\}$ and, by assumption, $-\beta \neq \varepsilon$, then $-\beta \in \{w_5, \ldots, w_8\}$. Therefore, without loss of generality, $U_1 \in \mathcal{F}_{i\beta\varepsilon u}$, with $u \in \{w_5, \ldots, w_8\}$. Consequently, considering $W_3, W_4 \in \mathcal{G}_{i\varepsilon}$ we conclude that Lemma 1.5 is contradicted. □

Proposition 5.12 *Let $|\mathcal{J}| = 4$, with $\mathcal{J} = \{\beta, \gamma, \delta, \varepsilon\}$, and $W_1, \ldots, W_4 \in \mathcal{G}_i$ satisfying:*

W_1	i	β	γ	w_1	w_2
W_2	i	β	δ	w_3	w_4
W_3	i	δ	ε	w_5	w_6
W_4	i	γ	ε	w_7	w_8

where $\beta, \gamma, \delta, \varepsilon, w_1, \ldots, w_8 \in \mathcal{I}\backslash\{i, -i\}$ are pairwise distinct. In these conditions, there exist $x, y \in \mathcal{J}$ such that $|\mathcal{G}_{ixy}| = 0$ and $x = -y$.

Proof. Let us consider $W_1, \ldots, W_4 \in \mathcal{G}_i$ satisfying the stated conditions. By contradiction, suppose that for any $x, y \in \mathcal{J}$ satisfying $|\mathcal{G}_{ixy}| = 0$ we have $x \neq -y$. We note that, for each $x \in \mathcal{J}$ there exists a unique $y \in \mathcal{J}$ so that $|\mathcal{G}_{ixy}| = 0$. Then, taking into account Proposition 5.11, we conclude that $|\mathcal{F}_{i\alpha}| \neq 2$ for all $\alpha \in \mathcal{J}$, which contradicts Proposition 5.4. □

Considering $W_1, \ldots, W_4 \in \mathcal{G}_i$ satisfying the conditions in Table 5.4 and taking into account Proposition 5.12, there are two possible hypotheses:

i) $\beta = -\varepsilon$ and $\gamma \neq -\delta$;

ii) $\beta = -\varepsilon$ and $\gamma = -\delta$.

Without loss of generality, let us suppose $W_1 \in \mathcal{G}_{ijklm}$. Then, if *i*) or *ii*) are satisfied, we get, respectively:

W_1	i	j	k	l	m
W_2	i	j	δ	w_1	w_2
W_3	i	δ	$-j$	w_3	w_4
W_4	i	k	$-j$	w_5	w_6

Table 5.6: $\mathcal{G}_i$ satisfies *i*).

W_1	i	j	k	l	m
W_2	i	j	$-k$	w_1	w_2
W_3	i	$-k$	$-j$	w_3	w_4
W_4	i	k	$-j$	w_5	w_6

Table 5.7: $\mathcal{G}_i$ satisfies *ii*).

If $W_1, \ldots, W_4 \in \mathcal{G}_i$ verify the conditions listed in Table 5.6, then, up to equivalent cases, $\delta = -l$ or $\delta = n$. That is:

W_1	i	j	k	l	m
W_2	i	j	$-l$	w_1	w_2
W_3	i	$-l$	$-j$	w_3	w_4
W_4	i	k	$-j$	w_5	w_6

W_1	i	j	k	l	m
W_2	i	j	n	w_1	w_2
W_3	i	n	$-j$	w_3	w_4
W_4	i	k	$-j$	w_5	w_6

Table 5.8: Codewords of $\mathcal{G}_i$ satisfying $i)$ with $\delta = -l$ or $\delta = n$, respectively.

Therefore, in Tables 5.7 and 5.8 are listed all possibilities for the elements of $\mathcal{J}$ when the codewords of $\mathcal{G}_i$ satisfy the conditions in Table 5.4.

Now consider that $W_1, \ldots, W_4 \in \mathcal{G}_i$ satisfy:

W_1	i	β	γ	δ	w_1
W_2	i	β	ε	w_2	w_3
W_3	i	γ	ε	w_4	w_5
W_4	i	δ	w_6	w_7	w_8

Table 5.9: Index distribution structure of $W_1, \ldots, W_4 \in \mathcal{G}_i$.

where $\beta, \gamma, \delta, \varepsilon, w_1, \ldots, w_8 \in \mathcal{I} \backslash \{i, -i\}$ are pairwise distinct.

As in the previous case, we begin by presenting a result which will help us to characterize the different hypotheses for the indices $\beta, \gamma, \delta, \varepsilon \in \mathcal{J}$.

Proposition 5.13 *If $|\mathcal{J}| = 4$, with $\mathcal{J} = \{\beta, \gamma, \delta, \varepsilon\}$, and the codewords of $\mathcal{G}_i$ satisfy*

W_1	i	β	γ	δ	w_1
W_2	i	β	ε	w_2	w_3
W_3	i	γ	ε	w_4	w_5
W_4	i	δ	w_6	w_7	w_8

with $\beta, \gamma, \delta, \varepsilon, w_1, \ldots, w_8 \in \mathcal{I} \backslash \{i, -i\}$ pairwise distinct, then $|\mathcal{F}_{i\varepsilon}| \leq 1$ and *i)* $|\mathcal{F}_{i\beta}| = 2$ *implies* $W_4 \in \mathcal{G}_{-\beta}$*;*

ii) $|\mathcal{F}_{i\gamma}| = 2$ *implies* $W_4 \in \mathcal{G}_{-\gamma}$*;*

iii) $|\mathcal{F}_{i\delta}| = 2$ *implies* $\delta = -\varepsilon$*.*

Proof. Let us consider that $W_1, \ldots, W_4 \in \mathcal{G}_i$ satisfy the stated conditions. We note that, any $\alpha \in \mathcal{J}$ is such that $|\mathcal{G}_{i\alpha}| = 2$, then, by Lemma 2.2, $|\mathcal{F}_{i\alpha}| \leq 2$ for any $\alpha \in \mathcal{J}$.

Suppose that $|\mathcal{F}_{i\beta}| = 2$. Let $U_1, U_2 \in \mathcal{F}_{i\beta}$ so that $U_1 \in \mathcal{F}_{i\beta u_1 u_2}$ and $U_2 \in \mathcal{F}_{i\beta u_3 u_4}$, with $u_1,...,u_4 \in \mathcal{I}\backslash\{i, -i, \beta, -\beta\}$. We note that, $\mathcal{I}\backslash\{i, -i\} = \{\beta, \gamma, \delta, \varepsilon, w_1,...,w_8\}$. Considering $W_1, W_2 \in \mathcal{G}_{i\beta}$ and Lemma 1.5, we must impose $u_1,...,u_4 \in \{w_4,...,w_8\}$ pairwise distinct. Accordingly, in these conditions, $-\beta \in \{w_4, \ldots, w_8\}$. Therefore, $u_1, \ldots, u_4 \in \{w_4, \ldots, w_8\}\backslash\{-\beta\}$. If $-\beta \in \{w_4, w_5\}$, then W_4 and one of the codewords of $\mathcal{F}_{i\beta}$ contradict Lemma 1.5. Accordingly, $-\beta \in \{w_6, w_7, w_8\}$ and $W_4 \in \mathcal{G}_{-\beta}$.

Assuming $|\mathcal{F}_{i\gamma}| = 2$ we may conclude, by a similar reasoning, that $W_4 \in \mathcal{G}_{-\gamma}$.

Now consider that $|\mathcal{F}_{i\delta}| = 2$, with $U_1, U_2 \in \mathcal{F}_{i\delta}$ satisfying $U_1 \in \mathcal{F}_{i\delta u_1 u_2}$ and $U_2 \in \mathcal{F}_{i\delta u_3 u_4}$. Considering the codewords $W_1, W_4 \in \mathcal{G}_{i\delta}$ and Lemma 1.5 we must impose $u_1, \ldots, u_4 \in \{\varepsilon, w_2, \ldots, w_5\}$ pairwise distinct. Note that, $-\delta \in \{\varepsilon, w_2, \ldots, w_5\}$. If $\delta \neq -\varepsilon$, then one of the codewords in $\{W_2, W_3\}$ contradicts Lemma 1.5 with one of the codewords in $\mathcal{F}_{i\delta}$. Therefore, $\delta = -\varepsilon$.

Next, we prove that it is not possible to have $|\mathcal{F}_{i\varepsilon}| = 2$. Suppose, by contradiction, that $|\mathcal{F}_{i\varepsilon}| = 2$, with $U_1, U_2 \in \mathcal{F}_{i\varepsilon}$ so that $U_1 \in \mathcal{F}_{i\varepsilon u_1 u_2}$ and $U_2 \in \mathcal{F}_{i\varepsilon u_3 u_4}$, where $u_1, \ldots, u_4 \in \{\delta, w_1, w_6, w_7, w_8\}$ are pairwise distinct. Therefore, we have $-\varepsilon \in \{\delta, w_1, w_6, w_7, w_8\}$. If $-\varepsilon = \delta$, then W_4 and one of the codewords of $\mathcal{F}_{i\varepsilon}$ contradict Lemma 1.5. On the other hand, if $-\varepsilon \in \{w_1, w_6, w_7, w_8\}$, then there exists $U \in \mathcal{F}_{i\varepsilon\delta u}$, with $u \in \{w_1, w_6, w_7, w_8\}$, and Lemma 1.5 is once again contradicted. □

Since we are assuming $|\mathcal{J}| = 4$, by Proposition 5.4 there are, at least, two elements $\alpha \in \mathcal{J}$ satisfying $|\mathcal{F}_{i\alpha}| = 2$. Having into account the previous proposition, if $W_1, \ldots, W_4 \in \mathcal{G}_i$ satisfy the conditions listed in Table 5.9, then two of the three following conditions must be satisfied: $W_4 \in \mathcal{G}_{-\beta}$; $W_4 \in \mathcal{G}_{-\gamma}$; $\delta = -\varepsilon$.

Therefore, considering $W_1, \ldots, W_4 \in \mathcal{G}_i$ satisfying the conditions listed in Table 5.9 and assuming that $W_1 \in \mathcal{G}_{ijklm}$, from Propositions 5.4 and 5.13 we get, up to an equivalent index distribution, the following possible partial index distributions for the codewords of $\mathcal{G}_i$:

W_1	i	j	k	l	m
W_2	i	j	ε	w_1	w_2
W_3	i	k	ε	w_3	w_4
W_4	i	l	$-j$	$-k$	w_5

Table 5.10: $|\mathcal{F}_{ij}| = |\mathcal{F}_{ik}| = 2$.

W_1	i	j	k	l	m
W_2	i	j	$-l$	w_1	w_2
W_3	i	k	$-l$	w_3	w_4
W_4	i	l	$-j$	w_5	w_6

Table 5.11: $|\mathcal{F}_{ij}| = |\mathcal{F}_{il}| = 2$.

If the codewords of $\mathcal{G}_i$ satisfy the conditions in Table 5.10, then there are three distinct hypotheses for ε to be considered. Namely, $\varepsilon = -l$, $\varepsilon = -m$ or $\varepsilon = n$, see bellow.

W_1	i	j	k	l	m
W_2	i	j	$-l$	w_1	w_2
W_3	i	k	$-l$	w_3	w_4
W_4	i	l	$-j$	$-k$	w_5

W_1	i	j	k	l	m
W_2	i	j	$-m$	w_1	w_2
W_3	i	k	$-m$	w_3	w_4
W_4	i	l	$-j$	$-k$	w_5

W_1	i	j	k	l	m
W_2	i	j	n	w_1	w_2
W_3	i	k	n	w_3	w_4
W_4	i	l	$-j$	$-k$	w_5

Table 5.12: Partial index distributions for $W_1, \ldots, W_4 \in \mathcal{G}_i$.

Therefore, if $W_1, \ldots, W_4 \in \mathcal{G}_i$ satisfy the conditions listed in Table 5.9, then the elements of $\mathcal{J}$ satisfy one of the hypotheses presented in Tables 5.11 and 5.12.

We have presented a partial index distributions for the codewords of $\mathcal{G}_i$ when $|\mathcal{J}| = 4$. However, we are interested in their complete characterization. To show how we do it, we will consider one of the presented partial index distributions of the codewords of $\mathcal{G}_i$.

Let $W_1, \ldots, W_4 \in \mathcal{G}_i$ be so that:

W_1	i	j	k	l	m
W_2	i	j	$-l$	w_1	w_2
W_3	i	$-l$	$-j$	w_3	w_4
W_4	i	k	$-j$	w_5	w_6

Table 5.13: Partial index distribution of $W_1, \ldots, W_4 \in \mathcal{G}_i$

with $w_1, \ldots, w_6 \in \{-k, -m, n, -n, o, -o\}$ pairwise distinct. Focusing our attention on w_1 and w_2, we consider, up to an equivalent cases, the following hypotheses:

i) $w_1 = -k$ and $w_2 = -m$;

ii) $w_1 = -k$ and $w_2 = n$;

iii) $w_1 = -m$ and $w_2 = n$;

iv) $w_1 = n$ and $w_2 = o$.

If the conditions *i)* or *iii)* are satisfied, we get respectively:

W_1	i	j	k	l	m
W_2	i	j	$-l$	$-k$	$-m$
W_3	i	$-l$	$-j$	n	o
W_4	i	k	$-j$	$-n$	$-o$

Table 5.14: Hypothesis *i)*.

W_1	i	j	k	l	m
W_2	i	j	$-l$	$-m$	n
W_3	i	$-l$	$-j$	$-k$	o
W_4	i	k	$-j$	$-n$	$-o$

Table 5.15: Hypothesis *iii)*.

If $W_1, \ldots, W_4 \in \mathcal{G}_i$ satisfy the condition *ii)*, then we get the two following possible index distributions presented in Table 5.16.

W_1	i	j	k	l	m
W_2	i	j	$-l$	$-k$	n
W_3	i	$-l$	$-j$	$-m$	o
W_4	i	k	$-j$	$-n$	$-o$

W_1	i	j	k	l	m
W_2	i	j	$-l$	$-k$	n
W_3	i	$-l$	$-j$	$-n$	o
W_4	i	k	$-j$	$-m$	$-o$

Table 5.16: Hypothesis *ii*).

Now considering that the codewords of $\mathcal{G}_i$ verify the condition *iv*), we obtain the two following index distributions:

W_1	i	j	k	l	m
W_2	i	j	$-l$	n	o
W_3	i	$-l$	$-j$	$-k$	$-m$
W_4	i	k	$-j$	$-n$	$-o$

W_1	i	j	k	l	m
W_2	i	j	$-l$	n	o
W_3	i	$-l$	$-j$	$-k$	$-n$
W_4	i	k	$-j$	$-m$	$-o$

Table 5.17: Hypothesis *iv*).

We obtain all possible index distributions for the codewords of $\mathcal{G}_i$ when $|\mathcal{J}| = 4$ applying the same reasoning to each one of the other presented partial index distributions for the codewords $W_1, \ldots, W_4 \in \mathcal{G}_i$.

- **Index distribution of the codewords of $\mathcal{G}_i$ considering $|\mathcal{J}| = 5$**

Let us suppose that $W_1, \ldots, W_4 \in \mathcal{G}_i$ are such that $|\mathcal{J}| = 5$, with $\mathcal{J} = \{\beta, \gamma, \delta, \varepsilon, \theta\}$. In this case, considering the way of how the elements of $\mathcal{J}$ are "shared" by the codewords of $\mathcal{G}_i$, there exists only one possibility:

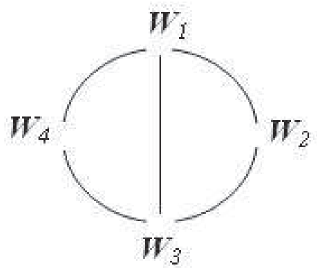

Figure 5.2: Structure for the index distribution of $W_1, \ldots, W_4 \in \mathcal{G}_i$.

That is, $W_1, \ldots, W_4 \in \mathcal{G}_i$ satisfy:

W_1	i	β	γ	δ	w_1
W_2	i	β	ε	w_2	w_3
W_3	i	γ	ε	θ	w_4
W_4	i	δ	θ	w_5	w_6

Table 5.18: Structure for the index distribution of $W_1, \ldots, W_4 \in \mathcal{G}_i$.

with $w_1, \ldots, w_6 \in \mathcal{I} \backslash \{i, -i, \beta, \gamma, \delta, \varepsilon, \theta\}$ and pairwise distinct.

Considering $\mathcal{I} = \{i, -i, j, -j, k, -k, l, -l, m, -m, n, -n, o, -o\}$ and $W_1 \in \mathcal{G}_{ijklm}$, we have, up to an equivalent case, five distinct hypotheses for the elements $\beta, \gamma, \delta, \varepsilon, \theta \in \mathcal{J}$:

W_1	i	j	k	l	m
W_2	i	j	$-l$	w_1	w_2
W_3	i	k	$-l$	$-j$	w_3
W_4	i	l	$-j$	w_4	w_5

W_1	i	j	k	l	m
W_2	i	j	$-m$	w_1	w_2
W_3	i	k	$-m$	$-j$	w_3
W_4	i	l	$-j$	w_4	w_5

W_1	i	j	k	l	m
W_2	i	j	n	w_1	w_2
W_3	i	k	n	$-j$	w_3
W_4	i	l	$-j$	w_4	w_5

W_1	i	j	k	l	m
W_2	i	j	n	w_1	w_2
W_3	i	k	n	$-m$	w_3
W_4	i	l	$-m$	w_4	w_5

W_1	i	j	k	l	m
W_2	i	j	o	w_1	w_2
W_3	i	k	o	n	w_3
W_4	i	l	n	w_4	w_5

Table 5.19: Partial index distributions of $W_1, \ldots, W_4 \in \mathcal{G}_i$.

As before, when we have considered $|\mathcal{J}| = 4$, we are interested in the complete characterization of the index distribution of the codewords of $\mathcal{G}_i$.

Let us consider, for example, $W_1, \ldots, W_4 \in \mathcal{G}_i$ satisfying:

W_1	i	j	k	l	m
W_2	i	j	$-l$	w_1	w_2
W_3	i	k	$-j$	$-l$	w_3
W_4	i	l	$-j$	w_4	w_5

Table 5.20: Partial index distribution of $W_1, \ldots, W_4 \in \mathcal{G}_i$.

with $w_1, \ldots, w_5 \in \{-k, -m, n, -n, o, -o\}$ pairwise distinct. Concentrating our attention on w_3 we get as possibilities, up to an equivalent index: $w_3 = -m$ or $w_3 = n$. For each one of these cases we will present next, up to an equivalent case, all possible index characterizations.

Assuming $w_3 = -m$, we get the following possibilities for w_1, w_2, w_4 and w_5:

i) $w_1 = -k$, $w_2 = n$, $w_4 = -n$ and $w_5 = o$;

ii) $w_1 = n$, $w_2 = o$, $w_4 = -n$ and $w_5 = -o$.

In the case of $w_3 = n$, then w_1, w_2, w_4 and w_5 satisfy one of the following conditions:

i) $w_1 = -k$, $w_2 = -m$, $w_4 = -n$ and $w_5 = o$;

ii) $w_1 = -k$, $w_2 = -n$, $w_4 = -m$ and $w_5 = o$;

iii) $w_1 = -k$, $w_2 = o$, $w_4 = -m$ and $w_5 = -n$;

iv) $w_1 = -k$, $w_2 = o$, $w_4 = -m$ and $w_5 = -o$;

v) $w_1 = -k$, $w_2 = o$, $w_4 = -n$ and $w_5 = -o$;

vi) $w_1 = -m$, $w_2 = o$, $w_4 = -n$ and $w_5 = -o$.

We get all possible index distributions for the codewords of $\mathcal{G}_i$ when $|\mathcal{J}| = 5$ applying a similar reasoning to each one of the other partial index distributions presented in Table 5.19.

- **Index distribution of the codewords of $\mathcal{G}_i$ considering $|\mathcal{J}| = 6$**

Now assume that the codewords of $\mathcal{G}_i$ are such that $|\mathcal{J}| = 6$. In these conditions there exists a unique possible structure for the distribution of the "shared" indices of $W_1, \ldots, W_4 \in \mathcal{G}_i$:

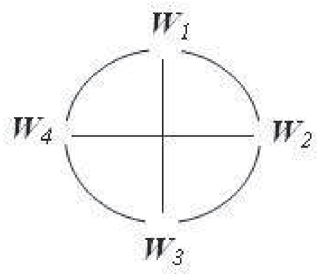

Figure 5.3: Structure for the index distribution of $W_1, \ldots, W_4 \in \mathcal{G}_i$.

Therefore, considering $\mathcal{J} = \{\beta, \gamma, \delta, \varepsilon, \theta, \mu\}$, the codewords $W_1, \ldots, W_4 \in \mathcal{G}_i$ satisfy:

W_1	i	β	γ	δ	w_1
W_2	i	β	ε	θ	w_2
W_3	i	γ	ε	μ	w_3
W_4	i	δ	θ	μ	w_4

Table 5.21: Structure for the index distribution of $W_1, \ldots, W_4 \in \mathcal{G}_i$.

with $w_1, \ldots, w_4 \in \mathcal{I} \backslash \{i, -i, \beta, \gamma, \delta, \varepsilon, \theta, \mu\}$ pairwise distinct.

Assuming $W_1 \in \mathcal{G}_{ijklm}$, we get:

W_1	i	j	k	l	m
W_2	i	j	ε	θ	w_2
W_3	i	k	ε	μ	w_3
W_4	i	l	θ	μ	w_4

Table 5.22: Partial index distribution of $W_1, \ldots, W_4 \in \mathcal{G}_i$.

where $\varepsilon, \theta, \mu, w_2, w_3, w_4 \in \mathcal{I} \backslash \{i, -i, j, k, l, m\}$.

Considering $\varepsilon, \theta, \mu \in \mathcal{J}$, we obtain, up to an equivalent partial index distribution, the following hypotheses:

i) $\varepsilon = -l$, $\theta = -k$ and $\mu = -j$;

ii) $\varepsilon = -l$, $\theta = -k$ and $\mu = -m$;

iii) $\varepsilon = -l$, $\theta = -k$ and $\mu = n$;

iv) $\varepsilon = -l$, $\theta = -m$ and $\mu = n$;

v) $\varepsilon = -l$, $\theta = n$ and $\mu = o$;

vi) $\varepsilon = -m$, $\theta = n$ and $\mu = o$.

For each one of the hypotheses which we have just to present we characterize completely the index distribution of all codewords of $\mathcal{G}_i$ by the concretization of w_2, w_3 and w_4. Let us consider, for instance, the case in which $W_1, \ldots, W_4 \in \mathcal{G}_i$ satisfy:

W_1	i	j	k	l	m
W_2	i	j	$-l$	$-k$	w_2
W_3	i	k	$-l$	$-j$	w_3
W_4	i	l	$-k$	$-j$	w_4

Table 5.23: Partial index distribution of $W_1, \ldots, W_4 \in \mathcal{G}_i$.

We note that $w_2, w_3, w_4 \in \{-m, n, -n, o, -o\}$ and are pairwise distinct. Thus, for $W_1, \ldots, W_4 \in \mathcal{G}_i$ satisfying the conditions in Table 5.23 there are, up to an equivalent index distribution, three possible hypotheses:

– $w_2 = -m$, $w_3 = n$ and $w_4 = -n$;

– $w_2 = -m$, $w_3 = n$ and $w_4 = o$;

– $w_2 = n$; $w_3 = -n$ and $w_4 = o$.

Considering each one of the other possible partial index distributions concentrated on the elements of $\mathcal{J}$, see the conditions $i)$ to $vi)$, proceeding in the same way we get all possible complete index distributions for the codewords of $\mathcal{G}_i$.

5.2.2 Index distribution of the codewords of $\mathcal{F}_i$

In the previous subsection we have described the method to get all possible index distributions for the codewords of $\mathcal{G}_i$. Here, we will concentrate our attention on the codewords of $\mathcal{F}_i$.

We are assuming $|\mathcal{G}_i| = 4$, accordingly, by Lemma 2.10, we have $10 \leq |\mathcal{F}_i| \leq 11$. Our aim is to characterize all possible index distributions for the codewords of $\mathcal{F}_i$, when it is given a certain index distribution of the codewords of $\mathcal{G}_i$.

Applying the method described in the last subsection, we obtain all possible index characterizations for $W_1, \ldots, W_4 \in \mathcal{G}_i$. We intend to analyze each one of these index distributions, identifying the respective possible index distributions for the codewords of $\mathcal{F}_i$. For many of the obtained index distributions for the codewords of $\mathcal{G}_i$ it is not possible to describe completely the index distribution of all codewords of $\mathcal{F}_i$ without contradictions on the definition of PL$(7, 2)$ code. However, there exist cases in which it is possible to characterize all codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ without contradictions.

Here, we present a method which, given a certain index distribution for the codewords $W_1, \ldots, W_4 \in \mathcal{G}_i$, allow us to verify if it is or not possible to characterize completely all codewords of $\mathcal{F}_i$ and, in the cases in which such characterization is possible, to identify all possible index distributions for the codewords of $\mathcal{F}_i$. We note that, the index distributions of $W_1, \ldots, W_4 \in \mathcal{G}_i$ for which it is not possible to identify all codewords of $\mathcal{F}_i$ are not valid, since lead to contradictions.

As said before, there are many possible index distributions for the codewords of $\mathcal{G}_i$, thus we will take only some representative cases to illustrate how the analysis of the codewords of $\mathcal{F}_i$ is done.

In Section 5.1 we have derived results which help us to characterize the index distribution of the codewords of $\mathcal{F}_i$ when the codewords of $\mathcal{G}_i$ satisfy certain conditions, in particular, when $|\mathcal{J}|$ assumes, respectively, one of the values in $\{4, 5, 6\}$. Thus, for having the application of each one of the presented results we will consider examples in which $|\mathcal{J}| = 4$, $|\mathcal{J}| = 5$ and $|\mathcal{J}| = 6$, respectively.

- **Characterization of $\mathcal{F}_i$ when $|\mathcal{J}| = 4$**

Let us consider $W_1, \ldots, W_4 \in \mathcal{G}_i$ so that:

W_1	i	j	k	l	m
W_2	i	j	$-k$	$-l$	n
W_3	i	$-k$	$-j$	$-n$	o
W_4	i	k	$-j$	$-m$	$-o$

Table 5.24: Index distribution of $W_1, \ldots, W_4 \in \mathcal{G}_i$.

In this case we have $|\mathcal{J}| = 4$ with $\mathcal{J} = \{j, -j, k, -k\}$.

By Proposition 5.4, there are, at least, two elements $\alpha \in \mathcal{J}$ so that $|\mathcal{F}_{i\alpha}| = 2$. In accordance, we begin the characterization of the codewords of $\mathcal{F}_i$ identifying, for each $\alpha \in \mathcal{J}$, the possible codewords in $\mathcal{F}_{i\alpha}$ assuming that $|\mathcal{F}_{i\alpha}| = 2$.

Taking into account the codewords of $\mathcal{G}_i$ and Lemma 1.5 we get:

i	j	o	$-m$
i	j	$-o$	$-n$

Table 5.25: $\mathcal{F}_{ij}$.

i	k	n	o
i	k	$-n$	$-l$

Table 5.26: $\mathcal{F}_{ik}$.

i	$-k$	m	$-o$
i	$-k$	$-m$	l

Table 5.27: $\mathcal{F}_{i,-k}$.

i	$-j$	l	n
i	$-j$	$-l$	m

Table 5.28: $\mathcal{F}_{i,-j}$.

First assume that $|\mathcal{F}_{i\alpha}| = 2$ for all $\alpha \in \mathcal{J}$. Accordingly, codewords satisfying the index distributions presented above, must exist. By Proposition 5.3 we have $|\mathcal{F}_i| = 10$, and so we must identify in $\mathcal{F}_i$ more two codewords. We note that the remaining codewords $U \in \mathcal{F}_i$ are such that $U \in \mathcal{F}_{i\alpha}$ with $\alpha \in \mathcal{I}\backslash(\{i, -i\} \cup \mathcal{J})$. Taking into account the index distribution of all codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ already known and Lemma 1.5, we conclude that it is not possible to identify in $\mathcal{F}_i$ any other codeword without facing a contradiction.

Therefore, we will focus our attention on the following two hypotheses: there are exactly two elements $\alpha \in \mathcal{J}$ satisfying $|\mathcal{F}_{i\alpha}| = 2$; there are exactly three elements $\alpha \in \mathcal{J}$ satisfying $|\mathcal{F}_{i\alpha}| = 2$.

We recall that from Proposition 5.4 we have:

i) if $\beta, \gamma \in \mathcal{J}$ are the unique elements in $\mathcal{J}$ satisfying $|\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = 2$, then $|\mathcal{F}_{i\delta}| = |\mathcal{F}_{i\varepsilon}| = 1$, with $\delta, \varepsilon \in \mathcal{J}\backslash\{\beta,\gamma\}$, and $|\mathcal{F}_{i\alpha}| = 3$ for all $\alpha \in \mathcal{I}\backslash(\{i,-i\} \cup \mathcal{J})$;

ii) if $\beta, \gamma, \delta \in \mathcal{J}$ are the unique elements in $\mathcal{J}$ satisfying $|\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = |\mathcal{F}_{i\delta}| = 2$, then there are seven elements $\alpha \in \mathcal{I}\backslash(\{i,-i\} \cup \mathcal{J})$ such that $|\mathcal{F}_{i\alpha}| = 3$.

In any one of the two given hypotheses there are two elements $\alpha \in \mathcal{J}$ so that $|\mathcal{F}_{i\alpha}| = 2$. Thus, combining the elements of $\mathcal{J}$, one of the following conditions must occurs: $|\mathcal{F}_{ij}| = |\mathcal{F}_{ik}| = 2$; $|\mathcal{F}_{ij}| = |\mathcal{F}_{i,-k}| = 2$; $|\mathcal{F}_{ij}| = |\mathcal{F}_{i,-j}| = 2$; $|\mathcal{F}_{ik}| = |\mathcal{F}_{i,-k}| = 2$; $|\mathcal{F}_{ik}| = |\mathcal{F}_{i,-j}| = 2$; $|\mathcal{F}_{i,-k}| = |\mathcal{F}_{i,-j}| = 2$.

As $|\mathcal{I}\backslash(\{i,-i\} \cup \mathcal{J})| = 8$, considering the conditions *i)* and *ii)* referred before and obtained from Proposition 5.4, in both cases, we may conclude that there exists, at most, one element $\alpha \in \mathcal{I}\backslash(\{i,-i\} \cup \mathcal{J})$ such that $|\mathcal{F}_{i\alpha}| \leq 2$.

Suppose that $|\mathcal{F}_{ij}| = |\mathcal{F}_{ik}| = 2$. Then, there exist in $\mathcal{F}_i$ codewords satisfying the index distributions presented in Tables 5.25 and 5.26. Taking into account these codewords, the codewords of $\mathcal{G}_i$ and Lemma 1.5, considering $-l, -m \in \mathcal{I}\backslash(\{i,-i\} \cup \mathcal{J})$, we conclude that the unique possible index distributions for codewords in $\mathcal{F}_{i,-l}$ and $\mathcal{F}_{i,-m}$ are, respectively:

i	$-l$	m	$-j$
i	$-l$	m	o
i	$-l$	m	$-o$

Table 5.29: $\mathcal{F}_{i,-l}$.

i	$-m$	l	$-k$
i	$-m$	l	n
i	$-m$	l	$-n$

Table 5.30: $\mathcal{F}_{i,-m}$.

Consequently, by Lemma 1.5, and considering the known codewords in $\mathcal{F}_{ij} \cup \mathcal{F}_{ik}$, we conclude that $|\mathcal{F}_{i,-l}| \leq 2$ and $|\mathcal{F}_{i,-m}| \leq 2$. That is, there are two elements $\alpha \in \mathcal{I}\backslash(\{i,-i\} \cup \mathcal{J})$ satisfying $|\mathcal{F}_{i\alpha}| \leq 2$, which contradicts Proposition 5.4. Thus, the condition $|\mathcal{F}_{ij}| = |\mathcal{F}_{ik}| = 2$ is not valid.

By a similar reasoning we come to the same conclusion for:

– $|\mathcal{F}_{ij}| = |\mathcal{F}_{i,-k}| = 2$, since $|\mathcal{F}_{il}|, |\mathcal{F}_{i,-n}| \leq 2$;

– $|\mathcal{F}_{ik}| = |\mathcal{F}_{i,-j}| = 2$, since $|\mathcal{F}_{im}|, |\mathcal{F}_{io}| \leq 2$;

– $|\mathcal{F}_{i,-k}| = |\mathcal{F}_{i,-j}| = 2$, since $|\mathcal{F}_{in}|, |\mathcal{F}_{i,-o}| \leq 2$.

Let us now assume $|\mathcal{F}_{ij}| = |\mathcal{F}_{i,-j}| = 2$. By what was done before, we must impose $|\mathcal{F}_{i,k}| \neq 2$ and $|\mathcal{F}_{i,-k}| \neq 2$. Consequently, by Proposition 5.4, for any $\alpha \in \mathcal{I} \backslash (\{i, -i\} \cup \mathcal{J})$ we must have $|\mathcal{F}_{i\alpha}| = 3$.

Consider in $\mathcal{F}_i$ the codewords satisfying the index distributions presented in Tables 5.25 and 5.28. Focusing our attention on $o \in \mathcal{I} \backslash (\{i, -i\} \cup \mathcal{J})$, and considering the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ already known and Lemma 1.5, we conclude that the remaining codewords $U_1, U_2 \in \mathcal{F}_{io}$ must satisfy: $U_1 \in \mathcal{F}_{i,o,k,-l}$ and $U_2 \in \mathcal{F}_{iomn}$. Considering $m \in \mathcal{I} \backslash (\{i, -i\} \cup \mathcal{J})$ we verify that the remaining codeword $U_3 \in \mathcal{F}_{im}$ must satisfy $U_3 \in \mathcal{F}_{i,m,-k,-o}$. However, when we focus our attention on $-o \in \mathcal{I} \backslash (\{i, -i\} \cup \mathcal{J})$ we conclude that it is not possible to characterize any other codeword in $\mathcal{F}_{i,-o}$ without facing a contradiction.

The assumption of $|\mathcal{F}_{ik}| = |\mathcal{F}_{i,-k}| = 2$ led us to the same conclusion, by a similar reasoning.

Therefore, for $W_1, \ldots, W_4 \in \mathcal{G}_i$ satisfying the index distribution presented in Table 5.24 it is not possible to characterize completely all codewords of $\mathcal{F}_i$ without facing a contradiction. Thus, the considered index distribution for the codewords of $\mathcal{G}_i$ is not valid.

- **Characterization of $\mathcal{F}_i$ when $|\mathcal{J}| = 5$**

In this case will be analyze two possible index distributions for the codewords of $\mathcal{G}_i$ satisfying $|\mathcal{J}| = 5$.

Example 1

Let us consider now $W_1, \ldots, W_4 \in \mathcal{G}_i$ such that:

W_1	i	j	k	l	m
W_2	i	j	$-l$	n	o
W_3	i	k	$-l$	$-j$	$-m$
W_4	i	l	$-j$	$-n$	$-o$

Table 5.31: Index distribution of $W_1, \ldots, W_4 \in \mathcal{G}_i$.

By the analysis of the index distribution of $W_1, \ldots, W_4 \in \mathcal{G}_i$ we verify that $|\mathcal{J}| = 5$ where $\mathcal{J} = \{j, -j, k, l, -l\}$.

As $|\mathcal{G}_i| = 4$, by Lemma 2.10 we get $|\mathcal{F}_i| = 10$ or $|\mathcal{F}_i| = 11$.

Suppose that $|\mathcal{F}_i| = 10$. From Lemma 2.10 it follows that $|\mathcal{F}_i^{(2)}| = 4$. By Lemma 2.14, we have $|\mathcal{F}_i^{(2)} \cap \mathcal{F}_\alpha| = 1$ for any $\alpha \in \mathcal{I} \backslash \{i, -i\}$. Thus, there are in $\mathcal{F}_i^{(2)}$ codewords U_1, U_2, U_3 and U_4 so that $U_1 \in \mathcal{F}_{ij}$, $U_2 \in \mathcal{F}_{ik}$, $U_3 \in \mathcal{F}_{il}$ and $U_4 \in \mathcal{F}_{im}$. Considering Lemma 1.5 and $W_1 \in \mathcal{G}_i$ we conclude that $U_1, \ldots, U_4$ must be pairwise distinct and, therefore, $\mathcal{F}_i^{(2)} = \{U_1, \ldots, U_4\}$.

Let us consider $U_1, \ldots, U_4 \in \mathcal{F}_i^{(2)}$:

U_1	i	j	u_1	u_2
U_2	i	k	u_3	u_4
U_3	i	l	u_5	u_6
U_4	i	m	u_7	u_8

Table 5.32: Partial index distribution of $U_1, \ldots, U_4 \in \mathcal{F}_i^{(2)}$.

We note that, by Lemma 2.14, $u_1, \ldots, u_8$ must be pairwise distinct. Furthermore, considering the codewords of $\mathcal{G}_i$ we must impose:

- $u_1, u_2 \in \{-k, -m, -n, -o\}$;
- $u_3, u_4 \in \{n, -n, o, -o\}$;

– $u_5, u_6 \in \{-k, -m, n, o\}$;

– $u_7, u_8 \in \{-j, -k, -l, n, -n, o, -o\}$.

By Lemma 2.14 there exist $U, U' \in \mathcal{F}_i^{(2)}$ so that $U \in \mathcal{F}_{i,-j}$ and $U' \in \mathcal{F}_{i,-l}$. Taking into account what was said before, we must consider $u_7 = -j$ and $u_8 = -l$. That is, $U_4 \in \mathcal{F}_{i,m,-j,-l}$ which is an absurdity since W_3 and U_4 contradict Lemma 1.5.

Now suppose that $|\mathcal{F}_i| = 11$. By Proposition 5.6 we get: $|\mathcal{F}_{i\alpha}| = 2$ for any $\alpha \in \mathcal{J}$; $|\mathcal{F}_{i,-k}| = 5$; $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i, -k\} \cup \mathcal{J})$.

Let us concentrate our attention on the codewords of $\mathcal{F}_{i,-k}$. Since $|\mathcal{F}_{i,-k}| = 5$, by Lemma 2.5 we must impose $|\mathcal{F}_{i,-k,\alpha}| = 1$ for each $\alpha \in \mathcal{I}\backslash\{i, -i, k, -k\}$. Thus, there exist in $\mathcal{F}_{i,-k}$ codewords U_1,...,U_4 so that $U_1 \in \mathcal{F}_{i,-k,j}$, $U_2 \in \mathcal{F}_{i,-k,l}$, $U_3 \in \mathcal{F}_{i,-k,-l}$ and $U_4 \in \mathcal{F}_{i,-k,-j}$. Considering the codewords of $\mathcal{G}_i$ and Lemma 1.5 we conclude that $U_1, \ldots, U_4$ must be pairwise distinct. Therefore, the codewords of $\mathcal{F}_{i,-k}$ are such that:

U_1	i	$-k$	j	u_1
U_2	i	$-k$	l	u_2
U_3	i	$-k$	$-l$	u_3
U_4	i	$-k$	$-j$	u_4
U_5	i	$-k$	u_5	u_6

Table 5.33: Partial index distribution of the codewords of $\mathcal{F}_{i,-k}$.

with $u_1, \ldots, u_6 \in \{m, -m, n, -n, o, -o\}$ and pairwise distinct.

Considering $W_1 \in \mathcal{G}_{ijklm}$, and taking into account Lemma 1.5, we must impose $u_1, u_2 \neq m$. Suppose that $u_3 = m$, that is, $U_3 \in \mathcal{F}_{i,-k,-l,m}$. We note that, by Proposition 5.6 we get $|\mathcal{F}_{i,-l}| = 2$, so we must identify one more codeword in $\mathcal{F}_{i,-l}$. Taking into account $W_2, W_3 \in \mathcal{G}_{i,-l}$, $U_3 \in \mathcal{F}_{i,-l}$ and Lemma 1.5 we conclude that $U_6 \in \mathcal{F}_{i,-l}$ is such that $U_6 \in \mathcal{F}_{i,-l,-n,-o}$, contradicting Lemma 1.5 with W_4. If we assume $u_4 = m$, that is, $U_4 \in \mathcal{F}_{i,-k,-j,m}$, since $|\mathcal{F}_{i,-j}| = 2$ we get, as in the previous case, a contradiction, since $U_6 \in \mathcal{F}_{i,-j}$ would verify $U_6 \in \mathcal{F}_{i,-j,n,o}$. Therefore, we must have $u_5 = m$. In these conditions and focusing our attention on $-m$, considering W_3 we must impose $u_3, u_4, u_6 \neq -m$. Assuming $u_1 = -m$, as $|\mathcal{F}_{ij}| = 2$, the remaining codeword in $\mathcal{F}_{ij}$

must verify $U_6 \in \mathcal{F}_{i,j,-n,-o}$, contradicting Lemma 1.5 with W_4. On the other hand, if $u_2 = -m$, since $|\mathcal{F}_{il}| = 2$ then the remaining codeword of $\mathcal{F}_{il}$ must satisfy $U_6 \in \mathcal{F}_{ilno}$, obtaining again a contradiction. Thus, we conclude that it is not possible to describe all the codewords of $\mathcal{F}_{i,-k}$ and, consequently, the considered index distribution of the codewords of $\mathcal{G}_i$ contradicts the definition of PL$(7,2)$ code.

Example 2

Until now we have presented examples for which it is not possible to characterize the index distribution of all codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ without contradictions on definition of perfect 2-error correcting Lee code. Next, we present an example in which such characterization is possible.

Consider $W_1, \ldots, W_4 \in \mathcal{G}_i$ satisfying:

W_1	i	j	k	l	m
W_2	i	j	n	$-m$	o
W_3	i	k	n	$-j$	$-l$
W_4	i	l	$-j$	$-n$	$-o$

Table 5.34: Index distribution of $W_1, \ldots, W_4 \in \mathcal{G}_i$.

with $\mathcal{J} = \{j, -j, k, l, n\}$.

By Lemma 2.10 we obtain $10 \leq |\mathcal{F}_i| \leq 11$. Let us begin considering $|\mathcal{F}_i| = 10$. From Lemma 2.10 one gets $|\mathcal{F}_i^{(2)}| = 4$. Accordingly, by Lemma 2.14, $|\mathcal{F}_i^{(2)} \cap \mathcal{F}_\alpha| = 1$ for each $\alpha \in \mathcal{I}\backslash\{i, -i\}$. Thus, considering the codewords of $\mathcal{G}_i$ and Lemma 1.5, U_1,...,$U_4 \in \mathcal{F}_i^{(2)}$ must satisfy the conditions presented in Table 5.35, where $u_1, u_2 \in \{-n, -o\}$. As an immediate consequence, we get the two possible index distributions for the codewords of $\mathcal{F}_i^{(2)}$ presented in Tables 5.36 and 5.37, respectively.

U_1	i	j	$-l$	u_1
U_2	i	k	$-m$	u_2
U_3	i	l	n	$-k$
U_4	i	m	$-j$	o

Table 5.35: Index distribution of the codewords of $\mathcal{F}_i^{(2)}$.

U_1	i	j	$-l$	$-n$
U_2	i	k	$-m$	$-o$
U_3	i	l	n	$-k$
U_4	i	m	$-j$	o

Table 5.36: $u_1 = -n$; $u_2 = -o$.

U_1	i	j	$-l$	$-o$
U_2	i	k	$-m$	$-n$
U_3	i	l	n	$-k$
U_4	i	m	$-j$	o

Table 5.37: $u_1 = -o$; $u_2 = -n$.

Let us consider that $U_1, \ldots, U_4 \in \mathcal{F}_i^{(2)}$ have the index distribution presented in Table 5.37.

By Proposition 5.7, there are, at least, two elements $\alpha \in \mathcal{J}$ so that $|\mathcal{F}_{i\alpha}| = 2$. Thus, we must scrutinize for which elements $\alpha \in \{j, -j, k, l, n\}$ it is possible to have $|\mathcal{F}_{i\alpha}| = 2$. Considering the codewords of $\mathcal{G}_i \cup \mathcal{F}_i^{(2)}$ and Lemma 1.5, it is possible to include in $\mathcal{F}_{ij}$, $\mathcal{F}_{i,-j}$ and $\mathcal{F}_{in}$, respectively, one more codeword: $U \in \mathcal{F}_{i,j,-k,-n}$; $U' \in \mathcal{F}_{i,-j,-k,-m}$; $U'' \in \mathcal{F}_{i,n,m,-o}$. However, when we consider the indices $l, k \in \mathcal{J}$ we can not do it without contradicting Lemma 1.5. Thus, we may conclude that, if $\alpha \in \mathcal{J}$ is such that $|\mathcal{F}_{i\alpha}| = 2$, then $\alpha \in \{j, -j, n\}$. We will analyze separately the following two hypotheses:

– there are exactly two elements $\alpha \in \{j, -j, n\}$ so that $|\mathcal{F}_{i\alpha}| = 2$;

– $|\mathcal{F}_{i\alpha}| = 2$ for any $\alpha \in \{j, -j, n\}$.

First assume that there are exactly two elements $\alpha \in \{j, -j, n\}$ so that $|\mathcal{F}_{i\alpha}| = 2$. Taking into account Proposition 5.7, $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I} \backslash (\{i, -i, -k\} \cup \mathcal{J})$, consequently, $|\mathcal{F}_{i,-m}| = 3$. At this moment, we have only identified one codeword of $\mathcal{F}_{i,-m}$, $U_2 \in \mathcal{F}_i^{(2)}$, thus we must include two more codewords in $\mathcal{F}_{i,-m}$. Taking into account the codewords of $\mathcal{G}_i \cup \mathcal{F}_i^{(2)}$ already known, and Lemma 1.5, if $U \in \mathcal{F}_{i,-m} \backslash \{U_2\}$, then

$$U \in \mathcal{F}_{i,-m,-k,-j} \cup \mathcal{F}_{i,-m,-k,-l} \cup \mathcal{F}_{i,-m,-k,-o}.$$

Considering Lemma 1.5 we conclude that $|\mathcal{F}_{i,-m} \backslash \{U_2\}| \leq 1$, a contradiction.

Let us now consider that $|\mathcal{F}_{ij}| = |\mathcal{F}_{i,-j}| = |\mathcal{F}_{in}| = 2$. In these conditions we must consider in $\mathcal{F}_i$ the codewords $U_5 \in \mathcal{F}_{i,j,-k,-n}$, $U_6 \in \mathcal{F}_{i,-j,-k,-m}$ and $U_7 \in \mathcal{F}_{i,n,m,-o}$. We have seen before that $|\mathcal{F}_{i,-m}| \leq 2$, thus, from Proposition 5.7, it follows that $|\mathcal{F}_{i,-k}| = 5$. We have now described three codewords of $\mathcal{F}_{i,-k}$: U_3, U_5 and U_6. Therefore, we must identify two more codewords in $\mathcal{F}_{i,-k}$. Taking into account all codewords of $\mathcal{G}_i \cup \mathcal{F}_i$, already known, as well as Lemma 1.5, we conclude that if $U \in \mathcal{F}_{i,-k} \backslash \{U_3, U_5, U_6\}$, then $U \in \mathcal{F}_{i,-k,-l,m} \cup \mathcal{F}_{i,-k,-l,o}$. Consequently, considering Lemma 1.5, we have $|\mathcal{F}_{i,-k} \backslash \{U_3, U_5, U_6\}| \leq 1$, a contradiction.

Now consider that the codewords of $\mathcal{F}_i^{(2)}$ satisfy the conditions in Table 5.36. In this case, considering the elements of $\mathcal{J}$, it is possible to have $|\mathcal{F}_{i\alpha}| = 2$ for $\alpha \in \{j, -j, k, n\}$, in fact, it is possible consider in $\mathcal{F}_i$ subsets of codewords satisfying: $\mathcal{F}_{i,j,-k,-o}$; $\mathcal{F}_{i,-j,-k,-m}$; $\mathcal{F}_{i,k,-n,o}$; $\mathcal{F}_{i,n,m,-o}$. Thus, taking into account Proposition 5.7, we must consider each one of the following hypotheses:

- there exist exactly two elements $\alpha \in \mathcal{J}$ verifying $|\mathcal{F}_{i\alpha}| = 2$;
- there exist exactly three elements $\alpha \in \mathcal{J}$ verifying $|\mathcal{F}_{i\alpha}| = 2$;
- there exist exactly four elements $\alpha \in \mathcal{J}$ verifying $|\mathcal{F}_{i\alpha}| = 2$.

Suppose that there exist exactly two elements $\alpha \in \mathcal{J}$ satisfying $|\mathcal{F}_{i\alpha}| = 2$. By Proposition 5.7 we have $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I} \backslash (\{i, -i, -k\} \cup \mathcal{J})$. Let us consider $-m \in \mathcal{I} \backslash (\{i, -i, -k\} \cup \mathcal{J})$. Until now we have the knowledge of only one codeword of $\mathcal{F}_{i,-m} : U_2 \in \mathcal{F}_{i,k,-m,-o}$. Thus, we must identify in $\mathcal{F}_{i,-m}$ two more codewords. Considering the codewords of $\mathcal{G}_i \cup \mathcal{F}_i^{(2)}$ already known and Lemma 1.5 we conclude that if $U \in \mathcal{F}_{i,-m} \backslash \{U_2\}$, then

$$U \in \mathcal{F}_{i,-m,-k,-j} \cup \mathcal{F}_{i,-m,-k,-l} \cup \mathcal{F}_{i,-m,-k,-n}.$$

Once again, by Lemma 1.5, we conclude that $|\mathcal{F}_{i,-m} \backslash \{U_2\}| \leq 1$ and, consequently, $|\mathcal{F}_{i,-m}| \leq 2$, which is a contradiction.

Accordingly, there exist, at least, three elements $\alpha \in \mathcal{J}$ such that $|\mathcal{F}_{i\alpha}| = 2$. Considering Proposition 5.7, we verify that in some cases the hypothesis $|\mathcal{F}_{i,-k}| = 5$ may occurs. Let us analyze this possibility. We note that, at this stage, it is known one codeword of $\mathcal{F}_{i,-k}$: $U_3 \in \mathcal{F}_{i,l,n,-k}$. If $|\mathcal{F}_{i,-k}| = 5$, then, considering Lemma 2.5, $|\mathcal{F}_{i,-k,\alpha}| = 1$ for each $\alpha \in \mathcal{I}\backslash\{i,-i,k,-k\}$. Therefore, $|\mathcal{F}_{i,-k,j}| = |\mathcal{F}_{i,-k,m}| = 1$. Taking into account the codeword $W_1 \in \mathcal{G}_{ijklm}$, we must impose $\mathcal{F}_{i,-k,j} \cap \mathcal{F}_{i,-k,m} = \varnothing$. So, let us consider $U_5 \in \mathcal{F}_{i,-k,j,u_1}$ and $U_6 \in \mathcal{F}_{i,-k,m,u_2}$. Having in mind the codewords of $\mathcal{G}_i \cup \mathcal{F}_i^{(2)}$ and Lemma 1.5 we must consider $u_1 = -o$ and $u_2 \in \{-l,-n\}$. Thus, the codewords of $\mathcal{F}_{i,-k}$ must satisfy one of the following conditions:

U_3	i	$-k$	l	n
U_5	i	$-k$	j	$-o$
U_6	i	$-k$	m	$-l$
U_7	i	$-k$	$-j$	$-m$
U_8	i	$-k$	$-n$	o

Table 5.38: $u_2 = -l$.

U_3	i	$-k$	l	n
U_5	i	$-k$	j	$-o$
U_6	i	$-k$	m	$-n$
U_7	i	$-k$	$-j$	$-m$
U_8	i	$-k$	$-l$	o

Table 5.39: $u_2 = -n$.

We note that, under the assumption $|\mathcal{F}_{i,-k}| = 5$ are already known eight codewords in $\mathcal{F}_i$. Since we are assuming $|\mathcal{F}_i| = 10$, then we must identify two more codewords. If the codewords of $\mathcal{F}_{i,-k}$ satisfy the index distribution presented in Table 5.38, having in mind all codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ already known and Lemma 1.5, we can identify only one codeword in $\mathcal{F}_i$: $U_9 \in \mathcal{F}_{i,m,n,-o}$. However, is not satisfied what is being supposed. On the other hand, if the codewords of $\mathcal{F}_{i,-k}$ satisfy the conditions in Table 5.39 the remaining codewords $U_9, U_{10} \in \mathcal{F}_i$ must verify one of the following two hypotheses:

1) $U_9 \in \mathcal{F}_{i,k,-n,o}$ and $U_{10} \in \mathcal{F}_{i,-l,m,-o}$;

2) $U_9 \in \mathcal{F}_{i,k,-n,o}$ and $U_{10} \in \mathcal{F}_{i,m,n,-o}$.

We note that, if $U_9, U_{10} \in \mathcal{F}_i$ satisfy the conditions in 1), by the analysis of all codewords $U_1, \ldots, U_{10} \in \mathcal{F}_i$ we conclude that there exist only three elements $\alpha \in \mathcal{J}$ satisfying $|\mathcal{F}_{i\alpha}| = 2$: $|\mathcal{F}_{ij}| = |\mathcal{F}_{i,-j}| = |\mathcal{F}_{ik}| = 2$. Furthermore, there are five elements

$\alpha \in \mathcal{I}\backslash(\{i, -i, -k\} \cup \mathcal{J})$ verifying $|\mathcal{F}_{i\alpha}| = 3$, namely:

$$|\mathcal{F}_{i,-l}| = |\mathcal{F}_{im}| = |\mathcal{F}_{i,-n}| = |\mathcal{F}_{io}| = |\mathcal{F}_{i,-o}| = 3.$$

On the other hand, if $U_9, U_{10} \in \mathcal{F}_i$ have the index distribution presented in 2), then $|\mathcal{F}_{i\alpha}| = 2$ for any $\alpha \in \{j, -j, k, n\}$ and $|\mathcal{F}_{i\alpha}| = 3$ for $\alpha \in \{m, -n, o, -o\}$.

If, by assumption, there exist only three elements $\alpha \in \mathcal{J}$ verifying $|\mathcal{F}_{i\alpha}| = 2$, then from Proposition 5.7 we get as possibilities: $|\mathcal{F}_{i,-k}| = 5$ or $|\mathcal{F}_{i,-k}| = 4$. If $|\mathcal{F}_{i,-k}| = 4$, by the same proposition we must impose $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i, -k\} \cup \mathcal{J})$. However, when we have assumed $|\mathcal{F}_{i\alpha}| = 2$ for only two elements $\alpha \in \mathcal{J}$ we have concluded that $|\mathcal{F}_{i,-m}| \leq 2$. So, we must impose $|\mathcal{F}_{i,-k}| = 5$. In these conditions, taking into account the analysis of the hypothesis $|\mathcal{F}_{i,-k}| = 5$ done before, we can identify only one possible index distribution for the codewords of $\mathcal{F}_i$: U_1,...,$U_4 \in \mathcal{F}_i^{(2)}$ described in Table 5.36; $U_5, \ldots, U_8 \in \mathcal{F}_{i,-k}$ presented in Table 5.39; $U_9 \in \mathcal{F}_{i,k,-n,o}$ and $U_{10} \in \mathcal{F}_{i,-l,m,-o}$.

Suppose that there are exactly four elements $\alpha \in \mathcal{J}$ so that $|\mathcal{F}_{i\alpha}| = 2$. We note that, we have already seen that, in this case,necessarily $|\mathcal{F}_{ij}| = |\mathcal{F}_{i,-j}| = |\mathcal{F}_{ik}| = |\mathcal{F}_{in}| = 2$. Thus, in addition to considering in $\mathcal{F}_i$ the codewords of $\mathcal{F}_i^{(2)}$ presented in Table 5.36, we should also consider the codewords U_5,...,$U_8 \in \mathcal{F}_i$ satisfying: $U_5 \in \mathcal{F}_{i,j,-k,-o}$; $U_6 \in \mathcal{F}_{i,-j,-k,-m}$; $U_7 \in \mathcal{F}_{i,k,-n,o}$; $U_8 \in \mathcal{F}_{i,n,m,-o}$. By Proposition 5.7 it follows that $|\mathcal{F}_{i,-k}| = 5$, $|\mathcal{F}_{i,-k}| = 4$ or $|\mathcal{F}_{i,-k}| = 3$. We note that we can not have $|\mathcal{F}_{i,-k}| = 3$ since, by the same proposition, we would get $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i, -k\} \cup \mathcal{J})$ and, as we have seen in the previous cases, $|\mathcal{F}_{i,-m}| \leq 2$. If we consider $|\mathcal{F}_{i,-k}| = 4$, since we have only described three codewords of $\mathcal{F}_{i,-k}$, $U_3 \in \mathcal{F}_{i,l,n,-k}$, $U_5 \in \mathcal{F}_{i,j,-k,-o}$ and $U_6 \in \mathcal{F}_{i,-j,-k,-m}$, then we must consider in $\mathcal{F}_{i,-k}$ only one more codeword. As we are assuming $|\mathcal{F}_i| = 10$, we must identify one more codeword $U \in \mathcal{F}_i$ so that $U \notin \mathcal{F}_{i,-k}$. Taking into account the codewords of $\mathcal{F}_i^{(2)}$, see Table 5.36, and U_5,...,U_8 presented before as well as the codewords of $\mathcal{G}_i$ and Lemma 1.5 we conclude that if $U \in \mathcal{F}_i\backslash\{U_1, \ldots, U_8\}$ then

$$U \in \mathcal{F}_{i,-k,m,-n} \cup \mathcal{F}_{i,-k,-l,m} \cup \mathcal{F}_{i,-k,-l,o},$$

that is, $U \in \mathcal{F}_{i,-k}$, contradicting what was said before. So, we must have $|\mathcal{F}_{i,-k}| = 5$. In this case, having in mind the analysis of the hypothesis $|\mathcal{F}_{i,-k}| = 5$ done before, we may conclude that there exists a unique possible index distribution for the codewords of $\mathcal{F}_i$: $U_1, \ldots, U_4 \in \mathcal{F}_i^{(2)}$ presented in Table 5.36; $U_5, \ldots, U_8 \in \mathcal{F}_{i,-k}$ satisfying the conditions in Table 5.39; $U_9 \in \mathcal{F}_{i,k,-n,o}$ and $U_{10} \in \mathcal{F}_{i,m,n,-o}$.

Therefore, considering $W_1, \ldots, W_4 \in \mathcal{G}_i$ satisfying the conditions in Table 5.34 and assuming $|\mathcal{F}_i| = 10$, there are two possible index distributions for the codewords of $\mathcal{F}_i$ which have in common the codewords $U_1, \ldots, U_9$ satisfying:

U_1	i	j	$-l$	$-n$
U_2	i	k	$-m$	$-o$
U_3	i	l	n	$-k$
U_4	i	m	$-j$	o
U_5	i	$-k$	j	$-o$

U_6	i	$-k$	m	$-n$
U_7	i	$-k$	$-j$	$-m$
U_8	i	$-k$	$-l$	o
U_9	i	k	$-n$	o

Table 5.40: Possible index distributions for codewords of $\mathcal{F}_i$.

and differing in only one codeword, U_{10}, existing two hypothesis: $U_{10} \in \mathcal{F}_{i,-l,m,-o}$ or $U_{10} \in \mathcal{F}_{i,m,n,-o}$.

Let us now assume that $|\mathcal{F}_i| = 11$. In these conditions, from Proposition 5.6 it follows that $|\mathcal{F}_{i\alpha}| = 2$ for any $\alpha \in \mathcal{J} = \{j, -j, k, l, n\}$. Considering the possible index distributions for the codewords in $\mathcal{F}_{il}$, taking into account the codewords of $\mathcal{G}_i$ described in Table 5.34 and Lemma 1.5, we verify that if $U \in \mathcal{F}_{il}$, then

$$U \in \mathcal{F}_{i,l,-k,-m} \cup \mathcal{F}_{i,l,-k,n} \cup \mathcal{F}_{i,l,-k,o}.$$

Consequently, by Lemma 1.5, $|\mathcal{F}_{il}| \leq 1$, which contradicts Proposition 5.6.

- **Characterization of $\mathcal{F}_i$ when $|\mathcal{J}| = 6$**

Consider $W_1, \ldots, W_4 \in \mathcal{G}_i$ satisfying the following index distribution:

W_1	i	j	k	l	m
W_2	i	j	$-l$	$-k$	$-n$
W_3	i	k	$-l$	n	$-m$
W_4	i	l	$-k$	n	o

Table 5.41: Index distribution for $W_1, \ldots, W_4 \in \mathcal{G}_i$.

with $\mathcal{J} = \{j, k, -k, l, -l, n\}$. As, by Lemma 2.10, $|\mathcal{F}_i| = 10$ or $|\mathcal{F}_i| = 11$, we will analyze separately these two hypotheses. Let us begin considering $|\mathcal{F}_i| = 10$. By Lemma 2.10 we get $|\mathcal{F}_i^{(2)}| = 4$. Taking into account Lemma 2.14 as well as Lemma 1.5 and the codewords of $\mathcal{G}_i$, we must impose U_1,...,$U_4 \in \mathcal{F}_i^{(2)}$ satisfying $U_1 \in \mathcal{F}_{ij}$, $U_2 \in \mathcal{F}_{ik}$, $U_3 \in \mathcal{F}_{il}$ and $U_4 \in \mathcal{F}_{im}$, with $U_1, \ldots, U_4$ pairwise distinct. Considering, in particular, $U_1 \in \mathcal{F}_{ij}$, $U_2 \in \mathcal{F}_{ik}$ and $U_3 \in \mathcal{F}_{il}$, we present, taking into account the codewords of $\mathcal{G}_i$ and Lemma 1.5, all possible index distributions for these codewords:

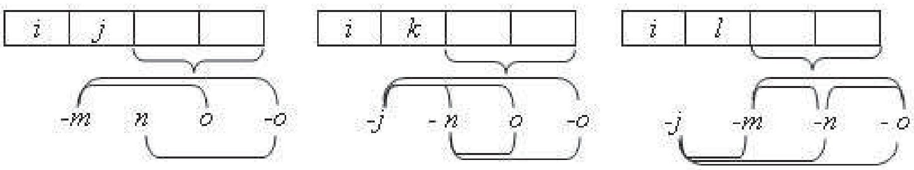

Figure 5.4: Possible index distributions for $U_1, U_2, U_3 \in \mathcal{F}_i^{(2)}$.

By the above schemes we conclude that:

– if $U_1 \in \mathcal{F}_{ij}$, then

$$U_1 \in \mathcal{F}_{i,j,-m,o} \cup \mathcal{F}_{i,j,-m,-o} \cup \mathcal{F}_{i,j,n,-o};$$

– if $U_2 \in \mathcal{F}_{ik}$, then

$$U_2 \in \mathcal{F}_{i,k,-j,-n} \cup \mathcal{F}_{i,k,-j,o} \cup \mathcal{F}_{i,k,-j,-o} \cup \mathcal{F}_{i,k,-n,o} \cup \mathcal{F}_{i,k,-n,-o};$$

– if $U_3 \in \mathcal{F}_{il}$, then

$$U_3 \in \mathcal{F}_{i,l,-j,-m} \cup \mathcal{F}_{i,l,-j,-n} \cup \mathcal{F}_{i,l,-j,-o} \cup \mathcal{F}_{i,l,-m,-n} \cup \mathcal{F}_{i,l,-m,-o} \cup \mathcal{F}_{i,l,-n,-o}.$$

Let $U_1, \ldots, U_4 \in \mathcal{F}_i^{(2)}$ such that $U_1 \in \mathcal{F}_{iju_1u_2}$, $U_2 \in \mathcal{F}_{iku_3u_4}$, $U_3 \in \mathcal{F}_{ilu_5u_6}$ and $U_4 \in \mathcal{F}_{imu_7u_8}$, with $u_1, \ldots, u_8 \in \mathcal{I} \backslash \{i, -i, j, k, l, m\}$. Considering Lemma 2.14 we must impose $u_1, \ldots, u_8$ pairwise distinct.

If $U_1 \in \mathcal{F}_{i,j,-m,o}$, then $U_2 \in \mathcal{F}_{i,k,-j} \cup \mathcal{F}_{i,k,-n,-o}$ and $U_3 \in \mathcal{F}_{i,l,-j} \cup \mathcal{F}_{i,l,-n,-o}$. Supposing $U_2 \in \mathcal{F}_{i,k,-j}$, then $U_3 \in \mathcal{F}_{i,l,-n,-o}$. Analyzing the hypotheses for $U_2 \in \mathcal{F}_{i,k,-j}$, we can not guarantee that $u_1, \ldots, u_6$ are pairwise distinct. On the other hand, if $U_2 \in \mathcal{F}_{i,k,-n,-o}$ any one of the possibilities for $U_3 \in \mathcal{F}_{il}$ will end up in a contradiction.

Now assume that $U_1 \in \mathcal{F}_{i,j,-m,-o}$. Under this assumption we must impose that $U_3 \in \mathcal{F}_{i,l,-j,-n}$. But considering all possible index distributions for $U_2 \in \mathcal{F}_{ik}$, we conclude that $u_1, \ldots, u_6$ are not pairwise distinct, which is a contradiction.

Let us now consider that $U_1 \in \mathcal{F}_{i,j,n,-o}$. In this case, $U_2 \in \mathcal{F}_{ik}$ and $U_3 \in \mathcal{F}_{il}$ satisfy one of the following index distributions:

1) $U_2 \in \mathcal{F}_{i,k,-n,o}$ and $U_3 \in \mathcal{F}_{i,l,-j,-m}$;

2) $U_2 \in \mathcal{F}_{i,k,-j,o}$ and $U_3 \in \mathcal{F}_{i,l,-m,-n}$.

Each one of the presented hypotheses implies $U_4 \in \mathcal{F}_{i,m,-k,-l}$, which is not possible since U_4 and W_2 contradict Lemma 1.5.

Therefore, we conclude that $|\mathcal{F}_i^{(2)}| \neq 4$ and so $|\mathcal{F}_i| \neq 10$.

Now consider $|\mathcal{F}_i| = 11$. By Proposition 5.9 there are, at least, five elements $\alpha \in \mathcal{J}$ satisfying $|\mathcal{F}_{i\alpha}| = 2$. Considering the codewords of $\mathcal{G}_i$ and Lemma 1.5, we get the following possible index distributions for the codewords of $\mathcal{F}_{i\alpha}$, assuming $|\mathcal{F}_{i\alpha}| = 2$, for $\alpha \in \mathcal{J} \backslash \{l\}$:

j				n				k				-l				-k			
i	j	-m	o	i	n	j	-o	i	k	-j	o	i	-l	-j	o	i	-k	-j	m
i	j	n	-o	i	n	-j	m	i	k	-n	-o	i	-l	m	-o	i	-k	-m	-o
								or				or				or			
								i	k	-n	o	i	-l	m	o	i	-k	m	-o
								i	k	-j	-o	i	-l	-j	-o	i	-k	-j	-m

Table 5.42: Possible index distributions for $\mathcal{F}_{i\alpha}$, with $\alpha \in \mathcal{J} \backslash \{l\}$ and $|\mathcal{F}_{i\alpha}| = 2$.

If $|\mathcal{F}_{il}| = 2$, then, unlike the other cases, there are more than two possibilities for $U, U' \in \mathcal{F}_{il}$. In this case, $U, U' \in \mathcal{F}_{il}$ are such that $U \in \mathcal{F}_{ilu_1u_2}$ and $U' \in \mathcal{F}_{ilu_3u_4}$, with $u_1, \ldots, u_4 \in \{-j, -m, -n, -o\}$ pairwise distinct.

Suppose that $|\mathcal{F}_{i\alpha}| = 2$ for any $\alpha \in \mathcal{J}$. By the analysis of the index distributions in Table 5.42, and taking into account Lemma 1.5, we conclude that the codewords $U_1, \ldots, U_9 \in \mathcal{F}_{ij} \cup \mathcal{F}_{ik} \cup \mathcal{F}_{i,-k} \cup \mathcal{F}_{i,-l} \cup \mathcal{F}_{in}$ whose index distribution satisfies:

U_1	i	j	$-m$	o
U_2	i	j	n	$-o$
U_3	i	n	$-j$	m
U_4	i	$-k$	m	$-o$
U_5	i	$-k$	$-j$	$-m$

U_6	i	$-l$	m	o
U_7	i	$-l$	$-j$	$-o$
U_8	i	k	$-j$	o
U_9	i	k	$-n$	$-o$

Table 5.43: Index distribution of $U_1, \ldots, U_9 \in \mathcal{F}_i$.

must be in $\mathcal{F}_i$. Consequently, $U_{10}, U_{11} \in \mathcal{F}_{il}$ are such that $U_{10} \in \mathcal{F}_{i,l,-j,-n}$ and $U_{11} \in \mathcal{F}_{i,l,-m,-o}$.

We have described a possible index distribution for the codewords of $\mathcal{F}_i$ assuming $|\mathcal{F}_i| = 11$ and considering that $|\mathcal{F}_{i\alpha}| = 2$ for any $\alpha \in \mathcal{J}$.

Next, we assume the existence of only five elements $\alpha \in \mathcal{J}$ satisfying $|\mathcal{F}_{i\alpha}| = 2$. In this case, by Proposition 5.9, we have $|\mathcal{F}_{i,-j}| = |\mathcal{F}_{i,-o}| = 5$. We note that, if $|\mathcal{F}_{i,-j}| = 5$, by Lemma 2.5 we must impose $|\mathcal{F}_{i,-j,\alpha}| = 1$ for each $\alpha \in \mathcal{I} \backslash \{i, -i, j, -j\}$. A similar conclusion is obtained when we consider $|\mathcal{F}_{i,-o}| = 5$, that is, $|\mathcal{F}_{i,-o,\alpha}| = 1$ for any $\alpha \in \mathcal{I} \backslash \{i, -i, o, -o\}$. Taking into account what was already said as well as Lemma 1.5 and the codewords of $\mathcal{G}_i$, $U_1, \ldots, U_5 \in \mathcal{F}_{i,-o}$ must satisfy the conditions presented in next table

U_1	i	$-o$	k	u_1
U_2	i	$-o$	$-l$	u_2
U_3	i	$-o$	n	u_3
U_4	i	$-o$	$-k$	u_4
U_5	i	$-o$	l	u_5

Table 5.44: Partial index distribution of $U_1, \ldots, U_5 \in \mathcal{F}_{i,-o}$.

with $u_1, \ldots, u_5$ pairwise distinct and satisfying:

- $u_1 \in \{-j, -n\}$;
- $u_2 \in \{-j, m\}$;
- $u_3 \in \{j, -j, m\}$;
- $u_4 \in \{-j, m, -m\}$;
- $u_5 \in \{-j, -m, -n\}$.

Then, we get the following possible index distributions for the codewords of $\mathcal{F}_{i,-o}$:

U_1	i	$-o$	k	$-j$
U_2	i	$-o$	$-l$	m
U_3	i	$-o$	n	j
U_4	i	$-o$	$-k$	$-m$
U_5	i	$-o$	l	$-n$

U_1	i	$-o$	k	$-n$
U_2	i	$-o$	$-l$	m
U_3	i	$-o$	n	j
U_4	i	$-o$	$-k$	$-m$
U_5	i	$-o$	l	$-j$

U_1	i	$-o$	k	$-n$
U_2	i	$-o$	$-l$	m
U_3	i	$-o$	n	j
U_4	i	$-o$	$-k$	$-j$
U_5	i	$-o$	l	$-m$

U_1	i	$-o$	k	$-n$
U_2	i	$-o$	$-l$	$-j$
U_3	i	$-o$	n	j
U_4	i	$-o$	$-k$	m
U_5	i	$-o$	l	$-m$

Table 5.45: Possible index distributions for the codewords of $\mathcal{F}_{i,-o}$.

However, for only one of these hypotheses it is possible to characterize the index distribution of all codewords of $\mathcal{F}_{i-j}$ without contradictions on definition of PL$(7, 2)$ code.

Namely, when: $U_1 \in \mathcal{F}_{i,-o,k,-n}$; $U_2 \in \mathcal{F}_{i,-o,-l,-j}$; $U_3 \in \mathcal{F}_{i,-o,n,j}$; $U_4 \in \mathcal{F}_{i,-o,-k,m}$; $U_5 \in \mathcal{F}_{i,-o,l,-m}$. In this case, considering the codewords of $\mathcal{G}_i \cup \mathcal{F}_{i,-o}$ and Lemma 1.5, $U_6, \ldots, U_9 \in \mathcal{F}_{i,-j}$ must verify:

U_6	i	$-j$	l	$-n$
U_7	i	$-j$	$-k$	$-m$
U_8	i	$-j$	n	m
U_9	i	$-j$	k	o

Table 5.46: Index distribution of $U_6, \ldots, U_9 \in \mathcal{F}_{i,-j}$.

As we are assuming $|\mathcal{F}_i| = 11$, we must identify in $\mathcal{F}_i$ two more codewords. Taking into account the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_{i,-j} \cup \mathcal{F}_{i,-o}$ and Lemma 1.5, we get three possible index distributions for $U_{10}, U_{11} \in \mathcal{F}_i \backslash (\mathcal{F}_{-j} \cup \mathcal{F}_{-o})$:

- $U_{10} \in \mathcal{F}_{i,j,-m,o}$ and $U_{11} \in \mathcal{F}_{i,-l,m,o}$;
- $U_{10} \in \mathcal{F}_{i,j,-m,o}$ and $U_{11} \in \mathcal{F}_{i,m,-n,o}$;
- $U_{10} \in \mathcal{F}_{i,-m,-n,o}$ and $U_{11} \in \mathcal{F}_{i,-l,m,o}$.

As we see, for the index distribution of the codewords of $\mathcal{G}_i$ presented in Table 5.41 it is possible characterize completely different index distributions for the codewords of $\mathcal{F}_i$.

Apparently, it seems that the presented examples for which we have described completely the index distributions of all the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ satisfy the definition of PL$(7, 2)$ code. However, when we consider other sets, $\mathcal{G}_\alpha \cup \mathcal{F}_\alpha$ with $\alpha \in \mathcal{I} \backslash \{i\}$, we verify that we always end up contradicting the definition of perfect 2-error correcting Lee code, as we will see in the next section.

5.3 Analysis of the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$

In this section we present the method which we have applied having in account all index distributions obtained for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$. Since the referred index distributions do not contradict the definition of PL$(7,2)$ code, the characterization of index distributions of other codewords of $\mathcal{G} \cup \mathcal{F}$ will be carried out. In this sense, we will analyze the codewords of $\mathcal{G}_\alpha \cup \mathcal{F}_\alpha$ for a certain $\alpha \in \mathcal{I}\backslash\{i\}$. We will concentrate our attention on elements $\alpha \in \mathcal{I}\backslash\{i,-i\}$ for which $|\mathcal{G}_{i\alpha}|$ is minimum and the respective value of $|\mathcal{F}_{i\alpha}|$ is maximum, since, in these conditions, it is necessary to characterize more codewords in $\mathcal{G}_\alpha$, being the number of possible index distributions for these codewords restricted by the known codewords of $\mathcal{F}_{i\alpha}$.

As we have identified many possible index distributions for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$, we will show throughout some illustrative examples how we always get a contradiction.

Example 1

Suppose that the index distributions of $W_1, \ldots, W_4 \in \mathcal{G}_i$ and $U_1, \ldots, U_{11} \in \mathcal{F}_i$ are the ones listed bellow:

W_1	i	j	k	l	m
W_2	i	j	$-l$	$-k$	n
W_3	i	k	$-l$	$-m$	$-n$
W_4	i	l	$-k$	$-m$	o

U_1	i	j	$-n$	o
U_2	i	j	$-m$	$-o$
U_3	i	$-m$	$-j$	n
U_4	i	l	n	$-o$
U_5	i	l	$-j$	$-n$
U_6	i	k	n	o
U_7	i	k	$-j$	$-o$
U_8	i	$-l$	$-j$	o
U_9	i	$-l$	m	$-o$
U_{10}	i	$-k$	$-j$	m
U_{11}	i	$-k$	$-n$	$-o$

Table 5.47: Index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$.

For showing that the presented index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ contradicts the definition of PL$(7,2)$ code, we will concentrate our attention on an element $\alpha \in \mathcal{I}\backslash\{i,-i\}$ for which $|\mathcal{G}_{i\alpha}|$ is minimum and $|\mathcal{F}_{i\alpha}|$ is maximum. As $|\mathcal{G}_{i,-o}| = 0$ and $|\mathcal{F}_{i,-o}| = 5$, we will consider the element $-o \in \mathcal{I}$.

We begin by analyzing the codewords of $\mathcal{G}_{-o}$. By Corollary 4.3 we know that $4 \leq |\mathcal{G}_{-o}| \leq 7$. Thus, we must characterize the index distribution of, at least, four codewords of $\mathcal{G}_{-o}$. We note that, until now we have not identified any codeword of $\mathcal{G}_{-o}$ since $|\mathcal{G}_{i,-o}| = 0$.

If $|\mathcal{G}_{-o}| = 4$, then from Proposition 5.1 we conclude that $|\mathcal{G}_{-o,-i}| \leq 2$ and, consequently, $|\mathcal{G}_{-o}\backslash(\mathcal{G}_i \cup \mathcal{G}_{-i})| \geq 2$. On the other hand, if $|\mathcal{G}_{-o}| \geq 5$, by Lemma 2.2 we have $|\mathcal{G}_{-o,-i}| \leq 3$ and, as a consequence, $|\mathcal{G}_{-o}\backslash(\mathcal{G}_i \cup \mathcal{G}_{-i})| \geq 2$. That is, under any assumption, we get always $|\mathcal{G}_{-o}\backslash(\mathcal{G}_i \cup \mathcal{G}_{-i})| \geq 2$. So, we will focus our attention on the codewords of $\mathcal{G}_{-o}\backslash(\mathcal{G}_i \cup \mathcal{G}_{-i})$.

To characterize the possible index distributions for the codewords of $\mathcal{G}_{-o}\backslash(\mathcal{G}_i \cup \mathcal{G}_{-i})$ it will be helpful to consider the partition $\mathcal{P}$ of $\mathcal{I}\backslash\{i,-i,o,-o\}$ induced by $U_2, U_4, U_7, U_9, U_{11} \in \mathcal{F}_{i,-o}$:

$$\mathcal{P}_1 = \{j,-m\};\ \ \mathcal{P}_2 = \{l,n\};\ \ \mathcal{P}_3 = \{k,-j\};\ \ \mathcal{P}_4 = \{-l,m\};\ \ \mathcal{P}_5 = \{-k,-n\}. \tag{5.8}$$

In fact, for $W \in \mathcal{G}_{-o}\backslash(\mathcal{G}_i \cup \mathcal{G}_{-i})$ satisfying $W \in \mathcal{G}_{-o,w_1,w_2,w_3,w_4}$ we must impose $w_1, \ldots, w_4 \in \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_5$ be such that $|\{w_1, \ldots, w_4\} \cap \mathcal{P}_p| \leq 1$ for $p = 1, \ldots, 5$, otherwise Lemma 1.5 is contradicted. Thus, taking into account the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$, see Table 5.47, and Lemma 1.5, we conclude that there exists only one possible index distribution for $W \in \mathcal{G}_{-o}\backslash(\mathcal{G}_i \cup \mathcal{G}_{-i})$: $W \in \mathcal{G}_{-o,-j,-k,-l,-m}$. That is, $|\mathcal{G}_{-o}\backslash(\mathcal{G}_i \cup \mathcal{G}_{-i})| \leq 1$, which is a contradiction. Therefore, the considered index distribution for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ contradicts the definition of PL$(7,2)$ code.

In this example it was simple to concluded that the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ contradict the necessary conditions for the existence of PL$(7,2)$ codes already established. However, in the majority of the cases this conclusion is obtained by a more complicated process of exhaustion, as we will see in next example.

Example 2

Now consider the following index distribution of $\mathcal{G}_i \cup \mathcal{F}_i$:

W_1	i	j	k	l	m
W_2	i	j	$-l$	n	$-k$
W_3	i	k	$-l$	o	$-n$
W_4	i	l	n	o	$-j$

U_1	i	j	$-m$	o
U_2	i	j	$-n$	$-o$
U_3	i	m	n	$-o$
U_4	i	k	$-m$	n
U_5	i	$-k$	m	o
U_6	i	$-j$	$-l$	m
U_7	i	$-l$	$-m$	$-o$
U_8	i	$-j$	k	$-o$
U_9	i	l	$-m$	$-n$
U_{10}	i	$-k$	l	$-o$
U_{11}	i	$-j$	$-k$	$-n$

Table 5.48: Index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$.

As $|\mathcal{G}_{i,-o}| = 0$ and $|\mathcal{F}_{i,-o}| = 5$, we will consider $-o \in \mathcal{I}$, beginning our analysis by the study of the codewords of $\mathcal{G}_{-o}$.

Such as in the previous example, $4 \leq |\mathcal{G}_{-o}| \leq 7$ and $|\mathcal{G}_{-o} \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})| \geq 2$.

Firstly, we are going to characterize the possible index distributions for the codewords of $\mathcal{G}_{-o} \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})$. With this purpose, we consider the following partition $\mathcal{P}$ of $\mathcal{I} \backslash \{i, -i, o, -o\}$ induced by the codewords $U_2, U_3, U_7, U_8, U_{10} \in \mathcal{F}_{i,-o}$:

$$\mathcal{P}_1 = \{j, -n\}; \;\; \mathcal{P}_2 = \{m, n\}; \;\; \mathcal{P}_3 = \{-l, -m\}; \;\; \mathcal{P}_4 = \{-j, k\}; \;\; \mathcal{P}_5 = \{-k, l\}. \quad (5.9)$$

As we have seen in the previous example, if $W \in \mathcal{G}_{-o} \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})$, then $W \in \mathcal{G}_{-o,w_1,w_2,w_3,w_4}$ with $w_1, \ldots, w_4 \in \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_5$ and $|\{w_1, \ldots, w_4\} \cap \mathcal{P}_p| \leq 1$ for all $p = 1, \ldots, 5$. Taking into account the considered index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ and Lemma 1.5, if $W \in \mathcal{G}_{-o} \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})$, then W is such that:

$$W \in \mathcal{G}_{-o,j,l,-m,n} \cup \mathcal{G}_{-o,-j,l,m,-n} \cup \mathcal{G}_{-o,-k,-l,m,-n} \cup \mathcal{G}_{-o,-j,-k,-m,n}.$$

Since $|\mathcal{G}_{-o} \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})| \geq 2$, having in view the possible index distributions for the codewords $W \in \mathcal{G}_{-o} \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})$ and Lemma 1.5, we conclude that $|\mathcal{G}_{-o} \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})| = 2$, with $W_5, W_6 \in \mathcal{G}_{-o} \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})$ satisfying one of the following index distributions.

W_5	$-o$	j	l	$-m$	n
W_6	$-o$	$-j$	l	m	$-n$

W_5	$-o$	j	l	$-m$	n
W_6	$-o$	$-k$	$-l$	m	$-n$

W_5	$-o$	$-j$	l	m	$-n$
W_6	$-o$	$-j$	$-k$	$-m$	n

W_5	$-o$	$-k$	$-l$	m	$-n$
W_6	$-o$	$-j$	$-k$	$-m$	n

Table 5.49: Possible index distributions for $W_5, W_6 \in \mathcal{G}_{-o} \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})$.

We have concluded that $|\mathcal{G}_{-o} \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})| = 2$. By Corollary 4.3, $4 \leq |\mathcal{G}_{-o}| \leq 7$, furthermore we have $|\mathcal{G}_{-o,i}| = 0$, then $|\mathcal{G}_{-o,-i}| \geq 2$. Thus, for each one of the possible index distributions for the codewords $W_5, W_6 \in \mathcal{G}_{-o} \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})$ we will analyze possible index distributions for the codewords $W \in \mathcal{G}_{-o,-i}$, having in mind that $|\mathcal{G}_{-o,-i}| \geq 2$.

Let us suppose that $W_5, W_6 \in \mathcal{G}_{-o} \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})$ are such that $W_5 \in \mathcal{G}_{-o,j,l,-m,n}$ and $W_6 \in \mathcal{G}_{-o,-j,l,m,-n}$. Considering the partition $\mathcal{P}$ of $\mathcal{I} \backslash \{i, -i, o, -o\}$, see (5.9), if $W \in \mathcal{G}_{-o,-i}$ is such that $W \in \mathcal{G}_{-o,-i,w_1,w_2,w_3}$, then $w_1, w_2, w_3 \in \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_5$ with $|\{w_1, w_2, w_3\} \cap \mathcal{P}_p| \leq 1$ for $p = 1, \ldots, 5$. Taking into account $W_5, W_6 \in \mathcal{G}_{-o} \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})$ as well as the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ and Lemma 1.5, we conclude that for $|\mathcal{G}_{-o,-i}| = 2$, $W_7, W_8 \in \mathcal{G}_{-o,-i}$ satisfy one of the following index distributions:

W_7					W_8				
-o	-i	k	-l	n	-o	-i	-j	-k	-m
					-o	-i	j	-k	m
					-o	-i	-k	-m	-n
-o	-i	-j	-l	n	-o	-i	j	-k	m
					-o	-i	k	-m	-n
					-o	-i	-k	-m	-n
-o	-i	k	-l	m	-o	-i	-j	-k	n
					-o	-i	-j	-k	-m
					-o	-i	-k	-m	-n
-o	-i	-k	-l	m	-o	-i	k	-m	-n

W_7					W_8				
-o	-i	-j	-k	n	-o	-i	j	k	-l
					-o	-i	j	-l	m
					-o	-i	k	-m	-n
-o	-i	-j	-k	-m	-o	-i	j	k	-l
					-o	-i	j	-l	m
-o	-i	j	-l	m	-o	-i	k	-m	-n
					-o	-i	-k	-m	-n
-o	-i	-j	-k	-l	-o	-i	k	-m	-n
-o	-i	j	-k	m	-o	-i	k	-m	-n
-o	-i	j	k	-l	-o	-i	-k	-m	-n

Table 5.50: Possible index distributions for $W_7, W_8 \in \mathcal{G}_{-o,-i}$.

If $|\mathcal{G}_{-o,-i}| = 3$, there are only two hypotheses for the index distribution of the codewords $W_7, W_8, W_9 \in \mathcal{G}_{-o,-i}$:

– $W_7 \in \mathcal{G}_{-o,-i,-j,-l,n}$, $W_8 \in \mathcal{G}_{-o,-i,j,-k,m}$, $W_9 \in \mathcal{G}_{-o,-i,k,-m,-n}$;

– $W_7 \in \mathcal{G}_{-o,-i,-j,-k,n}$, $W_8 \in \mathcal{G}_{-o,-i,j,-l,m}$, $W_9 \in \mathcal{G}_{-o,-i,k,-m,-n}$.

We note that, by Lemma 2.2, $|\mathcal{G}_{-o,-i}| \leq 3$. Therefore, all possible index distributions for the codewords of $\mathcal{G}_{-o,-i}$ are described when $W_5, W_6 \in \mathcal{G}_{-o} \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})$ assume the considered index distribution. In this case, unlike what has happened in the Example 1, we can characterize different index distributions for all codewords of $\mathcal{G}_{-o}$. Next step consists in the identification of all possible index distributions for all codewords of $\mathcal{F}_{-o}$, for each one of the presented index distributions of $\mathcal{G}_{-o}$. We will do it for some of the presented index distributions of the codewords of $\mathcal{G}_{-o}$, since the remaining cases follow the same reasoning.

Suppose that $|\mathcal{G}_{-o}| = 4$, with $W_5, W_6 \in \mathcal{G}_{-o} \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})$ satisfying the index distribution in consideration, that is, $W_5 \in \mathcal{G}_{-o,j,l,-m,n}$ and $W_6 \in \mathcal{G}_{-o,-j,l,m,-n}$, and $W_7, W_8 \in \mathcal{G}_{-o,-i}$ such that $W_7 \in \mathcal{G}_{-o,-i,k,-l,n}$ and $W_8 \in \mathcal{G}_{-o,-i,-j,-k,-m}$. We may analyze the index distribution of the codewords of $\mathcal{F}_{-o}$ applying equivalent results to the ones derived in Section 5.1, since by assumption $|\mathcal{G}_{-o}| = 4$, however, since the index distributions of some codewords, in particular, of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i \cup \mathcal{G}_{-o}$ are known, it is simpler to do it analyzing all possible index distributions for the codewords of each one of the following sets: $\mathcal{F}_{-o,-i}$, $\mathcal{F}_{-o,j}$, $\mathcal{F}_{-o,-j}$, $\mathcal{F}_{-o,k}$, $\mathcal{F}_{-o,-k}$, $\mathcal{F}_{-o,l}$, $\mathcal{F}_{-o,-l}$, $\mathcal{F}_{-o,m}$, $\mathcal{F}_{-o,-m}$, $\mathcal{F}_{-o,n}$, $\mathcal{F}_{-o,-n}$. As $|\mathcal{F}_{-o,i}| = 5$, taking into account Lemma 2.2 we should not consider any more codewords in $\mathcal{F}_{-o,i}$. The following schemes represent all possible index distributions for the codewords of $\mathcal{F}_{-o} \backslash \mathcal{F}_i$. These index distributions were obtained taking into account the codewords of $\mathcal{G}_i \cup \mathcal{F}_i \cup \mathcal{G}_{-o}$ and Lemma 1.5.

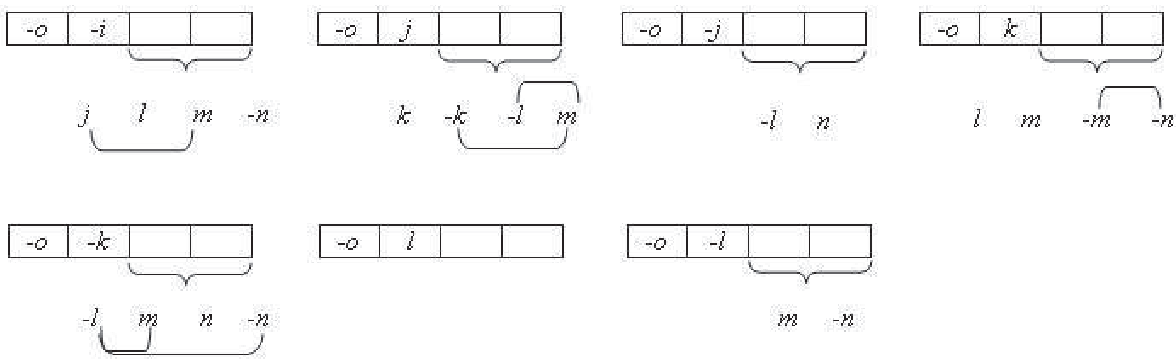

Figure 5.5: Possible index distributions for $U \in \mathcal{F}_{-o} \backslash \mathcal{F}_i$.

In the previous schemes we are saying that if $U \in \mathcal{F}_{-o,-i}$ is such that $U \in \mathcal{F}_{-o,-i,u_1,u_2}$, then $u_1, u_2 \in \{j, l, m, -n\}$. However, there exists only one possible index distribution for this codeword, $U \in \mathcal{F}_{-o,-i,j,m}$, since considering the remaining hypotheses we will always end up with a contradiction. In fact, if:

- $u_1 = j$ and $u_2 = l$, U contradicts Lemma 1.5 with W_5;
- $u_1 = j$ and $u_2 = -n$, U contradicts Lemma 1.5 with U_2;
- $u_1 = l$ and $u_2 = m$, U contradicts Lemma 1.5 with W_6;
- $u_1 = l$ and $u_2 = -n$, U contradicts Lemma 1.5 with W_6;
- $u_1 = m$ and $u_2 = -n$, U contradicts Lemma 1.5 with W_6.

If $U \in \mathcal{F}_{-o,j} \backslash (\mathcal{F}_i \cup \mathcal{F}_{-i})$, then $U \in \mathcal{F}_{-o,j,-k,m}$ or $\mathcal{F}_{-o,j,-l,m}$, and so on.

We note that, we have not presented possible index distributions for the codewords of $\mathcal{F}_{-o,m} \cup \mathcal{F}_{-o,-m} \cup \mathcal{F}_{-o,n} \cup \mathcal{F}_{-o,-n}$ since these codewords were analyzed when considered the sets $\mathcal{F}_{-o,\alpha}$ for $\alpha \in \{-i, j, -j, k, -k, l, -l\}$.

By the analysis of the schemes in Figure 5.5 we may conclude that if $U \in \mathcal{F}_{-o} \backslash \mathcal{F}_i$, then

$$U \in \mathcal{F}_{-o,j,m} \cup \mathcal{F}_{-o,k,-m,-n} \cup \mathcal{F}_{-o,-k,-l}.$$

Taking into account Lemma 1.5 we conclude that $|\mathcal{F}_{-o} \backslash \mathcal{F}_i| \leq 3$ and, as $|\mathcal{F}_{-o,i}| = 5$, we get $|\mathcal{F}_{-o}| \leq 8$, which is an absurdity since we are assuming $|\mathcal{G}_{-o}| = 4$ and from Lemma 2.10 it follows that $|\mathcal{F}_{-o}| = 10$ or $|\mathcal{F}_{-o}| = 11$. Therefore, the considered index distribution for the codewords of $\mathcal{G}_{-o}$ contradicts necessary conditions for the existence of PL$(7, 2)$ codes.

In the majority of the remaining possibilities for the index distributions of the codewords of $\mathcal{G}_{-o}$ we conclude, applying the same reasoning, that $|\mathcal{F}_{-o}| \leq 9$. However, there are some cases in which this does not happen, as we will see in the next characterizations of the codewords of $\mathcal{G}_{-o}$.

Now consider that $|\mathcal{G}_{-o}| = 4$ with $W_7 \in \mathcal{G}_{-o,-i,-j,-l,n}$ and $W_8 \in \mathcal{G}_{-o,-i,k,-m,-n}$. In these conditions, proceeding as in the previous case, we get the following possible index distributions for the codewords $U \in \mathcal{F}_{-o} \backslash \mathcal{F}_i$.

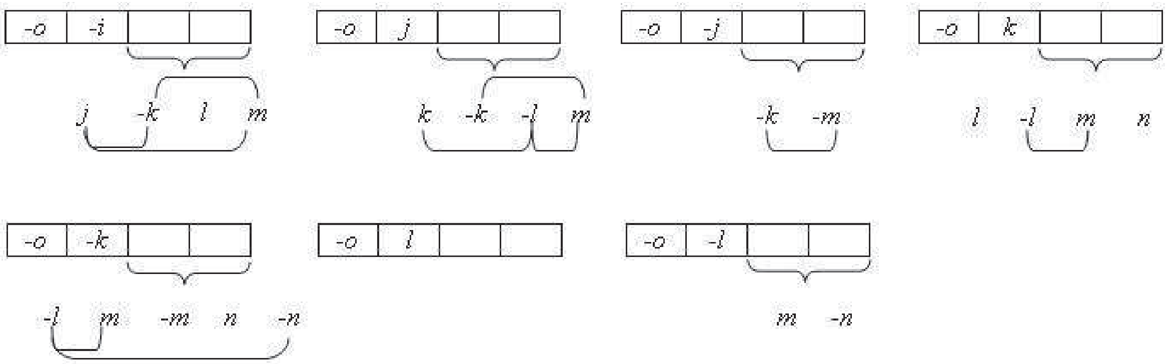

Figure 5.6: Possible index distributions for $U \in \mathcal{F}_{-o} \backslash \mathcal{F}_i$.

Analyzing the above schemes we conclude that if $U \in \mathcal{F}_{-o} \backslash \mathcal{F}_i$, then

$$U \in \mathcal{F}_{-o,-i,-k} \cup \mathcal{F}_{-o,j,m} \cup \mathcal{F}_{-o,k,-l} \cup \mathcal{F}_{-o,-j,-k,-m} \cup \mathcal{F}_{-o,-k,-l}.$$

Consequently, considering Lemma 1.5 and taking into account that $|\mathcal{F}_{-o,i}| = 5$, we get $|\mathcal{F}_{-o}| \leq 10$. However, by Lemma 2.10 we must impose $|\mathcal{F}_{-o}| = 10$ and, by the same lemma, $|\mathcal{F}_{-o}^{(2)}| = 4$. Considering Lemma 2.14 and the possible index distributions for the codewords of $\mathcal{F}_{-o}$, we conclude that if $U, U', U'', U''' \in \mathcal{F}_{-o}^{(2)}$, then we must have: $U \in \mathcal{F}_{-o,i,u_1,u_2}$; $U' \in \mathcal{F}_{-o,-k,u_3,u_4}$; $U'' \in \mathcal{F}_{-o,j,m,u_5}$; $U''' \in \mathcal{F}_{-o,k,-l,u_6}$; with $u_1, \ldots, u_6 \in \{-i, -j, l, -m, n, -n\}$ pairwise distinct. Consider $U''' \in \mathcal{F}_{-o,k,-l,u_6}$, looking at the schemes in Figure 5.6, we conclude that $u_6 = j$ or $u_6 = m$, contradicting what was said before and so the considered index distribution for the codewords of $\mathcal{G}_{-o}$ contradicts the definition of PL$(7,2)$ code.

There exist few hypotheses for $\mathcal{G}_{-o}$ for which it is possible to characterize completely all codewords of $\mathcal{G}_{-o} \cup \mathcal{F}_{-o}$ without contradictions on definition of PL$(7,2)$ code, being necessary, in these cases, to analyze other sets $\mathcal{G}_\alpha \cup \mathcal{F}_\alpha$ for $\alpha \in \mathcal{I} \backslash \{i, -o\}$. We present next one of these cases.

Now suppose that $|\mathcal{G}_{-o}| = 5$ with $W_7, W_8, W_9 \in \mathcal{G}_{-o,-i}$ such that $W_7 \in \mathcal{G}_{-o,-i,-j,-l,n}$, $W_8 \in \mathcal{G}_{-o,-i,j,-k,m}$ and $W_9 \in \mathcal{G}_{-o,-i,k,-m,-n}$. Taking into account the codewords of $\mathcal{G}_i \cup \mathcal{F}_i \cup \mathcal{G}_{-o}$ and Lemma 1.5, we obtain the following possible index distributions for the codewords of $\mathcal{F}_{-o} \backslash \mathcal{F}_i$.

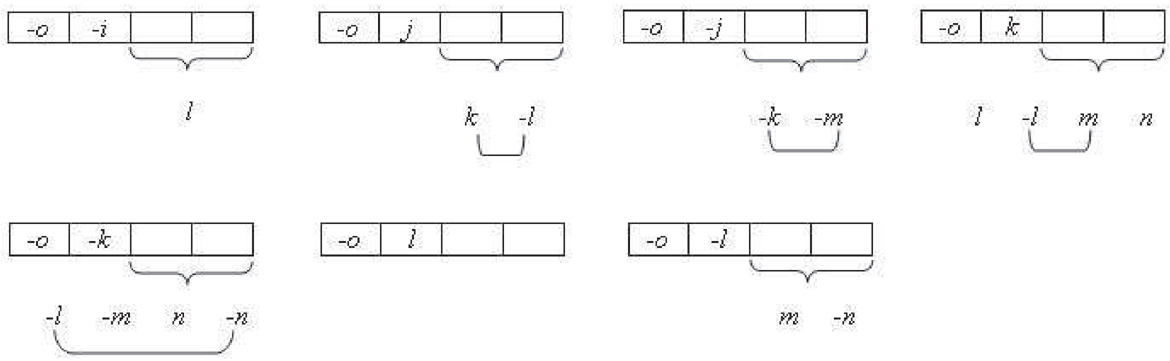

Figure 5.7: Possible index distributions for $U \in \mathcal{F}_{-o} \backslash \mathcal{F}_i$.

That is, if $U \in \mathcal{F}_{-o} \backslash \mathcal{F}_i$, then

$$U \in \mathcal{F}_{-o,k,-l} \cup \mathcal{F}_{-o,-j,-k,-m} \cup \mathcal{F}_{-o,-k,-l,-n}.$$

By assumption $|\mathcal{G}_{-o}| = 5$, accordingly, by Lemma 2.11 we get $7 \leq |\mathcal{F}_{-o}| \leq 10$. If $|\mathcal{F}_{-o}| = 7$, then from Lemma 2.11 it follows that $|\mathcal{F}_{-o}^{(2)}| = 4$. However, the possible index distributions for the codewords of $\mathcal{F}_{-o}$ contradicts Lemma 2.14 since if $U \in \mathcal{F}_{-o}$, then

$$U \in \mathcal{F}_{-o,i} \cup \mathcal{F}_{-o,k} \cup \mathcal{F}_{-o,-k}.$$

Thus, in these conditions, we must impose $|\mathcal{F}_{-o}| = 8$, with $U_{12} \in \mathcal{F}_{-o,-j,-k,-m}$, $U_{13} \in \mathcal{F}_{-o,-k,-l,-n}$ and $U_{14} \in \mathcal{F}_{-o,k,-l,u}$, with $u \in \{j, m\}$. Therefore, we can have a complete admissible characterization of all codewords of $\mathcal{G}_{-o} \cup \mathcal{F}_{-o}$. Then, we must analyze one other set $\mathcal{G}_\alpha \cup \mathcal{F}_\alpha$ for $\alpha \in \mathcal{I} \backslash \{i, -o\}$.

Let us consider the index $-k \in \mathcal{I}$. Until now the index distributions of the following codewords of $\mathcal{G}_{-k} \cup \mathcal{F}_{-k}$ are characterized:

W_2	i	j	$-l$	n	$-k$
W_8	$-o$	$-i$	j	$-k$	m

U_5	i	$-k$	m	o
U_{10}	i	$-k$	l	$-o$
U_{11}	i	$-j$	$-k$	$-n$
U_{12}	$-o$	$-j$	$-k$	$-m$
U_{13}	$-o$	$-k$	$-l$	$-n$

Table 5.51: Index distribution of codewords of $\mathcal{G}_{-o} \cup \mathcal{F}_{-o}$.

By Corollary 4.3 it follows that $4 \leq |\mathcal{G}_{-k}| \leq 7$ and so we must identify in $\mathcal{G}_{-k}$, at least, two more codewords. To do it, we will consider the partition $\mathcal{Q}$ of $\mathcal{I}\backslash\{i,k,-k,-o\}$ induced by the codewords $W_8, U_{10}, U_{12}, U_{13} \in \mathcal{G}_{-k,-o} \cup \mathcal{F}_{-k,-o}$:

$$\mathcal{Q}_1 = \{-i, j, m\}; \quad \mathcal{Q}_2 = \{l\}; \quad \mathcal{Q}_3 = \{-j, -m\}; \quad \mathcal{Q}_4 = \{-l, -n\}; \quad \mathcal{Q}_5 = \{n\}; \quad \mathcal{Q}_6 = \{o\}.$$

If $W \in \mathcal{G}_{-k}\backslash(\mathcal{G}_i \cup \mathcal{G}_{-o})$ is such that $W \in \mathcal{G}_{-k,w_1,w_2,w_3,w_4}$, then $w_1,\ldots,w_4 \in \mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_6$ and $|\{w_1,\ldots,w_4\} \cap \mathcal{Q}_q| \leq 1$ for $q = 1,\ldots,6$. We note that, $|\mathcal{G}_{-k,i}| = 1$, $|\mathcal{F}_{-k,i}| = 3$, $|\mathcal{G}_{-k,-o}| = 1$ and $|\mathcal{F}_{-k,-o}| = 3$. Accordingly, by Lemma 2.2 the codewords presented in Table 5.51 are the unique ones in $\mathcal{G}_{-k,i} \cup \mathcal{F}_{-k,i} \cup \mathcal{G}_{-k,-o} \cup \mathcal{F}_{-k,-o}$. Taking into account the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i \cup \mathcal{G}_{-o} \cup \mathcal{F}_{-o}$ and Lemma 1.5 we get the following hypotheses for the codewords $W \in \mathcal{G}_{-k}\backslash(\mathcal{G}_i \cup \mathcal{G}_{-o})$:

$-k$	$-i$	l	$-m$	o
$-k$	$-i$	l	$-n$	o
$-k$	j	l	$-n$	o

$-k$	$-i$	$-l$	$-m$	o
$-k$	$-i$	$-j$	$-n$	o
$-k$	$-i$	$-m$	n	o

Table 5.52: Possible index distributions for $W \in \mathcal{G}_{-k}\backslash(\mathcal{G}_i \cup \mathcal{G}_{-o})$.

Taking into account Lemma 1.5, the analysis of the possible index distributions for $W \in \mathcal{G}_{-k}\backslash(\mathcal{G}_i \cup \mathcal{G}_{-o})$ described above, leads us to conclude that $|\mathcal{G}_{-k}\backslash(\mathcal{G}_i \cup \mathcal{G}_{-o})| \leq 2$ and, consequently, $|\mathcal{G}_{-k}| = 4$. Furthermore, the codewords $W_{10}, W_{11} \in \mathcal{G}_{-k}\backslash(\mathcal{G}_i \cup \mathcal{G}_{-o})$ satisfy one of the following conditions:

1) $W_{10} \in \mathcal{G}_{-k,j,l,-n,o}$ and $W_{11} \in \mathcal{G}_{-k,-i,-l,-m,o}$;

2) $W_{10} \in \mathcal{G}_{-k,j,l,-n,o}$ and $W_{11} \in \mathcal{G}_{-k,-i,-m,n,o}$.

Next, for each one of these hypotheses we will concentrate our attention in the codewords of $\mathcal{F}_{-k}$. We note that, until now, only five codewords of $\mathcal{F}_{-k}$ are characterized. Since $|\mathcal{G}_{-k}| = 4$, by Lemma 2.10 we have $|\mathcal{F}_{-k}| = 10$ or $|\mathcal{F}_{-k}| = 11$. Therefore, we must identify in $\mathcal{F}_{-k}$, at least, five more codewords.

Suppose that $W_{10}, W_{11} \in \mathcal{G}_{-k}\backslash(\mathcal{G}_i \cup \mathcal{G}_{-o})$ verify the conditions in 1), that is, $W_{10} \in \mathcal{G}_{-k,j,l,-n,o}$ and $W_{11} \in \mathcal{G}_{-k,-i,-l,-m,o}$. Having in mind all known index distributions of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i \cup \mathcal{G}_{-o} \cup \mathcal{F}_{-o} \cup \mathcal{G}_{-k}$ and Lemma 1.5 we get the following possible index distributions for $U \in \mathcal{F}_{-k}\backslash(\mathcal{F}_i \cup \mathcal{F}_{-o})$.

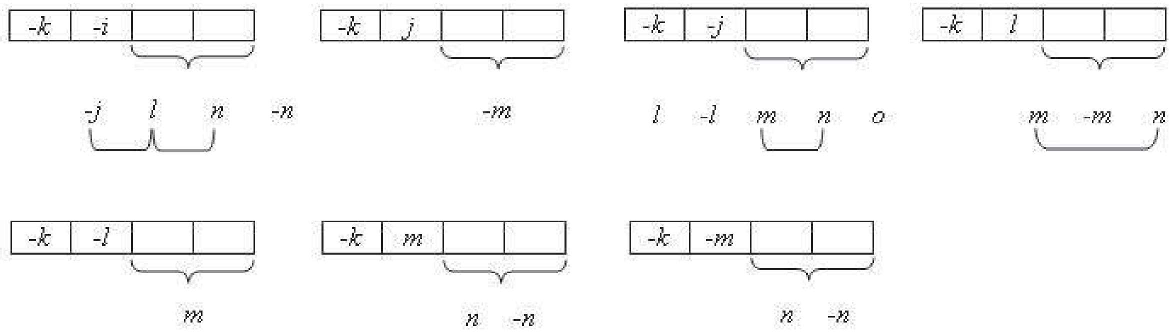

Figure 5.8: Possible index distributions for $U \in \mathcal{F}_{-k} \backslash (\mathcal{F}_i \cup \mathcal{F}_{-o})$.

That is, if $U \in \mathcal{F}_{-k} \backslash (\mathcal{F}_i \cup \mathcal{F}_{-o})$, then $U \in \mathcal{F}_{-k,-i,l} \cup \mathcal{F}_{,-k,m,n}$. Considering Lemma 1.5 we get $|\mathcal{F}_{-k} \backslash (\mathcal{F}_i \cup \mathcal{F}_{-o})| \leq 2$. Consequently, since $|\mathcal{F}_{-k,i} \cup \mathcal{F}_{-k,-o}| = 5$, we get $|\mathcal{F}_{-k}| \leq 7$ which is an absurdity. Assuming that $W_{10}, W_{11} \in \mathcal{G}_{-k} \backslash (\mathcal{G}_i \cup \mathcal{G}_{-o})$ satisfy the conditions in 2), following a similar reasoning we obtain again $|\mathcal{F}_{-k}| < 10$. Therefore, we conclude that although we can characterize the index distribution of all codewords of $\mathcal{G}_i \cup \mathcal{F}_i \cup \mathcal{G}_{-o} \cup \mathcal{F}_{-o}$, when we consider one other element in $\mathcal{I} \backslash \{i, -o\}$, in this case, $-k \in \mathcal{I}$, we get contradictions on the definition of PL$(7, 2)$ code.

Considering the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ presented in Table 5.48, for any possible index distribution of the codewords of $\mathcal{G}_{-o}$ we always end up, as in the presented examples, with one of the following conclusions:

– it is not possible to characterize the index distribution of all codewords of $\mathcal{F}_{-o}$;

– we can describe the index distribution of all codewords of $\mathcal{F}_{-o}$ but when we consider other element $\alpha \in \mathcal{I} \backslash \{i, -o\}$ it is impossible to characterize the index distribution of the codewords of $\mathcal{G}_\alpha \cup \mathcal{F}_\alpha$.

Consequently, the index distribution for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ presented in Table 5.48 contradicts the definition of PL$(7, 2)$ code.

Example 3

As we have said at the beginning of this section, we would analyze all obtained index distributions for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ considering other sets $\mathcal{G}_\alpha \cup \mathcal{F}_\alpha$ for $\alpha \in \mathcal{I}\backslash\{i\}$. We have also said that we would give preference to the elements $\alpha \in \mathcal{I}\backslash\{i\}$ so that $|\mathcal{G}_{i\alpha}|$ is minimum and $|\mathcal{F}_{i\alpha}|$ is maximum. In the previous examples, considering the codewords of $\mathcal{G}_i$, we have verified the existence of elements $\alpha \in \mathcal{I}\backslash\{i\}$ satisfying $|\mathcal{G}_{i\alpha}| = 0$, concentrating our attention on these elements. However, there are index distributions for $\mathcal{G}_i \cup \mathcal{F}_i$ in which $|\mathcal{G}_{i\alpha}| \geq 1$ for all $\alpha \in \mathcal{I}\backslash\{i, -i\}$, in particular, when $|\mathcal{J}| = 4$. In these cases, the criterion to choose the element $\alpha \in \mathcal{I}\backslash\{i, -i\}$ is the same, that is, we give preference to the elements for which $|\mathcal{G}_{i\alpha}|$ is minimum, concentrating our attention on the elements which satisfy $|\mathcal{G}_{i\alpha}| = 1$ and $|\mathcal{F}_{i\alpha}| = 3$. We note that, by Lemma 2.2, if $|\mathcal{G}_{i\alpha}| = 1$, then $|\mathcal{F}_{i\alpha}| \leq 3$.

The following example describe one of these cases.

Suppose that the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ satisfy:

W_1	i	j	k	l	m
W_2	i	j	$-k$	$-l$	n
W_3	i	$-k$	$-j$	$-m$	o
W_4	i	k	$-j$	$-n$	$-o$

U_1	i	j	$-m$	$-o$
U_2	i	k	$-l$	o
U_3	i	$-j$	l	n
U_4	i	$-k$	m	$-n$
U_5	i	k	$-m$	n
U_6	i	l	$-k$	$-o$
U_7	i	m	n	o
U_8	i	l	$-n$	o
U_9	i	$-l$	$-m$	$-n$
U_{10}	i	$-l$	m	$-o$

Table 5.53: Index distribution for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$.

In this case for all $\alpha \in \mathcal{I}\backslash\{i, -i\}$ we have $|\mathcal{G}_{i\alpha}| \geq 1$. Thus, as said before, we will focus our attention on an element $\alpha \in \mathcal{I}\backslash\{i, -i\}$ satisfying $|\mathcal{G}_{i\alpha}| = 1$ and $|\mathcal{F}_{i\alpha}| = 3$. There are several elements in these conditions, let us consider the element $-o \in \mathcal{I}$. Since we have characterized the index distribution of one codeword of $\mathcal{G}_{-o}$, $W_4 \in \mathcal{G}_{i,k,-j,-n,-o}$, taking into account Corollary 4.3 we must identify, at least, three more codewords in

$\mathcal{G}_{-o}$. We note that, if $|\mathcal{G}_{-o}| = 4$, then by Proposition 5.1 we get $|\mathcal{G}_{-o,-i}| \leq 2$ and, consequently, $|\mathcal{G}_{-o} \backslash \mathcal{G}_{-i}| \geq 2$. On the other hand, if $|\mathcal{G}_{-o}| \geq 5$, since from Lemma 2.2 it follows that $|\mathcal{G}_{-o,-i}| \leq 3$, then $|\mathcal{G}_{-o} \backslash \mathcal{G}_{-i}| \geq 2$. As $|\mathcal{G}_{-o,i}| = 1$, in both cases we have $|\mathcal{G}_{-o} \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})| \geq 1$. This is one of the differences when we compare this example with the previous examples. In the examples presented before we had to impose $|\mathcal{G}_{-o} \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})| \geq 2$. We could think that in this example there are more possible hypotheses for the index distribution of $W \in \mathcal{G}_{-o} \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})$, however since we have identified one codeword in $\mathcal{G}_{-o,i}$, its index distribution contributes to restrict the possibilities for the codewords of $\mathcal{G}_{-o} \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})$, as we will see.

As in the previous examples, to simplify the characterization of possible index distributions for $W \in \mathcal{G}_{-o} \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})$, we will consider the partition $\mathcal{P}$ of $\mathcal{I} \backslash \{i, -i, o, -o\}$ induced by the codewords $W_4 \in \mathcal{G}_{i,-o}$ and $U_1, U_6, U_{10} \in \mathcal{F}_{i,-o}$:

$$\mathcal{P}_1 = \{k, -j, -n\}; \;\; \mathcal{P}_2 = \{j, -m\}; \;\; \mathcal{P}_3 = \{l, -k\}; \;\; \mathcal{P}_4 = \{-l, m\}; \;\; \mathcal{P}_5 = \{n\}.$$

Considering this partition as well as the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ and Lemma 1.5, we conclude that if $W \in \mathcal{G}_{-o} \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})$, then

$$W \in \mathcal{G}_{-o,-j,-l,-m,n} \cup \mathcal{G}_{-o,-j,-k,m,n}.$$

Taking into account Lemma 1.5, we must impose $|\mathcal{G}_{-o} \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})| \leq 1$. Since $|\mathcal{G}_{-o,i}| = 1$, we get $|\mathcal{G}_{-o,-i}| \geq 2$. By Lemma 2.2, $|\mathcal{G}_{-o,-i}| \leq 3$, consequently, $|\mathcal{G}_{-o}| = 4$ or $|\mathcal{G}_{-o}| = 5$. Next, we proceed as in the previous examples, we characterize all possible index distributions for the codewords of $\mathcal{G}_{-o,-i}$ and for each one of the possible index distributions of the codewords of $\mathcal{G}_{-o}$ we try to characterize all codewords of $\mathcal{F}_{-o}$. Similarly to the other presented examples, we always get contradictions on the necessary conditions for the existence of PL$(7, 2)$ codes, concluding again that the index distribution for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ presented in Table 5.53 are not allowed.

Although we have presented here only some examples, we have scrutinized all possible index distributions for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$, applying the reasoning presented in this section, ending always up in a contradiction. In fact, for all of them it is always possible to find an element $\alpha \in \mathcal{I} \backslash \{i\}$ for which it is not possible to characterize completely the index distribution of all codewords of $\mathcal{G}_\alpha \cup \mathcal{F}_\alpha$. Thus, we are in conditions to enunciate the following theorem.

Theorem 5.1 *For any* $\alpha \in \mathcal{I}$, $|\mathcal{G}_\alpha| \neq 4$.

As an immediate consequence of Theorem 5.1 and Corollary 4.3 we get:

Corollary 5.1 *For any* $\alpha \in \mathcal{I}$, $5 \leq |\mathcal{G}_\alpha| \leq 7$.

Chapter 6

Proof of $|\mathcal{G}_i| \neq 5$ for any $i \in \mathcal{I}$

In the previous chapters it is proved that the assumption of being a perfect 2-error correcting Lee code of word length 7 over $\mathbb{Z}^n$ is contradicted when $|\mathcal{G}_\alpha| \in \{3, 4, 8\}$, $\alpha \in \mathcal{I}$. As an immediate consequence we get $5 \leq |\mathcal{G}_\alpha| \leq 7$ for any $\alpha \in \mathcal{I}$. Here, we analyze the hypothesis $|\mathcal{G}_\alpha| = 5$ for some $\alpha \in \mathcal{I}$.

Let us then assume $|\mathcal{G}_i| = 5$ for $i \in \mathcal{I}$.

Since, from Lemma 2.2, $|\mathcal{G}_{i\alpha}| \leq 3$ for any $\alpha \in \mathcal{I} \backslash \{i, -i\}$, we will distinguish the cases:

1) $|\mathcal{G}_{i\alpha}| = 3$ for some $\alpha \in \mathcal{I} \backslash \{i, -i\}$;

2) $|\mathcal{G}_{i\alpha}| \leq 2$ for any $\alpha \in \mathcal{I} \backslash \{i, -i\}$.

In this chapter we study, separately, these two hypotheses. For each one of them we derive, initially, some conditions that must be satisfied by elements of $\mathcal{I}$, which will be useful for the characterization of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$. As we will see, in both cases there are different possible index distributions for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ satisfying the definition of PL$(7, 2)$ code, however, considering other elements of $\mathcal{I} \backslash \{i\}$, we will verify that any concretization of $\mathcal{G}_i \cup \mathcal{F}_i$ contradicts the definition of perfect Lee code, proving that $|\mathcal{G}_\alpha| \neq 5$ for any $\alpha \in \mathcal{I}$.

6.1 $|\mathcal{G}_{i\alpha}| = 3$ for some $\alpha \in \mathcal{I}\backslash\{i, -i\}$

Throughout this section we assume the existence of an element $\alpha \in \mathcal{I}\backslash\{i, -i\}$ satisfying $|\mathcal{G}_{i\alpha}| = 3$. Since we have considered

$$\mathcal{I} = \{i, -i, j, -j, k, -k, l, -l, m, -m, n, -n, o, -o\},$$

we assume, without loss of generality, $|\mathcal{G}_{ij}| = 3$.

This section is divided into three subsections. In the first one we present some results which help us to get all possible index distributions for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$, that will be given in the second subsection. In last subsection we prove through illustrative cases that any set $\mathcal{G}_i \cup \mathcal{F}_i$ does not satisfy the definition of PL$(7, 2)$ code.

6.1.1 Necessary conditions for the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$

Let us consider $|\mathcal{G}_i| = 5$ and $|\mathcal{G}_{ij}| = 3$. Then, by Lemma 2.11, $7 \leq |\mathcal{F}_i| \leq 10$.

The following proposition restricts even more the variation of $|\mathcal{F}_i|$.

Proposition 6.1 *If* $|\mathcal{G}_i| = 5$ *and* $|\mathcal{G}_{ij}| = 3$*, then* $|\mathcal{F}_{ij}| = 0$ *and* $8 \leq |\mathcal{F}_i| \leq 9$.

Proof. Since $|\mathcal{G}_{ij}| = 3$, from Lemma 2.2 it follows that

$$|\mathcal{D}_{ij} \cup \mathcal{E}_{ij}| + 2|\mathcal{F}_{ij}| + 9 = 10,$$

implying $|\mathcal{F}_{ij}| = 0$ and $|\mathcal{D}_{ij} \cup \mathcal{E}_{ij}| = 1$.

As $|\mathcal{G}_i| = 5$ and $|\mathcal{G}_{ij}| = 3$, by Lemma 2.15 we get $8 \leq |\mathcal{F}_i| \leq 10$. Supposing $|\mathcal{F}_i| = 10$, by Lemma 2.1 we must impose $|\mathcal{D}_i \cup \mathcal{E}_i| = 0$, which contradicts $|\mathcal{D}_{ij} \cup \mathcal{E}_{ij}| = 1$. Therefore, $8 \leq |\mathcal{F}_i| \leq 9$. □

It is possible, up to an equivalent index distributions, to characterize all codewords of $\mathcal{G}_{ij}$, as we will see in the next proposition.

Proposition 6.2 *The index distribution of the codewords $W_1, W_2, W_3 \in \mathcal{G}_{ij}$ satisfies:*

W_1	i	j	k	l	m
W_2	i	j	$-k$	$-l$	n
W_3	i	j	$-m$	$-n$	o

Proof. Let W_1, W_2 and W_3 be the codewords of $\mathcal{G}_{ij}$ satisfying $W_1 \in \mathcal{G}_{ijw_1w_2w_3}$, $W_2 \in \mathcal{G}_{ijw_4w_5w_6}$ and $W_3 \in \mathcal{G}_{ijw_7w_8w_9}$, with $w_1, \ldots, w_9 \in \mathcal{I}\backslash\{i, -i, j, -j\}$. By Lemma 1.5, $w_1, \ldots, w_9$ must be pairwise distinct. Since $\{w_1, \ldots, w_9\} \subset \mathcal{I}\backslash\{i, -i, j, -j\}$ and $|\mathcal{I}\backslash\{i, -i, j, -j\}| = 10$, with $\mathcal{I}\backslash\{i, -i, j, -j\} = \{k, -k, l, -l, m, -m, n, -n, o, -o\}$, there exists a unique element $\alpha \in \{k, -k, l, -l, m, -m, n, -n, o, -o\}$ so that $|\mathcal{G}_{ij\alpha}| = 0$.

Suppose, without loss of generality, that $W_1 \in \mathcal{G}_{ijklm}$. In these conditions, we get $w_4, w_5, \ldots, w_9 \in \{-k, -l, -m, n, -n, o, -o\}$. Considering $W_2 \in \mathcal{G}_{ijw_4w_5w_6}$, we note that $|\{w_4, w_5, w_6\} \cap \{-k, -l, -m\}| \neq 0$, otherwise, there are $w, w' \in \{w_4, w_5, w_6\}$ such that $w = -w'$, which is not possible. Considering $W_3 \in \mathcal{G}_{ijw_7w_8w_9}$, applying the same reason, we conclude that $|\{w_7, w_8, w_9\} \cap \{-k, -l, -m\}| \neq 0$. Thus, $1 \leq |\{w_4, w_5, w_6\} \cap \{-k, -l, -m\}| \leq 2$. Next, we analyze, separately, the hypotheses $|\{w_4, w_5, w_6\} \cap \{-k, -l, -m\}| = 2$ and $|\{w_4, w_5, w_6\} \cap \{-k, -l, -m\}| = 1$, supposing, respectively:

i) $W_2 \in \mathcal{G}_{i,j,-k,-l,n}$;

ii) $W_2 \in \mathcal{G}_{i,j,-k,n,o}$.

If $W_2 \in \mathcal{G}_{i,j,-k,-l,n}$, the codewords of $\mathcal{G}_{ij}$ have the following index distribution:

W_1	i	j	k	l	m
W_2	i	j	$-k$	$-l$	n
W_3	i	j	$-m$	$-n$	o

Table 6.1: Index distribution of the codewords of $\mathcal{G}_{ij}$ supposing $W_2 \in \mathcal{G}_{i,j,-k,-l,n}$.

Let us now assume that $W_2 \in \mathcal{G}_{i,j,-k,n,o}$. In this case, focusing our attention on the index distribution of W_1, we must, in principle, distinguish the following two possibilities:

W_1	i	j	k	l	m
W_2	i	j	$-k$	n	o
W_3	i	j	$-l$	$-m$	$-n$

W_1	i	j	k	l	m
W_2	i	j	$-k$	n	o
W_3	i	j	$-l$	$-n$	$-o$

Table 6.2: Index distributions of the codewords of $\mathcal{G}_{ij}$ supposing $W_2 \in \mathcal{G}_{i,j,-k,n,o}$.

But analyzing carefully the index distributions presented in Tables 6.1 and 6.2, we verify that they are equivalent. In fact, any one of them induces a partition of $\mathcal{I}\backslash\{i,-i,j,-j\}$ of the type:

$$\{\alpha,\beta,\gamma\};\ \ \{-\alpha,-\beta,\delta\};\ \ \{-\gamma,-\delta,\theta\};\ \ \{-\theta\};$$

with $\{\alpha,\beta,\gamma,\delta,\theta\} = \{k,l,m,n,o\}$. $\square$

Taking into account the previous proposition, we will assume for the index distribution of the codewords $W_1, W_2, W_3 \in \mathcal{G}_{ij}$ the one given in:

W_1	i	j	k	l	m
W_2	i	j	$-k$	$-l$	n
W_3	i	j	$-m$	$-n$	o

Table 6.3: Index distribution of the codewords of $\mathcal{G}_{ij}$.

The index distribution of the codewords of $\mathcal{G}_{ij}$ induces a partition $\mathcal{P}$ of $\mathcal{I}\backslash\{i,-i,j\}$:

$$\mathcal{P}_1 = \{k,l,m\};\ \mathcal{P}_2 = \{-k,-l,n\};\ \mathcal{P}_3 = \{-m,-n,o\};\ \mathcal{P}_4 = \{-o\};\ \mathcal{P}_5 = \{-j\}. \quad (6.1)$$

This partition will be useful in the characterization of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$. In fact, taking into account Lemma 1.5, if $W \in \mathcal{G}_i\backslash\mathcal{G}_j$, with $W \in \mathcal{G}_{iw_1w_2w_3w_4}$ and $w_1,\ldots,w_4 \in \mathcal{I}\backslash\{i,-i,j\}$, then $|\{w_1,\ldots,w_4\}\cap\mathcal{P}_1| \leq 1$, $|\{w_1,\ldots,w_4\}\cap\mathcal{P}_2| \leq 1$ and

$|\{w_1, \ldots, w_4\} \cap \mathcal{P}_3| \leq 1$. The same reasoning is valid for the index distribution of the codewords of $\mathcal{F}_i$.

Until now $-o$ and $-j$ are the unique elements in $\mathcal{I}\backslash\{-i\}$ which are not used in the index distribution of the codewords of $\mathcal{G}_{ij}$. Particular attention will be given to these elements in the characterization of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$. Let us consider the subsets of, respectively, $\mathcal{G}_i$ and $\mathcal{F}_i$:

$$\mathcal{H} = \{W \in \mathcal{G}_{iw_1w_2w_3w_4} : w_1 \in \mathcal{P}_1 \ \wedge \ w_2 \in \mathcal{P}_2 \ \wedge \ w_3 \in \mathcal{P}_3 \ \wedge \ w_4 \in \{-o, -j\}\}$$

and

$$\mathcal{J} = \{U \in \mathcal{F}_{iu_1u_2u_3} : u_1 \in \mathcal{P}_1 \ \wedge \ u_2 \in \mathcal{P}_2 \ \wedge \ u_3 \in \mathcal{P}_3\}.$$

Taking into account the partition of $\mathcal{I}\backslash\{i, -i, j\}$, see (6.1), we get

$$\mathcal{G}_i\backslash\mathcal{G}_j = \mathcal{H} \cup \mathcal{G}_{i,-o,-j}$$

and

$$\mathcal{F}_i = \mathcal{J} \cup \mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}.$$

We recall that $|\mathcal{G}_i\backslash\mathcal{G}_j| = 2$, since we are assuming $|\mathcal{G}_i| = 5$ and $|\mathcal{G}_{ij}| = 3$. Besides, by Proposition 6.1, $8 \leq |\mathcal{F}_i| \leq 9$.

Next, we present results which impose conditions on the index distribution of the codewords of $(\mathcal{G}_i\backslash\mathcal{G}_j) \cup \mathcal{F}_i$ by the establishment of relations between the cardinality of the sets $\mathcal{H}$, $\mathcal{J}$, $\mathcal{F}_{i,-o}$ and $\mathcal{F}_{i,-j}$. The following proposition will be useful to obtain the refereed relations.

Proposition 6.3 *The set $\mathcal{D}_{i,j,-o} \cup \mathcal{E}_{i,j,-o}$ satisfies $|\mathcal{D}_{i,j,-o} \cup \mathcal{E}_{i,j,-o}| = 1$.*

Proof. Let $V = (v_1,...,v_7)$ be a word of type $[\pm 1^3]$ so that $|v_i| = |v_j| = |v_{-o}| = 1$. This word must be covered by a codeword of $\mathcal{D}_{i,j,-o} \cup \mathcal{E}_{i,j,-o} \cup \mathcal{F}_{i,j,-o} \cup \mathcal{G}_{i,j,-o}$.

Taking into account the index distribution of the codewords of $\mathcal{G}_{ij}$, $|\mathcal{G}_{i,j,-o}| = 0$ and, by Proposition 6.1, $|\mathcal{F}_{ij}| = 0$. Consequently, $|\mathcal{D}_{i,j,-o} \cup \mathcal{E}_{i,j,-o}| \geq 1$. Considering Lemma 1.5 we conclude that $|\mathcal{D}_{i,j,-o} \cup \mathcal{E}_{i,j,-o}| = 1$. □

The following proposition restricts the variation of $|\mathcal{H} \cup \mathcal{J}|$, $|\mathcal{H}|$ and $|\mathcal{J}|$.

Proposition 6.4 *The sets $\mathcal{H}$ and $\mathcal{J}$ satisfy $|\mathcal{H} \cup \mathcal{J}| \leq 6$. Furthermore, $1 \leq |\mathcal{H}| \leq 2$ and $|\mathcal{J}| \leq 5$.*

Proof. Let $W \in \mathcal{H}$ and $U \in \mathcal{J}$ be such that $W \in \mathcal{G}_{iw_1w_2w_3}$ and $U \in \mathcal{F}_{iu_1u_2u_3}$, with $w_1, u_1 \in \mathcal{P}_1$, $w_2, u_2 \in \mathcal{P}_2$ and $w_3, u_3 \in \mathcal{P}_3$, where:

$$\mathcal{P}_1 = \{k, l, m\}; \;\; \mathcal{P}_2 = \{-k, -l, n\}; \;\; \mathcal{P}_3 = \{-m, -n, o\}.$$

Assuming by contradiction that $|\mathcal{H} \cup \mathcal{J}| \geq 7$, there exists, at least, one element $\alpha \in \mathcal{P}_1$ such that $|\mathcal{H}_{i\alpha} \cup \mathcal{J}_{i\alpha}| \geq 3$. We note that, $\mathcal{H}_{i\alpha}$ is denoting the set $\{W \in \mathcal{H} : W \in \mathcal{G}_{i\alpha}\}$. On the other hand, $\mathcal{J}_{i\alpha}$ represents the set $\{U \in \mathcal{J} : U \in \mathcal{F}_{i\alpha}\}$. Let $V_1, V_2, V_3 \in \mathcal{H}_{i\alpha} \cup \mathcal{J}_{i\alpha}$ satisfying:

V_1	i	α	v_1	v_2
V_2	i	α	v_3	v_4
V_3	i	α	v_5	v_6

Table 6.4: Index distribution of the codewords $V_1, V_2, V_3 \in \mathcal{H}_{i\alpha} \cup \mathcal{J}_{i\alpha}$.

where $v_1, v_3, v_5 \in \mathcal{P}_2$ and $v_2, v_4, v_6 \in \mathcal{P}_3$. By Lemma 1.5, we must impose $v_1, \ldots, v_6$ pairwise distinct, consequently, $\{v_1, \ldots, v_6\} = \mathcal{P}_2 \cup \mathcal{P}_3$, which is an absurdity since $-\alpha \in \mathcal{P}_2 \cup \mathcal{P}_3$. Therefore, $|\mathcal{H} \cup \mathcal{J}| \leq 6$.

The condition $|\mathcal{H}| \leq 2$ is obtained immediately, since $\mathcal{H} \subset \mathcal{G}_i\backslash\mathcal{G}_j$ and $|\mathcal{G}_i\backslash\mathcal{G}_j| = 2$.

Assume, by contradiction, that $|\mathcal{H}| = 0$. Taking into account the partition $\mathcal{P}$ of $\mathcal{I}\backslash\{i, -i, j\}$, see (6.1), $W, W' \in \mathcal{G}_i\backslash\mathcal{G}_j$ must satisfy $W, W' \in \mathcal{G}_{i,-o,-j}$, contradicting Lemma 1.5. Therefore, $1 \leq |\mathcal{H}| \leq 2$.

Since $|\mathcal{H} \cup \mathcal{J}| \leq 6$, $|\mathcal{H}| \geq 1$ and $\mathcal{H} \cap \mathcal{J} = \varnothing$, we get $|\mathcal{J}| \leq 5$. $\square$

By the above proposition we get $1 \leq |\mathcal{H}| \leq 2$. The next two results allow us to obtain conditions for the codewords of $\mathcal{F}_i$ when $|\mathcal{H}| = 1$ and $|\mathcal{H}| = 2$, respectively. In particular, as $\mathcal{F}_i = \mathcal{J} \cup \mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$, more precisely, we get conditions for the cardinality of $\mathcal{J}$, $\mathcal{F}_{i,-o}$, $\mathcal{F}_{i,-j}$ and $\mathcal{F}_{i,-o,-j}$.

Proposition 6.5 *If $|\mathcal{H}| = 1$, then $3 \leq |\mathcal{J}| \leq 5$ and $|\mathcal{F}_{i,-o,-j}| = 0$. In particular, considering $W_4 \in \mathcal{H}$ one has:*

i) if $W_4 \in \mathcal{G}_{i,-o}$, then $4 \leq |\mathcal{J}| \leq 5$. Moreover, if $|\mathcal{J}| = 4$, then $|\mathcal{F}_{i,-o}| = 1$, $|\mathcal{F}_{i,-j}| = 3$ and $|\mathcal{F}_i| = 8$;

ii) if $W_4 \in \mathcal{G}_{i,-j}$, then $3 \leq |\mathcal{J}| \leq 5$. Moreover, if $|\mathcal{J}| = 3$, then $|\mathcal{F}_{i,-o}| = 3$, $|\mathcal{F}_{i,-j}| = 2$ and $|\mathcal{F}_i| = 8$.

Proof. Assuming $|\mathcal{H}| = 1$, let us denote by W_4 the only codeword of $\mathcal{H}$. As $\mathcal{G}_i \backslash \mathcal{G}_j = \mathcal{H} \cup \mathcal{G}_{i,-o,-j}$ and $|\mathcal{G}_i \backslash \mathcal{G}_j| = 2$, there exists $W_5 \in \mathcal{G}_{i,-o,-j}$ and so, by Lemma 1.5, $|\mathcal{F}_{i,-o,-j}| = 0$.

Consider the partition $\mathcal{P}$ of $\mathcal{I} \backslash \{i, -i, j\}$:

$$\mathcal{P}_1 = \{k, l, m\};\ \mathcal{P}_2 = \{-k, -l, n\};\ \mathcal{P}_3 = \{-m, -n, o\};\ \mathcal{P}_4 = \{-o\};\ \mathcal{P}_5 = \{-j\}.$$

By definition of $\mathcal{H}$, $W_4 \in \mathcal{G}_{iw_1w_2w_3w_4}$, with $w_1 \in \mathcal{P}_1$, $w_2 \in \mathcal{P}_2$, $w_3 \in \mathcal{P}_3$ and $w_4 \in \{-o, -j\}$. Then, we must consider the two following hypotheses:

i) $W_4 \in \mathcal{G}_{i,-o}$;

ii) $W_4 \in \mathcal{G}_{i,-j}$.

In what follows it will be useful to recall the following equation obtained from Lemma 2.2:

$$|\mathcal{D}_{i\alpha} \cup \mathcal{E}_{i\alpha}| + 2|\mathcal{F}_{i\alpha}| + 3|\mathcal{G}_{i\alpha}| = 10, \tag{6.2}$$

for any $\alpha \in \mathcal{I} \backslash \{i, -i\}$.

Suppose that $W_4 \in \mathcal{G}_{i,-o}$. Concentrating our attention on $-o \in \mathcal{I}$, we verify that, since $W_5 \in \mathcal{G}_{i,-o}$ and $|\mathcal{G}_{i,j,-o}| = 0$, $|\mathcal{G}_{i,-o}| = 2$. By Proposition 6.3, $|\mathcal{D}_{i,-o} \cup \mathcal{E}_{i,-o}| \geq 1$. Having into account (6.2), we conclude that $|\mathcal{F}_{i,-o}| \leq 1$.

Let us now focus our attention on $-j \in \mathcal{I}$. Note that, W_5 is the unique codeword in $\mathcal{G}_{i,-j}$ ($|\mathcal{G}_{i,-j}| = 1$). Consequently, by (6.2), $|\mathcal{F}_{i,-j}| \leq 3$.

Taking into account that we have concluded before, $|\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}| \leq 4$.

Now, by Proposition 6.1, $8 \leq |\mathcal{F}_i| \leq 9$. As $\mathcal{F}_i = \mathcal{J} \cup \mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$, it follows that $|\mathcal{J}| \geq 4$. Having in mind Proposition 6.4, $4 \leq |\mathcal{J}| \leq 5$. If, in particular, $|\mathcal{J}| = 4$, then

$|\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}| = 4$, with $|\mathcal{F}_{i,-o}| = 1$, $|\mathcal{F}_{i,-j}| = 3$ and $|\mathcal{F}_{i,-o,-j}| = 0$. In these conditions $|\mathcal{F}_i| = 8$.

Now consider that $W_4 \in \mathcal{G}_{i,-j}$. Since $W_5 \in \mathcal{G}_{i,-o,-j}$, then $|\mathcal{G}_{i,-j}| = 2$ and $|\mathcal{G}_{i,-o}| = 1$. Taking into account Proposition 6.3, $|\mathcal{D}_{i,-o} \cup \mathcal{E}_{i,-o}| \geq 1$ and by (6.2) it follows that $|\mathcal{F}_{i,-o}| \leq 3$. Furthermore, $|\mathcal{F}_{i,-j}| \leq 2$. Consequently, $|\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}| \leq 5$ and, since $|\mathcal{F}_i| \geq 8$, we get $|\mathcal{J}| \geq 3$. Now considering Proposition 6.4, one has $3 \leq |\mathcal{J}| \leq 5$.

If $|\mathcal{J}| = 3$, then $|\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}| = 5$, with $|\mathcal{F}_{i,-o}| = 3$, $|\mathcal{F}_{i,-j}| = 2$ and $|\mathcal{F}_{i,-o,-j}| = 0$, and so $|\mathcal{F}_i| = 8$.

Thus, independently of the index distribution of $W_4 \in \mathcal{H}$, we conclude that $3 \leq |\mathcal{J}| \leq 5$. □

Proposition 6.6 *If* $|\mathcal{H}| = 2$*, then* $|\mathcal{G}_{i,-o,-j}| = 0$ *and* $3 \leq |\mathcal{J}| \leq 4$*. In particular, if* $|\mathcal{J}| = 3$*, then* $|\mathcal{F}_i| = 8$ *and considering* $W_4, W_5 \in \mathcal{H}$*:*

i) if $W_4, W_5 \in \mathcal{G}_{i,-o}$*, then either* $|\mathcal{F}_{i,-j}| = 5$*, or,* $|\mathcal{F}_{i,-j}| = 4$*,* $|\mathcal{F}_{i,-o}| = 1$ *and* $|\mathcal{F}_{i,-o,-j}| = 0$*;*

ii) if $W_4, W_5 \in \mathcal{G}_{i,-j}$*, then either* $|\mathcal{F}_{i,-o}| = 4$*,* $|\mathcal{F}_{i,-j}| = 2$ *and* $|\mathcal{F}_{i,-o,-j}| = 1$*, or,* $|\mathcal{F}_{i,-o}| = 4$*,* $|\mathcal{F}_{i,-j}| = 1$ *and* $|\mathcal{F}_{i,-o,-j}| = 0$*;*

iii) if $W_4 \in \mathcal{G}_{i,-o}$ *and* $W_5 \in \mathcal{G}_{i,-j}$*, then either* $|\mathcal{F}_{i,-o}| = 3$*,* $|\mathcal{F}_{i,-j}| = 3$ *and* $|\mathcal{F}_{i,-o,-j}| = 1$*, or,* $|\mathcal{F}_{i,-o}| = 2$*,* $|\mathcal{F}_{i,-j}| = 3$ *and* $|\mathcal{F}_{i,-o,-j}| = 0$*.*

Proof. Let W_4, W_5 be codewords in $\mathcal{H}$. The condition $|\mathcal{G}_{i,-o,-j}| = 0$ comes immediately from the definition of $\mathcal{H}$.

By Proposition 6.4, $|\mathcal{H} \cup \mathcal{J}| \leq 6$. As we are assuming $|\mathcal{H}| = 2$, then $|\mathcal{J}| \leq 4$. The proof of $|\mathcal{J}| \geq 3$ is obtained from the analysis of each one of the hypotheses for $W_4, W_5 \in \mathcal{H}$:

i) $W_4, W_5 \in \mathcal{G}_{i,-o}$;

ii) $W_4, W_5 \in \mathcal{G}_{i,-j}$;

iii) $W_4 \in \mathcal{G}_{i,-o}$ and $W_5 \in \mathcal{G}_{i,-j}$.

Let us suppose that $W_4, W_5 \in \mathcal{G}_{i,-o}$. In these conditions, since $|\mathcal{G}_{i,j,-o}| = 0$, we have $|\mathcal{G}_{i,-o}| = 2$ and $|\mathcal{G}_{i,-j}| = 0$. Focusing our attention on $-o \in \mathcal{I}$ and taking into account that, by Proposition 6.3, $|\mathcal{D}_{i,-o} \cup \mathcal{E}_{i,-o}| \geq 1$, considering (6.2) we conclude that $|\mathcal{F}_{i,-o}| \leq 1$. Now concentrating our attention on $-j \in \mathcal{I}$ and considering again (6.2), we get $|\mathcal{F}_{i,-j}| \leq 5$.

If $|\mathcal{F}_{i,-j}| = 5$, by Lemma 2.5, $|\mathcal{F}_{i,-j,-o}| = 1$ and, consequently, $|\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}| \leq 5$. On the other hand, if $|\mathcal{F}_{i,-j}| \leq 4$, we conclude also $|\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}| \leq 5$. Thus, taking into account that $\mathcal{F}_i = \mathcal{J} \cup \mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$ and that, by Proposition 6.1, $8 \leq |\mathcal{F}_i| \leq 9$, in both cases we must impose $|\mathcal{J}| \geq 3$. If $|\mathcal{J}| = 3$, then $|\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}| = 5$ and $|\mathcal{F}_i| = 8$, furthermore one of the following conditions must occurs:

- $|\mathcal{F}_{i,-j}| = 5$;
- $|\mathcal{F}_{i,-j}| = 4$, $|\mathcal{F}_{i,-o}| = 1$ and $|\mathcal{F}_{i,-o,-j}| = 0$.

Now assume that $W_4, W_5 \in \mathcal{G}_{i,-j}$. In these conditions, $|\mathcal{G}_{i,-j}| = 2$ and $|\mathcal{G}_{i,-o}| = 0$. Considering $-j \in \mathcal{I}$, from (6.2) it follows that $|\mathcal{F}_{i,-j}| \leq 2$. Focusing our attention on $-o \in \mathcal{I}$, taking into account Proposition 6.3 and (6.2), we conclude that $|\mathcal{F}_{i,-o}| \leq 4$.

If $|\mathcal{F}_{i,-j}| = 2$, then the codewords $W_4, W_5 \in \mathcal{G}_{i,-j}$ and $U_1, U_2 \in \mathcal{F}_{i,-j}$ are such that $W_4 \in \mathcal{G}_{i,-j,w_1,w_2,w_3}$, $W_5 \in \mathcal{G}_{i,-j,w_4,w_5,w_6}$, $U_1 \in \mathcal{F}_{i,-j,u_1,u_2}$ and $U_2 \in \mathcal{F}_{i,-j,u_3,u_4}$, with $w_1, \ldots, w_6, u_1, \ldots, u_4 \in \mathcal{I}\backslash\{i, -i, j, -j\}$ pairwise distinct. As $|\mathcal{I}\backslash\{i, -i, j, -j\}| = 10$ and $|\mathcal{G}_{i,-o,-j}| = 0$, it follows that $|\mathcal{F}_{i,-j,-o}| = 1$. In these conditions, $|\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}| \leq 5$. Assuming $|\mathcal{F}_{i,-j}| \leq 1$, we get again $|\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}| \leq 5$. Thus, independently of the cardinality of $\mathcal{F}_{i,-j}$, $|\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}| \leq 5$. As, by Proposition 6.1, $8 \leq |\mathcal{F}_i| \leq 9$, and $\mathcal{F}_i = \mathcal{J} \cup \mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$, then we must impose $|\mathcal{J}| \geq 3$. If $|\mathcal{J}| = 3$, then $|\mathcal{F}_i| = 8$ and one of the following conditions must occurs:

- $|\mathcal{F}_{i,-j}| = 2$, $|\mathcal{F}_{i,-o}| = 4$ and $|\mathcal{F}_{i,-o,-j}| = 1$;
- $|\mathcal{F}_{i,-j}| = 1$ and $|\mathcal{F}_{i,-o}| = 4$ and $|\mathcal{F}_{i,-o,-j}| = 0$.

Consider $W_4 \in \mathcal{G}_{i,-o}$ and $W_5 \in \mathcal{G}_{i,-j}$. Accordingly with what is being assumed, $|\mathcal{G}_{i,-o}| = |\mathcal{G}_{i,-j}| = 1$. Consequently, by (6.2), we get $|\mathcal{F}_{i,-o}| \leq 3$ and $|\mathcal{F}_{i,-j}| \leq 3$. Suppose that $|\mathcal{F}_{i,-o}| = 3$. Since $|\mathcal{G}_{i,-o,-j}| = 0$ and, by Proposition 6.3, there exists $V \in \mathcal{D}_{i,-o,j} \cup \mathcal{E}_{i,-o,j}$, then the codewords $U_1, U_2, U_3 \in \mathcal{F}_{i,-o}$ and $W_4 \in \mathcal{G}_{i,-o}$ are such that $U_1 \in \mathcal{F}_{i,-o,u_1,u_2}$, $U_2 \in \mathcal{F}_{i,-o,u_3,u_4}$, $U_3 \in \mathcal{F}_{i,-o,u_5,u_6}$ and $W_4 \in \mathcal{G}_{i,-o,w_1,w_2,w_3}$ with $u_1, \ldots, u_6, w_1, w_2, w_3 \in \mathcal{I}\backslash\{i, -i, o, -o, j\}$. Since $|\mathcal{I}\backslash\{i, -i, o, -o, j\}| = 9$ and $w_1, w_2, w_3 \neq -j$, then $|\mathcal{F}_{i,-o,-j}| = 1$. Then, independently of the value of $|\mathcal{F}_{i,-o}|$ we conclude that $|\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}| \leq 5$ and, consequently, $|\mathcal{J}| \geq 3$. If $|\mathcal{J}| = 3$, then $|\mathcal{F}_i| = 8$ and one of the following conditions is satisfied:

- $|\mathcal{F}_{i,-o}| = 3$, $|\mathcal{F}_{i,-j}| = 3$ and $|\mathcal{F}_{i,-o,-j}| = 1$;
- $|\mathcal{F}_{i,-o}| = 2$, $|\mathcal{F}_{i,-j}| = 3$ and $|\mathcal{F}_{i,-o,-j}| = 0$.

□

The index characterization of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ is mostly based in Propositions 6.5 and 6.6, as we will see next.

6.1.2 Index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$

Here our aim is to show a possible method to obtain all possible index distributions for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$, taking into account that, by assumption, $|\mathcal{G}_i| = 5$ and, by Proposition 6.1, $8 \leq |\mathcal{F}_i| \leq 9$. Applying the results derived in the previous subsection, manly Propositions 6.5 and 6.6, we identify many possible index distributions for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$. By this reason, in this subsection we present only some illustrative and representative cases in which the described method is used to obtain them.

In last subsection we have considered the subsets $\mathcal{H} \subset \mathcal{G}_i\backslash\mathcal{G}_j$ and $\mathcal{J} \subset \mathcal{F}_i$ defined by

$$\mathcal{H} = \{W \in \mathcal{G}_{iw_1w_2w_3w_4} : w_1 \in \mathcal{P}_1 \ \wedge \ w_2 \in \mathcal{P}_2 \ \wedge \ w_3 \in \mathcal{P}_3 \ \wedge \ w_4 \in \{-o, -j\}\}$$

and

$$\mathcal{J} = \{U \in \mathcal{F}_{iu_1u_2u_3} : u_1 \in \mathcal{P}_1 \ \wedge \ u_2 \in \mathcal{P}_2 \ \wedge \ u_3 \in \mathcal{P}_3\},$$

with $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ elements of the partition $\mathcal{P}$ of $\mathcal{I}\backslash\{i, -i, j\}$ induced by the codewords of $\mathcal{G}_{ij}$:

$$\mathcal{P}_1 = \{k, l, m\}; \ \mathcal{P}_2 = \{-k, -l, n\}; \ \mathcal{P}_3 = \{-m, -n, o\}; \ \mathcal{P}_4 = \{-o\}; \ \mathcal{P}_5 = \{-j\}.$$

By Proposition 6.4, $1 \leq |\mathcal{H}| \leq 2$. If $|\mathcal{H}| = 1$, with $W_4 \in \mathcal{H}$, then $W_4 \in \mathcal{G}_{i,-o}$ or $W_4 \in \mathcal{G}_{i,-j}$. On the other hand, if $|\mathcal{H}| = 2$, then $W_4, W_5 \in \mathcal{H}$ satisfy one of the following conditions: $W_4, W_5 \in \mathcal{G}_{i,-o}$; $W_4, W_5 \in \mathcal{G}_{i,-j}$; $W_4 \in \mathcal{G}_{i,-o}$ and $W_5 \in \mathcal{G}_{i,-j}$. We shall present some representative cases of possible index distributions for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ in which all of these possibilities for the codewords of $\mathcal{H}$ are satisfied.

The characterization of codewords of $\mathcal{H}$ and $\mathcal{J}$ is based on the elements of $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$. The following table presents all possible combinations between the elements of these sets:

k	$-l$	$-m$	1
		$-n$	2
		o	3
	n	$-m$	4
		o	5
l	$-k$	$-m$	6
		$-n$	7
		o	8
	n	$-m$	9
		o	10
m	$-n$	$-k$	11
		$-l$	12
	o	$-k$	13
		$-l$	14
		n	15

Table 6.5: Combinations between the elements of $\mathcal{P}_1$, $\mathcal{P}_2$ e $\mathcal{P}_3$.

From the above table we obtain all possible index distributions for the codewords of $\mathcal{H}$ and $\mathcal{J}$. In fact, if $W \in \mathcal{H}$, then $W \in \mathcal{G}_{i,-o,w_1,w_2,w_3}$ or $W \in \mathcal{G}_{i,-j,w_1,w_2,w_3}$, with w_1, w_2 and w_3 satisfying one of the conditions presented in the table. If $U \in \mathcal{J}$, then $U \in \mathcal{F}_{iu_1u_2u_3}$, with u_1, u_2 and u_3 satisfying one of the hypotheses described.

We recall that, by Lemma 1.5,

$$|\mathcal{D}_{i\alpha\beta} \cup \mathcal{E}_{i\alpha\beta} \cup \mathcal{F}_{i\alpha\beta} \cup \mathcal{G}_{i\alpha\beta}| = 1,$$

for any $\alpha, \beta \in \mathcal{I}\backslash\{i, -i\}$ satisfying $|\alpha| \neq |\beta|$. This means that, for example, we can not consider $W \in \mathcal{H}$ satisfying $W \in \mathcal{G}_{i,-o,k,-l,-m}$ and $U \in \mathcal{J}$ such that $U \in \mathcal{F}_{i,k,-l,o}$, since $W, U \in \mathcal{G}_{i,k,-l} \cup \mathcal{F}_{i,k,-l}$. In what follows we will apply frequently this lemma does not being many times referenced.

- **$|\mathcal{H}| = 1$ and $W \in \mathcal{H}$ satisfies $W \in \mathcal{G}_{i,-o}$**

Let $W_4 \in \mathcal{H}$ such that $W_4 \in \mathcal{G}_{i,-o}$. By definition of $\mathcal{H}$, $W_4 \in \mathcal{G}_{i,-o,w_1,w_2,w_3}$ with $w_1 \in \mathcal{P}_1$, $w_2 \in \mathcal{P}_2$ and $w_3 \in \mathcal{P}_3$. Taking into account $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$, we get, up to an equivalent index distributions, the following hypotheses for $W_4 \in \mathcal{H}$: $W_4 \in \mathcal{G}_{i,-o,-m,k,-l}$; $W_4 \in \mathcal{G}_{i,-o,-m,k,n}$. In fact, when we chose an element of $\mathcal{P}_3$ to be index of W_4, observing $\mathcal{P}_1$ and $\mathcal{P}_2$ we verify that it is indifferent to select $-m$ or $-n$, on the other hand, being chosen $-m$, when we get an element of $\mathcal{P}_1$, it is also indifferent to chose k or l. Thus, for example, if we consider $W_4 \in \mathcal{G}_{i,-o,-n,-k,l}$ this index distribution is equivalent to $W_4 \in \mathcal{G}_{i,-o,-m,k,-l}$. In fact, permuting m with n, k with $-k$ and l with $-l$ we get the same partition $\mathcal{P}$ of $\mathcal{I}\backslash\{i, -i, j\}$.

It is possible to characterize $W_5 \in \mathcal{G}_i\backslash(\mathcal{G}_j \cup \mathcal{H})$ when W_4 satisfies each one of the referred hypotheses. Since $W_5 \notin \mathcal{H}$, then $W_5 \in \mathcal{G}_{i,-o,-j,w_1,w_2}$ with $w_1, w_2 \in \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$. We note that, by Lemma 1.5, $|\{w_1, w_2\} \cap \mathcal{P}_p| \leq 1$ for $p \in \{1, 2, 3\}$. Taking into account $W_1, \ldots, W_4 \in \mathcal{G}_i$ described at this moment, we get the following possibilities for W_5 when $W_4 \in \mathcal{G}_{i,-o,-m,k,-l}$ and $W_4 \in \mathcal{G}_{i,-o,-m,k,n}$, respectively:

1	W_5	i	$-o$	$-j$	l	$-k$
2	W_5	i	$-o$	$-j$	l	n
3	W_5	i	$-o$	$-j$	m	$-k$
4	W_5	i	$-o$	$-j$	m	n
5	W_5	i	$-o$	$-j$	l	$-n$
6	W_5	i	$-o$	$-j$	m	$-n$
7	W_5	i	$-o$	$-j$	$-k$	$-n$

Table 6.6: $W_4 \in \mathcal{G}_{i,-o,-m,k,-l}$.

1	W_5	i	$-o$	$-j$	l	$-k$
2	W_5	i	$-o$	$-j$	m	$-k$
3	W_5	i	$-o$	$-j$	m	$-l$
4	W_5	i	$-o$	$-j$	l	$-n$
5	W_5	i	$-o$	$-j$	m	$-n$
6	W_5	i	$-o$	$-j$	$-k$	$-n$
7	W_5	i	$-o$	$-j$	$-l$	$-n$

Table 6.7: $W_4 \in \mathcal{G}_{i,-o,-m,k,n}$.

Since we have characterized all possible codewords of $\mathcal{G}_i$, next step consists in the description of all possible codewords of $\mathcal{F}_i$. Recall that, by Lemma 6.1, $8 \leq |\mathcal{F}_i| \leq 9$. Furthermore, from Proposition 6.5, the codewords of $\mathcal{F}_i$ must satisfy:

- $4 \leq |\mathcal{J}| \leq 5$;

- $|\mathcal{F}_{i,-o,-j}| = 0$;

– if $|\mathcal{J}| = 4$, then $|\mathcal{F}_{i,-o}| = 1$, $|\mathcal{F}_{i,-j}| = 3$ and $|\mathcal{F}_i| = 8$.

Considering Proposition 6.5, we will characterize the codewords of $\mathcal{F}_i$ distinguishing the cases: $|\mathcal{J}| = 5$ and $|\mathcal{J}| = 4$. Next we present examples in which for a certain characterization of $\mathcal{G}_i$:

1) it is not possible to describe all codewords of $\mathcal{F}_i$;

2) the characterization of $\mathcal{F}_i$ depends on the cardinality of $\mathcal{J}$;

3) the characterization of $\mathcal{F}_i$ does not depend on the cardinality of $\mathcal{J}$.

Example 1: It is not possible to describe all codewords of $\mathcal{F}_i$.

Let us consider $W_4 \in \mathcal{G}_{i,-o,-m,k,-l}$ and $W_5 \in \mathcal{G}_{i,-o,-j,l,-k}$. Since, by Proposition 6.5, $4 \leq |\mathcal{J}| \leq 5$, we begin by analyzing possible codewords in $\mathcal{J}$. From the analysis of Table 6.5 and taking into account Lemma 1.5, if $U \in \mathcal{J}$, then $U \in \mathcal{F}_{ikno} \cup \mathcal{F}_{iln} \cup \mathcal{F}_{i,m,-n} \cup \mathcal{F}_{imo}$. Thus, considering again Lemma 1.5, we conclude that $|\mathcal{J}| = 4$. Accordingly, we get two possibilities for the codewords $U_1, \ldots, U_4 \in \mathcal{J}$:

U_1	i	k	n	o
U_2	i	l	n	$-m$
U_3	i	m	$-n$	$-k$
U_4	i	m	o	$-l$

U_1	i	k	n	o
U_2	i	l	n	$-m$
U_3	i	m	$-n$	$-l$
U_4	i	m	o	$-k$

Table 6.8: Possible index distributions for the codewords of $\mathcal{J}$.

To complete the characterization of the codewords of $\mathcal{F}_i$, we recall that, by Proposition 6.5, since $|\mathcal{J}| = 4$, we have $|\mathcal{F}_i| = 8$, in particular, $|\mathcal{F}_{i,-o}| = 1$, $|\mathcal{F}_{i,-j}| = 3$ and $|\mathcal{F}_{i,-o,-j}| = 0$.

Consider $U_1, \ldots, U_4 \in \mathcal{J}$ such that $U_1 \in \mathcal{F}_{ikno}$, $U_2 \in \mathcal{F}_{i,l,n,-m}$, $U_3 \in \mathcal{F}_{i,m,-n,-k}$ and $U_4 \in \mathcal{F}_{i,m,o,-l}$. Since $|\mathcal{F}_{i,-j}| = 3$, we begin by describing the index distribution of the codewords of $\mathcal{F}_{i,-j}$. For that, we will consider the following scheme in which all possibilities for these codewords not contradicting the definition of PL$(7, 2)$ code are presented, considering the codewords of $\mathcal{G}_i \cup \mathcal{J}$ described until now.

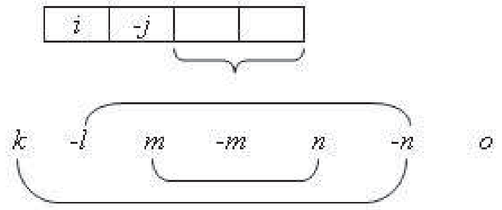

Figure 6.1: Possible index distributions for $U \in \mathcal{F}_{i,-j}$.

By the analysis of this scheme, if $U \in \mathcal{F}_{i,-j}$, then $U \in \mathcal{F}_{i,-j,-n} \cup \mathcal{F}_{i,-j,m,n}$. Consequently, $|\mathcal{F}_{i,-j}| \leq 2$, contradicting Proposition 6.5.

Now suppose that the codewords U_1,...,$U_4 \in \mathcal{J}$ satisfy: $U_1 \in \mathcal{F}_{ikno}$, $U_2 \in \mathcal{F}_{i,l,n,-m}$, $U_3 \in \mathcal{F}_{i,m,-n,-l}$ and $U_4 \in \mathcal{F}_{i,m,o,-k}$. Likewise in the previous case, we begin by describing the codewords of $\mathcal{F}_{i,-j}$:

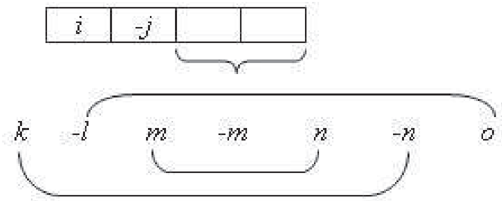

Figure 6.2: Possible index distributions for $U \in \mathcal{F}_{i,-j}$.

By the analysis of the above scheme, the codewords $U_5, U_6, U_7 \in \mathcal{F}_{i,-j}$ must satisfy:

U_5	i	$-j$	$-l$	o
U_6	i	$-j$	m	n
U_7	i	$-j$	k	$-n$

Table 6.9: Index distribution of the codewords of $\mathcal{F}_{i,-j}$.

To complete the characterization of all codewords of $\mathcal{F}_i$ we must identify the unique codeword in $\mathcal{F}_{i,-o}$. Considering the codewords $W_4, W_5 \in \mathcal{G}_{i,-o}$, if $U_8 \in \mathcal{F}_{i,-o,u_1,u_2}$, then $u_1, u_2 \in \{m, n, -n\}$. That is, $U_8 \in \mathcal{F}_{i,-o,m,n}$ or $U_8 \in \mathcal{F}_{i,-o,m,-n}$. In both cases Lemma 1.5 is contradicted when we consider, respectively, U_6 and U_3.

Therefore, the codewords $W_4 \in \mathcal{G}_{i,-o,-m,k,-l}$ and $W_5 \in \mathcal{G}_{i,-o,-j,l,-k}$ contradict the definition of PL$(7,2)$ code.

There are many index distributions for the codewords of $\mathcal{G}_i$ for which it is not possible to describe all codewords of $\mathcal{F}_i$ without contradictions. When $W_4 \in \mathcal{G}_{i,-o,-m,k,-l}$ and W_5 assumes, respectively, the index distributions represented by $1,2,3,6$ and 7 in Table 6.6, it is not possible to characterize all codewords of $\mathcal{F}_i$ without facing an absurdity. We came to the same conclusion when $W_4 \in \mathcal{G}_{i,-o,-m,k,n}$ and W_5 satisfies the conditions $1,5$ and 7 in Table 6.7.

Example 2: The characterization of $\mathcal{F}_i$ depends on the cardinality of $\mathcal{J}$.

Consider $W_4 \in \mathcal{G}_{i,-o,-m,k,-l}$ and $W_5 \in \mathcal{G}_{i,-o,-j,l,-n}$. We begin by identifying all possible codewords for $\mathcal{J}$. By the analysis of Table 6.5 and taking into account the codewords $W_4, W_5 \in \mathcal{G}_i \backslash \mathcal{G}_j$, if $U \in \mathcal{J}$, then

$$U \in \mathcal{F}_{ikno} \cup \mathcal{F}_{i,l,-k} \cup \mathcal{F}_{iln} \cup \mathcal{F}_{i,m,-n} \cup \mathcal{F}_{imo}.$$

Suppose that $|\mathcal{J}| = 5$. Then, $U_1, \ldots, U_5 \in \mathcal{J}$ must satisfy:

U_1	i	k	n	o
U_2	i	l	n	$-m$
U_3	i	l	$-k$	o
U_4	i	m	o	$-l$
U_5	i	m	$-n$	$-k$

Table 6.10: Index distribution of the codewords of $\mathcal{J}$.

By Proposition 6.1, $8 \leq |\mathcal{F}_i| \leq 9$, As $\mathcal{F}_i = \mathcal{J} \cup \mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$, then $|\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}| \geq 3$. Considering the codewords of $\mathcal{G}_i \cup \mathcal{J}$, if $U \in \mathcal{F}_{i,-o}$, then $U \in \mathcal{F}_{i,-o,m,n}$. Consequently, $|\mathcal{F}_{i,-j} \backslash \mathcal{F}_{i,-o}| \geq 2$. In the following scheme all possible index distributions for the codewords of $\mathcal{F}_{i,-j}$ are described.

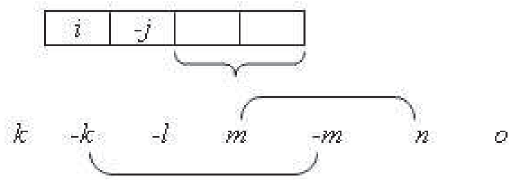

Figure 6.3: Possible index distributions for $U \in \mathcal{F}_{i,-j}$.

Thus, $|\mathcal{F}_{i,-j}| \leq 2$. Since $|\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}| \geq 3$ and $|\mathcal{F}_{i,-o}| \leq 1$, we must impose $|\mathcal{F}_{i,-o}| = 1$, with $U_6 \in \mathcal{F}_{i,-o,m,n}$, and $|\mathcal{F}_{i,-j}| = 2$, with $U_7 \in \mathcal{F}_{i,-j,-k,-m}$ and $U_8 \in \mathcal{F}_{i,-j,m,n}$. However, the codewords U_6 and U_8 contradict Lemma 1.5.

Now suppose that $|\mathcal{J}| = 4$. By Proposition 6.5, $|\mathcal{F}_{i,-o}| = 1$. So, we begin by the identification of this codeword. Taking into account $W_4, W_5 \in \mathcal{G}_i\backslash\mathcal{G}_j$ and Lemma 1.5, if $U \in \mathcal{F}_{i,-o}$, then $U \in \mathcal{F}_{i,-o,-k,m} \cup \mathcal{F}_{i,-o,m,n}$.

Let us assume that $U_1 \in \mathcal{F}_{i,-o,-k,m}$. From the analysis of Table 6.5, we get the following possible index distributions for the codewords of $\mathcal{J}$:

U_2	i	k	n	o
U_3	i	m	$-n$	$-l$
U_4	i	l	n	$-m$
U_5	i	l	$-k$	o

U_2	i	k	n	o
U_3	i	m	o	$-l$
U_4	i	l	n	$-m$
U_5	i	l	$-k$	o

U_2	i	m	o	n
U_3	i	m	$-n$	$-l$
U_4	i	l	n	$-m$
U_5	i	l	$-k$	o

Table 6.11: Possible index distributions for the codewords of $\mathcal{J}$.

By Proposition 6.5, for each one of these possibilities we must identify three codewords in $\mathcal{F}_{i,-j}$. Proceeding as in the scheme of Figure 6.3, we conclude that if the codewords $U_2, \ldots, U_5 \in \mathcal{J}$ are such that $U_2 \in \mathcal{F}_{ikno}$, $U_3 \in \mathcal{F}_{i,m,o,-l}$, $U_4 \in \mathcal{F}_{i,l,n,-m}$ and $U_5 \in \mathcal{F}_{i,l,-k,o}$, then it is not possible to characterize all codewords of $\mathcal{F}_{i,-j}$, without facing a contradiction.

On the other hand, for $U_2, \ldots, U_5 \in \mathcal{J}$ satisfying one of the following conditions:

i) $U_2 \in \mathcal{F}_{ikno}$, $U_3 \in \mathcal{F}_{i,m,-n,-l}$, $U_4 \in \mathcal{F}_{i,l,n,-m}$ and $U_5 \in \mathcal{F}_{i,l,-k,o}$;

ii) $U_2 \in \mathcal{F}_{imno}$, $U_3 \in \mathcal{F}_{i,m,-n,-l}$, $U_4 \in \mathcal{F}_{i,l,n,-m}$ and $U_5 \in \mathcal{F}_{i,l,-k,o}$;

we get, respectively, the following index characterization for $U_6, U_7, U_8 \in \mathcal{F}_{i,-j}$:

U_6	i	$-j$	$-k$	$-m$
U_7	i	$-j$	m	n
U_8	i	$-j$	o	$-l$

Table 6.12: If *i)* is satisfied.

U_6	i	$-j$	$-k$	$-m$
U_7	i	$-j$	k	n
U_8	i	$-j$	o	$-l$

Table 6.13: If *ii)* is satisfied.

In these cases we are able to describe completely all codewords of $\mathcal{G}_i \cup \mathcal{F}_i$, without facing an absurdity as we show next.

If $U_1 \in \mathcal{F}_{i,-o,m,n}$, considering Table 6.5, and taking into account $W_4, W_5 \in \mathcal{G}_i \backslash \mathcal{G}_j$, we identify eleven possible index distributions for the codewords of $\mathcal{J}$, however for each one of them we conclude, applying a similar reasoning to that one described in the scheme of Figure 6.3, that it is not possible to describe all codewords of $\mathcal{F}_{i,-j}$ without contradicting the definition of PL$(7,2)$ code.

Thus, for $W_4 \in \mathcal{G}_{i,-o,-m,k,-l}$ and $W_5 \in \mathcal{G}_{i,-o,-j,l,-n}$, there exist only two possible index distributions for the codewords of $\mathcal{F}_i$:

U_1	i	$-o$	$-k$	m
U_2	i	k	n	o
U_3	i	m	$-n$	$-l$
U_4	i	l	n	$-m$
U_5	i	l	$-k$	o
U_6	i	$-j$	$-k$	$-m$
U_7	i	$-j$	m	n
U_8	i	$-j$	o	$-l$

U_1	i	$-o$	$-k$	m
U_2	i	m	o	n
U_3	i	m	$-n$	$-l$
U_4	i	l	n	$-m$
U_5	i	l	$-k$	o
U_6	i	$-j$	$-k$	$-m$
U_7	i	$-j$	k	n
U_8	i	$-j$	o	$-l$

Table 6.14: Index distributions for the codewords of $\mathcal{F}_i$.

Example 3: The characterization of $\mathcal{F}_i$ does not depend on the cardinality of $\mathcal{J}$.

Let us consider $W_4 \in \mathcal{G}_{i,-o,-m,k,-l}$ and $W_5 \in \mathcal{G}_{i,-o,-j,m,n}$. In this case, as we will see, it is possible to characterize all codewords of $\mathcal{F}_i$ satisfying the definition of perfect Lee code when $|\mathcal{J}| = 4$ as well as when $|\mathcal{J}| = 5$.

First assume that $|\mathcal{J}| = 5$. From the analysis of Table 6.5, we get the two possible index distributions for the codewords of $\mathcal{J}$:

U_1	i	k	n	o
U_2	i	l	n	$-m$
U_3	i	l	$-k$	$-n$
U_4	i	m	$-n$	$-l$
U_5	i	m	o	$-k$

U_1	i	k	n	o
U_2	i	l	n	$-m$
U_3	i	l	$-k$	o
U_4	i	m	$-n$	$-k$
U_5	i	m	o	$-l$

Table 6.15: Possible index distributions for the codewords of $\mathcal{J}$

By Proposition 6.1, $8 \leq |\mathcal{F}_i| \leq 9$. Since $\mathcal{F}_i = \mathcal{J} \cup \mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$ and $|\mathcal{J}| = 5$, then $|\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}| \geq 3$.

Suppose that the codewords U_1,...,$U_5 \in \mathcal{J}$ are such that: $U_1 \in \mathcal{G}_{ikno}$; $U_2 \in \mathcal{G}_{i,l,n,-m}$; $U_3 \in \mathcal{G}_{i,l,-k,-n}$; $U_4 \in \mathcal{G}_{i,m,-n,-l}$; $U_5 \in \mathcal{G}_{i,m,o,-k}$. Let us analyze the set $\mathcal{F}_{i,-o}$. Considering the codewords of $\mathcal{G}_i \cup \mathcal{J}$ and Lemma 1.5, we conclude that if $U \in \mathcal{F}_{i,-o}$, then $U \in \mathcal{F}_{i,-o,u_1,u_2}$ with $u_1, u_2 \in \{-k, l, -n\}$. Taking into account the codeword U_3, we must impose $|\mathcal{F}_{i,-o}| = 0$, consequently, $|\mathcal{F}_{i,-j}| \geq 3$. Following a similar reasoning like the one applied in the scheme of Figure 6.3, we obtain the following hypotheses for the codewords of $\mathcal{F}_{i,-j}$:

U_6	i	$-j$	$-k$	$-m$
U_7	i	$-j$	k	$-n$
U_8	i	$-j$	o	l

U_6	i	$-j$	$-k$	$-m$
U_7	i	$-j$	k	$-n$
U_8	i	$-j$	o	$-l$

Table 6.16: Index distributions of the codewords of $\mathcal{F}_{i,-j}$

And so all codewords of $\mathcal{F}_i$ are characterized.

Now assume that the codewords of $\mathcal{J}$ satisfy: $U_1 \in \mathcal{G}_{ikno}$; $U_2 \in \mathcal{G}_{i,l,n,-m}$; $U_3 \in \mathcal{G}_{i,l,-k,o}$; $U_4 \in \mathcal{G}_{i,m,-n,-k}$; $U_5 \in \mathcal{G}_{i,m,o,-l}$. In this case it is possible to identify a unique codeword in $\mathcal{F}_{i,-o}$: $U_6 \in \mathcal{F}_{i,-o,l,-n}$. Let us consider the set $\mathcal{F}_{i,-j}$. Note that, as $|\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}| \geq 3$ and $|\mathcal{F}_{i,-o}| \leq 1$, then $|\mathcal{F}_{i,-j}| \geq 2$. In the following scheme all possible index distributions for the codewords of $\mathcal{F}_{i,-j}$ are presented:

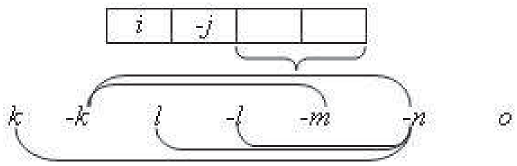

Figure 6.4: Possible index distribution for $U \in \mathcal{F}_{i,-j}$.

Thus, if $U \in \mathcal{F}_{i,-j}$, then $U \in \mathcal{F}_{i,-j,-k,-m} \cup \mathcal{F}i, -j, -n$. Consequently, $|\mathcal{F}_{i,-j}| \leq 2$. Accordingly, there exist two possible index distributions for the codewords of $\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$:

U_6	i	$-o$	l	$-n$
U_7	i	$-j$	$-k$	$-m$
U_8	i	$-j$	$-n$	$-l$

U_6	i	$-o$	l	$-n$
U_7	i	$-j$	$-k$	$-m$
U_8	i	$-j$	k	$-n$

Table 6.17: Index distributions of the codewords of $\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$

Once again it was possible to characterize all codewords of $\mathcal{F}_i$.

Let us now assume that $|\mathcal{J}| = 4$. Applying the same strategy used in the analysis of the condition $|\mathcal{J}| = 4$ when $W_4 \in \mathcal{G}_{i,-o,-m,k,-l}$ and $W_5 \in \mathcal{G}_{i,-o,-j,l,-n}$ (Example 2), we obtain possible index distributions for all codewords of $\mathcal{F}_i$. In fact, for $W_4 \in \mathcal{G}_{i,-o,-m,k,-l}$ and $W_5 \in \mathcal{G}_{i,-o,-j,m,n}$ we get, under the assumption $|\mathcal{J}| = 4$, twenty three distinct index distributions for the codewords of $\mathcal{F}_i$.

In Example 2, when considered $W_4 \in \mathcal{G}_{i,-o,-m,k,-l}$ and $W_5 \in \mathcal{G}_{i,-o,-j,l,-n}$, we have characterized only two possible index distributions for the codewords of $\mathcal{F}_i$. However, for certain index distributions of $W_4, W_5 \in \mathcal{G}_i \backslash \mathcal{G}_j$ there are many hypotheses for the codewords of $\mathcal{F}_i$. The example which is being now considered is one of these cases.

- **$|\mathcal{H}| = 1$ and $W \in \mathcal{H}$ satisfies $W \in \mathcal{G}_{i,-j}$**

Assume that $|\mathcal{H}| = 1$ and $W_4 \in \mathcal{H}$ is such that $W_4 \in \mathcal{G}_{i,-j}$. By definition of $\mathcal{H}$, $W_4 \in \mathcal{G}_{i,-j,w_1,w_2,w_3}$ with $w_1 \in \mathcal{P}_1$, $w_2 \in \mathcal{P}_2$ and $w_3 \in \mathcal{P}_3$. We recall that $\mathcal{P}_1 = \{k, l, m\}$, $\mathcal{P}_2 = \{-k, -l, n\}$ and $\mathcal{P}_3 = \{-m, -n, o\}$. Up to an equivalent index distributions, W_4 satisfies one of the following conditions:

W_4	i	$-j$	k	$-l$	$-m$
W_4	i	$-j$	k	n	$-m$
W_4	i	$-j$	k	$-l$	o
W_4	i	$-j$	k	n	o
W_4	i	$-j$	m	n	o

Table 6.18: Possible index distributions for $W_4 \in \mathcal{H}$.

Next step consists in the characterization of all possible index distributions for $W_5 \in \mathcal{G}_i\backslash(\mathcal{G}_j \cup \mathcal{H})$ when W_4 assumes one of the conditions presented in Table 6.18. We note that, since $W_5 \notin \mathcal{H}$, then $W_5 \in \mathcal{G}_{i,-o,-j}$. In Tables 6.19, 6.20, 6.21, 6.22 and 6.23 all possible index distributions for W_5 are described, when $W_4 \in \mathcal{G}_{i,-j,k,-l,-m}$, $W_4 \in \mathcal{G}_{i,-j,k,n,-m}$, $W_4 \in \mathcal{G}_{i,-j,k,-l,o}$, $W_4 \in \mathcal{G}_{i,-j,k,n,o}$ and $W_4 \in \mathcal{G}_{i,-j,m,n,o}$, respectively.

For each one of the presented index distributions of $W_4, W_5 \in \mathcal{G}_i\backslash\mathcal{G}_j$ we try to characterize $\mathcal{F}_i$. We recall that, by Proposition 6.1, $8 \leq |\mathcal{F}_i| \leq 9$, furthermore, from Proposition 6.5 it follows that:

- $3 \leq |\mathcal{J}| \leq 5$;
- $|\mathcal{F}_{i,-o,-j}| = 0$;
- if $|\mathcal{J}| = 3$, then $|\mathcal{F}_{i,-o}| = 3$, $|\mathcal{F}_{i,-j}| = 2$ and $|\mathcal{F}_i| = 8$.

As in the previous case, for certain index distributions of $W_4, W_5 \in \mathcal{G}_i\backslash\mathcal{G}_j$ it is not possible to describe all codewords of $\mathcal{F}_i$ without facing a contradiction. In some cases it is only possible to find out possible codewords for $\mathcal{F}_i$ when $|\mathcal{J}|$ assumes certain values and, in other situations, we can characterize $\mathcal{F}_i$ independently of the cardinality of $\mathcal{J}$.

1	W_5	i	$-o$	$-j$	l	$-k$
2	W_5	i	$-o$	$-j$	l	n
3	W_5	i	$-o$	$-j$	m	$-k$
4	W_5	i	$-o$	$-j$	m	n
5	W_5	i	$-o$	$-j$	l	$-n$
6	W_5	i	$-o$	$-j$	m	$-n$
7	W_5	i	$-o$	$-j$	$-k$	$-n$

Table 6.19: $W_4 \in \mathcal{G}_{i,-j,k,-l,-m}$.

1	W_5	i	$-o$	$-j$	l	$-k$
2	W_5	i	$-o$	$-j$	m	$-k$
3	W_5	i	$-o$	$-j$	m	$-l$
4	W_5	i	$-o$	$-j$	l	$-n$
5	W_5	i	$-o$	$-j$	m	$-n$
6	W_5	i	$-o$	$-j$	$-k$	$-n$
7	W_5	i	$-o$	$-j$	$-l$	$-n$

Table 6.20: $W_4 \in \mathcal{G}_{i,-j,k,n,-m}$.

1	W_5	i	$-o$	$-j$	l	$-k$
2	W_5	i	$-o$	$-j$	l	n
3	W_5	i	$-o$	$-j$	m	$-k$
4	W_5	i	$-o$	$-j$	m	n
5	W_5	i	$-o$	$-j$	l	$-m$
6	W_5	i	$-o$	$-j$	l	$-n$
7	W_5	i	$-o$	$-j$	m	$-n$
8	W_5	i	$-o$	$-j$	$-k$	$-m$
9	W_5	i	$-o$	$-j$	$-k$	$-n$
10	W_5	i	$-o$	$-j$	n	$-m$

Table 6.21: $W_4 \in \mathcal{G}_{i,-j,k,-l,o}$.

1	W_5	i	$-o$	$-j$	l	$-k$
2	W_5	i	$-o$	$-j$	m	$-k$
3	W_5	i	$-o$	$-j$	m	$-l$
4	W_5	i	$-o$	$-j$	l	$-m$
5	W_5	i	$-o$	$-j$	l	$-n$
6	W_5	i	$-o$	$-j$	m	$-n$
7	W_5	i	$-o$	$-j$	$-k$	$-m$
8	W_5	i	$-o$	$-j$	$-k$	$-n$
9	W_5	i	$-o$	$-j$	$-l$	$-m$
10	W_5	i	$-o$	$-j$	$-l$	$-n$

Table 6.22: $W_4 \in \mathcal{G}_{i,-j,k,n,o}$.

1	W_5	i	$-o$	$-j$	k	$-l$
2	W_5	i	$-o$	$-j$	l	$-k$
3	W_5	i	$-o$	$-j$	k	$-m$
4	W_5	i	$-o$	$-j$	k	$-n$
5	W_5	i	$-o$	$-j$	l	$-m$
6	W_5	i	$-o$	$-j$	l	$-n$
7	W_5	i	$-o$	$-j$	$-k$	$-m$
8	W_5	i	$-o$	$-j$	$-k$	$-n$
9	W_5	i	$-o$	$-j$	$-l$	$-m$
10	W_5	i	$-o$	$-j$	$-l$	$-n$

Table 6.23: $W_4 \in \mathcal{G}_{i,-j,m,n,o}$.

Next, we will present the methodology applied, which is mostly based in Proposition 6.5, considering examples in which:

1) it is not possible to describe $\mathcal{F}_i$;

2) there exist possible index distributions for all codewords of $\mathcal{F}_i$.

Example 1: It is not possible to describe $\mathcal{F}_i$.

Consider $W_4 \in \mathcal{G}_{i,-j,k,-l,-m}$ and $W_5 \in \mathcal{G}_{i,-o,-j,l,-k}$. By Proposition 6.5 we get $3 \leq |\mathcal{J}| \leq 5$. Let us describe the codewords of $\mathcal{J}$. Taking into account the codewords $W_4, W_5 \in \mathcal{G}_i\backslash\mathcal{G}_j$ and Lemma 1.5, the analysis of Table 6.5 allows us to conclude that if $U \in \mathcal{J}$, then

$$U \in \mathcal{F}_{ikno} \cup \mathcal{F}_{iln} \cup \mathcal{F}_{i,m,-n} \cup \mathcal{F}_{imo},$$

and so $|\mathcal{J}| \leq 4$.

If $|\mathcal{J}| = 4$, the codewords $U_1, \ldots, U_4 \in \mathcal{J}$ must satisfy one of the following conditions:

U_1	i	k	n	o
U_2	i	l	n	$-m$
U_3	i	m	$-n$	$-k$
U_4	i	m	o	$-l$

U_1	i	k	n	o
U_2	i	l	n	$-m$
U_3	i	m	$-n$	$-l$
U_4	i	m	o	$-k$

Table 6.24: Possible index distributions for the codewords of $\mathcal{J}$.

By Proposition 6.1, $8 \leq |\mathcal{F}_i| \leq 9$. As $\mathcal{F}_i = \mathcal{J} \cup \mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$ and $|\mathcal{J}| = 4$, then $|\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}| \geq 4$. The following schemes allow us to identify, for both hypotheses of $\mathcal{J}$, all possible index distributions for the codewords of $\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$:

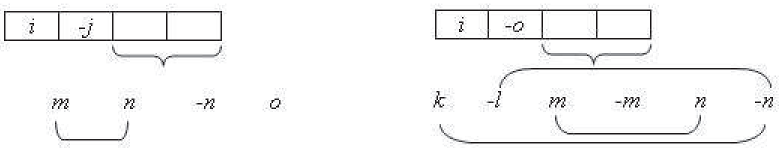

Figure 6.5: Possible index distributions for the codewords of $\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$.

By the analysis of the schemes in Figure 6.5 we conclude that if $U \in \mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$, then $U \in \mathcal{F}_{i,m,n} \cup \mathcal{F}_{i,-o,-n}$. Consequently, $|\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}| \leq 2$, which is a contradiction.

Now suppose that $|\mathcal{J}| = 3$. By Proposition 6.5, $|\mathcal{F}_{i,-o}| = 3$, $|\mathcal{F}_{i,-j}| = 2$ and $|\mathcal{F}_{i,-o,-j}| = 0$.

We begin by characterizing the codewords of $\mathcal{F}_{i,-j}$ taking into account the codewords $W_4, W_5 \in \mathcal{G}_i \backslash \mathcal{G}_j$ and Lemma 1.5:

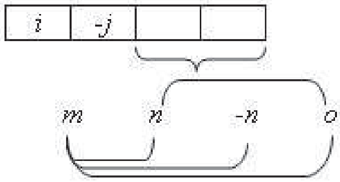

Figure 6.6: Possible index distributions for $U \in \mathcal{F}_{i,-j}$.

By the analysis of the above scheme, $U_1, U_2 \in \mathcal{F}_{i,-j}$ must satisfy: $U_1 \in \mathcal{F}_{i,-j,n,o}$ and $U_2 \in \mathcal{F}_{i,-j,m,-n}$. However, when we try to characterize the codewords of $\mathcal{J}$ we verify, considering Table 6.5, that if $U \in \mathcal{J}$, then $U \in \mathcal{F}_{i,l,n,-m} \cup \mathcal{F}_{imo}$ and so $|\mathcal{J}| \leq 2$, which is a contradiction.

Therefore, the considered index distribution for $W_4, W_5 \in \mathcal{G}_i \backslash \mathcal{G}_j$ does not satisfy the definition of perfect error correcting Lee code.

There exist other index distributions for the codewords of $\mathcal{G}_i \backslash \mathcal{G}_j$ for which, such as in the presented example, it is not possible to characterize all codewords of $\mathcal{F}_i$. Namely,

- $1, 2, 3, 5, 6$ and 7 in Table 6.19;

- $5, 6$ and 7 in Table 6.20;

- $1, 5, 7, 9$ and 10 in Table 6.21;

- $1, 2, 4, 6, 8$ and 10 in Table 6.22;

- $1, 2, 3, 5, 8$ and 10 in Table 6.23.

Example 2: There exist possible index distributions for all codewords of $\mathcal{F}_i$.

Suppose that $W_4, W_5 \in \mathcal{G}_i\backslash\mathcal{G}_j$ are such that $W_4 \in \mathcal{G}_{i,-j,k,n,o}$ and $W_5 \in \mathcal{G}_{i,-o,-j,l,-n}$. Since, by Proposition 6.5, $3 \leq |\mathcal{J}| \leq 5$, we will consider the three possible hypotheses: $|\mathcal{J}| = 5$, $|\mathcal{J}| = 4$ and $|\mathcal{J}| = 3$.

Let us begin by assuming $|\mathcal{J}| = 5$. By the analysis of Table 6.5, having in mind the codewords of $\mathcal{G}_i\backslash\mathcal{G}_j$ and Lemma 1.5, we verify that if $U \in \mathcal{J}$, then U is such that $U \in \mathcal{F}_{i,l,n,-m} \cup \mathcal{F}_{i,k,-l} \cup \mathcal{F}_{i,l,-k} \cup \mathcal{F}_{i,m,-n} \cup \mathcal{F}_{imo}$, more precisely, $U_1, \ldots, U_5 \in \mathcal{J}$ satisfy one of the following index distributions:

U_1	i	l	n	$-m$
U_2	i	l	$-k$	o
U_3	i	m	o	$-l$
U_4	i	m	$-n$	$-k$
U_5	i	k	$-l$	$-m$

U_1	i	l	n	$-m$
U_2	i	l	$-k$	o
U_3	i	m	o	$-l$
U_4	i	m	$-n$	$-k$
U_5	i	k	$-l$	$-n$

Table 6.25: Possible index distributions for the codewords of $\mathcal{J}$.

Since $\mathcal{F}_i = \mathcal{J} \cup \mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$ and, by assumption, $|\mathcal{J}| = 5$, taking into account Proposition 6.1, we must impose $|\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}| \geq 3$.

Assume that the codewords of $\mathcal{J}$ are such that: $U_1 \in \mathcal{F}_{i,l,n,-m}$; $U_2 \in \mathcal{F}_{i,l,-k,o}$; $U_3 \in \mathcal{F}_{i,m,o,-l}$; $U_4 \in \mathcal{F}_{i,m,-n,-k}$; $U_5 \in \mathcal{F}_{i,k,-l,-m}$. Let us identify the possible index distributions for the codewords of $\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$:

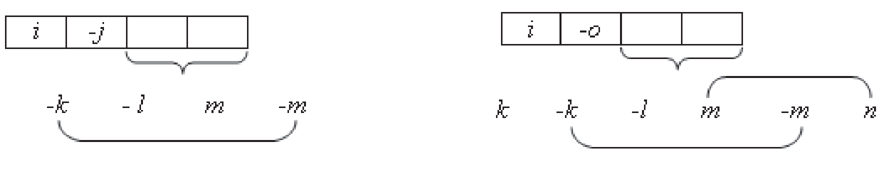

Figure 6.7: Possible index distributions for $U \in \mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$.

Through the above schemes, we conclude that if $U \in \mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$, then

$$U \in \mathcal{F}_{i,-k,-m} \cup \mathcal{F}_{i,-o,m,n}.$$

Accordingly, considering Lemma 1.5, we get $|\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}| \leq 2$, which is a contradiction.

Now consider that the codewords of $\mathcal{J}$ satisfy: $U_1 \in \mathcal{F}_{i,l,n,-m}$; $U_2 \in \mathcal{F}_{i,l,-k,o}$; $U_3 \in \mathcal{F}_{i,m,o,-l}$; $U_4 \in \mathcal{F}_{i,m,-n,-k}$; $U_5 \in \mathcal{F}_{i,k,-l,-n}$. Following a similar reasoning that one applied in the other hypothesis for $\mathcal{J}$, we get possible index distributions for the codewords of $\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$:

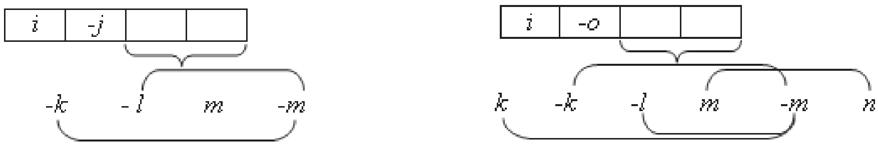

Figure 6.8: Possible index distributions for $U \in \mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$.

Accordingly, $|\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}| = 3$ and the codewords $U_6, U_7, U_8 \in \mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$ must satisfy one of the following conditions:

U_6				U_7				U_8			
i	m	n	-o	i	-j	-l	-m	i	k	-m	-o
								i	-k	-m	-o
				i	-j	-k	-m	i	k	-m	-o
								i	-l	-m	-o

Table 6.26: Possible index distributions for the codewords of $\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$.

Now consider $|\mathcal{J}| = 4$. By the analysis of Table 6.5, we get the following possible index distributions for the codewords $U_1, \ldots, U_4 \in \mathcal{J}$:

U_1				U_2				U_3				U_4				
i	l	n	-m	i	l	-k	o	i	k	-l	-n	i	m	o	-l	**1**
												i	m	-n	-k	**2**
								i	k	-l	-m	i	m	o	-l	**3**
												i	m	-n	-k	**4**
												i	m	-n	-l	**5**
								i	m	o	-l	i	m	-n	-k	**6**
				i	k	-l	-m	i	m	-n	-k	i	m	o	-l	**7**
								i	m	-n	-l	i	m	o	-k	**8**
				i	k	-l	-n	i	m	-n	-k	i	m	o	-l	**9**
i	k	-l	-m	i	l	-k	-m	i	m	-n	-k	i	m	o	-l	**10**
								i	m	-n	-l	i	m	o	-k	**11**
				i	l	-k	o	i	m	o	-l	i	m	-n	-k	**12**
i	k	-l	-n	i	m	-n	-k	i	m	o	-l	i	l	-k	-m	**13**
												i	l	-k	o	**14**

Table 6.27: Possible index distributions for the codeword of $\mathcal{J}$.

As we are assuming $|\mathcal{J}| = 4$, by Proposition 6.1, we must impose $|\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}| \geq 4$. For each one of the conditions presented in Table 6.27, using the same strategy as in the previous case, we identify all possible index distributions for codewords of $\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$. However, in the majority of the cases it is not possible to complete the characterization of $\mathcal{F}_i$ without contradictions. In fact, we are only allowed to do it when the codewords of $\mathcal{J}$ satisfy, respectively, the conditions 1, 2, 5 and 6, presented in Table 6.27. The respective codewords $U_5, U_6, U_7, U_8 \in \mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$ for each one of the referred conditions are presented in the following table:

<table>
<tr><th colspan="4">U_5</th><th colspan="4">U_6</th><th colspan="4">U_7</th><th colspan="4">U_8</th><th></th></tr>
<tr><td rowspan="2">i</td><td rowspan="2">m</td><td rowspan="2">n</td><td rowspan="2">$-o$</td><td rowspan="2">i</td><td rowspan="2">$-j$</td><td rowspan="2">$-l$</td><td rowspan="2">$-m$</td><td rowspan="2">i</td><td rowspan="2">$-j$</td><td rowspan="2">$-k$</td><td rowspan="2">m</td><td>i</td><td>k</td><td>$-m$</td><td>$-o$</td><td rowspan="2">1</td></tr>
<tr><td>i</td><td>$-k$</td><td>$-m$</td><td>$-o$</td></tr>
<tr><td rowspan="2">i</td><td rowspan="2">$-j$</td><td rowspan="2">$-k$</td><td rowspan="2">$-m$</td><td rowspan="2">i</td><td rowspan="2">$-j$</td><td rowspan="2">$-l$</td><td rowspan="2">m</td><td rowspan="2">i</td><td rowspan="2">m</td><td rowspan="2">n</td><td rowspan="2">$-o$</td><td>i</td><td>k</td><td>$-m$</td><td>$-o$</td><td rowspan="2">2</td></tr>
<tr><td>i</td><td>$-l$</td><td>$-m$</td><td>$-o$</td></tr>
<tr><td>i</td><td>$-j$</td><td>$-l$</td><td>$-m$</td><td>i</td><td>$-j$</td><td>$-k$</td><td>m</td><td>i</td><td>$-k$</td><td>$-m$</td><td>$-o$</td><td>i</td><td>m</td><td>n</td><td>$-o$</td><td>5</td></tr>
<tr><td>i</td><td>m</td><td>n</td><td>$-o$</td><td>i</td><td>$-j$</td><td>$-l$</td><td>$-m$</td><td>i</td><td>k</td><td>$-l$</td><td>$-o$</td><td>i</td><td>$-k$</td><td>$-m$</td><td>$-o$</td><td>6</td></tr>
</table>

Table 6.28: Possible index distributions for the codewords of $\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$.

Suppose that $|\mathcal{J}| = 3$. As, by Proposition 6.5, $|\mathcal{F}_{i,-o}| = 3$, $|\mathcal{F}_{i,-j}| = 2$ and $|\mathcal{F}_{i,-o,-j}| = 0$, then $|\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}| = 5$. Beginning the characterization of all possible codewords of $\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$ and proceeding as in the schemes illustrated in Figure 6.8, we conclude that $U_1, \ldots, U_5 \in \mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$ must satisfy: $U_1 \in \mathcal{F}_{i,-j,-k,m}$; $U_2 \in \mathcal{F}_{i,-j,-l,-m}$; $U_3 \in \mathcal{F}_{i,-o,-m,-k}$; $U_4 \in \mathcal{F}_{i,-o,m,n}$; $U_5 \in \mathcal{F}_{i,-o,k,-l}$. By the analysis of Table 6.5, we get the two possible index distributions for the codewords of $\mathcal{J}$, completing the characterization of $\mathcal{F}_i$:

U_6	i	l	$-k$	o
U_7	i	l	n	$-m$
U_8	i	m	$-n$	$-l$

U_6	i	l	$-k$	o
U_7	i	l	n	$-m$
U_8	i	m	$-n$	$-l$

Table 6.29: Possible index distributions for the codewords of $\mathcal{J}$.

- **$|\mathcal{H}| = 2$ and $W_4, W_5 \in \mathcal{H}$ satisfy $W_4, W_5 \in \mathcal{G}_{i,-o}$**

Let us assume $|\mathcal{H}| = 2$ with $W_4, W_5 \in \mathcal{H}$ satisfying $W_4, W_5 \in \mathcal{G}_{i,-o}$. Up to an equivalent index distributions, there are four distinct possible characterizations for $W_4, W_5 \in \mathcal{H}$:

W_4	i	$-o$	k	$-l$	$-m$
W_5	i	$-o$	l	$-k$	$-n$

W_4	i	$-o$	k	$-l$	$-m$
W_5	i	$-o$	m	$-n$	$-k$

W_4	i	$-o$	k	n	$-m$
W_5	i	$-o$	m	$-n$	$-k$

W_4	i	$-o$	k	n	$-m$
W_5	i	$-o$	m	$-n$	$-l$

Table 6.30: Possible index distributions for the codewords $W_4, W_5 \in \mathcal{H}$.

For all possible index distributions of the codewords $W_4, W_5 \in \mathcal{H}$ presented above it is possible to characterize $\mathcal{F}_i$. We identify those possible index distributions taking into account, mainly, Proposition 6.6 from which we know that:

- $3 \leq |\mathcal{J}| \leq 4$;
- if $|\mathcal{J}| = 3$, then $|\mathcal{F}_i| = 8$ and one of the following conditions must occurs
  - ◇ $|\mathcal{F}_{i,-j}| = 5$;
  - ◇ $|\mathcal{F}_{i,-j}| = 4$, $|\mathcal{F}_{i,-o}| = 1$ and $|\mathcal{F}_{i,-o,-j}| = 0$.

To exemplify how was done the characterization of the possible codewords of $\mathcal{F}_i$, we will consider the case where $W_4 \in \mathcal{G}_{i,-o,k,-l,-m}$ and $W_5 \in \mathcal{G}_{i,-o,l,-k,-n}$.

Let us suppose that $|\mathcal{J}| = 4$. From the analysis of Table 6.5, and taking into account $W_4, W_5 \in \mathcal{H}$ as well as Lemma 1.5, we verify that the codewords $U_1, \ldots, U_4 \in \mathcal{J}$ must satisfy the index distribution presented in Table 6.31.

U_1	i	k	n	o
U_2	i	l	n	$-m$
U_3	i	m	$-n$	$-l$
U_4	i	m	o	$-k$

Table 6.31: Index distribution of the codewords of $\mathcal{J}$.

Next step consists in the characterization of the remaining codewords of $\mathcal{F}_i$, that is, the codewords of $\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$. We note that, since $|\mathcal{J}| = 4$ and that, by Proposition 6.1, $8 \leq |\mathcal{F}_i| \leq 9$, we must impose $|\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}| \geq 4$. Let us identify the possible index distributions for the codewords of $\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$:

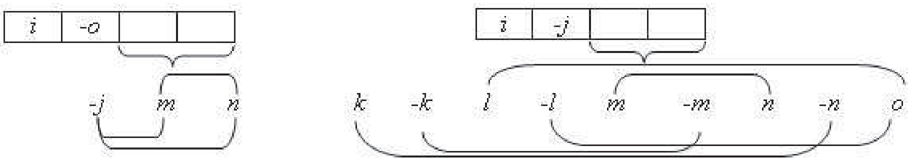

Figure 6.9: Possible index distributions for $U \in \mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$.

Considering Lemma 1.5 and analyzing the schemes in Figure 6.9, we conclude that $|\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}| = 4$ and $U_5, \ldots, U_8 \in \mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$ must satisfy one of the following conditions:

U_5				U_6				U_7				U_8			
i	-j	o	l	i	-j	-k	-m	i	-j	k	-n	i	-j	m	n
												i	-j	m	-o
												i	-j	n	-o
												i	m	n	-o
i	-j	o	-l	i	-j	-k	-m	i	-j	k	-n	i	-j	m	n
												i	-j	m	-o
												i	-j	n	-o
												i	m	n	-o

Table 6.32: Possible index distributions for the codewords of $\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$.

Let us now consider $|\mathcal{J}| = 3$. Taking into account Proposition 6.6, we will study separately the following hypotheses:

- ⋄ $|\mathcal{F}_{i,-j}| = 5$;
- ⋄ $|\mathcal{F}_{i,-j}| = 4$, $|\mathcal{F}_{i,-o}| = 4$ and $|\mathcal{F}_{i,-o,-j}| = 0$.

Suppose that $|\mathcal{F}_{i,-j}| = 5$. By Lemma 2.5, for each $\alpha \in \mathcal{I}\backslash\{i,-i,j,-j\}$ there exists a unique $U \in \mathcal{F}_{i,-j}$ such that $U \in \mathcal{F}_{i,-j,\alpha}$. Consider $-o, k, -l, -m \in \mathcal{I}$, the elements in the index distribution of W_4. Taking into account Lemma 2.5, we get

$$|\mathcal{F}_{i,-j,-o}| = |\mathcal{F}_{i,-j,k}| = |\mathcal{F}_{i,-j,-l}| = |\mathcal{F}_{i,-j,-m}| = 1.$$

Since $W_4 \in \mathcal{G}_{i,-o,k,-l,-m}$, by Lemma 1.5, $U_1 \in \mathcal{F}_{i,-j,-o,u_1}$, $U_2 \in \mathcal{F}_{i,-j,k,u_2}$, $U_3 \in \mathcal{F}_{i,-j,-l,u_3}$ and $U_4 \in \mathcal{F}_{i,-j,-m,u_4}$ are such that $u_1, \dots, u_4$ are pairwise distinct and $u_1, \dots, u_4 \notin \{-o, k, -l, -m\}$. Considering what was said before and taking into account Lemma 1.5 as well as the index distribution of the codewords of $\mathcal{G}_i$, we conclude that: $u_1 \in \{m, n\}$; $u_2 \in \{n, -n, o\}$; $u_3 \in \{m, -n, o\}$; $u_4 \in \{-k, l, n\}$. Consequently, the codewords $U_1, \dots, U_5 \in \mathcal{F}_{i,-j}$ must satisfy one of the following conditions:

i	-j	-o	m	i	-j	-l	-n	i	-j	k	n	i	-j	-m	-k	i	-j	l	o	1
i	-j	-o	m	i	-j	-l	-n	i	-j	k	n	i	-j	-m	l	i	-j	-k	o	2
i	-j	-o	m	i	-j	-l	-n	i	-j	k	o	i	-j	-m	-k	i	-j	l	n	3
i	-j	-o	m	i	-j	-l	o	i	-j	k	-n	i	-j	-m	-k	i	-j	l	n	4
i	-j	-o	n	i	-j	k	-n	i	-j	-m	l	i	-j	-l	m	i	-j	-k	o	5
i	-j	-o	n	i	-j	k	-n	i	-j	-m	l	i	-j	-l	o	i	-j	-k	m	6
i	-j	-o	n	i	-j	k	-n	i	-j	-l	m	i	-j	-m	-k	i	-j	l	o	7
i	-j	-o	n	i	-j	k	o	i	-j	-l	-n	i	-j	-m	l	i	-j	-k	m	8

Table 6.33: Index distributions of the codewords of $\mathcal{F}_{i,-j}$.

The majority of the hypotheses for $\mathcal{F}_{i,-j}$ presented in Table 6.33 imply, by the analysis of Table 6.5, $|\mathcal{J}| \leq 2$, contradicting Proposition 6.6. There are only two cases in which this does not happen, those, correspond to the cases where $U_1, \dots, U_5 \in \mathcal{F}_{i,-j}$ satisfy the conditions in 4) or in 7). If the codewords of $\mathcal{F}_{i,-j}$ satisfy the conditions in 4), then $U_6, U_7, U_8 \in \mathcal{J}$ are such that: $U_6 \in \mathcal{F}_{ikno}$; $U_7 \in \mathcal{F}_{i,m,-n,-l}$; $U_8 \in \mathcal{F}_{i,m,o,-k}$. On the other hand, if $\mathcal{F}_{i,-j}$ satisfy the conditions in 7), then: $U_6 \in \mathcal{F}_{ikno}$; $U_7 \in \mathcal{F}_{i,l,n,-m}$; $U_8 \in \mathcal{F}_{i,m,o,-k}$.

Next we will analyze the condition $|\mathcal{F}_{i,-j}| = 4$, $|\mathcal{F}_{i,-o}| = 1$ and $|\mathcal{F}_{i,-o,-j}| = 0$. We begin by identifying the unique codeword in $\mathcal{F}_{i,-o}$. Considering Lemma 1.5 and the codewords of $\mathcal{G}_i$ we get $U_1 \in \mathcal{F}_{i,-o,m,n}$. From Table 6.5 it follows the distinct index distributions for $U_2, U_3, U_4 \in \mathcal{J}$:

<table>
<tr><th colspan="4">U2</th><th colspan="4">U3</th><th colspan="4">U4</th></tr>
<tr><td rowspan="4">i</td><td rowspan="4">k</td><td rowspan="4">n</td><td rowspan="4">o</td><td rowspan="3">i</td><td rowspan="3">l</td><td rowspan="3">n</td><td rowspan="3">-m</td><td>i</td><td>m</td><td>-n</td><td>-l</td></tr>
<tr><td>i</td><td>m</td><td>o</td><td>-k</td></tr>
<tr><td>i</td><td>m</td><td>o</td><td>-l</td></tr>
<tr><td>i</td><td>m</td><td>-n</td><td>-l</td><td>i</td><td>m</td><td>o</td><td>-k</td></tr>
<tr><td colspan="4"></td><td colspan="4"></td><td colspan="4"></td></tr>
<tr><td rowspan="2">i</td><td rowspan="2">m</td><td rowspan="2">-n</td><td rowspan="2">-l</td><td rowspan="2">i</td><td rowspan="2">m</td><td rowspan="2">o</td><td rowspan="2">-k</td><td>i</td><td>l</td><td>n</td><td>-m</td></tr>
<tr><td>i</td><td>l</td><td>n</td><td>o</td></tr>
</table>

Table 6.34: Possible index distributions for the codewords of $\mathcal{J}$.

It is only possible to characterize all codewords of $\mathcal{F}_{i,-j}$, taking into account that $|\mathcal{F}_{i,-j}| = 4$, when $U_2, U_3, U_4 \in \mathcal{J}$ satisfy, respectively:

i) $U_2 \in \mathcal{F}_{ikno}$, $U_3 \in \mathcal{F}_{i,l,n,-m}$; $U_4 \in \mathcal{F}_{i,m,o,-k}$;

ii) $U_2 \in \mathcal{F}_{ikno}$, $U_3 \in \mathcal{F}_{i,m,-n,-l}$; $U_4 \in \mathcal{F}_{i,m,o,-k}$.

For each one of these conditions we get, respectively, the following index distribution for the codewords of $\mathcal{F}_{i,-j}$:

U_5	i	$-j$	$-k$	$-m$
U_6	i	$-j$	$-n$	k
U_7	i	$-j$	o	l
U_8	i	$-j$	$-l$	m

Table 6.35: $\mathcal{J}$ satisfies *i)*.

U_5	i	$-j$	$-m$	$-k$
U_6	i	$-j$	l	n
U_7	i	$-j$	$-n$	k
U_8	i	$-j$	o	$-l$

Table 6.36: $\mathcal{J}$ satisfies *ii)*.

- $|\mathcal{H}| = 2$ **and** $W_4, W_5 \in \mathcal{H}$ **satisfy** $W_4, W_5 \in \mathcal{G}_{i,-j}$

Suppose that $|\mathcal{H}| = 2$ and $W_4, W_5 \in \mathcal{H}$ satisfy $W_4, W_5 \in \mathcal{G}_{i,-j}$. Let $W_4 \in \mathcal{G}_{i,-j,w_1,w_2,w_3}$ and $W_5 \in \mathcal{G}_{i,-j,w_4,w_5,w_6}$ with $w_1, w_4 \in \mathcal{P}_1$, $w_2, w_5 \in \mathcal{P}_2$ and $w_3, w_6 \in \mathcal{P}_3$, where $\mathcal{P}_1 = \{k, l, m\}$, $\mathcal{P}_2 = \{-k, -l, n\}$ and $\mathcal{P}_3 = \{-m, -n, o\}$. Up to an equivalent index distributions, W_4 and W_5 verify one of the conditions presented in the following tables where for certain index distributions of W_4, that is for $W_4 \in \mathcal{G}_{i,-j,k,-l,o}$, $W_4 \in \mathcal{G}_{i,-j,k,n,o}$, $W_4 \in \mathcal{G}_{i,-j,k,-l,-m}$ and $W_4 \in \mathcal{G}_{i,-j,k,n,-m}$, are presented the respective possible index distributions for W_5.

W_5	i	$-j$	l	n	$-m$
W_5	i	$-j$	m	$-n$	$-k$

Table 6.37: $W_4 \in \mathcal{G}_{i,-j,k,-l,o}$.

W_5	i	$-j$	m	$-n$	$-k$
W_5	i	$-j$	m	$-n$	$-l$

Table 6.38: $W_4 \in \mathcal{G}_{i,-j,k,n,o}$.

W_5	i	$-j$	l	$-k$	$-n$
W_5	i	$-j$	l	$-k$	o
W_5	i	$-j$	l	n	o
W_5	i	$-j$	m	$-n$	$-k$
W_5	i	$-j$	m	o	$-k$
W_5	i	$-j$	m	o	n

Table 6.39: $W_4 \in \mathcal{G}_{i,-j,k,-l,-m}$.

W_5	i	$-j$	m	$-n$	$-k$
W_5	i	$-j$	m	$-n$	$-l$
W_5	i	$-j$	m	o	$-k$
W_5	i	$-j$	m	o	$-l$

Table 6.40: $W_4 \in \mathcal{G}_{i,-j,k,n,-m}$.

Such as seen in previous cases, for some of the possible index distributions of $W_4, W_5 \in \mathcal{H}$ it is not possible to describe all codewords of $\mathcal{F}_i$ without facing a contradiction. In fact, this happens when $W_4 \in \mathcal{G}_{i,-j,k,n,-m}$ and $W_5 \in \mathcal{G}_{i,-j,m,-n,-k}$ as well as when $W_4 \in \mathcal{G}_{i,-j,k,-l,-m}$ and W_5 assumes each one of the following index distributions: $W_5 \in \mathcal{G}_{i,-j,m,-n,-k}$; $W_5 \in \mathcal{G}_{i,-j,m,o,-k}$; $W_5 \in \mathcal{G}_{i,-j,m,o,n}$.

Next, we will focus our attention on $W_4, W_5 \in \mathcal{H}$ satisfying, respectively, $W_4 \in \mathcal{G}_{i,-j,k,n,-m}$ and $W_5 \in \mathcal{G}_{i,-j,m,-n,-l}$ to exemplify how we obtain all the index distributions for the codewords of $\mathcal{F}_i$.

The characterization of $\mathcal{F}_i$ will be done having in view the Proposition 6.6 from which we get:

- $3 \leq |\mathcal{J}| \leq 4$;
- if $|\mathcal{J}| = 3$, then $|\mathcal{F}_i| = 8$ and one of the following conditions must occurs
  - ◇ $|\mathcal{F}_{i,-o}| = 4$, $|\mathcal{F}_{i,-j}| = 2$ and $|\mathcal{F}_{i,-o,-j}| = 1$;
  - ◇ $|\mathcal{F}_{i,-o}| = 4$, $|\mathcal{F}_{i,-j}| = 1$ and $|\mathcal{F}_{i,-o,-j}| = 0$.

We begin by considering the hypothesis $|\mathcal{J}| = 4$. Analyzing Table 6.5, and taking into account the codewords of $\mathcal{G}_i$ as well as Lemma 1.5, we identify two possible index distributions for the codewords of $\mathcal{J}$:

U_1	i	k	$-l$	o
U_2	i	l	n	o
U_3	i	m	o	$-k$
U_4	i	l	$-k$	$-m$

U_1	i	k	$-l$	o
U_2	i	l	n	o
U_3	i	m	o	$-k$
U_4	i	l	$-k$	$-n$

Table 6.41: Possible index distributions for the codewords of $\mathcal{J}$

As we are assuming $|\mathcal{J}| = 4$ and by Proposition 6.1 we get $8 \leq |\mathcal{F}_i| \leq 9$, then $|\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}| \geq 4$.

Considering the first distribution, that is, $U_1 \in \mathcal{F}_{i,k,-l,o}$, $U_2 \in \mathcal{F}_{ilno}$, $U_3 \in \mathcal{F}_{i,m,o,-k}$ and $U_4 \in \mathcal{F}_{i,l,-k,-m}$, let us identify possible index distributions for the codewords of $\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$:

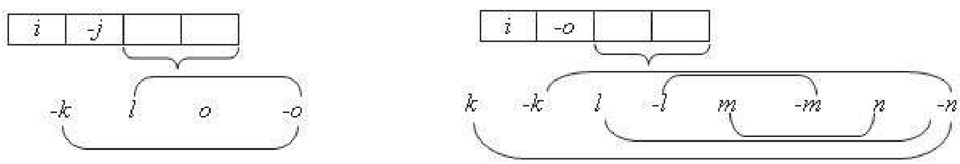

Figure 6.10: Possible index distributions for $U \in \mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$.

Looking at the above schemes we conclude that if $U \in \mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$, then

$$U \in \mathcal{F}_{i,-o,-l,-m} \cup \mathcal{F}_{i,-o,m,n} \cup \mathcal{F}_{i,-j,-o} \cup \mathcal{F}_{i,-o,-n}.$$

Accordingly, taking into account Lemma 1.5, we get $|\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}| = 4$. From the analysis of the schemes in Figure 6.10, and considering again Lemma 1.5, we obtain the following possible index distributions for the codewords $U_5, \ldots, U_8 \in \mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$:

<table>
<tr><th colspan="4">U5</th><th colspan="4">U6</th><th colspan="4">U7</th><th colspan="4">U8</th></tr>
<tr><td rowspan="4">i</td><td rowspan="4">-o</td><td rowspan="4">-l</td><td rowspan="4">-m</td><td rowspan="4">i</td><td rowspan="4">-o</td><td rowspan="4">m</td><td rowspan="4">n</td><td rowspan="2">i</td><td rowspan="2">-o</td><td rowspan="2">-j</td><td rowspan="2">-k</td><td>i</td><td>-o</td><td>-n</td><td>l</td></tr>
<tr><td>i</td><td>o</td><td>-n</td><td>k</td></tr>
<tr><td rowspan="2">i</td><td rowspan="2">-o</td><td rowspan="2">-j</td><td rowspan="2">l</td><td>i</td><td>-o</td><td>-n</td><td>k</td></tr>
<tr><td>i</td><td>-o</td><td>-n</td><td>-k</td></tr>
</table>

Table 6.42: Possible index distributions for the codewords of $\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$.

Applying the usual strategy we get the following possible index distributions for the remaining codewords of $\mathcal{F}_i$ when $U_1, \ldots, U_4 \in \mathcal{J}$ satisfy: $U_1 \in \mathcal{F}_{i,k,-l,o}$, $U_2 \in \mathcal{F}_{ilno}$, $U_3 \in \mathcal{F}_{i,m,o,-k}$ and $U_4 \in \mathcal{F}_{i,l,-k,-n}$.

<table>
<tr><th colspan="4">U5</th><th colspan="4">U6</th><th colspan="4">U7</th><th colspan="4">U8</th></tr>
<tr><td rowspan="4">i</td><td rowspan="4">-o</td><td rowspan="4">k</td><td rowspan="4">-n</td><td rowspan="4">i</td><td rowspan="4">-o</td><td rowspan="4">m</td><td rowspan="4">n</td><td rowspan="2">i</td><td rowspan="2">-o</td><td rowspan="2">-j</td><td rowspan="2">-k</td><td>i</td><td>-o</td><td>-m</td><td>l</td></tr>
<tr><td>i</td><td>-o</td><td>-m</td><td>-l</td></tr>
<tr><td rowspan="2">i</td><td rowspan="2">-o</td><td rowspan="2">-j</td><td rowspan="2">l</td><td>i</td><td>-o</td><td>-m</td><td>-l</td></tr>
<tr><td>i</td><td>-o</td><td>-m</td><td>-k</td></tr>
</table>

Table 6.43: Possible index distributions for the codewords of $\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$.

Let us now suppose $|\mathcal{J}| = 3$. Taking into account Proposition 6.6, we begin by assuming that $|\mathcal{F}_{i,-o}| = 4$, $|\mathcal{F}_{i,-j}| = 2$ and $|\mathcal{F}_{i,-o,-j}| = 1$. Considering the codewords $W_4, W_5 \in \mathcal{G}_i \backslash \mathcal{G}_j$ and having in view Lemma 1.5, we get only two possible index distributions for the codewords $U_1, U_2 \in \mathcal{F}_{i,-j}$:

– $U_1 \in \mathcal{F}_{i,-j,l,o}$ and $U_2 \in \mathcal{F}_{i,-j,-k,-o}$;

– $U_1 \in \mathcal{F}_{i,-j,l,-o}$ and $U_2 \in \mathcal{F}_{i,-j,-k,o}$.

Assume that $U_1 \in \mathcal{F}_{i,-j,l,o}$ and $U_2 \in \mathcal{F}_{i,-j,-k,-o}$. By the analysis of Table 6.5, the codewords $U_3, U_4, U_5 \in \mathcal{J}$ must satisfy one of the following conditions:

<table>
<tr><th colspan="4">U3</th><th colspan="4">U4</th><th colspan="4">U5</th><th></th></tr>
<tr><td rowspan="4">i</td><td rowspan="4">k</td><td rowspan="4">-l</td><td rowspan="4">o</td><td rowspan="2">i</td><td rowspan="2">l</td><td rowspan="2">-k</td><td rowspan="2">-m</td><td>i</td><td>m</td><td>o</td><td>-k</td><td>1</td></tr>
<tr><td>i</td><td>m</td><td>o</td><td>n</td><td>2</td></tr>
<tr><td rowspan="2">i</td><td rowspan="2">l</td><td rowspan="2">-k</td><td rowspan="2">-n</td><td>i</td><td>m</td><td>o</td><td>-k</td><td>3</td></tr>
<tr><td>i</td><td>m</td><td>o</td><td>n</td><td>4</td></tr>
</table>

Table 6.44: Possible index distributions for the codewords of $\mathcal{J}$.

For each one of the hypotheses described in Table 6.44 we can find possible index distributions for the remaining codewords of $\mathcal{F}_i$, that is, for the codewords of $\mathcal{F}_{i,-o}$, see Table 6.45. We do it considering, for each case, all codewords of $\mathcal{G}_i \cup \mathcal{F}_{i,-j} \cup \mathcal{J}$ already known as well as Lemma 1.5. We note that, in Table 6.45 the numbers 1, 2, 3 and 4 identify the respective codewords of $\mathcal{F}_{i,-o}$ for $\mathcal{J}$ satisfying the conditions identified in Table 6.44 by the same numbers.

<table>
<tr><th></th><th colspan="4">U6</th><th colspan="4">U7</th><th colspan="4">U8</th></tr>
<tr><td rowspan="3">1</td><td rowspan="3">i</td><td rowspan="3">-o</td><td rowspan="3">-l</td><td rowspan="3">-m</td><td rowspan="2">i</td><td rowspan="2">-o</td><td rowspan="2">k</td><td rowspan="2">-n</td><td>i</td><td>-o</td><td>l</td><td>n</td></tr>
<tr><td>i</td><td>-o</td><td>m</td><td>n</td></tr>
<tr><td>i</td><td>-o</td><td>l</td><td>-n</td><td>i</td><td>-o</td><td>m</td><td>n</td></tr>
<tr><td>2</td><td>i</td><td>-o</td><td>k</td><td>-n</td><td>i</td><td>-o</td><td>-l</td><td>-m</td><td>i</td><td>-o</td><td>l</td><td>n</td></tr>
<tr><td rowspan="3">3</td><td rowspan="3">i</td><td rowspan="3">-o</td><td rowspan="3">k</td><td rowspan="3">-n</td><td rowspan="2">i</td><td rowspan="2">-o</td><td rowspan="2">-l</td><td rowspan="2">-m</td><td>i</td><td>-o</td><td>l</td><td>n</td></tr>
<tr><td>i</td><td>-o</td><td>n</td><td>m</td></tr>
<tr><td>i</td><td>-o</td><td>l</td><td>-m</td><td>i</td><td>-o</td><td>m</td><td>n</td></tr>
<tr><td>4</td><td>i</td><td>-o</td><td>l</td><td>n</td><td>i</td><td>-o</td><td>-l</td><td>-m</td><td>i</td><td>-o</td><td>k</td><td>-n</td></tr>
</table>

Table 6.45: Possible index distributions for the codewords of $\mathcal{F}_{i,-o}$.

Now suppose that $U_1 \in \mathcal{F}_{i,-j,l,-o}$ and $U_2 \in \mathcal{F}_{i,-j,-k,o}$. We then get the following possible index distributions for the codewords of $\mathcal{J}$:

<table>
<tr><th colspan="4">U3</th><th colspan="4">U4</th><th colspan="4">U5</th><th></th></tr>
<tr><td rowspan="4">i</td><td rowspan="4">k</td><td rowspan="4">-l</td><td rowspan="4">o</td><td rowspan="2">i</td><td rowspan="2">m</td><td rowspan="2">o</td><td rowspan="2">n</td><td>i</td><td>l</td><td>-k</td><td>-m</td><td>5</td></tr>
<tr><td>i</td><td>l</td><td>-k</td><td>-n</td><td>6</td></tr>
<tr><td rowspan="2">i</td><td rowspan="2">l</td><td rowspan="2">n</td><td rowspan="2">o</td><td>i</td><td>l</td><td>-k</td><td>-m</td><td>7</td></tr>
<tr><td>i</td><td>l</td><td>-k</td><td>-n</td><td>8</td></tr>
</table>

Table 6.46: Possible index distributions for the codewords of $\mathcal{J}$.

As in the previous case, for each one of the hypotheses presented in Table 6.46 we can complete the characterization of all codewords of $\mathcal{F}_i$:

<table>
<tr><th></th><th colspan="4">U6</th><th colspan="4">U7</th><th colspan="4">U8</th></tr>
<tr><td>5</td><td>i</td><td>-o</td><td>-k</td><td>m</td><td>i</td><td>-o</td><td>-l</td><td>-m</td><td>i</td><td>-o</td><td>k</td><td>-n</td></tr>
<tr><td>6</td><td>i</td><td>-o</td><td>-k</td><td>m</td><td>i</td><td>-o</td><td>-l</td><td>-m</td><td>i</td><td>-o</td><td>k</td><td>-n</td></tr>
<tr><td rowspan="3">7</td><td rowspan="3">i</td><td rowspan="3">-o</td><td rowspan="3">-l</td><td rowspan="3">-m</td><td rowspan="2">i</td><td rowspan="2">-o</td><td rowspan="2">k</td><td rowspan="2">-n</td><td>i</td><td>-o</td><td>m</td><td>n</td></tr>
<tr><td>i</td><td>-o</td><td>-k</td><td>m</td></tr>
<tr><td>i</td><td>-o</td><td>-k</td><td>-n</td><td>i</td><td>-o</td><td>m</td><td>n</td></tr>
<tr><td rowspan="3">8</td><td rowspan="3">i</td><td rowspan="3">-o</td><td rowspan="3">k</td><td rowspan="3">-n</td><td rowspan="2">i</td><td rowspan="2">-o</td><td rowspan="2">m</td><td rowspan="2">n</td><td>i</td><td>-o</td><td>-k</td><td>-m</td></tr>
<tr><td>i</td><td>-o</td><td>-l</td><td>-m</td></tr>
<tr><td>i</td><td>-o</td><td>-k</td><td>m</td><td>i</td><td>-o</td><td>-l</td><td>-m</td></tr>
</table>

Table 6.47: Possible index distributions for the codewords of $\mathcal{F}_{i,-o}$.

To conclude the analysis of the hypothesis $|\mathcal{J}| = 3$, we now assume that $|\mathcal{F}_{i,-o}| = 4$, $|\mathcal{F}_{i,-j}| = 1$ and $|\mathcal{F}_{i,-o,-j}| = 0$. Let us begin by characterizing all possible codewords $U_1, \ldots, U_4 \in \mathcal{F}_{i,-o}$:

<table>
<tr><th colspan="4">U1</th><th colspan="4">U2</th><th colspan="4">U3</th><th colspan="4">U4</th></tr>
<tr><td rowspan="3">i</td><td rowspan="3">-o</td><td rowspan="3">k</td><td rowspan="3">-l</td><td rowspan="2">i</td><td rowspan="2">-o</td><td rowspan="2">m</td><td rowspan="2">n</td><td>i</td><td>-o</td><td>-k</td><td>-m</td><td>i</td><td>-o</td><td>l</td><td>-n</td></tr>
<tr><td>i</td><td>-o</td><td>l</td><td>-m</td><td>i</td><td>-o</td><td>-k</td><td>-n</td></tr>
<tr><td>i</td><td>-o</td><td>l</td><td>n</td><td>i</td><td>-o</td><td>-k</td><td>-m</td><td>i</td><td>-o</td><td>m</td><td>-n</td></tr>
<tr><td rowspan="2">i</td><td rowspan="2">-o</td><td rowspan="2">k</td><td rowspan="2">-n</td><td rowspan="2">i</td><td rowspan="2">-o</td><td rowspan="2">-l</td><td rowspan="2">-m</td><td>i</td><td>-o</td><td>l</td><td>n</td><td>i</td><td>-o</td><td>-k</td><td>m</td></tr>
<tr><td>i</td><td>-o</td><td>m</td><td>n</td><td>i</td><td>-o</td><td>-k</td><td>l</td></tr>
</table>

Table 6.48: Possible index distributions for the codewords of $\mathcal{F}_{i,-o}$.

Considering the hypotheses presented above for the codewords of $\mathcal{F}_{i,-o}$, there exists only one for which it is possible to characterize all codewords of $\mathcal{F}_i$, this happens when $U_1 \in \mathcal{F}_{i,-o,k,-n}$; $U_2 \in \mathcal{F}_{i,-o,-l,-m}$; $U_3 \in \mathcal{F}_{i,-o,l,n}$; $U_4 \in \mathcal{F}_{i,-o-k,m}$. In this case we get the following possible index distributions for $U_5, U_6, U_7 \in \mathcal{J}$ and $U_8 \in \mathcal{F}_{i,-j}$:

<table>
<tr><th colspan="4">U5</th><th colspan="4">U6</th><th colspan="4">U7</th><th colspan="4">U8</th></tr>
<tr><td rowspan="4">i</td><td rowspan="4">m</td><td rowspan="4">o</td><td rowspan="4">n</td><td rowspan="4">i</td><td rowspan="4">k</td><td rowspan="4">-l</td><td rowspan="4">o</td><td rowspan="2">i</td><td rowspan="2">l</td><td rowspan="2">-k</td><td rowspan="2">-m</td><td>i</td><td>-j</td><td>l</td><td>o</td></tr>
<tr><td>i</td><td>-j</td><td>-k</td><td>o</td></tr>
<tr><td rowspan="2">i</td><td rowspan="2">l</td><td rowspan="2">-k</td><td rowspan="2">-n</td><td>i</td><td>-j</td><td>l</td><td>o</td></tr>
<tr><td>i</td><td>-j</td><td>-k</td><td>o</td></tr>
</table>

Table 6.49: Possible index distributions for the codewords of $\mathcal{J} \cup \mathcal{F}_{i,-j}$.

- $|\mathcal{H}| = 2$ **and** $W_4, W_5 \in \mathcal{H}$ **satisfy** $W_4 \in \mathcal{G}_{i,-o}$ **and** $W_5 \in \mathcal{G}_{i,-j}$

Suppose that $|\mathcal{H}| = 2$ and $W_4, W_5 \in \mathcal{H}$ are such that $W_4 \in \mathcal{G}_{i,-o,w_1,w_2,w_3}$ and $W_5 \in \mathcal{G}_{i,-j,w_4,w_5,w_6}$, with $w_1, w_4 \in \mathcal{P}_1$, $w_2, w_5 \in \mathcal{P}_2$ and $w_3, w_6 \in \mathcal{P}_3$. Up to an equivalent index distributions, W_5 must verify one of the following conditions when W_4 satisfies, respectively, $W_4 \in \mathcal{G}_{-i,-o,k,-l,-m}$ and $W_4 \in \mathcal{G}_{-i,-o,k,n,-m}$:

1	W_5	i	$-j$	k	n	o	1	W_5	i	$-j$	k	$-l$	$-n$
2	W_5	i	$-j$	l	$-k$	$-m$	2	W_5	i	$-j$	k	$-l$	o
3	W_5	i	$-j$	l	$-k$	$-n$	3	W_5	i	$-j$	l	$-k$	$-m$
4	W_5	i	$-j$	l	$-k$	o	4	W_5	i	$-j$	l	$-k$	$-n$
5	W_5	i	$-j$	l	n	$-m$	5	W_5	i	$-j$	l	$-k$	o
6	W_5	i	$-j$	l	n	o	6	W_5	i	$-j$	l	n	o
7	W_5	i	$-j$	m	$-n$	$-k$	7	W_5	i	$-j$	m	$-n$	$-k$
8	W_5	i	$-j$	m	$-n$	$-l$	8	W_5	i	$-j$	m	$-n$	$-l$
9	W_5	i	$-j$	m	o	$-k$	9	W_5	i	$-j$	m	o	$-k$
10	W_5	i	$-j$	m	o	$-l$	10	W_5	i	$-j$	m	o	$-l$
11	W_5	i	$-j$	m	o	n	11	W_5	i	$-j$	m	o	n

Table 6.50: $W_4 \in \mathcal{G}_{-i,-o,k,-l,-m}$. Table 6.51: $W_4 \in \mathcal{G}_{-i,-o,k,n,-m}$.

The characterization of all codewords of $\mathcal{F}_i$ for each one of the index distributions for $W_4, W_5 \in \mathcal{H}$ presented in the tables above is mainly based in Proposition 6.6 from which it follows:

- $3 \leq |\mathcal{J}| \leq 4$;
- if $|\mathcal{J}| = 3$, then $|\mathcal{F}_i| = 8$ and one of the following conditions must occurs
  - ◇ $|\mathcal{F}_{i,-o}| = 3$, $|\mathcal{F}_{i,-j}| = 3$ and $|\mathcal{F}_{i,-o,-j}| = 1$;
  - ◇ $|\mathcal{F}_{i,-o}| = 2$, $|\mathcal{F}_{i,-j}| = 3$ and $|\mathcal{F}_{i,-o,-j}| = 0$.

For some of the possible index distributions of $W_4, W_5 \in \mathcal{H}$ it is not possible to characterize completely all codewords of $\mathcal{F}_i$, this happens when $W_4 \in \mathcal{G}_{i,-o,k,-l,-m}$ and W_5 satisfies the conditions 3, 4, 6, 7 and 11 in Table 6.50; and when $W_4 \in \mathcal{G}_{i,-o,k,n,-m}$ and W_5 satisfies the conditions 1, 2, 4, 5, 8 and 10 in Table 6.51.

Let us present a case in which is possible identify index distributions for all codewords of $\mathcal{F}_i$. Consider, as an illustrative example, the case where $W_4 \in \mathcal{G}_{i,-o,k,-l,-m}$ and $W_5 \in \mathcal{G}_{i,-j,k,n,o}$.

First suppose that $|\mathcal{J}| = 4$. From the analysis of Table 6.5, it follows two possible index distributions for $U_1, \ldots, U_4 \in \mathcal{J}$:

U_1	i	l	n	$-m$
U_2	i	l	$-k$	$-n$
U_3	i	m	$-n$	$-l$
U_4	i	m	o	$-k$

U_1	i	l	n	$-m$
U_2	i	l	$-k$	o
U_3	i	m	o	$-l$
U_4	i	m	$-n$	$-k$

Table 6.52: Possible index distributions for the codewords of $\mathcal{J}$.

Next step consists in the characterization of the remaining codewords of $\mathcal{F}_i$, that is, the codewords of $\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$. As $|\mathcal{J}| = 4$, we must impose $|\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}| \geq 4$.

Suppose that the codewords of $\mathcal{J}$ are such that: $U_1 \in \mathcal{F}_{i,l,n,-m}$; $U_2 \in \mathcal{F}_{i,l,-k,-n}$; $U_3 \in \mathcal{F}_{i,m,-n,-l}$; $U_4 \in \mathcal{F}_{i,m,o,-k}$. In this case we obtain the following possible index distributions for the codewords of $\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$:

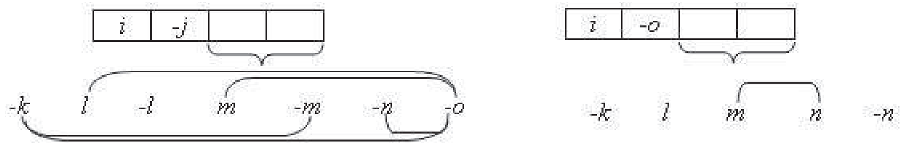

Figure 6.11: Possible index distributions for $U \in \mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$.

By the analysis of the above schemes, we conclude that if $U \in \mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$, then

$$U \in \mathcal{F}_{i,-j,-o} \cup \mathcal{F}_{i,-j,-k,-m} \cup \mathcal{F}_{i,-o,m,n}.$$

Accordingly, considering Lemma 1.5, it follows that $|\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}| \leq 3$, which is an absurdity.

Now consider that the codewords of $\mathcal{J}$ satisfy: $U_1 \in \mathcal{F}_{i,l,n,-m}$; $U_2 \in \mathcal{F}_{i,l,-k,o}$; $U_3 \in \mathcal{F}_{i,m,o,-l}$; $U_4 \in \mathcal{F}_{i,m,-n,-k}$. In this case we obtain the following possible index distributions for the codewords of $\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$:

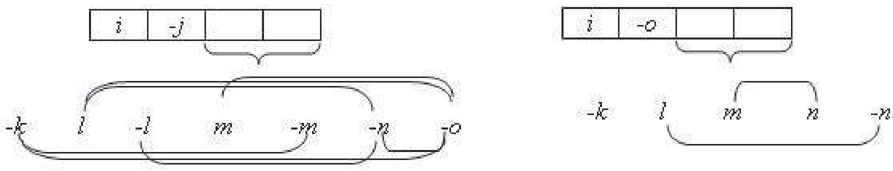

Figure 6.12: Possible index distributions for $U \in \mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$.

Analyzing the schemes presented in Figure 6.12 we get as possibilities for the index distribution of the codewords $U_5, \ldots, U_8 \in \mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$:

U_5	i	$-j$	$-k$	$-m$
U_6	i	l	$-n$	$-o$
U_7	i	m	n	$-o$
U_8	i	$-j$	$-l$	$-n$

U_5	i	l	$-n$	$-o$
U_6	i	m	n	$-o$
U_7	i	$-j$	$-k$	$-o$
U_8	i	$-j$	$-l$	$-n$

U_5	i	$-j$	$-k$	$-m$
U_6	i	m	n	$-o$
U_7	i	$-j$	l	$-o$
U_8	i	$-j$	$-l$	$-n$

U_5	i	$-j$	$-k$	$-m$
U_6	i	l	$-n$	$-o$
U_7	i	$-j$	m	$-o$
U_8	i	$-j$	$-l$	$-n$

Table 6.53: Possible index distributions for the codewords of $\mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$.

Now, we assume $|\mathcal{J}| = 3$. Taking into account Proposition 6.6, we begin by supposing that $|\mathcal{F}_{i,-o}| = |\mathcal{F}_{i,-j}| = 3$ and $|\mathcal{F}_{i,-o,-j}| = 1$. Let us identify possible codewords in $\mathcal{F}_{i,-o}$. Consider $U_1 \in \mathcal{F}_{i,-o,u_1,u_2}$, $U_2 \in \mathcal{F}_{i,-o,u_3,u_4}$ and $U_3 \in \mathcal{F}_{i,-o,u_5,u_6}$. We note that, taking into account that $W_4 \in \mathcal{G}_{i,-o,k,-l,-m}$ and Lemma 1.5, we must impose $u_1, \ldots, u_6 \in \{-j, -k, l, m, n, -n\}$ with $u_1, \ldots, u_6$ pairwise distinct. Thus, the codewords $U_1, U_2, U_3 \in \mathcal{F}_{i,-o}$ must satisfy one of the conditions presented in Table 6.54.

U_1				U_2				U_3			
i	$-o$	n	l	i	$-o$	$-j$	$-k$	i	$-o$	m	$-n$
				i	$-o$	$-j$	m	i	$-o$	$-k$	$-n$
				i	$-o$	$-j$	$-n$	i	$-o$	$-k$	m
i	$-o$	n	m	i	$-o$	$-j$	$-k$	i	$-o$	l	$-n$
				i	$-o$	$-j$	l	i	$-o$	$-k$	$-n$
				i	$-o$	$-j$	$-n$	i	$-o$	$-k$	l

Table 6.54: Possible index distributions for the codewords of $\mathcal{F}_{i,-o}$.

When the codewords of $\mathcal{F}_{i,-o}$ are such that $U_1 \in \mathcal{F}_{i,-o,n,l}$, it is not possible to characterize the remaining codewords of $\mathcal{F}_i$, that is, the codewords of $\mathcal{J} \cup \mathcal{F}_{i,-j}$, without facing a contradiction.

If $U_1 \in \mathcal{F}_{i,-o,n,m}$, for each possibility presented in Table 6.54 we try to find out, firstly, all possible index distributions for the codewords of $\mathcal{F}_{i,-j}$, taking into account that $|\mathcal{F}_{i,-j}| = 3$, after we will identify the codewords of $\mathcal{J}$.

For the codewords $U_1, U_2, U_3 \in \mathcal{F}_{i,-o}$, $U_4, U_5 \in \mathcal{F}_{i,-j}$ and $U_6, U_7, U_8 \in \mathcal{J}$ we get the following index distributions:

U_1	i	$-o$	n	m
U_2	i	$-o$	$-j$	$-k$
U_3	i	$-o$	l	$-n$
U_4	i	$-j$	l	$-m$
U_5	i	$-j$	$-l$	$-n$
U_6	i	l	$-k$	o
U_7	i	m	$-n$	$-k$
U_8	i	m	o	$-l$

U_1	i	$-o$	n	m
U_2	i	$-o$	$-j$	l
U_3	i	$-o$	$-k$	$-n$
U_4	i	$-j$	m	$-n$
U_5	i	$-j$	$-k$	$-m$
U_6	i	l	$-k$	o
U_7	i	l	n	$-m$
U_8	i	m	o	$-l$

U_1	i	$-o$	n	m
U_2	i	$-o$	$-j$	l
U_3	i	$-o$	$-k$	$-n$
U_4	i	$-j$	$-k$	m
U_5	i	$-j$	$-l$	$-n$
U_6	i	m	o	$-l$
U_7	i	l	n	$-m$
U_8	i	l	$-k$	o

U_1	i	$-o$	n	m
U_2	i	$-o$	$-j$	l
U_3	i	$-o$	$-k$	$-n$
U_4	i	$-j$	$-k$	$-m$
U_5	i	$-j$	$-l$	$-n$
U_6	i	l	$-k$	o
U_7	i	l	n	$-m$
U_8	i	m	o	$-l$

Table 6.55: Possible index distributions for the codewords of $\mathcal{F}_i$.

To conclude the characterization of all possible index distribution of the codewords of $\mathcal{F}_i$, we will consider $|\mathcal{J}| = 3$ with $|\mathcal{F}_{i,-o}| = 2$, $|\mathcal{F}_{i,-j}| = 3$ and $|\mathcal{F}_{i,-o,-j}| = 0$.

Let us identify the possible index distributions for the codewords of $\mathcal{F}_{i,-j}$:

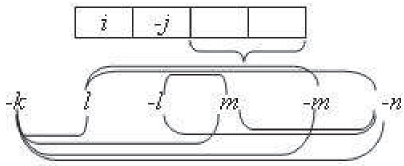

Figure 6.13: Possible index distributions for $U \in \mathcal{F}_{i,-j}$.

Accordingly, taking into account Lemma 1.5, the codewords of $U_1, U_2, U_3 \in \mathcal{F}_{i,-j}$ satisfy one of the following conditions:

U_1	i	$-j$	$-l$	m
U_2	i	$-j$	$-m$	$-k$
U_3	i	$-j$	l	$-n$

U_1	i	$-j$	$-l$	m
U_2	i	$-j$	l	$-m$
U_3	i	$-j$	$-k$	$-n$

U_1	i	$-j$	$-l$	$-n$
U_2	i	$-j$	l	$-m$
U_3	i	$-j$	$-k$	m

Table 6.56: Possible index distribution for the codewords of $\mathcal{F}_{i,-j}$.

However, for each one of the hypotheses presented above it is not possible to get all codewords of $\mathcal{J} \cup \mathcal{F}_{i,-o}$ without contradicting the definition of PL$(7, 2)$ code.

6.1.3 Analysis of the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$

In the previous subsection we have identified possible index distributions for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ when $|\mathcal{G}_i| = 5$, with $|\mathcal{G}_{ij}| = 3$, and $8 \leq |\mathcal{F}_i| \leq 9$. Although we have presented few examples, we have analyzed all cases applying always similar strategies to identify all possible index distributions for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$, having obtained many possibilities. The question is: considering each one of the obtained hypotheses for $\mathcal{G}_i \cup \mathcal{F}_i$, is it possible to describe the remaining codewords necessary to cover all words of $\mathbb{Z}^7$ without contradicting the definition of perfect 2-error correcting Lee code? To answer to this question we have focused our attention on codewords of the other sets $\mathcal{G}_\alpha \cup \mathcal{F}_\alpha$, with $\alpha \in \mathcal{I}\backslash\{i\}$. Since we have identified all codewords of $\mathcal{G}_i \cup \mathcal{F}_i$, it would be natural to verify what happens when we consider codewords of $\mathcal{A}_i \cup \mathcal{B}_i \cup \mathcal{C}_i \cup \mathcal{D}_i \cup \mathcal{E}_i$. However, we have decided to analyze again sets of type $\mathcal{G}_\alpha \cup \mathcal{F}_\alpha$, $\alpha \in \mathcal{I}\backslash\{i\}$, since the codewords of $\mathcal{G}$ and $\mathcal{F}$ are the ones with more nonzero coordinates.

As we have said before, we have identified many possible hypotheses for $\mathcal{G}_i \cup \mathcal{F}_i$. Let us consider, as a representative example, one of these cases.

Consider the codewords $W_1, \ldots, W_5 \in \mathcal{G}_i$ and $U_1, \ldots, U_8 \in \mathcal{F}_i$ satisfying:

W_1	i	j	k	l	m
W_2	i	j	$-k$	$-l$	n
W_3	i	j	$-m$	$-n$	o
W_4	i	$-o$	k	$-l$	$-m$
W_5	i	$-j$	k	n	o

U_1	i	$-o$	m	n
U_2	i	$-o$	$-j$	$-k$
U_3	i	$-o$	l	$-n$
U_4	i	$-j$	l	$-m$
U_5	i	$-j$	$-l$	$-n$
U_6	i	l	$-k$	o
U_7	i	m	$-n$	$-k$
U_8	i	m	o	$-l$

Table 6.57: Index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$.

Our aim is to find out an element $\alpha \in \mathcal{I}\backslash\{i\}$ for which it is not possible to describe all codewords of $\mathcal{G}_\alpha \cup \mathcal{F}_\alpha$.

By the analysis of the index distribution of the codewords of $\mathcal{G}_i$, we verify that for any $\alpha \in \mathcal{I}\backslash\{i,-i\}$ we get: $|\mathcal{G}_{i\alpha}| = 3$, $|\mathcal{G}_{i\alpha}| = 2$ or $|\mathcal{G}_{i\alpha}| = 1$. Since, by Corollary 5.1, $5 \leq |\mathcal{G}_\alpha| \leq 7$, for each element $\alpha \in \mathcal{I}\backslash\{i,-i\}$ it will be necessary to identify, at least, two more codewords of $\mathcal{G}_\alpha$. Let us concentrate our attention on the elements $\alpha \in \mathcal{I}\backslash\{i,-i\}$ satisfying $|\mathcal{G}_{i\alpha}| = 1$, since in this case we must characterize more codewords of $\mathcal{G}_\alpha$. As we have seen before, in the description of codewords it is useful to see $\mathcal{I}$ partitioned in subsets. For that reason in the set of the elements $\alpha \in \mathcal{I}\backslash\{i,-i\}$ satisfying $|\mathcal{G}_{i\alpha}| = 1$, we will give preference to the elements which verify also $|\mathcal{F}_{i\alpha}| = 3$. That is, we are interested in the indices $\alpha \in \mathcal{I}\backslash\{i,-i\}$ such that $|\mathcal{G}_{i\alpha}| = 1$ and $|\mathcal{F}_{i\alpha}| = 3$. In these cases, the codewords of $\mathcal{G}_{i\alpha} \cup \mathcal{F}_{i\alpha}$ induce a partition on $\mathcal{I}$ with few elements.

Observing Table 6.57 we verify that $-j, -k, l, m, -n$ and $-o$ are in these conditions.

Let us consider, for example, $m \in \mathcal{I}$. We will try to characterize all possible index distributions for all codewords of $\mathcal{G}_m \cup \mathcal{F}_m$. Now $5 \leq |\mathcal{G}_m| \leq 7$ and from Propositions 2.11, 2.12 and 2.13 it follows, respectively:

- if $|\mathcal{G}_m| = 5$, then $7 \leq |\mathcal{F}_m| \leq 10$;
- if $|\mathcal{G}_m| = 6$, then $4 \leq |\mathcal{F}_m| \leq 8$;
- if $|\mathcal{G}_m| = 7$, then $2 \leq |\mathcal{F}_m| \leq 5$.

Let us consider $W_1 \in \mathcal{G}_{im}$ and $U_1, U_7, U_8 \in \mathcal{F}_{im}$ (see Table 6.57). These codewords induce the following partition $\mathcal{Q}$ of $\mathcal{I}\backslash\{i,m,-m\}$:

$$\mathcal{Q}_1 = \{j,k,l\},\ \mathcal{Q}_2 = \{-o,n\},\ \mathcal{Q}_3 = \{-n,-k\},\ \mathcal{Q}_4 = \{o,-l\},\ \mathcal{Q}_5 = \{-j\},\ \mathcal{Q}_6 = \{-i\}. \tag{6.3}$$

As seen before, this type of partition help us to characterize the index distribution of the codewords of $\mathcal{G}_m \cup \mathcal{F}_m$. In fact, by Lemma 1.5 and taking into account the codewords of $\mathcal{G}_{im} \cup \mathcal{F}_{im}$, if $V \in (\mathcal{G}_m \cup \mathcal{F}_m)\backslash(\mathcal{G}_i \cup \mathcal{F}_i)$, then $V \notin \mathcal{G}_{mv_1v_2} \cup \mathcal{F}_{mv_1v_2}$ with $v_1, v_2 \in \mathcal{Q}_q$, for $q \in \{1, \dots, 4\}$.

We begin by characterizing all codewords of $\mathcal{G}_m$. As $|\mathcal{G}_{im}| = 1$ and $|\mathcal{G}_m| \geq 5$, we must identify in $\mathcal{G}_m$, at least, four more codewords. That is, $|\mathcal{G}_m\backslash\mathcal{G}_i| \geq 4$. We note that, considering the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$, the element $-i$ is the unique element in $\mathcal{I}$ that, until now, it is not being used in the characterization of the index distribution of any codeword. Thus, when we consider a codeword $V \in \mathcal{G}_{m,-i} \cup \mathcal{F}_{m,-i}$ there exists less probability of getting a contradiction on the definition of being a PL$(7,2)$ code than when we consider a codeword $V \in (\mathcal{G}_m \cup \mathcal{F}_m)\backslash(\mathcal{G}_{-i} \cup \mathcal{F}_{-i})$. So, in the characterization of the codewords of $\mathcal{G}_m\backslash\mathcal{G}_i$ we begin by characterizing the codewords $W \in \mathcal{G}_m\backslash(\mathcal{G}_i \cup \mathcal{G}_{-i})$.

By Lemma 2.2, $|\mathcal{G}_{m\alpha}| \leq 3$ for all $\alpha \in \mathcal{I}\backslash\{m,-m\}$. As $|\mathcal{G}_m\backslash\mathcal{G}_i| \geq 4$, it follows that $|\mathcal{G}_m\backslash(\mathcal{G}_i \cup \mathcal{G}_{-i})| \geq 1$. Let us begin by identifying all possible index distributions for the codewords $W \in \mathcal{G}_m\backslash(\mathcal{G}_i \cup \mathcal{G}_{-i})$. Taking into account the partition $\mathcal{Q}$, all codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ presented in Table 6.57 and Lemma 1.5, we conclude that if $W \in \mathcal{G}_m\backslash(\mathcal{G}_i \cup \mathcal{G}_{-i})$, then W must satisfy one of the conditions presented in Table 6.58.

W	m	j	$-o$	$-l$	$-n$
W	m	$-j$	k	$-n$	$-o$
W	m	$-j$	l	n	$-k$
W	m	$-j$	l	o	$-n$

Table 6.58: Possible index distributions for $W \in \mathcal{G}_m \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})$

Analyzing the above table and considering Lemma 1.5, we conclude that $|\mathcal{G}_m \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})| \leq 2$. Furthermore, if $|\mathcal{G}_m \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})| = 2$ then $W_6, W_7 \in \mathcal{G}_m \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})$ must satisfy one of the following conditions:

W_6	m	j	$-o$	$-l$	$-n$
W_7	m	$-j$	l	n	$-k$

W_6	m	j	$-o$	$-l$	$-n$
W_7	m	$-j$	l	o	$-n$

W_6	m	$-j$	k	$-n$	$-o$
W_7	m	$-j$	l	n	$-k$

Table 6.59: Possible index distributions for $W_6, W_7 \in \mathcal{G}_m \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})$.

We will analyze the following hypotheses:

1) $|\mathcal{G}_m \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})| = 1$;

2) $|\mathcal{G}_m \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})| = 2$.

1) Suppose $|\mathcal{G}_m \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})| = 1$.

Let us consider $W_6 \in \mathcal{G}_m \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})$ satisfying $W_6 \in \mathcal{G}_{m,j,-o,-l,-n}$.

In these conditions we have $|\mathcal{G}_{mi}| = 1$ and $|\mathcal{G}_m \backslash (\mathcal{G}_i \cup \mathcal{G}_{-i})| = 1$. Since $|\mathcal{G}_m| \geq 5$, we get $|\mathcal{G}_{m,-i}| \geq 3$. However, by Lemma 2.2, we conclude that $|\mathcal{G}_{m,-i}| = 3$ and, consequently, $|\mathcal{G}_m| = 5$.

Let us characterize the possible index distributions for $W_7, W_8, W_9 \in \mathcal{G}_{m,-i}$. Considering the partition $\mathcal{Q}$ as well as the codewords already known and Lemma 1.5, we get the following possible index distributions for the codewords of $\mathcal{G}_{m,-i}$, presented in Table 6.60.

W_7					W_8					W_9				
m	-i	-k	l	-o	m	-i	j	n	o	m	-i	-j	k	-l
										m	-i	-j	k	-n
					m	-i	k	n	-l	m	-i	-j	-n	o
					m	-i	k	-n	o	m	-i	-j	-l	n
m	-i	k	-l	n	m	-i	j	-k	o	m	-i	-j	l	-o
										m	-i	-j	l	-n

Table 6.60: Possible index distributions for the codewords of $\mathcal{G}_{m,-i}$.

Next step consists in finding out all possible index distributions for the respective codewords of $\mathcal{F}_m$. We note that, since the codewords of $\mathcal{G}_m$ are such that $|\mathcal{G}_m| = 5$ and $|\mathcal{G}_{m,-i}| = 3$, we could adapt the results derived in Subsection 6.1.1 and apply them in the analysis of the codewords of $\mathcal{F}_m$. However, as at this moment there exists many information about the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i \cup \mathcal{G}_m$, we can quickly analyze the set $\mathcal{F}_m$ using another strategy, as we will see.

Suppose, for instance, that $W_7, W_8, W_9 \in \mathcal{G}_{m,-i}$ are such that: $W_7 \in \mathcal{G}_{m,-i,-k,l,-o}$; $W_8 \in \mathcal{G}_{m,-i,j,n,o}$; $W_9 \in \mathcal{G}_{m,-i,-j,k,-n}$. Let us characterize the codewords of $\mathcal{F}_m$. As $|\mathcal{G}_{m,-i}| = 3$, by Lemma 2.15, $8 \leq |\mathcal{F}_m| \leq 10$. Since $|\mathcal{F}_{mi}| = 3$, we get $|\mathcal{F}_m\backslash\mathcal{F}_i| \geq 5$. In the schemes presented bellow, all possible index distributions for the codewords $U \in \mathcal{F}_m\backslash\mathcal{F}_i$ are given. These index distributions were obtained having in mind Lemma 1.5 and all codewords of $\mathcal{G}_i \cup \mathcal{F}_i \cup \mathcal{G}_m$ already known.

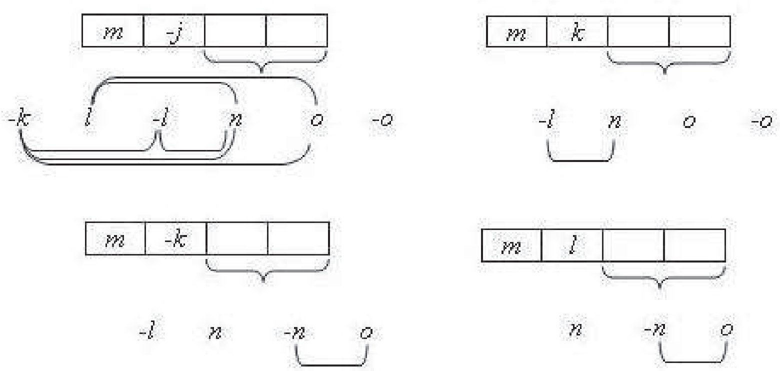

Figure 6.14: Possible index distributions for $U \in \mathcal{F}_m\backslash\mathcal{F}_i$.

Analyzing the schemes in Figure 6.14 we conclude that if $U \in \mathcal{F}_m \backslash \mathcal{F}_i$, then

$$U \in \mathcal{F}_{m,-j,-k} \cup \mathcal{F}_{m,-j,l} \cup \mathcal{F}_{m,-l,n} \cup \mathcal{F}_{m,-n,o}.$$

Taking into account Lemma 1.5, it follows that $|\mathcal{F}_m \backslash \mathcal{F}_i| \leq 4$, which is a contradiction. So, the considered index distribution for the codewords of $\mathcal{G}_m$ contradicts the definition of PL$(7,2)$ code.

In the majority of the cases, such as shown in this illustrative example, it is not possible to describe all codewords of $\mathcal{F}_m$. Considering all the hypotheses presented in Table 6.60, for only one of them it is possible to characterize completely the index distribution of the codewords of $\mathcal{F}_m$. This happens when: $W_7 \in \mathcal{G}_{m,-i,k,-l,n}$; $W_8 \in \mathcal{G}_{m,-i,j,-k,o}$; $W_9 \in \mathcal{G}_{m,-i,-j,l,-n}$. Producing the corresponding schemes we conclude that, in this case, $|\mathcal{F}_m| = 8$ and the codewords $U_9, \ldots, U_{13} \in \mathcal{F}_m \backslash \mathcal{F}_i$ must satisfy one of the following index distributions:

U_9	m	$-j$	k	$-o$
U_{10}	m	k	$-n$	o
U_{11}	m	$-k$	l	$-o$
U_{12}	m	l	n	o
U_{13}	m	$-j$	$-k$	$-l$

U_9	m	$-j$	k	$-o$
U_{10}	m	k	$-n$	o
U_{11}	m	$-k$	l	$-o$
U_{12}	m	l	n	o
U_{13}	m	$-j$	$-k$	n

Table 6.61: Possible index distributions for the codewords of $\mathcal{F}_m \backslash \mathcal{F}_i$.

Since we can characterize completely the codewords of $\mathcal{G}_i \cup \mathcal{F}_i \cup \mathcal{G}_m \cup \mathcal{F}_m$, we must continue our analysis verifying what happens with another element of $\mathcal{I} \backslash \{i, m\}$. For that purpose, it will be helpful to analyze an element $\alpha \in \mathcal{I} \backslash \{i, m\}$ for which $|\mathcal{G}_{i\alpha} \cup \mathcal{G}_{m\alpha}|$ is the lowest possible, implying the identification of more codewords of $\mathcal{G}_\alpha$ and helping in the search of a contradiction. As $-j \in \mathcal{I}$ is one of the elements $\alpha \in \mathcal{I} \backslash \{i, m\}$ for which $|\mathcal{G}_{i\alpha} \cup \mathcal{G}_{m\alpha}|$ is the lowest possible, with $|\mathcal{G}_{i,-j} \cup \mathcal{G}_{m,-j}| = 2$, we will concentrate our attention on this element. We will analyze simultaneously both distributions presented in Table 6.61, considering only the common codewords. We begin by characterizing all possible index distributions for the remaining codewords of $\mathcal{G}_{-j}$. As, by Corollary 5.1 $|\mathcal{G}_{-j}| \geq 5$, we get $|\mathcal{G}_{-j} \backslash (\mathcal{G}_i \cup \mathcal{G}_m)| \geq 3$. To identify the codewords of $\mathcal{G}_{-j} \backslash (\mathcal{G}_i \cup \mathcal{G}_m)$, we will consider the partition $\mathcal{R}$ of $\mathcal{I}\backslash\{i, j, -j, m\}$ induced by the codewords $W_5 \in \mathcal{G}_{i,-j}$ and $U_2, U_4, U_5 \in \mathcal{F}_{i,-j}$:

$$\mathcal{R}_1 = \{k, n, o\},\ \mathcal{R}_2 = \{-o, -k\},\ \mathcal{R}_3 = \{l, -m\},\ \mathcal{R}_4 = \{-l, -n\},\ \mathcal{R}_5 = \{-i\}. \quad (6.4)$$

Combining the elements of the partition $\mathcal{R}$, and taking into account Lemma 1.5 and all codewords of $\mathcal{G}_i \cup \mathcal{F}_i \cup \mathcal{G}_m \cup \mathcal{F}_m$, we conclude that if $W \in \mathcal{G}_{-j}\backslash(\mathcal{G}_i \cup \mathcal{G}_m)$, then

$$W \in \mathcal{G}_{-j,o,-k,-m,-l} \cup \mathcal{G}_{-j,-i,-m,-o,n} \cup \mathcal{G}_{-j,-i,-m,-k,n}.$$

By Lemma 1.5, $|\mathcal{G}_{-j}\backslash(\mathcal{G}_i \cup \mathcal{G}_m)| \leq 2$ and, consequently, $|\mathcal{G}_{-j}| \leq 4$, which is a contradiction.

Letting $W_6 \in \mathcal{G}_m\backslash(\mathcal{G}_i \cup \mathcal{G}_{-i})$ assume any other index distribution described in Table 6.58 and applying the reasoning described in the presented illustrative example, we will always end up in a contradiction.

2) Suppose $|\mathcal{G}_m\backslash(\mathcal{G}_i \cup \mathcal{G}_{-i})| = 2$.

Let $W_6, W_7 \in \mathcal{G}_m\backslash(\mathcal{G}_i \cup \mathcal{G}_{-i})$. Considering the hypotheses presented in Table 6.59, we are going to assume that $W_6 \in \mathcal{G}_{m,j,-o,-l,-n}$ and $W_7 \in \mathcal{G}_{m,-j,l,n,-k}$.

As $|\mathcal{G}_m\backslash\mathcal{G}_{-i}| = 3$, we must impose $|\mathcal{G}_{m,-i}| \geq 2$. To characterize all possible index distributions for the codewords of $\mathcal{G}_{m,-i}$ we take into account the partition $\mathcal{Q}$, see (6.3), and all codewords of $\mathcal{G}_i \cup \mathcal{F}_i \cup \mathcal{G}_m\backslash\mathcal{G}_{-i}$ already known. We then may conclude that $|\mathcal{G}_{m,-i}| = 2$ and $W_8, W_9 \in \mathcal{G}_{m,-i}$ satisfy one of the following conditions:

<table>
<tr><th colspan="5">W8</th><th colspan="5">W9</th></tr>
<tr><td rowspan="3">m</td><td rowspan="3">-i</td><td rowspan="3">j</td><td rowspan="3">n</td><td rowspan="3">o</td><td>m</td><td>-i</td><td>-j</td><td>k</td><td>-l</td></tr>
<tr><td>m</td><td>-i</td><td>-j</td><td>k</td><td>-n</td></tr>
<tr><td>m</td><td>-i</td><td>-j</td><td>k</td><td>-o</td></tr>
<tr><td rowspan="3">m</td><td rowspan="3">-i</td><td rowspan="3">k</td><td rowspan="3">-l</td><td rowspan="3">n</td><td>m</td><td>-i</td><td>j</td><td>-k</td><td>o</td></tr>
<tr><td>m</td><td>-i</td><td>l</td><td>-n</td><td>o</td></tr>
<tr><td>m</td><td>-i</td><td>-j</td><td>-n</td><td>o</td></tr>
<tr><td rowspan="3">m</td><td rowspan="3">-i</td><td rowspan="3">j</td><td rowspan="3">-k</td><td rowspan="3">o</td><td>m</td><td>-i</td><td>-j</td><td>k</td><td>-l</td></tr>
<tr><td>m</td><td>-i</td><td>-j</td><td>k</td><td>-n</td></tr>
<tr><td>m</td><td>-i</td><td>-j</td><td>k</td><td>-o</td></tr>
<tr><td rowspan="2">m</td><td rowspan="2">-i</td><td rowspan="2">l</td><td rowspan="2">o</td><td rowspan="2">-n</td><td>m</td><td>-i</td><td>-j</td><td>k</td><td>-l</td></tr>
<tr><td>m</td><td>-i</td><td>-j</td><td>k</td><td>-o</td></tr>
</table>

Table 6.62: Possible index distributions for the codewords of $\mathcal{G}_{m,-i}$.

Let us consider $W_8, W_9 \in \mathcal{G}_{m,-i}$ satisfying $W_8 \in \mathcal{G}_{m,-i,k,-l,n}$ and $W_9 \in \mathcal{G}_{m,-i,-j,-n,o}$. We note that, for all $\alpha \in \mathcal{I}\backslash\{m,-m\}$, $|\mathcal{G}_{m\alpha}| \leq 2$. Since $|\mathcal{G}_m| = 5$, by Proposition 2.11, $7 \leq |\mathcal{F}_m| \leq 10$. As $|\mathcal{F}_{mi}| = 3$, then $|\mathcal{F}_m\backslash\mathcal{F}_i| \geq 4$. Next schemes characterize all possible index distributions for the codewords $U \in \mathcal{F}_m\backslash\mathcal{F}_i$:

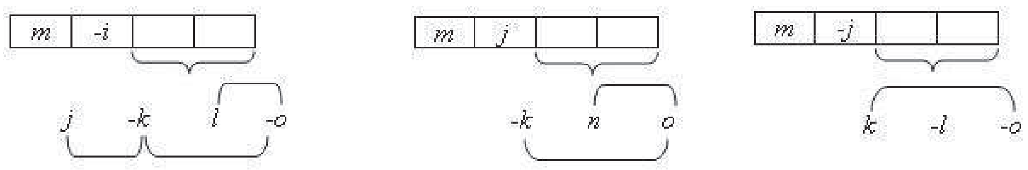

Figure 6.15: Possible index distributions for $U \in \mathcal{F}_m\backslash\mathcal{F}_i$.

Analyzing these schemes, we verify that $|\mathcal{F}_m\backslash\mathcal{F}_i| = 4$ with $U_9, \ldots, U_{12} \in \mathcal{F}_m\backslash\mathcal{F}_i$ satisfying one of the following index distributions:

U_9	m	$-i$	j	$-k$
U_{10}	m	$-j$	k	$-o$
U_{11}	m	j	n	o
U_{12}	m	$-i$	l	$-o$

U_9	m	$-i$	j	$-k$
U_{10}	m	$-j$	k	$-o$
U_{11}	m	j	n	o
U_{12}	m	$-i$	$-k$	$-o$

Table 6.63: Possible index distributions for the codewords of $\mathcal{F}_m\backslash\mathcal{F}_i$

In both cases we have $|\mathcal{F}_m| = 7$. Accordingly, by Lemma 2.11 we get $|\mathcal{F}_m^{(2)}| = 4$. Taking into account Lemma 2.14, the codewords $V_1, \ldots, V_4 \in \mathcal{F}_m^{(2)}$ satisfy $V_1 \in \mathcal{F}_{mv_1v_2v_3}, \ldots, V_4 \in \mathcal{F}_{mv_{10}v_{11}v_{12}}$, with $v_1, \ldots, v_{12} \in \mathcal{I}\backslash\{m,-m\}$ pairwise distinct. However, in both cases, analyzing all codewords of $\mathcal{F}_m$, we conclude that if $U \in \mathcal{F}_m$, then

$$U \in \mathcal{F}_{mi} \cup \mathcal{F}_{mj} \cup \mathcal{F}_{m,-o},$$

contradicting Lemma 2.14.

Assuming that $W_8, W_9 \in \mathcal{G}_{m,-i}$ satisfy any other hypothesis presented in Table 6.62, we would also get a contradiction.

We have applied this same strategy to verify that all index distributions obtained for $\mathcal{G}_i \cup \mathcal{F}_i$ lead us to a contradiction. We have just prove the following theorem:

Theorem 6.1 *If $|\mathcal{G}_\alpha| = 5$, for $\alpha \in \mathcal{I}$, then $|\mathcal{G}_{\alpha\beta}| \leq 2$ for any $\beta \in \mathcal{I}\backslash\{\alpha,-\alpha\}$.*

6.2 $|\mathcal{G}_{i\alpha}| \leq 2$ for any $\alpha \in \mathcal{I}\backslash\{i,-i\}$

We have just proved that if $|\mathcal{G}_i| = 5$ then $|\mathcal{G}_{i\alpha}| \leq 2$ for any $\alpha \in \mathcal{I}\backslash\{i,-i\}$. Here we analyze the hypothesis $|\mathcal{G}_i| = 5$ and $|\mathcal{G}_{i\alpha}| \leq 2$ for any $\alpha \in \mathcal{I}\backslash\{i,-i\}$. Our aim is to show that under such assumption the definition of PL$(7,2)$ code will be contradicted.

Initially, we present some results which impose conditions on the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$. These results allow us to get all possible index distributions for $\mathcal{G}_i \cup \mathcal{F}_i$ which, apparently, do not contradict the definition of perfect error correcting Lee code. Finally, we show that any obtained index distribution for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ implies contradictions on the definition of PL$(7,2)$ code when considered one other set $\mathcal{G}_\alpha \cup \mathcal{F}_\alpha$ with $\alpha \in \mathcal{I}\backslash\{i\}$.

6.2.1 Necessary conditions for the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$

Let us consider $\mathcal{O} \subset \mathcal{I}\backslash\{i,-i\}$ such that

$$\mathcal{O} = \{\alpha \in \mathcal{I}\backslash\{i,-i\} : |\mathcal{G}_{i\alpha}| = 2\}.$$

The following result restricts the variation of $|\mathcal{O}|$.

Proposition 6.7 *The cardinality of $\mathcal{O}$ satisfies $8 \leq |\mathcal{O}| \leq 10$.*

Proof. Since

$$|\mathcal{G}_i| = \frac{1}{4} \sum_{\alpha \in \mathcal{I}\backslash\{i,-i\}} |\mathcal{G}_{i\alpha}|$$

and we are considering $|\mathcal{G}_i| = 5$, it follows that

$$\sum_{\alpha \in \mathcal{I}\backslash\{i,-i\}} |\mathcal{G}_{i\alpha}| = 20. \tag{6.5}$$

We are assuming $|\mathcal{G}_{i\alpha}| \leq 2$ for any $\alpha \in \mathcal{I}\backslash\{i,-i\}$. On the other hand, $|\mathcal{I}\backslash\{i,-i\}| = 12$. Then, from (6.5), we conclude that $8 \leq |\mathcal{O}| \leq 10$. □

Note that, by Lemma 2.2, if $\alpha \in \mathcal{O}$, that is, if $|\mathcal{G}_{i\alpha}| = 2$, then $|\mathcal{F}_{i\alpha}| \leq 2$. The following proposition guarantees the existence of an element $\alpha \in \mathcal{O}$ so that $|\mathcal{F}_{i\alpha}| = 2$.

Proposition 6.8 *There exists $\alpha \in \mathcal{O}$ such that $|\mathcal{F}_{i\alpha}| = 2$.*

Proof. By Lemma 2.11, if $|\mathcal{G}_i| = 5$, then $7 \leq |\mathcal{F}_i| \leq 10$. As

$$|\mathcal{F}_i| = \frac{1}{3} \sum_{\alpha \in \mathcal{I} \backslash \{i,-i\}} |\mathcal{F}_{i\alpha}|,$$

then

$$\sum_{\alpha \in \mathcal{I} \backslash \{i,-i\}} |\mathcal{F}_{i\alpha}| \geq 21. \tag{6.6}$$

On the other hand, since

$$|\mathcal{G}_i| = \frac{1}{4} \sum_{\alpha \in \mathcal{I} \backslash \{i,-i\}} |\mathcal{G}_{i\alpha}|$$

and $|\mathcal{G}_i| = 5$, then

$$\sum_{\alpha \in \mathcal{I} \backslash \{i,-i\}} |\mathcal{G}_{i\alpha}| = 20. \tag{6.7}$$

Recall that from Lemma 2.2 we know that

$$|\mathcal{D}_{i\alpha} \cup \mathcal{E}_{i\alpha}| + 2|\mathcal{F}_{i\alpha}| + 3|\mathcal{G}_{i\alpha}| = 10, \ \forall\, \alpha \in \mathcal{I} \backslash \{i, -i\}. \tag{6.8}$$

By Proposition 6.7 it follows that $8 \leq |\mathcal{O}| \leq 10$. We will analyze separately what happens when $|\mathcal{O}|$ assumes each one of these possible values.

If $|\mathcal{O}| = 8$, by (6.7) and taking into account that $|\mathcal{I} \backslash \{i, -i\}| = 12$, we must impose $|\mathcal{G}_{i\alpha}| = 1$ for any $\alpha \in \mathcal{I} \backslash (\{i, -i\} \cup \mathcal{O})$. Consequently, by (6.8), $|\mathcal{F}_{i\alpha}| \leq 3$ for any $\alpha \in \mathcal{I} \backslash (\{i, -i\} \cup \mathcal{O})$. As $|\mathcal{I} \backslash (\{i, -i\} \cup \mathcal{O})| = 4$, we conclude that

$$\sum_{\alpha \in \mathcal{I} \backslash (\{i,-i\} \cup \mathcal{O})} |\mathcal{F}_{i\alpha}| \leq 12.$$

Accordingly, from (6.6) we get

$$\sum_{\alpha \in \mathcal{O}} |\mathcal{F}_{i\alpha}| \geq 9. \tag{6.9}$$

Since $|\mathcal{G}_{i\alpha}| = 2$ for any $\alpha \in \mathcal{O}$, considering (6.8) it follows that $|\mathcal{F}_{i\alpha}| \leq 2$ for any $\alpha \in \mathcal{O}$. As $|\mathcal{O}| = 8$, by (6.9) there exists, at least, one element $\alpha \in \mathcal{O}$ such that $|\mathcal{F}_{i\alpha}| = 2$.

Now suppose that $|\mathcal{O}| = 9$. Then, taking into account (6.7), we must impose the existence of $\beta, \gamma, \delta \in \mathcal{I} \backslash (\{i, -i\} \cup \mathcal{O})$ satisfying $|\mathcal{G}_{i\beta}| = |\mathcal{G}_{i\gamma}| = 1$ and $|\mathcal{G}_{i\delta}| = 0$. Thus, considering (6.8) we get

$$\sum_{\alpha \in \mathcal{I} \backslash (\{i,-i\} \cup \mathcal{O})} |\mathcal{F}_{i\alpha}| \leq 2 \times 3 + 5 = 11.$$

Consequently, by (6.6) we get

$$\sum_{\alpha\in\mathcal{O}} |\mathcal{F}_{i\alpha}| \geq 10.$$

As we are assuming $|\mathcal{O}| = 9$, there exists $\alpha \in \mathcal{O}$ satisfying $|\mathcal{F}_{i\alpha}| = 2$.

Finally, assume $|\mathcal{O}| = 10$. Considering (6.7), the elements $\beta, \gamma \in \mathcal{I}\backslash(\{i,-i\} \cup \mathcal{O})$ are such that $|\mathcal{G}_{i\beta}| = |\mathcal{G}_{i\gamma}| = 0$. In these conditions

$$\sum_{\alpha\in\mathcal{I}\backslash(\{i,-i\}\cup\mathcal{O})} |\mathcal{F}_{i\alpha}| \leq 2 \times 5 = 10.$$

Taking into account (6.6) it follows that

$$\sum_{\alpha\in\mathcal{O}} |\mathcal{F}_{i\alpha}| \geq 11.$$

Thus, there exists $\alpha \in \mathcal{O}$ satisfying $|\mathcal{F}_{i\alpha}| = 2$. □

Considering the previous proposition, let $j \in \mathcal{I}\backslash\{i,-i\}$ be such that $j \in \mathcal{O}$ and $|\mathcal{F}_{ij}| = 2$. That is, $|\mathcal{G}_{ij}| = |\mathcal{F}_{ij}| = 2$. Let $W_1, W_2 \in \mathcal{G}_{ij}$ and $U_1, U_2 \in \mathcal{F}_{ij}$ so that:

W_1	i	j	w_1	w_2	w_3
W_2	i	j	w_4	w_5	w_6

U_1	i	j	u_1	u_2
U_2	i	j	u_3	u_4

Table 6.64: Partial index distribution for the codewords of $\mathcal{G}_{ij} \cup \mathcal{F}_{ij}$.

with $w_1, \ldots, w_6, u_1, \ldots, u_4 \in \mathcal{I}\backslash\{i,-i,j,-j\}$. We note that, taking into account Lemma 1.5, $w_1, \ldots, w_6, u_1, \ldots, u_4$ must be pairwise distinct.

Since $|\mathcal{I}\backslash\{i,-i,j,-j\}| = 10$, it follows that

$$\mathcal{I}\backslash\{i,-i,j,-j\} = \{w_1, \ldots, w_6, u_1, \ldots, u_4\}.$$

The codewords $W_1, W_2 \in \mathcal{G}_{ij}$ and $U_1, U_2 \in \mathcal{F}_{ij}$ induce a partition $\mathcal{S}$ of $\mathcal{I}\backslash\{i,-i,j\}$:

$$\mathcal{S}_1 = \{w_1, w_2, w_3\};\ \mathcal{S}_2 = \{w_4, w_5, w_6\};\ \mathcal{S}_3 = \{u_1, u_2\};\ \mathcal{S}_4 = \{u_3, u_4\};\ \mathcal{S}_5 = \{-j\}. \tag{6.10}$$

As $|\mathcal{G}_{ij}| = |\mathcal{F}_{ij}| = 2$, then, by Lemma 2.2, the codewords $W_1, W_2, U_1, U_2 \in \mathcal{G}_{ij} \cup \mathcal{F}_{ij}$, described in Table 6.64, are the unique codewords in $\mathcal{G}_{ij} \cup \mathcal{F}_{ij}$.

As we have said before, what we have in view is the characterization of all possible index distributions of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$. For that, taking into account the partial index distribution of the codewords of $\mathcal{G}_{ij} \cup \mathcal{F}_{ij}$, presented in Table 6.64, we will state in the following proposition conditions which must be satisfied by the codewords of $\mathcal{G}_i$.

Proposition 6.9 *The cardinality of* $(\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}$ *satisfies*

$$1 \leq |(\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}| \leq 3.$$

Furthermore, $W_3, W_4, W_5 \in \mathcal{G}_i \backslash \mathcal{G}_j$ *are such that:*

i) if $|(\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}| = 1$, *with* $\{\alpha\} = (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}$, *then*

W_3	i	α	x_1	x_2	x_3
W_4	i	α	x_4	x_5	x_6
W_5	i	x_7	x_8	x_9	x_{10}

with $x_1, \ldots, x_{10} \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \backslash \{\alpha\}$ *pairwise distinct and* $\{x_1, \ldots, x_{10}\} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \backslash \{\alpha\}$;

ii) if $|(\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}| = 2$, *with* $\{\alpha, \beta\} = (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}$, *then*

W_3	i	α	β	x_1	x_2
W_4	i	α	x_3	x_4	x_5
W_5	i	β	x_6	x_7	x_8

with $x_1, \ldots, x_8 \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \backslash \{\alpha, \beta\}$ *and pairwise distinct;*

iii) if $|(\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}| = 3$, *with* $\{\alpha, \beta, \gamma\} = (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}$, *then*

W_3	i	α	β	x_1	x_2
W_4	i	α	γ	x_3	x_4
W_5	i	β	γ	x_5	x_6

with $x_1, \ldots, x_6 \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \backslash \{\alpha, \beta, \gamma\}$ *and pairwise distinct.*

Proof. By Proposition 6.7 we know that $|\mathcal{O}| \geq 8$. By assumption, $j \in \mathcal{O}$ and considering the partition $\mathcal{S}$ of $\mathcal{I}\backslash\{i,-i,j\}$, see (6.10), we conclude that

$$|(\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}| \geq 7$$

and, consequently, $|(\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}| \geq 1$.

If $\alpha \in (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}$, by the characterization of $W_1, W_2 \in \mathcal{G}_{ij}$ we conclude that $W, W' \in \mathcal{G}_{i\alpha}$ are such that $\{W, W'\} \cap \{W_1, W_2\} = \varnothing$ since $W_1 \in \mathcal{G}_{ijw_1w_2w_3}$ and $W_2 \in \mathcal{G}_{ijw_4w_5w_6}$ with $\{w_1, \ldots, w_6\} \cap (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) = \varnothing$.

Suppose, by contradiction, that $|(\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}| \geq 4$. Thus, let us assume $\{\alpha, \beta, \gamma, \delta\} = (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}$. Let $\{W_3, W_4, W_5\} = \mathcal{G}_i\backslash\mathcal{G}_j$. Taking into account what was said before, $\mathcal{G}_{i\alpha} \cup \mathcal{G}_{i\beta} \cup \mathcal{G}_{i\gamma} \cup \mathcal{G}_{i\delta} \subset \{W_3, W_4, W_5\}$. As, by definition of $\mathcal{O}$, $|\mathcal{G}_{i\alpha}| = |\mathcal{G}_{i\beta}| = |\mathcal{G}_{i\gamma}| = |\mathcal{G}_{i\delta}| = 2$, Lemma 1.5 is contradicted.

Suppose that $|(\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}| = 1$ with $\{\alpha\} = (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}$. Let $W_3, W_4, W_5 \in \mathcal{G}_i\backslash\mathcal{G}_j$. The partial index distribution of these codewords satisfies:

W_3	i	α	x_1	x_2	x_3
W_4	i	α	x_4	x_5	x_6
W_5	i	x_7	x_8	x_9	x_{10}

Table 6.65: Partial index distribution of the codewords of $\mathcal{G}_i\backslash\mathcal{G}_j$.

with $x_1, \ldots, x_{10} \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{\alpha\}$. We recall that, $|\mathcal{G}_{ix}| \leq 2$ for any $x \in \mathcal{I}\backslash\{i,-i\}$. Since $|\mathcal{G}_{ijx}| = 1$ for any $x \in \mathcal{S}_1 \cup \mathcal{S}_2$, we get $|\mathcal{G}_{ix}\backslash\mathcal{G}_j| \leq 1$ for any $x \in \mathcal{S}_1 \cup \mathcal{S}_2$. On the other hand, as we are supposing $\{\alpha\} = (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}$, then $|\mathcal{G}_{ix}| \leq 1$ for any $x \in (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{\alpha\}$. Consequently, $x_1, \ldots, x_{10}$ are pairwise distinct. As

$$|\mathcal{S}_1 \cup \mathcal{S}_2 \cup (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{\alpha\}| = 10,$$

it follows that $\{x_1, \ldots, x_{10}\} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{\alpha\}$.

Now consider $|(\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}| = 2$ with $\{\alpha, \beta\} = (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}$. Since $|(\mathcal{G}_{i\alpha} \cup \mathcal{G}_{i\beta}) \cap \mathcal{G}_{ij}| = 0$, having in view Lemma 1.5, the partial index distribution of the codewords W_3, W_4 and W_5 in $\mathcal{G}_i \backslash \mathcal{G}_j$ must satisfy the conditions presented in the following table:

W_3	i	α	β	x_1	x_2
W_4	i	α	x_3	x_4	x_5
W_5	i	β	x_6	x_7	x_8

Table 6.66: Partial index distribution of the codewords of $\mathcal{G}_i \backslash \mathcal{G}_j$.

with $x_1, \ldots, x_8 \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \backslash \{\alpha, \beta\}$. Following a similar reasoning to the one applied in the previous case, we conclude that $|\mathcal{G}_{ix} \backslash \mathcal{G}_j| \leq 1$ for any $x \in \mathcal{S}_1 \cup \mathcal{S}_2$ and $|\mathcal{G}_{ix}| \leq 1$ for any $x \in (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \backslash \{\alpha, \beta\}$. Therefore, $x_1, \ldots, x_8$ are pairwise distinct.

Now assume that $|(\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}| = 3$. Let $\{\alpha, \beta, \gamma\} = (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}$. Since $|(\mathcal{G}_{i\alpha} \cup \mathcal{G}_{i\beta} \cup \mathcal{G}_{i\gamma}) \cap \mathcal{G}_{ij}| = 0$, then, having in mind Lemma 1.5, the partial index distribution of the codewords $W_3, W_4, W_5 \in \mathcal{G}_i \backslash \mathcal{G}_j$ satisfy:

W_3	i	α	β	x_1	x_2
W_4	i	α	γ	x_3	x_4
W_5	i	β	γ	x_5	x_6

Table 6.67: Partial index distribution of the codewords of $\mathcal{G}_i \backslash \mathcal{G}_j$.

with $x_1, \ldots, x_6 \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \backslash \{\alpha, \beta, \gamma\}$. By the same reasons referred in the previous cases, $x_1, \ldots, x_6$ are pairwise distinct. □

We have in view the characterization of all possible index distributions of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$. Until now, taking into account the previous proposition, we have identified conditions which must be necessary satisfied by all codewords of $\mathcal{G}_i$. Next, we will also concentrate our attention in the codewords of $\mathcal{F}_i$.

Since $|\mathcal{G}_i| = 5$, then by Lemma 2.11 we get $7 \leq |\mathcal{F}_i| \leq 10$. If $|\mathcal{F}_i| = 7$, then, by the same lemma, $|\mathcal{F}_i^{(2)}| = 4$. Taking into account Lemma 2.14, the condition

$|\mathcal{F}_i^{(2)}| = 4$ restricts significantly the number of hypotheses for the index distributions of the codewords of $\mathcal{F}_i$. If $8 \leq |\mathcal{F}_i| \leq 10$, it will be helpful to find out conditions which restrict the number of possible index distributions for the codewords of $\mathcal{F}_i$. Next, we present a result which establishes a relation between the index distribution of the codewords of $\mathcal{G}_i$ and $\mathcal{F}_i$ when $8 \leq |\mathcal{F}_i| \leq 10$.

Proposition 6.10 *If* $8 \leq |\mathcal{F}_i| \leq 10$, *then there exist, at least, three elements* $\alpha \in \mathcal{O}\backslash\{j\}$ *satisfying* $|\mathcal{G}_{i\alpha}| = |\mathcal{F}_{i\alpha}| = 2$.

Proof. Suppose that $8 \leq |\mathcal{F}_i| \leq 10$. Since

$$|\mathcal{F}_i| = \frac{1}{3} \sum_{\alpha \in \mathcal{I}\backslash\{i,-i\}} |\mathcal{F}_{i\alpha}|,$$

we get

$$24 \leq \sum_{\alpha \in \mathcal{I}\backslash\{i,-i\}} |\mathcal{F}_{i\alpha}| \leq 30. \tag{6.11}$$

From Lemma 2.2 we know that:

$$|\mathcal{D}_{i\alpha} \cup \mathcal{E}_{i\alpha}| + 2|\mathcal{F}_{i\alpha}| + 3|\mathcal{G}_{i\alpha}| = 10, \ \forall\, \alpha \in \mathcal{I}\backslash\{i,-i\}. \tag{6.12}$$

By Proposition 6.9 we get $1 \leq |(\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}| \leq 3$.

Let us verify what happens when $|(\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}|$ assumes each one of the possible values.

Suppose that $|(\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}| = 1$, with $\{\beta\} = (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}$. Taking into account Proposition 6.9, the codewords $W_3, W_4, W_5 \in \mathcal{G}_i\backslash\mathcal{G}_j$ satisfy:

W_3	i	β	x_1	x_2	x_3
W_4	i	β	x_4	x_5	x_6
W_5	i	x_7	x_8	x_9	x_{10}

Table 6.68: Partial index distribution of the codewords of $\mathcal{G}_i\backslash\mathcal{G}_j$.

with $x_1, \ldots, x_{10} \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{\beta\}$ pairwise distinct and $\{x_1, \ldots, x_{10}\} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{\beta\}$. Thus, considering the codewords of $\mathcal{G}_{ij} \cup \mathcal{F}_{ij}$

as well as the codewords of $\mathcal{G}_i \backslash \mathcal{G}_j$, we conclude that $|\mathcal{G}_{i\alpha}| = 2$ for any $\alpha \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \{\beta\}$ and, on the other hand, $|\mathcal{G}_{i\alpha}| = 1$ for any $\alpha \in (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{\beta\}$. Taking into account (6.12), we verify that $|\mathcal{F}_{i\alpha}| \leq 3$ for each $\alpha \in (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{\beta\}$. Consequently, as $|(\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{\beta\}| = 4$, we get

$$\sum_{\alpha \in (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{\beta\}} |\mathcal{F}_{i\alpha}| \leq 12.$$

Having in mind that $|\mathcal{F}_{ij}| = 2$, by (6.11) we must impose

$$\sum_{\alpha \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \{\beta\}} |\mathcal{F}_{i\alpha}| \geq 10. \tag{6.13}$$

Since $|\mathcal{G}_{i\alpha}| = 2$ for any $\alpha \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \{\beta\}$, by (6.12) we get $|\mathcal{F}_{i\alpha}| \leq 2$ for any $\alpha \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \{\beta\}$. As $|\mathcal{S}_1 \cup \mathcal{S}_2 \cup \{\beta\}| = 7$, by (6.13) we conclude that there are, at least, three elements $\alpha \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \{\beta\}$ satisfying $|\mathcal{F}_{i\alpha}| = 2$.

Let us now suppose that $|(\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}| = 2$, with $\{\beta, \gamma\} = (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}$. By Proposition 6.9, the codewords $W_3, W_4, W_5 \in \mathcal{G}_i \backslash \mathcal{G}_j$ must verify:

W_3	i	β	γ	x_1	x_2
W_4	i	β	x_3	x_4	x_5
W_5	i	γ	x_6	x_7	x_8

Table 6.69: Partial index distribution of the codewords of $\mathcal{G}_i \backslash \mathcal{G}_j$.

with $x_1, \ldots, x_8 \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{\beta, \gamma\}$ and pairwise distinct. Since $|\mathcal{S}_1 \cup \mathcal{S}_2 \cup (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{\beta, \gamma\}| = 9$, let us consider

$$\{r\} = [\mathcal{S}_1 \cup \mathcal{S}_2 \cup (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{\beta, \gamma\}]\backslash\{x_1, \ldots, x_8\}.$$

If $r \in \mathcal{S}_1 \cup \mathcal{S}_2$, then $|\mathcal{G}_{i\alpha}| = 2$ for any $\alpha \in (\mathcal{S}_1 \cup \mathcal{S}_2)\backslash\{r\} \cup \{\beta, \gamma\}$. On the other hand, $|\mathcal{G}_{i\alpha}| = 1$ for any $\alpha \in \{r\} \cup (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{\beta, \gamma\}$. In these conditions, considering (6.12),

$$\sum_{\alpha \in \{r\} \cup (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{\beta, \gamma\}} |\mathcal{F}_{i\alpha}| \leq 12.$$

Since $|\mathcal{F}_{ij}| = 2$, taking into account (6.11) we must impose

$$\sum_{\alpha \in (\mathcal{S}_1 \cup \mathcal{S}_2)\backslash\{r\}\cup\{\beta,\gamma\}} |\mathcal{F}_{i\alpha}| \geq 10.$$

As, by (6.12), $|\mathcal{F}_{i\alpha}| \leq 2$ for all $\alpha \in (\mathcal{S}_1 \cup \mathcal{S}_2)\backslash\{r\}\cup\{\beta, \gamma\}$, and $|(\mathcal{S}_1 \cup \mathcal{S}_2)\backslash\{r\}\cup\{\beta, \gamma\}| = 7$, we conclude that there are, at least, three elements $\alpha \in (\mathcal{S}_1 \cup \mathcal{S}_2)\backslash\{r\}\cup\{\beta, \gamma\}$ satisfying $|\mathcal{F}_{i\alpha}| = 2$.

Now suppose that $r \in (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{\beta, \gamma\}$. In these conditions, $|\mathcal{G}_{i\alpha}| = 2$ for any $\alpha \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \{\beta, \gamma\}$. On the other hand, $|\mathcal{G}_{i\alpha}| = 1$ for $\alpha \in (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{\beta, \gamma, r\}$. We note that, $|\mathcal{G}_{ir}| = 0$. Taking into account (6.12),

$$\sum_{\alpha \in (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{\beta,\gamma\}} |\mathcal{F}_{i\alpha}| \leq 5 + 2 \times 3 = 11.$$

Since $|\mathcal{F}_{ij}| = 2$, by (6.11) it follows that

$$\sum_{\alpha \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \{\beta,\gamma\}} |\mathcal{F}_{i\alpha}| \geq 11. \tag{6.14}$$

By (6.12), $|\mathcal{F}_{i\alpha}| \leq 2$ for any $\alpha \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \{\beta, \gamma\}$ furthermore $|\mathcal{S}_1 \cup \mathcal{S}_2 \cup \{\beta, \gamma\}|$=8, Thus, by (6.14), we must impose the existence of, at least, three elements $\alpha \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \{\beta, \gamma\}$ satisfying $|\mathcal{F}_{i\alpha}| = 2$.

Finally, consider $|(\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}| = 3$, with $\{\beta, \gamma, \delta\} = (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}$. By Proposition 6.9, we know that $W_3, W_4, W_5 \in \mathcal{G}_i\backslash\mathcal{G}_j$ must satisfy:

W_3	i	β	γ	x_1	x_2
W_4	i	β	δ	x_3	x_4
W_5	i	γ	δ	x_5	x_6

Table 6.70: Partial index distribution of the codewords of $\mathcal{G}_i\backslash\mathcal{G}_j$.

with $x_1, \ldots, x_6 \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{\beta, \gamma, \delta\}$ and pairwise distinct. As $|\mathcal{S}_1 \cup \mathcal{S}_2 \cup (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{\beta, \gamma, \delta\}| = 8$, consider

$$\{r, s\} = [\mathcal{S}_1 \cup \mathcal{S}_2 \cup (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{\beta, \gamma, \delta\}]\backslash\{x_1, \ldots, x_6\}.$$

We will analyze the following hypotheses separately:

◇ $r, s \in \mathcal{S}_1 \cup \mathcal{S}_2$;

◇ $r \in \mathcal{S}_1 \cup \mathcal{S}_2$ and $s \in (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{\beta, \gamma, \delta\}$;

◇ $r, s \in (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{\beta, \gamma, \delta\}$.

Assume that $r, s \in \mathcal{S}_1 \cup \mathcal{S}_2$. Thus, $|\mathcal{G}_{i\alpha}| = 2$ for any $\alpha \in (\mathcal{S}_1 \cup \mathcal{S}_2)\backslash\{r, s\} \cup \{\beta, \gamma, \delta\}$ and, on the other hand, $|\mathcal{G}_{i\alpha}| = 1$ for any $\alpha \in \{r, s\} \cup (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{\beta, \gamma, \delta\}$. Taking into account (6.12), we conclude that

$$\sum_{\alpha \in \{r,s\} \cup (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{\beta,\gamma,\delta\}} |\mathcal{F}_{i\alpha}| \leq 12.$$

Consequently, by (6.11) and having in view that $|\mathcal{F}_{ij}| = 2$,

$$\sum_{\alpha \in (\mathcal{S}_1 \cup \mathcal{S}_2)\backslash\{r,s\} \cup \{\beta,\gamma,\delta\}} |\mathcal{F}_{i\alpha}| \geq 10.$$

This implies the existence of, at least, three elements $\alpha \in (\mathcal{S}_1 \cup \mathcal{S}_2)\backslash\{r, s\} \cup \{\beta, \gamma, \delta\}$ satisfying $|\mathcal{F}_{i\alpha}| = 2$.

Let us suppose that $r \in \mathcal{S}_1 \cup \mathcal{S}_2$ and $s \in (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{\beta, \gamma, \delta\}$. In these conditions, $|\mathcal{G}_{i\alpha}| = 2$ for any $\alpha \in (\mathcal{S}_1 \cup \mathcal{S}_2)\backslash\{r\} \cup \{\beta, \gamma, \delta\}$. On the other hand, $|\mathcal{G}_{i\alpha}| = 1$ for any $\alpha \in \{r\} \cup (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{s, \beta, \gamma, \delta\}$. We note that $|\mathcal{G}_{is}| = 0$. Considering (6.12) we get

$$\sum_{\alpha \in \{r\} \cup (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{\beta,\gamma,\delta\}} |\mathcal{F}_{i\alpha}| \leq 5 + 2 \times 3 = 11.$$

Consequently, by (6.11),

$$\sum_{\alpha \in (\mathcal{S}_1 \cup \mathcal{S}_2)\backslash\{r\} \cup \{\beta,\gamma,\delta\}} |\mathcal{F}_{i\alpha}| \geq 11.$$

Thus, there are, at least, three elements $\alpha \in (\mathcal{S}_1 \cup \mathcal{S}_2)\backslash\{r\} \cup \{\beta, \gamma, \delta\}$ so that $|\mathcal{F}_{i\alpha}| = 2$.

If $r, s \in (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{\beta, \gamma, \delta\}$, then $|\mathcal{G}_{i\alpha}| = 2$ for any $\alpha \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \{\beta, \gamma, \delta\}$. Furthermore, $|\mathcal{G}_{ir}| = |\mathcal{G}_{is}| = 0$. Accordingly, taking into account (6.12),

$$\sum_{\alpha \in \{r,s\}} |\mathcal{F}_{i\alpha}| \leq 2 \times 5 = 10.$$

Consequently, from (6.11) it follows that

$$\sum_{\alpha \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \{\beta,\gamma,\delta\}} |\mathcal{F}_{i\alpha}| \geq 12.$$

Therefore, there are, at least, three elements $\alpha \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \{\beta, \gamma, \delta\}$ satisfying $|\mathcal{F}_{i\alpha}| = 2$.
$\square$

6.2.2 Index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$

In this subsection we describe how we get the possible index distributions for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$, taking into account that, by assumption, $|\mathcal{G}_i| = 5$ and, consequently, by Lemma 2.11, $7 \leq |\mathcal{F}_i| \leq 10$.

We begin by considering $W_1, W_2 \in \mathcal{G}_{ij}$ and $U_1, U_2 \in \mathcal{F}_{ij}$ and by characterizing the different possible index distributions for these codewords. Later, taking into account Proposition 6.9 and considering one of the possible index distributions of the codewords of $\mathcal{G}_{ij} \cup \mathcal{F}_{ij}$, we exemplify how we characterize the remaining codewords of $\mathcal{G}_i$. In the last part of this subsection, throughout illustrative examples we show how we have analyzed the index distributions of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ when is known the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_{ij}$. We note that, due to the large number of possible index distributions for the codewords of $\mathcal{G}_i \cup \mathcal{F}_{ij}$, we present only some representative examples in which we describe the methodology that we have applied in all cases.

Taking into account what was proven in the previous subsection, let us consider $W_1, W_2 \in \mathcal{G}_{ij}$ and $U_1, U_2 \in \mathcal{F}_{ij}$ so that:

W_1	i	j	w_1	w_2	w_3
W_2	i	j	w_4	w_5	w_6

U_1	i	j	u_1	u_2
U_2	i	j	u_3	u_4

Table 6.71: Partial index distribution of the codewords of $\mathcal{G}_{ij} \cup \mathcal{F}_{ij}$.

with $w_1, \ldots, w_6, u_1, \ldots, u_4 \in \mathcal{I}\backslash\{i, -i, j, -j\}$ pairwise distinct. We note that, $\mathcal{I}\backslash\{i, -i, j, -j\} = \{w_1, \ldots, w_6, u_1, \ldots, u_4\}$.

Recall that we are considering

$$\mathcal{I} = \{i, -i, j, -j, k, -k, l, -l, m, -m, n, -n, o, -o\}.$$

Since $|\mathcal{G}_{ij}| = |\mathcal{F}_{ij}| = 2$, from Lemma 2.2 we conclude that W_1, W_2, U_1 and U_2 are the unique codewords in $\mathcal{G}_{ij} \cup \mathcal{F}_{ij}$.

We begin by identifying different possible index distributions for the codewords of $\mathcal{G}_{ij} \cup \mathcal{F}_{ij}$.

Suppose, without loss of generality, that $W_1 \in \mathcal{G}_{ijklm}$. Taking into account Lemma 1.5, there are three possible distinct index distributions for $W_2 \in \mathcal{G}_{ij}$:

1) $W_2 \in \mathcal{G}_{i,j,-k,-l,-m}$;

2) $W_2 \in \mathcal{G}_{i,j,-k,-l,n}$;

3) $W_2 \in \mathcal{G}_{i,j,-k,n,o}$.

Considering the elements of $\mathcal{I}$ as well as Lemma 1.5, up to an equivalent index distribution, we get for each one of these hypotheses the following index distributions for the codewords of $\mathcal{F}_{ij}$:

U_1	i	j	n	o
U_2	i	j	$-n$	$-o$

Table 6.72: If $W_2 \in \mathcal{G}_{i,j,-k,-l,-m}$.

U_1	i	j	o	$-m$
U_2	i	j	$-o$	$-n$

Table 6.73: If $W_2 \in \mathcal{G}_{i,j,-k,-l,n}$.

U_1	i	j	$-l$	$-m$
U_2	i	j	$-n$	$-o$

Table 6.74: If $W_2 \in \mathcal{G}_{i,j,-k,n,o}$.

U_1	i	j	$-l$	$-n$
U_2	i	j	$-m$	$-o$

Table 6.75: If $W_2 \in \mathcal{G}_{i,j,-k,n,o}$.

Since we have characterized all possible index distributions for the codewords of $\mathcal{G}_{ij} \cup \mathcal{F}_{ij}$, next step consists in the description, for each one of the presented hypotheses, of the remaining codewords of $\mathcal{G}_i$, that is, $W_3, W_4, W_5 \in \mathcal{G}_i \backslash \mathcal{G}_j$.

• Characterization of the index distribution of the codewords of $\mathcal{G}_i$

Here we present the method we have used to identify the possible index distributions for all codewords of $\mathcal{G}_i$ when considered a certain index distribution for the codewords of $\mathcal{G}_{ij} \cup \mathcal{F}_{ij}$. As example, let us consider $W_1, W_2 \in \mathcal{G}_{ij}$ and $U_1, U_2 \in \mathcal{F}_{ij}$ satisfying:

W_1	i	j	k	l	m
W_2	i	j	$-k$	$-l$	$-m$

U_1	i	j	n	o
U_2	i	j	$-n$	$-o$

Table 6.76: Index distribution for the codewords of $\mathcal{G}_{ij} \cup \mathcal{F}_{ij}$.

Let us consider the partition $\mathcal{S}$ of $\mathcal{I}\backslash\{i,-i,j\}$ given by:

$$\mathcal{S}_1 = \{k,l,m\};\ \mathcal{S}_2 = \{-k,-l,-m\};\ \mathcal{S}_3 = \{n,o\};\ \mathcal{S}_4 = \{-n,-o\};\ \mathcal{S}_5 = \{-j\}. \quad (6.15)$$

The characterization of the possible index distributions for the codewords W_3, W_4 and W_5 in $\mathcal{G}_i\backslash\mathcal{G}_j$ is mainly based in Proposition 6.9 from which we know that $1 \leq |(\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}| \leq 3$. Thus, in the study of the index distribution of the codewords W_3, W_4 and W_5 we will consider the following hypotheses:

a) $|(\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}| = 1$;

b) $|(\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}| = 2$;

c) $|(\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}| = 3$.

We recall that $\mathcal{O} \subset \mathcal{I}$ is such that $\alpha \in \mathcal{O}$ if and only if $|\mathcal{G}_{i\alpha}| = 2$.

a) **Suppose that $|(\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}| = 1$.**

Let $\{\alpha\} = (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}$. In these conditions, by Proposition 6.9, the codewords $W_3, W_4, W_5 \in \mathcal{G}_i\backslash\mathcal{G}_j$ must satisfy the conditions presented in Table 6.77, where $\{x_1, \ldots, x_{10}\} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{\alpha\}$.

W_3	i	α	x_1	x_2	x_3
W_4	i	α	x_4	x_5	x_6
W_5	i	x_7	x_8	x_9	x_{10}

Table 6.77: Partial index distribution for $W_3, W_4, W_5 \in \mathcal{G}_i \backslash \mathcal{G}_j$.

Considering the sets $\mathcal{S}_3$, $\mathcal{S}_4$ and $\mathcal{S}_5$, see (6.15), we distinguish, without loss of generality, the cases:

i) $\alpha = n$;

ii) $\alpha = -j$.

If $\alpha = n$, taking into account the codewords of $\mathcal{G}_{ij} \cup \mathcal{F}_{ij}$ and Lemma 1.5, up to an equivalent index distribution, the codewords $W_3, W_4, W_5 \in \mathcal{G}_i \backslash \mathcal{G}_j$ satisfy:

W_3	i	n	$-j$	k	$-l$
W_4	i	n	$-o$	l	$-m$
W_5	i	$-n$	o	m	$-k$

Table 6.78: Index distribution for $W_3, W_4, W_5 \in \mathcal{G}_i \backslash \mathcal{G}_j$ when $\alpha = n$.

If $\alpha = -j$, then, as in the previous case, up to an equivalent index distribution, W_3, W_4 and W_5 verify:

W_3	i	$-j$	$-n$	k	$-l$
W_4	i	$-j$	o	l	$-m$
W_5	i	n	$-o$	m	$-k$

Table 6.79: Index distribution for $W_3, W_4, W_5 \in \mathcal{G}_i \backslash \mathcal{G}_j$ when $\alpha = -j$.

b) **Suppose that** $|(\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}| = 2$**.**

Let $\{\alpha, \beta\} = (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}$. By Proposition 6.9, the index distribution of $W_3, W_4, W_5 \in \mathcal{G}_i \backslash \mathcal{G}_j$ must satisfy the conditions presented in Table 6.80, where $x_1, \ldots, x_8 \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \backslash \{\alpha, \beta\}$ are pairwise distinct.

W_3	i	α	β	x_1	x_2
W_4	i	α	x_3	x_4	x_5
W_5	i	β	x_6	x_7	x_8

Table 6.80: Partial index distribution for $W_3, W_4, W_5 \in \mathcal{G}_i \backslash \mathcal{G}_j$.

Considering the partition $\mathcal{S}$ of $\mathcal{I} \backslash \{i, -i, j\}$, see (6.15), without loss of generality, there exist two possible hypotheses:

i) $\alpha = n$ and $\beta = -o$;

ii) $\alpha = n$ and $\beta = -j$.

If $\alpha = n$ and $\beta = -o$, then we get the following partial index distribution:

W_3	i	n	$-o$	x_1	x_2
W_4	i	n	x_3	x_4	x_5
W_5	i	$-o$	x_6	x_7	x_8

Table 6.81: Partial index distribution for $W_3, W_4, W_5 \in \mathcal{G}_i \backslash \mathcal{G}_j$.

Considering the codewords of $\mathcal{G}_{ij} \cup \mathcal{F}_{ij}$ as well as Lemma 1.5 we must impose $x_1, \ldots, x_8 \neq -n, o$. Since

$$|\mathcal{S}_1 \cup \mathcal{S}_2 \cup (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \backslash \{n, -o, -n, o\}| = 7,$$

we get a contradiction.

Let us now consider $\alpha = n$ and $\beta = -j$. That is:

W_3	i	n	$-j$	x_1	x_2
W_4	i	n	x_3	x_4	x_5
W_5	i	$-j$	x_6	x_7	x_8

Table 6.82: Partial index distribution for $W_3, W_4, W_5 \in \mathcal{G}_i \backslash \mathcal{G}_j$.

Since $|\mathcal{S}_1 \cup \mathcal{S}_2 \cup (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{n, -j\}| = 9$, we distinguish the cases:

- $\mathcal{S}_1 \cup \mathcal{S}_2 \not\subset \{x_1, ..., x_8\}$;

- $\mathcal{S}_1 \cup \mathcal{S}_2 \subset \{x_1, ..., x_8\}$.

Suppose that $\mathcal{S}_1 \cup \mathcal{S}_2 \not\subset \{x_1, \ldots, x_8\}$. Under this condition we get, up to an equivalent index distribution, the following hypotheses for $W_3, W_4, W_5 \in \mathcal{G}_i\backslash\mathcal{G}_j$:

W_3	i	n	$-j$	k	$-l$
W_4	i	n	$-o$	l	$-k$
W_5	i	$-j$	$-n$	o	m

W_3	i	n	$-j$	k	$-m$
W_4	i	n	$-o$	l	$-k$
W_5	i	$-j$	$-n$	o	m

W_3	i	n	$-j$	k	$-l$
W_4	i	n	$-o$	l	$-m$
W_5	i	$-j$	$-n$	o	m

Table 6.83: Possible index distributions for the codewords of $\mathcal{G}_i\backslash\mathcal{G}_j$.

If $\mathcal{S}_1 \cup \mathcal{S}_2 \subset \{x_1, \ldots, x_8\}$, then there are two different possibilities for the codewords W_3, W_4 and W_5:

W_3	i	n	$-j$	k	$-l$
W_4	i	n	$-o$	l	$-m$
W_5	i	$-j$	$-n$	m	$-k$

W_3	i	n	$-j$	k	$-l$
W_4	i	n	$-o$	l	$-m$
W_5	i	$-j$	o	m	$-k$

Table 6.84: Possible index distributions for the codewords of $\mathcal{G}_i\backslash\mathcal{G}_j$.

c) **Suppose that** $|(\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}| = 3$**.**

Without loss of generality, suppose that $\alpha, \beta, \gamma \in (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5) \cap \mathcal{O}$ are such that $\alpha = n$, $\beta = -o$ and $\gamma = -j$. Thus, considering Proposition 6.9, the codewords $W_3, W_4, W_5 \in \mathcal{G}_i\backslash\mathcal{G}_j$ must satisfy the conditions presented in Table 6.85, where $x_1, \ldots, x_6 \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup (\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5)\backslash\{n, -o, -j\}$ are pairwise distinct.

W_3	i	n	$-o$	x_1	x_2
W_4	i	n	$-j$	x_3	x_4
W_5	i	$-o$	$-j$	x_5	x_6

Table 6.85: Partial index distribution for the codewords of $\mathcal{G}_i\backslash\mathcal{G}_j$.

Taking into account the codewords of $\mathcal{G}_{ij} \cup \mathcal{F}_{ij}$ and Lemma 1.5, we must impose $x_1, \ldots, x_6 \neq -n, o$. Consequently, $\{x_1, \ldots, x_6\} = \mathcal{S}_1 \cup \mathcal{S}_2$. Accordingly, up to an equivalent index distribution, $W_3, W_4, W_5 \in \mathcal{G}_i\backslash\mathcal{G}_j$ satisfy:

W_3	i	n	$-o$	k	$-l$
W_4	i	n	$-j$	l	$-m$
W_5	i	$-o$	$-j$	m	$-k$

Table 6.86: Index distribution for the codewords of $\mathcal{G}_i\backslash\mathcal{G}_j$.

Considering $W_2 \in \mathcal{G}_{i,j,-k,-l,n}$ and $U_1, U_2 \in \mathcal{F}_{ij}$ satisfying the index distribution presented in Table 6.73, as well as, $W_2 \in \mathcal{G}_{i,j,-k,n,o}$ and $U_1, U_2 \in \mathcal{F}_{ij}$ satisfying, respectively, the conditions presented in Tables 6.74 and 6.75, following a similar reasoning to the one done before we get all possible index distributions for the codewords W_3, W_4 and W_5 in $\mathcal{G}_i\backslash\mathcal{G}_j$.

• Characterization of the index distribution of the codewords of $\mathcal{F}_i$

For a certain index distribution of the codewords of $\mathcal{G}_{ij} \cup \mathcal{F}_{ij}$ we have shown how we have obtained possible index distributions for all codewords of $\mathcal{G}_i$. Here we describe the method which allows us to characterize the remaining codewords of $\mathcal{F}_i$ starting from the knowledge of a certain index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_{ij}$.

We recall that, by Lemma 2.11, $7 \leq |\mathcal{F}_i| \leq 10$. In the characterization of all codewords of $\mathcal{F}_i$ we analyze, separately, the hypotheses:

i) $|\mathcal{F}_i| = 7$;

ii) $8 \leq |\mathcal{F}_i| \leq 10$.

The hypothesis $|\mathcal{F}_i| = 7$ will be analyzed taking into account that, by Lemma 2.11, $|\mathcal{F}_i^{(2)}| = 4$. The analysis of $8 \leq |\mathcal{F}_i| \leq 10$ will be based in Proposition 6.10.

To exemplify how we characterize all codewords of $\mathcal{F}_i$, we will consider some possible index distributions for the codewords of $\mathcal{G}_i \cup \mathcal{F}_{ij}$. We present examples in which:

1) it is not possible to describe $\mathcal{F}_i$;

2) it is only possible to describe $\mathcal{F}_i$ when $|\mathcal{F}_i| = 7$;

3) it is only possible to describe $\mathcal{F}_i$ when $8 \leq |\mathcal{F}_i| \leq 10$;

4) it is possible to describe $\mathcal{F}_i$ when $|\mathcal{F}_i| = 7$ and $8 \leq |\mathcal{F}_i| \leq 10$.

Example 1: It is not possible to describe $\mathcal{F}_i$.

Let us consider $W_1, \ldots, W_5 \in \mathcal{G}_i$ and $U_1, U_2 \in \mathcal{F}_{ij}$ satisfying:

W_1	i	j	k	l	m
W_2	i	j	$-k$	$-l$	$-m$
W_3	i	n	$-j$	k	$-l$
W_4	i	n	$-o$	l	$-m$
W_5	i	$-n$	o	m	$-k$

U_1	i	j	n	o
U_2	i	j	$-n$	$-o$

Table 6.87: Codewords of $\mathcal{G}_i \cup \mathcal{F}_{ij}$.

Suppose that $|\mathcal{F}_i| = 7$. By Lemma 2.11, $|\mathcal{F}_i^{(2)}| = 4$. Consequently, from Lemma 2.14 it follows that $V_1, \ldots, V_4 \in \mathcal{F}_i^{(2)}$ are such that $V_1 \in \mathcal{F}_{iy_1y_2y_3}, \ldots, V_4 \in \mathcal{F}_{iy_{10}y_{11}y_{12}}$ with $y_1, \ldots, y_{12} \in \mathcal{I}\backslash\{i, -i\}$ pairwise distinct. Thus, there exists $V \in \mathcal{F}_i^{(2)}$ so that $V \in \mathcal{F}_{ij}$. Since U_1 and U_2 are the unique codewords in $\mathcal{F}_{ij}$, either $U_1 \in \mathcal{F}_i^{(2)}$ or $U_2 \in \mathcal{F}_i^{(2)}$. Taking into account the known codewords of $\mathcal{G}_i \cup \mathcal{F}_{ij}$ and Lemma 1.5, we verify that if $U_2 \in \mathcal{F}_i^{(2)}$, it is not possible to characterize all codewords of $\mathcal{F}_i^{(2)}$ without contradicting the definition of PL$(7, 2)$ code. On the other hand, considering $U_1 \in \mathcal{F}_i^{(2)}$, there exists a unique possible index distribution for the codewords of $\mathcal{F}_i^{(2)}$, presented in the following table.

$V_1 = U_1$	i	j	n	o
V_2	i	k	$-m$	$-n$
V_3	i	l	$-j$	$-k$
V_4	i	m	$-l$	$-o$

Table 6.88: Codewords of $\mathcal{F}_i^{(2)}$.

We have characterized five of the seven codewords of $\mathcal{F}_i$. However, considering the codewords of $\mathcal{G}_i \cup \mathcal{F}_{ij} \cup \mathcal{F}_i^{(2)}$ already known and Lemma 1.5, it is possible to add to the set $\mathcal{F}_i$ only one more codeword $U \in \mathcal{F}_i$ satisfying: $U \in \mathcal{F}_{i,-j,-m,o}$. Any other hypothesis for $U \in \mathcal{F}_i$ contradicts Lemma 1.5. That is, $|\mathcal{F}_i| \leq 6$, contradicting Lemma 2.11.

Consider $8 \leq |\mathcal{F}_i| \leq 10$. By Proposition 6.10, there are, at least, three elements $\alpha \in \mathcal{O}\backslash\{j\}$ satisfying $|\mathcal{F}_{i\alpha}| = 2$, that is, there are, at least, three elements $\alpha \in \mathcal{O}\backslash\{j\}$ such that $|\mathcal{G}_{i\alpha}| = |\mathcal{F}_{i\alpha}| = 2$. We begin by identifying the elements $\alpha \in \mathcal{O}\backslash\{j\}$:

$$\mathcal{O}\backslash\{j\} = \{k, -k, l, -l, m, -m, n\}.$$

Next, we check which of these elements $\alpha \in \mathcal{O}\backslash\{j\}$ can satisfy the condition $|\mathcal{F}_{i\alpha}| = 2$. For that, consider the following schemes where are presented all possible index distributions for codewords of $\mathcal{F}_{i\alpha}$ with $\alpha \in \mathcal{O}\backslash\{j\}$:

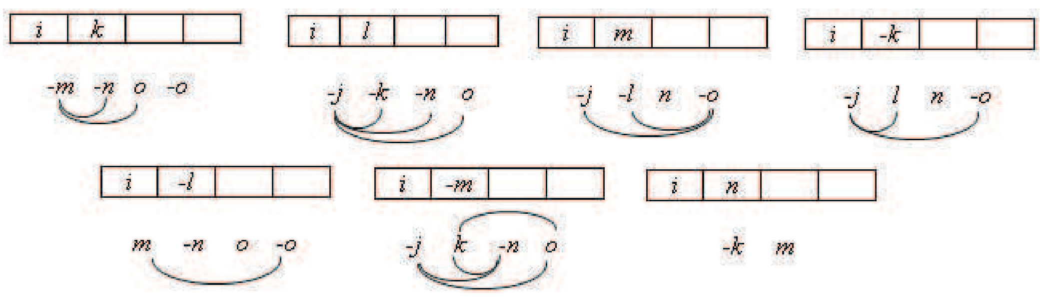

Figure 6.16: Possible index distributions for codewords of $\mathcal{F}_i$.

We note that, the different index distributions for codewords of $\mathcal{F}_i$ presented in the schemes of Figure 6.16 come from the analysis of the codewords of $\mathcal{G}_i \cup \mathcal{F}_{ij}$, taking into account Lemma 1.5. Looking at the schemes, we verify, for example, that $|\mathcal{F}_{ik}| \leq 1$, in fact, as the unique possible index distributions for codewords of $\mathcal{F}_{ik}$ are

$U \in \mathcal{F}_{i,k,-m,-n}$ and $U' \in \mathcal{F}_{i,k,-m,o}$, by Lemma 1.5 we conclude that is not possible consider both codewords, U and U', in $\mathcal{F}_{ik}$. We may also conclude that $|\mathcal{F}_{il}|$, $|\mathcal{F}_{im}|$, $|\mathcal{F}_{i,-k}|$, $|\mathcal{F}_{i,-l}|$, $|\mathcal{F}_{in}| \leq 1$.

There is only one element $-m \in \mathcal{O}\backslash\{j\}$ for which it is possible to have $|\mathcal{F}_{i,-m}| = 2$. In fact, we can consider $\mathcal{F}_{i,-m} = \{U, U'\}$ with $U \in \mathcal{F}_{i,-m,-j,-n}$ and $U' \in \mathcal{F}_{i,-m,k,o}$, or, $U \in \mathcal{F}_{i,-m,-j,o}$ and $U' \in \mathcal{F}_{i,-m,k,-n}$. Since $-m \in \mathcal{O}\backslash\{j\}$ is the unique element satisfying the condition $|\mathcal{F}_{i,-m}| = 2$, this contradicts Proposition 6.10.

Consequently, the considered index distribution for the codewords of $\mathcal{G}_i \cup \mathcal{F}_{ij}$ contradicts the definition of PL(7, 2) code.

In the majority of the cases, likewise this example, given the codewords of $\mathcal{G}_i \cup \mathcal{F}_{ij}$, we can not characterize all codewords of $\mathcal{F}_i$ without contradicting the definition of perfect error correcting Lee code. However, there exist index distributions for the codewords of $\mathcal{G}_i \cup \mathcal{F}_{ij}$ in which it is possible to have a complete admissible characterization, as we will see in next examples.

Example 2: It is only possible to describe $\mathcal{F}_i$ when $|\mathcal{F}_i| = 7$.

Suppose that the codewords of $\mathcal{G}_i \cup \mathcal{F}_{ij}$ are such that:

W_1	i	j	k	l	m
W_2	i	j	$-k$	n	o
W_3	i	$-l$	$-m$	$-j$	o
W_4	i	$-l$	$-o$	n	k
W_5	i	$-m$	$-n$	$-k$	l

U_1	i	j	$-l$	$-n$
U_2	i	j	$-m$	$-o$

Table 6.89: Codewords of $\mathcal{G}_i \cup \mathcal{F}_{ij}$.

We begin by characterizing the codewords of $\mathcal{F}_i$, assuming $|\mathcal{F}_i| = 7$. In these conditions, by Lemma 2.11, we have $|\mathcal{F}_i^{(2)}| = 4$. Considering Lemma 2.14, since U_1 and U_2 are the unique codewords in $\mathcal{F}_{ij}$, we must impose $U_1 \in \mathcal{F}_i^{(2)}$ or $U_2 \in \mathcal{F}_i^{(2)}$. From Lemma 2.14 we know that there is no $\alpha \in \mathcal{I}\backslash\{i, -i\}$ so that $|\mathcal{F}_{i\alpha} \cap \mathcal{F}_i^{(2)}| \geq 2$. Taking into account the codewords of $\mathcal{G}_i \cup \mathcal{F}_{ij}$ and Lemma 1.5, if we assume $U_1 \in \mathcal{F}_i^{(2)}$ we can not describe all codewords of $\mathcal{F}_i^{(2)}$ without facing a contradiction. On the other hand, if $U_2 \in \mathcal{F}_i^{(2)}$, then the codewords of $\mathcal{F}_i^{(2)}$ are such that:

U_2	i	j	$-m$	$-o$
U_3	i	k	$-n$	o
U_4	i	l	$-j$	n
U_5	i	m	$-k$	$-l$

Table 6.90: Codewords of $\mathcal{F}_i^{(2)}$.

As we are under the assumption of $|\mathcal{F}_i| = 7$, then we must identify in $\mathcal{F}_i$ two more codewords. Considering the schemes in Figure 6.17 all possible index distributions for the remaining codewords of $\mathcal{F}_i$ are given. We note that, these schemes were obtained taking into account all codewords already known as well as Lemma 1.5.

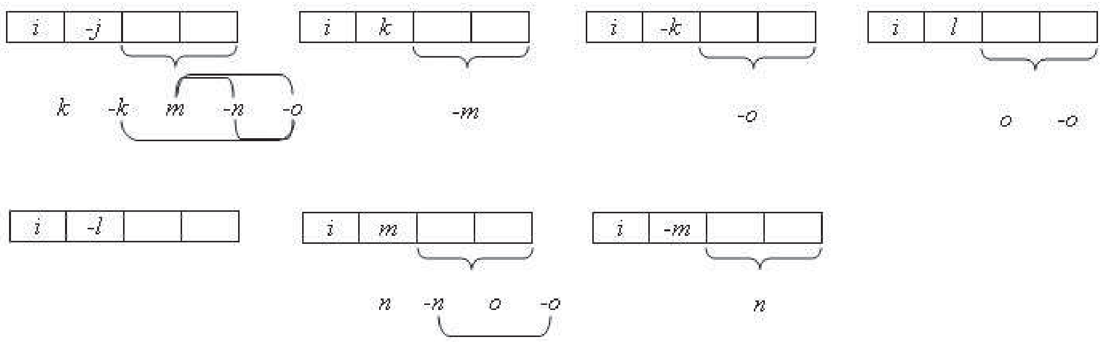

Figure 6.17: Possible index distributions for codewords $U \in \mathcal{F}_i$.

Therefore, $U_6, U_7 \in \mathcal{F}_i$ must satisfy one of the following conditions:

- $U_6 \in \mathcal{F}_{i,-j,m,-n}$ and $U_7 \in \mathcal{F}_{i,-j,-k,-o}$;
- $U_6 \in \mathcal{F}_{i,m,-n,-o}$ and $U_7 \in \mathcal{F}_{i,-j,-k,-o}$.

Now suppose that $8 \leq |\mathcal{F}_i| \leq 10$. By Proposition 6.10 we must identify in $\mathcal{O}\backslash\{j\}$, at least, three elements α satisfying $|\mathcal{F}_{i\alpha}| = 2$. Observing the index distribution of the codewords $W_1, \ldots, W_5 \in \mathcal{G}_i$ we verify that

$$\mathcal{O}\backslash\{j\} = \{k, -k, l, -l, -m, n, o\}.$$

In the next figure all possible index distributions for the codewords of $\mathcal{F}_{i\alpha}$, with $\alpha \in \mathcal{O}\backslash\{j\}$ are presented.

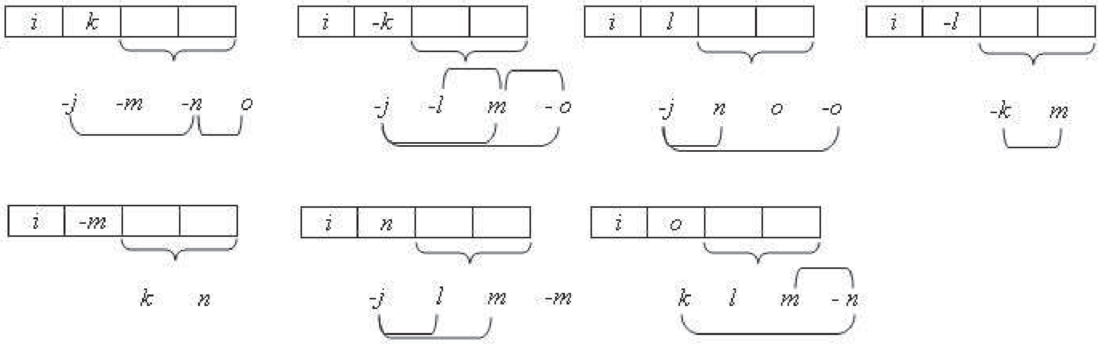

Figure 6.18: Possible index distributions for codewords of $\mathcal{F}_i$.

Taking into account the schemes in Figure 6.18 we conclude that there exist only two elements $\alpha \in \mathcal{O}\backslash\{j\}$ for which $|\mathcal{F}_{i\alpha}| = 2$: $-k$ and $-l$. Consequently, Proposition 6.10 is contradicted.

Example 3: It is only possible to describe $\mathcal{F}_i$ when $8 \leq |\mathcal{F}_i| \leq 10$.

Suppose that the codewords of $\mathcal{G}_i \cup \mathcal{F}_{ij}$ satisfy:

W_1	i	j	k	l	m
W_2	i	j	$-k$	n	o
W_3	i	$-l$	$-j$	m	$-k$
W_4	i	$-l$	$-o$	k	n
W_5	i	$-j$	$-n$	l	o

U_1	i	j	$-l$	$-n$
U_2	i	j	$-m$	$-o$

Table 6.91: Codewords of $\mathcal{G}_i \cup \mathcal{F}_{ij}$.

Proceeding as in the previous examples, we conclude that it is not possible to describe all codewords of $\mathcal{F}_i$ when $|\mathcal{F}_i| = 7$. So, we will assume $8 \leq |\mathcal{F}_i| \leq 10$. Taking into account Proposition 6.10, we are going to identify the elements $\alpha \in \mathcal{O}\backslash\{j\}$,

$$\mathcal{O}\backslash\{j\} = \{-j, k, -k, l, -l, m, n, o\},$$

for which it is possible to have $|\mathcal{F}_{i\alpha}| = 2$.

Consider the schemes bellow where are presented all possible index distributions for the codewords of $\mathcal{F}_{i\alpha}$ with $\alpha \in \mathcal{O}\backslash\{j\}$:

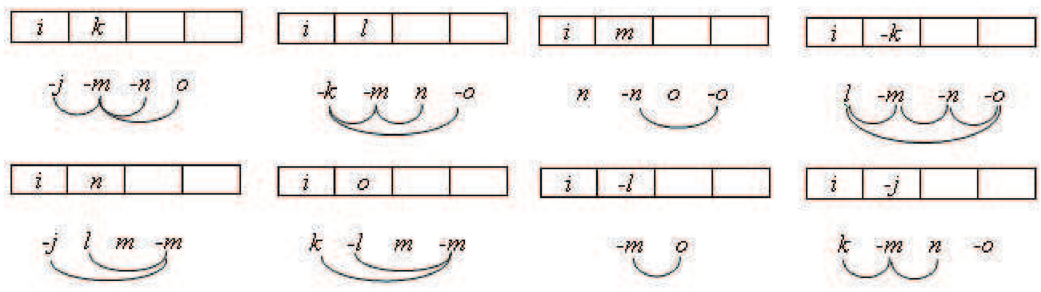

Figure 6.19: Possible index distributions for the codewords of $\mathcal{F}_i$.

By the analysis of the above schemes and taking into account the codewords of $\mathcal{F}_{ij}$ we conclude that there exist exactly three elements in $\mathcal{O}\backslash\{j\}$ satisfying the conditions in Proposition 6.10: $-k$, l and $-l$. Thus, to guarantee $|\mathcal{F}_{i,-k}| = |\mathcal{F}_{il}| = |\mathcal{F}_{i,-l}| = 2$, the codewords $U_3, \ldots, U_6$ described bellow must be in $\mathcal{F}_i$.

U_3	i	l	$-k$	$-o$
U_4	i	l	$-m$	n
U_5	i	$-k$	$-m$	$-n$
U_6	i	$-l$	$-m$	o

Table 6.92: Index distribution for codewords of $\mathcal{F}_i$.

As we are supposing $8 \leq |\mathcal{F}_i| \leq 10$, we must identify in $\mathcal{F}_i$, at least, two more codewords. Taking into account the index distribution of all codewords known at this moment as well as Lemma 1.5, we find out the remaining codewords of $\mathcal{F}_i$ identifying all possible index distributions for codewords in $\mathcal{F}_{i,-j}$, $\mathcal{F}_{ik}\backslash\mathcal{F}_{-j}$, $\mathcal{F}_{i,-k}\backslash\mathcal{F}_{-j}$, $\mathcal{F}_{il}\backslash(\mathcal{F}_{-j} \cup \mathcal{F}_k \cup \mathcal{F}_{-k})$ and so on, see Figure 6.20.

By the analysis of the schemes in Figure 6.20 we conclude that there exist only two possible index distributions for the remaining codewords of $\mathcal{F}_i$: $U_7 \in \mathcal{F}_{i,-j,k,-m}$ and $U_8 \in \mathcal{F}_{i,m,-n,-o}$. Thus, $|\mathcal{F}_i| = 8$ and all codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ are characterized.

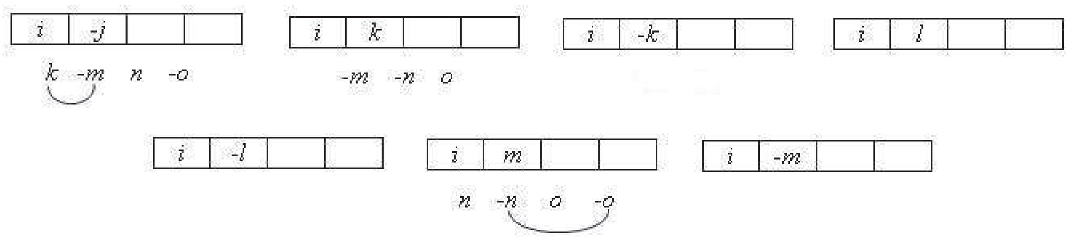

Figure 6.20: Possible index distributions for codewords of $\mathcal{F}_i$.

Example 4: It is possible to describe $\mathcal{F}_i$ when $|\mathcal{F}_i| = 7$ and $8 \leq |\mathcal{F}_i| \leq 10$.

Let us assume $W_1, \ldots, W_5 \in \mathcal{G}_i$ and $U_1, U_2 \in \mathcal{F}_{ij}$ satisfying:

W_1	i	j	k	l	m
W_2	i	j	$-k$	n	o
W_3	i	$-l$	$-o$	m	n
W_4	i	$-l$	$-j$	$-m$	$-k$
W_5	i	$-o$	$-j$	$-n$	k

U_1	i	j	$-l$	$-n$
U_2	i	j	$-m$	$-o$

Table 6.93: Index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_{ij}$.

Suppose that $|\mathcal{F}_i| = 7$. Since by Lemma 2.11 we get $|\mathcal{F}_i^{(2)}| = 4$, taking into account Lemma 2.14, either $U_1 \in \mathcal{F}_i^{(2)}$ or $U_2 \in \mathcal{F}_i^{(2)}$. Considering the hypotheses $U_1 \in \mathcal{F}_i^{(2)}$ and $U_2 \in \mathcal{F}_i^{(2)}$, having in view Lemmas 2.14 and 1.5 as well as the codewords of $\mathcal{G}_i \cup \mathcal{F}_{ij}$, we get, respectively, the following index distributions for the codewords of $\mathcal{F}_i^{(2)}$:

U_1	i	j	$-l$	$-n$
U_3	i	k	$-m$	n
U_4	i	l	$-k$	$-o$
U_5	i	m	$-j$	o

Table 6.94: $U_1 \in \mathcal{F}_i^{(2)}$.

U_2	i	j	$-m$	$-o$
U_3	i	k	$-l$	o
U_4	i	l	$-j$	n
U_5	i	m	$-k$	$-n$

Table 6.95: $U_2 \in \mathcal{F}_i^{(2)}$.

To complete the characterization of the codewords of $\mathcal{F}_i$, since we are assuming $|\mathcal{F}_i| = 7$, we must identify, for each one of the presented hypotheses, two more codewords.

Considering $U_1 \in \mathcal{F}_i^{(2)}$, see Table 6.94, taking into account all codewords described until now and Lemma 1.5, we get the following possible index distributions for the remaining codewords $U_6, U_7 \in \mathcal{F}_i$:

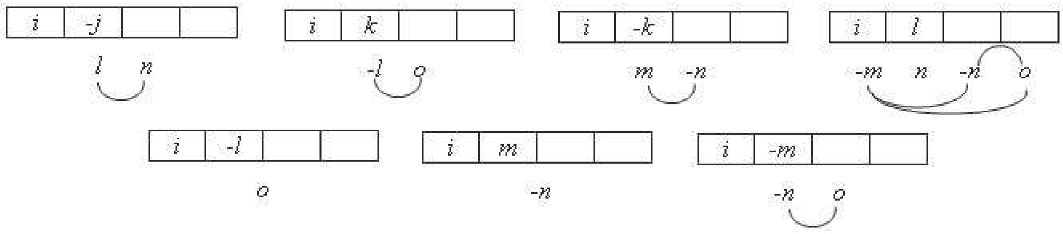

Figure 6.21: Possible index distributions for the codewords of $\mathcal{F}_i$.

By the analysis of the above schemes we verify that there are several possibilities for the index distribution of the codewords $U_6, U_7 \in \mathcal{F}_i$:

U_6				U_7			
i	-j	l	n	i	k	-l	o
				i	-k	m	-n
				i	l	-n	o
				i	l	-m	-n
				i	l	-m	o
				i	-m	-n	o
i	k	-l	o	i	-k	m	-n
				i	l	-n	o
				i	l	-m	-n
				i	l	-n	o
				i	-m	-n	o
i	-k	m	-n	i	l	-n	o
				i	l	-m	-n
				i	l	-m	o
				i	-m	-n	o

Table 6.96: Possible index distributions for $U_6, U_7 \in \mathcal{F}_i$.

If we assume that $U_2 \in \mathcal{F}_i^{(2)}$, see Table 6.95, we conclude, following a similar reasoning to the one applied in the previous case, that the codewords $U_6, U_7 \in \mathcal{F}_i$ must satisfy one of the conditions presented in Table 6.97.

U_6				U_7			
i	-j	m	o	i	k	-m	n
				i	-k	l	-o
				i	l	-n	o
				i	l	-m	-n
				i	l	-m	o
				i	-m	-n	o
i	k	-m	n	i	-k	l	-o
				i	l	-n	o
				i	l	-m	-n
				i	l	-m	o
				i	-m	-n	o
i	-k	l	-o	i	l	-n	o
				i	l	-m	-n
				i	l	-m	o
				i	-m	-n	o

Table 6.97: Possible index distributions for $U_6, U_7 \in \mathcal{F}_i$.

In this case we can characterize $\mathcal{F}_i$, under the assumption of $|\mathcal{F}_i| = 7$, getting several possible index distributions for the codewords of $\mathcal{F}_i$. Next, we show that it is also possible characterize all the codewords of $\mathcal{F}_i$ when we assume $8 \leq |\mathcal{F}_i| \leq 10$.

Let us then assume $8 \leq |\mathcal{F}_i| \leq 10$. Considering Proposition 6.10, we must identify the elements $\alpha \in \mathcal{O}\backslash\{j\}$ which verify $|\mathcal{F}_{i\alpha}| = 2$. By the analysis of $W_1, \ldots, W_5 \in \mathcal{G}_i$, see Table 6.93, we get $\mathcal{O}\backslash\{j\} = \{-j, k, -k, -l, m, n, -o\}$. Next we present the possible index distributions for the codewords of $\mathcal{F}_{i\alpha}$, with $\alpha \in \mathcal{O}\backslash\{j\}$:

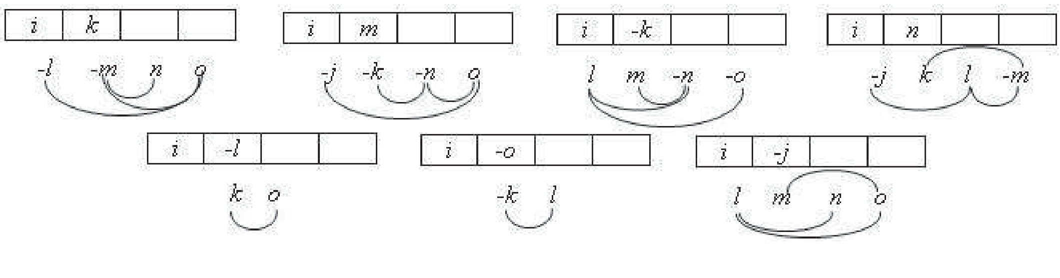

Figure 6.22: Possible index distributions for the codewords of $\mathcal{F}_i$.

From the analysis of the schemes in Figure 6.22 we conclude that for any $\alpha \in \mathcal{O}\backslash\{j\}$ it is possible to have $|\mathcal{F}_{i\alpha}| = 2$. By Proposition 6.10, it must exist, at least, three elements $\alpha \in \mathcal{O}\backslash\{j\}$ satisfying $|\mathcal{F}_{i\alpha}| = 2$. Thus, we should identify, considering the schemes in Figure 6.22, sets of codewords which must be in $\mathcal{F}_i$ guaranteing the conditions of

Proposition 6.10.

$\diamond$ If $|\mathcal{F}_{ik}| = 2$, then $U_3 \in \mathcal{F}_{i,k,-l,o}$ and $U_4 \in \mathcal{F}_{i,k,-m,n}$. We note that, in these conditions, $|\mathcal{F}_{i,-l}| = 2$. Thus, having in view Proposition 6.10, at least one of the following conditions must be satisfied:

1) $|\mathcal{F}_{im}| = 2$, with $U_5 \in \mathcal{F}_{i,m,-j,o}$, $U_6 \in \mathcal{F}_{i,m,-k,-n}$;

2) $|\mathcal{F}_{in}| = 2$, with $U_5 \in \mathcal{F}_{i,n,-j,l}$;

3) $|\mathcal{F}_{i,-o}| = 2$, with $U_5 \in \mathcal{F}_{i,-o,-k,l}$;

We have identified the codewords in $\mathcal{F}_i$ assuming $|\mathcal{F}_{ik}| = 2$. Next, we apply the same reasoning considering now the assumption $|\mathcal{F}_{i\alpha}| = 2$, with $\alpha \in \mathcal{O} \backslash \{j, m\}$. We impose $|\mathcal{F}_{ik}| \neq 2$ since we have just analyzed this hypothesis.

$\diamond$ If $|\mathcal{F}_{im}| = 2$, then $U_3 \in \mathcal{F}_{i,m,-j,o}$ and $U_4 \in \mathcal{F}_{i,m,-k,-n}$. In this case one of the following conditions must be satisfied:

4) $|\mathcal{F}_{i,-k}| = |\mathcal{F}_{i,-o}| = 2$, with $U_5 \in \mathcal{F}_{i,-k,l,-o}$;

5) $|\mathcal{F}_{in}| = |\mathcal{F}_{i,-j}| = 2$, with $U_5 \in \mathcal{F}_{i,n,-j,l}$ and $U_6 \in \mathcal{F}_{i,n,k,-m}$;

6) $|\mathcal{F}_{i,-l}| = |\mathcal{F}_{i,-j}| = 2$, with $U_5 \in \mathcal{F}_{i,n,-j,l}$ and $U_6 \in \mathcal{F}_{i,-l,k,o}$.

$\diamond$ Now suppose that $|\mathcal{F}_{i,-k}| = 2$. Then $U_3 \in \mathcal{F}_{i,-k,l,-o}$ and $U_4 \in \mathcal{F}_{i,-k,m,-n}$. We note that, in this case, $|\mathcal{F}_{i,-o}| = 2$, thus one of following conditions must be verified:

7) $|\mathcal{F}_{in}| = 2$, with $U_5 \in \mathcal{F}_{i,n,-j,l}$ and $U_6 \in \mathcal{F}_{i,n,k,-m}$;

8) $|\mathcal{F}_{i,-l}| = 2$, with $U_5 \in \mathcal{F}_{i,-l,k,o}$;

9) $|\mathcal{F}_{i,-j}| = 2$, with $U_5 \in \mathcal{F}_{i,-j,l,n}$ and $U_6 \in \mathcal{F}_{i,-j,m,o}$.

$\diamond$ If $|\mathcal{F}_{in}| = 2$, then $U_3 \in \mathcal{F}_{i,n,-j,l}$ and $U_4 \in \mathcal{F}_{i,n,k,-m}$. In these conditions one of the following hypotheses must be satisfied:

10) $|\mathcal{F}_{i,-l}| = |\mathcal{F}_{ik}| = 2$, with $U_5 \in \mathcal{F}_{i,-l,k,o}$;

11) $|\mathcal{F}_{i,-o}| = |\mathcal{F}_{i,-j}| = 2$, with $U_5 \in \mathcal{F}_{i,-o,-k,l}$ and $U_6 \in \mathcal{F}_{i,-j,m,o}$.

$\diamond$ Supposing $|\mathcal{F}_{i,-l}| = 2$, then $U_3 \in \mathcal{F}_{i,-l,k,o}$ and we must impose:

12) $|\mathcal{F}_{i,-o}| = |\mathcal{F}_{i,-j}| = 2$, with $U_4 \in \mathcal{F}_{i,-o,-k,l}$, $U_5 \in \mathcal{F}_{i,-j,m,o}$ and $U_6 \in \mathcal{F}_{i,-j,l,n}$.

We will not consider the hypotheses $|\mathcal{F}_{i,-o}| = 2$ and $|\mathcal{F}_{i,-j}| = 2$ since one of the previous conditions would occur.

Thus, if $8 \leq |\mathcal{F}_i| \leq 10$, one of the presented sets of codewords must be in $\mathcal{F}_i$. Since the previous conditions allow us to characterize at most six codewords in $\mathcal{F}_i$, considering the referred subsets of codewords, we must, for each one of them, to identify other codewords in $\mathcal{F}_i$. Next, considering one of the presented hypotheses, we exemplify how we can get the remaining codewords of $\mathcal{F}_i$.

Let us consider $U_3, \ldots, U_6 \in \mathcal{F}_i$ such that:

U_3	i	k	$-l$	o
U_4	i	k	$-m$	n
U_5	i	m	$-j$	o
U_6	i	$-k$	m	$-n$

Table 6.98: Index distribution of codewords of $\mathcal{F}_i$.

In this case, we must identify, at least, two more codewords in $\mathcal{F}_i$. For that we will consider the following schemes where all possible index distributions for the remaining codewords $U \in \mathcal{F}_i$ are presented :

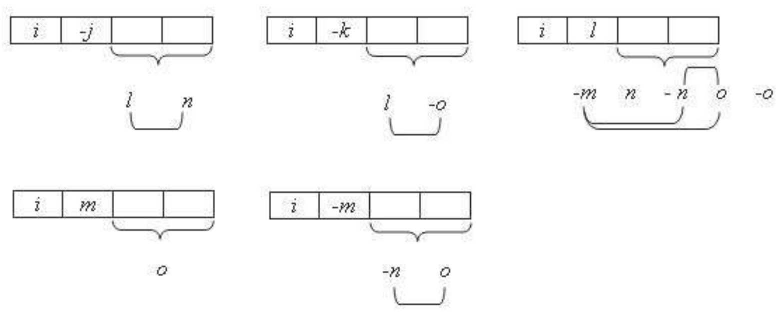

Figure 6.23: Possible index distributions for $U \in \mathcal{F}_i$.

Analyzing the previous schemes and considering Lemma 1.5 we conclude that it is possible to characterize all codewords of $\mathcal{F}_i$ when we assume $|\mathcal{F}_i| = 8$ or $|\mathcal{F}_i| = 9$.

If $|\mathcal{F}_i| = 8$, the remaining codewords $U_7, U_8 \in \mathcal{F}_i$ must satisfy one of the following conditions:

<table>
<tr><th colspan="4">U7</th><th colspan="4">U8</th></tr>
<tr><td rowspan="5">i</td><td rowspan="5">-j</td><td rowspan="5">l</td><td rowspan="5">n</td><td>i</td><td>-k</td><td>l</td><td>-o</td></tr>
<tr><td>i</td><td>l</td><td>-m</td><td>-n</td></tr>
<tr><td>i</td><td>l</td><td>-m</td><td>o</td></tr>
<tr><td>i</td><td>l</td><td>-n</td><td>o</td></tr>
<tr><td>i</td><td>-m</td><td>-n</td><td>o</td></tr>
<tr><td rowspan="4">i</td><td rowspan="4">-k</td><td rowspan="4">l</td><td rowspan="4">-o</td><td>i</td><td>l</td><td>-m</td><td>-n</td></tr>
<tr><td>i</td><td>l</td><td>-m</td><td>o</td></tr>
<tr><td>i</td><td>l</td><td>-n</td><td>o</td></tr>
<tr><td>i</td><td>-m</td><td>-n</td><td>o</td></tr>
</table>

Table 6.99: Possible index distributions for $U_7, U_8 \in \mathcal{F}_i$.

If $|\mathcal{F}_i| = 9$, then $U_7, U_8, U_9 \in \mathcal{F}_i$ must verify one of the following hypotheses:

<table>
<tr><th colspan="4">U7</th><th colspan="4">U8</th><th colspan="4">U9</th></tr>
<tr><td rowspan="4">i</td><td rowspan="4">-j</td><td rowspan="4">l</td><td rowspan="4">n</td><td rowspan="4">i</td><td rowspan="4">-k</td><td rowspan="4">l</td><td rowspan="4">-o</td><td>i</td><td>l</td><td>-m</td><td>-n</td></tr>
<tr><td>i</td><td>l</td><td>-m</td><td>o</td></tr>
<tr><td>i</td><td>l</td><td>-n</td><td>o</td></tr>
<tr><td>i</td><td>-m</td><td>-n</td><td>o</td></tr>
</table>

Table 6.100: Possible index distributions for $U_7, U_8, U_9 \in \mathcal{F}_i$.

If we had considered any other of the presented hypotheses for the index distribution of the codewords of $\mathcal{F}_i$, by a similar reasoning to the one applied in this example we would characterize all codewords of $\mathcal{F}_i$.

We note that, unlike the previous index distributions of $\mathcal{G}_i \cup \mathcal{F}_{ij}$ presented in Examples 2) and 3), in this case we have obtained many possible index distributions for the codewords of $\mathcal{F}_i$.

6.2.3 Analysis of the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$

In the previous subsection we have shown that although for certain possible index distributions of the codewords of $\mathcal{G}_i \cup \mathcal{F}_{ij}$ it is not possible to characterize completely the codewords of $\mathcal{F}_i$, there are cases where we can describe all codewords of $\mathcal{G}_i \cup \mathcal{F}_i$. Apparently, it seems that in these cases no contradiction will be achieved, however, as we will see here, such it is not true.

Here we present the method we have used to show that each one of the possible index distributions for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ leads to a contradiction. Since there exist many possible index distributions for $\mathcal{G}_i \cup \mathcal{F}_i$, we describe the applied methodology presenting some illustrative examples.

Example 1

Let us consider $W_1, \dots, W_5 \in \mathcal{G}_i$ and $U_1, \dots, U_8 \in \mathcal{F}_i$ satisfying the following index distribution:

W_1	i	j	k	l	m
W_2	i	j	$-k$	n	o
W_3	i	$-l$	$-j$	m	$-k$
W_4	i	$-l$	$-o$	k	n
W_5	i	$-j$	$-n$	l	o

U_1	i	j	$-l$	$-n$
U_2	i	j	$-m$	$-o$
U_3	i	l	$-k$	$-o$
U_4	i	l	$-m$	n
U_5	i	$-k$	$-m$	$-n$
U_6	i	$-l$	$-m$	o
U_7	i	$-j$	k	$-m$
U_8	i	m	$-n$	$-o$

Table 6.101: Index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$.

Since we have described all codewords of $\mathcal{G}_i \cup \mathcal{F}_i$, we will focus our attention on other element in $\mathcal{I}\backslash\{i\}$. Our aim is to achieve a contradiction, in particular, we have constantly in view possible contradictions of Lemma 1.5. We are interested in the choice of an element $\alpha \in \mathcal{I}\backslash\{i\}$ for which the number of known codewords of $\mathcal{G}_{i\alpha}$ is minimum, since this implies the characterization of other more codewords of $\mathcal{G}_\alpha$. We note that, the bigger is the number of the other codewords which must be characterized, more probability to contradict Lemma 1.5 exists. On the other hand, although at this moment it is not known any codeword of $\mathcal{G}_{-i}$, we do not give preference to this element of $\mathcal{I}$ since we do not have, throughout the known codewords of $\mathcal{G}_i \cup \mathcal{F}_i$, any information about it.

We will analyze the considered set $\mathcal{G}_i \cup \mathcal{F}_i$ concentrating our attention on $-m \in \mathcal{I}$. In fact, observing the codewords of $\mathcal{G}_i$, see Table 6.101, we verify that $-m$ is such that $|\mathcal{G}_{i,-m}| = 0$. Since, by Corollary 5.1, $5 \leq |\mathcal{G}_\alpha| \leq 7$ for any $\alpha \in \mathcal{I}$, we must identify, at least, five codewords in $\mathcal{G}_{-m}$. To simplify the characterization of the possible index distributions for the codewords of $\mathcal{G}_{-m}$ we will consider the partition $\mathcal{X}$ of $\mathcal{I}\backslash\{i,m,-m\}$ induced by the codewords $U_2, U_4, U_5, U_6, U_7 \in \mathcal{F}_{i,-m}$:

$$\mathcal{X}_1 = \{j,-o\};\ \mathcal{X}_2 = \{l,n\};\ \mathcal{X}_3 = \{-k,-n\};\ \mathcal{X}_4 = \{-l,o\};\ \mathcal{X}_5 = \{-j,k\};\ \mathcal{X}_6 = \{-i\}. \tag{6.16}$$

By Theorem 6.1, if $|\mathcal{G}_{-m}| = 5$, then $|\mathcal{G}_{-m,\alpha}| \leq 2$ for any $\alpha \in \mathcal{I}\backslash\{m,-m\}$. On the other hand, independently of the value of $|\mathcal{G}_{-m}|$, taking into account Lemma 2.2 we get $|\mathcal{G}_{-m,\alpha}| \leq 3$ for any $\alpha \in \mathcal{I}\backslash\{m,-m\}$. So, for any possible value of $|\mathcal{G}_{-m}|$ we must impose $|\mathcal{G}_{-m}\backslash\mathcal{G}_{-i}| \geq 3$. As we have said before, we give preference to the codewords which do not have $-i$ in their index distributions since we do not have any information about $-i$ being more difficult to get contradictions. Thus, we begin by analyzing possible index distributions for the codewords of $\mathcal{G}_{-m}\backslash\mathcal{G}_{-i}$.

Taking into account the partition $\mathcal{X}$, see (6.16), all codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ and Lemma 1.5, if $W \in \mathcal{G}_{-m}\backslash\mathcal{G}_{-i}$, then W must satisfies one of the following index distributions:

$-m$	k	$-n$	l	$-o$
$-m$	k	$-n$	j	o
$-m$	$-o$	$-j$	$-k$	n
$-m$	$-o$	$-j$	$-l$	$-n$

Table 6.102: Possible index distributions for $W \in \mathcal{G}_{-m}\backslash\mathcal{G}_{-i}$.

From the analysis of the Table 6.102, if $W \in \mathcal{G}_{-m}\backslash\mathcal{G}_{-i}$, then $W \in \mathcal{G}_{-m,k,-n} \cup \mathcal{G}_{-m,-o,-j}$. Thus, by Lemma 1.5 we get $|\mathcal{G}_{-m}\backslash\mathcal{G}_{-i}| \leq 2$, which is a contradiction. Therefore, the considered index distribution for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ does not satisfy the definition of PL(7,2) code.

This is an example in which, considering one other element $\alpha \in \mathcal{I}\backslash\{i\}$, it is not possible to characterize all codewords of $\mathcal{G}_\alpha$, concluding immediately that such index distribution for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ contradicts the definition of PL(7, 2) code. However, in many cases, when considered another element $\alpha \in \mathcal{I}\backslash\{i\}$, we can describe completely all codewords of $\mathcal{G}_\alpha$ being necessary to analyze what happens with the codewords of $\mathcal{F}_\alpha$, as we will see in the next example.

Example 2

Now consider the following index distribution for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$:

W_1	i	j	k	l	m
W_2	i	j	$-k$	$-l$	n
W_3	i	$-j$	o	n	l
W_4	i	$-j$	$-o$	$-m$	$-l$
W_5	i	o	$-n$	$-k$	m

U_1	i	j	o	$-m$
U_2	i	j	$-o$	$-n$
U_3	i	l	$-m$	$-n$
U_4	i	l	$-k$	$-o$
U_5	i	n	m	$-o$
U_6	i	n	k	$-m$
U_7	i	o	k	$-l$
U_8	i	$-j$	k	$-n$

Table 6.103: Index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$.

As in the previous example, let us analyze an element $\alpha \in \mathcal{I}\backslash\{i\}$ for which $|\mathcal{G}_{i\alpha}|$ is minimum. Looking at Table 6.103, we verify that $|\mathcal{G}_{i\alpha}| \geq 1$ for any $\alpha \in \mathcal{I}\backslash\{i, -i\}$. So, let us consider, for example, $-o \in \mathcal{I}$ for which $|\mathcal{G}_{i,-o}| = 1$ and $|\mathcal{F}_{i,-o}| = 3$.

By Corollary 5.1, $5 \leq |\mathcal{G}_{-o}| \leq 7$. Since $|\mathcal{G}_{i,-o}| = 1$, we have to characterize, at least, four more codewords in $\mathcal{G}_{-o}$. As in the previous example, independently of the value of $|\mathcal{G}_{-o}|$, we have $|\mathcal{G}_{-o}\backslash\mathcal{G}_{-i}| \geq 3$. As $W_4 \in \mathcal{G}_{-o}\backslash\mathcal{G}_{-i}$, we must identify, at least, two more codewords in $\mathcal{G}_{-o}\backslash\mathcal{G}_{-i}$. For that, we will consider the partition $\mathcal{X}$ of $\mathcal{I}\backslash\{i, o, -o\}$ induced by the codewords $W_4 \in \mathcal{G}_{i,-o}$ and $U_2, U_4, U_5 \in \mathcal{F}_{i,-o}$:

$$\mathcal{X}_1 = \{-j, -m, -l\};\ \mathcal{X}_2 = \{j, -n\};\ \mathcal{X}_3 = \{l, -k\};\ \mathcal{X}_4 = \{n, m\};\ \mathcal{X}_5 = \{k\};\ \mathcal{X}_6 = \{-i\}. \tag{6.17}$$

Taking into account this partition, the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ as well as Lemma 1.5, we conclude that $|\mathcal{G}_{-o}\backslash\mathcal{G}_{-i}| = 3$ with $W_6, W_7 \in \mathcal{G}_{-o}\backslash\mathcal{G}_{-i}$ satisfying one of the following index distributions:

W_6	$-o$	$-j$	$-n$	l	m
W_7	$-o$	$-m$	j	l	n

W_6	$-o$	$-m$	j	l	n
W_7	$-o$	k	$-l$	$-n$	m

Table 6.104: Possible index distributions for $W_6, W_7 \in \mathcal{G}_{-o}\backslash\mathcal{G}_{-i}$.

To complete the characterization of all codewords of $\mathcal{G}_{-o}$, for each one of the presented hypotheses we will identify possible codewords in $\mathcal{G}_{-o,-i}$. We note that, since $|\mathcal{G}_{-o}\backslash\mathcal{G}_{-i}| = 3$ and, by Lemma 2.2, $|\mathcal{G}_{-o,-i}| \leq 3$, it follows that $5 \leq |\mathcal{G}_{-o}| \leq 6$.

Suppose that $W_6 \in \mathcal{G}_{-o,-j,-n,l,m}$ and $W_7 \in \mathcal{G}_{-o,-m,j,l,n}$.

Let us assume $|\mathcal{G}_{-o}| = 5$. In these conditions, we must identify two codewords $W_8, W_9 \in \mathcal{G}_{-o,-i}$. In the characterization of $W_8, W_9 \in \mathcal{G}_{-o,-i}$ we must have into account Theorem 6.1, that is, the index distribution of W_8 and W_9 must be such that $|\mathcal{G}_{-o,\alpha}| \leq 2$ for any $\alpha \in \mathcal{I}\backslash\{o,-o\}$. Considering again the partition $\mathcal{X}$, see (6.17), the codewords already known and Lemma 1.5, the codewords $W_8, W_9 \in \mathcal{G}_{-o,-i}$ must verify one of the following conditions:

W_8	$-o$	$-i$	j	$-k$	m
W_9	$-o$	$-i$	k	$-l$	n

W_8	$-o$	$-i$	j	$-k$	m
W_9	$-o$	$-i$	k	$-l$	$-n$

Table 6.105: Possible index distributions for $W_8, W_9 \in \mathcal{G}_{-o,-i}$.

If we suppose $|\mathcal{G}_{-o}| = 6$, then $W_8, W_9, W_{10} \in \mathcal{G}_{-o,-i}$ must verify one of the conditions presented in Table 6.106.

W_8	$-o$	$-i$	$-m$	$-n$	$-k$
W_9	$-o$	$-i$	$-l$	j	m
W_{10}	$-o$	$-i$	k	$-j$	n

W_8	$-o$	$-i$	k	$-m$	$-n$
W_9	$-o$	$-i$	$-l$	j	m
W_{10}	$-o$	$-i$	$-j$	$-k$	n

Table 6.106: Index distributions for $W_8, W_9, W_{10} \in \mathcal{G}_{-o,-i}$.

If we consider $W_6, W_7 \in \mathcal{G}_{-o} \backslash \mathcal{G}_{-i}$ so that $W_6 \in \mathcal{G}_{-o,-m,j,l,n}$ and $W_7 \in \mathcal{G}_{-o,k,-l,-n,m}$, we necessarily have $|\mathcal{G}_{-o}| = 5$ with $W_8, W_9 \in \mathcal{G}_{-o,-i}$ satisfying one of the following index distributions:

W_8	$-o$	$-i$	$-j$	$-n$	l
W_9	$-o$	$-i$	j	$-k$	m

W_8	$-o$	$-i$	j	$-k$	m
W_9	$-o$	$-i$	k	$-j$	l

W_8	$-o$	$-i$	j	$-k$	m
W_9	$-o$	$-i$	k	$-j$	n

Table 6.107: Possible index distributions for $W_8, W_9 \in \mathcal{G}_{-o,-i}$.

Therefore, for the considered index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ there exist several possible index distributions for the codewords of $\mathcal{G}_{-o}$. Next step consist in to complete the characterization of all codewords of $\mathcal{F}_{-o}$ for each one of the obtained index distributions for $\mathcal{G}_{-o}$. We recall that, by Lemmas 2.11 and 2.12, we know, respectively:

– if $|\mathcal{G}_{-o}| = 5$, then $7 \leq |\mathcal{F}_{-o}| \leq 10$; furthermore, if $|\mathcal{F}_{-o}| = 7$, then $|\mathcal{F}_{-o}^{(2)}| = 4$;

– if $|\mathcal{G}_{-o}| = 6$, then $4 \leq |\mathcal{F}_{-o}| \leq 8$; furthermore, if $|\mathcal{F}_{-o}| = 4$, then $|\mathcal{F}_{-o}^{(2)}| = 4$.

Let us suppose that the codewords of $\mathcal{G}_{-o}\backslash\mathcal{G}_i$ satisfy:

W_6	$-o$	$-j$	$-n$	l	m
W_7	$-o$	$-m$	j	l	n
W_8	$-o$	$-i$	j	$-k$	m
W_9	$-o$	$-i$	k	$-l$	n

Table 6.108: Index distribution of the codewords of $\mathcal{G}_{-o}\backslash\mathcal{G}_i$.

We are assuming $|\mathcal{G}_{-o}| = 5$, accordingly, $7 \leq |\mathcal{F}_{-o}| \leq 10$. As $|\mathcal{F}_{i,-o}| = 3$, we must identify, at least, four more codewords in $\mathcal{F}_{-o}$. We can identify the index distributions of the codewords of $\mathcal{F}_{-o}$ applying the same strategy used in the previous subsection in the characterization of the codewords of $\mathcal{F}_i$, however, since we know the index distribution of many codewords we can do it easily recurring to the following schemes, where all possible index distributions for codewords of $\mathcal{F}_{-o}$ are presented, taking into account all codewords of $\mathcal{G}_i \cup \mathcal{F}_i \cup \mathcal{G}_{-o}$ already known and Lemma 1.5:

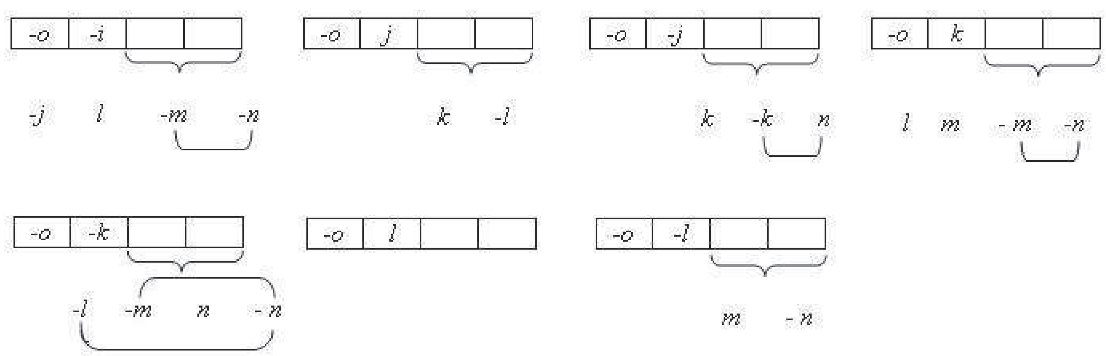

Figure 6.24: Possible index distributions for $U \in \mathcal{F}_{-o}$.

By the analysis of the above schemes, if $U \in \mathcal{F}_{-o}\backslash\mathcal{F}_i$, then

$$U \in \mathcal{F}_{-o,-m,-n} \cup \mathcal{F}_{-o,-j,-k,n} \cup \mathcal{F}_{-o,-k,-l,-n}.$$

Taking into account Lemma 1.5 we conclude that $|\mathcal{F}_{-o}\backslash\mathcal{F}_i| \leq 3$ which implies $|\mathcal{F}_{-o}| \leq 6$, contradicting Lemma 2.11. Thus, the considered index distribution for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i \cup \mathcal{G}_{-o}$ contradicts the definition of perfect error correcting Lee code.

Now suppose that $|\mathcal{G}_{-o}| = 6$, with $W_6, \ldots, W_{10} \in \mathcal{G}_{-o} \backslash \mathcal{G}_i$ such that:

W_6	$-o$	$-j$	$-n$	l	m
W_7	$-o$	$-m$	j	l	n
W_8	$-o$	$-i$	$-m$	$-n$	$-k$
W_9	$-o$	$-i$	$-l$	j	m
W_{10}	$-o$	$-i$	k	$-j$	n

Table 6.109: Index distribution of the codewords of $\mathcal{G}_{-o} \backslash \mathcal{G}_i$.

Since $|\mathcal{F}_{i,-o}| = 3$, by Lemma 2.12 we must identify in $\mathcal{F}_{-o}$, at least, one more codeword. Following a similar reasoning to the one described in Figure 6.24, we conclude that if $U \in \mathcal{F}_{-o} \backslash \mathcal{F}_i$, then $U \in \mathcal{F}_{-o,k,-l,-n}$. That is, $|\mathcal{F}_{-o}| = 4$. Accordingly, from Lemma 2.12 it follows that $|\mathcal{F}^{(2)}_{-o}| = 4$. As $\mathcal{F}_{-o} = \mathcal{F}_{-o,i} \cup \mathcal{F}_{-o,k,-l,-n}$, taking into account Lemma 2.14, we verify that the condition $|\mathcal{F}^{(2)}_{-o}| = 4$ cannot be satisfied, which is a contradiction. Therefore, similarly to the previous case, the index distribution considered for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i \cup \mathcal{G}_{-o}$ contradicts the definition of PL$(7,2)$ code.

In the previous index distributions for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i \cup \mathcal{G}_{-o}$, the complete description of the codewords of $\mathcal{F}_{-o}$ leads to contradictions on necessary conditions for the existence of PL$(7,2)$ codes. Next, we present an example in which we can characterize all codewords of $\mathcal{F}_{-o}$ being necessary to analyze one other element $\alpha \in \mathcal{I} \backslash \{i, -o\}$.

Let $W_6, \ldots, W_{10} \in \mathcal{G}_{-o} \backslash \mathcal{G}_i$ satisfying:

W_6	$-o$	$-j$	$-n$	l	m
W_7	$-o$	$-m$	j	l	n
W_8	$-o$	$-i$	k	$-m$	$-n$
W_9	$-o$	$-i$	$-l$	j	m
W_{10}	$-o$	$-i$	$-j$	$-k$	n

Table 6.110: Index distribution of the codewords of $\mathcal{G}_{-o} \backslash \mathcal{G}_i$.

As $|\mathcal{G}_{-o}| = 6$ and $|\mathcal{F}_{-o,i}| = 3$, considering Lemma 2.12, we must identify, at least, one codeword in $\mathcal{F}_{-o} \backslash \mathcal{F}_i$. Taking into account all codewords of $\mathcal{G}_i \cup \mathcal{F}_i \cup \mathcal{G}_{-o}$

already known as well as Lemma 1.5, we conclude that if $U \in \mathcal{F}_{-o}\backslash\mathcal{F}_i$, then $U \in \mathcal{F}_{-o,k,-l,n} \cup \mathcal{F}_{-o,-k,-l,-n}$. Thus, $4 \leq |\mathcal{F}_{-o}| \leq 5$. Supposing $|\mathcal{F}_{-o}| = 4$, as in the previous case, the condition $|\mathcal{F}_{-o}^{(2)}| = 4$ is not satisfied. Then, we must impose $|\mathcal{F}_{-o}| = 5$, with $U_8 \in \mathcal{F}_{-o,k,-l,n}$ and $U_9 \in \mathcal{F}_{-o,-k,-l,-n}$.

Until now we have characterize a possible index distribution for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i \cup \mathcal{G}_{-o} \cup \mathcal{F}_{-o}$ without contradictions on the definition of PL$(7,2)$ code. So, in this case, we must analyze what happens when another element $\alpha \in \mathcal{I}\backslash\{i,-o\}$ is considered.

We note that, considering the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$, see Table 6.103, $-m \in \mathcal{I}$ is such that $|\mathcal{G}_{i,-m}| = 1$ and $|\mathcal{F}_{i,-m}| = 3$. Furthermore, considering the codewords of $\mathcal{G}_{-o}\backslash\mathcal{G}_i$, see Table 6.110, we verify that $W_7, W_8 \in \mathcal{G}_{-o,-m}$. Let us consider $\mathcal{G}_{-m}$. As in the previous examples, we must impose $5 \leq |\mathcal{G}_{-m}| \leq 7$ with $|\mathcal{G}_{-m}\backslash\mathcal{G}_{-i}| \geq 3$. Taking into account the known codewords of $\mathcal{G}_{-m}$, we verify that only one of them is in $\mathcal{G}_{-m,-i}$, thus, we must identify in $\mathcal{G}_{-m}\backslash\mathcal{G}_{-i}$, at least, one codeword. For that, we will consider the following partition $\mathcal{Y}$ of $\mathcal{I}\backslash\{i,m,-m\}$ induced by $W_4 \in \mathcal{G}_{i,-m}$ and $U_1, U_3, U_6 \in \mathcal{F}_{i,-m}$:

$$\mathcal{Y}_1 = \{-j,-o,-l\};\ \ \mathcal{Y}_2 = \{j,o\};\ \ \mathcal{Y}_3 = \{l,-n\};\ \ \mathcal{Y}_4 = \{n,k\};\ \ \mathcal{Y}_5 = \{-k\};\ \ \mathcal{Y}_6 = \{-i\}. \tag{6.18}$$

Taking into account the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i \cup \mathcal{G}_{-o} \cup \mathcal{F}_{-o}$, Lemma 1.5 and the partition $\mathcal{Y}$ described above, we conclude that it is not possible characterize another codeword in $\mathcal{G}_{-m}\backslash\mathcal{G}_{-i}$ without contradictions on the definition of PL$(7,2)$ code.

Considering any other of the presented index distributions for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i \cup \mathcal{G}_{-o}$, and applying a similar reasoning, we verify that each one of them contradicts necessary conditions for the existence of PL$(7,2)$ codes.

In this section we have presented few examples to describe how to show that a possible index distribution for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ contradicts the definition of PL$(7,2)$ code. Although we have obtained many possible index distributions for the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$, applying the same strategy presented before, we have shown that each one of them implies contradictions on the definition of PL$(7,2)$ code. In fact, for any index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ there exists always an element

$\alpha \in \mathcal{I}\backslash\{i\}$ whose characterization of all codewords of $\mathcal{G}_\alpha \cup \mathcal{F}_\alpha$ contradicts necessary conditions for the existence of perfect error correcting Lee codes.

Therefore, we conclude that the condition $|\mathcal{G}_i| = 5$ and $|\mathcal{G}_{i\alpha}| \leq 2$ for any $\alpha \in \mathcal{I}\backslash\{i, -i\}$ contradicts the definition of perfect error correcting Lee code. Taking into account Theorem 6.1, we get immediately the following theorem:

Theorem 6.2 *For any $\alpha \in \mathcal{I}$, $|\mathcal{G}_\alpha| \neq 5$.*

As an immediate consequence of Theorem 6.2 and Corollary 5.1 we get:

Corollary 6.1 *For any $\alpha \in \mathcal{I}$, $6 \leq |\mathcal{G}_\alpha| \leq 7$.*

Chapter 7

Non-existence of PL$(7, 2)$ codes

7.1 Conclusion of the proof of the non-existence of PL$(7, 2)$ codes

In the previous chapters we have concluded, firstly, that if there exists a PL$(7, 2)$ code $\mathcal{M}$, then $\mathcal{G} \subset \mathcal{M}$ is such that $3 \leq |\mathcal{G}_\alpha| \leq 8$ for any $\alpha \in \mathcal{I}$. Later, we have analyzed, separately and by this order, the hypotheses $|\mathcal{G}_\alpha| = 8$, $|\mathcal{G}_\alpha| = 3$, $|\mathcal{G}_\alpha| = 4$ and $|\mathcal{G}_\alpha| = 5$, for $\alpha \in \mathcal{I}$, having verified that each one of them contradicts necessary conditions for the existence of PL$(7, 2)$ codes, concluding that the existence of such codes imposes $6 \leq |\mathcal{G}_\alpha| \leq 7$ for any $\alpha \in \mathcal{I}$.

Here, we show that the assumption $6 \leq |\mathcal{G}_\alpha| \leq 7$, for any $\alpha \in \mathcal{I}$, leads us to contradictions, proving thus the non-existence of PL$(7, 2)$ codes.

We begin by presenting results which will help us to characterize the index distribution of the codewords of $\mathcal{G}_\alpha$, $\alpha \in \mathcal{I}$.

Proposition 7.1 *There exists $\alpha \in \mathcal{I}$ such that $|\mathcal{G}_\alpha| = 7$. Furthermore, if $|\mathcal{G}_\alpha| = 7$, for some $\alpha \in \mathcal{I}$, then there exist, at least, four elements $\beta \in \mathcal{I} \backslash \{\alpha, -\alpha\}$ satisfying $|\mathcal{G}_{\alpha\beta}| = 3$.*

Proof. By Corollary 6.1 we know that $6 \leq |\mathcal{G}_\alpha| \leq 7$ for any $\alpha \in \mathcal{I}$.

We recall that

$$g = |\mathcal{G}| = \frac{1}{5} \sum_{\alpha \in \mathcal{I}} |\mathcal{G}_\alpha|. \tag{7.1}$$

Let us suppose, by contradiction, that $|\mathcal{G}_\alpha| = 6$ for any $\alpha \in \mathcal{I}$. As $|\mathcal{I}| = 14$, by (7.1) we conclude that $g = \frac{84}{5}$, which it is not possible since g must be an integer number. Therefore, there exists $\alpha \in \mathcal{I}$ such that $|\mathcal{G}_\alpha| = 7$.

Let $\alpha \in \mathcal{I}$ be such that $|\mathcal{G}_\alpha| = 7$. We note that $|\mathcal{G}_\alpha| = \frac{1}{4} \sum_{\beta \in \mathcal{I}\backslash\{\alpha,-\alpha\}} |\mathcal{G}_{\alpha\beta}|$, that is,

$$\sum_{\beta \in \mathcal{I}\backslash\{\alpha,-\alpha\}} |\mathcal{G}_{\alpha\beta}| = 28. \tag{7.2}$$

From Lemma 2.2 it follows that $|\mathcal{G}_{\alpha\beta}| \leq 3$ for any $\beta \in \mathcal{I}\backslash\{\alpha, -\alpha\}$. If we suppose, by contradiction, that, at most, there are three elements $\beta \in \mathcal{I}\backslash\{\alpha, -\alpha\}$ satisfying $|\mathcal{G}_{\alpha\beta}| = 3$, then from (7.2) it follows that $\sum_{\beta \in \mathcal{I}\backslash\{\alpha,-\alpha\}} |\mathcal{G}_{\alpha\beta}| \leq 27$, facing up a contradiction. Accordingly, there are, at least, four elements $\beta \in \mathcal{I}\backslash\{\alpha, -\alpha\}$ such that $|\mathcal{G}_{\alpha\beta}| = 3$. □

Consider $\mathcal{I} = \{i, -i, j, -j, k, -k, l, -l, m, -m, n, -n, o, -o\}$. Taking into account the previous proposition, let us assume $|\mathcal{G}_i| = 7$ and $|\mathcal{G}_{ij}| = 3$. From Proposition 6.2 it follows that the codewords $W_1, W_2, W_3 \in \mathcal{G}_{ij}$ satisfy the following index distribution:

W_1	i	j	k	l	m
W_2	i	j	$-k$	$-l$	n
W_3	i	j	$-m$	$-n$	o

Table 7.1: Index distribution of the codewords of $\mathcal{G}_{ij}$.

The index distribution of the codewords of $\mathcal{G}_{ij}$ induces the following partition $\mathcal{P}$ of $\mathcal{I}\backslash\{i, -i, j\}$:

$$\mathcal{P}_1 = \{k, l, m\};\ \ \mathcal{P}_2 = \{-k, -l, n\};\ \ \mathcal{P}_3 = \{-m, -n, o\};\ \ \mathcal{P}_4 = \{-j\};\ \ \mathcal{P}_5 = \{-o\}. \tag{7.3}$$

Having in view Proposition 7.1 and the partition of $\mathcal{P}$, next result imposes conditions on the elements $\alpha \in \mathcal{I}\backslash\{i, -i, j\}$ which satisfy $|\mathcal{G}_{i\alpha}| = 3$.

Proposition 7.2 *There are, at least, two elements* $\alpha, \beta \in \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ *satisfying* $|\mathcal{G}_{i\alpha}| = |\mathcal{G}_{i\beta}| = 3$.

Proof. By Proposition 7.1 we know that there are, at least, three elements $\alpha \in \mathcal{I} \backslash \{i, -i, j\}$ satisfying $|\mathcal{G}_{i\alpha}| = 3$.

Let us suppose that $|\mathcal{G}_{i,-j}| = |\mathcal{G}_{i,-o}| = 3$. Since, by Lemma 1.5, $|\mathcal{G}_{i,-j,-o}| \leq 1$, then $|\mathcal{G}_{i,-j} \cup \mathcal{G}_{i,-o}| \geq 5$. As $\mathcal{G}_{ij} \cap (\mathcal{G}_{i,-j} \cup \mathcal{G}_{i,-o}) = \varnothing$ and $\mathcal{G}_{ij} \cup \mathcal{G}_{i,-j} \cup \mathcal{G}_{i,-o} \subset \mathcal{G}_i$, then $|\mathcal{G}_i| \geq 8$, which is a contradiction.

Thus, there are, at least, two elements $\alpha, \beta \in \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ such that $|\mathcal{G}_{i\alpha}| = |\mathcal{G}_{i\beta}| = 3$.
□

By Proposition 7.2, let us consider $\alpha \in \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ such that $|\mathcal{G}_{i\alpha}| = 3$. Analyzing the partition $\mathcal{P}$ of $\mathcal{I} \backslash \{i, -i, j\}$, see (7.3), we distinguish, without loss of generality, the hypotheses:

- $\alpha = k$;
- $\alpha = m$;
- $\alpha = -m$;
- $\alpha = o$.

Our aim is to characterize all possible index distributions for the codewords of $\mathcal{G}_i$. For that, we will analyze each one of the referred hypotheses.

Let us suppose that $\alpha = k$, that is, $|\mathcal{G}_{ik}| = 3$. Since $W_1 \in \mathcal{G}_{ik}$, then to complete the characterization of the codewords of $\mathcal{G}_{ik}$ we must describe the index distribution of two more codewords of $\mathcal{G}_{ik}$. Taking into account the partition $\mathcal{P}$ of $\mathcal{I} \backslash \{i, -i, j\}$ and Lemma 1.5, we obtain all possible index distributions for $W_4, W_5 \in \mathcal{G}_{ik}$, see Table 7.2. We note that, in the table, W_4 on the left is matching to W_5 presented on the right.

W_4	i	k	$-j$	$-l$	$-n$
W_4	i	k	$-j$	$-l$	o
W_4	i	k	$-j$	n	$-m$
W_4	i	k	$-j$	n	o
W_4	i	k	$-j$	n	o

W_5	i	k	$-o$	$-m$	n
W_5	i	k	$-o$	$-m$	n
W_5	i	k	$-o$	$-n$	$-l$
W_5	i	k	$-o$	$-m$	$-l$
W_5	i	k	$-o$	$-n$	$-l$

Table 7.2: Possible index distributions for $W_4, W_5 \in \mathcal{G}_{ik}$.

If we consider $\alpha = m$, that is, $|\mathcal{G}_{im}| = 3$, such as in the previous case, having in view the partition $\mathcal{P}$ and Lemma 1.5, we get the following hypotheses for the index distribution of $W_4, W_5 \in \mathcal{G}_{im}$:

W_4	i	m	$-j$	o	$-k$
W_4	i	m	$-j$	o	$-l$
W_4	i	m	$-j$	o	n
W_4	i	m	$-j$	o	n

W_5	i	m	$-o$	$-n$	$-l$
W_5	i	m	$-o$	$-n$	$-k$
W_5	i	m	$-o$	$-n$	$-k$
W_5	i	m	$-o$	$-n$	$-l$

Table 7.3: Possible index distributions for $W_4, W_5 \in \mathcal{G}_{im}$.

Supposing that $|\mathcal{G}_{i,-m}| = 3$, proceeding as in the previous cases we get the following possible index distributions for $W_4, W_5 \in \mathcal{G}_{i,-m}$:

W_4	i	$-m$	$-j$	k	$-l$
W_4	i	$-m$	$-j$	k	$-l$
W_4	i	$-m$	$-j$	k	n
W_4	i	$-m$	$-j$	l	$-k$
W_4	i	$-m$	$-j$	l	$-k$
W_4	i	$-m$	$-j$	l	n

W_5	i	$-m$	$-o$	l	$-k$
W_5	i	$-m$	$-o$	l	n
W_5	i	$-m$	$-o$	l	$-k$
W_5	i	$-m$	$-o$	k	$-l$
W_5	i	$-m$	$-o$	k	n
W_5	i	$-m$	$-o$	k	$-l$

Table 7.4: Possible index distributions for $W_4, W_5 \in \mathcal{G}_{i,-m}$.

If we suppose $|\mathcal{G}_{io}| = 3$, following the same reasoning applied in the analysis of the previous cases, we conclude that the characterization of the index distribution of the remaining codewords of $\mathcal{G}_{io}$ contradicts Lemma 1.5. Therefore, $|\mathcal{G}_{io}| \leq 2$.

Thus, to characterize the possible index distributions of all codewords of $\mathcal{G}_i$ we consider, separately, the hypotheses $|\mathcal{G}_{ik}| = 3$, $|\mathcal{G}_{im}| = 3$ and $|\mathcal{G}_{i,-m}| = 3$, with $W_4, W_5 \in \mathcal{G}_i$ satisfying each one of the conditions presented, respectively, in Tables 7.2, 7.3 and 7.4. We note that, in any case we have characterized the index distribution of five codewords of $\mathcal{G}_i$. Since $|\mathcal{G}_i| = 7$, we must describe the index distribution of two more codewords of $\mathcal{G}_i$.

As an illustrative example let us consider $|\mathcal{G}_{ik}| = 3$, with $W_4 \in \mathcal{G}_{i,k,-j,-l,-n}$ and $W_5 \in \mathcal{G}_{i,k,-o,-m,n}$. Taking into account the partition $\mathcal{P}$ of $\mathcal{I}\backslash\{i, -i, j\}$, see (7.3), the index distribution of all known codewords and Lemma 1.5, the remaining codewords of $\mathcal{G}_i$, that is, $W_6, W_7 \in \mathcal{G}_i\backslash(\mathcal{G}_j \cup \mathcal{G}_k)$, satisfy, respectively, one of the following conditions:

W_6	i	$-j$	l	$-k$	$-m$
W_6	i	$-j$	l	$-k$	$-m$
W_6	i	$-j$	l	$-k$	o
W_6	i	$-j$	l	n	o
W_6	i	$-j$	l	n	o
W_6	i	$-j$	l	n	o
W_6	i	$-j$	m	o	$-k$
W_6	i	$-j$	m	o	n
W_6	i	$-j$	m	o	n
W_6	i	$-j$	m	o	n

W_7	i	$-j$	m	o	n
W_7	i	$-o$	$-n$	$-k$	m
W_7	i	$-o$	$-n$	$-k$	m
W_7	i	$-o$	$-n$	$-k$	l
W_7	i	$-o$	$-n$	$-k$	m
W_7	i	$-o$	$-j$	m	$-k$
W_7	i	$-o$	$-n$	$-k$	l
W_7	i	$-o$	$-n$	$-k$	l
W_7	i	$-o$	$-n$	$-k$	m
W_7	i	$-o$	$-j$	l	$-k$

Table 7.5: Possible index distributions for $W_6, W_7 \in \mathcal{G}_i\backslash(\mathcal{G}_j \cup \mathcal{G}_k)$.

We note that, in the above table the codeword W_6, on the left, is matching to the respective codeword W_7 presented on the right.

Until now we have characterized all possible index distributions for the codewords of $\mathcal{G}_i$. Next step could consist, as in other studied cases, into identify for each one of the presented hypotheses the index distribution of the codewords of $\mathcal{F}_i$. Since, by Lemma 2.13, $2 \leq |\mathcal{F}_i| \leq 5$, to characterize completely $\mathcal{F}_i$ we have to present the index distribution of, at least, two codewords. Accordingly, the characterization of $\mathcal{F}_i$ will not bring difficulties due to the small minimal number of codewords in $\mathcal{F}_i$. In fact, if we consider, for instance, $W_6 \in \mathcal{G}_{i,-j,l,-k,-m}$ and $W_7 \in \mathcal{G}_{i,-j,m,o,n}$, then taking into account the partition $\mathcal{P}$ of $\mathcal{I}\backslash\{i,-i,j\}$, the index distribution of all known codewords and Lemma 1.5, we verify that, in this case, we have $|\mathcal{F}_i| = 3$ or $|\mathcal{F}_i| = 2$. If $|\mathcal{F}_i| = 3$, then the index distribution of $U_1, U_2, U_3 \in \mathcal{F}_i$ must satisfy: $U_1 \in \mathcal{F}_{i,-n,-o,l}$; $U_2 \in \mathcal{F}_{i,-k,m,-n}$; $U_3 \in \mathcal{F}_{i,-l,m,-o}$. On the other hand, if $|\mathcal{F}_i| = 2$, then $U_1, U_2 \in \mathcal{F}_i$ must verify one of the following conditions:

U_1	i	$-l$	m	$-o$
U_1	i	$-l$	m	$-o$
U_1	i	$-l$	m	$-o$
U_1	i	$-k$	m	$-n$
U_1	i	$-k$	m	$-n$
U_1	i	$-k$	m	$-o$

U_2	i	$-k$	m	$-n$
U_2	i	$-k$	$-n$	$-o$
U_2	i	l	$-n$	$-o$
U_2	i	l	$-n$	$-o$
U_2	i	$-l$	m	$-o$
U_2	i	l	$-n$	$-o$

Table 7.6: Possible index distributions for $U_1, U_2 \in \mathcal{F}_i$.

As we have seen in the previous example, we can identify different possible index distributions for the codewords of $\mathcal{F}_i$ without contradict the definition of PL(7, 2) code. Since our aim is to show that each one of the possible index distributions for the codewords of $\mathcal{G}_i$ does not satisfy necessary conditions for the existence of these codes, instead of studying the set $\mathcal{F}_i$, we will focus our attention on other sets $\mathcal{G}_\alpha$ for $\alpha \in \mathcal{I}\backslash\{i\}$.

To show how we analyze other sets $\mathcal{G}_\alpha$, with $\alpha \in \mathcal{I}\backslash\{i\}$, having in view achieving contradictions, we will consider in what follows, as an illustrative example, $W_6, W_7 \in \mathcal{G}_i\backslash(\mathcal{G}_j \cup \mathcal{G}_k)$ satisfying: $W_6 \in \mathcal{G}_{i,-j,l,-k,-m}$ and $W_7 \in \mathcal{G}_{i,-j,m,o,n}$. The analysis of the remaining hypotheses presented in Table 7.5 is similar.

Thus, let us consider $\mathcal{G}_i$, with $W_1, \ldots, W_7 \in \mathcal{G}_i$ satisfying the index distribution presented in Table 7.7.

W_1	i	j	k	l	m
W_2	i	j	$-k$	$-l$	n
W_3	i	j	$-m$	$-n$	o
W_4	i	k	$-j$	$-l$	$-n$
W_5	i	k	$-o$	$-m$	n
W_6	i	$-j$	l	$-k$	$-m$
W_7	i	$-j$	m	o	n

Table 7.7: Index distribution of the codewords of $\mathcal{G}_i$.

We note that, in this case, we have $|\mathcal{G}_{ij}| = |\mathcal{G}_{i,-j}| = |\mathcal{G}_{ik}| = |\mathcal{G}_{i,-m}| = |\mathcal{G}_{in}| = 3$. Then, in the choice of the elements $\alpha \in \mathcal{I}\backslash\{i\}$ for which we will analyze $\mathcal{G}_\alpha$ we will give preference to these elements, that is, $\alpha \in \{j, -j, k, -m, n\}$.

We recall that, by Corollary 6.1, $6 \le |\mathcal{G}_\alpha| \le 7$ for any $\alpha \in \mathcal{I}$.

Let us begin by characterizing $\mathcal{G}_k$. Since, at this moment, we know the index distribution of only three codewords of $\mathcal{G}_k$, $W_1, W_4, W_5 \in \mathcal{G}_{ik}$, then we must characterize, at least, three more codewords of $\mathcal{G}_k$. For that, we will consider the partition $\mathcal{K}$ of $\mathcal{I}\backslash\{i, k, -k\}$ induced by $W_1, W_4, W_5 \in \mathcal{G}_{ik}$:

$$\mathcal{K}_1 = \{j, l, m\};\ \ \mathcal{K}_2 = \{-j, -l, -n\};\ \ \mathcal{K}_3 = \{-o, -m, n\};\ \ \mathcal{K}_4 = \{o\};\ \ \mathcal{K}_5 = \{-i\}.$$

Taking into account the partition $\mathcal{K}$, the index distribution of the codewords of $\mathcal{G}_i$ and Lemma 1.5, we verify that $|\mathcal{G}_k\backslash\mathcal{G}_i| = 3$ and $W_8, W_9, W_{10} \in \mathcal{G}_k\backslash\mathcal{G}_i$ must satisfy one of the conditions presented in the following tables:

W_8					W_9					W_{10}				
k	-i	j	-l	-o	k	-i	l	-j	n	k	-i	o	m	-n
k	-i	j	-l	-m	k	-i	l	-j	-o	k	-i	o	m	-n
					k	-i	l	-j	n	k	-i	o	m	-n
										k	-i	m	-o	-n
					k	-i	m	-o	-j	k	-i	o	l	-n
										k	-i	o	n	l
					k	-i	m	-o	-n	k	-i	o	l	-j
										k	-i	o	n	l

Table 7.8: Possible index distributions for $W_8, W_9, W_{10} \in \mathcal{G}_k\backslash\mathcal{G}_i$.

<table>
<tr><th colspan="5">W8</th><th colspan="5">W9</th><th colspan="5">W10</th></tr>
<tr><td rowspan="5">k</td><td rowspan="5">-i</td><td rowspan="5">j</td><td rowspan="5">-n</td><td rowspan="5">-o</td><td rowspan="2">k</td><td rowspan="2">-i</td><td rowspan="2">l</td><td rowspan="2">-j</td><td rowspan="2">n</td><td>k</td><td>-i</td><td>o</td><td>m</td><td>-l</td></tr>
<tr><td>k</td><td>-i</td><td>o</td><td>-m</td><td>-l</td></tr>
<tr><td rowspan="3">k</td><td rowspan="3">-i</td><td rowspan="3">m</td><td rowspan="3">n</td><td rowspan="3">-l</td><td>k</td><td>-i</td><td>o</td><td>l</td><td>-j</td></tr>
<tr><td>k</td><td>-i</td><td>o</td><td>-m</td><td>l</td></tr>
<tr><td>k</td><td>-i</td><td>o</td><td>-m</td><td>-j</td></tr>
<tr><td rowspan="2">k</td><td rowspan="2">-i</td><td rowspan="2">l</td><td rowspan="2">-j</td><td rowspan="2">n</td><td rowspan="2">k</td><td rowspan="2">-i</td><td rowspan="2">m</td><td rowspan="2">-o</td><td rowspan="2">-n</td><td>k</td><td>-i</td><td>o</td><td>j</td><td>-l</td></tr>
<tr><td>k</td><td>-i</td><td>o</td><td>-m</td><td>-l</td></tr>
<tr><td>k</td><td>-i</td><td>l</td><td>-n</td><td>-o</td><td>k</td><td>-i</td><td>m</td><td>n</td><td>-l</td><td>k</td><td>-i</td><td>o</td><td>-m</td><td>-j</td></tr>
<tr><td rowspan="4">k</td><td rowspan="4">-i</td><td rowspan="4">l</td><td rowspan="4">-n</td><td rowspan="4">-m</td><td rowspan="3">k</td><td rowspan="3">-i</td><td rowspan="3">m</td><td rowspan="3">-o</td><td rowspan="3">-j</td><td>k</td><td>-i</td><td>o</td><td>j</td><td>-l</td></tr>
<tr><td>k</td><td>-i</td><td>o</td><td>n</td><td>j</td></tr>
<tr><td>k</td><td>-i</td><td>o</td><td>n</td><td>-l</td></tr>
<tr><td>k</td><td>-i</td><td>m</td><td>-o</td><td>-l</td><td>k</td><td>-i</td><td>o</td><td>n</td><td>j</td></tr>
</table>

Table 7.9: Possible index distributions for $W_8, W_9, W_{10} \in \mathcal{G}_k \backslash \mathcal{G}_i$.

From the analysis of Tables 7.8 and 7.9 we verify that there are many possible index distributions for the codewords of $\mathcal{G}_k \backslash \mathcal{G}_i$. So, for each one of these hypotheses we proceed our study characterizing other sets $\mathcal{G}_\alpha$ with $\alpha \in \mathcal{I} \backslash \{i, k\}$. Since the reasoning applied in the analysis of each one of the conditions presented in Tables 7.8 and 7.9 is similar, we show how we have done it presenting three illustrative examples.

Example 1

Consider $W_8, W_9, W_{10} \in \mathcal{G}_k \backslash \mathcal{G}_i$ satisfying: $W_8 \in \mathcal{G}_{k,-i,j,-l,-o}$, $W_9 \in \mathcal{G}_{k,-i,l,-j,n}$ and $W_{10} \in \mathcal{G}_{k,-i,o,m,-n}$. Let us analyze $\mathcal{G}_{-m}$. Since $W_3, W_5, W_6 \in \mathcal{G}_{i,-m}$ are the unique codewords of $\mathcal{G}_{-m}$ already known, we must characterize, at least, three codewords of $\mathcal{G}_{-m} \backslash (\mathcal{G}_i \cup \mathcal{G}_k)$. Such as in the analysis of $\mathcal{G}_k$, we will consider the partition $\mathcal{Q}$ of $\mathcal{I} \backslash \{i, m, -m\}$ induced by W_3, W_5 and W_6:

$$\mathcal{Q}_1 = \{j, -n, o\}; \quad \mathcal{Q}_2 = \{k, -o, n\}; \quad \mathcal{Q}_3 = \{-j, l, -k\}; \quad \mathcal{Q}_4 = \{-l\}; \quad \mathcal{Q}_5\{-i\}. \qquad (7.4)$$

Taking into account this partition as well as the index distribution of all known codewords and Lemma 1.5, we conclude that it is not possible to characterize, at least, three codewords in $\mathcal{G}_{-m} \backslash (\mathcal{G}_i \cup \mathcal{G}_k)$ without contradictions.

Example 2

Let us consider $W_8, W_9, W_{10} \in \mathcal{G}_k \backslash \mathcal{G}_i$ such that: $W_8 \in \mathcal{G}_{k,-i,l,-n,-o}$, $W_9 \in \mathcal{G}_{k,-i,m,n,-l}$ and $W_{10} \in \mathcal{G}_{k,-i,o,-m,-j}$. We begin the analysis of this hypothesis by characterizing the remaining codewords of $\mathcal{G}_n$. We note that, at this moment are known four codewords of $\mathcal{G}_n$, $W_2, W_5, W_7 \in \mathcal{G}_{in}$ and $W_9 \in \mathcal{G}_{k,-i,m,n,-l}$. Then, we must characterize, at least, two codewords of $\mathcal{G}_n \backslash (\mathcal{G}_i \cup \mathcal{G}_k)$. For that, we consider the partition $\mathcal{N}$ of $\mathcal{I} \backslash \{i, n, -n\}$ induced by $W_2, W_5, W_7 \in \mathcal{G}_{in}$:

$$\mathcal{N}_1 = \{j, -k, -l\}; \;\; \mathcal{N}_2 = \{k, -o, -m\}; \;\; \mathcal{N}_3 = \{-j, m, o\}; \;\; \mathcal{N}_4 = \{l\}; \;\; \mathcal{N}_5 = \{-i\}. \tag{7.5}$$

Taking into account the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{G}_k$, we conclude that $|\mathcal{G}_n \backslash (\mathcal{G}_i \cup \mathcal{G}_k)| = 2$, with $W_{11}, W_{12} \in \mathcal{G}_n \backslash (\mathcal{G}_i \cup \mathcal{G}_k)$ satisfying one of the following conditions:

W_{11}	n	l	$-k$	$-o$	m
W_{12}	n	$-i$	l	j	$-m$

W_{11}	n	l	$-k$	$-o$	m
W_{12}	n	$-i$	l	j	o

W_{11}	n	$-i$	$-k$	$-o$	$-j$
W_{12}	n	$-i$	l	j	$-m$

W_{11}	n	$-i$	$-k$	$-o$	$-j$
W_{12}	n	$-i$	l	j	o

Table 7.10: Possible index distributions for $W_{11}, W_{12} \in \mathcal{G}_n \backslash (\mathcal{G}_i \cup \mathcal{G}_k)$.

In this example, unlike the previous one, we can characterize the index distribution of all codewords of three sets $\mathcal{G}_\alpha$, namely, $\mathcal{G}_i$, $\mathcal{G}_k$ and $\mathcal{G}_n$. Next step consists into analyze other sets $\mathcal{G}_\alpha$ with $\alpha \in \mathcal{I} \backslash \{i, k, n\}$.

Let us consider $W_{11} \in \mathcal{G}_{n,-i,-k,-o,-j}$ and $W_{12} \in \mathcal{G}_{n,-i,l,j,o}$. Let us analyze the index distribution of the remaining codewords of $\mathcal{G}_j$. Since $|\mathcal{G}_j \cap (\mathcal{G}_i \cup \mathcal{G}_k \cup \mathcal{G}_n)| = 4$, we must characterize, at least, two codewords in $\mathcal{G}_j \backslash (\mathcal{G}_i \cup \mathcal{G}_k \cup \mathcal{G}_n)$. From the analysis of the partition $\mathcal{J}$ of $\mathcal{I} \backslash \{i, j, -j\}$ induced by the codewords $W_1, W_2, W_3 \in \mathcal{G}_{ij}$:

$$\mathcal{J}_1 = \{k, l, m\}; \;\; \mathcal{J}_2 = \{-k, -l, n\}; \;\; \mathcal{J}_3 = \{-m, -n, o\}; \;\; \mathcal{J}_4 = \{-o\}; \;\; \mathcal{J}_5 = \{-i\}; \tag{7.6}$$

and considering the index distribution of all known codewords, we get the following two possible index distributions for the codewords $W_{13}, W_{14} \in \mathcal{G}_j \backslash (\mathcal{G}_i \cup \mathcal{G}_k \cup \mathcal{G}_n)$.

W_{13}	j	$-o$	$-n$	$-k$	m
W_{14}	j	$-i$	$-o$	$-m$	$-l$

W_{13}	j	$-i$	m	$-n$	$-k$
W_{14}	j	$-i$	$-o$	$-m$	$-l$

Table 7.11: Possible index distributions for $W_{13}, W_{14} \in \mathcal{G}_j \backslash (\mathcal{G}_i \cup \mathcal{G}_k \cup \mathcal{G}_n)$.

However, when we try to characterize all codewords of $\mathcal{G}_{-j}$, considering the partition $\mathcal{R}$ of $\mathcal{I} \backslash \{i, j, -j\}$ induced by $W_4, W_6, W_7 \in \mathcal{G}_{i,-j}$:

$$\mathcal{R}_1 = \{k, -l, -n\}; \;\; \mathcal{R}_2 = \{l, -k, -m\}; \; ; \; \mathcal{R}_3 = \{m, o, n\}; \;\; \mathcal{R}_4 = \{-o\}; \;\; \mathcal{R}_5 = \{-i\}. \tag{7.7}$$

we conclude that, in both cases, it is not possible to characterize one more codeword in $\mathcal{G}_{-j} \backslash (\mathcal{G}_i \cup \mathcal{G}_k \cup \mathcal{G}_n \cup \mathcal{G}_j)$ without contradictions.

If we had considered $W_{11}, W_{12} \in \mathcal{G}_n$ satisfying any other condition presented in Table 7.10, following a similar reasoning we conclude again that the characterization of the codewords of sets $\mathcal{G}_\alpha$, with $\alpha \in \{j, -j, -m\}$, contradicts necessary conditions for the existence of PL(7, 2) codes.

Example 3

In the previous examples we have verified that when we consider the elements $\alpha \in \mathcal{I} \backslash \{i, -i\}$ satisfying $|\mathcal{G}_{i\alpha}| = 3$, that is, the elements in $\{k, j, -j, -m, n\}$, although it is possible to describe completely the index distribution of all codewords of $\mathcal{G}_\alpha$ for some $\alpha \in \{k, j, -j, -m, n\}$, we have found, in the both presented examples, an element $\alpha \in \{k, j, -j, -m, n\}$ for which the characterization of the index distribution of all codewords of $\mathcal{G}_\alpha$ implies contradictions. However, there exist cases in which such does not happen, that is, there exist cases in which we can characterize completely the index distribution of all codewords of $\mathcal{G}_i \cup \mathcal{G}_k \cup \mathcal{G}_j \cup \mathcal{G}_{-j} \cup \mathcal{G}_{-m} \cup \mathcal{G}_n$, as we will see in next example.

Let us consider $W_8, W_9, W_{10} \in \mathcal{G}_k \backslash \mathcal{G}_i$ so that: $W_8 \in \mathcal{G}_{k,-i,j,-n,-o}$, $W_9 \in \mathcal{G}_{k,-i,m,n,-l}$ and $W_{10} \in \mathcal{G}_{k,-i,o,-m,-j}$.

We begin by characterizing the remaining codewords of $\mathcal{G}_j$. At this moment we know the index distribution of four codewords of $\mathcal{G}_j$, $W_1, W_2, W_3 \in \mathcal{G}_{ij}$ and $W_8 \in \mathcal{G}_{k,-i,j,-n,-o}$.

Therefore, we must describe, at least, the index distribution of two more codewords of $\mathcal{G}_j$. Considering the partition $\mathcal{J}$ of $\mathcal{I}\backslash\{i,j,-j\}$, see (7.6), and the index distribution of all known codewords, $W_{11}, W_{12} \in \mathcal{G}_j\backslash(\mathcal{G}_i \cup \mathcal{G}_k)$ must satisfy: $W_{11} \in \mathcal{G}_{j,-i,l,n,-m}$ and $W_{12} \in \mathcal{G}_{j,-i,m,o,-k}$.

Let us now characterize the remaining codewords of $\mathcal{G}_n$. Since we already have characterized five codewords of $\mathcal{G}_n$, namely $W_2, W_5, W_7 \in \mathcal{G}_{in}$, $W_9 \in \mathcal{G}_{kn}\backslash\mathcal{G}_i$ and $W_{11} \in \mathcal{G}_{jn}\backslash(\mathcal{G}_i \cup \mathcal{G}_k)$, we must describe the index distribution of, at least, one more codeword of $\mathcal{G}_n$. Taking into account the partition $\mathcal{N}$ of $\mathcal{I}\backslash\{i,n,-n\}$, see (7.5), and the index distribution of all known codewords, we conclude that $W_{13} \in \mathcal{G}_n\backslash(\mathcal{G}_i \cup \mathcal{G}_k \cup \mathcal{G}_j)$ satisfies one of the following possible index distributions:

1) $W_{13} \in \mathcal{G}_{n,l,-k,-o,m}$;

2) $W_{13} \in \mathcal{G}_{n,-i,-k,-o,-j}$.

If we consider $W_{13} \in \mathcal{G}_{n,l,-k,-o,m}$, from the analysis of the partition $\mathcal{R}$ of $\mathcal{I}\backslash\{i,j,-j\}$ induced by the codewords $W_4, W_6, W_7 \in \mathcal{G}_{i,-j}$, see (7.7), we conclude that the remaining codewords of $\mathcal{G}_{-j}$, that is, $W_{14}, W_{15} \in \mathcal{G}_{-j}\backslash(\mathcal{G}_i \cup \mathcal{G}_k \cup \mathcal{G}_j \cup \mathcal{G}_n)$ must satisfy, respectively: $W_{14} \in \mathcal{G}_{-j,-i,-n,m,l}$ and $W_{15} \in \mathcal{G}_{-j,-i,-o,-l,-k}$. However, when we try to characterize the index distribution of the remaining codewords of $\mathcal{G}_{-m}$, considering the partition $\mathcal{Q}$ of $\mathcal{I}\backslash\{i,m,-m\}$, see (7.4), we can not do it without contradictions.

Now assume that $W_{13} \in \mathcal{G}_{n,-i,-k,-o,-j}$. Since, at this moment, we know the index distribution of five codewords of $\mathcal{G}_{-j}$, proceeding as in the previous case we verify that there exist two possible index distributions for the remaining codeword $W_{14} \in \mathcal{G}_{-j}\backslash(\mathcal{G}_i \cup \mathcal{G}_k \cup \mathcal{G}_j \cup \mathcal{G}_n)$:

i) $W_{14} \in \mathcal{G}_{-j,-o,m,l,-n}$;

ii) $W_{14} \in \mathcal{G}_{-j,-i,-n,m,l}$.

If $W_{14} \in \mathcal{G}_{-j,-o,m,l,-n}$, then the remaining codeword $W_{15} \in \mathcal{G}_{-m}$ must satisfy one of the following hypotheses: $W_{15} \in \mathcal{G}_{-m,-l,-k,-o,-n}$ or $W_{15} \in \mathcal{G}_{-m,-i,-l,-k,-n}$. On the other hand, if we consider $W_{14} \in \mathcal{G}_{-j,-i,-n,m,l}$, then $W_{15} \in \mathcal{G}_{-m}$ satisfies one of the following conditions: $W_{15} \in \mathcal{G}_{-m,-l,-k,-o,-n}$ or $W_{15} \in \mathcal{G}_{-m,-i,-l,-k,-n}$.

In both cases we have characterized the index distribution of all codewords of $\mathcal{G}_i \cup \mathcal{G}_k \cup \mathcal{G}_j \cup \mathcal{G}_n \cup \mathcal{G}_{-j} \cup \mathcal{G}_{-m}$. So, next step consists into analyze other sets $\mathcal{G}_\alpha$ for $\alpha \in \mathcal{I}\backslash\{i, j, -j, k, -m, n\}$. To show how we do it, let us consider $W_{14} \in \mathcal{G}_{-j,-o,m,l,-n}$ and $W_{15} \in \mathcal{G}_{-m,-l,-k,-o,-n}$. Under this assumption, the following table recall the index distribution of all codewords of $(\mathcal{G}_k \cup \mathcal{G}_j \cup \mathcal{G}_n \cup \mathcal{G}_{-j} \cup \mathcal{G}_{-m})\backslash\mathcal{G}_i$ described at this moment. We note that the considered codewords of $\mathcal{G}_i$ are described in Table 7.7.

W_8	k	$-i$	j	$-n$	$-o$
W_9	k	$-i$	m	n	$-l$
W_{10}	k	$-i$	o	$-m$	$-j$
W_{11}	j	$-i$	l	n	$-m$
W_{12}	j	$-i$	m	o	$-k$
W_{13}	n	$-i$	$-k$	$-o$	$-j$
W_{14}	$-j$	$-o$	m	l	$-n$
W_{15}	$-m$	$-l$	$-k$	$-o$	$-n$

Table 7.12: Index distribution of the codewords of $(\mathcal{G}_k \cup \mathcal{G}_j \cup \mathcal{G}_n \cup \mathcal{G}_{-j} \cup \mathcal{G}_{-m})\backslash\mathcal{G}_i$.

Until now we know the index distribution of five codewords of $\mathcal{G}_{-k}$. Accordingly, we must describe, at least, one more codeword of $\mathcal{G}_{-k}$. For that, we will take into account the following partition $\mathcal{K}$ of $\mathcal{I}\backslash\{i, j, -j, k, -k, -m, n\}$ induced by the codewords $W_{12}, W_{13} \in \mathcal{G}_{-k,-i}$:

$$\mathcal{K}_1 = \{m, o\};\ \ \mathcal{K}_2 = \{-o\};\ \ \mathcal{K}_3 = \{l, -l\};\ \ \mathcal{K}_4 = \{-n\};\ \ \mathcal{K}_5 = \{-i\}.$$

Considering this partition as well as the index distribution of all known codewords and Lemma 1.5, we conclude that we can not characterize one more codeword of $\mathcal{G}_{-k}\backslash(\mathcal{G}_i \cup \mathcal{G}_j \cup \mathcal{G}_{-j} \cup \mathcal{G}_k \cup \mathcal{G}_{-m} \cup \mathcal{G}_n)$ without contradict necessary conditions for the existence of PL(7, 2) codes.

If we had considered the other referred hypotheses for W_{14} and W_{15}, analyzing the remaining codewords of $\mathcal{G}_{-k}$ we would concluded the same.

We have presented the analysis of only three possible index distributions for the codewords of $\mathcal{G}_k$ assuming that the codewords of $\mathcal{G}_i$ satisfy the conditions presented in Table 7.7. However, considering each one of the presented hypotheses in Tables 7.8 and 7.9 for the index distribution of the codewords of $\mathcal{G}_k \backslash \mathcal{G}_i$, proceeding as we have shown in the illustrative examples we get always contradictions, concluding, thus, that the considered index distribution for the codewords of $\mathcal{G}_i$ is not valid.

Considering all possible index distributions for the codewords of $\mathcal{G}_i$, following a similar reasoning we verify that each one of them leads us to contradictions. Thus, we are in conditions to establish the main theorem:

Theorem 7.1 *There exist no PL$(7,2)$ codes.*

7.2 Conclusions

As seen before, a code $\mathcal{M}$ is a perfect r-error correcting Lee code of word length n over $\mathbb{Z}$, shortly a PL(n,r) code, if and only if the Lee spheres of radius r centered at codewords of $\mathcal{M}$ tile $\mathbb{Z}^n$.

The Golomb-Welch conjecture states that there is no PL(n,r) code for $n \geq 3$ and $r \geq 2$. Many efforts have been made to prove the conjecture, however, until now, its validity it is only proved for some particular values of n and r: $3 \leq n \leq 5$ and $r \geq 2$; $n = 6$ and $r = 2$.

Here, we reinforce the conjecture proving the non-existence of PL$(7,2)$ codes.

The way how the proof was built reveals how difficult was to solve the case. We have focused our attention on words which dist three units from $O = (0, \ldots, 0)$. Actually, there exist many ways to try to cover all these words by codewords, and although we have obtained many results which restrict the number of such hypotheses, in many cases, to achieve contradictions we had to apply exhaustion methods to study a large number of cases. This was the major hard work of the proof. In some cases we have tried to use computational methods having in view a quick analysis of the many cases we had to deal with, however, it would be necessary to implement an algorithm requiring a lot of information, not being easy to do it, at least, with our knowledge.

The reasoning behind the presented proof is faithful to the geometric idea of the problem. We believe that the described line of thinking it is not the only one which allow us to conclude the non-existence of $\mathrm{PL}(7,2)$ codes and that in the future other faster proofs will arise.

Unfortunately, it seems to us, considering the difficulties experienced during the proof, that this method should not be extended to the proofs of other values for the parameters n and r.

In future we expect to continue to work on the Golomb-Welch conjecture, giving other contributions for the unsolved cases, in particular, we would first like to investigate the existence of $\mathrm{PL}(n,r)$ codes for $n=6,7$ and $r\geq 3$.

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