Far East Journal of Mathematical Sciences (FJMS)

http://www.pphmj.com
http://dx.doi.org/10.17654/MS109010057
Volume 109, Number 1, 2018, Pages 57-88 **ISSN: 0972-0871**

DISTANCE BETWEEN LINEAR VARIETIES IN $\mathbb{R}^n$. APPLICATION OF AN (IN)EQUALITY BY (FAN-TODD) BEESACK

M. A. Facas Vicente[1], José Vitória[1], Fernando Martins[2,3], Cecília Costa[4] and Pedro Tadeu[5]

[1]Department of Mathematics
Faculty of Sciences and Technology
University of Coimbra
Apartado 3008, EC Santa Cruz
3001-501 Coimbra, Portugal

[2]Instituto Politécnico de Coimbra
ESEC
Portugal

[3]Instituto de Telecomunicações
Av. Rovisco Pais, 1 – Torre
Norte, 1049-001 Lisboa, Portugal

[4]Department of Mathematics
School of Sciences and Technology
University of Trás-os-Montes e Alto Douro
Apartado 1013, 5001-801 Vila Real, Portugal

Received: October 24, 2017; Accepted: April 17, 2018
2010 Mathematics Subject Classification: 41A17, 52A40.
Keywords and phrases: analytic inequalities, Gram determinant, minimum norm vector of an affine set, closest points of two linear varieties.

[5]Instituto Politécnico da Guarda
UDI - Unidade de Investigação para o Desenvolvimento do Interior
Escola Superior de Educação
Comunicação e Desporto, 6300-559 Guarda, Portugal

Abstract

The closest point of a linear variety to an external point is found by using the equality case of an Ostrowski's type inequality. This point is given in a closed form as the quotient of a formal and a scalar Gram determinant. Then the best approximation pair of points onto two linear varieties is obtained, besides its characterization.

1. Introduction

In this paper, we answer an implicit open question of Fan and Todd [5, p. 63]. We give a determinantal formula for the point where the inequality of the referred one turns into equality. We obtain the point of least norm of the intersection of certain hyperplanes, and present a result, in terms of Gram determinants, for the minimum distance from a certain linear variety to the origin of coordinates (Proposition 1). We note that this formula generalizes the one by Mitrinovic [10, 11] for the case of two equations. This best approximation problem was dealt with in [13], where the centre of (degenerate) hyperquadrics plays a decisive role. In [13], no answer in closed form was given.

In this paper, we give a new proof of Beesack's inequality ([1, Theorem 1]; [12, Theorem 1.7]), by following arguments used in [5, p. 63, Lemma].

The Beesack's formula (Theorem 2) gives the point of a general linear variety closest to the origin of the coordinates. We extend the formula of Beesack [1, Theorem 1] in order to get the nearest point of a linear variety to an external point, in $\mathbb{R}^n$. When extending Theorem 2, we obtain the projection of an external point onto a general linear variety (Proposition 2).

Proposition 2 is used for getting the best approximation points of two linear varieties (Proposition 3). Also, a characterization of the best approximation pair of two linear varieties is presented (Proposition 4).

The approach we use in this paper has a constructive flavour. In [4] and [6], we present another way to obtain the Euclidean distance between two linear varieties and exhibit the optimal pair of points, respectively.

Our context is the Euclidean space $\mathbb{R}^n$, endowed with the standard unit basis

$$(\vec{e}_1, \vec{e}_2, ..., \vec{e}_n)$$

and the ordinary inner product

$$\vec{u} \bullet \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n,$$

where

$$\vec{a} = a_1 \vec{e}_1 + a_2 \vec{e}_2 + \cdots + a_n \vec{e}_n = (a_1, a_2, ..., a_n).$$

The Euclidean norm $\| \vec{a} \| = \sqrt{\vec{a} \bullet \vec{a}}$ is used and the Gram determinant is

$$G(\vec{p}_1, \vec{p}_2, ..., \vec{p}_r) = \det \begin{bmatrix} \vec{p}_1 \bullet \vec{p}_1 & \vec{p}_1 \bullet \vec{p}_2 & \cdots & \vec{p}_1 \bullet \vec{p}_r \\ \vec{p}_2 \bullet \vec{p}_1 & \vec{p}_2 \bullet \vec{p}_2 & \cdots & \vec{p}_2 \bullet \vec{p}_r \\ \vdots & \vdots & \ddots & \vdots \\ \vec{p}_r \bullet \vec{p}_1 & \vec{p}_r \bullet \vec{p}_2 & \cdots & \vec{p}_r \bullet \vec{p}_r \end{bmatrix}, \quad 1 \le r \le n. \quad (1)$$

It is well known that $G(\vec{p}_1, \vec{p}_2, ..., \vec{p}_r) \ge 0$ and $G(\vec{p}_1, \vec{p}_2, ..., \vec{p}_r) = 0$ if and only if the vectors $\vec{p}_1, \vec{p}_2, ..., \vec{p}_r$ are linearly dependent. See, for example, [3, p. 132].

Some abuse of notation, authorized by adequate isomorphisms, is to be declared, notably the identification of point, vector, ordered set, column-matrix.

This paper is organized in seven sections: In Section 2, we present and prove a result, Proposition 1, which answers an open question of Fan and Todd and make a remark concerning a formula of Mitrinovic. Section 3 is dedicated to a generalization of Proposition 1. This means that we study the projection of the origin onto a general linear variety. In Section 4, we deal with the projection of an external point onto a general linear variety. In Section 5, we treat the distance between two disjoint linear varieties. We characterize the best two points, one on each linear variety, that are the extremities of the straight line segment that materializes the distance between the two linear varieties. An illustrative numerical example is presented in Section 6. Finally, in Section 7, we draw some conclusions.

2. The Minimum Norm Vector of a Certain Linear Variety

In this section, we state Proposition 1, which solves an old open question of Fan and Todd. The proof makes use of a result of the mentioned by the authors.

The next result [5, p. 63, Lemma] gives the radius of the sphere tangent to a certain linear variety, as the quotient of two Gram determinants.

Theorem 1. *Let $\vec{a}_1, \vec{a}_2, ..., \vec{a}_m$ be m linearly independent vectors in $\mathbb{R}^n$, $2 \le m \le n$. If a vector $\vec{x} \in \mathbb{R}^n$ varies under the conditions*

$$\vec{a}_i \bullet \vec{x} = 0, \ \text{with } 1 \le i \le m-1,$$

$$\vec{a}_m \bullet \vec{x} = 1, \tag{2}$$

then

$$\vec{x} \bullet \vec{x} \ge \frac{G(\vec{a}_1, \vec{a}_2, ..., \vec{a}_{m-1})}{G(\vec{a}_1, \vec{a}_2, ..., \vec{a}_{m-1}, \vec{a}_m)}. \tag{3}$$

Furthermore, the minimum value is obtained if and only if $\vec{x}$ is a linear combination of $\vec{a}_1, \vec{a}_2, ..., \vec{a}_m$.

For the sake of completeness and for later use in the proof of our Proposition 1, we present here, essentially, the proof given by Fan and Todd [5, p. 63, Lemma].

Proof. For the vector $\vec{x}$ satisfying conditions (2), we have

$$G(\vec{a}_1, \vec{a}_2, ..., \vec{a}_m, \vec{x}) = -G(\vec{a}_1, \vec{a}_2, ..., \vec{a}_{m-1}) + (\vec{x} \bullet \vec{x})G(\vec{a}_1, \vec{a}_2, ..., \vec{a}_m) \geq 0.$$

Hence

$$\vec{x} \bullet \vec{x} \geq \frac{G(\vec{a}_1, \vec{a}_2, ..., \vec{a}_{m-1})}{G(\vec{a}_1, \vec{a}_2, ..., \vec{a}_{m-1}, \vec{a}_m)}.$$

By hypothesis, the vectors $\vec{a}_1, \vec{a}_2, ..., \vec{a}_m$ are linearly independent, so

$$G(\vec{a}_1, \vec{a}_2, ..., \vec{a}_m, \vec{x}) = 0$$

if and only if $\vec{x}$ is a linear combination of $\vec{a}_1, \vec{a}_2, ..., \vec{a}_m$. It follows that

$$\vec{x} \bullet \vec{x} = \frac{G(\vec{a}_1, \vec{a}_2, ..., \vec{a}_{m-1})}{G(\vec{a}_1, \vec{a}_2, ..., \vec{a}_{m-1}, \vec{a}_m)} \tag{4}$$

if and only if the vector $\vec{x}$ is of the form $\vec{x} = \alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \cdots + \alpha_m\vec{a}_m$. $\square$

Now we are in a position for stating the equality case. A determinantal formula for the closest vector to the origin lying in a certain linear variety is given.

Proposition 1. (1) *Let* $\vec{a}_1, \vec{a}_2, ..., \vec{a}_m$ *be linearly independent vectors in* $\mathbb{R}^n$. *The minimum Euclidean norm vector in* $\mathbb{R}^n$ *satisfying the equations*

$$\begin{aligned} \vec{a}_1 \bullet \vec{x} &= 0 \\ \vec{a}_2 \bullet \vec{x} &= 0 \\ &\vdots \\ \vec{a}_{m-1} \bullet \vec{x} &= 0 \\ \vec{a}_m \bullet \vec{x} &= 1 \end{aligned} \tag{5}$$

is given by

$$\vec{s} = \frac{\begin{vmatrix} \vec{a}_1 \bullet \vec{a}_1 & \vec{a}_1 \bullet \vec{a}_2 & \cdots & \vec{a}_1 \bullet \vec{a}_{m-1} & \vec{a}_1 \bullet \vec{a}_m \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vec{a}_{m-1} \bullet \vec{a}_1 & \vec{a}_{m-1} \bullet \vec{a}_2 & \cdots & \vec{a}_{m-1} \bullet \vec{a}_{m-1} & \vec{a}_{m-1} \bullet \vec{a}_m \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_{m-1} & \vec{a}_m \end{vmatrix}}{\begin{vmatrix} \vec{a}_1 \bullet \vec{a}_1 & \vec{a}_1 \bullet \vec{a}_2 & \cdots & \vec{a}_1 \bullet \vec{a}_{m-1} & \vec{a}_1 \bullet \vec{a}_m \\ \vec{a}_2 \bullet \vec{a}_1 & \vec{a}_2 \bullet \vec{a}_2 & \cdots & \vec{a}_2 \bullet \vec{a}_{m-1} & \vec{a}_2 \bullet \vec{a}_m \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vec{a}_{m-1} \bullet \vec{a}_1 & \vec{a}_{m-1} \bullet \vec{a}_2 & \cdots & \vec{a}_{m-1} \bullet \vec{a}_{m-1} & \vec{a}_{m-1} \bullet \vec{a}_m \\ \vec{a}_m \bullet \vec{a}_1 & \vec{a}_m \bullet \vec{a}_2 & \cdots & \vec{a}_m \bullet \vec{a}_{m-1} & \vec{a}_m \bullet \vec{a}_m \end{vmatrix}}, \quad (6)$$

where the determinant in the numerator is to be expanded by the last row, in order to yield a linear combination of the vectors $\vec{a}_1, \vec{a}_2, ..., \vec{a}_m$.

(2) *Furthermore,*

$$\| \vec{s} \|^2 = \vec{s} \bullet \vec{s} = \frac{G(\vec{a}_1, \vec{a}_2, ..., \vec{a}_{m-1})}{G(\vec{a}_1, \vec{a}_2, ..., \vec{a}_{m-1}, \vec{a}_m)}.$$

Proof. (1) We look for the scalars $\alpha_1, \alpha_2, ..., \alpha_m$, such that the vector $\vec{x} = \alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \cdots + \alpha_m\vec{a}_m$ satisfies the conditions (5).

For that end, we solve the system

$$\begin{bmatrix} \vec{a}_1 \bullet \vec{a}_1 & \vec{a}_1 \bullet \vec{a}_2 & \cdots & \vec{a}_1 \bullet \vec{a}_{m-1} & \vec{a}_1 \bullet \vec{a}_m \\ \vec{a}_2 \bullet \vec{a}_1 & \vec{a}_2 \bullet \vec{a}_2 & \cdots & \vec{a}_2 \bullet \vec{a}_{m-1} & \vec{a}_2 \bullet \vec{a}_m \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vec{a}_{m-1} \bullet \vec{a}_1 & \vec{a}_{m-1} \bullet \vec{a}_2 & \cdots & \vec{a}_{m-1} \bullet \vec{a}_{m-1} & \vec{a}_{m-1} \bullet \vec{a}_m \\ \vec{a}_m \bullet \vec{a}_1 & \vec{a}_m \bullet \vec{a}_2 & \cdots & \vec{a}_m \bullet \vec{a}_{m-1} & \vec{a}_m \bullet \vec{a}_m \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{m-1} \\ \alpha_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

As the vectors $\vec{a}_1, \vec{a}_2, ..., \vec{a}_m$ are, by hypothesis, linearly independent, the determinant of the matrix of the above system, which is the Gram determinant

$$G(\vec{a}_1, \vec{a}_2, ..., \vec{a}_{m-1}, \vec{a}_m),$$

is non-null.

So, by the Cramer's rule, we have

$$\alpha_i = \frac{\begin{vmatrix} \vec{a}_1 \bullet \vec{a}_1 & \cdots & \vec{a}_1 \bullet \vec{a}_{i-1} & 0 & \vec{a}_1 \bullet \vec{a}_{i+1} & \cdots & \vec{a}_1 \bullet \vec{a}_m \\ \vec{a}_2 \bullet \vec{a}_1 & \cdots & \vec{a}_2 \bullet \vec{a}_{i-1} & 0 & \vec{a}_2 \bullet \vec{a}_{i+1} & \cdots & \vec{a}_2 \bullet \vec{a}_m \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vec{a}_{m-1} \bullet \vec{a}_1 & \cdots & \vec{a}_{m-1} \bullet \vec{a}_{i-1} & 0 & \vec{a}_{m-1} \bullet \vec{a}_{i+1} & \cdots & \vec{a}_{m-1} \bullet \vec{a}_m \\ \vec{a}_m \bullet \vec{a}_1 & \cdots & \vec{a}_m \bullet \vec{a}_{i-1} & 1 & \vec{a}_m \bullet \vec{a}_{i+1} & \cdots & \vec{a}_m \bullet \vec{a}_m \end{vmatrix}}{\begin{vmatrix} \vec{a}_1 \bullet \vec{a}_1 & \vec{a}_1 \bullet \vec{a}_2 & \cdots & \vec{a}_1 \bullet \vec{a}_{m-1} & \vec{a}_1 \bullet \vec{a}_m \\ \vec{a}_2 \bullet \vec{a}_1 & \vec{a}_2 \bullet \vec{a}_2 & \cdots & \vec{a}_2 \bullet \vec{a}_{m-1} & \vec{a}_2 \bullet \vec{a}_m \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vec{a}_{m-1} \bullet \vec{a}_1 & \vec{a}_{m-1} \bullet \vec{a}_2 & \cdots & \vec{a}_{m-1} \bullet \vec{a}_{m-1} & \vec{a}_{m-1} \bullet \vec{a}_m \\ \vec{a}_m \bullet \vec{a}_1 & \vec{a}_m \bullet \vec{a}_2 & \cdots & \vec{a}_m \bullet \vec{a}_{m-1} & \vec{a}_m \bullet \vec{a}_m \end{vmatrix}},$$

with $i = 1, ..., m$.

Here, for brevity, we introduce some notations:

$$\alpha_i = \frac{G_i}{G}, \quad \alpha_i \vec{a}_i = \frac{G_i}{G} \vec{a}_i \coloneqq \frac{\vec{G}_i}{G},$$

where

$G = G(\vec{a}_1, \vec{a}_2, ..., \vec{a}_{m-1}, \vec{a}_m),$

$$G_i = \begin{vmatrix} \vec{a}_1 \bullet \vec{a}_1 & \cdots & \vec{a}_1 \bullet \vec{a}_{i-1} & 0 & \vec{a}_1 \bullet \vec{a}_{i+1} & \cdots & \vec{a}_1 \bullet \vec{a}_m \\ \vec{a}_2 \bullet \vec{a}_1 & \cdots & \vec{a}_2 \bullet \vec{a}_{i-1} & 0 & \vec{a}_2 \bullet \vec{a}_{i+1} & \cdots & \vec{a}_2 \bullet \vec{a}_m \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vec{a}_{m-1} \bullet \vec{a}_1 & \cdots & \vec{a}_{m-1} \bullet \vec{a}_{i-1} & 0 & \vec{a}_{m-1} \bullet \vec{a}_{i+1} & \cdots & \vec{a}_{m-1} \bullet \vec{a}_m \\ \vec{a}_m \bullet \vec{a}_1 & \cdots & \vec{a}_m \bullet \vec{a}_{i-1} & 1 & \vec{a}_m \bullet \vec{a}_{i+1} & \cdots & \vec{a}_m \bullet \vec{a}_m \end{vmatrix}$$

and the symbolic determinant

$$\vec{G}_i = \begin{vmatrix} \vec{a}_1 \bullet \vec{a}_1 & \cdots & \vec{a}_1 \bullet \vec{a}_{i-1} & 0 & \vec{a}_1 \bullet \vec{a}_{i+1} & \cdots & \vec{a}_1 \bullet \vec{a}_m \\ \vec{a}_2 \bullet \vec{a}_1 & \cdots & \vec{a}_2 \bullet \vec{a}_{i-1} & 0 & \vec{a}_2 \bullet \vec{a}_{i+1} & \cdots & \vec{a}_2 \bullet \vec{a}_m \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vec{a}_{m-1} \bullet \vec{a}_1 & \cdots & \vec{a}_{m-1} \bullet \vec{a}_{i-1} & 0 & \vec{a}_{m-1} \bullet \vec{a}_{i+1} & \cdots & \vec{a}_{m-1} \bullet \vec{a}_m \\ \vec{0} & \cdots & \vec{0} & \vec{a}_i & \vec{0} & \cdots & \vec{0} \end{vmatrix}.$$

Using these notations and rearranging in a suitable manner the terms of the determinants, we get

$$\vec{s} = \sum_{i=1}^{m} \alpha_i \vec{a}_i = \frac{\begin{vmatrix} \vec{a}_1 \bullet \vec{a}_1 & \vec{a}_1 \bullet \vec{a}_2 & \cdots & \vec{a}_1 \bullet \vec{a}_{m-1} & \vec{a}_1 \bullet \vec{a}_m \\ \vec{a}_2 \bullet \vec{a}_1 & \vec{a}_2 \bullet \vec{a}_2 & \cdots & \vec{a}_2 \bullet \vec{a}_{m-1} & \vec{a}_2 \bullet \vec{a}_m \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vec{a}_{m-1} \bullet \vec{a}_1 & \vec{a}_{m-1} \bullet \vec{a}_2 & \cdots & \vec{a}_{m-1} \bullet \vec{a}_{m-1} & \vec{a}_{m-1} \bullet \vec{a}_m \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_{m-1} & \vec{a}_m \end{vmatrix}}{\begin{vmatrix} \vec{a}_1 \bullet \vec{a}_1 & \vec{a}_1 \bullet \vec{a}_2 & \cdots & \vec{a}_1 \bullet \vec{a}_m \\ \vec{a}_2 \bullet \vec{a}_1 & \vec{a}_2 \bullet \vec{a}_2 & \cdots & \vec{a}_2 \bullet \vec{a}_m \\ \vdots & \vdots & \ddots & \vdots \\ \vec{a}_{m-1} \bullet \vec{a}_1 & \vec{a}_{m-1} \bullet \vec{a}_2 & \cdots & \vec{a}_{m-1} \bullet \vec{a}_m \\ \vec{a}_m \bullet \vec{a}_1 & \vec{a}_m \bullet \vec{a}_2 & \cdots & \vec{a}_m \bullet \vec{a}_m \end{vmatrix}}.$$

(2) It is just sufficient to use (4), in order to obtain $\| \vec{s} \|^2$. □

For computational purposes, we notice that, in the numerator of (6), the coefficients of the vectors $\vec{a}_1, \vec{a}_2, ..., \vec{a}_m$ are the co-factors of the elements in the last row of the matrix

$$\begin{bmatrix} \vec{a}_1 \bullet \vec{a}_1 & \vec{a}_1 \bullet \vec{a}_2 & \cdots & \vec{a}_1 \bullet \vec{a}_{m-1} & \vec{a}_1 \bullet \vec{a}_m \\ \vec{a}_2 \bullet \vec{a}_1 & \vec{a}_2 \bullet \vec{a}_2 & \cdots & \vec{a}_2 \bullet \vec{a}_{m-1} & \vec{a}_2 \bullet \vec{a}_m \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vec{a}_{m-1} \bullet \vec{a}_1 & \vec{a}_{m-1} \bullet \vec{a}_2 & \cdots & \vec{a}_{m-1} \bullet \vec{a}_{m-1} & \vec{a}_{m-1} \bullet \vec{a}_m \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

Remark 1. The particular case of Mitrinovic

The determinantal formula (6), for the least norm vector of the given linear variety, is a generalization of the formula of Mitrinovic [10, p. 67] and [11, p. 93], after the corrections in [2] of the misprints:

$$x_k = \frac{b_k \sum_{i=1}^{p} a_i^2 - a_k \sum_{i=1}^{p} a_i b_i}{\left(\sum_{i=1}^{p} a_i^2\right)\left(\sum_{i=1}^{p} b_i^2\right) - \left(\sum_{i=1}^{p} a_i b_i\right)^2}, \quad k = 1, 2, ..., p,$$

where $(a_1, a_2, ..., a_p)$ and $(b_1, b_2, ..., b_p)$ are two non-proportional sequences of real numbers satisfying

$$\sum_{i=1}^{p} a_i x_i = 0 \quad \text{and} \quad \sum_{i=1}^{p} b_i x_i = 1.$$

3. The Minimum Norm Vector of a General Linear Variety

Here we treat the projection of the origin of the coordinates onto a general linear variety extending Proposition 1. The point where the sphere centered at the origin is tangent to any linear variety is given in a closed form by relation (8) to follow. This result has been obtained in a different form and by another approach in [1].

Theorem 2 [1]. *Let* $\vec{a}_1, \vec{a}_2, ..., \vec{a}_m$ *be linearly independent vectors in* $\mathbb{R}^n$, *with* $m \geq 2$. *The minimum Euclidean norm vector in* $\mathbb{R}^n$ *satisfying the equations*

$$\begin{aligned} \vec{a}_1 \bullet \vec{x} &= c_1 \\ \vec{a}_2 \bullet \vec{x} &= c_2 \\ &\vdots \\ \vec{a}_{m-1} \bullet \vec{x} &= c_{m-1} \\ \vec{a}_m \bullet \vec{x} &= c_m, \end{aligned} \tag{7}$$

with, at least, one non-zero c_i, $i = 1, \ldots, m$, *is given by the relation*

$$\vec{s}\,' = \frac{\begin{vmatrix} \vec{a}_1' \bullet \vec{a}_1' & \vec{a}_1' \bullet \vec{a}_2' & \cdots & \vec{a}_1' \bullet \vec{a}_{m-1}' & \vec{a}_1' \bullet \vec{a}_m' \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vec{a}_{m-1}' \bullet \vec{a}_1' & \vec{a}_{m-1}' \bullet \vec{a}_2' & \cdots & \vec{a}_{m-1}' \bullet \vec{a}_{m-1}' & \vec{a}_{m-1}' \bullet \vec{a}_m' \\ \vec{a}_1' & \vec{a}_2' & \cdots & \vec{a}_{m-1}' & \vec{a}_m' \end{vmatrix}}{\begin{vmatrix} \vec{a}_1' \bullet \vec{a}_1' & \vec{a}_1' \bullet \vec{a}_2' & \cdots & \vec{a}_1' \bullet \vec{a}_{m-1}' & \vec{a}_1' \bullet \vec{a}_m' \\ \vec{a}_2' \bullet \vec{a}_1' & \vec{a}_2' \bullet \vec{a}_2' & \cdots & \vec{a}_2' \bullet \vec{a}_{m-1}' & \vec{a}_2' \bullet \vec{a}_m' \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vec{a}_{m-1}' \bullet \vec{a}_1' & \vec{a}_{m-1}' \bullet \vec{a}_2' & \cdots & \vec{a}_{m-1}' \bullet \vec{a}_{m-1}' & \vec{a}_{m-1}' \bullet \vec{a}_m' \\ \vec{a}_m' \bullet \vec{a}_1' & \vec{a}_m' \bullet \vec{a}_2' & \cdots & \vec{a}_m' \bullet \vec{a}_{m-1}' & \vec{a}_m' \bullet \vec{a}_m' \end{vmatrix}}, \qquad (8)$$

where

$$\vec{a}_i' = \vec{a}_i - \frac{c_i}{c_m}\vec{a}_m, \quad i = 1, \ldots, m-1, \qquad (9)$$

and

$$\vec{a}_m' = \frac{1}{c_m}\vec{a}_m. \qquad (10)$$

Furthermore,

$$\|\vec{s}\,'\|^2 = \vec{s}\,' \bullet \vec{s}\,' = \frac{G(\vec{a}_1', \vec{a}_2', \ldots, \vec{a}_{m-1}')}{G(\vec{a}_1', \vec{a}_2', \ldots, \vec{a}_{m-1}', \vec{a}_m')}. \qquad (11)$$

Proof. Performing elementary matrix operations, we turn into the form $[0\ 0\ \cdots\ 0\ 1]^T$ the last column of the augmented matrix of the system (7)

$$\begin{bmatrix} a_{1_1} & a_{1_2} & \cdots & a_{1_{n-1}} & a_{1_n} & c_1 \\ a_{2_1} & a_{2_2} & \cdots & a_{2_{n-1}} & a_{2_n} & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{m-1_1} & a_{m-1_2} & \cdots & a_{m-1_{n-1}} & a_{m-1_n} & c_{m-1} \\ a_{m_1} & a_{m_2} & \cdots & a_{m_{n-1}} & a_{m_n} & c_m \end{bmatrix},$$

where $\vec{a}_i = (a_{i_1}, a_{i_2}, \ldots, a_{i_{n-1}}, a_{i_n})$. □

4. Projection of a Point onto a Linear Variety

To deal with this problem by taking into account the result of the preceding section, we use the fact that Euclidean distance is preserved under translations.

We are given a linear variety V and an external point Q. We perform a translation towards the origin O of the coordinates: the pair (Q, V) turns into the pair (O, V'). We, then, apply Theorem 2 to the pair (O, V'). Finally, we undo the performed translation: we go back from the origin O to the point Q. We state the following:

Proposition 2. *Let* $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m$ *be linearly independent vectors in* $\mathbb{R}^n$, *with* $m \geq 2$. *Then*:

(1) *The projection S of the external point* $Q := \vec{q} = (q_1, q_2, \ldots, q_n)$ *onto the linear variety V defined by*

$$\begin{aligned} \vec{a}_1 \bullet \vec{x} &= c_1 \\ \vec{a}_2 \bullet \vec{x} &= c_2 \\ &\vdots \\ \vec{a}_{m-1} \bullet \vec{x} &= c_{m-1} \\ \vec{a}_m \bullet \vec{x} &= c_m, \end{aligned} \tag{12}$$

with, at least, one non-zero c_i, $i = 1, \ldots, m$, *is given by*

$$S := \vec{s} = \vec{s}'' + \vec{q}, \tag{13}$$

where

$$\vec{s}'' = \frac{\begin{vmatrix} \vec{a}_1'' \bullet \vec{a}_1'' & \vec{a}_1'' \bullet \vec{a}_2'' & \cdots & \vec{a}_1'' \bullet \vec{a}_{m-1}'' & \vec{a}_1'' \bullet \vec{a}_m'' \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vec{a}_{m-1}'' \bullet \vec{a}_1'' & \vec{a}_{m-1}'' \bullet \vec{a}_2'' & \cdots & \vec{a}_{m-1}'' \bullet \vec{a}_{m-1}'' & \vec{a}_{m-1}'' \bullet \vec{a}_m'' \\ \vec{a}_1'' & \vec{a}_2'' & \cdots & \vec{a}_{m-1}'' & \vec{a}_m'' \end{vmatrix}}{\begin{vmatrix} \vec{a}_1'' \bullet \vec{a}_1'' & \vec{a}_1'' \bullet \vec{a}_2'' & \cdots & \vec{a}_1'' \bullet \vec{a}_{m-1}'' & \vec{a}_1'' \bullet \vec{a}_m'' \\ \vec{a}_2'' \bullet \vec{a}_1'' & \vec{a}_2'' \bullet \vec{a}_2'' & \cdots & \vec{a}_2'' \bullet \vec{a}_{m-1}'' & \vec{a}_2'' \bullet \vec{a}_m'' \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vec{a}_{m-1}'' \bullet \vec{a}_1'' & \vec{a}_{m-1}'' \bullet \vec{a}_2'' & \cdots & \vec{a}_{m-1}'' \bullet \vec{a}_{m-1}'' & \vec{a}_{m-1}'' \bullet \vec{a}_m'' \\ \vec{a}_m'' \bullet \vec{a}_1'' & \vec{a}_m'' \bullet \vec{a}_2'' & \cdots & \vec{a}_m'' \bullet \vec{a}_{m-1}'' & \vec{a}_m'' \bullet \vec{a}_m'' \end{vmatrix}}, \quad (14)$$

with

$$\vec{a}_i'' = \vec{a}_i - \frac{c_i'}{c_m'}\vec{a}_m, \quad i = 1, \ldots, m-1,$$

$$\vec{a}_m'' = \frac{1}{c_m'}\vec{a}_m \quad (15)$$

and

$$c_i' = c_i - \vec{a}_i \bullet \vec{q}. \quad (16)$$

(2) *For the distance, we have*

$$d^2(Q, V) = d^2(O, V') = \| \vec{s}'' \|^2 = \frac{G(\vec{a}_1'', \vec{a}_2'', \ldots, \vec{a}_{m-1}'')}{G(\vec{a}_1'', \vec{a}_2'', \ldots, \vec{a}_{m-1}'', \vec{a}_m'')}. \quad (17)$$

Proof. (1) We perform a translation towards the origin of the coordinates of the pair (Q, V) in order to get the pair (O, V'). We have

$$\vec{x}' = \vec{x} + \overrightarrow{QO} = \vec{x} - \vec{q}.$$

Replacing, in equations (12), $\vec{x}$ with $\vec{x}' + \vec{q}$, we get

$$\vec{a}_1 \bullet \vec{x}' = c_1'$$

$$\vec{a}_2 \bullet \vec{x}' = c_2'$$

$$\vdots$$

$$\vec{a}_{m-1} \bullet \vec{x}' = c'_{m-1}$$

$$\vec{a}_m \bullet \vec{x}' = c'_m, \tag{18}$$

with, at least, one non-zero c'_i and $c'_i = c_i - \vec{a}_i \bullet \vec{q}$.

Now, by using relations (8)-(11), we obtain the relations (14)-(16).

Finally, undoing the translation, we have

$$\vec{s} = \vec{s}'' + \vec{q}.$$

(2) The Euclidean distance is translation invariant:

$$d^2(Q, V) = d^2(Q, S) = d^2(O, V') = \| \vec{s}'' \|^2$$

$$= \frac{G(\vec{a}''_1, \vec{a}''_2, ..., \vec{a}''_{m-1})}{G(\vec{a}''_1, \vec{a}''_2, ..., \vec{a}''_{m-1}, \vec{a}''_m)}. \qquad \square$$

5. Distance between Two Linear Varieties

In this section, we deal with the interesting problem of finding the best approximation pair of points of two given disjoint and non-parallel linear varieties V_1 and V_2. In other words, we are looking for the point S_1 on the linear variety V_1 and the point S_2 on the linear variety V_2 such that the vector $\overrightarrow{S_1S_2}$ is, to within a signal, the shortest one linking the referred to linear varieties. Here the main tool is Proposition 2. This result is applied twice, just bearing in mind that, in the present case, the external point is either the generic point $G_{V_1} := \overrightarrow{gV_1}$ of the linear variety V_1 or the generic point $G_{V_2} := \overrightarrow{gV_2}$ of the linear variety V_2.

Some notation is in order, for the sake of simplicity of the statement of our next result.

We write the vector $\vec{f} \in \mathbb{R}^n$ in the following manner:

$$\vec{f} = (f_1, f_2, ..., f_h, f_{h+1}, f_{h+2}, ..., f_n) := (f_1, f_2, ..., f_h, \vec{\varphi}) \in \mathbb{R}^h \times \mathbb{R}^{n-h}.$$

We state the main result of this paper.

Proposition 3. *Let us consider two disjoint and non-parallel linear varieties* V_1 *and* V_2 *given, respectively, by*

$$V_1 := \begin{cases} \vec{a}_1 \bullet \vec{x} = c_1 \\ \vec{a}_2 \bullet \vec{x} = c_2 \\ \quad \vdots \\ \vec{a}_{m_1-1} \bullet \vec{x} = c_{m_1-1} \\ \vec{a}_{m_1} \bullet \vec{x} = c_{m_1}, \end{cases} \tag{19}$$

where $\vec{a}_1, \vec{a}_2, ..., \vec{a}_{m_1}$ *are linearly independent vectors in* $\mathbb{R}^n$ *and with, at least, one non-zero scalar* c_i, $i = 1, ..., m_1$, $m_1 \geq 2$, *and*

$$V_2 := \begin{cases} \vec{b}_1 \bullet \vec{y} = d_1 \\ \vec{b}_2 \bullet \vec{y} = d_2 \\ \quad \vdots \\ \vec{b}_{m_2-1} \bullet \vec{y} = d_{m_2-1} \\ \vec{b}_{m_2} \bullet \vec{y} = d_{m_2}, \end{cases} \tag{20}$$

where $\vec{b}_1, \vec{b}_2, ..., \vec{b}_{m_2}$ *are linearly independent vectors in* $\mathbb{R}^n$ *and with, at least, one non-zero scalar* d_i, $i = 1, ..., m_2$, $m_2 \geq 2$.

Let us denote $\vec{x} = (x_1, ..., x_{m_1}, x_{m_1+1}, ..., x_n) \in V_1$ *as* $\vec{x} = (x_1, ..., x_{m_1}, \vec{\xi}) \in \mathbb{R}^{m_1} \times \mathbb{R}^{n-m_1}$ *and* $\vec{y} = (y_1, ..., y_{m_2}, y_{m_2+1}, ..., y_n) \in V_2$ *as* $\vec{y} = (y_1, ..., y_{m_2}, \vec{\eta}) \in \mathbb{R}^{m_2} \times \mathbb{R}^{n-m_2}$.

Let us denote by $[S_1 S_2]$ *the shortest straight line segment connecting the two linear varieties* V_1 *and* V_2.

Then

(1) *The points* $S_1 \in V_1$ *and* $S_2 \in V_2$ *are obtained through the unique solution of the overdetermined consistent system of linear algebraic*

equations

$$\begin{cases} S_1(\vec{\eta}) = G_{V_1}(\vec{\xi}), \\ S_2(\vec{\xi}) = G_{V_2}(\vec{\eta}), \end{cases} \tag{21}$$

where

(i)

$G_{V_1}(\vec{\xi}) := G_{V_1}(x_{m_1+1}, x_{m_1+2}, ..., x_n)$ *and* $G_{V_2}(\vec{\eta}) := G_{V_2}(y_{m_2+1}, y_{m_2+2}, ..., y_n)$

are the generic points of, respectively, the linear varieties V_1 *and* V_2;

(ii)

$S_1(\vec{\eta}) := S_1(y_{m_2+1}, y_{m_2+2}, ..., y_n)$ *and* $S_2(\vec{\xi}) := S_2(x_{m_1+1}, x_{m_1+2}, ..., x_n)$

are given, respectively, by

$$S_1(\vec{\eta}) = S_1''(\vec{\eta}) + G_{V_2}(\vec{\eta})$$

and

$$S_2(\vec{\xi}) = S_2'(\vec{\xi}) + G_{V_1}(\vec{\xi});$$

and where

(iii) $S_1''(\vec{\eta})$ *is given by*

$$\vec{s}_1''(\vec{\eta}) := S_1''(\vec{\eta}) = \frac{\begin{vmatrix} \vec{a}_1'' \bullet \vec{a}_1'' & \cdots & \vec{a}_1'' \bullet \vec{a}_{m_1-1}'' & \vec{a}_1'' \bullet \vec{a}_{m_1}'' \\ \vdots & \cdots & \vdots & \vdots \\ \vec{a}_{m_1-1}'' \bullet \vec{a}_1'' & \cdots & \vec{a}_{m_1-1}'' \bullet \vec{a}_{m_1-1}'' & \vec{a}_{m_1-1}'' \bullet \vec{a}_{m_1}'' \\ \vec{a}_1'' & \cdots & \vec{a}_{m_1-1}'' & \vec{a}_{m_1}'' \end{vmatrix}}{\begin{vmatrix} \vec{a}_1'' \bullet \vec{a}_1'' & \cdots & \vec{a}_1'' \bullet \vec{a}_{m_1-1}'' & \vec{a}_1'' \bullet \vec{a}_{m_1}'' \\ \vec{a}_2'' \bullet \vec{a}_1'' & \cdots & \vec{a}_2'' \bullet \vec{a}_{m_1-1}'' & \vec{a}_2'' \bullet \vec{a}_{m_1}'' \\ \vdots & \cdots & \vdots & \vdots \\ \vec{a}_{m_1-1}'' \bullet \vec{a}_1'' & \cdots & \vec{a}_{m_1-1}'' \bullet \vec{a}_{m_1-1}'' & \vec{a}_{m_1-1}'' \bullet \vec{a}_{m_1}'' \\ \vec{a}_{m_1}'' \bullet \vec{a}_1'' & \cdots & \vec{a}_{m_1}'' \bullet \vec{a}_{m_1-1}'' & \vec{a}_{m_1}'' \bullet \vec{a}_{m_1}'' \end{vmatrix}}, \tag{22}$$

being

$$\vec{a}_i'' = \vec{a}_i - \frac{c_i'}{c_{m_1}'}\vec{a}_{m_1}, \quad i = 1, ..., m_1 - 1,$$

$$\vec{a}_{m_1}'' = \frac{1}{c_{m_1}'}\vec{a}_{m_1}$$

with, at least, one non-zero $c_i' = c_i - \vec{a}_i \bullet \overrightarrow{g_{V_2}}$, $i = 1, ..., m_1$, *and*

$$\vec{s}_2''(\vec{\xi}) := S_2''(\vec{\xi}) = \frac{\begin{vmatrix} \vec{b}_1'' \bullet \vec{b}_1'' & \cdots & \vec{b}_1'' \bullet \vec{b}_{m_2-1}'' & \vec{b}_1'' \bullet \vec{b}_{m_2}'' \\ \vdots & \cdots & \vdots & \vdots \\ \vec{b}_{m_2-1}'' \bullet \vec{b}_1'' & \cdots & \vec{b}_{m_2-1}'' \bullet \vec{b}_{m_2-1}'' & \vec{b}_{m_2-1}'' \bullet \vec{b}_{m_2}'' \\ \vec{b}_1'' & \cdots & \vec{b}_{m_2-1}'' & \vec{b}_{m_2}'' \end{vmatrix}}{\begin{vmatrix} \vec{b}_1'' \bullet \vec{b}_1'' & \cdots & \vec{b}_1'' \bullet \vec{b}_{m_2-1}'' & \vec{b}_1'' \bullet \vec{b}_{m_2}'' \\ \vec{b}_2'' \bullet \vec{b}_1'' & \cdots & \vec{b}_2'' \bullet \vec{b}_{m_2-1}'' & \vec{b}_2'' \bullet \vec{b}_{m_2}'' \\ \vdots & \cdots & \vdots & \vdots \\ \vec{b}_{m_2-1}'' \bullet \vec{b}_1'' & \cdots & \vec{b}_{m_2-1}'' \bullet \vec{b}_{m_2-1}'' & \vec{b}_{m_2-1}'' \bullet \vec{b}_{m_2}'' \\ \vec{b}_{m_2}'' \bullet \vec{b}_1'' & \cdots & \vec{b}_{m_2}'' \bullet \vec{b}_{m_2-1}'' & \vec{b}_{m_2}'' \bullet \vec{b}_{m_2}'' \end{vmatrix}}, \quad (23)$$

being

$$\vec{b}_i'' = \vec{b}_i - \frac{d_i'}{d_{m_2}'}\vec{b}_{m_2}, \quad i = 1, ..., m_2 - 1,$$

$$\vec{b}_{m_2}'' = \frac{1}{d_{m_2}'}\vec{b}_{m_2}$$

with, at least, one non-zero $d_i' = d_i - \vec{b}_i \bullet \overrightarrow{g_{V_1}}$, $i = 1, ..., m_2$.

(2) *The distance* $d(V_1, V_2)$ *between the two linear varieties is given by*

$$d(V_1, V_2) = \| \overrightarrow{S_1 S_2} \|.$$

Proof. Essentially the proof consists on dealing once at a time with the two linear varieties V_1 and V_2:

(1) finding the generic point of each linear variety;

(2) applying Proposition 2.

In the following way:

(i) The generic points

From the underdetermined system (19), we can, without loss of generality, assume that the generic point $G_{V_1} := \overrightarrow{g_{V_1}}$ depends on the $n - m_1 + 1$ parameters $x_{m_1+1}, x_{m_1+2}, ..., x_n$.

We write

$$G_{V_1} = G_{V_1}(\vec{\xi}) = \begin{bmatrix} x_1(x_{m_1+1}, ..., x_n) \\ \vdots \\ x_{m_1}(x_{m_1+1}, ..., x_n) \\ x_{m_1+1} \\ \vdots \\ x_n \end{bmatrix} := \overrightarrow{g_{V_1}}. \tag{24}$$

Similarly, we write for the generic point $G_{V_2} := \overrightarrow{g_{V_2}}$ of the linear variety V_2:

$$G_{V_2} = G_{V_2}(\vec{\eta}) = \begin{bmatrix} y_1(y_{m_2+1}, ..., y_n) \\ \vdots \\ y_{m_2}(y_{m_2+1}, ..., y_n) \\ y_{m_2+1} \\ \vdots \\ y_n \end{bmatrix} := \overrightarrow{g_{V_2}}. \tag{25}$$

(ii) The application of Proposition 2

(a) Concerning the pair (G_{V_2}, V_1), we get

$$S_1''(y_{m_2+1}, y_{m_2+2}, ..., y_n) = \frac{\begin{vmatrix} \vec{a}_1'' \bullet \vec{a}_1'' & \cdots & \vec{a}_1'' \bullet \vec{a}_{m_1-1}'' & \vec{a}_1'' \bullet \vec{a}_{m_1}'' \\ \vdots & \cdots & \vdots & \vdots \\ \vec{a}_{m_1-1}'' \bullet \vec{a}_1'' & \cdots & \vec{a}_{m_1-1}'' \bullet \vec{a}_{m_1-1}'' & \vec{a}_{m_1-1}'' \bullet \vec{a}_{m_1}'' \\ \vec{a}_1'' & \cdots & \vec{a}_{m_1-1}'' & \vec{a}_{m_1}'' \end{vmatrix}}{\begin{vmatrix} \vec{a}_1'' \bullet \vec{a}_1'' & \cdots & \vec{a}_1'' \bullet \vec{a}_{m_1-1}'' & \vec{a}_1'' \bullet \vec{a}_{m_1}'' \\ \vec{a}_2'' \bullet \vec{a}_1'' & \cdots & \vec{a}_2'' \bullet \vec{a}_{m_1-1}'' & \vec{a}_2'' \bullet \vec{a}_{m_1}'' \\ \vdots & \cdots & \vdots & \vdots \\ \vec{a}_{m_1-1}'' \bullet \vec{a}_1'' & \cdots & \vec{a}_{m_1-1}'' \bullet \vec{a}_{m_1-1}'' & \vec{a}_{m_1-1}'' \bullet \vec{a}_{m_1}'' \\ \vec{a}_{m_1}'' \bullet \vec{a}_1'' & \cdots & \vec{a}_{m_1}'' \bullet \vec{a}_{m_1-1}'' & \vec{a}_{m_1}'' \bullet \vec{a}_{m_1}'' \end{vmatrix}},$$

where

$$\vec{a}_i'' = \vec{a}_i - \frac{c_i'}{c_{m_1}'}\vec{a}_{m_1}, \quad i = 1, ..., m_1 - 1,$$

$$\vec{a}_{m_1}'' = \frac{1}{c_{m_1}'}\vec{a}_{m_1}$$

with, at least, one non-zero $c_i' = c_i - \vec{a}_i \bullet \overrightarrow{g_{V_2}}$, $i = 1, ..., m_1$.

(b) Concerning the pair (G_{V_1}, V_2), we get

$$S_2''(x_{m_1+1}, x_{m_1+2}, ..., x_n) = \frac{\begin{vmatrix} \vec{b}_1'' \bullet \vec{b}_1'' & \cdots & \vec{b}_1'' \bullet \vec{b}_{m_2-1}'' & \vec{b}_1'' \bullet \vec{b}_{m_2}'' \\ \vdots & \cdots & \vdots & \vdots \\ \vec{b}_{m_2-1}'' \bullet \vec{b}_1'' & \cdots & \vec{b}_{m_2-1}'' \bullet \vec{b}_{m_2-1}'' & \vec{b}_{m_2-1}'' \bullet \vec{b}_{m_2}'' \\ \vec{b}_1'' & \cdots & \vec{b}_{m_2-1}'' & \vec{b}_{m_2}'' \end{vmatrix}}{\begin{vmatrix} \vec{b}_1'' \bullet \vec{b}_1'' & \cdots & \vec{b}_1'' \bullet \vec{b}_{m_2-1}'' & \vec{b}_1'' \bullet \vec{b}_{m_2}'' \\ \vec{b}_2'' \bullet \vec{b}_1'' & \cdots & \vec{b}_2'' \bullet \vec{b}_{m_2-1}'' & \vec{b}_2'' \bullet \vec{b}_{m_2}'' \\ \vdots & \cdots & \vdots & \vdots \\ \vec{b}_{m_2-1}'' \bullet \vec{b}_1'' & \cdots & \vec{b}_{m_2-1}'' \bullet \vec{b}_{m_2-1}'' & \vec{b}_{m_2-1}'' \bullet \vec{b}_{m_2}'' \\ \vec{b}_{m_2}'' \bullet \vec{b}_1'' & \cdots & \vec{b}_{m_2}'' \bullet \vec{b}_{m_2-1}'' & \vec{b}_{m_2}'' \bullet \vec{b}_{m_2}'' \end{vmatrix}},$$

where

$$\vec{b}_i'' = \vec{b}_i - \frac{d_i'}{d_{m_2}'}\vec{b}_{m_2}, \quad i = 1, ..., m_2 - 1,$$

$$\vec{b}_{m_2}'' = \frac{1}{d_{m_2}'}\vec{b}_{m_2}$$

with, at least, one non-zero $d_i' = d_i - \vec{b}_i \bullet \overrightarrow{g_{V_1}}$, $i = 1, ..., m_2$.

Essentially, the points S_1'' and S_2'' resulted from translations of the pairs (G_{V_2}, V_1) and (G_{V_1}, V_2). Undoing the translations, we have

$$S_1 = S_1'' + G_{V_2},$$

$$S_2 = S_2'' + G_{V_1}.$$

We must get the unique solution of the overdetermined system

$$\begin{cases} S_1(y_{m_2+1}, y_{m_2+2}, ..., y_n) = G_{V_1}(x_{m_1+1}, x_{m_1+2}, ..., x_n), \\ S_2(x_{m_1+1}, x_{m_1+2}, ..., x_n) = G_{V_2}(y_{m_2+1}, y_{m_2+2}, ..., y_n) \end{cases} \tag{26}$$

of $2n$ equations and the $(n - m_1 + 1) + (n - m_2 + 1)$ indeterminates

$$x_{m_1+1}, x_{m_1+2}, ..., x_n, y_{m_2+1}, y_{m_2+2}, ..., y_n.$$

This system is consistent and has the unique solution

$$(x_{m_1+1}^*, x_{m_1+2}^*, ..., x_n^*, y_{m_2+1}^*, y_{m_2+2}^*, ..., y_n^*).$$

Hence, we obtain

$$S_1 = G_{V_1}^* = G_{V_1}(\vec{\xi}^*) = \begin{bmatrix} x_1(x_{m_1+1}^*, ..., x_n^*) \\ \vdots \\ x_{m_1}(x_{m_1+1}^*, ..., x_n^*) \\ x_{m_1+1}^* \\ \vdots \\ x_n^* \end{bmatrix} = \begin{bmatrix} x_1^* \\ \vdots \\ x_{m_1}^* \\ x_{m_1+1}^* \\ \vdots \\ x_n^* \end{bmatrix}$$

and

$$S_2 = G_{V_2}^* = G_{V_2}(\vec{\eta}^*) = \begin{bmatrix} y_1(y_{m_2+1}^*, \ldots, y_n^*) \\ \vdots \\ y_{m_2}(y_{m_2+1}^*, \ldots, y_n^*) \\ y_{m_2+1}^* \\ \vdots \\ y_n^* \end{bmatrix} = \begin{bmatrix} y_1^* \\ \vdots \\ y_{m_2}^* \\ y_{m_2+1}^* \\ \vdots \\ y_n^* \end{bmatrix}.$$

The assertion on consistency of system (26) and uniqueness of the solution of system (26) are supported by results on existence and uniqueness of best approximation problems [8, p. 64, Theorem 1] [7, p. 45, Theorem 2.2.5]. □

Remark 2. Some attention must be paid to the formulas (22) and (23). In fact, we have

$$\vec{s}_1'' = \vec{s}_1''(\vec{\eta}) = \frac{1}{A}\sum_{i=1}^{m_1} A_i\vec{a}_i'', \tag{27}$$

where $A, \ A_i, \ i = 1, \ldots, m_1$ are higher-degree polynomials in several variables $y_{m_2+1}, y_{m_2+2}, \ldots, y_n$ and

$$\vec{s}_2'' = \vec{s}_2''(\vec{\xi}) = \frac{1}{B}\sum_{j=1}^{m_2} B_j\vec{b}_j'', \tag{28}$$

where $B, \ B_j, \ j = 1, \ldots, m_2$ are higher-degree polynomials in several variables $x_{m_1+1}, x_{m_2+2}, \ldots, x_n$. However, from (27) and (28), we have

$$\vec{s}_1'' = \vec{s}_1''(\vec{\eta}) = \sum_{i=1}^{n} L_{1i}(\vec{\eta})\vec{e}_i, \tag{29}$$

where $L_{1i}, \ i = 1, \ldots, n,$ are first degree polynomials in the variables $y_{m_2+1}, y_{m_2+2}, \ldots, y_n$ and

$$\vec{s}_2'' = \vec{s}_2''(\vec{\xi}) = \sum_{i=1}^{n} L_{2i}(\vec{\xi})\vec{e}_i, \tag{30}$$

where L_{2i}, $i = 1, \ldots, n$, are first degree polynomials in the variables $x_{m_2+1}, x_{m_2+2}, \ldots, x_n$.

This question is worth for a longer explanation as follows:

By performing the mentioned convenient translations on the systems (19) and (20), we obtain two systems where the right hand sides are vectors whose entries are linear expressions in the parameters that are coordinates of the vectors $\overrightarrow{G_{V_1}}$ and $\overrightarrow{G_{V_2}}$. By using arguments involving the uniqueness of (least squares) solution of a linear system by using the Moore-Penrose inverse, we assert that the solutions of the afore-referred to systems are given in terms of such parameters. The best solution in the least squares sense of the system $A\vec{x} = \vec{b}$ is given [9, p. 439] by $\vec{x} = A^{\dagger}\vec{b}$, where $A^{\dagger}$ stands for the Moore-Penrose inverse of matrix A. In our case, $A^{\dagger}$ is a constant matrix, so $\vec{x}$ depends on the parameters in vector $\vec{b}$.

Hence,

$$\vec{s}_1'' = \begin{bmatrix} L_1(y_{m_2+1}, y_{m_2+2}, \ldots, y_n) \\ L_2(y_{m_2+1}, y_{m_2+2}, \ldots, y_n) \\ \vdots \\ L_n(y_{m_2+1}, y_{m_2+2}, \ldots, y_n) \end{bmatrix}$$

and

$$\vec{s}_2'' = \begin{bmatrix} L_1(x_{m_1+1}, x_{m_1+2}, \ldots, x_n) \\ L_2(x_{m_1+1}, x_{m_1+2}, \ldots, x_n) \\ \vdots \\ L_n(x_{m_1+1}, x_{m_1+2}, \ldots, x_n) \end{bmatrix}.$$

For the sake of clarity, we synthesize:

Scholium. Regarding the given linear varieties and without loss of generality, we can write

$$V_1 = \left\{ \begin{bmatrix} x_1(x_{m_1+1}, \ldots, x_n) \\ \vdots \\ x_{m_1}(x_{m_1+1}, \ldots, x_n) \\ x_{m_1+1} \\ \vdots \\ x_n \end{bmatrix} : (x_{m_1+1}, \ldots, x_n) \in \mathbb{R}^{n-m_1} \right\}$$

and

$$V_2 = \left\{ \begin{bmatrix} y_1(y_{m_2+1}, \ldots, y_n) \\ \vdots \\ y_{m_2}(x_{m_2+1}, \ldots, y_n) \\ y_{m_2+1} \\ \vdots \\ y_n \end{bmatrix} : (y_{m_2+1}, \ldots, y_n) \in \mathbb{R}^{n-m_2} \right\}.$$

Hence, we may write

$$S_1 = G_{V_1}^* = \begin{bmatrix} x_1(x_{m_1+1}^*, \ldots, x_n^*) \\ \vdots \\ x_{m_1}(x_{m_1+1}^*, \ldots, x_n^*) \\ x_{m_1+1}^* \\ \vdots \\ x_n^* \end{bmatrix} = \begin{bmatrix} x_1^* \\ \vdots \\ x_{m_1}^* \\ x_{m_1+1}^* \\ \vdots \\ x_n^* \end{bmatrix}$$

and

$$S_2 = G_{V_2}^* = \begin{bmatrix} y_1(y_{m_2+1}^*, \ldots, y_n^*) \\ \vdots \\ y_{m_2}(y_{m_2+1}^*, \ldots, y_n^*) \\ y_{m_2+1}^* \\ \vdots \\ y_n^* \end{bmatrix} = \begin{bmatrix} y_1^* \\ \vdots \\ y_{m_2}^* \\ y_{m_2+1}^* \\ \vdots \\ y_n^* \end{bmatrix},$$

where

$$(x^*_{m_1+1},\ x^*_{m_1+2},\ \ldots,\ x^*_n,\ y^*_{m_2+1},\ y^*_{m_2+2},\ \ldots,\ y^*_n)$$

is the unique solution of the overdetermined system (26).

A classical projection theorem [8, p. 64, Theorem 1] [7, p. 45, Theorem 2.2.5] [3, p. 64, Exercise 2] concerning the case of a point and a linear variety, leads us to a result on the projection vector connecting two linear varieties. It is a characterization of the pair of best approximation points that may be useful when testing the accuracy of numerical examples.

Proposition 4. *Let* V_1 *and* V_2 *be two non-parallel linear varieties*: $V_1 = P_1 + M_1$ *and* $V_2 = P_2 + M_2$, *where* M_1 *and* M_2 *are subspaces of* $\mathbb{R}^n$ *and* P_1 *and* P_2 *are fixed points in* $\mathbb{R}^n$. *Then the unique points* $S_1 \in V_1$ *and* $S_2 \in V_2$ *form a best approximation pair* (S_1, S_2) *of the linear varieties* V_1 *and* V_2 *if and only if the two vectors whose extremities are* S_1 *and* S_2 *are orthogonal simultaneously to the subspaces* M_1 *and* M_2.

Proof. We need just two facts: the definition of a vector orthogonal to a set of $\mathbb{R}^n$, where a vector is said to be *orthogonal* to set if it is orthogonal to each vector of the set; and a projection theorem, where it is stated that the projection vector is orthogonal to the unique subspace associated to the given linear variety and not to the linear variety itself [8, p. 64, Theorem 1] [7, p. 45, Theorem 2.2.5] [3, p. 64, Exercise 2].

We have:

(1) $S_2 = \vec{s}_2$ is the projection of $S_1 := \vec{s}_1$ onto the linear variety V_2: hence $\overrightarrow{S_1S_2}$ is orthogonal to the subspace M_2;

(2) $S_1 = \vec{s}_1$ is the projection of $S_2 := \vec{s}_2$ onto the linear variety V_1: hence $\overrightarrow{S_1S_2}$ is orthogonal to the subspace M_1. □

Notice that the vector $\overrightarrow{S_1S_2}$ is not orthogonal either to the linear varieties V_1 or V_2.

Finally, we have a result concerning the separating hyperplanes [3, pp. 105-106] and the smallest sphere tangent to the two linear varieties simultaneously.

Corollary 1. *The smallest sphere* S *tangent to the linear varieties* V_1 *and* V_2 *is given by*

$$S = \left\{ \vec{x} \in \mathbb{R}^n : \left\| \vec{x} - \frac{\vec{s}_1 + \vec{s}_2}{2} \right\| = \left\| \frac{\vec{s}_1 - \vec{s}_2}{2} \right\| \right\}$$

and the supporting hyperplanes are

$$H_i = \{ \vec{x} \in \mathbb{R}^n : (\vec{s}_1 - \vec{s}_2) \bullet (\vec{x} - \vec{s}_i) = 0 \}, \quad i = 1, 2.$$

Corollary 2. *The point*

$$\vec{s} = -\frac{1}{8} \frac{\| \vec{s}_1 - \vec{s}_2 \|}{\| \vec{s}_1 + \vec{s}_2 \|} (\vec{s}_1 + \vec{s}_2)$$

is the point closest to the origin of the coordinates which belongs to the smallest sphere tangent to the linear varieties V_1 *and* V_2.

If the linear varieties V_1 and V_2 are translates of the same given subspace, then there are infinitely many best approximation pairs $(S_1, S_2) \in V_1 \times V_2$. In this context, the optimal approximation pair is (S_1^*, S_2^*), where S_1^* and S_2^* are the best approximation points with minimum norm.

Hence, we have

Corollary 3. *Let* $(S_1^*, S_2^*) \in V_1 \times V_2$ *be the optimal approximation pair of the parallel linear varieties* V_1 *and* V_2. *Then the sphere* S^* *tangent to the parallel linear varieties* V_1 *and* V_2 *which is nearest to the origin is*

$$S^* = \left\{ \vec{x} \in \mathbb{R}^n : \left\| \vec{x} - \frac{\vec{s}_1^* + \vec{s}_2^*}{2} \right\| = \left\| \frac{\vec{s}_1^* - \vec{s}_2^*}{2} \right\| \right\}.$$

Corollary 4. *The point*

$$\vec{s}^* = -\frac{1}{8}\frac{\| \vec{s}_1^* - \vec{s}_2^* \|}{\| \vec{s}_1^* + \vec{s}_2^* \|}(\vec{s}_1^* + \vec{s}_2^*)$$

is the point closest to the origin of the coordinates which belongs to the family of spheres that are tangent to the parallel linear varieties V_1 and V_2.

6. Illustrative Numerical Example

We are given two linear varieties. We exhibit the best two approximation points – one on each linear variety, and show that the vector $\overrightarrow{S_1S_2}$ is orthogonal to both the subspace M_1 and the subspace M_2 associated to the linear varieties V_1 and V_2, respectively, but not to the linear varieties themselves.

Let the two linear varieties V_1 and V_2 be defined as follows:

$$V_1 := \begin{cases} \vec{a}_1 \bullet \vec{x} = 1, \\ \vec{a}_2 \bullet \vec{x} = 2, \end{cases} \tag{31}$$

with $\vec{a}_1 = (1, -1, -2, 1, 1)$ and $\vec{a}_2 = (1, 1, -4, 1, 2)$;

$$V_2 := \begin{cases} \vec{b}_1 \bullet \vec{y} = -10, \\ \vec{b}_2 \bullet \vec{y} = -20, \\ \vec{b}_3 \bullet \vec{y} = 3, \end{cases} \tag{32}$$

with $\vec{b}_1 = (1, -1, -2, 1, 1)$, $\vec{b}_2 = (-1, 1, -4, 1, 2)$ and $\vec{b}_3 = (1, 1, -4, -1, 3)$.

In respect of Proposition 3:

(I) The generic points G_{V_1} and G_{V_2} of the linear varieties V_1 and V_2 are

$$G_{V_1} = \begin{bmatrix} \frac{3}{2} + 3x_3 - x_4 - \frac{3}{2}x_5 \\ \frac{1}{2} + x_3 - \frac{1}{2}x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

and

$$G_{V_2} = \begin{bmatrix} \frac{23}{2} + y_4 - \frac{1}{2}y_5 \\ \frac{23}{2} + \frac{4}{3}y_4 - \frac{1}{2}y_5 \\ 5 + \frac{1}{3}y_4 + \frac{1}{2}y_5 \\ y_4 \\ y_5 \end{bmatrix}.$$

(II) We perform a translation along the vector $\overrightarrow{G_{V_2}(y_4, y_5)O} = O - G_{V_2}(y_4, y_5)$; the linear variety V_1' is obtained by replacing $\vec{x}$ in relation (31) with $\vec{x}' + \overrightarrow{G_{V_2}}$:

$$V_1' := \begin{cases} \vec{a}_1 \bullet \vec{x}' = 11, \\ \vec{a}_2 \bullet \vec{x}' = -1 - 2y_4 + y_5 \end{cases} \tag{33}$$

with

$$S_1'(y_4, y_5) = \begin{bmatrix} \frac{15}{7} + \frac{2}{21}y_4 - \frac{1}{21}y_5 \\ -\frac{131}{21} - \frac{38}{63}y_4 + \frac{19}{63}y_5 \\ -\frac{4}{21} + \frac{20}{63}y_4 - \frac{10}{63}y_5 \\ \frac{15}{7} + \frac{2}{21}y_4 - \frac{1}{21}y_5 \\ \frac{2}{21} - \frac{10}{63}y_4 + \frac{5}{63}y_5 \end{bmatrix}$$

and

$$S_1(y_4,\ y_5) = S_1' + G_{V_2} = \begin{bmatrix} \frac{191}{14} + \frac{23}{21}y_4 - \frac{23}{42}y_5 \\ \frac{221}{42} + \frac{46}{63}y_4 - \frac{25}{126}y_5 \\ \frac{101}{21} + \frac{41}{63}y_4 + \frac{43}{126}y_5 \\ \frac{15}{7} + \frac{23}{21}y_4 - \frac{1}{21}y_5 \\ \frac{2}{21} - \frac{10}{63}y_4 + \frac{68}{63}y_5 \end{bmatrix}$$

analogously.

(III) We perform a translation along the vector $\overrightarrow{G_{V_1}(x_3,\ x_4,\ x_5)O} = O - G_{V_1}(x_3,\ x_4,\ x_5)$; the linear variety V_2' is obtained by replacing $\vec{y}$ in the relation (32) with $\vec{y}' + \overrightarrow{G_{V_1}}$:

$$V_2' := \begin{cases} \vec{b}_1 \bullet \vec{y}' = -11, \\ \vec{b}_2 \bullet \vec{y}' = -19 + 6x_3 - 2x_4 - 3x_5, \\ \vec{b}_3 \bullet \vec{y}' = 1 + 2x_4 - x_5 \end{cases} \tag{34}$$

with

$$S_2'(x_3,\ x_4,\ x_5) = \begin{bmatrix} \frac{633}{209} - \frac{366}{209}x_3 + \frac{192}{209}x_4 + \frac{148}{209}x_5 \\ \frac{35}{19} + \frac{12}{19}x_3 - \frac{1}{19}x_4 - \frac{15}{38}x_5 \\ \frac{674}{209} - \frac{150}{209}x_3 + \frac{41}{209}x_4 + \frac{159}{418}x_8 \\ -\frac{1371}{209} + \frac{240}{209}x_3 - \frac{191}{209}x_4 - \frac{129}{418}x_5 \\ \frac{172}{209} - \frac{42}{209}x_3 + \frac{70}{209}x_4 - \frac{7}{209}x_5 \end{bmatrix}$$

and

$$S_2(x_3,\ x_4,\ x_5) = S_2' + G_{V_1} = \begin{bmatrix} \frac{1893}{418} + \frac{261}{209}x_3 - \frac{17}{209}x_4 - \frac{331}{418}x_5 \\ \frac{89}{38} + \frac{31}{19}x_3 - \frac{1}{19}x_4 - \frac{17}{19}x_5 \\ \frac{674}{209} + \frac{59}{209}x_3 + \frac{41}{209}x_4 + \frac{159}{418}x_5 \\ -\frac{1371}{209} + \frac{240}{209}x_3 + \frac{18}{209}x_4 - \frac{129}{418}x_5 \\ \frac{172}{209} - \frac{42}{209}x_3 + \frac{70}{209}x_4 + \frac{202}{209}x_5 \end{bmatrix}.$$

(IV) Solving the system

$$\begin{cases} S_1(y_4,\ y_5) = G_{V_1}(x_3,\ x_4,\ x_5), \\ S_2(x_3,\ x_4,\ x_5) = G_{V_2}(y_4,\ y_5), \end{cases}$$

we obtain $x_3^* = \frac{837}{848}$, $x_4^* = -\frac{4765}{848}$, $x_5^* = \frac{1489}{424}$, $y_4^* = -\frac{3560}{509}$, $y_5^* = \frac{453}{212}$.

Hence, using

$$S_1 = G_{V_1}^* = \begin{bmatrix} \frac{3}{2} + 3x_3^* - x_4^* - \frac{3}{2}x_5^* \\ \frac{1}{2} + x_3^* - \frac{1}{2}x_5^* \\ x_3^* \\ x_4^* \\ x_5^* \end{bmatrix}$$

and

$$S_2 = G_{V_2}^* = \begin{bmatrix} \frac{23}{2} + y_4^* - \frac{1}{2}y_5^* \\ \frac{23}{2} + \frac{4}{3}y_4^* - \frac{1}{2}y_5^* \\ 5 + \frac{1}{3}y_4^* + \frac{1}{2}y_5^* \\ y_4^* \\ y_5^* \end{bmatrix},$$

we, finally, obtain

$$S_1 = \begin{bmatrix} \frac{77}{16} \\ -\frac{57}{212} \\ \frac{837}{848} \\ -\frac{4765}{848} \\ \frac{1489}{424} \end{bmatrix} \quad \text{and} \quad S_2 = \begin{bmatrix} \frac{55}{16} \\ \frac{469}{424} \\ \frac{3169}{848} \\ -\frac{3560}{509} \\ \frac{453}{212} \end{bmatrix}.$$

The distance between the two varieties is given by

$$d(V_1, V_2) = \| \overrightarrow{S_1S_2} \| = \frac{2174}{559}.$$

In respect of Proposition 4;

Let us consider

$$V_1 = P_1 + M_1 := \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix} + \left\{ \begin{bmatrix} 3x_3 - x_4 - \frac{3}{2}x_5 \\ x_3 - \frac{1}{2}x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} : x_3, x_4, x_5 \in \mathbb{R} \right\}$$

and

$$V_2 = P_2 + M_2 := \begin{bmatrix} \frac{23}{2} \\ \frac{23}{2} \\ 5 \\ 0 \\ 0 \end{bmatrix} + \left\{ \begin{bmatrix} y_4 - \frac{1}{2}y_5 \\ \frac{4}{3}y_4 - \frac{1}{2}y_5 \\ \frac{1}{3}y_4 + \frac{1}{2}y_5 \\ y_4 \\ y_5 \end{bmatrix} : y_4, y_5 \in \mathbb{R} \right\}.$$

(α_1) The vector $\overrightarrow{S_1S_2}$ is orthogonal to the unique subspace M_1 associated to the linear variety V_1. Consider the arbitrarily fixed vector

$$\vec{v}_1 = \begin{bmatrix} 3x_3 - x_4 - \frac{3}{2}x_5 \\ x_3 - \frac{1}{2}x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in M_1.$$

We have

$$\overrightarrow{S_1S_2} \bullet \vec{v}_1 = 0.$$

(α_2) The vector $\overrightarrow{S_1S_2}$ is orthogonal to the unique subspace M_2 associated to the linear variety V_2. Consider the arbitrarily fixed vector

$$\vec{v}_2 = \begin{bmatrix} y_4 - \frac{1}{2}y_5 \\ \frac{4}{3}y_4 - \frac{1}{2}y_5 \\ \frac{1}{3}y_4 + \frac{1}{2}y_5 \\ y_4 \\ y_5 \end{bmatrix} \in M_2.$$

We have

$$\overrightarrow{S_1S_2} \bullet \vec{v}_2 = 0.$$

(β_1) The vector $\overrightarrow{S_1S_2}$ is not orthogonal to the linear variety V_1.

Take the fixed vector

$$\vec{u}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \in V_1.$$

We have $\overrightarrow{S_1S_2} \bullet \vec{u}_1 = -1.3750 \neq 0.$

(β_2) The vector $\overrightarrow{S_1S_2}$ is not orthogonal to the linear variety V_2.

Take the fixed vector

$$\vec{u}_2 = \begin{bmatrix} \frac{23}{2} \\ \frac{71}{6} \\ \frac{24}{3} \\ 1 \\ 2 \end{bmatrix} \in V_2.$$

We have $\overrightarrow{S_1S_2} \bullet \vec{u}_2 = 13.7500 \neq 0.$

7. Conclusions

In this paper, we presented a determinantal formula for the point satisfying the equality condition in an inequality by Fan and Todd (we answered the implicit old open question in [5, p. 63]: to get a closed form for the minimum norm vector of the given linear variety). In a previous paper [13], we got, by using the center of convenient hyperquadrics, the point where the inequality (3) turns into the equality (4).

Here, we also restated a determinantal formula for the point of tangency between a sphere and any linear variety.

Furthermore, we obtained the projection of an external point onto a linear variety as a quotient of two determinants. Subsequently and consequently, this result was extended for getting the best approximation pair of two disjoint and non-parallel linear varieties. A characterization of this pair of best approximation points is offered.

Acknowledgements

M. A. Facas Vicente is supported by INESC Coimbra, DEEC, University of Coimbra, Pólo II, 3030-290 Coimbra, Portugal. M. A. Facas Vicente also has been supported by the Portuguese Foundation for Science and Technology (FCT) under project grant UID/Multi/00308/2019.

Fernando Martins work is funded by FCT/MEC through national funds and when applicable co-funder FEDER-PT2020 partnership agreement under the project UID/EEA/50008/2013.

References

[1] P. R. Beesack, On Bessel's inequality and Ostrowski's, Univ. Beograd. Publ. Elektohn. Fak. Ser. Mat. Fiz. 510 (1975), 69-71.

[2] C. Costa, P. Tadeu and J. Vitória, Geometric application of an analytic (in)equality of (Ostrowski) Mitrinovic, Inequality Theory and Applications, Yeol Je Cho, Jong Kyu Kim and Sever S. Dragomir, eds., Nova Science Publishers, Vol. 6, 2010, pp. 143-150.

[3] F. Deutsch, Best Approximation in Inner Product Spaces, Springer, New York, 2001.

[4] M. A. Facas Vicente, Armando Gonçalves and José Vitória, Euclidean distance between two linear varieties, Appl. Math. Sci. 8(21) (2014), 1039-1043.

[5] K. Fan and J. Todd, A determinantal inequality, J. London Math. Soc. 30 (1955), 58-64.

[6] Armando Gonçalves, M. A. Facas Vicente and José Vitória, Optimal pair of two linear varieties, Appl. Math. Sci. 9(12) (2015), 593-596.

[7] P.-J. Laurent, Approximation et Optimisation, Hermann, Paris, 1972.

[8] D. G. Luenberger, Optimization by Vector Space Methods, J. Wiley, New York, 1969.

[9] C. D. Meyer, Matrix Analysis and Applied Linear Algebra, SIAM, Philadelphia, 2000.

[10] D. S. Mitrinovic, Analytic Inequalities, Springer, Berlin, 1970.

[11] D. S. Mitrinovic, J. E. Pecaric and A. M. Fink, Classic and New Inequalities in Analysis, Kluwer, Dordrecht, 1993.

[12] S. Varosanec, History, generalizations and unified treatment of two Ostrowski's inequalities, J. Inequal. Pure Appl. Math. 5(2) (2004), Article 23, 20 pp.

[13] J. Vitória, M. A. Facas Vicente, J. M. F. Santos, C. Costa and P. Tadeu, On an inequality by Fan and Todd, Inequality Theory and Applications, Yeol Je Cho, Jong Kyu Kim and Sever S. Dragomir, eds., Nova Science Publishers, Vol. 6, 2010, pp. 121-132.