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CONJUGACY AND GEOMETRY I - FOOT OF THE PERPENDICULAR, DISTANCE AND GRAM DETERMINANT

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Abstract

In this note on space geometry, the Gram determinant is used for expressing distances, vectors whose magnitude equals those distances and best approximation points. Three cases are considered: distances from a point to a line and to a plane and distances between two skew lines. (Symbolic) determinants occur in the expressions of the feet of perpendiculars and in the representation of the vectors materializing the distances. Because best approximation problems often require the use of subspaces, in order to solve the general cases of the proposed problems, we make extensive use of the conjugacy principle much present in Mathematics. The main purpose of this paper, focused on the resolution of distance problems in tridimensional geometry, is to provide the acquisition of spatial abilities through the proposed constructive approach. The obtained results, which could be a starting point and give clues for solving more advanced geometry problems, are applicable in several fields of practical sciences, such as the Coordinate Metrology, for instance. Moreover, this paper may be a window for coming across with a diversity of scalar products.

1. Introduction

Undergraduate Science and Engineering students at Universities (classic and technical) are expected to deal with distance problems in space geometry, using cross, $\times$, and dot, $\bullet$, products of vectors. Usually textbooks just present formulas for distances. Not much insistence is put on exhibiting either the foot of the perpendicular or one vector which achieves the distance.

We get results on Approximation Theory, by presenting direct derivations for the best approximation points and vectors achieving the distance, thus casting some light on these elements and paving the way for someone to look for more generalized problems which, naturally, demand more sophisticated approaches. We work on the ordinary space. The presented results, are not only inspiring but also are applicable in very practical sciences and techniques, notably on Coordinate Geometric Metrology, Photogrammetry or, even, Surveying. We keep convinced of the didactical advantages, concerning the acquisition of spatial abilities, and of the needed constructive approach, as well.

Science and Engineering readers may feel comfortable when working in the ordinary space, as they may recognize it as an orthogonal positive-definite space, which most of them are used to deal with – in contrast to the negative-definite spaces and the isotropic ones [8, p. 149], which are easily accepted by more mathematically minded persons.

In this paper, we deal with distances from a point to a line, from a point to a plane and distances between two skew lines. Problems of best approximation involve, essentially, subspaces, which, in our case means that lines and planes must pass through the origin of coordinates. We have to appeal to the conjugacy principle much present, yet not transparent, in Mathematics. The principle of conjugacy has an enormous range of applicability and can be schematized by the following image [6, vol. II, p. 374]: "to cross an impassable wall we sink down a shaft *S*, then we make a horizontal tunnel *T*, finally we dig up the reverse shaft S^{-1}; the whole trip is schematized by $A = STS^{-1}$". In other words, to solve a difficult problem *A*, we get and solve an easier one *T*, using a transformation *S* and its inverse S^{-1}. Usually, the relation of *A* and *T* is called conjugacy and it is an equivalence relation.

We also perform some convenient translations, legitimated by the conjugacy principle and by the fact [4] that distances are invariant under translations. When using the conjugacy principle, we consider a translation *t* (which is a nonlinear application) defined by

$$t_{\vec{v}} : \mathbb{R}^3 \to \mathbb{R}^3,$$

$$\vec{x} \to \vec{x} + \vec{v},$$

where, of course, we have

$$(t_{\vec{v}})^{-1} = t_{(-\vec{v})}.$$

The main tool we use is the Gram determinant which, in an inner product space and given a sequence of elements $\overrightarrow{x_1}, \overrightarrow{x_2}, \overrightarrow{x_3},$ is defined by

$$G(\overrightarrow{x_1}) = \overrightarrow{x_1} \bullet \overrightarrow{x_1}; \quad G(\overrightarrow{x_1}, \overrightarrow{x_2}) = \begin{vmatrix} \overrightarrow{x_1} \bullet \overrightarrow{x_1} & \overrightarrow{x_1} \bullet \overrightarrow{x_2} \\ \overrightarrow{x_2} \bullet \overrightarrow{x_1} & \overrightarrow{x_2} \bullet \overrightarrow{x_2} \end{vmatrix}$$

and

$$G(\overrightarrow{x_1}, \overrightarrow{x_2}, \overrightarrow{x_3}) = \begin{vmatrix} \overrightarrow{x_1} \bullet \overrightarrow{x_1} & \overrightarrow{x_1} \bullet \overrightarrow{x_2} & \overrightarrow{x_1} \bullet \overrightarrow{x_3} \\ \overrightarrow{x_2} \bullet \overrightarrow{x_1} & \overrightarrow{x_2} \bullet \overrightarrow{x_2} & \overrightarrow{x_2} \bullet \overrightarrow{x_3} \\ \overrightarrow{x_3} \bullet \overrightarrow{x_1} & \overrightarrow{x_3} \bullet \overrightarrow{x_2} & \overrightarrow{x_3} \bullet \overrightarrow{x_3} \end{vmatrix}.$$

In this paper, we state and prove the results we are going to use, by constructing the solutions, instead of simply enunciating, and then particularizing, the results known in Approximation Theory. Of course, in this constructive approach we establish the well known formulae for distances in terms of dot and cross products of vectors.

Some abuse of notation is patent in this paper: we use the sign := to identify these situations. We consider an orthonormal referential $\{O; (\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3})\}$ and – under the umbrella of adequate isomorphisms – we write points and vectors in several ways, according to our needs in each moment.

In Sections 2 and 3, we consider first the case of the objects passing through the origin. This option is due to three main reasons: the fact that in Approximation Theory most of the results are linked to subspaces; didactical preoccupations; and the need to pass through the origin in the case of two skew lines, in Section 4.

2. Distance from a Point to a Line

Given a point and a line, we construct the foot of the perpendicular, one (out of two) vector linking the point and the foot of the perpendicular and, finally, we find the distance between the point and the line.

In order to fit in the cadre of results of Approximation Theory, we have to displace the given line to the origin of the coordinates. We present first the particular case of the line through the origin and later, in Subsection 2.2, we treat the general case.

2.1. Distance from a point to a line through the origin

In the next result we establish a formula to the distance from a point to a line through the origin, using the Gram determinant. We also present formulae to the foot of the perpendicular and a vector which represents this distance.

Proposition 1. *Let* l_0 *be a line* (*containing the origin O*) *given by*

$$X = O + \alpha\vec{u}, \quad (\alpha \in \mathbb{R})$$

and $P := \vec{p}$ *be a point such that* $P \notin l_0$. *Then*

(i) *the distance* $d(P, l_0)$ *between point P and line* l_0 *is given by*

$$d^2(P, l_0) = \frac{G(\vec{u}, \vec{p})}{G(\vec{u})};$$

(ii) *the foot of the perpendicular S is given by the formula*

$$S = -\frac{\begin{vmatrix} \vec{u} \bullet \vec{u} & \vec{u} \bullet \vec{p} \\ \vec{u} & \vec{0} \end{vmatrix}}{G(\vec{u})};$$

(iii) *a vector achieving the distance* $\overrightarrow{SP}$ *is given by*

$$\overrightarrow{SP} = \frac{\begin{vmatrix} \vec{u} \bullet \vec{u} & \vec{u} \bullet \vec{p} \\ \vec{u} & \vec{p} \end{vmatrix}}{G(\vec{u})},$$

where G stands for the Gram determinant.

Proof. Let us consider a point $P := \vec{p}$ and a line l_0 (containing the origin *O*) given by $X = O + \alpha\vec{u}$, $(\alpha \in \mathbb{R})$.

We look for the foot $S := \vec{s}$ of the perpendicular drawn from the point *P* onto the line l_0. Then, we form the vector $\overrightarrow{SP}$ and, finally, we get

$$\|\overrightarrow{SP}\| = d(P, l_0).$$

We build this proof in three steps:

(a) consider a plane π through the point *P* and perpendicular to the line l_0: π is given by $\overrightarrow{PX} \bullet \vec{u} = 0$;

(b) intersect the constructed plane π and the given line l_0, thus obtaining the foot S, $S = l_0 \cap \pi$; we have, successively,

$$\begin{cases} X = O + \alpha\vec{u} \\ \overrightarrow{PX} \bullet \vec{u} = 0 \end{cases} \text{ or } \begin{cases} X = \alpha\vec{u} \\ (X - P) \bullet \vec{u} = 0 \end{cases} \text{ or } (\alpha\vec{u} - \vec{p}) \bullet \vec{u} = 0,$$

thus getting

$$\alpha = \frac{\vec{p} \bullet \vec{u}}{\vec{u} \bullet \vec{u}}.$$

So, we get the foot of the perpendicular

$$S := \vec{s} = \frac{\vec{p} \bullet \vec{u}}{\vec{u} \bullet \vec{u}}\vec{u}$$

which we can write in the form

$$S := \vec{s} = -\frac{\begin{vmatrix} \vec{u} \bullet \vec{u} & \vec{u} \bullet \vec{p} \\ \vec{u} & \vec{0} \end{vmatrix}}{G(\vec{u})},$$

where $G(\vec{u})$ is the Gram determinant of $\vec{u}$ and the (symbolic) determinant on the numerator is obtained by formally expanding by cofactors of the second row.

One vector materializing the distance is

$$\overrightarrow{SP} = P - S := \vec{p} - \vec{s} = \vec{p} - \frac{\vec{p} \bullet \vec{u}}{\vec{u} \bullet \vec{u}}\vec{u}$$

and taking into account the evident equality

$$\vec{p} = \frac{\vec{u} \bullet \vec{u}}{\vec{u} \bullet \vec{u}}\vec{p} = \frac{\begin{vmatrix} \vec{u} \bullet \vec{u} & 0 \\ \vec{u} & \vec{p} \end{vmatrix}}{G(\vec{u})},$$

we have

$$\vec{p} - \vec{s} = \frac{\begin{vmatrix} \vec{u} \bullet \vec{u} & \vec{u} \bullet \vec{p} \\ \vec{u} & \vec{p} \end{vmatrix}}{G(\vec{u})},$$

where, again, the numerator is a symbolic determinant to be expanded along the second row.

(c) Now, in order to measure the vector $\overrightarrow{SP}$ and to find the distance $d(P, l_0)$, we have, successively,

$$\begin{aligned}
\|\overrightarrow{SP}\|^2 &= d^2(P, l_0) = \|\vec{p} - \vec{s}\|^2 = (\vec{p} - \vec{s}) \bullet (\vec{p} - \vec{s}) \\
&= \vec{p} \bullet \vec{p} - 2\vec{p} \bullet \vec{s} + \vec{s} \bullet \vec{s} \\
&= \vec{p} \bullet \vec{p} - 2\frac{\vec{p} \bullet \vec{u}}{\vec{u} \bullet \vec{u}}(\vec{p} \bullet \vec{u}) + \left(\frac{\vec{p} \bullet \vec{u}}{\vec{u} \bullet \vec{u}}\vec{u}\right) \bullet \left(\frac{\vec{p} \bullet \vec{u}}{\vec{u} \bullet \vec{u}}\vec{u}\right) \\
&= \vec{p} \bullet \vec{p} - \frac{2}{\vec{u} \bullet \vec{u}}(\vec{u} \bullet \vec{p})^2 + \frac{(\vec{u} \bullet \vec{p})^2}{(\vec{u} \bullet \vec{u})^2}(\vec{u} \bullet \vec{u}) \\
&= \vec{p} \bullet \vec{p} - \frac{(\vec{u} \bullet \vec{p})^2}{\vec{u} \bullet \vec{u}} = \frac{(\vec{p} \bullet \vec{p})(\vec{u} \bullet \vec{u}) - (\vec{u} \bullet \vec{p})^2}{\vec{u} \bullet \vec{u}} \\
&= \frac{\begin{vmatrix} \vec{u} \bullet \vec{u} & \vec{u} \bullet \vec{p} \\ \vec{p} \bullet \vec{u} & \vec{p} \bullet \vec{p} \end{vmatrix}}{\vec{u} \bullet \vec{u}} := \frac{G(\vec{u}, \vec{p})}{G(\vec{u})},
\end{aligned}$$

where $G(\vec{u}, \vec{p})$ stands for the Gram determinant of $\vec{u}$ and $\vec{p}$. □

Remark 2. Note that

$$\|\vec{u} \times \vec{p}\|^2 = \begin{vmatrix} \vec{u} \bullet \vec{u} & \vec{u} \bullet \vec{p} \\ \vec{p} \bullet \vec{u} & \vec{p} \bullet \vec{p} \end{vmatrix},$$

so, we have

$$d(P, l_0) = \frac{\|\vec{p} \times \vec{u}\|}{\|\vec{u}\|} := \frac{\|\overrightarrow{OP} \times \vec{u}\|}{\|\vec{u}\|},$$

the well known formula for the distance from a point P to a line through the origin O.

2.2. General case. Distance from a point to a line

In this subsection we present the general result.

Proposition 3. *Let l be a line given by*

$$X = M + \alpha\vec{u}, \quad (\alpha \in \mathbb{R})$$

with $M := \vec{m}$ and let $P := \vec{p}$ be a point such that $P \notin l$. Then

(i) *the distance* $d(P, l)$ *between point P and line l is given by*

$$d^2(P, l) = \frac{G(\vec{u}, \vec{p} - \vec{m})}{G(\vec{u})};$$

(ii) *the foot of the perpendicular S is given by the formula*

$$S = -\frac{\begin{vmatrix} \vec{u} \bullet \vec{u} & \vec{u} \bullet (\vec{p} - \vec{m}) \\ \vec{u} & \vec{0} \end{vmatrix}}{G(\vec{u})} + M;$$

(iii) *a vector achieving the distance* $\overrightarrow{SP}$ *is given by*

$$\overrightarrow{SP} = \frac{\begin{vmatrix} \vec{u} \bullet \vec{u} & \vec{u} \bullet (\vec{p} - \vec{m}) \\ \vec{u} & (\vec{p} - \vec{m}) \end{vmatrix}}{G(\vec{u})}.$$

Proof. We have a pair (P, l) formed by a point $P := \vec{p}$ and a line l with equation

$$X = M + \alpha\vec{u}, \quad (\alpha \in \mathbb{R}),$$

such that $P \notin l$. In order to approaching by Gram determinants, we have to displace the line l to the origin. We make the translation $\overrightarrow{MO} = O - M := -M := -\vec{m}$. So – see Figure 1 – the pair (P', l') enters into consideration, where

$$P' = P - M := \vec{p}\,' = \vec{p} - \vec{m}, \text{ and } l' := X' = O + \alpha\vec{u}, \quad (\alpha \in \mathbb{R}).$$

We have

$$d(P, l) = d(P', l')$$

and we are in conditions to apply Proposition 1 to the pair (P', l'). From (i), we have

$$d^2(P', l') = \frac{G(\vec{u}, \vec{p}\,')}{G(\vec{u})} = \frac{G(\vec{u}, \vec{p} - \vec{m})}{G(\vec{u})};$$

and so the distance $d(P, l)$ is given by

$$d^2(P, l) = \frac{G(\vec{u}, \vec{p} - \vec{m})}{G(\vec{u})}.$$

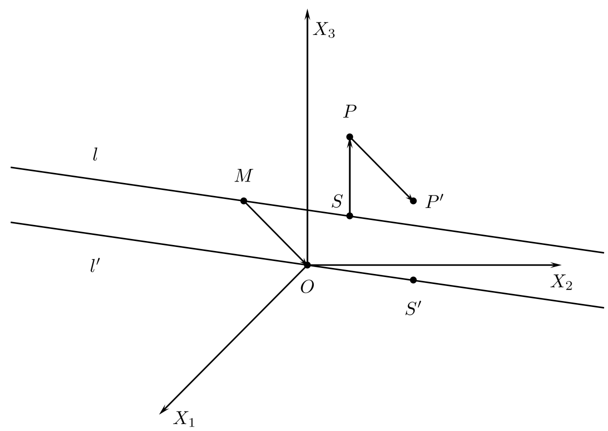

Figure 1. The pair (P, l) moves to the pair (P', l').

From (ii), we know that the foot of the perpendicular S' is given by

$$S' = -\frac{\begin{vmatrix} \vec{u} \bullet \vec{u} & \vec{u} \bullet \vec{p}' \\ \vec{u} & \vec{0} \end{vmatrix}}{G(\vec{u})} = -\frac{\begin{vmatrix} \vec{u} \bullet \vec{u} & \vec{u} \bullet (\vec{p} - \vec{m}) \\ \vec{u} & \vec{0} \end{vmatrix}}{G(\vec{u})}.$$

After being done the reverse translation $\overrightarrow{OM} = M - O = \vec{m}$, we get the foot of the perpendicular $S = S' + M$.

Finally by applying (iii) the vector materializing the distance between P' and l' is

$$\overrightarrow{S'P'} = \frac{\begin{vmatrix} \vec{u} \bullet \vec{u} & \vec{u} \bullet \vec{p}' \\ \vec{u} & \vec{p}' \end{vmatrix}}{G(\vec{u})} = \frac{\begin{vmatrix} \vec{u} \bullet \vec{u} & \vec{u} \bullet (\vec{p} - \vec{m}) \\ \vec{u} & \vec{p} - \vec{m} \end{vmatrix}}{G(\vec{u})},$$

and as $\overrightarrow{SP}$ is such that $\overrightarrow{S'P'} = \overrightarrow{SP}$, we get also a formula to the vector materializing the distance between *P* and *l*. □

2.3. Example

We illustrate our proposal to compute the distance from a point to a line with a numerical example. For didactical reasons we present two (obviously related) processes: (I) applying directly Proposition 3; (II) remaking the process of proof of Proposition 3.

Consider the point $P = (3, 2, 1)$ and the line $l := X = M + \alpha\vec{u}$ given by

$$X = (1, 2, 0) + \alpha(\overrightarrow{e_1} + \overrightarrow{e_2} + \overrightarrow{e_3}), \quad \alpha \in \mathbb{R}.$$

(I) By Proposition 3(i), we have the distance $d(P, l)$ between point P and line l given by

$$d^2(P, l) = \frac{G(\vec{u}, \vec{p} - \vec{m})}{G(\vec{u})} = \frac{\begin{vmatrix} 3 & 3 \\ 3 & 5 \end{vmatrix}}{3} = 2.$$

Applying Proposition 3(ii), we obtain the formula to the foot S of the perpendicular:

$$S = -\frac{\begin{vmatrix} 3 & 3 \\ \vec{u} & \vec{0} \end{vmatrix}}{3} + M = (2, 3, 1).$$

Finally, using Proposition 3(iii), we find a vector $\overrightarrow{SP}$ that materializes the distance:

$$\overrightarrow{SP} = \frac{\begin{vmatrix} 3 & 3 \\ \vec{u} & \vec{p} - \vec{m} \end{vmatrix}}{3} = \overrightarrow{e_1} - \overrightarrow{e_2}.$$

(II) We have the pair (P, l), with $P, O \notin l$. We have to displace the line l to the origin; the translation to be made is associated to the vector $\overrightarrow{MO} := -\vec{m} = -\overrightarrow{e_1} - 2\overrightarrow{e_2}$. After that, we obtain the pair (P', l'), where

$$P' = (3, 2, 1) - (1, 2, 0) = (2, 0, 1) := \vec{p}' = 2\overrightarrow{e_1} + \overrightarrow{e_3}$$

$$l' := X' = \alpha(\overrightarrow{e_1} + \overrightarrow{e_2} + \overrightarrow{e_3}), \quad (\alpha \in \mathbb{R}),$$

so the problem now concerns the distance $d(P', l')$; thus we are in conditions to apply Proposition 1 to this pair. Hence

$$S' := \vec{s}' = -\frac{\begin{vmatrix} 3 & 3 \\ \vec{u} & \vec{0} \end{vmatrix}}{3} = \overrightarrow{e_1} + \overrightarrow{e_2} + \overrightarrow{e_3} = (1, 1, 1)$$

is the foot of the perpendicular onto the line l'.

The vector which materializes the distance

$$\overrightarrow{S'P'} = P' - S' := \vec{p}\,' - \vec{s}\,' = \frac{\begin{vmatrix} 3 & 3 \\ \vec{u} & \vec{p}\,' \end{vmatrix}}{3} = \frac{-3\vec{u} + 3\vec{p}\,'}{3} = \overrightarrow{e_1} - \overrightarrow{e_2}.$$

Making the reverse translation $\overrightarrow{OM} = \vec{m}$, we get the foot of the perpendicular onto the line l

$$\vec{s} = \vec{s}\,' + \vec{m}, \text{ i.e., } S = S' + M = (1, 1, 1) + (1, 2, 0) = (2, 3, 1)$$

and the vector $\overrightarrow{SP}$ which materializes the distance is

$$\overrightarrow{SP} = \vec{p} - \vec{s} = (\vec{p}\,' + \vec{m}) - (\vec{s}\,' + \vec{m}) = (\vec{p}\,' - \vec{s}\,') = \overrightarrow{e_1} - \overrightarrow{e_2}.$$

Finally, the distance between point P and line l is given by

$$d^2(P, l) = \| \vec{p} - \vec{s} \|^2 = \frac{G(\vec{u},\ \vec{p} - \vec{m})}{G(\vec{u})} = \frac{\begin{vmatrix} 3 & 3 \\ 3 & 5 \end{vmatrix}}{3} = 2.$$

3. Distance from a Point to a Plane

In this section, we are interested in computing the distance from point to plane, using Gram determinants. We consider first a plane containing the origin of the coordinates. By a constructive procedure, the foot of the perpendicular is found, the vector linking a given point to this foot on the plane is exhibited and the distance is computed. In Subsection 3.2 some adequate translations are done, in order to put the problem in all generality.

3.1. Distance from a point to a plane through the origin

In the next results we establish formulae to the distance from a point to a plane passing through the origin, using vector products and the Gram determinant. We also present formulae to the foot of the perpendicular and a vector which achieves this distance.

Proposition 4. *Let* π_0 *be a plane* (*containing the origin O*) *given by*

$$X = O + \alpha\vec{u} + \beta\vec{v}, \quad (\alpha, \beta \in \mathbb{R})$$

with $\vec{u}$ *and* $\vec{v}$ *noncollinear vectors and let* $P := \vec{p}$ *be a point such that* $P \notin \pi_0$. *Then*

(i) *the distance* $d(P, \pi_0)$ *between point P and plane* π_0 *is given by*

$$d(P, \pi_0) = \frac{|\vec{u} \times \vec{v} \bullet \vec{p}|}{\|\vec{u} \times \vec{v}\|};$$

(ii) *the foot of the perpendicular S is given by the formula*

$$S := \vec{s} = P - \frac{\vec{u} \times \vec{v} \bullet \vec{p}}{\|\vec{u} \times \vec{v}\|^2}(\vec{u} \times \vec{v});$$

(iii) *a vector achieving the distance* $\overrightarrow{SP}$ *is given by*

$$\overrightarrow{SP} = \frac{\vec{u} \times \vec{v} \bullet \vec{p}}{\|\vec{u} \times \vec{v}\|^2}(\vec{u} \times \vec{v}).$$

Proof. Let us consider a point $P := \vec{p}$ and a plane π_0 through the origin of coordinates *O*, given by

$$X = O + \alpha\vec{u} + \beta\vec{v}, \quad (\alpha, \beta \in \mathbb{R}).$$

We intersect the given plane π_0 with a line *l* perpendicular to it and drawn from the point *P*. So we get the foot of the perpendicular $S := \vec{s}$. Then, we form the vector $\overrightarrow{SP}$ and measure it.

Let us start the constructing procedure of the foot of the perpendicular and of a vector which materializes the distance. We have

$$S = l \cap \pi_0 = \begin{cases} X = P + \gamma(\vec{u} \times \vec{v}), \\ X = O + \alpha\vec{u} + \beta\vec{v} \end{cases}$$

which gives, successively,

$$P + \gamma(\vec{u} \times \vec{v}) = O + \alpha\vec{u} + \beta\vec{v},$$

$$\alpha\vec{u} + \beta\vec{v} - \gamma(\vec{u} \times \vec{v}) = P - O :=,$$

$$\alpha\vec{u} + \beta\vec{v} - \gamma\vec{w} = \vec{p} :=,$$

$$[\vec{u} \quad \vec{v} \quad -\vec{w}]\begin{bmatrix}\alpha \\ \beta \\ \gamma\end{bmatrix} = \begin{bmatrix}p_1 \\ p_2 \\ p_3\end{bmatrix},$$

with $\vec{w} = \vec{u} \times \vec{v}$. So, from the previous system, we get

$$\begin{cases} \alpha = \dfrac{\begin{vmatrix} p_1 & v_1 & -w_1 \\ p_2 & v_2 & -w_2 \\ p_3 & v_3 & -w_3 \end{vmatrix}}{\begin{vmatrix} u_1 & v_1 & -w_1 \\ u_2 & v_2 & -w_2 \\ u_3 & v_3 & -w_3 \end{vmatrix}} \\ \beta = \dfrac{\begin{vmatrix} u_1 & p_1 & -w_1 \\ u_2 & p_2 & -w_2 \\ u_3 & p_3 & -w_3 \end{vmatrix}}{\begin{vmatrix} u_1 & v_1 & -w_1 \\ u_2 & v_2 & -w_2 \\ u_3 & v_3 & -w_3 \end{vmatrix}} \\ \gamma = \dfrac{\begin{vmatrix} u_1 & v_1 & p_1 \\ u_2 & v_2 & p_2 \\ u_3 & v_3 & p_3 \end{vmatrix}}{\begin{vmatrix} u_1 & v_1 & -w_1 \\ u_2 & v_2 & -w_2 \\ u_3 & v_3 & -w_3 \end{vmatrix}} \end{cases}$$

equivalent to

$$\begin{cases} \alpha = \dfrac{\vec{p} \times \vec{v} \bullet \vec{w}}{\vec{u} \times \vec{v} \bullet \vec{w}} \\ \beta = \dfrac{\vec{u} \times \vec{p} \bullet \vec{w}}{\vec{u} \times \vec{v} \bullet \vec{w}} \\ \gamma = -\dfrac{\vec{u} \times \vec{v} \bullet \vec{p}}{\vec{u} \times \vec{v} \bullet \vec{w}} \end{cases}$$

that is

$$\begin{cases} \alpha = \dfrac{(\vec{p} \times \vec{v}) \bullet (\vec{u} \times \vec{v})}{\| \vec{u} \times \vec{v} \|^2} \\ \beta = \dfrac{(\vec{u} \times \vec{p}) \bullet (\vec{u} \times \vec{v})}{\| \vec{u} \times \vec{v} \|^2} \\ \gamma = -\dfrac{\vec{u} \times \vec{v} \bullet \vec{p}}{\| \vec{u} \times \vec{v} \|^2}. \end{cases}$$

Hence we have an expression for the foot of the perpendicular

$$S := \vec{s} = P + \gamma(\vec{u} \times \vec{v}) = P - \frac{\vec{u} \times \vec{v} \bullet \vec{p}}{\| \vec{u} \times \vec{v} \|^2} (\vec{u} \times \vec{v})$$

and for the vector (within a signal) materializing the distance

$$\overrightarrow{SP} = P - S := \vec{p} - \vec{s} = \frac{\vec{u} \times \vec{v} \bullet \vec{p}}{\| \vec{u} \times \vec{v} \|^2} (\vec{u} \times \vec{v}).$$

Hence it follows the well known formula for the distance from a point to a plane through the origin *O*:

$$d(P, \pi_0) = \| \vec{p} - \vec{s} \| = \frac{| \vec{u} \times \vec{v} \bullet \vec{p} |}{\| \vec{u} \times \vec{v} \|^2} \| \vec{u} \times \vec{v} \|$$

$$= \frac{| \vec{u} \times \vec{v} \bullet \vec{p} |}{\| \vec{u} \times \vec{v} \|}. \qquad \square$$

So far, we worked only with products of vectors. It remains getting expressions in terms of Gram determinant and (symbolic) determinants. We establish the needed

Proposition 5. *Let* π_0 *be a plane* (*containing the origin O*) *given by*

$$X = O + \alpha\vec{u} + \beta\vec{v}, \quad (\alpha, \beta \in \mathbb{R})$$

with $\vec{u}$ *and* $\vec{v}$ *noncollinear vectors and let* $P := \vec{p}$ *be a point such that* $P \notin \pi_0$. *Then*

(i) *the distance* $d(P, \pi_0)$ *between point P and plane* π_0 *is given by*

$$d^2(P, \pi_0) = \frac{G(\vec{u}, \vec{v}, \vec{p})}{G(\vec{u}, \vec{v})};$$

(ii) *the foot of the perpendicular S is given by the formula*

$$S = -\frac{\begin{vmatrix} \vec{u} \bullet \vec{u} & \vec{u} \bullet \vec{v} & \vec{u} \bullet \vec{p} \\ \vec{v} \bullet \vec{u} & \vec{v} \bullet \vec{v} & \vec{v} \bullet \vec{p} \\ \vec{u} & \vec{v} & \vec{0} \end{vmatrix}}{G(\vec{u}, \vec{v})};$$

(iii) *a vector achieving the distance* $\overrightarrow{SP}$ *is given by*

$$\overrightarrow{SP} = \frac{\begin{vmatrix} \vec{u} \bullet \vec{u} & \vec{u} \bullet \vec{v} & \vec{u} \bullet \vec{p} \\ \vec{v} \bullet \vec{u} & \vec{v} \bullet \vec{v} & \vec{v} \bullet \vec{p} \\ \vec{u} & \vec{v} & \vec{p} \end{vmatrix}}{G(\vec{u}, \vec{v})}.$$

Proof. Concerning distance, using the previous proposition, it is straightforward and we have

$$d^2(P, \pi_0) = \frac{|\vec{u} \times \vec{v} \bullet \vec{p}|^2}{\|\vec{u} \times \vec{v}\|^2}$$

$$= \frac{\left(\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ p_1 & p_2 & p_3 \end{vmatrix}\right)^2}{\begin{vmatrix} \vec{u} \bullet \vec{u} & \vec{u} \bullet \vec{v} \\ \vec{v} \bullet \vec{u} & \vec{v} \bullet \vec{v} \end{vmatrix}}.$$

Since $|A|^2 = |A||A| = |A||A^T| = |AA^T|$, follows

$$d^2(P, \pi_0) = \frac{\left|\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ p_1 & p_2 & p_3 \end{bmatrix}\begin{bmatrix} u_1 & v_1 & p_1 \\ u_2 & v_2 & p_2 \\ u_3 & v_3 & p_3 \end{bmatrix}\right|}{\begin{vmatrix} \vec{u} \bullet \vec{u} & \vec{u} \bullet \vec{v} \\ \vec{v} \bullet \vec{u} & \vec{v} \bullet \vec{v} \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} \vec{u} \bullet \vec{u} & \vec{u} \bullet \vec{v} & \vec{u} \bullet \vec{p} \\ \vec{v} \bullet \vec{u} & \vec{v} \bullet \vec{v} & \vec{v} \bullet \vec{p} \\ \vec{p} \bullet \vec{u} & \vec{p} \bullet \vec{v} & \vec{p} \bullet \vec{p} \end{vmatrix}}{\begin{vmatrix} \vec{u} \bullet \vec{u} & \vec{u} \bullet \vec{v} \\ \vec{v} \bullet \vec{u} & \vec{v} \bullet \vec{v} \end{vmatrix}}$$

$$:= \frac{G(\vec{u}, \vec{v}, \vec{p})}{G(\vec{u}, \vec{v})}.$$

With respect to the foot of the perpendicular, we use α and β of the equation of plane π_0. It follows:

$$S := \vec{s} = \alpha\vec{u} + \beta\vec{v}$$

$$= \frac{-[(\vec{u}\times\vec{v})\bullet(\vec{v}\times\vec{p})]\vec{u} + [(\vec{u}\times\vec{v})\bullet(\vec{u}\times\vec{p})]\vec{v} - \vec{0}\|\vec{u}\times\vec{v}\|^2}{\|\vec{u}\times\vec{v}\|^2}$$

$$= -\frac{[(\vec{u}\times\vec{v})\bullet(\vec{v}\times\vec{p})]\vec{u} - [(\vec{u}\times\vec{v})\bullet(\vec{u}\times\vec{p})]\vec{v} + \vec{0}\|\vec{u}\times\vec{v}\|^2}{\|\vec{u}\times\vec{v}\|^2}$$

$$= -\frac{\begin{vmatrix} \vec{u}\bullet\vec{v} & \vec{u}\bullet\vec{p} \\ \vec{v}\bullet\vec{v} & \vec{v}\bullet\vec{p} \end{vmatrix}\vec{u} - \begin{vmatrix} \vec{u}\bullet\vec{u} & \vec{u}\bullet\vec{p} \\ \vec{v}\bullet\vec{u} & \vec{v}\bullet\vec{p} \end{vmatrix}\vec{v} + \vec{0}\begin{vmatrix} \vec{u}\bullet\vec{u} & \vec{u}\bullet\vec{v} \\ \vec{v}\bullet\vec{u} & \vec{v}\bullet\vec{v} \end{vmatrix}}{\|\vec{u}\times\vec{v}\|^2}.$$

Hence

$$\vec{s} = -\frac{\begin{vmatrix} \vec{u}\bullet\vec{u} & \vec{u}\bullet\vec{v} & \vec{u}\bullet\vec{p} \\ \vec{v}\bullet\vec{u} & \vec{v}\bullet\vec{v} & \vec{v}\bullet\vec{p} \\ \vec{u} & \vec{v} & \vec{0} \end{vmatrix}}{\begin{vmatrix} \vec{u}\bullet\vec{u} & \vec{u}\bullet\vec{v} \\ \vec{v}\bullet\vec{u} & \vec{v}\bullet\vec{v} \end{vmatrix}},$$

where the numerator is a symbolic determinant, to be expanded along the third row. So, we can write

$$\vec{s} := -\frac{\begin{vmatrix} \vec{u}\bullet\vec{u} & \vec{u}\bullet\vec{v} & \vec{u}\bullet\vec{p} \\ \vec{v}\bullet\vec{u} & \vec{v}\bullet\vec{v} & \vec{v}\bullet\vec{p} \\ \vec{u} & \vec{v} & \vec{0} \end{vmatrix}}{G(\vec{u}, \vec{v})}.$$

Taking into account the evident equalities

$$\vec{p} = \frac{G(\vec{u}, \vec{v})}{G(\vec{u}, \vec{v})}\vec{p}$$

$$= \frac{\begin{vmatrix} \vec{u}\bullet\vec{u} & \vec{u}\bullet\vec{v} & 0 \\ \vec{v}\bullet\vec{u} & \vec{v}\bullet\vec{v} & 0 \\ \vec{u} & \vec{v} & \vec{p} \end{vmatrix}}{\begin{vmatrix} \vec{u}\bullet\vec{u} & \vec{u}\bullet\vec{v} \\ \vec{v}\bullet\vec{u} & \vec{v}\bullet\vec{v} \end{vmatrix}}$$

and by the linearity on the third column of the (symbolic) determinant, follows a vector realizing the distance

$$\overrightarrow{SP} = P - S = \vec{p} - \vec{s} = \frac{\begin{vmatrix} \vec{u} \bullet \vec{u} & \vec{u} \bullet \vec{v} & \vec{u} \bullet \vec{p} \\ \vec{v} \bullet \vec{u} & \vec{v} \bullet \vec{v} & \vec{v} \bullet \vec{p} \\ \vec{u} & \vec{v} & \vec{p} \end{vmatrix}}{\begin{vmatrix} \vec{u} \bullet \vec{u} & \vec{u} \bullet \vec{v} \\ \vec{v} \bullet \vec{u} & \vec{v} \bullet \vec{v} \end{vmatrix}},$$

where the determinant in the numerator is to be expanded algebraically to yield a linear combination of the vectors $\vec{u}$, $\vec{v}$ and $\vec{p}$. □

3.2. General case. Distance from a point to a plane

In this subsection we present the general results, in terms of dot and cross products and in terms of Gram determinants, in the following two propositions.

Proposition 6. *Let π be a plane given by*

$$X = M + \alpha\vec{u} + \beta\vec{v}, \quad (\alpha, \beta \in \mathbb{R})$$

with $M := \vec{m}$ *and let* $P := \vec{p}$ *be a point such that* $P \notin \pi$. *Then*

(i) *the distance* $d(P, \pi)$ *between point P and plane π is given by*

$$d(P, \pi) = \frac{|\, \vec{u} \times \vec{v} \bullet (\vec{p} - \vec{m})\,|}{\|\, \vec{u} \times \vec{v} \,\|};$$

(ii) *the foot of the perpendicular S is given by the formula*

$$S := \vec{s} = P - \frac{\vec{u} \times \vec{v} \bullet (\vec{p} - \vec{m})}{\|\, \vec{u} \times \vec{v} \,\|^2}(\vec{u} \times \vec{v});$$

(iii) *a vector achieving the distance* $\overrightarrow{SP}$ *is given by*

$$\overrightarrow{SP} = \frac{\vec{u} \times \vec{v} \bullet (\vec{p} - \vec{m})}{\|\, \vec{u} \times \vec{v} \,\|^2}(\vec{u} \times \vec{v}).$$

Proof. Let us consider a point $P := \vec{p}$ and a plane π given by

$$X = M + \alpha\vec{u} + \beta\vec{v}, \quad (\alpha, \beta \in \mathbb{R}).$$

As in the proof of Proposition 4, we intersect the given plane π with a line l perpendicular to it and drawn from the point P. So we get the foot of the perpendicular $S := \vec{s}$. Then, again, we form the vector $\overrightarrow{SP}$ and measure it.

Let us start the constructing procedure of the foot of the perpendicular and of a vector which achieves the distance. We have

$$S = l \cap \pi = \begin{cases} X = P + \gamma(\vec{u} \times \vec{v}), \\ X = M + \alpha\vec{u} + \beta\vec{v}. \end{cases}$$

From the proof of Proposition 4, mutatis mutandis, we get

$$\begin{cases} \alpha = \dfrac{[(\vec{p} - \vec{m}) \times \vec{v}] \bullet (\vec{u} \times \vec{v})}{\| \vec{u} \times \vec{v} \|^2}, \\ \beta = \dfrac{[\vec{u} \times (\vec{p} - \vec{m})] \bullet (\vec{u} \times \vec{v})}{\| \vec{u} \times \vec{v} \|^2}, \\ \gamma = -\dfrac{\vec{u} \times \vec{v} \bullet (\vec{p} - \vec{m})}{\| \vec{u} \times \vec{v} \|^2}. \end{cases}$$

Thusly, we obtain

$$S := \vec{s} = P + \gamma(\vec{u} \times \vec{v}) = P - \frac{\vec{u} \times \vec{v} \bullet (\vec{p} - \vec{m})}{\| \vec{u} \times \vec{v} \|^2}(\vec{u} \times \vec{v}),$$

$$\overrightarrow{SP} = P - S := \vec{p} - \vec{s} = \frac{\vec{u} \times \vec{v} \bullet (\vec{p} - \vec{m})}{\| \vec{u} \times \vec{v} \|^2}(\vec{u} \times \vec{v})$$

and

$$d(P, \pi) = \| \vec{p} - \vec{s} \| = \frac{| \vec{u} \times \vec{v} \bullet (\vec{p} - \vec{m}) |}{\| \vec{u} \times \vec{v} \|^2} \| \vec{u} \times \vec{v} \|$$

$$= \frac{| \vec{u} \times \vec{v} \bullet (\vec{p} - \vec{m}) |}{\| \vec{u} \times \vec{v} \|}.$$

We remark that, in the previous results, the scalars are quotients involving volumes and areas. □

Now, it remains getting expressions in terms of Gram determinant and (symbolic) determinants. For that, we have the following:

Proposition 7. *Let* π *be a plane given by*

$$X = M + \alpha\vec{u} + \beta\vec{v}, \quad (\alpha, \beta \in \mathbb{R})$$

with $M := \vec{m}$ *and let* $P := \vec{p}$ *be a point such that* $P \notin \pi$. *Then*

(i) *the distance* $d(P, \pi)$ *between point P and plane* π *is given by*

$$d^2(P, \pi) = \frac{G(\vec{u}, \vec{v}, \vec{p} - \vec{m})}{G(\vec{u}, \vec{v})};$$

(ii) *the foot of the perpendicular S is given by the formula*

$$S = -\frac{\begin{vmatrix} \vec{u} \bullet \vec{u} & \vec{u} \bullet \vec{v} & \vec{u} \bullet (\vec{p} - \vec{m}) \\ \vec{v} \bullet \vec{u} & \vec{v} \bullet \vec{v} & \vec{v} \bullet (\vec{p} - \vec{m}) \\ \vec{u} & \vec{v} & \vec{0} \end{vmatrix}}{G(\vec{u}, \vec{v})} + M;$$

(iii) *a vector* $\overrightarrow{SP}$ *achieving the distance is given by*

$$\overrightarrow{SP} = \frac{\begin{vmatrix} \vec{u} \bullet \vec{u} & \vec{u} \bullet \vec{v} & \vec{u} \bullet (\vec{p} - \vec{m}) \\ \vec{v} \bullet \vec{u} & \vec{v} \bullet \vec{v} & \vec{v} \bullet (\vec{p} - \vec{m}) \\ \vec{u} & \vec{v} & \vec{p} - \vec{m} \end{vmatrix}}{G(\vec{u}, \vec{v})}.$$

Proof. Now our geometric pair (P, π) is formed by a point P and a plane π given by $X = M + \alpha\vec{u} + \beta\vec{v}$.

We look for the usual information: foot S of the perpendicular and a vector $\overrightarrow{SP} = P - S$ which materialize the distance $d(P, \pi)$.

For entering into the Gram context, the plane π has to be displaced to the origin, by a translation $\overrightarrow{MO} = O - M = -\vec{m}$.

So – see Figure 2 – we have, after translation,

$$P' = P - M, \text{ i.e., } \vec{p}' = \vec{p} - \vec{m},$$

$$\pi' := X' = O + \alpha\vec{u} + \beta\vec{v}.$$

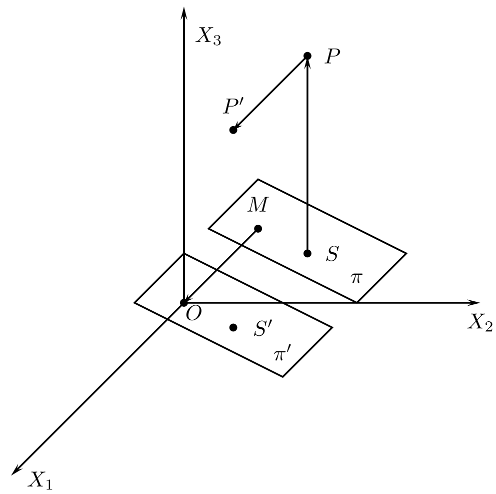

Figure 2. The pair (P, π) turns into the pair (P', π').

We have, concerning the pair (P', π'), using Proposition 5,

$$S' = \vec{s}\,' = -\frac{\begin{vmatrix} \vec{u} \bullet \vec{u} & \vec{u} \bullet \vec{v} & \vec{u} \bullet \vec{p}\,' \\ \vec{v} \bullet \vec{u} & \vec{v} \bullet \vec{v} & \vec{v} \bullet \vec{p}\,' \\ \vec{u} & \vec{v} & \vec{0} \end{vmatrix}}{G(\vec{u}, \vec{v})},$$

with $\vec{p}\,' = \vec{p} - \vec{m}$. Moreover,

$$\vec{p}\,' - \vec{s}\,' = \frac{\begin{vmatrix} \vec{u} \bullet \vec{u} & \vec{u} \bullet \vec{v} & \vec{u} \bullet (\vec{p} - \vec{m}) \\ \vec{v} \bullet \vec{u} & \vec{v} \bullet \vec{v} & \vec{v} \bullet (\vec{p} - \vec{m}) \\ \vec{u} & \vec{v} & \vec{p} - \vec{m} \end{vmatrix}}{G(\vec{u}, \vec{v})}$$

and

$$d^2(P', \pi') = \frac{G(\vec{u}, \vec{v}, \vec{p}\,')}{G(\vec{u}, \vec{v})} = \frac{G(\vec{u}, \vec{v}, \vec{p} - \vec{m})}{G(\vec{u}, \vec{v})} = d^2(P, \pi).$$

By making the reversing translation $-\vec{m}$, we get

$$\vec{s} = \vec{s}\,' + \vec{m}, \text{ i.e., } S = S' + M,$$

$$\overrightarrow{SP} = \vec{p} - \vec{s} = (\vec{p}\,' + \vec{m}) - (\vec{s}\,' + \vec{m}). \qquad \square$$

3.3. Example

Consider the point $P = (3, 2, 1)$ and the plane

$$\pi := x_1 + x_2 + x_3 = 1,$$

which we write in the form

$$\pi := X = M + \alpha\vec{u} + \beta\vec{v}, \quad (\alpha, \beta \in \mathbb{R}),$$

where $M = (1, 0, 0)$, $\vec{u} = (-1, 1, 0)$, $\vec{v} = (-1, 0, 1)$.

Applying Proposition 7, we have the distance from point P to plane π

$$d^2(P, \pi) = \frac{G(\vec{u}, \vec{v}, \vec{p} - \vec{m})}{G(\vec{u}, \vec{v})} = \frac{\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 9 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}} = \frac{25}{3};$$

the foot of the perpendicular

$$S = -\frac{\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ \vec{u} & \vec{v} & \vec{0} \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}} + M = \left(\frac{4}{3}, \frac{1}{3}, -\frac{2}{3}\right);$$

and a vector representing the distance

$$\overrightarrow{SP} = \frac{5}{3}(\overrightarrow{e_1} + \overrightarrow{e_2} + \overrightarrow{e_3}).$$

4. Distance between two Skew Lines

In this section, we present a process to determine the distance between two skew lines and the corresponding best approximation pair. The key idea is that we invoke twice the distance from a point to a line through the origin. We have two translation movements: each line has, once, to be displaced to the origin. Afterwards, two corresponding reverse translations have to be done, as well. The feet of the perpendiculars (one foot on each line) depend on parameters. The vector whose extremities are the feet of the perpendiculars also depends on these parameters.

Let us consider the skew lines l_1 and l_2 given, respectively, by

$$X = P + \alpha\vec{u}, \quad X = Q + \beta\vec{v}, \quad (\alpha, \beta \in \mathbb{R}).$$

We look for the foot of the perpendicular S_1, onto l_1 and the foot of the perpendicular S_2, onto l_2. Afterwards, we form a vector which achieves the distance between the two lines, for example, $\overrightarrow{S_1S_2}$. The distance is given by $d(l_1, l_2) = \|\overrightarrow{S_1S_2}\|$.

We may follow the next steps.

(a) Translation of line l_1 to the origin.

We translate – see Figure 3 – the pair (l_1, l_2), so obtaining the pair (l_1', l_2') where

$$l_1' := X' = P - P + \alpha\vec{u} = O + \alpha\vec{u},$$

$$l_2' := X' = Q' + \beta\vec{v},$$

with $Q' = Q - P$.

We took the current point, $Q_2'(\beta)$, on the line l_2', which is

$$Q_2'(\beta) = (q_1 - p_1 + \beta v_1, q_2 - p_2 + \beta v_2, q_3 - p_3 + \beta v_3) := \overrightarrow{q_2'}(\beta).$$

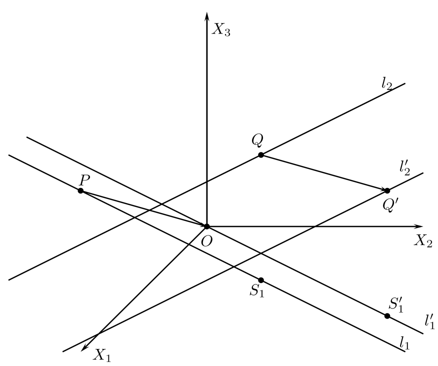

Figure 3. The pair (l_1, l_2) turns into the pair (l_1', l_2').

Our problem, now, is to determine the distance, $d(Q_2'(\beta), l_1')$, between the point $Q_2'(\beta)$ and the line l_1'.

Applying Proposition 1(ii) we have, $S_1'(\beta)$, the foot of the perpendicular onto the line l_1'

$$S_1'(\beta) = -\frac{\begin{vmatrix} \vec{u} \bullet \vec{u} & \vec{u} \bullet \overrightarrow{q_2'}(\beta) \\ \vec{u} & \vec{0} \end{vmatrix}}{G(\vec{u})}.$$

(b)Translation of line l_2 to the origin.

We do another translation of the pair (l_1, l_2), so – see Figure 4 – obtaining the pair (l_1'', l_2''), where

$$l_2'' := X'' = Q - Q + \beta\vec{v} = O + \beta\vec{v},$$

$$l_1'' := X'' = P'' + \alpha\vec{u}, \text{ with } P'' = P - Q.$$

The current point on the line l_1'' is

$$P_1''(\alpha) = (p_1 - q_1 + \alpha u_1,\ p_2 - q_2 + \alpha u_2,\ p_3 - q_3 + \alpha u_3) := \overrightarrow{p_1''}(\alpha)$$

and applying again Proposition 1(ii) we obtain the foot, $S_2''(\alpha)$, of the perpendicular onto l_2''

$$S_2''(\alpha) = -\frac{\begin{vmatrix} \vec{v} \bullet \vec{v} & \vec{v} \bullet \overrightarrow{p_1''}(\alpha) \\ \vec{v} & \vec{0} \end{vmatrix}}{G(\vec{v})}.$$

(c) Doing the reverse translations.

Turning back to the original pair (l_1, l_2), we have

(1) $S_1(\beta) = S_1'(\beta) + P$, onto the line l_1,

(2) $S_2(\alpha) = S_2''(\alpha) + Q$, onto the line l_2

and

(3) $P(\alpha) = P_1''(\alpha) + Q$, onto the line l_1,

(4) $Q(\beta) = Q_2'(\beta) + P$, onto the line l_2.

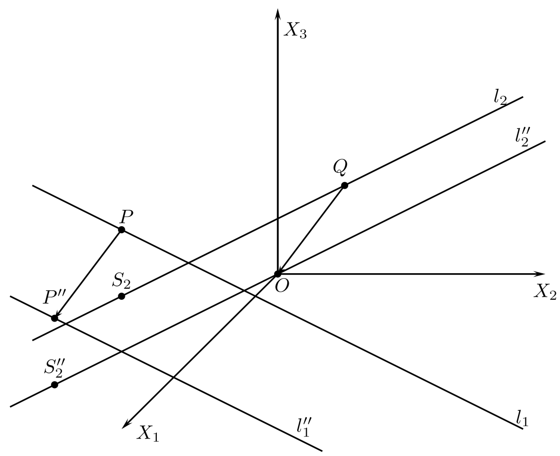

Figure 4. The pair (l_1, l_2) turns into the pair (l_1'', l_2'').

The relations (1), (2), (3) and (4) form a system with six equations and two unknowns α, β. This system is consistent, by geometrical reasons.

Solving this system, we get

$$\alpha = \alpha^*, \quad \beta = \beta^*.$$

(d) Final step.

Once we have the needed concretization, α^*, β^*, of the parameters α and β, we obtain the feet of the perpendicular

$$S_1 = S_1(\beta^*), \quad S_2 = S_2(\alpha^*)$$

and – see Figure 5 – one (out of two) vector achieving the distance $d(l_1, l_2)$ is

$$\overrightarrow{S_1S_2} = S_2 - S_1.$$

Hence, the distance is given by

$$d(l_1, l_2) = \| S_2 - S_1 \|.$$

[Just for verification of the numerical results, we have, due to geometrical considerations,

$$S_1 = P(\alpha^*), S_2 = Q(\beta^*).]$$

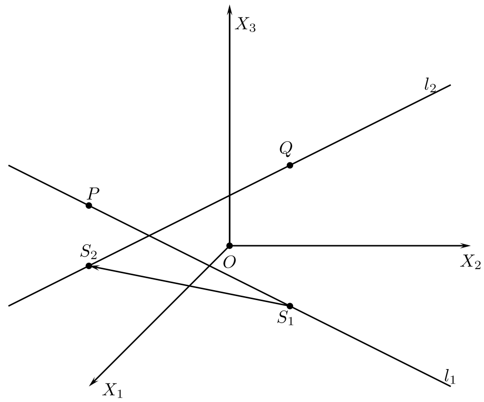

Figure 5. The vector $\overrightarrow{S_1S_2}$ achieves the distance between the skew lines l_1 and l_2.

4.1. Example

Consider the lines l_1 and l_2 given by

$$l_1 := X = P + \alpha\vec{u} = (0, 1, 0) + \alpha(-\overrightarrow{e_1} + \overrightarrow{e_2}), \quad (\alpha \in \mathbb{R}),$$

$$l_2 := X = Q + \beta\vec{v} = (0, 1, 3) + \beta(-\overrightarrow{e_1} + \overrightarrow{e_3}), \quad (\beta \in \mathbb{R}).$$

We are going to determine the distance between lines l_1 and l_2 and the pair of best approximation points, following the steps:

(a) Force the line l_1 through the origin. We have the pair (l_1', l_2'), where

$$l_1' := X' = P - P + \alpha\vec{u} = O + \alpha\vec{u} = \alpha(-\overrightarrow{e_1} + \overrightarrow{e_2}),$$

$$l_2' := X' = Q - P + \beta\vec{v} = Q' + \beta\vec{v} = (0, 0, 3) + \beta(-\overrightarrow{e_1} + \overrightarrow{e_3}).$$

Considering a current point $Q_2'(\beta) = (-\beta, 0, 3 + \beta) := \overrightarrow{q_2'}(\beta)$ of l_2', our problem turns to determining the distance, $d(Q_2'(\beta), l_1')$, between the point $Q_2'(\beta)$ and the line l_1'.

The foot of the perpendicular, $S_1'(\beta)$, onto l_1', is given by

$$S_1'(\beta) = -\frac{\begin{vmatrix} \vec{u} \bullet \vec{u} & \vec{u} \bullet \overrightarrow{q_2'}(\beta) \\ \vec{u} & \vec{0} \end{vmatrix}}{G(\vec{u})} = \left(-\frac{\beta}{2}, \frac{\beta}{2}, 0\right), \quad (\beta \in \mathbb{R}).$$

The vector which materializes the distance (concerning the pair (l'_1, l'_2)) is given by

$$Q'_2(\beta) - S'_1(\beta) = \left(-\frac{\beta}{2}, -\frac{\beta}{2}, 3 + \beta\right).$$

(b) Force the line l_2 through the origin.

We get the pair (l''_1, l''_2), where

$$l''_2 := X'' = Q - Q + \beta\vec{v} = O + \beta\vec{v},$$

$$l''_1 := X'' = P - Q + \alpha\vec{u} = P'' + \alpha\vec{u}.$$

By considering a current point $P''_1(\alpha) = (-\alpha, \alpha, -3) := \overrightarrow{p''_1}(\alpha), (\alpha \in \mathbb{R})$, our problem concerns now the distance, $d(P''_1(\alpha), l''_2)$, between the point $P''_1(\alpha)$ and the line l''_2.

The foot of the perpendicular $S''_2(\alpha)$, on l''_2, is given by

$$S''_2(\alpha) = -\frac{\begin{vmatrix} \vec{v} \bullet \vec{v} & \vec{v} \bullet \overrightarrow{p''_1}(\alpha) \\ \vec{v} & \vec{0} \end{vmatrix}}{G(\vec{v})} = \left(-\frac{\alpha}{2} + \frac{3}{2}, 0, \frac{\alpha}{2} - \frac{3}{2}\right), \quad (\alpha \in \mathbb{R}).$$

The vector materializing the distance $d(l''_1, l''_2)$ is given by

$$P''_1(\alpha) - S''_2(\alpha) = \left(-\frac{\alpha}{2} - \frac{3}{2}, \alpha, -\frac{\alpha}{2} - \frac{3}{2}\right), (\alpha \in \mathbb{R}).$$

(c) Making the corresponding reverse translations.

Turning back to the original pair of lines (l_1, l_2), we have

(1) $S_1(\beta) = S'_1(\beta) + P = \left(-\frac{\beta}{2}, 1 + \frac{\beta}{2}, 0\right)$, onto the line l_1,

(2) $S_2(\alpha) = S''_2(\alpha) + Q = \left(-\frac{\alpha}{2} + \frac{3}{2}, 1, \frac{\alpha}{2} + \frac{3}{2}\right)$, onto the line l_2

and

(3) $P(\alpha) = P''_1(\alpha) + Q = (-\alpha, 1 + \alpha, 0)$, belonging to the line l_1,

(4) $Q(\beta) = Q'_2(\beta) + P = (-\beta, 1, 3 + \beta)$, belonging to the line l_2.

So we obtain a system with six linear equations and the two unknowns α and β. This system, by geometric reasons, is consistent. We get

$$\alpha^* = -1 \quad \text{and} \quad \beta^* = -2.$$

(d) Final step.

Finally we have the best approximation pair that is to say the foot of the perpendicular S_1 on l_1 and the foot of the perpendicular S_2 on l_2, given by

$$S_1 = S_1(\beta^*) = S_1(-2) = (1, 0, 0),$$

$$S_2 = S_2(\alpha^*) = S_2(-1) = (2, 1, 1)$$

and the vector (within a signal) which represents the distance $d(l_1, l_2)$ is

$$\overrightarrow{S_1S_2} = S_2 - S_1 = (2, 1, 1) - (1, 0, 0) = \vec{e_1} + \vec{e_2} + \vec{e_3}$$

and the distance being given by

$$d(l_1, l_2) = \| S_2 - S_1 \| = \sqrt{3}.$$

5. Final Remarks and Conclusions

An important tool in Approximation Theory is the Gram determinant. Relevant references are [1], [3], [4] and [5].

We derived our formulae involving the Gram determinant just for the cases of line and plane and two skew lines. We aimed to arrive at results which we presented in terms of tools which constitute windows to more advanced best approximation problems. To construct the foot of the perpendicular [2], [8, p. 168] – or the solution to the minimum problem [3] – and to exhibit one vector which achieves the distance, in our terminology – or the remainder or error [3, p. 182]; or the optimal error [5, p. 63] – are, in our opinion, appropriate instances for stimulating visualization abilities. Furthermore, the need of translations – due to the fact that the referred to best approximation results are dealt with in the context of subspaces – begs the question as how to approach distance to affine sets, i.e., to translates of subspaces. This is an opportunity for stressing the importance of inner product spaces by referring to [4] and mentioning the approach by general projectors [7]. Also, an interesting question is the consideration of skewness of two lines in higher dimensional real vector spaces. The dot product plays an important role in the Gram determinant theory, nevertheless wariness to other scalar products [8, p. 149] may arise some interesting approximation questions.

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