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The units of a partial Burnside ring relative to the Young subgroups of a symmetric group

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Abstract

The unit group of a partial Burnside ring relative to the Young subgroups of the symmetric group S_n on n letters is included in the image by the tom Dieck homomorphism. As a consequence of this fact, the alternating character ν_n of S_n is expressed explicitly as a $\mathbb{Z}$-linear combinations of permutation characters associated with finite left S_n-sets S_n/Y for the Young subgroups Y.

1 Introduction

Let G be a finite group, and let $\mathrm{Cl}(G)$ be a full set of non-conjugate subgroups of G. For each $H \leq G$, G/H denotes the set of left cosets gH, $g \in G$, of H in G. The Burnside ring $\Omega(G)$ of G is the commutative ring consisting of all formal $\mathbb{Z}$-linear combinations of symbols $[G/H]$ corresponding to G/H, $H \in \mathrm{Cl}(G)$, with multiplication given by

$$[G/H] \cdot [G/U] = \sum_{HgU \in H\backslash G/U} [G/(H \cap {}^gU)]$$

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for all H, $U \in \mathrm{Cl}(G)$, where ${}^gU = gUg^{-1}$ and $[G/(H\cap {}^gU)] = [G/K]$ for a conjugate $K \in \mathrm{Cl}(G)$ of $H \cap {}^gU$ (see, *e.g.*, [5], [13, §2.1]). The identity of $\Omega(G)$ is $[G/G]$. For shortness' sake, we usually write $1 = [G/G]$. While the unit group of $\Omega(G)$ is an elementary abelian 2-group (cf. §2), it is quite interesting to analyze units of $\Omega(G)$.

Let S_n be the symmetric group on n letters $[n] := \{1, 2, \ldots, n\}$, and let $\mathfrak{Y}_n$ be the set of Young subgroups of S_n. Since the Young subgroups are closed under intersection and conjugation, there exists a subring of $\Omega(S_n)$ consisting of all formal $\mathbb{Z}$-linear combinations of symbols $[S_n/Y]$ for $Y \in \mathfrak{Y}_n$, which is denoted by $\Omega(S_n, \mathfrak{Y}_n)$ (see also [2]). (In short, this subring is a partial Burnside ring relative to the Young subgroups of a symmetric group.) Let $R(S_n)$ be the character ring of S_n. Then it is well-known that $\Omega(S_n, \mathfrak{Y}_n) \cong R(S_n)$ (see, *e.g.*, [13, Proposition 7.2]) and the unit group of $R(S_n)$ consists of $\pm 1_{S_n}$, $\pm\nu_n$, where 1_{S_n} is the trivial character of S_n and ν_n is the alternating character of S_n (see, *e.g.*, [11, Example 2]). In particular, the unit group of $\Omega(S_n, \mathfrak{Y}_n)$ is isomorphic to the Klein four-group.

Recently, in [7], Idei and the first author have given a formula of a non-identity unit of $\Omega(S_n, \mathfrak{Y}_n)$ which is described in terms of the Möbius function $\mu_{\mathfrak{Y}_n}$ on the poset $(\mathfrak{Y}_n, \leq)$ (see Eq. (2)). Such a unit is also a unit of $\Omega(S_n)$, and there seems to be some characterization of it as a unit of $\Omega(S_n)$. In general, however, there are many units of $\Omega(S_n)$ (see [3]). The purpose of this paper is to characterize a non-identity unit of $\Omega(S_n, \mathfrak{Y}_n)$ in terms of the tom Dieck homomorphism (see §2). Consequently, we have shown that the unit group of $\Omega(S_n, \mathfrak{Y}_n)$ is included in the image by the tom Dieck homomorphism. In the sequel, ν_n is expressed explicitly as a $\mathbb{Z}$-linear combinations of permutation characters associated with finite left S_n-sets S_n/Y for $Y \in \mathfrak{Y}_n$ (cf. Theorem 4.4), which is also a consequence of [6, Proposition 2.3.8, Exercise 3.15]. In order to show such a result, we see that each image of a permutation character associated with a finite left G-set by the tom Dieck homomorphism is a reduced Lefschetz invariant of a certain G-poset, which is essentially given in [10].

2 The tom Dieck homomorphism

Given $H \leq G$ and a finite left G-set X, we set

$$\mathrm{inv}_H(X) = \{x \in X \mid hx = x \quad \text{for all} \quad h \in H\}.$$

By [5, Proposition 1.2.2], the map $\varphi : \Omega(G) \to \widetilde{\Omega}(G) := \prod_{H\in \mathrm{Cl}(G)} \mathbb{Z}$ given by

$$[G/U] \mapsto (\sharp \mathrm{inv}_H(G/U))_{H\in\mathrm{Cl}(G)}$$

for all $U \in \mathrm{Cl}(G)$ is an injective ring homomorphism, which is called the Burnside homomorphism or the mark homomorphism. Obviously, the unit group of $\widetilde{\Omega}(G)$ is $\prod_{H\in\mathrm{Cl}(G)}\langle -1\rangle$, whence the unit group of $\Omega(G)$ is an elementary abelian 2-group.

We denote by $R_{\mathbb{R}}(G)$ the real representation ring of G, and denote by $\Omega(G)^\times$ the unit group of $\Omega(G)$. For each element x of $\Omega(G)$ with $\varphi(x) = (x_H)_{H\in\mathrm{Cl}(G)}$, we write $x = \varphi^{-1}((x_H)_{H\in \mathrm{Cl}(G)})$. By [5, Proposition 5.5.9], there exists a group homomorphism $u = u_G : R_{\mathbb{R}}(G) \to \Omega(G)^{\times}$ such that

$$M \mapsto \varphi^{-1}(((-1)^{\dim M^H})_{H\in \mathrm{Cl}(G)})$$

for all $\mathbb{R}G$-module M, where M^H is the space of H-invariants of M.

Let $H \leq G$. We set $W_G(H) = N_G(H)/H$, where $N_G(H)$ is the normalizer of H in G. Assume that a finitely generated left $\mathbb{C}G$-module M affords a $\mathbb{C}$-character χ of G. For each $H \leq G$, M^H is viewed as a $\mathbb{C}W_G(H)$-module, which affords the $\mathbb{C}$-character $\overline{\chi}$ of $W_G(H)$ given by

$$\overline{\chi}(gH) = \frac{1}{|H|}\sum_{h\in H}\chi(gh)$$

for all $gH \in W_G(H)$ (see, *e.g.*, [1, Lemma 3.1]). In particular, $\dim M^H$ is equal to the inner product $\langle \chi|_H, 1_H\rangle_H$ of the $\mathbb{C}$-character $\chi|_H$ of H and the trivial character 1_H of H, where $\chi|_H$ is the restriction of χ into H.

Let $\overline{R}_{\mathbb{R}}(G)$ be the ring of real valued virtual $\mathbb{C}$-characters of G. Then it follows from the preceding argument and [12, Theorem A] that $u : R_{\mathbb{R}}(G) \to \Omega(G)^{\times}$ is extended to the group homomorphism $\overline{u} = \overline{u}_G : \overline{R}_{\mathbb{R}}(G) \to \Omega(G)^{\times}$ given by

$$\chi \mapsto \varphi^{-1}(((-1)^{\langle \chi|_H, 1_H\rangle_H})_{H\in \mathrm{Cl}(G)})$$

for all $\chi \in \overline{R}_{\mathbb{R}}(G)$, which is called the tom Dieck homomorphism. According to [13, Corollary 4.3], we have

$$\overline{u}(\chi) = \sum_{U\in \mathrm{Cl}(G)} \frac{1}{|W_G(U)|}\left(\sum_{H\leq G}\mu(U,H)(-1)^{\langle \chi|_H, 1_H\rangle_H}\right)[G/U] \qquad (1)$$

for all $\chi \in \overline{R}_{\mathbb{R}}(G)$, where μ is the Möbius function on the poset $(\mathfrak{S}(G), \leq)$ of all subgroups of G.

Example 2.1 Obviously, $\overline{u}(1_G) = -1$.

Example 2.2 Let A_n be the alternating group on $[n]$. Then $1 - [S_n/A_n]$ is a unit of $\Omega(S_n)$ and is the image of ν_n by the tom Dieck homomorphism.

Remark 2.3 If G is not solvable, then by [9, Theorem 5.4], $u : R_{\mathbb{R}}(G) \to \Omega(G)^{\times}$ is not surjective. In particular, if $n \geq 4$, then $\overline{u} : \overline{R}_{\mathbb{R}}(S_n) \to \Omega(S_n)^{\times}$ is not surjective; $\overline{u}_{S_2}$ and $\overline{u}_{S_3}$ are surjective, however (see [9]). (Note that $2|\mathrm{Im}\,\overline{u}_{S_4}| = |\Omega(S_4)^{\times}| = 2^6$.)

3 The units of $\Omega(S_n, \mathfrak{Y}_n)$

We denote by $R(G)$ the character ring of G. The permutation character π_X associated with a finite left G-set X is given by

$$\pi_X(g) = \sharp\{x \in X \mid gx = x\}$$

for all $g \in G$. We define a ring homomorphism $\mathrm{char}_G : \Omega(G) \to R(G)$ by

$$[X] \mapsto \pi_X$$

for all finite left G-sets X (cf. [12, §6]).

By [13, Proposition 7.2], the ring homomorphism $\mathrm{char}_{S_n} : \Omega(S_n) \to R(S_n)$ induces an isomorphism

$$\overline{\mathrm{char}}_{S_n} : \Omega(S_n, \mathfrak{Y}_n) \to R(S_n)$$

(see also [8, 2.3]). Hence there exists a unique unit, say α, of $\Omega(S_n, \mathfrak{Y}_n)$ satisfying $\overline{\mathrm{char}}_{S_n}(\alpha) = \nu_n$. The unit group of $\Omega(S_n, \mathfrak{Y}_n)$ consists of ± 1, $\pm\alpha$, which are also units of $\Omega(S_n)$.

For each $H \le S_n$, we define a Young subgroup Y_H to be the intersection of all Young subgroups containing H. Each Young subgroup Y of S_n with respect to a partition $(n_1, n_2, \dots, n_r)$ includes a product σ_Y of pairwise disjoint n_i-cycles for $i = 1, 2, \dots, r$ satisfying $Y = Y_{\langle \sigma_Y \rangle}$. Under these notations, we are now in a position to state the following lemma (cf. [13, §7.1]).

Lemma 3.1 *If $\varphi(\alpha) = (\alpha_H)_{H \in \mathrm{Cl}(S_n)}$, then $\alpha_H = \alpha_{Y'_H}$ for all $H \in \mathrm{Cl}(S_n)$, where $Y'_H \in \mathrm{Cl}(S_n)$ is a conjugate of Y_H, and $\alpha_Y = \nu_n(\sigma_Y)$ for all $Y \in \mathrm{Cl}(S_n) \cap \mathfrak{Y}_n$.*

Proof. Suppose that $\alpha = \sum_{j=1}^{s} a_j [S_n/Y_j]$ with $a_j \in \mathbb{Z}$ and $Y_j \in \mathrm{Cl}(S_n) \cap \mathfrak{Y}_n$. If $H \le S_n$, then by the definition of Y_H,

$$\mathrm{inv}_{Y_H}(S_n/Y_j) = \{\sigma Y_j \mid Y_H \le {}^{\sigma}Y_j\} = \{\sigma Y_j \mid H \le {}^{\sigma}Y_j\} = \mathrm{inv}_H(S_n/Y_j)$$

for $j = 1, 2, \dots, s$. Hence it turns out that $\alpha_H = \alpha_{Y'_H}$ for all $H \in \mathrm{Cl}(S_n)$. If $Y \in \mathfrak{Y}_n$, then by assumption, $Y = Y_{\langle \sigma_Y \rangle}$, whence $\mathrm{inv}_Y(S_n/Y_j) = \mathrm{inv}_{\langle \sigma_Y \rangle}(S_n/Y_j)$ for $j = 1, 2, \dots, s$. Since $\overline{\mathrm{char}}_{S_n}(\alpha) = \nu_n$, we conclude that for each $Y \in \mathrm{Cl}(S_n) \cap \mathfrak{Y}_n$,

$$\nu_n(\sigma_Y) = \sum_{j=1}^{s} a_j \pi_{[S_n/Y_j]}(\sigma_Y) = \sum_{j=1}^{s} a_j \sharp \mathrm{inv}_{\langle \sigma_Y \rangle}(S_n/Y_j) = \sum_{j=1}^{s} a_j \sharp \mathrm{inv}_Y(S_n/Y_j) = \alpha_Y.$$

This completes the proof. □

By using [13, Corollary 4.3] and Lemma 3.1, α is expressed in the form

$$\alpha = \sum_{Y \in \mathrm{Cl}(S_n) \cap \mathfrak{Y}_n} \frac{1}{|W_{S_n}(Y)|} \left(\sum_{H \in \mathfrak{Y}_n} \mu_{\mathfrak{Y}_n}(Y, H) \nu_n(\sigma_H) \right) [S_n/Y], \tag{2}$$

where $\sigma_H \in S_n$ with $H = Y_{\langle \sigma_H \rangle}$. This formula is presented in [7, Corollary 5.2].

We aim to show that α is included in the image by the tom Dieck homomorphism. The permutation character $\pi_{[n]}$ associated with the S_n-set $[n]$ is given by

$$\sigma \mapsto \sharp\{k \in [n] \mid \sigma(k) = k\}$$

for all $\sigma \in S_n$. For each $H \leq S_n$, let $\mathrm{Orb}_H\big([n]\big)$ be the set of H-orbits in $[n]$. By the Cauchy-Frobenius lemma (see, *e.g.*, [13, Lemma 2.7]), $\langle \pi_{[n]}|_H, 1_H \rangle_H = \sharp\mathrm{Orb}_H\big([n]\big)$ for all $H \leq S_n$. Set $\chi_n = \pi_{[n]} - 1_{S_n}$. Then it is easily verified that χ_n is an irreducible $\mathbb{C}$-character of S_n. Obviously, $\pi_{[n]} \in \overline{R}_{\mathbb{R}}(S_n)$. We define a unit β of $\Omega(S_n)$ by

$$\begin{aligned}\varphi(\beta) &= ((-1)^{\sharp\mathrm{Orb}_H([n])})_{H \in \mathrm{Cl}(S_n)} = ((-1)^{\langle \pi_{[n]}|_H, 1_H \rangle_H})_{H \in \mathrm{Cl}(S_n)} \\ &= -((-1)^{\langle \chi_n|_H, 1_H \rangle_H})_{H \in \mathrm{Cl}(S_n)},\end{aligned}$$

so that β is the image of $\pi_{[n]}$ by the tom Dieck homomorphism. The fact that $\alpha = (-1)^n \beta$ (cf. Theorem 3.4) is obtained by a combination of Lemma 3.1 and the following lemmas.

Lemma 3.2 *For each $H \leq S_n$, $\sharp\mathrm{Orb}_H([n]) = \sharp\mathrm{Orb}_{Y_H}([n])$. In particular, for each $Y \in \mathfrak{Y}_n$, $\sharp\mathrm{Orb}_Y\big([n]\big) = \sharp\mathrm{Orb}_{\langle \sigma_Y \rangle}\big([n]\big)$, where $\sigma_Y \in S_n$ with $Y = Y_{\langle \sigma_Y \rangle}$.*

Proof. Evidently, the H-orbits in $[n]$ coincide with the Y_H-orbits in $[n]$, completing the proof. □

Lemma 3.3 *For each $\sigma \in S_n$, $(-1)^{\sharp\mathrm{Orb}_{\langle \sigma \rangle}([n])} = (-1)^n \nu_n(\sigma)$.*

Proof. Assume that σ is a product of pairwise disjoint n_i-cycles for $i = 1, 2, \ldots, r$ such that $\sum_i n_i = n$. Then it is obvious that $\sharp\mathrm{Orb}_{\langle \sigma \rangle}\big([n]\big) = r$. On the other hand, if $\ell = \sharp\{i \mid n_i \text{ is odd}\}$, then $\nu_n(\sigma) = (-1)^{r-\ell} = (-1)^{r+n}$, because $\ell \equiv n \pmod 2$, completing the proof. □

We are now successful in characterizing the units of $\Omega(S_n, \mathfrak{Y}_n)$ in terms of the tom Dieck homomorphism.

Theorem 3.4 *The unit group of $\Omega(S_n, \mathfrak{Y}_n)$ is included in the image by the tom Dieck homomorphism. In particular, $\alpha = (-1)^n \beta$.*

Proof. Since $-1 = \overline{u}(1_{S_n})$, it suffices to verify that $\alpha = (-1)^n \beta$. Suppose now that $\varphi(\alpha) = (\alpha_H)_{H \in \mathrm{Cl}(S_n)}$ and $\varphi(\beta) = (\beta_H)_{H \in \mathrm{Cl}(S_n)}$. If $\alpha_Y = (-1)^n \beta_Y$ for all $Y \in \mathrm{Cl}(S_n) \cap \mathfrak{Y}_n$, then by Lemmas 3.1 and 3.2, $\alpha_H = (-1)^n \beta_H$ for all $H \in \mathrm{Cl}(S_n)$, whence $\alpha = (-1)^n \beta$. Now let $Y \in \mathrm{Cl}(S_n) \cap \mathfrak{Y}_n$. Then by virtue of Lemma 3.1, $\alpha_Y = \nu_n(\sigma_Y)$, where $\sigma_Y \in S_n$ with $Y = Y_{\langle \sigma_Y \rangle}$. Hence it follows from Lemmas 3.2 and 3.3 that

$$\alpha_Y = (-1)^{\sharp\mathrm{Orb}_{\langle \sigma_Y \rangle}([n]) + n} = (-1)^{\sharp\mathrm{Orb}_Y([n]) + n} = (-1)^n \beta_Y.$$

We have thus completed the proof. □

4 The reduced Lefschetz invariant of a G-poset

There is a valuable application of Theorem 3.4. The expression of ν_n as a $\mathbb{Z}$-linear combinations of permutation characters $\pi_{S_n/Y}$ for $Y \in \mathfrak{Y}_n$ is implicit in Eq. (2), while it is worth studying the explicit descriptions.

A finite left G-set equipped with order relation $\leq$ is called a G-poset if $\leq$ is invariant under the action of G. Let P be a G-poset, and let $Sd_i(P)$ be the set of chains $x_0 < x_1 < \cdots < x_i$ of elements of P of cardinality $i+1$. Recall that $\Omega(G)$ is the Grothendieck group of the category of finite left G-sets and is an abelian group generated by the isomorphism classes $[X]$ of finite left G-sets X (cf. [5, 13]). The Lefschetz invariant Λ_P of P is defined by

$$\Lambda_P = \sum_{i=0}^{\infty} (-1)^i [Sd_i(P)] \in \Omega(G),$$

and the reduced Lefschetz invariant $\widetilde{\Lambda}_P$ of P is defined by $\widetilde{\Lambda}_P = \Lambda_P - 1$ (cf. [4, 10]). In particular, for the poset $P(X)$ consisting of nonempty and proper subsets of a finite left G-set X, the K-component of $\varphi(\widetilde{\Lambda}_{P(X)})$ with $K \in \mathrm{Cl}(G)$ is equal to the reduced Euler-Poincaré characteristic of $P(X)^K (= \mathrm{inv}_K(P(X)))$:

$$\sum_{i=0}^{\infty} (-1)^i |Sd_i(P(X)^K)| - 1.$$

We next give a combinatorial proof of the following proposition, which is essentially proved by [10, Proposition 5.1].

Proposition 4.1 *Let X be a finite left G-set. The reduced Lefschetz invariant $\widetilde{\Lambda}_{P(X)}$ of $P(X)$ is the image of π_X by the tom Dieck homomorphism.*

To prove Proposition 4.1, we require the following combinatorial lemma.

Lemma 4.2 *For each positive integer j, set*

$$c_j = \sum_{i=1}^{j} (-1)^i \sum_{(n_1, n_2, \ldots, n_i) \in A(i,j)} \binom{j}{n_1, n_2, \ldots, n_i},$$

where $A(i,j) = \{(n_1, n_2, \ldots, n_i) \mid \sum_k n_k = j$ and $n_1, n_2, \ldots, n_i \in \mathbb{N}\}$ and

$$\binom{j}{n_1, n_2, \ldots, n_i} = \frac{j!}{n_1! n_2! \cdots n_i!} \quad (\textit{multinomial coefficients}).$$

Then $c_j = (-1)^j$ for any positive integers j.

Proof. We use induction on j. Obviously, $c_1 = -1$. Assume that $j \geq 2$ and

$$c_\ell = \sum_{i=1}^{\ell}(-1)^i \sum_{(n_1, n_2, \ldots, n_i) \in A(i,\ell)} \binom{\ell}{n_1,\, n_2, \ldots,\, n_i} = (-1)^\ell$$

for any positive integer ℓ less than j. Clearly, there exists a bijection

$$\dot{\bigcup_{\ell=i,\ldots,j-1}} A(i,\ell) \to A(i+1,j),\ (n_1,\, n_2, \ldots,\, n_i) \mapsto (n_1,\, n_2, \ldots,\, n_i,\, j-\ell)$$

for each positive integer i less than j. Hence the inductive assumption yields

$$\begin{aligned} c_j &= -\binom{j}{j} + \sum_{i=2}^{j}(-1)^i \sum_{\ell=i-1}^{j-1} \sum_{(n_1, n_2, \ldots, n_{i-1}) \in A(i-1,\ell)} \frac{j!}{n_1!n_2!\cdots n_{i-1}!(j-\ell)!} \\ &= -\binom{j}{j} - \sum_{\ell=1}^{j-1}\binom{j}{\ell,\, j-\ell} c_\ell \\ &= -1 - (1-1)^j + 1 + (-1)^j = (-1)^j, \end{aligned}$$

as desired. This completes the proof. □

Proof of Proposition 4.1. For each $K \leq G$, we denote by m_K the number of K-orbits in X. Then it follows from the Cauchy-Frobenius lemma that $m_K = \langle \pi_X|_K, 1_K\rangle_K$. Hence the assertion is equivalent to the equality $\varphi(\widetilde{\Lambda}_{P(X)}) = ((-1)^{m_K})_{K\in \mathrm{Cl}(G)}$. Let $K \in \mathrm{Cl}(G)$. Every chain $x_0 < x_1 < \cdots < x_i$ of elements of $P(X)^K$ is built on pairwise disjoint unions y_j of ℓ_j K-orbits for $j = 0,\, 1, \ldots,\, i$ such that $\sum_j \ell_j < m_K$ and $x_k = \dot\cup_{j=0}^{k} y_j$ for $k = 0,\, 1, \ldots,\, i$. Hence a simple observation enables us to get

$$|Sd_i(P(X)^K)| = \sum_{(m_1, m_2, \ldots, m_{i+2}) \in A(i+2, m_K)} \binom{m_K}{m_1,\, m_2, \ldots,\, m_{i+2}}$$

for each integer i with $0 \leq i \leq m_K - 2$. This, together with Lemma 4.2, shows that

$$\begin{aligned} \sum_{i=0}^{\infty}(-1)^i|Sd_i(P(X)^K)| &= \sum_{i=2}^{m_K}(-1)^i \sum_{(m_1, m_2, \ldots, m_i) \in A(i, m_K)} \binom{m_K}{m_1,\, m_2, \ldots,\, m_i} \\ &= (-1)^{m_K} + 1. \end{aligned}$$

Thus the reduced Euler-Poincaré characteristic of $P(X)^K$ is $(-1)^{m_K}$ (see also [10, Proposition 5.1]). Consequently, we have $\varphi(\widetilde{\Lambda}_{P(X)}) = ((-1)^{m_K})_{K\in \mathrm{Cl}(G)}$. This completes the proof. □

We turn to the study of an explicit description of β (cf. Eq. (3)).

Corollary 4.3 *The reduced Lefschetz invariant* $\widetilde{\Lambda}_{P([n])}$ *of* $P([n])$ *coincides with* β.

Proof. By definition, β is the image of $\pi_{[n]}$ by the tom Dieck homomorphism. Hence the assertion is an immediate consequence of Proposition 4.1 with $G = S_n$ and $X = [n]$. □

Given $H \leq G$, we denote by ${1_H}^G$ the $\mathbb{C}$-character of G induced from 1_H, which coincides with $\pi_{G/H}$.

For each cycle type $\lambda = (1^{m_1}, \dots, n^{m_n})$ of a permutation on $[n]$, let S_λ be a Young subgroup of S_n isomorphic to $S_1^{(m_1)} \times \cdots \times S_n^{(m_n)}$, where each $S_j^{(m_j)}$ is the direct product of m_j copies of S_j.

We are now ready to express ν_n explicitly as a $\mathbb{Z}$-linear combinations of ${1_Y}^{S_n}$ for $Y \in \mathfrak{Y}_n$. The following result simplifies the expression of ν_n in [8, Theorem 2.3.15], and is also a consequence of [6, Proposition 2.3.8, Exercise 3.15].

Theorem 4.4 *The alternating character* ν_n *of* S_n *is expressed explicitly in the form*

$$\nu_n = \sum_{\lambda=(1^{m_1}, \dots, n^{m_n})} (-1)^{n+m_1+\cdots+m_n} \frac{(m_1 + \cdots + m_n)!}{m_1! \cdots m_n!} {1_{S_\lambda}}^{S_n},$$

where the sum runs over all cycle types of permutations on $[n]$.

Proof. Since $\beta = (-1)^n \alpha$ by Theorem 3.4 and $\nu_n = \overline{\mathrm{char}}_{S_n}(\alpha)$, the assertion is equivalent to the formula

$$((-1)^n \alpha =) \beta = \sum_{\lambda=(1^{m_1}, \dots, n^{m_n})} (-1)^{m_1+\cdots+m_n} \frac{(m_1 + \cdots + m_n)!}{m_1! \cdots m_n!} [S_n/S_\lambda]. \quad (3)$$

Let i be a nonnegative integer. Then $[Sd_i(P([n]))] = \sum_t [O_t]$, where O_t's are the S_n-orbits in $Sd_i(P([n]))$. Observe that each representative $x_0 < x_1 < \cdots < x_i$ of an S_n-orbit O_t determines $\{y_0, y_1, \dots, y_i, y_{i+1}\} \subset P([n])$ such that $x_k = \dot\bigcup_{j=0}^k y_j$ for $k = 0, 1, \dots, i$ and $[n] = \dot\bigcup_{j=0}^{i+1} y_j$, which corresponds to a cycle type $(1^{m_1}, \dots, n^{m_n})$ with $m_\ell = \sharp\{k \mid \sharp y_k = \ell\}$ for $\ell = 1, \dots, n$. Conversely, if $\lambda = (1^{m_1}, \dots, n^{m_n})$ with $\sum_\ell m_\ell = i + 2 \geq 2$ is a cycle type of a permutation on $[n]$, then we can make up

$$\frac{(m_1 + \cdots + m_n)!}{m_1! \cdots m_n!}$$

S_n-orbits in $Sd_i(P([n]))$, which are isomorphic to S_n/S_λ as S_n-sets. Hence Eq. (3) follows form Corollary 4.3. This completes the proof. □

Remark 4.5 By the Cauchy-Frobenius lemma, $\langle {1_H}^G, 1_G \rangle_G = \langle \pi_{G/H}, 1_G \rangle_G = 1$ for all $H \leq G$. Hence it follows from Theorem 4.4 that

$$\sum_{\lambda=(1^{m_1}, \dots, n^{m_n})} (-1)^{m_1+\cdots+m_n} \frac{(m_1 + \cdots + m_n)!}{m_1! \cdots m_n!} = (-1)^n \langle \nu_n, 1_{S_n} \rangle_{S_n} = 0.$$

Example 4.6 By Eq. (3), the coefficients of $[S_n/U]$ in Eq. (1) with $G = S_n$ and $\chi = \pi_{[n]}$ are completely determined, and so are the coefficients of $[S_n/Y]$ in Eq. (2). If $n = 3$, then

$$\alpha = [S_3/S_{(1^3)}] - 2[S_3/S_{(1^1,\,2^1)}] + [S_3/S_{(3^1)}]$$

(see also [8, pp. 41–42 and Theorem 2.3.15]). If $n = 4$, then

$$\alpha = [S_4/S_{(1^4)}] - 3[S_4/S_{(1^2,\,2^1)}] + [S_4/S_{(2^2)}] + 2[S_4/S_{(1^1,\,3^1)}] - [S_4/S_{(4^1)}].$$

5 Concluding remarks

If e is an idempotent of $\mathbb{Q} \otimes_{\mathbb{Z}} \Omega(S_n, \mathfrak{Y}_n)$ and if $2e \in \Omega(S_n, \mathfrak{Y}_n)$, then $1 - 2e$ and $-1 + 2e$ are units of $\Omega(S_n, \mathfrak{Y}_n)$. Since α is a unit of $\Omega(S_n, \mathfrak{Y}_n)$, it follows that $(1 - \alpha)/2$ is an idempotent of $\mathbb{Q} \otimes_{\mathbb{Z}} \Omega(S_n, \mathfrak{Y}_n)$. However, $(1 - \alpha)/2 \notin \Omega(S_n, \mathfrak{Y}_n)$, because $(1_{S_n} - \nu_n)/2 \notin R(S_n)$. Thus we have the following.

Proposition 5.1 *The idempotents of* $\Omega(S_n, \mathfrak{Y}_n)$ *are only* 0, 1.

What about *Proof* of Proposition 5.1 in terms of S_n-sets? Of course, in Eq. (3), the coefficient of $[S_n/S_{(1^n)}]$ is always $(-1)^n$, which implies that $(1 - \alpha)/2 \notin \Omega(S_n)$. This fact is also a consequence of Theorem 3.4, because it follows from [9, (5.4.1)] that, if e is a non-trivial idempotent, *i.e.*, $e \neq 0$, 1, of $\Omega(G)$, then the unit $1 - 2e$ is not included in the image by the tom Dieck homomorphism.

The following lemma is [12, Lemma 2.1] (see also [5, Proposition 1.3.5]), which is used for the proof of existence of the tom Dieck homomorphism in [12].

Lemma 5.2 *An element* $(x_H)_{H \in \mathrm{Cl}(G)}$ *of* $\widetilde{\Omega}(G)$ *is included in the image* $\mathrm{Im}\varphi$ *by the Burnside homomorphism if and only if*

$$\sum_{gU \in W_G(U)} x_{\langle g \rangle U} \equiv 0 \pmod{|W_G(U)|}$$

for all $U \in \mathrm{Cl}(G)$*, where* $x_{\langle g \rangle U} = x_K$ *for a conjugate* $K \in \mathrm{Cl}(G)$ *of* $\langle g \rangle U$*.*

We end this paper by giving a direct proof of the fact that $(1 - \alpha)/2 \notin \Omega(S_n)$.

Proof of Proposition 5.1. Suppose that $\varphi(\alpha) = (\alpha_H)_{H \in \mathrm{Cl}(S_n)}$. Then it follows from Lemma 3.3 and Theorem 3.4 (or the definition of α) that

$$\sum_{\sigma \in S_n} \frac{1 - \alpha_{\langle \sigma \rangle}}{2} = \sum_{\sigma \in S_n} \frac{1 - \nu_n(\sigma)}{2} = |A_n| \not\equiv 0 \pmod{|S_n|},$$

where $\alpha_{\langle \sigma \rangle} = \alpha_{\langle \tau \rangle}$ if σ is a conjugate of $\tau \in S_n$ with $\langle \tau \rangle \in \mathrm{Cl}(S_n)$. This, combined with Lemma 5.2, shows that $((1 - \alpha_H)/2)_{H \in \mathrm{Cl}(S_n)} \notin \mathrm{Im}\varphi$. Thus $(1 - \alpha)/2 \notin \Omega(S_n)$, completing the proof. □

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