416. Rotations of a dumbbell equipped with 'the leier constraint'

A. V. Rodnikov
Bauman Moscow State Technical University, Russia
e-mail: *avrodnikov@yandex.ru*

(Received: 23 September; accepted: 02 December)

Abstract. We consider a special space tethered system consisting of a dumbbell-shaped rigid body and a particle. The particle coast along on the cable. The cable ends are placed in the dumbbell endpoints. We call such system 'the system with leier constraint (the Dutch term 'leier' means the rope with both fixed ends). We assume that the system mass center moves along the circular orbit in the Newtonian Central Force Field. We study the dumbbell's relative motion caused by the particle of small mass in the orbital frame of reference. We deduce a sufficient condition for librations of the dumbbell about its stable equilibrium. We find a family of the dumbbell's asymptotic motions tending to librations about unstable equilibrium. The surface of such asymptotic motions is an interstream separating the areas of the dumbbell's right-hand and left-hand rotations. We deduce an equation of this surface.

Key words: space tethered system, leier constraint, asymptotic solution, circular orbit

Introduction

Space tethered systems are one of the most interesting topics in dynamics. For the first time the motion of a particle tethered to a spacecraft has been suggested in [1, 2]. Presently there are hundreds papers devoted to various aspects of the motion of tethered satellites. In this paper we study some generalization of the classic couple.

We consider the system that moves in the Newtonian Central Force Field and consists of a dumbbell-shaped rigid body and a particle. The particle coasts along on the cable with ends placed in the dumbbell endpoints. We call such cable 'a leier'. (the Dutch maritime term 'leier' means the rope with both fixed ends).

We assume the system mass center describes circular orbit, the cable length is small in comparison with orbit radius, the particle mass is small in comparison with the dumbbell mass, the cable do not leave the orbit plane. We study the dumbbell rotation caused by the small particle in the orbital frame of reference. It is well known that the dumbbell-shaped satellite has two types of relative equilibria. There are the stable 'vertical' equilibria and unstable 'horizontal' equilibria. We claim that the small particle sufficiently influence the dumbbell relative motion only if the dumbbell is initially quasi-horizontal. We prove that if the system Jacobi's integral less than some constant then only librations about the 'vertical' equilibrium are possible. We note that there exist a set of the dumbbell relative motions tending to librations about the 'horizontal' equilibria. Factually, these asymptotic motions form the surface being an interstream between areas of left-hand and right-hand rotations of the dumbbell. We deduce the equation of this interstream.

Designations and parameters

Consider a mechanical system consisting of a rigid body and a particle with mass m_3. Assume that the body is a dumbbell, i.e. it is composed of particles with masses m_1 and m_2 connecting by weightless rod of length $2c$.

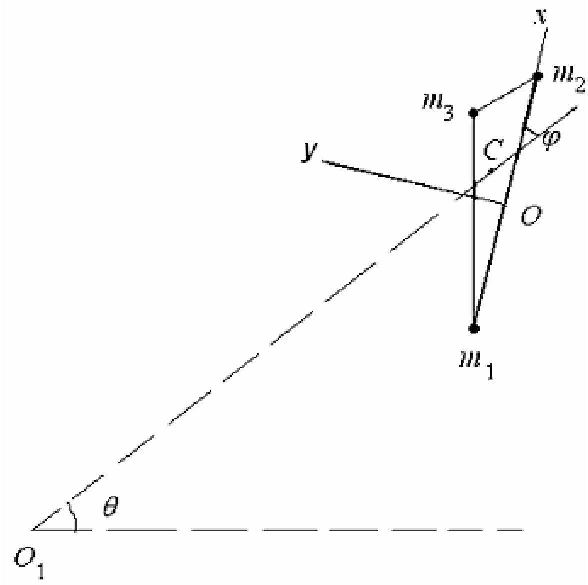

Fig. 1.

Without loss of generality, $m_2 \geq m_1$. Suppose the particle m_3 coast along on the cable with ends fixed to the dumbbell endpoints (Fig.1). This cable can be called 'a leier'. Denote by $2a$ the cable length. Let C be the mass center for considered system and O_1 be the attracting center. Suppose C moves along the circular orbit, i.e. $O_1C=r=$const and the particles m_1, m_2, m_3 do not leave the plane of this orbit. Moreover assume $a<<r$. Denote by φ the angle between O_1C and the rod.

Evidently, the particle m_3 cannot leave the ellipse with foci in the dumbbell endpoints. The ellipse has eccentricity

$e = c/a$ and semi-axises a and $b = \sqrt{a^2 - c^2}$. Let Oxy be a coordinate system with origin in the dumbbell midpoint (see fig.1). Clearly, if x and y is the coordinates of the particle m_3 the inequality

$$x^2 + dy^2 - a^2 \le 0;\; d = a^2/b^2 \qquad (1)$$

is valid. The motion of m_3 is called the constrained one if (1) is equality. In this case the coordinates of m_3 can be determine by formulae

$$x = a\cos\gamma,\; y = b\sin\gamma \qquad (2)$$

where γ is an eccentric anomaly of the mentioned ellipse. If m_3 moves inside the ellipse then we say that the motion is the unconstrained one (or the free one).

Let $\mu = (m_2 - m_1)/(m_2 + m_1)$ and $\nu = m_3/(m_2 + m_1)$. Trivially, $0 < \mu < 1,\; 0 < e < 1,\; \nu > 0$,

It is clear that the dimensionless parameters μ, ν, e and the variables φ, γ determine the considered system dynamics completely in the case of constrained motion.

Lagrangian and Jacobi's integral

Lagrangian for relative motion of the considered couple has a form [3,4]

$$L = L_2 + L_1 + L_0 \qquad (3)$$

where

$$L_2 = \frac{1}{2}\Big\{\varphi'^2 + k[(1 - 2e\mu\cos\gamma + e^2\cos^2\gamma)\varphi'^2 + \sqrt{1-e^2}(1 - 2e\mu\cos\gamma)\varphi'\gamma' + (1 - e^2\cos^2\gamma)\gamma'^2$$

$$L_1 = ke\cos\gamma(e\cos\gamma - 2\mu)\varphi'$$

$$L_0 = -W = \frac{3}{2}\cos^2\varphi +$$

$$+ k\Big\{\frac{9}{8}e^2\cos\varphi - \frac{3}{2}e\mu\cos\gamma + \frac{3}{8}e^2\cos 2\gamma -$$

$$-\frac{3}{4}e\mu\Big[(1 - \sqrt{1-e^2})\cos(2\varphi - \gamma) +$$

$$+(1 + \sqrt{1-e^2})\cos(2\varphi + \gamma)\Big] +$$

$$+\frac{3}{16}\Big[(1 - \sqrt{1-e^2})^2\cos(2\varphi - \gamma) +$$

$$+(1 + \sqrt{1-e^2})^2\cos(2\varphi + \gamma)\Big]\Big\}.$$

$$k = \frac{\nu}{e^2(1-\mu^2)}.$$

Hence we have **Jacobi's integral** $L_2 + W = h$.

The prime $'$ denotes the derivative w.r.t. dimensionless time $\tau = G^{1/2}M^{1/2}r^{-3/2}t$, where G is the gravity constant, M is the mass of the attracting center.

The constrained motion condition

Note also that the constrained motion is possible only if [3,4]

$$\sqrt{1-e^2}(1 - e\mu\cos\gamma)(\varphi' + 1)^2 +$$

$$+2(1 - e^2\cos^2\gamma)(\varphi' + 1)\gamma' +$$

$$+\sqrt{1-e^2}\gamma'^2 - \frac{3}{2}(1 - e^2)\sin 2\gamma\sin 2\varphi + \frac{\sqrt{1-e^2}}{2}\cdot$$

$$\cdot(3\cos 2\varphi\cos 2\gamma + 1 - e\mu\cos\gamma(1 + 3\cos 2\varphi)) \ge 0.$$

The dumbbell rotations caused by the small particle

Let the mass m_3 be small in comparison with the dumbbell mass, i.e. $k << 1$.

It is well-known that there exist two types of stationary motions of the dumbbell-shaped satellite. There are 'the vertical' equilibria ($\varphi = 0$ or $\varphi = \pi$) and 'the horizontal' equilibria ($\varphi = \pm\pi/2$).

Obviously 'the vertical' equilibria are stable. The particle motion does not destroy these equilibria. Only some librations of the dumbbell about 'vertical' position are possible in this case.

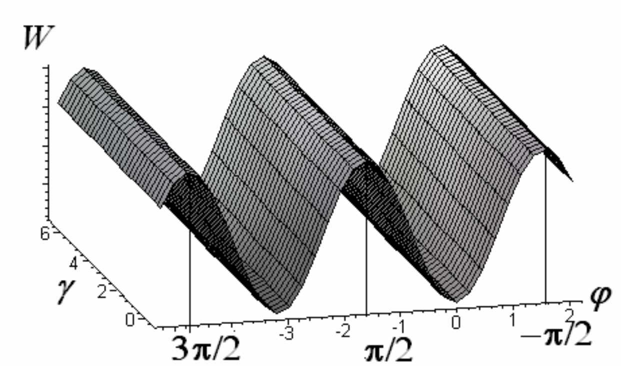

Fig. 2.

It can easily be checked that if the dumbbell is 'quasi-horizontal' initially then the particle motion along the leier force the upturning of the dumbbell. The further motion of the dumbbell belongs to one of three types. There are 'the libratory motion' about the 'vertical' equilibria, 'the rotary motion' about mass center, the complicated 'tumbling motion' consisting of libratory and rotary segments.

Let us remark that the dumbbell tends to librations about its 'horizontal' equilibria for some singular initial values of (γ, γ').

A sufficient condition for the dumbbell libration

It is not hard to prove that if Jacobi's integral constant h is smaller than $h^* = 3/8 \cdot k(5e^2 - 2)$ then only `the libratory motion' is possible. Consider a plot of W (Fig. 2). We see a mountain country consisting of parallel ridges $\varphi = \pi/2 + \pi k$ and valleys $\varphi = \pi k$, where k is integer. The ridge is the sequence of 'peaks' $\gamma = \pi k$ and saddle-points $\gamma = \pi/2 + \pi k$. In the saddle-point $W = h^*$.

Therefore if $h < h^*$ then the dumbbell 'cannot pass through the ridge' and rotations on complete angle are impossible. For instance, the libratory motion is observed for any initial value of φ and zero initial velocities if initial value of γ is about $\pi/2$. It can be shown numerically that 'the rotary motion' is guaranteed only if the initial value of $|\gamma'|$ is sufficiently big.

The motion equations reduction for the symmetric dumbbell

Note that 'the tumbling motion' is a set of right-hand and left-hand rotations with close to 180^0 angles. Factually, chaotic rotations of the dumbbell are obtained.

Consider a single rotation from this set. Let $(\gamma_1, \gamma_1', \varphi_1, \varphi_1')$ be values of $(\gamma, \gamma', \varphi, \varphi')$ in the beginning of this rotation. It is clear that $\varphi' \approx 0$ and $\varphi_1 \approx \pm\pi/2$. (Without loss of generality it can be assumed that $\varphi_1 \approx -\pi/2$). It is obvious that the motion in the vicinity of `horizontal' equilibrium determine the direction of the considered rotation. Substituting $\varphi \approx -\pi/2 + \sqrt{k}\psi$ in the dumbbell's motion equation we obtain

$$\psi' - 3\psi + D\sqrt{k} = 0\,, \tag{4}$$

$$2(1 - e^2 \cos^2\gamma)\gamma' + e^2\gamma' \sin 2\gamma - \\ - 3(1 - e^2)\sin\gamma = 0 \tag{5}$$

where $D = \sqrt{1-e^2}\gamma'' - e^2\gamma' \sin 2\gamma - 3/2\sqrt{1-e^2}\sin 2\gamma$. Here we are restricted to a case of symmetric dumbbell $\mu = 0 \Leftrightarrow m_1 = m_2$ and neglect the terms of order higher than $\sqrt{k}$.

Note that (5) is equation of motion for the particle if the dumbbell *is fixed* in the 'horizontal' position [3]. Equality

$$(1 - e^2\cos^2\gamma)\gamma'^2 + 3(1 - e^2)\cos^2\gamma = h_2 \tag{6}$$

is the Jacobi's integral for (5). Analyzing phase portrait of (6) we see that there exist three types of equation (5) solutions. Solutions of the first type correspond to librations about $\pm\pi/2$. They are periodic functions of τ with period

$$T = \oint \frac{d\gamma}{\gamma'(\gamma, h_2)}.$$

Solutions of the second type correspond to the asymptotic motions tending to $\gamma = 0$ or $\gamma = \pi$. It can easily be checked that in this case the cable weakens. Let us remark that such effect is also observed for the motions in some vicinity of the separatrix. Solutions of the third type correspond to rotations about the dumbbell. In this case derivative of γ w.r.t. τ is the periodic function with period

$$T = \int_0^{\pi} \frac{d\gamma}{\gamma'(\gamma, h_2)}.$$

Moreover, solutions of (5) can be represented in a form

$$\gamma = \gamma(\tau, \gamma_1, h_2(\gamma_1, \gamma_1')) = \frac{\pi}{T}\tau + \sigma(\tau, \gamma_1, h_2)$$

where $\sigma(\tau, \gamma_1, h_2)$ is T-periodic function of τ. Thus if the motion is constrained then $D = D(\gamma''(\tau,, \gamma_1, h_2), \gamma'(\tau,, \gamma_1, h_2), \gamma(\tau,, \gamma_1, h_2) = D_1(\tau)$ is T-periodic function of τ. (Here h_2 depends on (γ_1, γ_1')).

The reduced equations' solutions

Solutions of (4) can be represented in a form $\psi(\tau) = p(\tau) + q(\tau)$, where

$$p(\tau) = \frac{\sqrt{k}}{2\sqrt{3}}\Big(exp(\tau\sqrt{3})\int_{\tau}^{+\infty} exp(-\xi\sqrt{3})D_1(\xi)d\xi + \\ + exp(-\tau\sqrt{3})\int_{-\infty}^{\tau} exp(\xi\sqrt{3})D_1(\xi)d\xi\Big),$$

$$q(\tau) = \frac{1}{2\sqrt{3}}\Big(C_1 exp(\tau\sqrt{3}) + C_2 exp(-\tau\sqrt{3})\Big) \tag{7}$$

From equalities

$$p(\tau + T) = \\ = \frac{\sqrt{k}}{2\sqrt{3}}\Big(exp((\tau + T)\sqrt{3})\int_{\tau+T}^{+\infty} exp(-\xi\sqrt{3})D_1(\xi)d\xi + \\ + exp(-(\tau + T)\sqrt{3})\int_{-\infty}^{\tau+T} exp(\xi\sqrt{3})D_1(\xi)d\xi\Big) = \\ = \frac{\sqrt{k}}{2\sqrt{3}}\Big(exp(\tau + T)\sqrt{3}\Big)\cdot \\ \cdot\int_{\tau}^{+\infty} exp(-(\zeta + T)\sqrt{3})D_1(\zeta + T)d\zeta + \\ + exp(-(\tau + T)\sqrt{3})\cdot \\ \cdot\int_{-\infty}^{\tau+T} exp((\zeta + T)\sqrt{3})D_1(\zeta + T)d\zeta\Big) = \\ = \frac{\sqrt{k}}{2\sqrt{3}}\Big(exp\,\tau\sqrt{3})\int_{\tau}^{+\infty} exp(-\xi\sqrt{3})D_1(\xi)d\xi + \\ + exp(-\tau\sqrt{3})\int_{-\infty}^{\tau} exp(\xi\sqrt{3})D_1(\xi)d\xi\Big) = p(\tau)$$

it follows that $p(\tau)$ is T-periodic function of τ. Constants C_1 and C_2 are defined by formulae

$$C_1 = k^{-1/2}\left(\sqrt{3}\varphi_1 + \frac{\sqrt{3}\pi}{2} + \varphi_1'\right) - k^{1/2}A,$$

$$C_2 = k^{-1/2}\left(\sqrt{3}\varphi_1 + \frac{\sqrt{3}\pi}{2} - \varphi_1'\right) - k^{1/2}B,$$

where

$$A = \int_0^{+\infty} e^{-\xi\sqrt{3}} D_1(\xi)d\xi,\ B = \int_{-\infty}^{\xi} e^{\xi\sqrt{3}} D_1(\xi)d\xi.$$

A surface of asymptotic motions as an interstream

Clearly, if $C_1>0$ then the dumbbell will turn counterclockwise and if $C_1<0$ then the dumbbell will turn clockwise. Certainly, this criterion is valid only for the constrained motion.

If $C_1=0$ then the dumbbell remain in the vicinity of horizontal equilibrium, i.e. we have the dumbbell asymptotic motion tending to librations about $\varphi = -\pi/2$ (or $\varphi = \pi/2$). Clearly, this asymptotic motion is unstable. Thus the equation

$$\sqrt{3}\varphi_1 + \frac{\sqrt{3}\pi}{2} + \varphi'_1 = k\int_0^{+\infty} e^{-\xi\sqrt{3}} D_1(\xi)d\xi \qquad (8)$$

define a surface of asymptotic motions in the four-dimensional space of $(\gamma_1, \gamma'_1, \varphi_1, \varphi'_1)$. In other words, (8) is the equation of an original **interstream** dividing the space of initial values into the areas of rotations clockwise and rotations counterclockwise. Note also that if $C_1 = C_2 = 0$ then we have the dumbbell periodic motion about horizontal equilibrium.

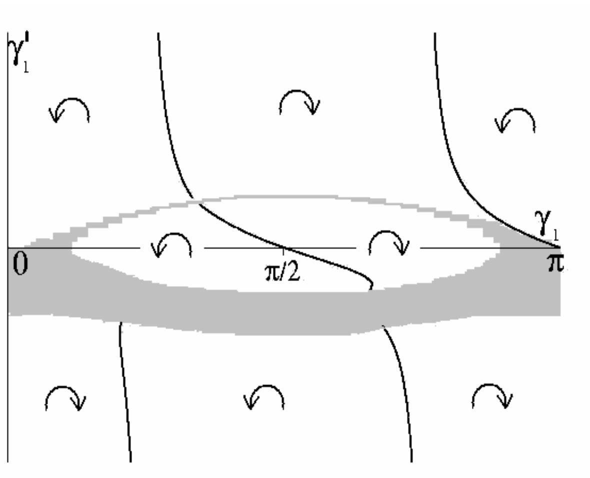

Fig. 3.

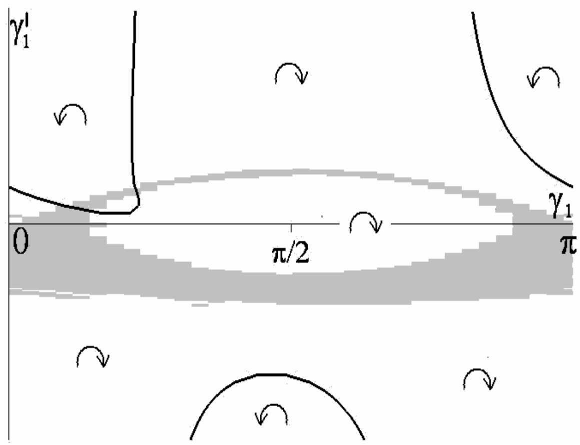

Fig. 4.

Examples of interstreams

The right side of (8) depends only on γ_1, γ'_1 and left side depends only on φ_1, φ'_1. Therefore the interstreams can be depicted in the plane (γ_1, γ'_1) for fixed values of φ_1, φ'_1.

In particular, if the dumbbell is precisely horizontal ($\varphi_1 = -\pi/2$; $\varphi'_1 = 0$ at the beginning of considered rotation then (8) is reduced up to the equality $A=0$. The corresponding interstream is depicted in Fig. 3 for $e=1/2$. In this figure the areas of right-hand and left-hand rotations are marked by the circular arrows.

The similar interstream for $\varphi_1 = -90°3'$; $\varphi_1 = 0$ is depicted in Figure 4. Here also $e=1/2$.

In figures 2 and 3 the shadowed area corresponds to the motion with the weakened cable.

On integral *A* computation

Finally note that the infinite integral A is reduced up to definite. It follows from equalities

$$A = \int_0^{+\infty} exp(-\xi\sqrt{3})D_1(\xi)d\xi =$$
$$= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} exp(-\xi\sqrt{3})D_1(\xi)d\xi =$$
$$= \sum_{n=0}^{\infty} \int_0^T exp(-(\zeta + nT)\sqrt{3})D_1(\zeta + nT)d\xi =$$
$$= \int_0^T exp(-\zeta\sqrt{3})D_1(\zeta)d\zeta \sum_{n=0}^{\infty} exp(-nT\sqrt{3}) =$$
$$= \frac{1}{1 - exp(-T\sqrt{3})} \int_0^T exp(-\tau\sqrt{3})D_1(\tau)d\tau$$

Further, using $d\tau = \gamma'(\tau,, \gamma_1, h_2)d\gamma$ we can change the variable in the last integral. For instance, consider the area of the particle 'positive rotations'. In this area $h_2 > 3(1-e^2)$ and $\gamma' > 0$. Here using (6) we get

$$\gamma' = \sqrt{r(h_2, \gamma)}, \qquad (9)$$

where

$$r(h_2, \gamma) = \frac{h_2 - 3(1-e^2)}{1 - e^2 cos^2 \gamma},$$
$$h_2 = (1 - e^2 cos^2 \gamma_1)\gamma_1'^2 + 3(1-e^2)cos^2 \gamma_1 .$$

From (9) it follows that

$$T = \int_{\gamma 1}^{\gamma} \frac{d\gamma}{\sqrt{r(h_2, \gamma)}}$$

and

$$T = \int_0^{\pi} \frac{d\gamma}{\sqrt{r(h_2, \gamma)}}.$$

Hence

$$A = \frac{1}{1 - exp(-T\sqrt{3})} \cdot$$

$$\cdot \int_{\gamma 1}^{\pi + \gamma 1} exp\left(-\sqrt{3}\int_{\gamma 1}^{\gamma} \frac{d\xi}{\sqrt{r(h_2, \gamma)}}\right) \frac{D(h_2, \gamma)}{\sqrt{r(h_2, \gamma)}} d\gamma ,$$

where

$$D_2(h_2, \gamma) = -e^2 sin\, 2\gamma \cdot$$
$$\cdot \left(\frac{\sqrt{1-e^2}\left(3 sin^2 \gamma + \sqrt{r(h_2, \gamma)}\right)}{2(1 - e^2 cos^2 \gamma)} + \sqrt{r(h_2, \gamma)} \right).$$

Conclusions

In this paper the space tethered system consisting of the dumbbell-shaped rigid body and the particle of small mass is considered. The particle moves along the cable with ends fixed in the body. The dumbbell rotations caused by the particle are studied. The sufficient condition of the dumbbell librations about its stable equilibrium is obtained. The family of asymptotic motions tending to librations about unstable equilibria is found. This family forms the interstream separating the area of the dumbbell rotations clockwise from the area of rotations counterclockwise. The equation of the interstream is deduced.

Acknowledgment

The author thanks V.V.Beletsky, Yu.F.Golubev, I.I.Kosenko and V.V.Sazonov for useful discussions.

References

[1] **Beletsky, V. V. and Novikova, E. T.** On the Relative Motion of Two Tethered Bodies on an Orbit. //Cosmic Research, v.7, No 3, pp.377-384. (1969).

[2] **Beletsky, V. V.** On the Relative Motion of Two Tethered Bodies on an Orbit II. //Cosmic Reseach ,v.7,No 6. (1969).

[3] **Rodnikov, A. V.** Equilibrium Positions of a Weight on a Cable Fixed to a Dumbbell-Shaped Space Station moving along a circular geocentric orbit.// Cosmic Reseach,v. 44,No 1, pp. 58-68. (2006).

[4] **Rodnikov A. V.** The algorithms for capture of the space garbage using 'leier constraint'.: Regular and Chaotic Dynamics, v.11, 4, pp. 483-489. (2006).