

On C-Reducibility in Cartan Space

その他（別言語等） のタイトル	カルタン空間におけるC-分解可能性について
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On C-Reducibility in Cartan Space

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We introduce the conception of C-reducibility in Cartan space, and show that a Cartan space endowed with (α, β) -metric is C-reducible, if and only if the metric is only of Randers' or Kropina's type. This is the main theorem in this paper. Moreover, we consider C-reducibility in Cartan space endowed with a generalized Randers metric which derived from a usual Cartan metric by β -change. We prove that the space with the generalized metric is C-reducible if the space with original metric is C-reducible, and that the converse is true.

Keywords : Cartan space, C-reducibility, (α, β) -metric

1. PRELIMINARIES

In this paper, we consider two remarkable classes of Cartan spaces, that is, a C-reducible type and a generalized Randers type.

Let M be a real n -dimensional manifold, (T^*M, π^*, M) the cotangent bundle of M and $H : U^* \rightarrow R$ a regular Hamiltonian (i.e. real smooth function on a domain $U^* \subset T^*M$ positively homogeneous of degree two in p). The pair of $\mathcal{H}^n = (M, H(x, p))$ is called a *Hamilton space* and the function $H(x, p)$ is *fundamental (metric) function* such that the nondegenerate matrix with the entries

$$g^{ij}(x, p) = \hat{\partial}^i \hat{\partial}^j H \tag{1}$$

is defined on U^* , hereafter in this paper, we denote $\hat{\partial}^i = \frac{\partial}{\partial p_i}$, while $\partial_i = \frac{\partial}{\partial x^i}$ and indices i, j, \dots run over $1, 2, \dots, n$. Of course, g^{ij} is component of a contravariant d -tensor field, named as *metric d -tensor* of \mathcal{H}^n , which is symmetric, positive definite and its reciprocal component $g_{ij}(x, p)$ is given by $g^{ij}g_{jk} = \delta_k^i$.

In this paper, the term *d -tensor field $T(x, p)$* of \mathcal{H}^n , for instance, of type $(1, 1)$, means a collection of n^2 functions $T_j^i(x, p)$ of variables x^i and p_i which obey the usual transformation law of components of tensor of M

such that

$$\bar{T}_b^a(\bar{x}, \bar{p}) = \frac{\partial \bar{x}^a}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^b} T_j^i(x, p)$$

under the coordinate transformation $x^i \rightarrow \bar{x}^a$.

We can construct the operators $\delta_i = \partial_i + N_{ji} \hat{\partial}^j$ which form a local basis of horizontal distribution N supplementary to the vertical distribution V of T^*M , i.e. $T_u(T^*M) = N_u \oplus V_u$, $u \in T^*M$.

We can obtain on a domain U of the tangent bundle (TM, π, M) of M , a regular Lagrangian $L(x, y)$ by *Legendre transformation*

$$L(x, y) = p_i y^i - H(x, p)$$

of H , where p_i is the solution of the system $y^i = \hat{\partial}^i H(x, p)$. $L(x, y)$ is positively homogeneous of degree two and behaves as the fundamental function of *Lagrange space* $\mathcal{L}^n = (M, L(x, y))$, which is seemed the dual of Hamilton space (ref. to [4]). More precisely, the geometric structure of the cotangent bundle T^*M or of Hamilton space is referred to our previous papers[1][2] or R. Miron's[6].

A *Cartan space* $\mathcal{C}^n = (M, F(x, p))$ is a special Hamilton space with $H = \frac{1}{2}F^2$, where F is called (*fundamental*) *metric function* of \mathcal{C}^n and positively homogeneous of degree one in $p = (p_i)$.

A *canonical nonlinear connection* in \mathcal{C}_n is given by

$$N_{ij} = \gamma_{ij}^o - \frac{1}{2} \gamma_{ro}^o \hat{\partial}^r g_{ij}$$

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where

$$\gamma_{ij}^o = \gamma_{ij}^h \rho_h, \quad \gamma_{r\circ}^o = \gamma_{rs}^o g^{sh} p_h$$

and

$$\gamma_{jk}^i = \frac{1}{2} g^{ih} (\partial_j g_{hk} + \partial_k g_{jh} - \partial_h g_{jk}).$$

A canonical d -connection $C\Gamma(N) = (H_{jk}^i, C_i^{jk})$ in C_n is defined by [7] as follows:

$$H_{jk}^i = \frac{1}{2} g^{ih} (\delta_j g^{hk} + \delta_k g_{jh} - \delta_h g_{jk}),$$

$$C_i^{jk} = -\frac{1}{2} g_{ih} (\partial^j g^{hk} + \partial^k g^{jh} - \partial^h g^{jk}).$$

By the differentiability and homogeneity of $F(x, p)$, we can easily show properties about the d -tensor $C^{ijk} = g^{ih} C_h^{jk}$ such that

$$C^{ijk} = -\frac{1}{2} \partial^i \partial^j \partial^k \left(\frac{1}{2} F^2 \right), \quad C^{\circ jk} = C^{ijk} p_i = 0. \quad (2)$$

The torsion tensor C^{ijk} is completely symmetric in indices, so we consider the case it splits in the form:

$$C^{ijk} = A^{ij} Q^k + A^{jk} Q^i + A^{ki} Q^j \quad (3)$$

for some symmetric tensor A^{ij} and vector Q^i . By reason of $C^{ijk} p_k = 0$, which follows from the homogeneity of fundamental function $F(x, p)$, we have

$$A^{i\circ} = 0, \quad Q^\circ = 0$$

for non-Riemannian ($C^{ijk} \neq 0$) Cartan space, where the index \circ means the result of contracting by p . Thus, we put the angular metric tensor

$$h^{ij} = F \partial^i \partial^j F = g^{ij} - l^i l^j, \quad l^i = \partial^i F \quad (4)$$

as most suitable one for A^{ij} , because this tensor is symmetric and has the properties such that

$$h^{ij} p_j = 0, \quad h^{ij} g_{ij} = n - 1, \quad i.e. \text{ rank}(h^{ij}) = 0. \quad (5)$$

Therefore, on account of (5), we can easily prove;

Proposition 1. If the tensor C^{ijk} satisfies the relation

$$C^{ijk} = h^{ij} Q^k + h^{jk} Q^i + h^{ki} Q^j,$$

then the vector Q^i has the form $Q^i = \frac{1}{n+1} C^i$, where $C^i = g_{jk} C^{ijk}$ is non-zero vector.

And we set;

Definition 1. A Cartan space of dimension $n > 2$ is called C -reducible, if the torsion tensor C^{ijk} is expressed in the form

$$C^{ijk} = \frac{1}{n+1} (h^{ij} C^k + h^{jk} C^i + h^{ki} C^j)$$

Cartan space of dimension two is always C -reducible, because there exists a scalar l such that $FC^{ijk} = l m^i m^j m^k$, where m^i is orthogonal to the unit vector l^i .

In the previous paper[3], we introduced an (α, β) metric in Cartan space, as analogous one in Finsler space (ref. to [5]). Its outline is as follows:

Definition 2. We call a Cartan space $C^n = (M, F(x, p))$ endowed with (α, β) -metric, if its metric function $F(x, p)$ is a function of $\alpha(x, p), \beta(x, p)$ only;

$$F(x, p) = \check{F}(\alpha(x, p), \beta(x, p)),$$

$$\text{with} \quad \alpha(x, p) = (a^{ij}(x) p_i p_j)^{1/2}, \quad \beta(x, p) = b^i(x) p_i,$$

where $a^{ij}(x)$ is a Riemannian metric on the base manifold M and $b^i(x)$ is a vector field on M such that $\beta \neq 0$ on a domain of $T^*M \setminus \{0\}$. The space (M, α) is called associated Riemannian space of C^n .

We need in this paper two special Cartan space with (α, β) -metric, that is as follows:

Definition 3. A metric function $F(x, p)$ of Cartan space is called of Randers' type if it is given by

$$F(x, p) = \sqrt{a^{ij}(x) p_i p_j} + b^i p_i = \alpha + \beta \quad (6)$$

and of Kropina's type if given by

$$F(x, p) = \frac{a^{ij}(x) p_i p_j}{b^i p_i} = \frac{\alpha^2}{\beta}. \quad (7)$$

These metrics were introduced by R. Miron[7] into Cartan space analogously to the case of Finsler space (ref. to [5]). Clearly, the function $\check{F}(\alpha, \beta)$ in Definition 2 should satisfy the conditions imposed to the function $F(x, p)$ as a fundamental one for C^n , so it is positively homogeneous of degree one in α and β . By this reason, there would be no confusion if we adopt the notation $F(\alpha, \beta)$ instead of $\check{F}(\alpha, \beta)$. Putting $H = F^2/2$, for convenience, we remark the following homogeneities of $H(x, p)$:

$$\left. \begin{aligned} \alpha H_\alpha + \beta H_\beta &= 2H \\ \alpha H_{\alpha\alpha} + \beta H_{\alpha\beta} &= H_\alpha, \quad \alpha H_{\alpha\beta} + \beta H_{\beta\beta} = H_\beta \\ \alpha^2 H_{\alpha\alpha} + 2\alpha\beta H_{\alpha\beta} + \beta^2 H_{\beta\beta} &= 2H \\ \alpha H_{\alpha\alpha\alpha} + \beta H_{\beta\alpha\alpha} &= \alpha H_{\alpha\alpha\beta} + \beta H_{\alpha\beta\beta} \\ &= \alpha H_{\alpha\beta} + \beta H_{\beta\beta} = 0 \end{aligned} \right\} \quad (8)$$

where the subscripts α, β of H mean its partial derivatives with respect to them. In the previous paper[3], we obtained the concrete expression for the metric tensor $g^{ij}(x, p)$ and its reciprocal component g_{ij} in Cartan space with (α, β) metric such that

$$g^{ij} = \rho a^{ij} + \rho_0 b^i b^j + \rho_1 (b^i P^j + b^j P^i) + \rho_2 P^i P^j$$

$$g_{ij} = \sigma a_{ij} - \sigma_0 B_i B_j + \sigma_1 (B_i p_j + B_j p_i) + \sigma_2 p_i p_j$$

and also for the tensor C^{ijk} ,

$$C^{ijk} = -\frac{1}{2} [r_1 b^i b^j b^k + \mathfrak{S}_{ijk} \{ \rho_1 a^{ij} b^k + \rho_2 a^{ij} P^k + r_2 b^i b^j P^k + r_3 b^i P^j P^k \} + r_4 P^i P^j P^k].$$

In the above three expressions, the notation

$$\left. \begin{aligned} P^i(x, p) &= a^{ij}(x) p_j, \quad B_i(x) = a^{ij}(x) b^j(x) \\ B^2(x) &= a_{ij} b^i b^j = a^{ij} B_i B_j \end{aligned} \right\} \quad (9)$$

are used and the coefficients ρ 's, σ 's and r 's are certain functions of partial derivatives of $H(\alpha, \beta)$, at most, of third order, and the operator \mathfrak{S}_{ijk} plays the rôle of abbreviation of completely symmetric summation with respect to the indices i, j, k for each term in the brackets, for example,

$$A^{ij}Q^k + A^{jk}Q^i + A^{ki}Q^j = \mathfrak{S}_{ijk}\{A^{ij}Q^k\} \quad (10)$$

in place of the right hand of (3).

In the next chapter, however, we use more direct calculation without above formulas for g^{ij} , g_{ij} and C^{ijk} .

2. MAIN THEOREM

Partial differentiation of $\alpha(x, p)$ and $\beta(x, p)$ yields

$$\hat{\partial}^i \alpha = \alpha^{-1} P^i, \quad \hat{\partial}^j \beta = b^j(x).$$

The vector $P^i(x, p)$ in (9) satisfies the relation

$$P^i p_i = \alpha^2, \quad \hat{\partial}^j P^i = a^{ij}(x), \quad \hat{\partial}^k \hat{\partial}^j P^i = 0.$$

$$\hat{\partial}^j \hat{\partial}^i \alpha = \alpha^{-1} (a^{ij}(x) - \alpha^{-2} P^i P^j).$$

By means of the quantities

$$k^{ij} = a^{ij} - \alpha^{-1} P^i P^j,$$

the partial derivative of alpha of third order is expressed such that

$$\hat{\partial}^j \hat{\partial}^i \hat{\partial}^k \alpha = -\alpha^{-3} \mathfrak{S}_{ijk}\{k^{ij} P^k\}. \quad (11)$$

Direct differentiation of $H(x, p)$ gives expressions for the metric tensor such that

$$\begin{aligned} g^{ij} &= \hat{\partial}^j \hat{\partial}^i H \\ &= H_{\alpha\alpha\alpha}(\hat{\partial}^i \alpha)(\hat{\partial}^j \alpha) + H_{\alpha\beta}\{(\hat{\partial}^i \alpha)b^j + (\hat{\partial}^j \alpha)b^i\} \\ &\quad + H_{\beta\beta}b^i b^j + H_{\alpha}\{\hat{\partial}^i \hat{\partial}^j \alpha\} \\ &= \alpha^{-1} H_{\alpha} k^{ij} + H_{\beta\beta} b^i b^j + \alpha^{-1} H_{\alpha\beta} (b^i P^j + b^j P^i) \\ &\quad + \alpha^{-2} H_{\alpha\alpha} P^i P^j \end{aligned} \quad (12)$$

using (11) and for the torsion tensor C^{ijk} such that

$$\begin{aligned} -2C^{ijk} &= \hat{\partial}^k g^{ij} \\ &= \mathfrak{S}_{ijk}\{(\alpha^{-2} H_{\alpha\alpha} - \alpha^{-3} H_{\alpha})k^{ij} P^k + \alpha^{-1} H_{\alpha\beta} k^{ij} b^k \\ &\quad + \alpha^{-1} H_{\beta\beta} P^i b^j b^k + \alpha^{-2} H_{\alpha\beta} P^i P^j b^k\} \\ &\quad + \alpha^{-3} H_{\alpha\alpha\alpha} P^i P^j P^k + H_{\beta\beta\beta} b^i b^j b^k. \end{aligned}$$

Substituting the new quantity $Q^i = b^i - \alpha^{-2} \beta P^i$ and using the homogeneity (8) for the above expression, we have a very simple form;

$$-2C^{ijk} = \alpha^{-1} H_{\alpha\beta} \mathfrak{S}_{ijk}\{k^{ij} Q^k\} + H_{\beta\beta\beta} Q^i Q^j Q^k. \quad (13)$$

If we put here

$$\alpha^{-1} H_{\alpha\beta} k^{ij} + 3H_{\beta\beta\beta} Q^i Q^j = -2A^{ij},$$

then (13) yields the equation

$$C^{ijk} = A^{ij}Q^k + A^{jk}Q^i + A^{ki}Q^j$$

which is nothing but (3), hence we get

Proposition 2. In the Cartan space with (α, β) -metric, the torsion tensor C^{ijk} splits in the form (3).

On the other hand, if we use the unit vector

$$l^i = \hat{\partial}^i F = (\sqrt{2H})^{-1} (\alpha^{-1} H_{\alpha} P^i + H_{\beta} b^i) \quad (14)$$

of C_n , the angular metric tensor in (4) is rewritten as

$$h^{ij} = \alpha^{-1} H_{\alpha} k^{ij} + (H_{\beta\beta} - (2H)^{-1} H_{\beta}^2) Q^i Q^j, \quad (15)$$

where we also substituted the above quantity Q^i and using the homogeneity (8). Therefore, by means of

$$k^{ij} = \alpha^{-1} H_{\alpha} \{h^{ij} - (H_{\beta\beta} - (2H)^{-1} H_{\beta}^2) Q^i Q^j\}$$

which follows from (15), the expression (11) is changed to the final form;

$$\begin{aligned} -2C^{ijk} &= \frac{H_{\alpha\beta}}{H_{\alpha}} \mathfrak{S}_{ijk}\{h^{ij} Q^k\} + \{H_{\beta\beta\beta} - \frac{3H_{\alpha\beta}}{H_{\alpha}} \times \\ &\quad (H_{\beta\beta} - \frac{1}{2H} H_{\beta}^2)\} Q^i Q^j Q^k. \end{aligned} \quad (16)$$

Picking up the last term, we put

$$Q = H_{\beta\beta\beta} - \frac{3H_{\alpha\beta}}{H_{\alpha}} (H_{\beta\beta} - \frac{1}{2H} H_{\beta}^2). \quad (17)$$

We are interested in the case $Q = 0$, that is,

$$C^{ijk} = \mathfrak{S}_{ijk}\{h^{ij} A^k\}$$

with $A^k p_k = 0$. Contracting g_{jk} to the bothside of the above equation, we have

$$C^i = g_{jk} C^{ijk} = (n+1)A^i, \quad i.e., \quad A^i = \frac{1}{n+1} C^i$$

by means of (5), which implies

$$C^{ijk} = \frac{1}{n+1} \mathfrak{S}_{ijk}\{h^{ij} C^k\}.$$

On account of Proposition 2, we can conclude

Proposition 3. When the quantity Q in (17) vanishes the space is C-reducible.

Obviously, for the cases of Randers' and Kropina's metric presented in last section the quantity Q vanishes, because, for the former $H = (\alpha + \beta)^2/2$, $H_{\alpha} = H_{\beta} = \alpha + \beta$, $H_{\alpha\beta} = 0$, $H_{\beta\beta} = 1$, $H_{\beta\beta\beta} = 0$, and for the later metric $H = \alpha^4/(2\beta^2)$, $H_{\alpha} = 2\alpha^3\beta^{-2}$, $H_{\beta} = -\alpha^4\beta^{-3}$, $H_{\alpha\beta} = -4\alpha^3\beta^{-3}$, $H_{\beta\beta} = 3\alpha^4\beta^{-4}$, $H_{\beta\beta\beta} = -12\alpha^4\beta^{-5}$, hence in both cases $Q = 0$ holds.

Proposition 4. The Cartan spaces with Randers' and Kropina's type metric are both C-reducible.

Let us obtain the necessary condition for Q to vanish. In order to replace H and its partial derivatives in the quantity Q by F and its ones, we use the relations

$$H_{\alpha\beta} = FF_{\alpha\beta} + F_{\alpha}F_{\beta}, \quad H_{\beta\beta} = FF_{\beta\beta} + F_{\beta}^2$$

$$H_{\beta\beta\beta} = -\alpha^3\beta^{-3}H_{\alpha\alpha\alpha} = -\alpha^3\beta^{-3}(FF_{\alpha\alpha\alpha} + 3F_{\alpha}F_{\alpha\alpha})$$

and the homogeneity about $F(x, p)$ such that

$$\alpha F_{\alpha} + \beta F_{\beta} = F$$

$$\alpha F_{\alpha\alpha} + \beta F_{\alpha\beta} = \alpha F_{\alpha\beta} + \beta F_{\beta\beta} = 0.$$

Therefore, (17) is rewritten as

$$Q = -\frac{\alpha^3}{\beta^3} F \left(\frac{3F_{\alpha\alpha}}{\alpha} + F_{\alpha\alpha\alpha} - 3\frac{F_{\alpha\alpha}^2}{F_{\alpha}} \right).$$

We start from $Q = 0$, then case

(i) $F_{\alpha\alpha} = 0$ reduces to the metric of Randers' type, because $F_\alpha = c_1$ and $F = c_1 + \phi(\beta) = c_1\alpha + c_2\beta$, by the homogeneity of $\phi(\beta)$ of degree one in β . The case

(ii) $F_{\alpha\alpha} \neq 0$ follows a differential equation such that

$$\frac{3}{\alpha} + \frac{F_{\alpha\alpha\alpha}}{F_{\alpha\alpha}} - \frac{3F_{\alpha\alpha}}{F_\alpha} = 0.$$

Paying attention to logarithmic differentiation, we obtain at first,

$$\frac{\alpha^3 F_{\alpha\alpha}}{F_\alpha^3} = e^{\psi(\beta)}, \quad \alpha^3 F_{\alpha\alpha} = k\beta^2 F_\alpha^3,$$

where the coefficient $e^{\psi(\beta)} = c\beta^2$, $c = \text{const.} \neq 0$ is caused by that the left side of the first expression is homogeneous of degree two in p , then we have

$$\frac{F_{\alpha\alpha}}{F_\alpha^3} = \frac{c\beta^2}{\alpha^3}$$

Integration this by α yields

$$\frac{1}{F_\alpha^2} = \frac{c\beta^2 + c_1\alpha^2}{\alpha^2}, \quad F_\alpha^2 = \frac{\alpha^2}{c\beta^2 + c_1\alpha^2}, \quad c_1 = \text{const.}$$

where we used the homogeneity of F again. Hence, the following two cases occur.

(iia) If $c_1 \neq 0$, then

$$F_\alpha = \pm \frac{\alpha}{\sqrt{c_1\alpha^2 + c\beta^2}}, \quad F = \pm \frac{1}{c_1} \sqrt{c_1\alpha^2 + c\beta^2} + c_0\beta,$$

($c_0 = \text{const.}$), F is rewritten as

$$\begin{aligned} F(x, p) &= \pm \sqrt{\left(\frac{1}{c_1}a^{ij} + \frac{c}{c_1^2}b^i b^j\right) p_i p_j + (c_0 b^k) p_k} \\ &= \bar{\alpha} + \bar{\beta} \end{aligned}$$

which is the metric of Randers' type.

(iib) If $c_1 = 0$, then

$$F_\alpha^2 = \frac{\alpha^2}{k\beta}, \quad F_\alpha = \pm \frac{1}{k} \frac{\alpha}{\beta}, \quad F = \pm \frac{1}{2k} \frac{\alpha^2}{\beta} + k_1\beta$$

($k, k_1 = \text{const.}$). F is rewritten as

$$F(x, p) = \frac{(\frac{1}{2k}a^{ij} + k_1 b^i b^j) p_i p_j}{\pm b^k p_k} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$$

that is, the metric of Kropina's type is obtained.

Summarizing the consideration in this chapter, we conclude

Theorem 1. *The Cartan space with (α, β) -metric is C-reducible if and only if the metric is of Randers' or Kropina's type.*

3. C-REDUCIBILITY IN THE GENERALIZED SPACE

In this chapter, we need to generalize the metric of Randers' type as follows:

Definition 4. For a given Cartan metric $F(x, p)$ in C^n , a metric function of $\tilde{F}(x, p)$ of the Cartan space \tilde{C}^n is called of *generalized Randers' type* if it has the form such that

$$\tilde{F}(x, p) = F(x, p) + \beta(x, p), \quad F^2 = g^{ij}(x, p) p_i p_j.$$

It should be remarked here that $F(x, p)$ is not Riemannian metric as α in Chapter 2 and $\beta(x, p)$ is the same one. We call the original space C^n as *associated Cartan space of \tilde{C}^n* and the deformation of the metric $F \rightarrow \tilde{F}$ as β -change of metric.

Actually, $\tilde{F}(x, p)$ given above is homogeneous of degree one in p . We distinguish the quantities in this section by attaching notation " \sim " to the top of ones of \tilde{C}^n , if there are the same ones appear in the associated space C^n . We take the quantities such as

$$\begin{aligned} g^{ij} &= \partial^i \partial^j (F^2/2) = h^{ij} + l^i l^j, \quad l^i = \partial^i F \quad (\text{in } \tilde{C}^n) \\ \tilde{g}^{ij} &= \partial^i \partial^j (\tilde{F}^2/2) = \tilde{h}^{ij} + \tilde{l}^i \tilde{l}^j \quad (\text{in } C^n) \\ \tilde{l}^i &= \partial^i \tilde{F} = l^i + b^i, \quad \partial^i \tilde{l}^i = \partial^i l^i = F^{-1} h^{ij}. \end{aligned} \quad (18)$$

And denoting

$$\tau = F^{-1} \tilde{F} = F^{-1}(F + \beta) = 1 + F^{-1}\beta, \quad (19)$$

$$\tilde{h}^{ij} = \tau h^{ij}, \quad \partial^i \tau = F^{-1}(b^i - F^{-1}\beta l^i), \quad (\partial^i \tau) p_i = 0 \quad (20)$$

are obtained. And for the metric tensor \tilde{g}^{ij} in (18) is given by two manners such that of \tilde{C}^n ,

$$\tilde{g}^{ij} = \tau g^{ij} + b^i b^j + b^i l^j + b^j l^i - F^{-1}\beta l^i l^j \quad (21)$$

$$= \tau g^{ij} + \tilde{l}^i \tilde{l}^j - \tau l^i l^j = \tau h^{ij} + \tilde{l}^i \tilde{l}^j. \quad (22)$$

On account of the useful Proposition 30.1 of Matsumoto[5], we can obtain the reciprocal component \tilde{g}_{ij} of \tilde{g}^{ij} in (22) such that

$$\tilde{g}_{ij} = \frac{1}{\tau} \{g_{ij} - B_i \tilde{l}_j - B_j \tilde{l}_i + (B^2 + \beta F^{-1}) \tilde{l}^i \tilde{l}^j\} \quad (23)$$

where g_{ij} is the reciprocal component of g^{ij} in C^n of (18) and notations

$$\tilde{l}_i = \tilde{F}^{-1} p_i, \quad B_i = g_{ij} b^j, \quad B^2 = g_{ij} b^i b^j = g^{ij} B^i B^j$$

are used instead of (9). In fact, we can verify $\tilde{g}^{ij} \tilde{g}_{jk} = \delta_k^i$ using the relations

$$\tilde{l}^i l_i = \tau, \quad \tilde{l}_i l^i = 0, \quad \tilde{l}_i b^i = \beta \tilde{F}^{-1},$$

$$\tilde{l}^i B_i = B^2 + \beta F^{-1}, \quad b_i l^i = B_i l^i = \beta F^{-1}.$$

We are concerned with the torsion tensor \tilde{C}^{ijk} (resp. C^{ijk}) in the space \tilde{C}^n (resp. C^n). Differentiating the both side of (22), as the followings;

$$\begin{aligned} \partial^k \tilde{g}^{ij} &= -2\tilde{C}^{ijk} \\ &= F^{-1}(b^k - \beta F^{-1} l^k) g^{ij} - 2\tau C^{ijk} + b^i F^{-1} h^{jk} \\ &\quad + b^j F^{-1} h^{ik} + F^{-2} l^k \beta l^i l^j - F^{-1} b^k l^i l^j \\ &\quad - F^{-2} \beta h^{ik} l^j - F^{-2} \beta l^i h^{jk} \\ &= -2\tau C^{ijk} + \mathfrak{S}_{ijk} \{h^{ij} F^{-1}(b^k - \beta F^{-1} l^k)\}, \end{aligned}$$

we obtain the relation between \tilde{C}^{ijk} and C^{ijk} such that

$$\tilde{C}^{ijk} = \tau C^{ijk} + \mathfrak{S}_{ijk} \{h^{ij} \gamma^k\} \quad (24)$$

$$\text{with} \quad \gamma^k = -\frac{1}{2}(b^k - \beta F^{-1} l^k), \quad \gamma^0 = \gamma^k p_k = 0$$

where quantities (18) and (19) are used.

At last of this chapter, we consider C-reducibility in this generalized space. Transvecting the expression (24) by (23), we have

$$\tilde{C}^i = \tilde{g}^{jk} \tilde{C}^{ijk} = g^{jk} C^{ijk} + \frac{n+1}{\tau} \gamma^i = C^i + \frac{n+1}{\tau} \gamma^i$$

because of (5) and $h^{ik} \tilde{l}_k = \tilde{F}^{-1} h^{i0} = 0$.

Furthermore, we know $\tilde{C}^i p_i = 0$ as well as $C_i p_i = 0$.

In the other words, if there exists a vector \tilde{C}^k in \tilde{C}^n such that $C^{ijk} = \mathfrak{S}_{ijk} \{h^{ij} \tilde{C}^k\} / (n+1)$, then there also exists a vector

$$\tilde{C}^k = C^k - \frac{n+1}{2\tau} (b^k - \beta F^{-1} l^k)$$

in \tilde{C}^n such that $\tilde{C}^{ijk} = \mathfrak{S}_{ijk} \{h^{ij} \tilde{C}^k\} / (n+1)$, and conversely. Hence, we conclude

Theorem 2. Let $n > 2$. Cartan space \tilde{C}^n with the generalized Randers' metric is C-reducible if and only if its associated Cartan space C^n is C-reducible.

References

- [1] Igarashi, T. : Lie derivative of d -object fields in the cotangent bundle, Tensor,N.S., Vol.52,(1992), p255-267.
- [2] Igarashi, T. : Remarkable connections in Hamilton spaces. Tensor,N.S., Vol.51,(1991), p156-161.
- [3] Igarashi, T. : (α, β) metric in Cartan spaces. Tensor,N.S., Vol.55,(1994), p74-83.
- [4] Kern, J.: Lagrange geometry, Arch. Math., Vol.25,(1974), p438-443.
- [5] Matsumoto, M. : Foundations of Finsler geometry and special Finsler spaces. Kaiseisha Press. ISBN 4-106165-11-7 (1986).
- [6] Miron, R. : Hamilton geometry, An. Ştiinţ. Univ. Al. I. Cuza, Iaşi,Ser.Nouă, Mat., Vol.35,(1989), p33-67.
- [7] Miron, R. : The geometry of Cartan spaces, Progress of Math., Vol.22,(1988), p1-38.

カルタン空間におけるC-分解可能性について

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概要

カルタン空間に「C-分解可能性」という概念を導入し、 (α, β) -計量を与えられたカルタン空間がC-分解可能であるのは、その計量がランダース型かクロピナ型るときであり、かつ、そのときに限る」ことを明らかにする。これが本論文の主定理である。さらに、通常のカルタン空間の計量から導出された一般ランダース計量をもつ空間を考える。そして、一般化された計量をもつ空間がC-分解可能なのは最初の空間がC-分解可能なときであり、また、逆も成り立つことを示す。

キーワード：カルタン空間，C-分解可能性， (α, β) -計量

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