Numerical Estimation on Correlation Inequalities for Holley-Liggett Bounds

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Numerical Estimation on Correlation Inequalities for Holley-Liggett Bounds

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Holley and Liggett gave good bounds on critical value and survival probability for the basic contact process in one dimension by a suitable renewal measure. On the other hand, the same bounds can be obtained by assuming a class of correlation inequalities holds. In this paper we give numerical evidence for validity of some of the above correlation inequalities.

Keywords: Correlation Inequality, Contact Process, Critical Value, Holley-Liggett Method

1 INTRODUCTION

In this paper we consider correlation inequalities which give upper bound on the critical value and lower bound on the survival probability of the basic contact process in one dimension. These bounds were obtained by Holley and Liggett[(1)] using a different method based on the Harris lemma.[(2)] Concerning the Harris lemma, see Chapter 3 in Konno,[(3)] for example. In the present paper we call their method the *Holley-Liggett method*.

First we introduce some notations. The dynamics of the basic contact process in one dimension is as follows. For any $x \in \xi$ with $\xi \subset \mathbf{Z}^1$,

$$\begin{aligned} \xi &\to \xi \cup \{x-1\} && \text{at rate } \lambda, \\ \xi &\to \xi \cup \{x+1\} && \text{at rate } \lambda, \\ \xi &\to \xi \setminus \{x\} && \text{at rate } 1. \end{aligned}$$

Let ξ_t^0 be the basic contact process starting from the origin. We define the *survival probability* for this process by

$$\rho_\lambda = P\{\xi_t^0 \neq \phi \quad \text{for any } t \geq 0\}.$$

The *critical value* on the basic contact process can be defined by

$$\lambda_c = \inf\{\lambda \geq 0 : \rho_\lambda > 0\}.$$

The best estimated value of this critical value is 1.648912 given by Jensen and Dickmann.[(4)] Concerning upper bound on λ_c and lower bound on ρ_λ, Holley and Liggett[(1)] gave the following good bounds by choosing a suitable renewal measure:

$$\lambda_c \leq \lambda_c^{(HL)} = 2,$$

and

$$\rho_\lambda \geq \rho_\lambda^{(HL)} = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{2\lambda}} \quad \text{for } \lambda \geq 2.$$

We call these bounds *Holley-Liggett bounds* from now on. On the other hand, if we assume the following correlation inequalities hold: for $m, n \geq 1$, $J(0,0)J(m,n) \leq J(m,0)J(n,0)$, then we can obtain the same Holley-Liggett bounds, where $J(m,n)$ is the probability of having 1 at m and 0's at all other sites in $[1, m+n+1]$ with

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respect to ν_λ which is the invarinat measure of the basic contact process starting from the state $\mathbf{Z}^1$. If $\circ$ and $\bullet$ represent 0 and 1 respectively, then we can write

$$J(m,n) = \nu_\lambda(\overbrace{\circ\cdots\circ}^{m}\bullet\overbrace{\circ\cdots\circ}^{n}).$$

For example,

$$\begin{aligned} J(0,0) &= \nu_\lambda\{\eta : \eta(1) = 1\} = \nu_\lambda(\bullet) = \rho_\lambda, \\ J(1,2) &= \nu_\lambda\{\eta : \eta(1) = 0, \eta(2) = 1, \\ &\quad \eta(3) = \eta(4) = 0\} = \nu_\lambda(\circ\bullet\circ\circ). \end{aligned}$$

The above mentioned approach is different from the Holley-Liggett method. The problem is that it is not known whether the above correlation inequalities hold or not for any $m, n \geq 1$. At the present stage, the only known thing is that there are $m, n \geq 1$ such that $J(0,0)J(m,n) \leq J(m,0)J(n,0)$, as shown by Liggett.[5] So the main purpose of this paper is to check whether $J(0,0)J(m,n) \leq J(m,0)J(n,0)$ holds for small $m, n \geq 1$ by Monte Carlo simulation. In fact, even when $m = n = 1$, this correlation inequality is interesting. The reason is as follows. From the Harris-FKG inequality, we have

$$\nu_\lambda(\bullet\bullet) \geq \nu_\lambda(\bullet)^2, \tag{1.1}$$

and

$$\nu_\lambda(\circ\circ) \geq \nu_\lambda(\circ)^2. \tag{1.2}$$

Concerning the Harris-FKG inequality, see Chapter II in Liggett,[6] or page 21 in Konno.[3] We can rewrite Eqs.(1.1) and (1.2) by using the conditional probability:

$$\nu_\lambda(\bullet|\bullet) \geq \nu_\lambda(\bullet), \tag{1.3}$$

and

$$\nu_\lambda(\circ|\circ) \geq \nu_\lambda(\circ), \tag{1.4}$$

where

$$\nu_\lambda(\bullet|\bullet) = \nu_\lambda\{\eta : \eta(1) = 1|\eta(0) = 1\},$$

$$\nu_\lambda(\circ|\circ) = \nu_\lambda\{\eta : \eta(1) = 0|\eta(0) = 0\}.$$

Moreover, in our setting, the following correlation inequalities were proved by Belitsky, Ferrari, Konno and Liggett recently:[7] for any $A, B \subset \mathbf{Z}^1$,

$$\overline{\rho}_\lambda(A\cap B)\overline{\rho}_\lambda(A\cup B) \geq \overline{\rho}_\lambda(A)\overline{\rho}_\lambda(B), \tag{1.5}$$

where $\overline{\rho}_\lambda(A) = \nu_\lambda\{\eta(x) = 0 \text{ for any } x \in A\}$. In particular, if we take $A = \{-1, 0\}$ and $B = \{0, 1\}$, then we have

$$\nu_\lambda(\circ)\nu_\lambda(\circ\circ\circ) \geq \nu_\lambda(\circ\circ)^2, \tag{1.6}$$

so this becomes

$$\nu_\lambda(\circ|\circ\circ) \geq \nu_\lambda(\circ|\circ), \tag{1.7}$$

where

$$\nu_\lambda(\circ|\circ\circ) = \nu_\lambda\{\eta : \eta(1) = 0|\eta(0) = \eta(-1) = 0\}.$$

On the other hand, when $m = n = 1$ for the correlation inequalities we consider in this paper, we have

$$\nu_\lambda(\bullet)\nu_\lambda(\bullet\bullet\bullet) \leq \nu_\lambda(\bullet\bullet)^2, \tag{1.8}$$

that is,

$$\nu_\lambda(\bullet|\bullet\bullet) \leq \nu_\lambda(\bullet|\bullet), \tag{1.9}$$

where

$$\nu_\lambda(\bullet|\bullet\bullet) = \nu_\lambda\{\eta : \eta(1) = 1|\eta(0) = \eta(-1) = 1\}.$$

The interesting thing is that a direction of inequality (1.9) is different from those of inequalities (1.3), (1.4) and (1.7). From the attractiveness (see page 72 in Liggett[6]), we can easily expect that inequalities (1.3) and (1.4) hold, moreover, inequality (1.7) also holds. However, concerning inequality (1.9), we can not easily conclude which direction of inequality is correct. Our estimation by Monte Carlo simulation suggests that inequality (1.9) holds. This is one of the interesting conclusions of this paper.

This paper is organized as follows. In Section 2, we briefly review the results of the Holley-Liggett method. In Section 3, we show how we can derive Holley-Liggett bounds from correlation inequalities. Section 4 treats numerical evidences on some of correlation inequalities. Moreover we give results on oriented percolation. Section 5 is devoted to conclusions.

2 HOLLEY-LIGGETT METHOD

First we introduce the following renewal measure μ on $\{0,1\}^{\mathbf{Z}}$ with density f is given by

$$\mu(\bullet\overbrace{\circ\cdots\circ}^{n_1}\bullet\overbrace{\circ\cdots\circ}^{n_2}\bullet\circ\ldots\circ\bullet\overbrace{\circ\cdots\circ}^{n_k}\bullet) = \frac{f(n_1+1)f(n_2+1)\cdots f(n_k+1)}{\sum_{m=1}^{\infty} mf(m)},$$

where $\mu(n_1, n_2, \ldots, n_k)$ be the probability of having 1's at $n_1+1, n_1+n_2+2, \ldots, n_1+n_2+\cdots+n_{k-1}+k-1$ and 0's at all other sites in $[1, n_1+n_2+\cdots+n_k+k-1]$ with respect to the renewal measure μ, for $k \geq 2$ and $n_i \geq 0$ $(i = 1, \ldots, k)$. Let Y the collection of all finite subsets of $\mathbf{Z}^1$ and for any $A \in Y$,

$$\sigma_\lambda(A) = P\{\xi_t^A \neq \phi \quad \text{for any } t \geq 0\},$$

where ξ_t^A be the basic contact process starting from A. By using the Harris lemma, the following theorem is obtained, see Chapter VI of Liggett,[6] for example.

Theorem 2.1. *Let $\lambda_c^{(HL)} = 2$. Then for $\lambda \geq \lambda_c^{(HL)}$,*

$$h_\lambda^{(HL)}(A) \leq \sigma_\lambda(A) \qquad \text{for all } A \in Y,$$

where

$$h_\lambda^{(HL)}(A) = \mu\{\eta : \eta(x) = 1 \text{ for some } x \in A\},$$

for a renewal measure μ on $\{0,1\}^{\mathbf{Z}}$ whose density f is given by $\Omega^* h_\lambda^{(HL)}(A) = 0$ for all A of the form $\{1, 2, \ldots, n\}$ $(n \geq 1)$, and Ω^* is the generator of dual process for the basic contact process in one dimension.

Note that the above f is given as follows: for $n \geq 1$,

$$f(n) = F(n) - F(n+1), \qquad \text{and}$$

$$F(n) = \frac{(2(n-1))!}{(n-1)!n!(2\lambda)^{n-1}}.$$

Applying Theorem 2.1 to $A = \{1\}$ gives Holley-Liggett bounds:

Corollary 2.2.

$$\lambda_c \le \lambda_c^{(HL)} = 2,$$

$$\rho_\lambda \ge \rho_\lambda^{(HL)} = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{2\lambda}} \quad (\lambda \ge 2).$$

3 CORRELATION INEQUALITIES

In this section, we review how we can derive Holley-Liggett bounds from the correlation inequalities. A similar argument appeared in Chapter 4 of Konno.[3] Compared with the Holley-Liggett method, we can easily obtain Holley-Liggett bounds by assuming these correlation identities hold.

Here we introduce the following correlation functions: for $m, n \ge 0$, we let $J(m,n)$ be the probability of having 1's at m and 0's at all other sites in $[1, m+n+1]$ with respect to ν_λ which is an invarinat measure starting from all sites are covered by 1's. That is,

$$J(m,n) = \nu_\lambda(\overbrace{\circ\cdots\circ}^{m}\bullet\overbrace{\circ\cdots\circ}^{n}).$$

From the definition of the basic contact process in one dimension, we have the following correlation identities:

$$\lambda\left[\overline{\rho}_\lambda(\{0,1,\ldots,n\}) - \overline{\rho}_\lambda(\{1,\ldots,n\})\right] +$$
$$\lambda\left[\overline{\rho}_\lambda(\{1,\ldots,n,n+1\}) - \overline{\rho}_\lambda(\{1,\ldots,n\})\right] +$$
$$\sum_{k=1}^{n}\left[\overline{\rho}_\lambda(\{1,\ldots,n\}\setminus\{k\}) - \overline{\rho}_\lambda(\{1,\ldots,n\})\right] = 0,$$

where $\overline{\rho}_\lambda(A) = \nu_\lambda\{\eta(x) = 0 \text{ for any } x \in A\}$. So the definition of $J(m,n)$ gives

Lemma 3.1. *For $n \ge 1$,*

$$2\lambda J(n,0) = \sum_{k=1}^{n} J(k-1, n-k),$$

that is,

$$2\lambda\nu_\lambda(\overbrace{\circ\cdots\circ}^{n}\bullet) = \sum_{k=1}^{n}\nu_\lambda(\overbrace{\circ\cdots\circ}^{k-1}\bullet\overbrace{\circ\cdots\circ}^{n-k}).$$

Next we introduce the following conjecture to get Holley-Liggett bounds. This was conjectured by Konno[3] (see Conjecture 4.5.2).

Conjecture 3.2. *For $m, n \ge 1$,*

$$J(0,0)J(m,n) \le J(m,0)J(n,0),$$

that is,

$$\nu_\lambda(\bullet)\nu_\lambda(\overbrace{\circ\cdots\circ}^{m}\bullet\overbrace{\circ\cdots\circ}^{n}) \le \nu_\lambda(\overbrace{\circ\cdots\circ}^{m}\bullet)\nu_\lambda(\overbrace{\circ\cdots\circ}^{n}\bullet).$$

When m or n is equal to 0, this inequality becomes equality. Define

$$\overline{J}(m,n) = \frac{\nu_\lambda(\overbrace{\circ\cdots\circ}^{m}\bullet\overbrace{\circ\cdots\circ}^{n})}{\nu_\lambda(\bullet)} = \frac{J(m,n)}{J(0,0)},$$

for $\lambda > \lambda_c$ and $m, n \ge 0$. Note that the definition of λ_c gives $J(0,0) = \nu_\lambda(\bullet) = \rho_\lambda > 0$ for $\lambda > \lambda_c$. By using $\overline{J}(m,n)$, we can rewrite the above conjecture as follows.

Conjecture 3.3. *For $\lambda > \lambda_c$ and $m, n \ge 1$,*

$$\overline{J}(m,n) \le \overline{J}(m,0)\overline{J}(n,0).$$

Here we introduce the generating function of the sequence $\overline{J}(n,0)$:

$$\varphi(u) = \sum_{n=0}^{\infty}\overline{J}(n,0)u^{n+1}.$$

By using this, Lemma 3.1 can be rewritten as

$$2\lambda\overline{J}(n,0) = \sum_{k=1}^{n}\overline{J}(k-1,n-k). \tag{3.1}$$

From the definition of $\varphi(u)$, we see that

$$\sum_{n=1}^{\infty}\left[\overline{J}(n,0)u^{n+1}\right] = \varphi(u) - u. \tag{3.2}$$

Assume Conjecture 3.3. So we have

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}\overline{J}(k-1,n-k)u^{n+1} \le$$
$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}\overline{J}(k-1,0)\overline{J}(n-k,0)u^{n+1}. \tag{3.3}$$

The definition of $\varphi(u)$ and Eq.(3.3) give

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}\overline{J}(k-1,n-k)u^{n+1} \le \varphi(u)^2. \tag{3.4}$$

By Eqs.(3.1), (3.2) and (3.4), we have

$$\varphi^2(u) - 2\lambda\varphi(u) + 2\lambda u \ge 0. \tag{3.5}$$

Note that if $\lambda > \lambda_c$, then

$$\varphi(1) = \sum_{n=0}^{\infty}\overline{J}(n,0) = \frac{1}{J(0,0)} = \frac{1}{\rho_\lambda}. \tag{3.6}$$

For $\lambda > \lambda_c$, the second equality follows from

$$\sum_{n=0}^{\infty}\nu_\lambda(\overbrace{\circ\cdots\circ}^{n}\bullet) =$$
$$\sum_{n=0}^{\infty}\left\{\nu_\lambda(\overbrace{\circ\cdots\circ}^{n}) - \nu_\lambda(\overbrace{\circ\cdots\circ}^{n+1})\right\} =$$
$$1 - \lim_{n\to\infty}\nu_\lambda(\overbrace{\circ\cdots\circ}^{n}) = 1.$$

From Eq.(3.5) with $u = 1$ and Eq.(3.6), we have

$$2\lambda\rho_\lambda^2 - 2\lambda\rho_\lambda + 1 \ge 0. \tag{3.7}$$

So the properties of continuity and monotonicity for ρ_λ imply

$$\rho_\lambda \ge \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{2\lambda}} \quad \text{for } \lambda \ge 2. \tag{3.8}$$

We should remark that the continuity comes from Theorem 1.6 in Chapter VI of Liggett[6] and the result by Bezuidenhout and Grimmett.[8] Concerning the monotonicity, it can be easily obtained, for example, see page 265 in Liggett.[(6)] The lower bound on ρ_λ in Eq.(3.8) is nothing but the Holley-Liggett one on the survival probability. Moreover, Eq.(3.8) gives Holley-Liggett bound on the critical value:

$$\lambda_c \le \lambda_c^{(HL)} = 2.$$

Therefore from the argument in this section we can obtain the following theorem:

Theorem 3.4. *Assume that for any $m, n \ge 1$,*

$$\nu_\lambda(\bullet)\nu_\lambda(\overbrace{\circ\cdots\circ}^{m}\bullet\overbrace{\circ\cdots\circ}^{n}) \le \nu_\lambda(\overbrace{\circ\cdots\circ}^{m}\bullet)\nu_\lambda(\overbrace{\circ\cdots\circ}^{n}\bullet).$$

Then we have

$$\lambda_c \le \lambda_c^{(HL)} = 2,$$

$$\rho_\lambda \ge \rho_\lambda^{(HL)} = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{2\lambda}} \quad (\lambda \ge 2).$$

4 NUMERICAL RESULTS

The simulations were conducted as follows: particles, occupying sites of a lattice in one dimension, correspond to a distribution of 1's (particle) and 0's (empty site) on $\mathbf{Z}^1$. At each step we take a particle, and either try to create another particle on a randomly chosen nearest neighboring site (with probability $p = \lambda/(\lambda+1)$, succeeds if the neighboring site is empty) or remove the particle leaving an empty site (probability $1 - p = 1/(\lambda+1)$).

The system size L was 10000 sites, with the initial configuration taken as all sites occupied. 4000*L steps were taken, the numbers of 01 (n_{01}), 10 (n_{10}), 010 (n_{010}) and 1 (n_1, corresponds to the number of occupied sites) have been summed over the last 2000*L steps. We checked that for all p values considered after the initial 2000*L there are no systematic changes in the concentration of particles, indicating that the system is in a stationary state.

The results are plotted in 2 ways: as $r_{010} \equiv (n_{01} * n_{10})/(n_{010} * n_1)$ versus p (Fig.1), and as $d_{010} \equiv (n_{01} * n_{10})/(n_1 * n_1) - n_{010}/n_1$ versus p (Fig.2).

Taking n_{01}/L as an approximation of $J(1,0) = \nu_\lambda(\circ\bullet)$, and making similar assumptions for n_1,n_{01} and n_{10} ane can write the statement of Conjecture 3.2 with $m = n = 1$ as

$$n_{010} * n_1 \le n_{01} * n_{10},$$

which implies that $r_{010} \ge 1$ and $d_{010} \ge 0$.

When interpreting the graphs note, that because the density of empty sites is close to 0 for big enough p, the numbers of configurations involving 1's are low and the relative error is high, so that plots for r look rather bumpy as p approaches 1. Plots for d are much smoother, but hide some information, since for big enough p d approaches zero much faster than r.

Other combinations were treated in a similar way (Fig.1,2). No violations of the inequalities were detected, apart from the cases when non-compliance can clearly be attributed to statistical error. For obvious symmetry considerations, configurations 0100 and 0010 should lead to the same r and p values, deviations in the graphs reflect the computational error involved.

Similar calculations were performed for the oriented site percolation, with p as the occupation probability. For oriented percolation the system size was 10000, 10000 steps were taken, with statistics summed over the last 5000 steps (Fig.3,4). For oriented percolation, similar conclusions as for basic contact process can be drawn, although at the moment we are unable to provide a theoretical explanation.

Note that for the basic contact the threshold point is at $p = 0.767324$ (Ref.[(4)]) while for oriented site percolation it is at $p = 0.705489$ (Ref.[(9)]), which defined the range of the p values used in the simulation.

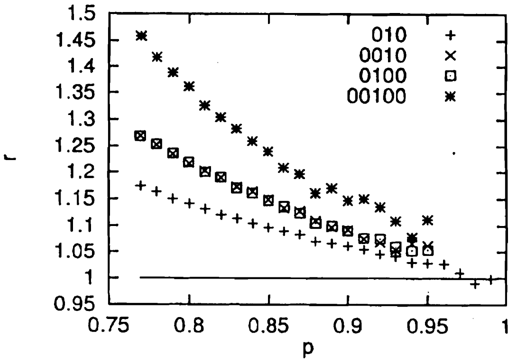

Fig 1. Basic contact process, r versus p. For 0100, 0010 and 00100 only points with $p \le 0.95$ are shown, for bigger values of p the error is too big, so that data looses all meaning.

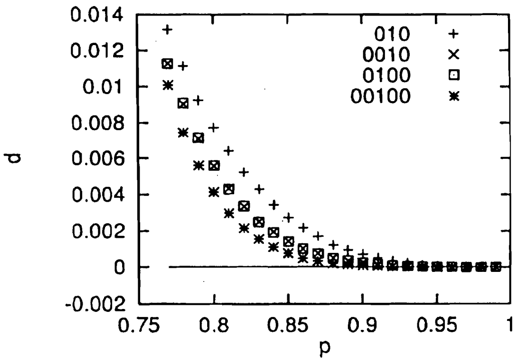

Fig 2. Basic contact process, d versus p.

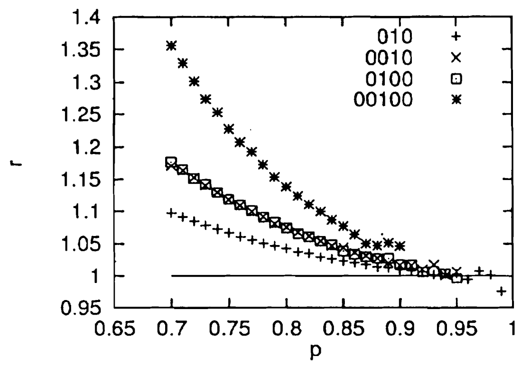

Fig 3. Oriented site percolation, r versus p. As in Fig.1 and Fig.3, for 0100, 0010 and 00100, points too close to $p = 1$ have a huge statistical error and have been ommitted.

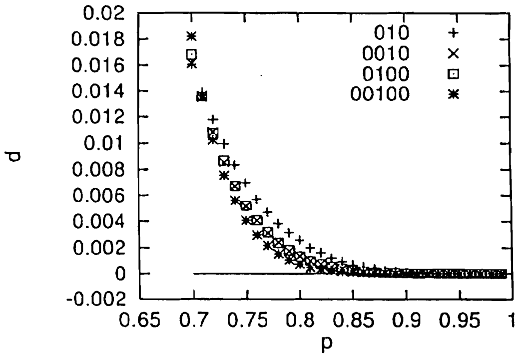

Fig 4. Oriented site percolation, d versus p.

5 CONCLUSIONS

In this paper we consider the correlation inequalities (Conjecture 3.2) which give the same bounds on critical value and survival probability for the one-dimensional basic contact process, as those derived by the Holley-Liggett method. Our results by Monte Carlo simulations suggest that Conjecture 3.2 holds for $(m,n) = (1,1),(1,2),(2,1),(2,2)$. Moreover a similar conclusion is obtained in the case of oriented site percolation. The next stage could be to give a proof of Conjecture 3.2 for any $m,n \geq 1$.

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ホリー・リゲット法に対応する相関不等式の数値的評価

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概要

ホリーとリゲットは、更新測度を用いることにより、1次元コンタクトプロセスの臨界値の上限と生存確率の下限を求めることに成功した。一方、同じ臨界値の上限と生存確率の下限を相関不等式のあるクラスを仮定することによって、ホリーとリゲットの証明に比べ、比較的容易に求められることが知られている。本論文では、上記相関不等式のクラスの代表的な幾つかの相関不等式に対し、モンテカルロ・シミュレーションによりその正当性を示した。

キーワード： 相関不等式、コンタクトプロセス、臨界値、ホリー・リゲット法

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