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Hirofumi HIRAYAMA, Shintaro KIKUCHI

Summary

We have established a mathematical model of arterial system. This paper expand theoretical analysis of the mechanical dynamical structure of the arterial wall. The general deformation theory of dynamical analysis was applied to establish the balancing equations of the forces and moments that operate on the arterial wall surface. To generalyze the dynamical problem, we brought the shell theory of the curved surface into the analysis of the arterial wall surface. To associate and identify the directions of the forces and moments before and after the deformation, we firstly analyzed the relative rotation angles between each lines of the micro surface elements around the 3 axies which were founded on the elements. Utilyzing these parameters of the relative rotation, movements, we induced the balancing equations of the forces. Since we Assume more general case, we also studied the balancing equations of bending, twisting moments and transverse shear. Then we have obtained 6 equilibrium equation in 3 directions. This paper is one of the vital points of the mathematical expansion of our theory.

Ⅱ The constructive dynamic analysis of the arterial wall Ⅱ − 1 The rotation angle of the arterial wall and the equilibrium equations for the stress and moments operating on the wall.

Introduction

The fluid dynamical interactions which develope between the blood pressure, flow and pulsatile changes which occur within the arterial system are controled macroscopically by the cardiac ejection and the geometric and mechanical properties of the arterial system. It is not sufficient to appreciate the pulsatile transmission phenomenone of blood as a simple conduction of change of the biological properties of the system. Rather those phenomenons should be accepted as one of the form of information transformation for maintaining the life activity.

Thus to understand the pulsatile transmission of the blood flow is the first step for recognizing the cardiovascular circulation.

In the first series of a modeling of the cardiovascular system, we have established a distributed parameter model of the human arterial system. In the previous paper, we had induced blood flow

Key words : Dynamical analysis – Rotation angle – Forces – Moments.

velocities in the longitudinal and radial direction based on the Womersleys elastic tube theory of arterial system. [1] To obtain the transmission line equation for distributed parameter model, one should constitute not only the fluid dynamical equations but also the structural dynamical equations of the arterial wall and its motion equations. Before precede to the arterial wall motion equations, we must analyze the dynamical equilibrium problems of the stress and moments. Traditionally many researchers analyzed the arterial wall stress or deformations based upon the assumptions that the distribution of the stress, moments and the deformations were axisymmetric. However in the actual arterial wall such as the femoral artery, the wall thickness has certain value and the effects of change of ratio of the wall thickness to the radius would cause developments of the transverse shear and moment. As a result the forces and moments naturally operate on the wall unevenly and cause non – axisymmetric deformation. So no longer the axisymmetric analysis can be applicable. Therefore it is needed to analyze the dynamical equilibrium problem of more general case as the non uniform distribution of the forces and moments on the arterial wall.

In this paper as the second chapter of the mathematical expansion, based on the strict dynamical theory [2, 3, 4, 5], which all assumed that the displacements are small and the stress – strain relation is liner, we analyzed firstly the relative rotation angle of the surface element of the arterial wall, then induced the equilibrium equations of the forces and moments for general non – axisymmetric case.

MATHEMATICAL EXPANSION II

The non axisymmetric loading problem of the arterial wall can be reduced to the mechanical equilibrium problem of the cylindrical shell receiving arbitrary distributed stress. (here shell means that the all wall are shaped to curved surface.). Therefore we expand general case of the stress distributions.

I] Formation of the coordinate system on the surface of the arterial wall.

Because of the extenstion of the middle surface in three dimentions, we should construct two coordinates to represent the position of the focused point. About the coordinates following definitions are made.

The generator : a straigt line along a curve while maintaining it parallel to its original direction.

The profiles : all planes which are normal to the generators. It is this profiles that designate the shape of a section of the deformed arterial wall.

The generators and profiles constitute sets of coordinates lines. We choose an arbitrary profile as the datum line and from this, measure the coordinate x along the generators.

The angle ϕ : the angle which a tangental plane to the cylinder makes with the holizontal plane.

We cut off from it an element bounded by two pairs of the adjucent neighbooring generators, and by two adjucent profiles x, x + dx. The four sides of element of the cut sectioned surface are depicted in Fig 1.

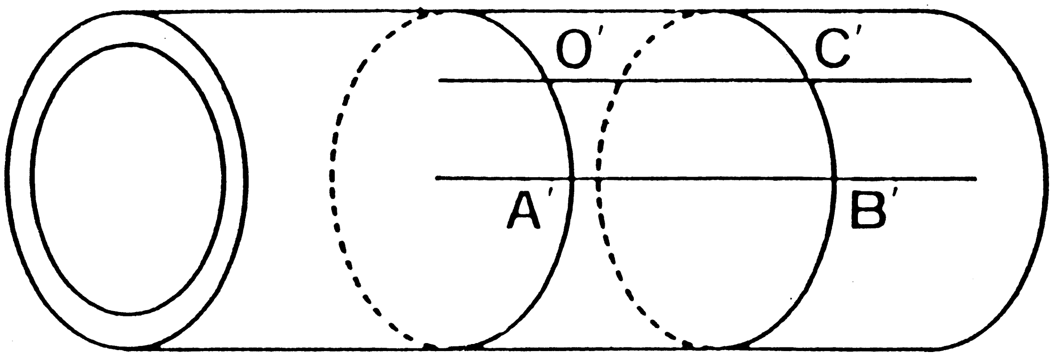

Fig 1 Arterial tube

Fig. 1 The schematic illustration of the cross section of the arterial segment. The line O' C', A' B' are generators. The curve O' A', C' B' are the profiles (the circumferential direction). The arterial tube is assumed to have the constant radius for a given compartment.
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The displacements of the original point due to the deformation are defined as followings

u : the displacement along the axis of the cylindricai tube (parallel circule displacement)

v : the displacement along a circule of the radius a + z (meridian circumferential displacement)

w : the displacement along the normal line (the radial displacement)

Those displacements are expressed utilizing the matrixs as (u, v, w) on the (x, y, z) coordinate on the arterial wall surface.

The element we consider is revealed in Fig 1 as area ABCO and A' B' C' O' which is the deformed surface of ABCO. The pattern of the deformed surface is arbitrary pictured in Fig 2.

II] The relative rotation angle.

Since the surface element which we concern is very small and can be treated as a plane, we can construct the x, y, z axies (the Gausian coordinates) on it.As a result taking the arbitrary point O on the middle surface as the original point, the longitudinal axies (parallel with the generator) can be made to coincide with the x axis, the tangent of the circumferential profile with y axis and the normal with the z axis.

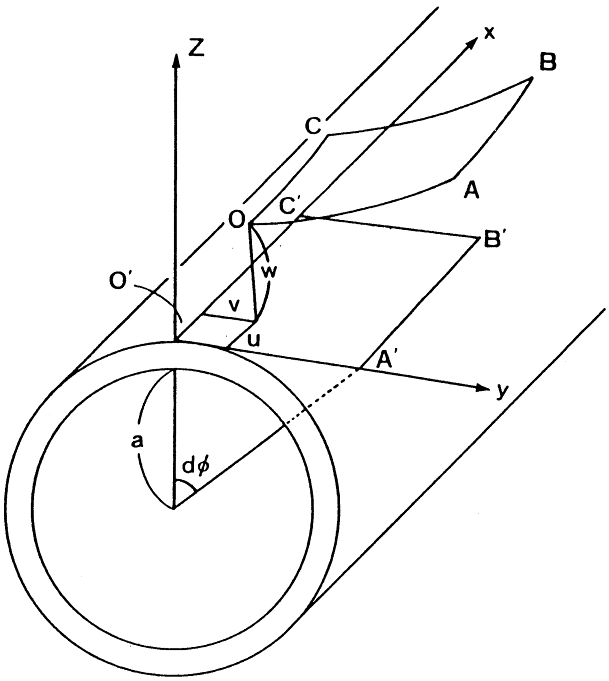

Fig 2 Micro-Surface of the artery

Fig. 2 The element of the arterial wall surface as a shell. Strictly the area O' C' B' A' on the surface have a curvature, but this surface is assumed to be too micro. So one can regard this area as a plane. The OCBA is the curved surface after the deformation had developed. The x (the longitudinal direction) coinsides with the tangent of the generator, the y axis (the circumferential direction) with the profile, the z axis (the radial direction) with the normal. (U, V, W) are the displacements of the original point O' in the longitudinal, circumferential and radial direction respectively.
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However after the deformation had been developed, these x, y, z axies would deviate from what previously established (the coordinate before the deformation has developed). Fortunately the change of the axies could be restricted to only one axis within the x, y, z axies. Since the z axis can be permitted to coincide with the Normal on the middle surface even after the deformation, only the x axis should be treated as the changed axis. Here as an ordinally way, the deformed x axis can be set as the longitudinal tangent line to the previous non deformed x axis (prallel with the generator). Then automatically the y axis after the deformation is setteled as vertical line to the x − z plane. Naturally the newly constructed y axis is different from the old, nondeformed y axis. So each line on the element deviate each other and make the rotative movements around each other. Due to these mentioned circumstances, no simple equilibrium balancing equations hold and one should analyze these rotation angle before establishing the forces and moments equilibrium equations.

1. The relative rotation angle between the side CB to the side OA.

This rotation is dissolved into 3 components which are the longitudinal displacement along the x axis (tangential to the parallel circule), circumferential displacement along the y axis (meridian to the shell), normal displacement along the z axis.

A. The relative rotation angle around the x axis.

This rotation is induced by the displacement v and w.

1. The v produces circumferential revolution of side OA and CB. The side OA rotates v / a around the x axis ($v : 2\pi a = \Delta\phi : 2\cdot\pi$, then $\Delta\phi = v / a$).

It seems that side CB also revolve parallely with OA around the xaxis. But in this general case, we consider about the non axisymmetric deformation. So the rotation is not completely identical in strict sense. During the distance of dx (between the side OA and CB), the material (arterial wall) exactly deforms. Therefore the rotation angle of OA (v / a) changes at the ratio of $\alpha(v/a) / \alpha x \cdots$ per unit length of the section along the x axis, accordingly at the CB which is distanced dx the side CB rotate $\alpha(v/a) / \alpha x \cdot dx$ surplus. As a result the rotation angle of CB around the x axis by displacement v is $v / a + \frac{\alpha(v/a)}{\alpha x}\cdot dx$.

Therefore the relative rotation angle between OA and CB around the x axis is

$$\frac{\alpha(v/a)}{\alpha x}dx$$

2. The w makes the side OA and CB rotate normally. The rotation of OA around x axis by w is w / a ($w : 2\pi a = \Delta\xi : 2\pi$.). Further more the micro central angle $d\phi$ also participates this rotation. As a result the rotation of OA around the x axis by w per unit central angle is $\alpha\cdot(w/a) / \alpha\phi$. The side CB distance the side OA by dx. Then the $\alpha\cdot(w/a) / \alpha\phi$ change $\frac{\alpha[\alpha(w/a)]}{\alpha x\ \alpha\phi}$ per unit lentgth of the x axis.

Therefore CB revolutes $\alpha[\alpha(w/a) / \alpha\phi] / \alpha x\cdot dx$ more than OA. Consequently the relative rotation angle between the OA and CB around the x axis produced by w and v is given as

$$\frac{\alpha(v/a)}{\alpha x}dx + \frac{\alpha^2(w/a)}{\alpha x \alpha\phi}dx \tag{1}$$

B. The relative rotation angle around the y axis.

Now we consider the orthogonal projection of side OC on the y axial plane (the plane which is verical to y axis.) Fig 3. The displacement ratio of w at O along the x axis is $\alpha w / \alpha x$. Since during dx, this displacement ratio changes $\frac{\alpha (\alpha w / \alpha x)}{\alpha x}$ per unit length of the x axis, so at C the displacement ratio is added by $\alpha (\alpha w / \alpha x^{*}$ dx. Therefore at C, true displacement ratio is $\alpha w / \alpha x + \frac{\alpha^2 w}{\alpha x^2} \cdot dx$. Accordingly the relative rotation angle is

$$-\frac{\alpha^2 w}{\alpha x^2} dx \qquad (2)$$

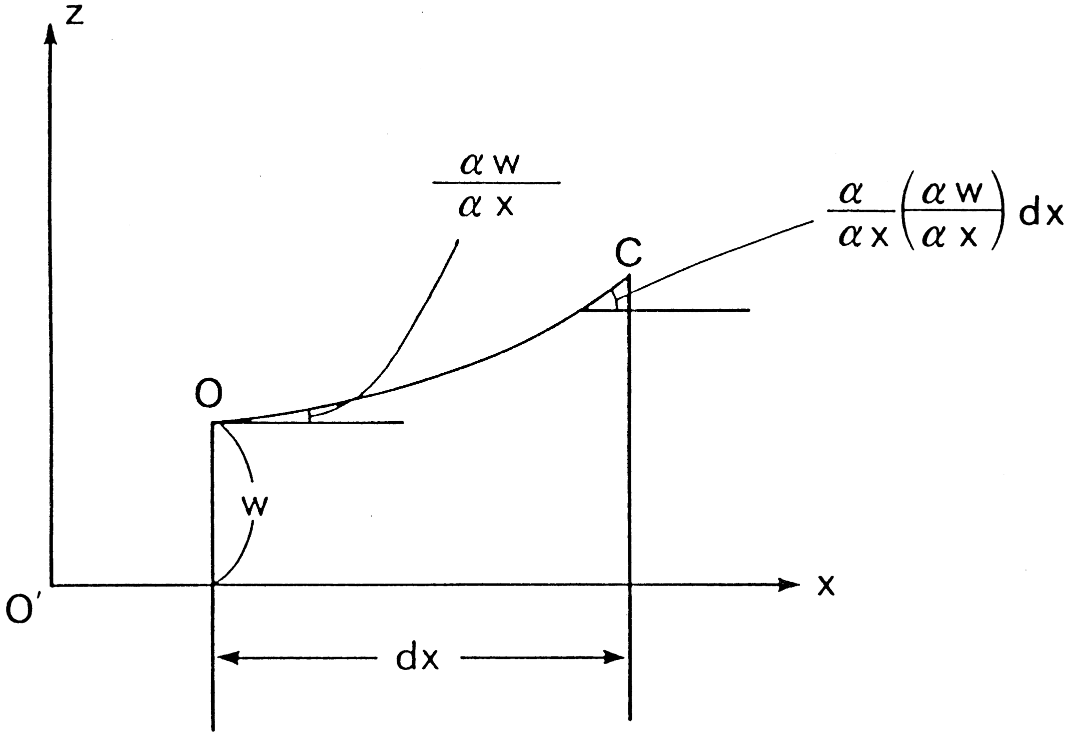

Fig 3 Rotation angle I

Fig. 3 This illustrate the orthogonal projection of the deformed line element OC on the x − z plane (the y axial plane). The strain $\alpha w / \alpha x$ changes along the x axis. At the position distanced for dx, the change of the strain is $\alpha^2 w / \alpha x^2$ dx.

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C. The relative rotation angle around the z axis. Fig 4.

Simmilary as B, we set the orthogonal projection of side OC on the x − y plane which is vertical to the z axis, the displacement ratio of v at point O is $\alpha v / \alpha x$. The same consideration about B brings us to the conclution of $\alpha v / \alpha x + \frac{\alpha^2 v}{\alpha x^2} \cdot dx$. Therefore the relative rotation angle around the z axis is

$$\frac{\alpha^2 v}{\alpha x^2} dx$$

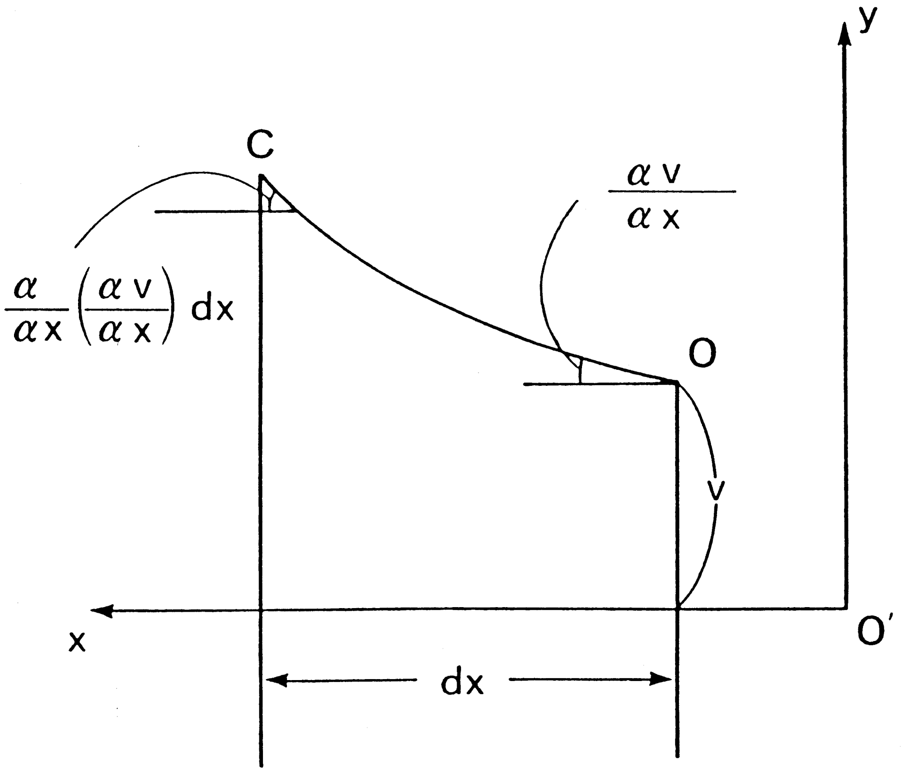

Fig4 Rotation angle II

Fig. 4 This figure illustrate the orthogonal projection of the line element OC on the x − y plane (the z axial plane). The circumferential strain $\alpha u / \alpha x$ changes in the longitudinal direction at the rate of $\alpha^2 u / \alpha x^2$.
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2. The relative rotation angle between the side OC and side AB.

The central angle composed by these two element is $d\phi$. But the displacements v and w are inevitably modified by $d\phi$.

A. The relative rotation angle around the x axis is the vector difference of following two factors.

1. The revolution of side OC around the x axis is produced by v and w. The contribution from v is v / a. The contribution from w is w / a. This angle is modified by the central angle $d\phi$ into α

$(w / a) / \alpha \phi$. Therefore the rotation angle is obtained as

$$\frac{v}{a} + \frac{\alpha (w / a)}{\alpha \phi}$$

2. The rotation of AB is easily established considering that the side AB is distanced at central angle $d\phi$ between side OC. Therefore the AB rotates $\frac{\alpha [v / a + \alpha (w / a) / \alpha \phi]}{d \phi} d\phi$ more than side OC. Accordingly the rotation angle of side AB around x axis is

$$\frac{v}{a} + \frac{\alpha (w / a)}{\alpha \phi} + \frac{\alpha [v / a + \alpha (w / a) / \alpha \phi]}{\alpha \phi} d\phi$$

Consequently the relative rotation between AB and OC around the x axis is sum of 1 and 2, then

$$d\phi + \frac{[\alpha (v / a + \alpha w / a) / \alpha \phi]}{\alpha \phi} d\phi \qquad (4)$$

B. The relative rotation angle around the y axis.

1. The rotation of side OC produced by w around the y axis at the original point O is $- \alpha w / \alpha x$. (the right rotation positive). The side AB is modified by central angle $d\phi$ more. Therefore AB rotates $\frac{\alpha (- \alpha w / \alpha x)}{\alpha \phi} \cdot d\phi$ more. As a result the rotation around the y axis of AB is

$$- \frac{\alpha w}{\alpha x} + \frac{\alpha (- \alpha w / \alpha x)}{\alpha \phi} d\phi$$

Consequently the relative rotation angle between O'A and OC by w around y axis is

$$- \frac{\alpha^2 w}{\alpha \phi \ \alpha x} d\phi$$

2. The rotation produced by v around the y axis.

At the original point O, the rotation angle $- \alpha v / \alpha x$ is modified by the degree of central angle $d\phi$. Consequently the rotation angle is $- \alpha v / \alpha x \cdot d\phi$. As with respect to the point A, the rotation angle $(- \alpha v / \alpha x \cdot d\phi)$ changes $\frac{\alpha (- \alpha v / \alpha x \cdot d\phi)}{\alpha \phi}$ per unit central angle. So totally the point A rotates $\alpha (- \alpha v / \alpha x \cdot d\phi) / \alpha \phi$ more. Neglecting the higher order of $(d\phi)^2$s relative rotation angle between OC and AB around the y axis is

$$- \frac{\alpha^2 w}{\alpha \phi \ \alpha x} d\phi - \frac{\alpha v}{\alpha x} d\phi \qquad (5)$$

C. The relative rotation angle around the z axis.

1. The w produces the rotation of the side AB $- \alpha w / \alpha x$ on the $x - y$ plane which is orthogonal to the z axis at the point O. Because of central angle $d\phi$, the side AB rotates truely $- \alpha w / \alpha x \cdot d\phi$ around the z axis.

2. The OC rotates around the z axis $\alpha v / \alpha x$ by displacement v. The side AB rotates $\frac{\alpha^{*}(\alpha v / \alpha x)}{\alpha \phi} d\phi$ more because of the central angle $d\phi$ Consequently the rotation around the z axis of the side AB is

$$\frac{\alpha v}{\alpha x} + \frac{\alpha (\alpha v / \alpha x)}{\alpha \phi} d\phi$$

Accordingly the relative rotation angle between AB and OC produced by displacement v around z is

$$\frac{\alpha^{*}(\alpha v / \alpha x)}{\alpha \phi} d\phi$$

The associated relative rotation angle between OC and AB is composed from the contribution of v and w independently and these results are expressed by the vector sum as

$$\frac{\alpha^{2} v}{\alpha \phi \ \alpha x} d\phi - \frac{\alpha w}{\alpha x} d\phi \qquad (6)$$

Ⅲ] The equilibrium equations of the forces operating the elements of the arteridlwall.

In this section we induce the equilibrium equations of resultant forces which operate on the sides of element based on the strucural dynamical theory of S. Timoshenko. We define the stress as followings

Tji : the i axis component of the stress T which operates on the surface that cross at right angle with j axis..

Nx : The normal force in a section x = const, the force in direction x transmitted by a unit length of section. It is positive if tensile.

N ϕ : The normal force in a section y = const, the force in direction y (circunferential) transmitted by a unit length of section. It is negative if compressive.

N ϕ x : The shearing force in a section x = const, the force transmitted by a unit length of this section and directed tangent to dy. It is positive if it points in the direction of increasing y on the same side of the shell element where a tensile force Nx point in direction of increasing x.

Nx ϕ : The shearing force in a section y = const, the force transmitted by a unit length of this section and directed tangent to dx.

Qx : The transverese force in a section x = const, the force normal to the middle surface transmitted by a unit length of each side.

Q ϕ : The transverese force in a section y = const, the force normal to the middle surface transmitted by a unit length of each side.

The forces which operate on the 4 edges (side line) should all lie in the tangential planes to the middle surface. The load per unit area of the shell element is composed of 3 forces Px, Py, Pz in direction of increasing x, y, z (outward) respectively. Fig 5.

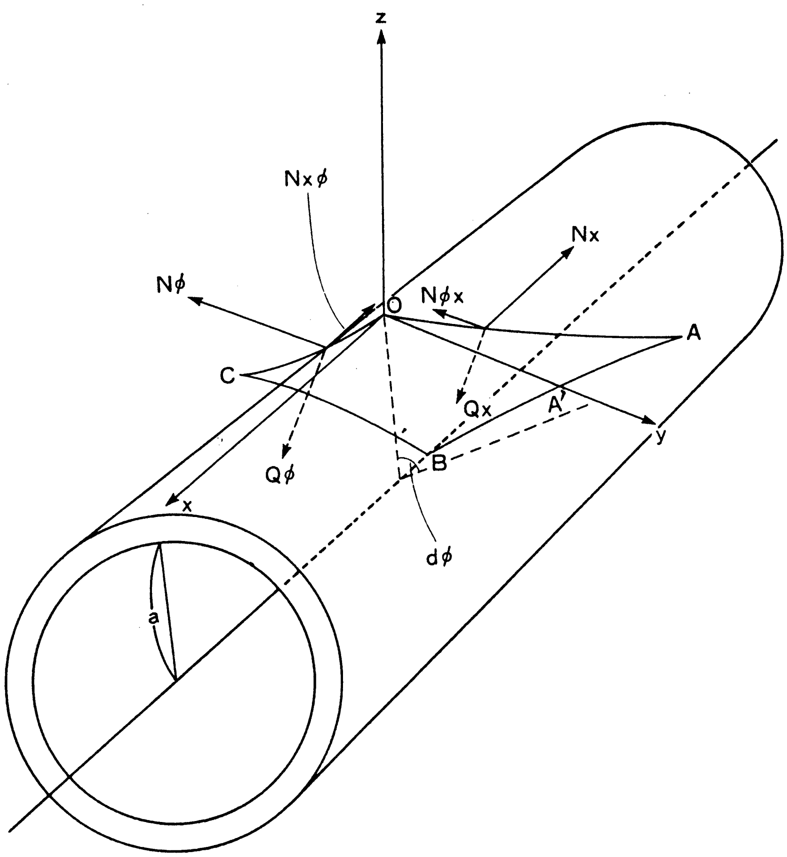

Fig 5 Forces on the micro surface

Fig. 5 This figure describes the distribution pattern of the orthogonal forces Nx, N ϕ, the shearing forces Nx ϕ, N ϕ x and the transverse forces Qx, Q ϕ on just deformed micro surface element OABC.

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I] The equilibrium equation for the longitudinal direction

1. Nx : Because OA is extremely small, Nx can be treated as constant on the length of OA ($= a^{\cdot} d\phi$). However on the CB which distance at dx from OA, the stress distributes differently from that on the CB. The Nx alters $\alpha Nx / \alpha x$ per unit length of the x axis. Therefore on CB, Nx is added by $\alpha (Nx / \alpha x)^{\cdot} dx$. As a result on CB force

$(Nx + \alpha Nx / \alpha x^{\cdot} dx)^{\cdot} a^{\cdot} d\phi$.

Consequently the net force along the x axis of Nx is

$$\frac{\alpha Nx}{\alpha x} dx\, a\, d\phi$$

2. $N\phi x$: The x component of the stress $N\phi$. Since the OC is very small, the $N\phi x$ is constant along the dx. But on the AB which distances for circumferentially $a^{\cdot} d\phi$ from OC, the $N\phi x$ changes $\alpha N\phi x / \alpha\phi$ per unit angle.

Therefore on AB, the $N\phi x$ suffers additional $\alpha N\phi x / \alpha\phi^{\cdot} d\phi$, then actually $(N\phi x + \frac{\alpha N\phi x}{\alpha\phi}{}^{\cdot} d\phi)^{\cdot} dx$ acts on the AB. As a result the net effect of $N\phi x$ on the longitudinal direction is

$$\frac{\alpha N\phi x}{\alpha\phi} dx\, d\phi$$

Seemingly the stress which contributes to the x axis resultant are onlyNx and $N\phi x$.

Nevertheless the forces $N\phi$, $Nx\phi$, Qx, $Q\phi$ also produce certain effects on the x axis direction because of the relative rotation around the axies between the each sides of the surface element. By multiplying the relative rotation angles around the z axis, we can deduce the effect of contributions of $Nx\phi$, $N\phi$ to the x axis stress component and multiplying the relative rotation angle around the y axis, we also obtain the contributions from Qx, $Q\phi$ on the longitudinal direction.

3. $N\phi$: which operates on OC should be multiplied by the relative rotation angle around the z axis between OC and AB

$(\alpha^2 v / \alpha\phi\ \alpha x - \alpha w / \alpha x)^{\cdot} d\phi$

In this case, this component operates constantly along the OC ($=$ dx). Therefore the contribution is

$$-N\phi \left(\frac{\alpha^2 v}{\alpha\phi\ \alpha x} - \frac{\alpha w}{\alpha x}\right) dx\, d\phi$$

4. $Nx\phi$: which acts on the OA should be multiplied by the relative rotation angle around the z axis between OA and CB $\alpha^2 v / \alpha^2 x^{\cdot} dx$. The $Nx\phi$ can be regarded as constant on OA ($= a^{\cdot} d\phi$). Then the contribution to the x axis is

$$- Nx\phi \frac{\alpha^2 v}{\alpha x^2} dx\, a\, d\phi$$

5. $Q\phi$ on OC must be multiplied by the relative rotation angle around the y axis between OC and AB

$- (\alpha^w / \alpha\phi\ \alpha x + \alpha v / \alpha x)^* d\phi$,

therefore the participation of $Q\phi$ to the x axis is

$$Q\phi x(-1)\left(\frac{\alpha^2 w}{\alpha\phi\ \alpha x} + \frac{\alpha v}{\alpha x}\right) d\phi\, dx$$

6. Qx on OA must be multiplied by the relative rotation angle around the y axis between OA and CB $(- \alpha^2 w / \alpha x^{2*} dx)$. The participation of Qx to the x axis is

$$Qx \frac{-\alpha^2 w}{\alpha x^2} dx\, a\, d\phi$$

Summing up these contribution and balancing the load Tx along the x axis direction, one should obtain the following equilibrium equation.

$$\frac{\alpha Nx}{\alpha x} dx\, a\, d\phi + \frac{\alpha N\phi x}{\alpha\phi} d\phi\, dx + (-N\phi)^* \left(\frac{\alpha^2 v}{\alpha\phi\ \alpha x} - \frac{\alpha w}{\alpha x}\right) dx\, d\phi + (-Q\phi)^*$$

$$\left(\frac{\alpha^2 w}{\alpha\phi\ \alpha x} + \frac{\alpha v}{\alpha x}\right) dx$$

$$^* d\phi - N\phi x \frac{\alpha^2 v}{\alpha x^2} dx\, a\, d\phi + Qx\left(-\frac{\alpha^2 w}{\alpha x^2}\right) dx$$

$$a d\phi = Tx^* d\phi^* a^* dx \tag{7}$$

II] The equilibrium equation for the circumferential direction.

1. $N\phi$: which operates on the OC changes at the rate of $\alpha N\phi / \alpha\phi$ during the transition from OC to AB traveling circumferentially around the central angle $d\phi$. Then the force on AB is given by

$$(N\phi + \alpha N\phi / \alpha\phi^* d\phi)^* dx.$$

The not effect of $N\phi$ on AB is

$$\frac{\alpha N\phi}{\alpha\phi} d\phi\, dx$$

2. $Nx\phi$: which operates on the OA changes at the rate of $\alpha Nx\phi / \alpha x$ per unit length of the x axis. Therefore it acts on CB as

$$(Nx\phi + \alpha Nx\phi / \alpha x^{\cdot} dx)^{\cdot} a^{\cdot} d\phi$$

As a result the net contribution of the $Nx\phi$ on circumferential direction is

$$\frac{\alpha Nx\phi}{\alpha x} dx\ a\ d\phi$$

Simmilary as the case of the longitudinal direction, not only the $N\phi$ and Nx, but also Nx, $N\phi x$, Qx, $Q\phi$ certainly participate to the circumferential resultants.

3. Nx : contributes to y axis resultant by being multiplied the relative rotation angle around the z axis between OA and CB ($\alpha^2 v / \alpha x^{2\cdot} dx$). Therefore the participation from Nx to the circumferential direction is

$$Nx\ a\ d\phi \frac{\alpha^2 v}{\alpha x^2} dx$$

4. $N\phi x$: which acts on the OC must be multiplied by the relative rotation angle around the z axis between OC and AB

$$\left(\frac{\alpha^2 v}{\alpha \phi\ \alpha x} - \frac{\alpha w}{\alpha x}\right) d\phi$$

Therefore the contribution to the circumferential direction is

$$N\phi \times dx \left(\frac{\alpha^2 v}{\alpha \phi\ \alpha x} - \frac{\alpha w}{\alpha x}\right) d\phi$$

5. Qx : should participate in the circumferential resultant by being multiplied the relative rotation angle around the x axis between OA and CB. Then the result is

$$(-1)\ Qx\ a\ d\phi \left(\frac{1}{a} - \frac{\alpha v}{\alpha x} + \frac{\alpha^2 w}{\alpha \phi\ \alpha x}\right) dx$$

6. $Q\phi$: takes part in the y axis direction by being multiplied the relative rotation angle around the x axis between OC and AB. Then the contribution is

$$(-1)^{\cdot} Q\phi^{\cdot} dx^{\cdot} \left(1 + \frac{\alpha v}{a \alpha \phi} + \frac{\alpha^2 w}{a \alpha \phi^2}\right) d\phi$$

Organiging these terms and balancing the circumferential load Ty per unit area of the y axis, the next equation is obtained.

$$\frac{\alpha N\phi}{\alpha \phi} d\phi\ d\phi + \frac{\alpha Nx\phi}{\alpha x} dx^{\cdot} a^{\cdot} d\phi + Nx\ a^{\cdot} d\phi \frac{\alpha^2 v}{\alpha x^2} dx$$

$$- Qx\ a^{\cdot} d\phi \frac{1}{a} \left(\frac{\alpha v}{\alpha x} + \frac{\alpha^2 w}{\alpha \phi\ \alpha x}\right) dx + N\phi x\ dx\ d\phi$$

$$\left(\frac{\alpha^2 v}{\alpha\phi\ \alpha x} - \frac{\alpha w}{\alpha x}\right) - Q\phi\ \left(1 + \frac{\alpha v}{a\alpha\phi} + \frac{\alpha^2 w}{a\alpha\phi^2}\right)$$

$$d\phi\ dx = Ty\ dx\cdot a\cdot d\phi \qquad (8)$$

Ⅲ] The equilibrium equation for the normal direction

1. The Qx which operates on OA changes $\alpha Qx / \alpha x$ per unit length of the x axis along the longitudinal direction. Therefore the net effect due to Qx to the normal direction is

$$\frac{\alpha Qx}{\alpha x} dx\ a\ d\phi$$

2. $Q\phi$ which works on OC changes $\alpha Q\phi / \alpha\phi$ per unit angle. Preceiding circumferentially from OC to AB, the net force difference is

$$\frac{\alpha Q\phi}{\alpha\phi} dx\ d\phi$$

Equally as the x, y directions, Nx, $N\phi$ on OA and $N\phi$, $N\phi x$ on OC contribute to z direction by being multiplied the relative rotation angle around each corresponding axis.

3. The participation of the Nx to the z direction can be obtained by multiplying the relative rotation angle around the y axis between OA and CB $(-1)\cdot \alpha^2 w / \alpha x^2\cdot dx$. Then the contribution is

$$(-1)\cdot Nx\cdot a\cdot d\phi\cdot(-1)\frac{\alpha^2 w}{\alpha x^2} dx.$$

4. The contribution from Nx on OA to the normal direction can be calculated by being multiplied the relative rotation angle around the x axis between OA and CB

$$\frac{1\,\alpha v}{a\,\alpha x} + \frac{\alpha^2 w}{\alpha\phi\ \alpha x} dx$$

5. $N\phi$ takes part in z direction by being multiplied by the relative rotation angle around x axis between OC and AB. Therefore the participation to the z direction is

$$N\phi\cdot dx\cdot \left(1 + \frac{1}{a}\frac{\alpha v}{\alpha\phi} + \frac{\alpha^2 w}{\alpha\phi^2}\right) d\phi$$

6. $N\phi x$ contributes to the normal direction being multiplied by the relative rotation angle around the y axis between OC and AB.

Consequently the contribution is

$$(-1)\ N\phi x\ dx\ (-1)\ \left(\frac{\alpha^2 w}{\alpha\phi\ \alpha x} + \frac{\alpha v}{\alpha x}\right) d\phi$$

Associating these forces and putting equal to the normal axis loading Tz, then the following equation holds.

$$\frac{\alpha Qx}{\alpha x} dx\, a\, d\phi + \frac{\alpha Q\phi}{\alpha \phi} d\phi\, d\phi + Nx\, a\, d\phi \frac{\alpha^2 w}{\alpha x^2} dx$$

$$+ Nx\phi\; a\, d\phi \frac{1}{a} \left(\frac{\alpha v}{\alpha x} + \frac{\alpha^2 w}{\alpha \phi\, \alpha x}\right) dx + N\phi x\, d\phi\, dx$$

$$\cdot \left(\frac{\alpha v}{\alpha x} + \frac{\alpha^2 w}{\alpha \phi\, \alpha x}\right) + N\phi\, dx\, d\phi \left(1 + \frac{\alpha v}{a \alpha \phi} + \frac{\alpha^2 w}{a \alpha \phi^2}\right)$$

$$= Tz\, dx\, a\, d\phi \qquad (9)$$

Ⅳ] The equilibrium equation of the moments operating the elements.

There are two ways to regard the arterial wall in stand point of the wall thickness. First is to look the wall as thin shell and the second is thick shell. When one stand by the former view, the stress can be regarded to distribute uniformly across the wall thickness. On the other hand when one stand by the latter case, the distribution of the stress changes continuously along the wall thickness. As we will mention in the following paper precisely, even in the aortic arch which can be treated as thin wall, the stress distribution gradient does exist. So in the middle sized artery such as the femoral artery where the ratio of the thickness to the radius exceed 0.1 and should be treated as a thick wall, the stress would distribute unevenly along the wall thickness. Under such condition, some of the stress surely produce moments with respect to the center of the cross section. There will develope at least four moments on the element.

Define the moment symbols as following.

Mij : The j component of the moment M which acts on the surface that cross at right angle with i axis.

Define the moments of longitudinal, circumferential and the normal direction as before

Mx : The bending moment by the stress σx in a section x = const that is transmitted by a unit length of section toward the direction x (tangent to the generator)

M ϕ : The bending moment by the stress $\sigma \phi$ in a section y = const that is transmitted by a unit length of section toward the direction y (tangent to the circumferential profile)

Mx ϕ : The twisting moment by the shearing stress $\gamma x \phi$ in a section y = const that transmitted by a unit length of the section toward the direction y.

M ϕ x : The twisting moment by the shearing stress $\gamma x \phi$ in a section y = const that transmitted by a unit length of the section toward the direction x. Fig 6.

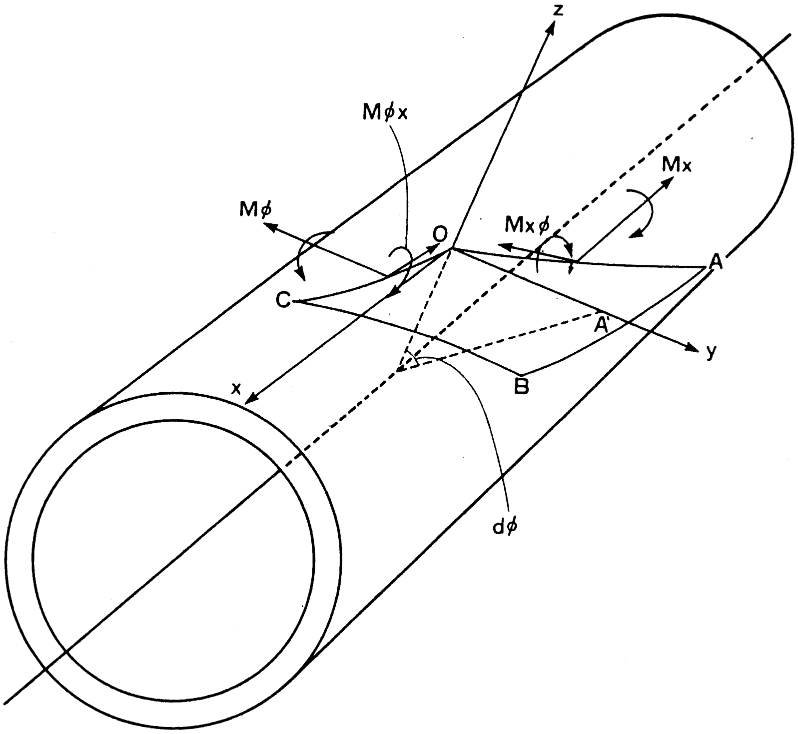

Fig 6 Moments on the micro surface

Fig. 6 This figure present the distribution pattern of the bending moments Mx, M ϕ and twisting moments Mx ϕ , M ϕ x on the micro surface element on the arterial wall. The moments are positive when it rotate in right.
author HIRAYAMA HIROHUMI

I] The moment equilibrium along the longitudinal direction.

1. The Mx which operates on OA changes α Mx / α x per unit length of the xaxis. Therefore on CB, the normal moments along the x axis is added by

$$\frac{\alpha \mathrm{Mx}}{\alpha \mathrm{x}} \mathrm{a\,dx\,d}\phi$$

This is the net effect of the Mx to the x axis direction.

2. M ϕ x which distributes evenly on OC changes at the rate of α M ϕ x / α ϕ circumferentially per unit central angle. Then on AB, the moment (α M ϕ x / α ϕ $\cdot$ d ϕ + M ϕ x)$\cdot$ dx operates. Consequently the net shearing moment is

$$\frac{\alpha \mathrm{N} \phi \mathrm{x}}{\alpha \phi} \mathrm{d}\phi \ \mathrm{dx}$$

According to the same reasons as forces, under the condition of the cubic deformation of the element, there should be some additional contribution from $Mx\phi$, $M\phi$ througth the relative rotation angle between theeach side of the element.

3. Mx should be multiplied by the relative rotation angle around the z axis between OA and CB $\alpha^2 v / \alpha x^{2} \cdot dx$. Then the contribution of $Mx\phi$ to the longitudinal direction is

$$Mx\phi \cdot a \cdot d\phi \cdot \frac{\alpha^2 v}{\alpha x^2} \cdot dx.$$

4. The participation of the $M\phi$ which operates on the OC to the x axis is obtained by being multiplied the relative rotation angle around the z axis between the OC and AB

$$(\alpha^2 v / \alpha\phi\,\alpha x - \alpha w / \alpha x) \cdot d\phi$$

The net effect is given as

$$(-1) \cdot M\phi \cdot dx \cdot \left(\frac{\alpha^2 v}{\alpha\phi\,\alpha x} + \frac{\alpha w}{\alpha x}\right) \cdot d\phi.$$

These moments are balanced with the Qx (the transverse shear) on OA. Therefore we get the following equation.

$$Mx\phi\, a\, d\phi\, dx \frac{\alpha^2 v}{\alpha x^2} - M\phi\, d\phi\, dx \left(\frac{\alpha^2 v}{\alpha\phi\,\alpha x} - \frac{\alpha w}{\alpha x}\right)$$

$$\frac{\alpha Mx}{\alpha x} dx\, a\, d\phi + \frac{\alpha M\phi x}{\alpha\phi} dx\, d\phi = Qx\, a\, dx\, d\phi \tag{10}$$

Ⅱ] The moment equilibrium along the circumferential direction.

1. The $M\phi$ which distributes evenly along the OC (= dx) alters at the ratio of $\alpha M\phi / \alpha\phi$ per unit central angle. As a result the total difference of $M\phi$ between OC and AB is

$$(-1) \frac{\alpha M\phi}{\alpha\phi} dx\, d\phi$$

$$\alpha\phi$$

that is the net effect.

2. The $Mx\phi$ changes $\alpha Mx\phi / \alpha x$ per unit length of the x axis. Therefore the net moment is given as

$$\frac{\alpha Mx\phi}{\alpha x} dx\, a\, d\phi$$

3. The contribution from Mx to the circumferential direction is established by multiplied the relative rotation angle around the z axis between OA and CB $\alpha^2 v / \alpha x^{2} \cdot dx$. Consequently the con-

tribution of the Mx

$$(-1)^{*}\ Mx^{*}\ a^{*}\ d\phi\ \frac{\alpha^2 v}{\alpha x^2}\ dx.$$

4. The contribution of M ϕ x is also calculated by utilyzing the relative rotation angle around the z axis between OC and AB. The result is given as

$$(-1)^{*}\ M\phi x^{*}\ dx^{*}\ \left(\frac{\alpha^2 v}{\alpha\phi\ \alpha x} - \frac{\alpha w}{\alpha x}\right)\ d\phi$$

These moments are equated to the transverse shearing force Q ϕ on OC. Then the following equation is formed.

$$-\frac{\alpha M\phi}{\alpha\phi}\,dx\,d\phi + \frac{\alpha Mx\phi}{\alpha x}\,dx\,a\,d\phi - Mx\,a\,dx\,d\phi\,\frac{\alpha^2 v}{\alpha x^2}$$

$$-Mx\phi\ dx\,d\phi\ \left(\frac{\alpha^2}{\alpha\phi\ \alpha x} - \frac{\alpha w}{\alpha x}\right) = -Q\phi\ a\,dx\,d\phi \qquad (11)$$

Ⅲ] The moment equilibrium along the normal direction.

1. The component of the contribution from Mx for the normal direction is obtained by multiplying the relative rotation angle around the y axis between OA and CB $(-1)^{*}(\alpha^2 w / \alpha x^2)^{*}\ dx$.

Then the contribution to the normal direction is

$$(-1)^{*}\ Mx^{*}\ a^{*}\ d\phi^{*}(-1)\frac{\alpha^2 w}{\alpha x^2}{}^{*}\ dx.$$

2. The contribution from Mx ϕ is calculated by multiplying the relative rotation angle around the x axis between OA and CB

$1 / a^{*}(\alpha v / \alpha x + \alpha^2 w / \alpha\phi\ \alpha x^{*}\ dx.$

As a result, the effect is

$$Mx\phi^{*}\ a^{*}\ d\phi^{*}\ 1 / a^{*}\ \left(\frac{\alpha v}{\alpha x} + \frac{\alpha^2 w}{\alpha\phi\ \alpha x}\right)^{*}\ dx.$$

3. The component through which M ϕ contributes in the normal direction is obtained by multiplying the relative rotation angle around the x axis between OC and AB $(d\phi + 1 / a^{*}(\alpha v / \alpha\phi + \alpha^2 w / \alpha\phi^2)\ d\phi)$

Consequently the contribution is given as

$$M\phi^{*}\ dx^{*}\ \left(1 + \frac{1}{a}\frac{\alpha v}{\alpha\phi} + \frac{\alpha^2 w}{\alpha\phi^2}\right)\ d\phi.$$

4. The participation of M ϕ x is given by being multiplying the relative rotation angle around they axis between OC and AB

$(-1)^{*}(\alpha v / \alpha x + \alpha^{2}w / \alpha x \alpha \phi^{*} d\phi$

Then the contribution is

$$(-1)^{*} M\phi x^{*} dx^{*} \left(\frac{\alpha v}{\alpha x} + \frac{\alpha^{2}w}{\alpha \phi \; \alpha x}\right)^{*} d\phi .$$

These moments are known to be balanced by the difference of N ϕ x and Nx ϕ . Therefore the equilibrium equationis expressed as following.

$$Mx\, a\, dx\, d\phi \frac{\alpha^{2}w}{\alpha x^{2}} + Mx\phi\; a\, d\phi \frac{1}{a} \left(\frac{\alpha v}{\alpha x} + \frac{\alpha^{2}w}{\alpha \phi \; \alpha x}\right)$$

$$+ M\phi\; dx\, d\phi \left(1 + \frac{1}{a} \left(\frac{\alpha v}{\alpha \phi} + \frac{\alpha^{2}w}{\alpha \phi^{2}}\right)\right) - Mx\phi\; d\phi$$

$$dx \left(\frac{\alpha v}{\alpha x} + \frac{\alpha^{2}w}{\alpha \phi \; \alpha x}\right) = (Nx\phi - N\phi x)\, dx\, a\, d\phi \tag{12}$$

However in many text books the equation (12) is written as

$$Mx\, a\, dx\, d\phi \left(\frac{\alpha v}{\alpha x} + \frac{\alpha^{2}w}{\alpha \phi \; \alpha x}\right) + Mx\phi\; a\, dx\, d\phi \frac{\alpha^{2}w}{\alpha x^{2}}$$

$$Mx\phi\; dx\, d\phi \left(1 + \frac{1}{a} \left(\frac{\alpha v}{\alpha \phi} + \frac{\alpha^{2}w}{\alpha \phi^{2}}\right)\right) - M\phi\; dx\, d\phi$$

$$\left(\frac{\alpha v}{\alpha x} + \frac{\alpha^{2}w}{\alpha \phi \; \alpha x}\right) = (N\phi x - Nx\phi)\, a\, dx\, d\phi \tag{13}$$

To obtain such a result, the distribution and the direction of the each moment should be given in the configuration that are expressed in the Fig 5 – 1 of the text book of Fllügge. The distribution of the moments are different from our one. Fig 7. As a result the components of the contributions is different from what we have deduced.

1. The contribution of Mx which acts on the OA is obtained by the relative rotation angle around the x axis between OA and CB

$1 /.a^{*}(\alpha v / \alpha x + \alpha^{2}w / \alpha x \alpha \phi)^{*} d\phi .$

Therefore the contribution to the normal direction from Mx is

$$Mx^{*} a^{*} d\phi^{*} 1 /^{*} \left(\frac{\alpha v}{\alpha x} + \frac{\alpha^{2}w}{\alpha \phi \; \alpha x}\right)^{*} d\phi .$$

2. The contribution of Mx ϕ which work on the OA is given by multiplying the relative rotation angle around the y axis between the OA and CB

$(-1)^{*} \alpha^{2}w / \alpha x^{2*} dx.$

Therefore

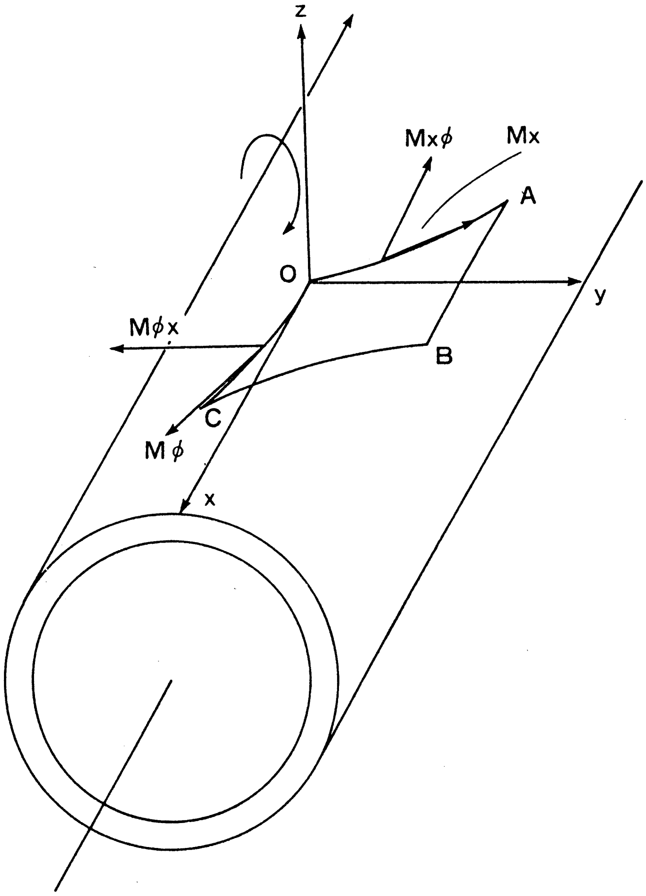

Fig 7 Moments on the micro surface
(by Flügge)

Fig. 7 The distribution of the moments according to the text of Flugge. Note the difference of the line elements on which the moments operate between our dynamical analysis.
author HIRAYAMA HIROHUMI

$$(-\ 1)^{*}\ Mx\ \phi\ ^{*}\ a^{*}\ d\ \phi\ ^{*}\ (-\ 1)^{*}\ \frac{\alpha\ ^{2}w}{\alpha\ x^{2}}{}^{*}\ dx.$$

3. The contribution of M ϕ which acts on the OC to the normal direction is calculated by multiplying the relative rotation angle around the y axis between OC and AB $(-\ 1)^{*}\ (\alpha\ ^{2}w\ /\ ^{*}\alpha\ \phi\ \ \alpha\ x$ $+\ \alpha\ v\ /\ \alpha\ x)^{*}\ d\ \phi$.

The result is

$$M\ \phi\ ^{*}\ dx^{*}\ (-\ 1)^{*}\ \left(\frac{\alpha\ ^{2}w}{\alpha\ \phi\ \ \alpha\ x} + \frac{\alpha\ v}{\alpha\ x}\right)\ ^{*}\ d\ \phi\ .$$

4. The contribution of $M\phi x$ which operates on OC is computed by multiplying the relative rotation angle around the x axis between OC and AB

$$d\phi + 1/a^{*}(\alpha v/\alpha\phi + \alpha^{2}w/\alpha\phi^{2})^{*}d\phi.$$

Then the contribution to the normal direction is expressed as

$$M\phi x^{*} dx^{*} d\phi^{*} \left(1 + \frac{1}{a}\left(\frac{\alpha v}{\alpha\phi} + \frac{\alpha^{2}w}{\alpha\phi^{2}}\right)\right)$$

Summing these 4 terms result in the equation (13) /

But practically we don't use the equation (12) or (13) which is soon recognized. Since the definition of Nx, $N\phi$, $Nx\phi$ $N\phi x$, Qx, $Q\phi$ in the textbook of Fllugge coincide with our definition, we use these equations. The difference of the direction of the moments between our one are not discussed further more in this paper.

Associating these mentioned theoretical expantion in this paper, we have reduced the relative rotation angle between each line on the element equations 1 – 6 and by utilyzing these angle we have established the static equilibrium equations of the forces (equation 7 8 9) and moments (equation 10 11 12) by strict dynamicalanalysis.

DISCUSSION

In this research we have disclosed the relative rotation angles that the arterial wall beared the cubic deformation on the curved three dimensional surface. Then utilyzing these parameters, we have induced the static equilibrium equations of the forces and moments. In this discussion we firstly refer to the significance of utilyzing the shell theory. Then discuss about the validity of applying the linear theory.

1. Artery as a shell

In this paper we analyzed the mechanical dynamic properties mathematically based on the geometric conception that regard the artery as a shell. Naturally speaking, all piecies of the construction of the arterial wall is a three dimensional body. On the other hand with respect to the geometric feature of the arterial segments, one is tempted to recognize the arterial structure as a shell conformation. In the strict mathematical treatment, the shell is defined as the structure which is enclosed by two curved surface whose distance are shorter than the principal radius of curvature. Ontologically the shell is interpreted as an object which is the substantiation of a curved surface, the arterial wall. This characterization does not necessary mean that the thickness of the shell must be extremely small in comparison with other parameters, nor the components of the shell must be all elastic solid material. So the arterial wall system is one of the most suitable object for applying the shell theory.

Except for the singular case of a plane plate in which the displacement of the strained middle surface is inextensional nor incompressive, generally after deformation, the shell structure has no developable surface already due to the extension or compression which had been spreaded around the section. [2]

In the membrane analysis theory it is sufficent to treate only the surface forces and in fact it can express the dynamical state in the shell appropriately. However as the thickness of the shell increases and the developability has disappeared, especially for the section where an abrupt change of the curvature would occure, the transverse forces or moments should operate. Therefore the membrane theory cannot describe the arterial wall dynamics sufficientlly. So the bending theory which imply not only the transverse forces but also the bending and twisting moments should be utilyzed.

For the conventional way of analyzing the forces and moments that operate on the shell,it is enough that one should confine the attention only to the small element of the surface. Generally a curved shell structure can be expressed geometrically by its middle surface, its edge line and its thickness. Those are the necessary and sufficient parameters. As a result one can establish the coordinate system on that surface.

2. About the rotation angle

In the thick walled shell to which the bending theory should be applied, there exist the extension or the compression in the middle surface. The generator, the circumferential tangent (profile) and the normal on the arbitrary point on the middle surface before the deformation coinsides with the x, y, z axies respectively. However when the deformation had developed, the configuration of the curved surface changes. So the different coordinates should be adopted.

Therefore after the deformation developes, expect for the normal (parallel with the z axies), not the generator but the tangent of the generator should coinside with the x axies. After establishing those x, z axies, then the y axies is fixed vertical to the x − z plane.

So essentially the generator is curved in comparison with the predeformed generator. Accordingly the direction of the forces would have changed. Especially with respect to the relative configuration of the each line on the element, above mentioned changes in the axies produe the changes of the position and arrangements of the line of the element. As a result there develope rotative movements around each axies and the relative rotation angle between the each line on the element would manifest.

Since the displacements are all extremely small in any direction, the higher order of the differentials could be neglected. However in the case where the cross sectional forces are not small in comparison with the bending forces, these higher order terms will become significant. [2]

3. About the modeling of the arterial wall.

The arterial wall is composed of elastic fiber, collagen fiber and smooth muscle. These components have different biophysical properties respectively and changes with the position of the artery and age. Therefore essentially it is inhomogeneous and the dynamical properties change in direction. Furthermore as will be mentioned later, the arterial wall have viscosity and the deformation attains to over 10% (the finite large deformation). Associating such properties, the arterial wall is never linear and it is almost impossible to establish a model which includes those mentioned conditions completely. However looking the arterial wall macroscopically as one member of the dynamic systemic circulation, the structure of the wall can be treated as homogeneous and within the plane right angle to the tube axis, the directional property of the wall is isotropic. Further more, under the kinds of limitting conditions, the steady and the dynamic stress – strain relations can be regarded as liner. Then we can treat the wall as a linear viscoelastic material.

To express the visco – elasticity of the material by physical model, we ordinary use the spring which describes the lumped elasticity abstractively and dash pot which expresses the lumped viscosity abstractively. Those elements represent the biophysical quality of the wall but do not signify the entity which produces the elasticity or viscosity. The parallel combination of these components is called the Voigt type model and series one is called the Maxwell type model. In many studies of the modeling of the arterial wall, the Voigt type model has been used frequently. However it does not express the dynamic properties of the arterial wall as followings

1. the bounded stress relaxation
2. the plateau of the elastic modulus
3. the frequency dependent damping.

Accordingly this model althought express the creep phenomenone except the initial elastic responce, the stress responce to the step strain input is constant and does not change with time. [6] The Maxwell model also does not express the following features of the wall

1. the bounded creep
2. the constancy of Young's modulus for frequency higher than 3Hz.
3. the non zero Young's modulus for lower frequency.
4. the stress relaxation to the non zero constant value.

Then this model can describe the stress relaxation in response to the constant strain and continuos irreversible development of the strain in reacting to the step stress but does not express the plateau part of the strain. So the simple model of either type is not suitable to represent the wall mechanics. Maxwell, J, A (1968) [7] examined the applicability of these elements to the expression of the attenuation (damping) of the pulse wave in canine carotid artery at frequency range 40Hz $<$ f $<$ 200Hz. To represent these attenuation, he concluded that the Voigt type model is favourable. However even utilyzing this model only the damping of the torsion wave in the wall could be expressed. Further more for the frequency range lower than 5Hz and in the small viscosity of the wall, the damping of the wave could be represented more properly by the Maxwell model. According to his data, the Voigt type model is applicable only for the high frequency range over 40Hz.

Westerhoff and Noodergraaf (1970) [8] following his previous model of the human systemic arterial tree, created a new mathematical and physical model for the wall propreties. Their model consisted of combination of two parallel Maxwell model with single spring and totally a five element Voigt model. Their model expressed the frequency dependency of the Youngs modulus, the stress – relaxation phenomenone, creep phenomenone and hysteresis quantitatively. They concluded the model covered all the known aspects of the visco elastic wall properties. However to incorporate such complex arterial mechanical properties which originate from the viscous property of the arterial wall did not affect the frequency – input impedance relation significantly in the total systemic circulation.

Cox, R. H (1972) [6] utilyzed the phenomenological model to represent the frequency dependence of the mechanical properties of the arteries. He founded that a model consisting a spring in series with a Voigt model, the 3 element model properly exhibit the data obtained from the canine femoral artery for heart rate larger than 2Hz. The 3 element model showed both the creep and stress relaxation phenomenone. The initial transient elastic response which the Voigt model cannot describe, was followed by the exponential creep. The stress relaxed to the non zero value and did not disappeared as the Maxwell model.

By such combination of these elements in complex form, we can make the degree of the approximation increase arbitrary. However in the complex model, the biophysical signficance of each elements in the model becomes obscure and the effects of change of the elements on the overall behaviour of the system cannot be detected in clear form. In addition for the actual arterial system in the body, such an unrealistic input form as the step stress input or the delta strain input does not exist nor operate to the arterial wall. The input pattern of the stress or strain is far more complex. So it is not sufficient to examine the aptness of the response to only the step functional strain or the delta form stress of these models.

By the help of engineering analyzing technique, recently some complex model of large finite deformative visco elastic model have been presented. However in our studies the most important purpose is to establish a comprehensive easy treatable model when one wished to understand the circulation system macroscopically. So we adopted only capacitance in this paper to represent the visco – elacticity of the wall. In the following papers we surely present the visco – elastic model that incldue the wall viscosity.

4. The linearity of the stress strain relation

In this paper we have analyzed the relative rotation angles and equilibrium conditions that forces and moments satisfy. To analyze such situations, we confined our attention to the middle surface of the element and assumed the Hookes law. That means the stress – strain relation is linear. However in the actual arterial wall, the stress – strain relation is complicated. Conventionally the stress strain relation of the biological materials had been investigated by applying the step functional stress or strain and analyzing the resulting strain or stress in Vitro. Therefore the experimental conditions are far from natural.

Wiederheim (1965) [9] utilyzed the step strain for the canine arteriales (60um $<$ D $<$ 150um) and analyzed the circumferential stress – strain relation. Althought the relation was marked nonlinear for the circumferential strain range $\Delta r / ro \leqq 0.23$, below this range, the relation was linear.

Attinger (1968) [10] examined the difference of the stress strain relation in different direction. He used the canine femoral artery and kept it in situ length. Inputting the step strain in the longitudinal direction and he measured resulting the stress in the longitudinal and circumferential direction. As a result in the both directions, the stress strain relations were nonlinear, yet the latter exhibited stronger nonlinearity.

To conform to the in Vivo state, converting the developed stress into the pressure dimensions (mmHg), the physiological pressure range 60 $<$ BPmmHg $<$ 175 corresponded to the developed stress for the range of 400g ~ 1200g. In this case the longitudinal strain was $0.3 < \Delta L / L < 0.6$. Within such strain range, the stress – strain relation was nonlinear. The linear stress – strain relation manifested only at the range of input stress over 1200g which corresponded to blood pressure over 175mmHg. Furthere more the nonlinearity in the circumferential direction was markedly influenced by the change of the tonus of the smooth muscle. However the relation in the longitudinal direction was almost linear for a wide range of the developed stress and was independent of the smooth muscle tonus. In the actual human arterial system, the longitudinal strain due to the pulsatile blood flow is small ($0.1 < \Delta L / L$) because of the longitudinal tetehring of the surrounding. That is far from the experimental artificial strain.

Dobrin, P. B. (1969) [11] used the canine carotid artery for the same experimental subject. Inputting the step wise circumferential strain and he has measured the consequential longitudinal and circumferential stress. Within the range of the circumferential strain $\Delta r / r < 0.4$, the circumferential stress – strain relation was almost linear. However for $0.4 < \Delta r / r < 1.2$ which corresponded to the transmural pressure about 50 – 200mmHg in his experimental instrument, the circumferential relation showed marked nonlinearity. Contrary to Attingers results, the relation between the stress in the longitudinal direction and the strain in the circumferential direction was nonlinear under the same circumferential strain. Yet when the strain $\Delta r / r$ was smaller than 0.2, the relation was linear.

Dobrin, P. B. (1973) [12] measured the examined the contribution of the smooth muscle to the arterial stress – strain relation. Utilyzig the KCI, he killed the smooth muscle and subtracted these effects from the stress obtained under the condition of maximum constriction that had been induced by Norepinephrine. Therefore the resultant stress – strain relation can be regarded as reflecting purely the effect of the smooth muscle. Under the isometric contraction of strain $\Delta D / D < 0.7$ which corresponded to the blood pressure $50 < BPmmHg < 150$, the circumferential stress – strain relation was linear.

Cox, R. H. (1975) [13] analyzed the stress – strain relation by employing the strain energy density function for the canine carotid artery. In any direction for circumferential stress – strain, longitudinal stress – circumferential strain, and radial stress – circumferential strain, these relations were all nonlinear. Nevertheless for the strain of $\Delta r / r < 0.2$, the relation could be treated as linear. He also (1976) [14] studied the stress – strain relation of canine iliac or carotid artery that was attributed exclusively to the mechanical property of the smooth muscle. The active stress – strain relation was almost linear untill the strain was $\Delta r / r < 0.56$. He suggested the collagenous fiber would have participated in the nonlinearity of the stress – strain relation.

As for the nonlinear analysis of the arterial wall, many researchers used the strain energy density function of variable type. Tanaka (1974) [15] studied the nonlinear stress – strain relation of the canine arterial arch within the physiological range of input stress. He presented the mathematical expression for the Tension (T) – strain (e) relation as $T = Ke^r$. This formula of course suggest the nonlinearity. The e and r were small especially in the peripheral artery. In the case of the femoral artery, r = 1.23 for the longitudinal direction and r = 1.59 for the circumferential direction. These values were small comparing with the data of the aortic arch ($1.75 < r < 2.05$). Associating these results, for the small range of e and r, one can regard the stress – strain relation is linear. Althought above mentioned nonlinear analysis were based on the least square method, the range which refered were beyond the physiological state. No one can apply the physiological significance to the least square method. Vaishnav (1972) [16], Fung (1979) [17] also applied the nonlinear analysis. But their analyzing procedures were extremely complex and does not give a practical advantage.

Admitting the nonlinear stress – strain relation, Patel, D. J (1967) [18] divided the stress – strain relation into two components in the canine descending aorta in Vivo. He measured the ratio of the incremental stress – strain component to the average stress – strain component within the physiological range. Under the static condition, the circumferential incremental stress was 19 – 20 per cent of the average stress, and for the longitudinal direction the incremental stress was 18 – 25% of the average stress. In addition, the incremental circumferential strain is 5.3 – 14% of the average strain and for the longitudinal direction, it was 4.2 – 7.9% of the average strain. About for the dynamic incremental stress – strain relation, the circumferential incremental stress was 11 – 14.9% of the average stress, the longitudinal one was 3.7 – 4.8% of the average stress. About for the strain for the circumferential direction, the incremental strain was 1 – 1.4% of the average strain, for the longitudinal incremental strain was 0.5 – 0.57% of the average strain. He concluded that the incremental components of the stress or strain are much small in comparison with the average stress, strain in the either direction. He also said that the nonlinear component of the stress – strain relation is small compared with linear one.

Conjoining these experimental results, one may be permitted to regard the stress – strain relation is linear in either direction within the physiological range of the pressure and strains. Thus the linear mechanical dynamics is applicable for analyzing the static equilibrium problems of the forces and moments. Althought the mathematical treatments were longsum, such process of analyzing the minute and exact mechnical dynamical properties of the arterial wall is one of the core part of the modeling of the arterial system. In the following paper, we expand the forces – displacements relation for a paving stone of construction of the arterial wall movement equations.

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