Chapter

Blow-up Solutions to Nonlinear Schrödinger Equation with a Potential

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Abstract

This is a sequel to the paper "Characterization of the ground state to the intercritical NLS with a linear potential by the virial functional" by the same authors. We continue to study the Cauchy problem for a nonlinear Schrödinger equation with a potential. In the previous chapter, we investigated some minimization problems and showed global existence of solutions to the equation with initial data, whose action is less than the value of minimization problems and positive virial functional. In particular, we saw that such solutions are bounded. In this chapter, we deal with solutions to the equation with initial data, whose virial functional is negative contrary to the previous paper and show that such solutions are unbounded.

Keywords: nonlinear Schrödinger equation, linear potential, standing wave, blow-up, grow-up, global existence

1. Introduction

In this chapter, we consider the Cauchy problem of the following nonlinear Schrödinger equation with a linear potential:

$$i\partial_t u + \Delta_V u = -|u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \tag{1}$$

where $d \geq 1$, $1 < p < 2^* - 1$,

$$2^* := \begin{cases} \infty & \text{if } d \in \{1,2\}, \\ \dfrac{2d}{d-2} & \text{if } d \geq 3, \end{cases} \tag{2}$$

and $\Delta_V := \Delta - V = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} - V$. In particular, we consider the Cauchy problem of Eq. (1) with initial condition

$$u(0,\cdot) = u_0 \in H^1(\mathbb{R}^d). \tag{3}$$

Eq. (1) with $V \in L^\infty(\mathbb{R}^d)$ is a model proposed to describe the local dynamics at a nucleation site (see [1]).

Eq. (1) is locally well-posed in the energy space $H^1(\mathbb{R}^d)$ under some assumptions, where Eq. (1) is called local well-posedness in $H^1(\mathbb{R}^d)$ if Eq. (1) satisfies all of the following conditions:

- There is uniqueness in $H^1(\mathbb{R}^d)$ for a solution to Eq. (1).
- For each $u_0 \in H^1(\mathbb{R}^d)$, there exists a solution to Eq. (1) with Eq. (3) defined on a maximal existence interval $(T_{\min}, T_{\max})$, where $T_{\max} = T_{\max}(u_0) \in (0, \infty]$ and $T_{\min} = T_{\min}(u_0) \in [-\infty, 0)$.
- There is the blow-up alternative. That is, if $T_{\max} < \infty$ (resp. $T_{\min} > -\infty$), then we have

$$\lim_{t \uparrow T_{\max}} \|u(t)\|_{H_x^1} = \infty \left(\text{resp.} \lim_{t \downarrow T_{\min}} \|u(t)\|_{H_x^1} = \infty \right). \tag{4}$$

- The solution depends on continuously on the initial condition. That is, if $u_{0,n} \to u_0$ in $H^1(\mathbb{R}^d)$, then for any closed interval $I \subset (T_{\min}, T_{\max})$, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, the solution u_n to Eq. (1) with $u_n(0,x) = u_{0,n}(x)$ is defined on $C_t(I; H^1(\mathbb{R}^d))$ and satisfies $u_n \to u$ in $C_t(I; H^1(\mathbb{R}^d))$ as $n \to \infty$, where u is the solution to Eq. (1) with $u(0,x) = u_0(x)$.

To state a local well-posedness result, we define the space

$$\mathcal{K}_0(\mathbb{R}^d) := \overline{\{f \in L^\infty(\mathbb{R}^d) : \text{supp} f \text{ is compact.}\}}^{\|\cdot\|_{\mathcal{K}}}, \tag{5}$$

where

$$\|f\|_{\mathcal{K}} := \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(y)|}{|x-y|^{d-2}} dy. \tag{6}$$

We note that

$$L^{\frac{d}{2}-\varepsilon}(\mathbb{R}^d) \cap L^{\frac{d}{2}+\varepsilon}(\mathbb{R}^d) \hookrightarrow L^{\frac{d}{2},1}(\mathbb{R}^d) \hookrightarrow \mathcal{K}(\mathbb{R}^d) := \{f : \|f\|_{\mathcal{K}} < \infty\} \tag{7}$$

for some $\varepsilon > 0$, where the space $L^{p,q}(\mathbb{R}^d)$ denotes the usual Lorentz space.

Theorem 1 (Local well-posedness, [2–4]) Let $d \geq 1$ and $1 < p < 2^* - 1$. If V satisfies one of the following, then Eq. (1) is locally well-posed in $H^1(\mathbb{R}^d)$.

- $V \in L^\eta(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ for $\eta \geq 1$ if $d = 1$ and $\eta > \frac{d}{2}$ if $d \geq 2$,
- $\|V_-\|_{\mathcal{K}} < 4\pi$ and $V \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap \mathcal{K}_0(\mathbb{R}^3)$, where $V_- := \min\{V(x), 0\}$.

Moreover, the solution u to Eq. (1) conserves its mass and energy with respect to time t, where they are defined as

$$\begin{aligned}&(\text{Mass}) \quad M[u(t)] := \|u(t)\|_{L_x^2}^2,\\ &(\text{Energy}) \quad E_V[u(t)] := \frac{1}{2}\|u(t)\|_{\dot{H}_x^1}^2 + \frac{1}{2}\int_{\mathbb{R}^d} V(x)|u(t,x)|^2 - \frac{1}{p+1}\|u(t)\|_{L_x^{p+1}}^{p+1}.\end{aligned} \tag{8}$$

We turn to time behaviors of the solution to Eq. (1). A solution to Eq. (1) has various kinds of time behaviors by the choice of initial data. For example, we can consider the following time behaviors.

- (Scattering) We say that the solution u to Eq. (1) scatters in positive time (resp. negative time) if $T_{\max} = \infty$ (resp. $T_{\min} = -\infty$) and there exists $\psi_+ \in H^1(\mathbb{R}^d)$ (resp. $\psi_- \in H^1(\mathbb{R}^d)$) such that

$$\lim_{t\to+\infty} \|u(t) - e^{it\Delta_V}\psi_+\|_{H_x^1} = 0 \quad \left(\text{resp.} \lim_{t\to-\infty} \|u(t) - e^{it\Delta_V}\psi_-\|_{H_x^1} = 0\right), \tag{9}$$

where $e^{it\Delta_V}f$ is a solution to the corresponding linear equation with Eq. (1)

$$i\partial_t u(t,x) + \Delta_V u(t,x) = 0, \quad u(0,x) = f(x). \tag{10}$$

We say that u scatters when u scatters in positive and negative time.

- (Blow-up) We say that the solution u to Eq. (1) blows up in positive time (resp. negative time) if $T_{\max} < \infty$ (resp. $T_{\min} > -\infty$). We say that u blows up when u blows up in positive and negative time.

- (Grow-up) We say that the solution u to Eq. (1) grows up in positive time (resp. negative time) if $T_{\max} = \infty$ (resp. $T_{\min} = -\infty$) and

$$\lim_{t\to\infty} \sup \|u(t)\|_{H_x^1} = \infty, \quad \left(\text{resp.} \lim_{t\to-\infty} \sup \|u(t)\|_{H_x^1} = \infty\right). \tag{11}$$

We say that u grows up when u grows up in positive and negative time.

- (Standing wave) We say that the solution u to Eq. (1) is a standing wave if $u = e^{i\omega t}Q_{\omega,V}$ for some $\omega \in \mathbb{R}$, where $Q_{\omega,V}$ satisfies the elliptic equation

$$-\omega Q_{\omega,V} + \Delta_V Q_{\omega,V} = -|Q_{\omega,V}|^{p-1} Q_{\omega,V}. \tag{12}$$

In particular, $Q_{\omega,V}$ is ground state to Eq. (12) if

$$Q_{\omega,V} \in \{\phi \in \mathcal{A}_{\omega,V} : S_{\omega,V}(\phi) \le S_{\omega,V}(\psi) \text{ for any } \psi \in \mathcal{A}_{\omega,V}\} =: \mathcal{G}_{\omega,V}, \tag{13}$$

where $S_{\omega,V}(f) := \frac{\omega}{2}M[f] + E_V[f]$ (and)

$$\mathcal{A}_{\omega,V} := \{\psi \in H^1(\mathbb{R}^d)\backslash\{0\} : S'_{\omega,V}(\psi) = 0\}. \tag{14}$$

We know the following results (Theorems 2 and 3) for time behaviors of the solutions to Eq. (1). For related results, we also list [5–38].

Theorem 2 (Hong, [3]) Let $d = p = 3$, $u_0 \in H^1(\mathbb{R}^3)$, and $Q_{1,0} \in \mathcal{G}_{1,0}$. Suppose that V satisfies $V \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap \mathcal{K}_0(\mathbb{R}^3)$, $V \geq 0$, $x \cdot \nabla V \in L^{\frac{3}{2}}(\mathbb{R}^3)$, and $x \cdot \nabla V \leq 0$. We also assume that

$$M[u_0]E_V[u_0] < M[Q_{1,0}]E_0[Q_{1,0}] \quad \text{and} \|u_0\|_{L^2}\|u_0\|_{\dot{H}^1_V} < \|Q_{1,0}\|_{L^2}\|Q_{1,0}\|_{\dot{H}^1}. \tag{15}$$

Then, the solution u to Eq. (1) with Eq. (3) scatters.

Theorem 3 (Hamano–Ikeda, [4]) Let $d = 3$, $\frac{7}{3} < p < 5$, $u_0 \in H^1(\mathbb{R}^3)$, and $Q_{1,0} \in \mathcal{G}_{1,0}$. Suppose that V satisfies $V \geq 0$ and $x \cdot \nabla V \in L^{\frac{3}{2}}(\mathbb{R}^3)$. We also assume that

$$M[u_0]^{\frac{1-s_c}{s_c}}E_V[u_0] < M[Q_{1,0}]^{\frac{1-s_c}{s_c}}E_0[Q_{1,0}], \tag{16}$$

where $s_c := \frac{d}{2} - \frac{2}{p-1}$.

1. (Scattering)

   If $V \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap \mathcal{K}_0(\mathbb{R}^3)$, $x \cdot \nabla V \leq 0$, and

   $$\|u_0\|_{L^2}^{\frac{1-s_c}{s_c}}\|u_0\|_{\dot{H}^1} < \|Q_{1,0}\|_{L^2}^{\frac{1-s_c}{s_c}}\|Q_{1,0}\|_{\dot{H}^1}, \tag{17}$$

   then $(T_{\min}, T_{\max}) = \mathbb{R}$, that is, exists globally in time. Moreover, if u_0 and V are radially symmetric, then u scatters.

2. (Blow-up or grow-up)

   If "$V \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap \mathcal{K}_0(\mathbb{R}^3)$ or $V \in L^\sigma(\mathbb{R}^3)$ for some $\frac{3}{2} < \sigma \leq \infty$," $2V + x \cdot \nabla V \geq 0$, and

   $$\|u_0\|_{L^2}^{\frac{1-s_c}{s_c}}\|u_0\|_{\dot{H}^1_V} > \|Q_{1,0}\|_{L^2}^{\frac{1-s_c}{s_c}}\|Q_{1,0}\|_{\dot{H}^1}, \tag{18}$$

   then u blows up or grows up. Furthermore, if one of the following holds:

   - "u_0 and V are radially symmetric," $x \cdot \nabla V \geq 0$, and $V \in L^\infty(\mathbb{R}^3)$,
   - $xu_0 \in L^2(\mathbb{R}^3)$,

then u blows up.

Remark 1 Mizutani [39] proved that for any $\psi \in H^1$, there exists $\phi_\pm \in H^1(\mathbb{R}^3)$ such that

$$\lim_{t\to\pm\infty} \|e^{it\Delta_V}\psi - e^{it\Delta}\phi_\pm\|_{H^1_x} = 0 \tag{19}$$

under the assumptions $V \in L^{\frac{3}{2}}(\mathbb{R}^3)$ and $V \geq 0$, where the double-sign corresponds. Combining this limit and scattering part in Theorem 3 (or Theorem 2), we can see that the nonlinear solution u to Eq. (1) approaches to a free solution $e^{it\Delta}\phi_\pm$ as $t \to \pm\infty$ for some $\phi_\pm \in H^1(\mathbb{R}^3)$.

We realize that there is no potential, which satisfies scattering and blow-up or grow-up parts in Theorem 1 at the same time. Indeed, if V satisfies $x \cdot \nabla V \leq 0$ and $2V + x \cdot \nabla V \geq 0$, then $V \notin L^{\frac{3}{2}}(\mathbb{R}^3)$. Then, we consider a minimization problem

$$n_{\omega,V} := \inf\{S_{\omega,V}(f) : f \in H^1(\mathbb{R}^d)\backslash\{0\},\ \ K_V(f) = 0\} \tag{20}$$

to get a potential V, which deduces scattering and blow-up or grow-up at the same time. It proved in [40] that the condition Eq. (16) can be rewritten as the following by using $n_{\omega,V}$.

Proposition 1 Let $d \geq 3$, $1 + \frac{4}{d} < p < 2^* - 1$, $f \in H^1(\mathbb{R}^d)$, and $Q_{1,0} \in \mathcal{G}_{1,0}$. Assume that V satisfies (A2) with $|\alpha| \leq 1$ and (A6) below. Then, the following two conditions are equivalent.

1. $M[f]^{\frac{1-s_c}{s_c}} E_V[f] < M[Q_{1,0}]^{\frac{1-s_c}{s_c}} E_0[Q_{1,0}]$,

2. There exists $\omega > 0$ such that $S_{\omega,V}(f) < n_{\omega,V}$.

Using $n_{\omega,V}$, we expect that if $S_{\omega,V}(u_0) < n_{\omega,V}$ and $K_V(u_0) \geq 0$, then the solution u scatters and if $S_{\omega,V}(u_0) < n_{\omega,V}$ and $K_V(u_0) < 0$, then the solution u blows up or grows up, where K_V is called virial functional and is defined as

$$\begin{aligned} K_V(f) &:= \frac{d}{d\lambda}\bigg|_{\lambda=0} S_{\omega,V}\left(e^{d\lambda} f\left(e^{2\lambda}\cdot\right)\right) \\ &= 2\|f\|_{\dot{H}^1}^2 - \int_{\mathbb{R}^d} (x \cdot \nabla V)|f(x)|^2 dx - \frac{(p-1)d}{p+1}\|f\|_{L^{p+1}}^{p+1}. \end{aligned} \tag{21}$$

It is well known that $K_V(u(t))$ denotes variance of the solution and if $xu_0 \in L^2(\mathbb{R}^d)$ then

$$K_V(u(t)) = \frac{1}{4} \cdot \frac{d^2}{dt^2}\|xu(t)\|_{L_x^2}^2 \tag{22}$$

for each $t \in (T_{\min}, T_{\max})$. We also consider a minimization problem $r_{\omega,V}$, which restricts $n_{\omega,V}$ to radial functions, that is,

$$r_{\omega,V} := \inf\{S_{\omega,V}(f) : f \in H^1_{\text{rad}}(\mathbb{R}^d)\backslash\{0\},\ \ K_V(f) = 0\} \tag{23}$$

and expect for radial initial data u_0 and radial potential V that if $S_{\omega,V}(u_0) < r_{\omega,V}$ and $K_V(u_0) \geq 0$, then the solution u scatters and if $S_{\omega,V}(u_0) < r_{\omega,V}$ and $K_V(u_0) < 0$, then the solution u blows up. For more general minimization problems

$$\begin{aligned} n_{\omega,V}^{\alpha,\beta} &:= \inf\left\{S_{\omega,V}(f) : f \in H^1(\mathbb{R}^d)\backslash\{0\},\ \ K_{\omega,V}^{\alpha,\beta}(f) = 0\right\}, \\ r_{\omega,V}^{\alpha,\beta} &:= \inf\left\{S_{\omega,V}(f) : f \in H^1_{\text{rad}}(\mathbb{R}^d)\backslash\{0\},\ \ K_{\omega,V}^{\alpha,\beta}(f) = 0\right\} \end{aligned} \tag{24}$$

with

$$\alpha > 0, \quad \beta \geq 0, \quad 2\alpha - d\beta \geq 0, \tag{25}$$

the authors showed in [40, 41] the following results (Theorems 4 and 5) Eq. (27), where the functional $K^{\alpha,\beta}_{\omega,V}$ is given as

$$K^{\alpha,\beta}_{\omega,V}(f) := \frac{d}{d\lambda}\Big|_{\lambda=0} S_{\omega,V}\left(e^{\alpha\lambda} f\left(e^{\beta\lambda}\cdot\right)\right). \tag{26}$$

Here, we realize $n_{\omega,V} = n^{d,2}_{\omega,V}$, $r_{\omega,V} = r^{d,2}_{\omega,V}$, and $K_V = K^{d,2}_{\omega,V}$.

To state the results, we give the assumptions of the potential V: Let $\mathfrak{a} \in (\mathbb{N} \cup \{0\})^d$.

A1. $V \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap \mathcal{K}_0(\mathbb{R}^3)$

A2. $x^{\mathfrak{a}}\partial^{\mathfrak{a}} V \in L^{\frac{d}{2}}(\mathbb{R}^d) + L^{\sigma}(\mathbb{R}^d)$ for some $\frac{d}{2} \le \sigma < \infty$

A3. $x^{\mathfrak{a}}\partial^{\mathfrak{a}} V \in L^{\frac{d}{2}}(\mathbb{R}^d) + L^{\infty}(\mathbb{R}^d)$

A4. $x^{\mathfrak{a}}\partial^{\mathfrak{a}} V \in L^{\eta}(\mathbb{R}^d) + L^{\sigma}(\mathbb{R}^d)$ for some $\frac{d}{2} < \eta \le \sigma < \infty$

A5. $x^{\mathfrak{a}}\partial^{\mathfrak{a}} V \in L^{\eta}(\mathbb{R}^d) + L^{\infty}(\mathbb{R}^d)$ for some $\frac{d}{2} < \eta < \infty$

A6. $V \ge 0$, $x \cdot \nabla V \le 0$, $2V + x \cdot \nabla V \ge 0$

A7. $V \ge 0$, $x \cdot \nabla V \le 0$, $\omega \ge \omega_0$ for

$$\omega_0 := -\frac{1}{2} \text{ess inf}_{x \in \mathbb{R}^d} (2V + x \cdot \nabla V). \tag{27}$$

We note that the third inequality implies $2V + x \cdot \nabla V + 2\omega \ge 0$ a.e. $x \in \mathbb{R}^d$.

Theorem 4 Let $d \ge 3$ and $1 + \frac{4}{d} < p < 2^* - 1$.

- (Non-radial case) Let V satisfy (A2) with $|\mathfrak{a}| \le 1$ and (A6). Then, for each (α, β) with Eq. (25) and $\omega > 0$, $n^{\alpha,\beta}_{\omega,V} = n^{\alpha,\beta}_{\omega,0}$ holds. Moreover, if $x \cdot \nabla V < 0$, then $n^{\alpha,\beta}_{\omega,V}$ is never attained.
- (Radial case) Let V satisfy (A3) with $|\mathfrak{a}| \le 1$ and (A7). Let V be radially symmetric. Then, $r^{\alpha,\beta}_{\omega,V}$ is attained for each (α, β) with Eq. (25). Moreover, if V satisfies (A3) with $|\mathfrak{a}| \le 2$ and $3x \cdot \nabla V + x\nabla^2 V x^T \le 0$, then $\mathcal{M}^{\alpha,\beta}_{\omega,V,\text{rad}} = \mathcal{G}_{\omega,V,\text{rad}}$ holds, where $\nabla^2 V$ denotes the Hessian matrix of V,

$$\begin{aligned}
\mathcal{M}^{\alpha,\beta}_{\omega,V,\text{rad}} &:= \left\{\phi \in H^1_{\text{rad}}(\mathbb{R}^d) : S_{\omega,V}(\phi) = r^{\alpha,\beta}_{\omega,V},\ \ K^{\alpha,\beta}_{\omega,V}(\phi) = 0\right\},\\
\mathcal{G}_{\omega,V,\text{rad}} &:= \left\{\phi \in \mathcal{A}_{\omega,V,\text{rad}} : S_{\omega,V}(\phi) \le S_{\omega,V}(\psi) \text{ for any } \psi \in \mathcal{A}_{\omega,V,\text{rad}}\right\},\\
\mathcal{A}_{\omega,V,\text{rad}} &:= \left\{\psi \in H^1_{\text{rad}}(\mathbb{R}^d) \backslash \{0\} : S'_{\omega,V}(\psi) = 0\right\}.
\end{aligned} \tag{28}$$

The inequality $n^{\alpha,\beta}_{\omega,V} \le r^{\alpha,\beta}_{\omega,V}$ holds by their definitions and the attainability of $n^{\alpha,\beta}_{\omega,V}$ and $r^{\alpha,\beta}_{\omega,V}$ deduces the following corollary.

Corollary 1 Under the all assumptions of (Non-radial case) in Theorem 4, we have

$$n_{\omega,V}^{\alpha,\beta} < r_{\omega,V}^{\alpha,\beta}. \tag{29}$$

Remark 2 In the case of $V = 0$, it is well known that $n_{\omega,0}^{\alpha,\beta}$ and $r_{\omega,0}^{\alpha,\beta}$ are attained by $Q_{\omega,0} \in \mathcal{G}_{\omega,0}$. That is, $n_{\omega,0}^{\alpha,\beta} = r_{\omega,0}^{\alpha,\beta} = S_{\omega,0}(Q_{\omega,0})$ holds.

Then, we investigate global existence of a solution to time-dependent Eq. (1).

Theorem 5 (Global well-posedness in H^1) Let $d \geq 3$ and $1 + \frac{4}{d} < p < 2^* - 1$.

- (Non-radial case) Let $u_0 \in H^1(\mathbb{R}^d)$ and $Q_{\omega,0} \in \mathcal{G}_{\omega,0}$. Suppose that V satisfies "(A1) or (A4) with $|\mathfrak{a}| = 0$," (A2) with $|\mathfrak{a}| = 1$, and (A6). We also assume that there exist (α, β) satisfying Eq. (25) and $\omega > 0$ such that

$$S_{\omega,V}(u_0) < S_{\omega,0}(Q_{\omega,0}) \quad \left(= n_{\omega,V}^{\alpha,\beta}\right), \quad K_{\omega,V}^{\alpha,\beta}(u_0) \geq 0. \tag{30}$$

  Then, the solution u to Eq. (1) with Eq. (3) exists globally in time. In particular, it follows that

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H_x^1} < \infty. \tag{31}$$

- (Radial case) Let $u_0 \in H_{\mathrm{rad}}^1(\mathbb{R}^d)$ and $Q_{\omega,V} \in \mathcal{G}_{\omega,V,\mathrm{rad}}$. Suppose that V is radially symmetric and satisfies "(A1) or (A5) with $|\mathfrak{a}| = 0$," (A3) with $|\mathfrak{a}| = 1, 2$, (A7), and $3x \cdot \nabla V + x\nabla^2 V x^T \leq 0$. If there exist (α, β) with Eq. (25) and $\omega > 0$ satisfying $\omega \geq \omega_0$ such that

$$S_{\omega,V}(u_0) < S_{\omega,V}(Q_{\omega,V}) \quad \left(= r_{\omega,V}^{\alpha,\beta}\right), \quad K_{\omega,V}^{\alpha,\beta}(u_0) \geq 0, \tag{32}$$

then the solution u to Eq. (1) with Eq. (3) exists globally in time.

1.1 Main theorem

In the previous paper, the authors handled the solution u to Eq. (1) with initial data u_0 satisfying $S_{\omega,V}(u_0) < m_{\omega,V}$ and $K_V(u_0) \geq 0$, where $m_{\omega,V}$ denotes $n_{\omega,V}$ or $r_{\omega,V}$. We note that $m_{\omega,V}$ is $m_{\omega,V}^{\alpha,\beta}$ with $(\alpha, \beta) = (d, 2)$ and $m_{\omega,V}^{\alpha,\beta}$ is independent of (α, β). In this chapter, we are interested in the solutions to Eq. (1) with initial data satisfying $S_{\omega,V}(u_0) < m_{\omega,V}$ and $K_V(u_0) < 0$. Our main theorem is the following:

Theorem 6 Let $d \geq 3$ and $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$.

- (Non-radial case) Let $u_0 \in H^1(\mathbb{R}^d)$ and $Q_{\omega,0} \in \mathcal{G}_{\omega,0}$. Suppose that V satisfy "(A1) or (A4) with $|\mathfrak{a}| = 0$," (A2) with $|\mathfrak{a}| = 1$, and (A6). We also assume that there exists $\omega > 0$ such that

$$S_{\omega,V}(u_0) < S_{\omega,0}(Q_{\omega,0}) \quad (= n_{\omega,V}), \quad K_V(u_0) < 0. \tag{33}$$

  Then, the solution u to Eq. (1) with Eq. (3) blows up or grows up. Moreover, u blows up under the additional assumption $xu_0 \in L^2(\mathbb{R}^d)$.

- (Radial case) Let $u_0 \in H^1_{\mathrm{rad}}(\mathbb{R}^d)$ and $Q_{\omega,V} \in \mathcal{G}_{\omega,V,\mathrm{rad}}$. Suppose that V is radially symmetric and satisfies "(A1) or (A5) with $|\alpha| = 0$," (A3) with $|\alpha| = 1, 2$, (A7), and $3x \cdot \nabla V + x\nabla^2 V x^T \leq 0$. We also assume that there exists $\omega > 0$ satisfying $\omega \geq \omega_0$ such that

$$S_{\omega,V}(u_0) < S_{\omega,V}(Q_{\omega,V}) \ \ (= r_{\omega,V}), \quad K_V(u_0) < 0. \tag{34}$$

Then, the solution u to Eq. (1) with Eq. (3) blows up.

Remark 3 Let V be a potential in Theorem 6. Combining Theorems 5 and 6, we complete bounded and unbounded dichotomy of $\{u_0 \in H^1(\mathbb{R}^d) : S_{\omega,V}(u_0) < S_{\omega,0}(Q_{\omega,0})\}$ and global existence and blow-up dichotomy of $\{u_0 \in H^1_{\mathrm{rad}}(\mathbb{R}^d) : S_{\omega,V}(u_0) < S_{\omega,V}(Q_{\omega,V})\}$ by using sign of the virial functional of initial data.

Remark 4 The following potential satisfies all of conditions in Theorem 6:

$$V(x) = \frac{\gamma\{\log(1+|x|)\}^\theta}{|x|^\mu}, \quad (\gamma > 0, 0 \leq \theta \leq \mu < 2, \mu > 0). \tag{35}$$

Theorem 6 with the potential Eq. (35) having $\theta = 0$ was considered in the previous paper [19] by the authors. As the other example, we put

$$V(x) := \frac{\gamma}{\langle x \rangle^\mu}, \quad (\gamma > 0, 0 < \mu < 2), \tag{36}$$

where $\langle \cdot \rangle$ is called the Japanese bracket and is defined as $\left(1 + |\cdot|^2\right)^{\frac{1}{2}}$.

1.2 Organization of the paper

The organization of the rest of this chapter is as follows. In Section 2, we collect some notations and tools used throughout this chapter. In Section 3, we prove non-radial case in Theorem 6 by using an argument in [13]. In Section 4, we show radial case in Theorem 6 by using an argument in [33].

2. Preliminaries

In this section, we define some notations and collect some tools, which are used throughout this chapter.

2.1 Notation and definition

For $1 \leq p \leq \infty$, $L^p = L^p(\mathbb{R}^d)$ denotes the usual Lebesgue space. For a Banach space X, we use $L^q(I;X)$ to denote the Banach space of functions $f : I \times \mathbb{R}^d \to \mathbb{C}$ whose norm is

$$\|f\|_{L^q(I;X)} := \| \|f(t)\|_X \|_{L^q(I)} < \infty. \tag{37}$$

We extend our notation as follows: If a time interval is not specified, then the t-norm is evaluated over $(-\infty, \infty)$. To indicate a restriction to a time subinterval $I \subset (-\infty, \infty)$, we will write as $L^q(I)$. $H^s(\mathbb{R}^d)$ and $\dot{H}^s(\mathbb{R}^d)$ are the usual Sobolev spaces, whose norms $\|f\|_{H^s} := \|(1-\Delta)^{\frac{s}{2}} f\|_{L^2}$ and $\|f\|_{\dot{H}^s} := \|(-\Delta)^{\frac{s}{2}} f\|_{L^2}$ respectively. We also define the Sobolev spaces $H^s_V(\mathbb{R}^d)$ and $\dot{H}^s_V(\mathbb{R}^d)$ with the potential V via norms $\|f\|_{H^s_V} := \|(1-\Delta_V)^{\frac{s}{2}} f\|_{L^2}$ and $\|f\|_{\dot{H}^s_V} := \|(-\Delta_V)^{\frac{s}{2}} f\|_{L^2}$ respectively.

2.2 Some tools

Proposition 2 Let $p \geq 1$. For $f \in H^1_{\mathrm{rad}}(\mathbb{R}^d)$, we have

$$\|f\|^{p+1}_{L^{p+1}(R \leq |x|)} \leq \frac{C}{R^{\frac{(d-1)(p-1)}{2}}} \|f\|^{\frac{p+3}{2}}_{L^2(R \leq |x|)} \|f\|^{\frac{p-1}{2}}_{\dot{H}^1(R \leq |x|)} \tag{38}$$

for any $R > 0$, where the implicit constant C is independent of R and f.

To state the next proposition, we define two functions:

$$\mathcal{X}_R := R^2 \mathcal{X}\left(\frac{|x|}{R}\right), \tag{39}$$

where $\mathcal{X} : [0,\infty) \to [0,\infty)$ (forms)

$$\mathcal{X}(r) := \begin{cases} r^2 & (0 \leq r \leq 1), \\ smooth & (1 \leq r \leq 3), \\ 0 & (3 \leq r) \end{cases} \tag{40}$$

and satisfies $\mathcal{X}''(r) \leq 2$.

$$\mathcal{Y}_R(x) := \mathcal{Y}\left(\frac{|x|}{R}\right), \tag{41}$$

where $\mathcal{Y} : [0,\infty) \to [0,\infty)$ (forms)

$$\mathcal{Y}(r) := \begin{cases} 0 & \left(0 \leq r \leq \frac{1}{2}\right), \\ smooth & \left(\frac{1}{2} \leq r \leq 1\right), \\ 1 & (1 \leq r) \end{cases} \tag{42}$$

and satisfies $0 \leq \mathcal{Y}'(r) \leq 3$.

Proposition 3 (Localized virial identity, [3]) Let w be $\mathcal{X}_R$ or $\mathcal{Y}_R$ defined as Eqs. (39) and (41) respectively. For the solution u to Eq. (1), we define

$$I_w(t) := \int_{\mathbb{R}^d} w(x)|u(t,x)|^2 dx. \tag{43}$$

Then, we have

$$
\begin{aligned}
I_{w'}(t) &= 2\mathrm{Im}\int_{\mathbb{R}^d} \frac{x \cdot \nabla u}{|x|} \overline{u} w' dx, \\
I_{w''}(t) &= \int_{\mathbb{R}^d} F_1 |x \cdot \nabla u|^2 dx + 4\int_{\mathbb{R}^d} \frac{w'}{|x|} |\nabla u|^2 dx - \int_{\mathbb{R}^d} F_2 |u|^{p+1} dx \\
&\quad - \int_{\mathbb{R}^d} F_3 |u|^2 dx - 2\int_{\mathbb{R}^d} \frac{w'}{|x|} (x \cdot \nabla V)|u|^2 dx.
\end{aligned} \tag{44}
$$

where

$$
\begin{aligned}
&F_1(w, |x|) := 4\left(\frac{w''}{|x|^2} - \frac{w'}{|x|^3}\right), \qquad F_2(w, |x|) := \frac{2(p-1)}{p+1}\left(w'' + \frac{d-1}{|x|} w'\right), \\
&F_3(w, |x|) := w^{(4)} + \frac{2(d-1)}{|x|} w^{(3)} + \frac{(d-1)(d-3)}{|x|^2} w'' + \frac{(d-1)(3-d)}{|x|^3} w'.
\end{aligned} \tag{45}
$$

3. Non-radial case of main theorem

In this section, we prove (Non-radial case) for Theorem 6. First, we recall rewriting of $n_{\omega,V}$, which is given in [40].

Lemma 1 Let $d \geq 3$, $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$, and $Q_{\omega,0} \in \mathcal{G}_{\omega,0}$. Assume that V satisfies (A2) with $|\alpha| \leq 1$ and (A6). Then,

$$
S_{\omega,0}(Q_{\omega,0}) = n_{\omega,V} = \inf\{T_{\omega,V}(f) : f \in H^1(\mathbb{R}^d)\backslash\{0\}, \ K_V(f) \leq 0\} \tag{46}
$$

holds, where the functional $T_{\omega,V}$ is defined as

$$
T_{\omega,V}(f) := S_{\omega,V}(f) - \frac{1}{4} K_V(f). \tag{47}
$$

Next, we give uniform estimate of the virial functional K_V.

Lemma 2 Under the all assumptions of (Non-radial) in Theorem 6, there exists $\delta > 0$ such that

$$
\sup_{t \in (T_{\min}, T_{\max})} K_V(u(t)) \leq -\delta < 0. \tag{48}
$$

Proof: Let $\delta := 4\{S_{\omega,V}(Q_{\omega,V}) - S_{\omega,V}(u_0)\} > 0$. Applying Lemma 1, we have

$$
\begin{aligned}
S_{\omega,V}(Q_{\omega,V}) \ &\leq T_{\omega,V}(u(t)) = S_{\omega,V}(u_0) - \frac{1}{4} K_V(u(t)) \\
&= S_{\omega,V}(Q_{\omega,V}) - \frac{1}{4}\delta - \frac{1}{4} K_V(u(t)),
\end{aligned} \tag{49}
$$

which implies the desired result.

The blow-up result with $xu_0 \in L^2(\mathbb{R}^d)$ of (Non-radial case) in Theorem 1.1 follows immediately from Lemma 2.

Proof of blow-up part in (Non-radial case) for Theorem 6: We assume that the solution u exists globally in time for contradiction. When $xu_0 \in L^2(\mathbb{R}^d)$, we have Eq. (22). Combining Eq. (22) and Lemma 2, there exists $\delta > 0$ such that

$$\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 = 4K_V(u(t)) < -4\delta < 0 \tag{50}$$

for any $t \in \mathbb{R}$. Therefore, we obtain $\|xu(t)\|_{L^2}^2 < 0$ if $|t|$ is sufficiently large. However, this is contradiction.

We consider Lemmas 3 and 4 to prove blow-up or grow-up part in (Non-radial case) for Theorem 6.

Lemma 3 Let $d \geq 3$ and $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$. We assume that $u \in C([0,\infty); H^1)$ be a solution to Eq. (1) satisfying $C_0 := \sup_{t\in[0,\infty)}\|u(t)\|_{\dot{H}_x^1} < \infty$. Then, it follows that

$$\|u(t)\|_{L^2(|x|\geq R)}^2 \leq o_R(1) + \eta \tag{51}$$

for any $\eta > 0$, $R > 0$, and $t \in \left[0, \frac{\eta R}{6C_0\|u\|_{L_x^2}}\right]$, where $o_R(1)$ goes to zero as $R \to \infty$ and is independent of t.

Proof: We consider $I_{\mathcal{Y}_R}$ given in Eq. (43). Using Proposition 3,

$$\begin{aligned} I(t) &= I(0) + \int_0^t I'(s)ds \leq I(0) + \int_0^t |I'(s)|\, ds \\ &\leq I(0) + \frac{2t}{R}\|\mathcal{Y}'\|_{L^\infty} \sup_{t\in[0,\infty)} \|u(t)\|_{\dot{H}_x^1}\|u\|_{L_x^2} \leq I(0) + \frac{6C_0\|u\|_{L_x^2} t}{R} \end{aligned} \tag{52}$$

for any $t \in [0, \infty)$. By the definition of $\mathcal{Y}_R$, we have

$$I(0) = \int_{\mathbb{R}^d} \mathcal{Y}_R(x)|u_0(x)|^2 dx \leq \|u_0\|_{L^2\left(|x|\geq \frac{R}{2}\right)}^2 = o_R(1) \tag{53}$$

and hence, we obtain

$$\|u(t)\|_{L^2(|x|\geq R)}^2 \leq I(t) \leq o_R(1) + \eta. \tag{54}$$

Lemma 4 Let $d \geq 3$ and $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$. Let $u \in C([0,\infty); H^1(\mathbb{R}^d))$ be a solution to Eq. (1). Then, for $q \in (p+1, 2^*)$, there exist constants $C = C(q, \|u_0\|_{L^2}, C_0) > 0$ and $\theta_q > 0$ such that the estimate

$$I_{\mathcal{X}_R}{}'{}'(t) \leq 4K_V(u(t)) + C\|u(t)\|_{L^2(R\leq|x|)}^{(p+1)\theta_q} + \frac{C}{R^2} \tag{55}$$

holds for any $R > 0$ and $t \in [0, \infty)$, where $\theta_q := \frac{2\{q-(p+1)\}}{(p+1)(q-2)} \in \left(0, \frac{2}{p+1}\right)$, C_0 is given in Lemma 3, and $I_{\mathcal{X}_R}$ is defined as Eq. (43).

Proof: Using Proposition 3, we have

$$I''_{\mathcal{X}_R}(t) = 4K_V(u(t)) + \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 + \mathcal{R}_4, \tag{56}$$

where $\mathcal{R}_k = \mathcal{R}_k(t)$ $(k = 1,2,3,4)$ are defined as

$$\begin{aligned} \mathcal{R}_1 := 4\int_{\mathbb{R}^d} &\left\{\frac{1}{|x|^2}\mathcal{X}''\left(\frac{r}{R}\right) - \frac{R}{|x|^3}\mathcal{X}'\left(\frac{|x|}{R}\right)\right\}|x\cdot\nabla u|^2 dx \\ +4\int_{\mathbb{R}^d} &\left\{\frac{R}{|x|}\mathcal{X}'\left(\frac{|x|}{R}\right) - 2\right\}|\nabla u(t,x)|^2 dx, \end{aligned} \tag{57}$$

$$\mathcal{R}_2 := -\frac{2(p-1)}{p+1}\int_{\mathbb{R}^d}\left\{\mathcal{X}''\left(\frac{|x|}{R}\right)+\frac{(d-1)R}{|x|}\mathcal{X}'\left(\frac{|x|}{R}\right)-2d\right\}|u(t,x)|^{p+1}dx, \tag{58}$$

$$\begin{aligned}\mathcal{R}_3 := &-\int_{\mathbb{R}^d}\left\{\frac{1}{R^2}\mathcal{X}^{(4)}\left(\frac{|x|}{R}\right)+\frac{2(d-1)}{R|x|}\mathcal{X}^{(3)}\left(\frac{|x|}{R}\right)+\frac{(d-1)(d-3)}{|x|^2}\mathcal{X}\prime'\left(\frac{|x|}{R}\right)\right.\\ &\left.+\frac{(d-1)(3-d)R}{|x|^3}\mathcal{X}'\left(\frac{|x|}{R}\right)\right\}|u(t,x)|^2dx,\end{aligned} \tag{59}$$

$$\mathcal{R}_4 := 2\int_{R\le|x|}\left\{2-\frac{R}{|x|}\mathcal{X}'\left(\frac{|x|}{R}\right)\right\}(x\cdot\nabla V)|u(t,x)|^2dx. \tag{60}$$

We set

$$\Omega := \left\{x\in\mathbb{R}^d : \frac{1}{|x|^2}\mathcal{X}''\left(\frac{|x|}{R}\right)-\frac{R}{|x|^3}\mathcal{X}'\left(\frac{|x|}{R}\right)\le 0\right\}. \tag{61}$$

By the inequality $\mathcal{X}'\left(\frac{|x|}{R}\right)\le\frac{2|x|}{R}$, we have

$$\mathcal{R}_1 \le 4\int_{\Omega^c}\left\{\mathcal{X}''\left(\frac{r}{R}\right)-2\right\}|\nabla u(t,x)|^2dx\le 0, \tag{62}$$

where Ω^c denotes a complement of Ω.

Next, we estimate $\mathcal{R}_2$. Applying Hölder's inequality and Sobolev's embedding, we have

$$\begin{aligned}\mathcal{R}_2 &\le C\|u(t)\|_{L^{p+1}(R\le|x|)}^{p+1}\le C\|u(t)\|_{L^q(R\le|x|)}^{(p+1)(1-\theta_q)}\|u(t)\|_{L^2(R\le|x|)}^{(p+1)\theta_q}\\ &\le C\|u(t)\|_{H^1}^{(p+1)(1-\theta_q)}\|u(t)\|_{L^2(R\le|x|)}^{(p+1)\theta_q}\le C\|u(t)\|_{L^2(R\le|x|)}^{(p+1)\theta_q}.\end{aligned} \tag{63}$$

Next, we estimate $\mathcal{R}_3$.

$$\mathcal{R}_3 \le \frac{C}{R^2}\|u(t)\|_{L^2(R\le|x|)}^2\le\frac{C}{R^2}. \tag{64}$$

Finally, $\mathcal{R}_4$ is estimated as $\mathcal{R}_4\le 0$ by $\mathcal{X}'\left(\frac{|x|}{R}\right)\le\frac{2|x|}{R}$ and $x\cdot\nabla V\le 0$, which completes the proof of the lemma.

Proof of blow-up or grow-up part in (Non-radial case) for Theorem 6. We assume that

$$T_{\max}=\infty \quad \text{and} \quad \sup_{t\in[0,\infty)}\|u(t)\|_{\dot{H}_x^1}<\infty \tag{65}$$

for contradiction. By Lemmas 2, 3, and 4, there exists $\delta>0$ such that

$$I''_{\mathcal{X}_R}(s)\le -4\delta + C\|u(s)\|_{L_x^2(R\le|x|)}^{(p+1)\theta_q}+\frac{C}{R^2}\le -4\delta + C\eta^{\frac{(p+1)\theta_q}{2}}+o_R(1) \tag{66}$$

for any $\eta>0$, $R>0$, and $s\in\left[0,\frac{\eta R}{6C_0\|u_0\|_{L^2}}\right]$. We take $\eta=\eta_0>0$ sufficiently small such as

$$C\eta_0^{\frac{(p+1)\theta_q}{2}} \le 2\delta. \tag{67}$$

and set

$$T = T(R) := \alpha_0 R := \frac{\eta_0 R}{6C_0 \|u_0\|_{L^2}}. \tag{68}$$

Applying Eq. (67), integrating Eq. (66) over $s \in [0, t]$, and integrating over $t \in [0, T]$, we have

$$\begin{aligned} I_{\mathcal{X}_R}(T) &\le I_{\mathcal{X}_R}(0) + I'_{\mathcal{X}_R}(0)T + \frac{1}{2}(-2\delta + o_R(1))T^2 \\ &= I_{\mathcal{X}_R}(0) + I'_{\mathcal{X}_R}(0)\alpha_0 R + \frac{1}{2}(-2\delta + o_R(1))\alpha_0^2 R^2. \end{aligned} \tag{69}$$

Here, we can see

$$I_{\mathcal{X}_R}(0) = o_R(1)R^2 \quad \text{and} \quad I'_{\mathcal{X}_R}(0) = o_R(1)R. \tag{70}$$

Indeed, we get

$$I_{\mathcal{X}_R}(0) \le R\|u_0\|^2_{L^2\left(|x| \le \sqrt{R}\right)} + cR^2\|u_0\|_{L^2\left(\sqrt{R} \le |x|\right)} = o_R(1)R^2, \tag{71}$$

and

$$I'_{\mathcal{X}_R}(0) \le 4\sqrt{R}\|u_0\|_{\dot{H}^1}\|u_0\|_{L^2\left(|x| \le \sqrt{R}\right)} + cR\|u_0\|_{\dot{H}^1}\|u_0\|_{L^2\left(\sqrt{R} \le |x|\right)} = o_R(1)R. \tag{72}$$

Combining Eqs. (69) and (70), we get

$$I_{\mathcal{X}_R}(T) \le \left(o_R(1) - \delta\alpha_0^2\right)R^2. \tag{73}$$

We take $R > 0$ such as $o_R(1) - \delta\alpha_0^2 < 0$. However, this contradicts $I_{\mathcal{X}_R}(T) \ge 0$.

4. Radial case of main theorem

In this section, we prove (Radial case) for Theorem 6. First, we introduce another characterization of $r_{\omega,V}$.

Lemma 5 Let $d \ge 3$, $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$, and $Q_{\omega,V} \in \mathcal{G}_{\omega,V,\mathrm{rad}}$. Assume that V is radially symmetric and satisfies (A3) with $|\alpha| \le 2$, (A7), and $3x \cdot \nabla V + x\nabla^2 V x^T \le 0$. Then,

$$S_{\omega,V}\left(Q_{\omega,V}\right) = r_{\omega,V} = \inf\left\{U_{\omega,V}(f) : f \in H^1_{\mathrm{rad}}\left(\mathbb{R}^d\right)\backslash\{0\},\ \ K_V(f) \le 0\right\} \tag{74}$$

holds, where the functional $U_{\omega,V}$ is defined as

$$U_{\omega,V}(f) := S_{\omega,V}(f) - \frac{1}{d(p-1)}K_V(f). \tag{75}$$

Proof: The lemma follows from proof of Lemma 1 (see [40], Lemma 4.3) combined $2\omega + 2V + x \cdot \nabla V \ge 0$.

Proof of (Radial case) for Theorem 6*.* Assume that the solution u to Eq. (1) exists globally in time for contradiction. We consider $I_{\mathcal{X}_R}$ again and recall

$$I''_{\mathcal{X}_R}(t) = 4K_V(u(t)) + \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 + \mathcal{R}_4, \tag{76}$$

where $\mathcal{R}_k$ $(1 \le k \le 4)$ are defined as Eqs. (57) $\sim$ (60). We use same estimates with proof of blow-up or grow-up for $\mathcal{R}_1$, $\mathcal{R}_3$, and $\mathcal{R}_4$. Applying Proposition 2 and the Young's inequality, we have

$$\begin{aligned}\mathcal{R}_2 &\le \frac{C}{R^{\frac{(d-1)(p-1)}{2}}} \|u(t)\|_{L^2(R\le|x|)}^{\frac{p+3}{2}} \|u(t)\|_{\dot{H}^1(R\le|x|)}^{\frac{p-1}{2}} \\ &\le \frac{C}{R^{\frac{2(d-1)(p-1)}{5-p}} \varepsilon^{\frac{4}{5-p}}} \|u\|_{L^2}^{\frac{2(p+3)}{5-p}} + 2\{d(p-1)-4\}\varepsilon\|u\|_{\dot{H}^1}^2 \\ &\le \frac{C}{R^{\frac{2(d-1)(p-1)}{5-p}} \varepsilon^{\frac{4}{5-p}}} \|u\|_{L^2}^{\frac{2(p+3)}{5-p}} + 4d(p-1)\varepsilon U_{\omega,V}(u)\end{aligned} \tag{77}$$

for each positive $\varepsilon > 0$, which is chosen later. Collecting these estimates, we have

$$\begin{aligned}&I''_{\mathcal{X}_R}(t) \\ &\le 4K_V(u) + 4d(p-1)\varepsilon U_{\omega,V}(u) + \frac{C}{R^{\frac{2(d-1)(p-1)}{5-p}} \varepsilon^{\frac{4}{5-p}}} + \frac{C}{R^2} \\ &= 4d(p-1)\{S_{\omega,V}(u) - U_{\omega,V}(u)\} + 4d(p-1)\varepsilon U_{\omega,V}(u) + \frac{C}{R^{\frac{2(d-1)(p-1)}{5-p}} \varepsilon^{\frac{4}{5-p}}} + \frac{C}{R^2} \\ &< 4d(p-1)(1-\delta)S_{\omega,V}(Q_{\omega,V}) + 4d(p-1)(\varepsilon-1)U_{\omega,V}(u) + \frac{C}{R^{\frac{2(d-1)(p-1)}{5-p}} \varepsilon^{\frac{4}{5-p}}} + \frac{C}{R^2} \\ &\le 4d(p-1)(\varepsilon-\delta)S_{\omega,V}(Q_{\omega,V}) + \frac{C}{R^{\frac{2(d-1)(p-1)}{5-p}} \varepsilon^{\frac{4}{5-p}}} + \frac{C}{R^2},\end{aligned} \tag{78}$$

where the second inequality is used $S_{\omega,V}(u) < (1-\delta)S_{\omega,V}(Q_{\omega,V})$ for some $\delta \in (0,1)$ and the third inequality is used $S_{\omega,V}(Q_{\omega,V}) \le U_{\omega,V}$ (see Lemma 5). Taking $\varepsilon \in (0,\delta)$ and sufficiently large $R > 0$, there exists $\eta > 0$ such that $I''_{\mathcal{X}_R}(t) < -\eta < 0$ for each $t \in \mathbb{R}$. However, this inequality implies that if $|t|$ is sufficiently large, then . This is contradiction and hence, we complete the proof.

5. Conclusions

In this chapter, our main result is Theorem 6. Combining the main result and a previous result (Theorem 5), we can classify time behavior of solutions to Eq. (1) with initial data below the ground state in the sense of their action $S_{\omega,V}$ by using sign of the virial functional for the initial data. More precisely, for the solution $u(t)$ with $S_{\omega,V}(u_0) < S_{\omega,0}(Q_{\omega,0})$, if $K_V(u_0) \ge 0$ then u is bounded in $H^1(\mathbb{R}^d)$ and if $K_V(u_0) < 0$ then u is unbounded in $H^1(\mathbb{R}^d)$. In addition, for the radial solution $u(t)$ with $S_{\omega,V}(u_0) < S_{\omega,V}(Q_{\omega,V})$, if $K_V(u_0) \ge 0$, then u exists globally in time and if $K_V(u_0) < 0$ then u blows up.

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Conflict of interest

The authors declare no conflict of interest.

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