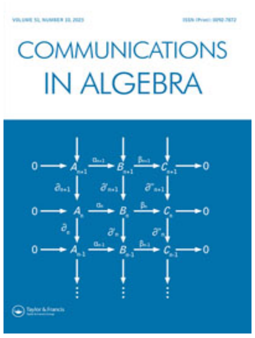


Communications in Algebra

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/lagb20

# Finite presentation of finitely determined modules

**Eero Hyry & Markus Klemetti**

**To cite this article:** Eero Hyry & Markus Klemetti (2023): Finite presentation of finitely determined modules, Communications in Algebra, DOI: 10.1080/00927872.2023.2234034

**To link to this article:** https://doi.org/10.1080/00927872.2023.2234034



Published online: 17 Jul 2023.

COMMUNICATIONS IN ALGEBRA®
https://doi.org/10.1080/00927872.2023.2234034



# Finite presentation of finitely determined modules

Eero Hyry[a] and Markus Klemetti[b]

[a]Faculty of Information Technology and Communication Sciences, Tampere University, Tampere, Finland; [b]Tampere University, Tampere, Finland

**ABSTRACT**
Motivated by topological data analysis, we study in this article certain notions of "tameness" for modules over posets. In particular, we show that after adding infinitary points the so called finitely determined modules become finitely presented.

**ARTICLE HISTORY**
Received 3 March 2023
Accepted 30 June 2023
Communicated by Alberto Facchini

**KEYWORDS**
Finitely determined; finitely presented; persistence module

**2020 MATHEMATICS SUBJECT CLASSIFICATION**
13E15; 55N31

## 1. Introduction

Miller defined in [6, p. 186, Def. 2.1] the notion of a "positively $a$-determined" module, where $a \in \mathbb{N}^n$. Positively $a$-determined modules are $\mathbb{N}^n$-graded modules over a polynomial ring in $n$ variables over a field. They are in a certain way determined by the homogenous components of degrees from the interval $[0, a] \subseteq \mathbb{N}^n$. Finitely determined modules are $\mathbb{Z}^n$-graded modules determined by the homogeneous components of degrees inside some interval in $\mathbb{Z}^n$. Whereas positively determined modules are always finitely generated, finitely determined modules need not be finitely generated in general.

Finitely determined modules have recently been studied by Miller in the context of topological data analysis (see [7]). Topological data analysis is a field of mathematics studying the shape of data by associating filtered topological spaces to data sets. The homological properties which "persist" along the filtration are considered important. By taking homology with coefficients in a field, one obtains a diagram of vector spaces and linear maps. This diagram is called a persistence module. In the standard case, the filtration is indexed by $\mathbb{Z}$ or $\mathbb{R}$, but the indexing set can be any poset.

We can think of the persistence module as a module over a poset. More formally, a persistence module over a poset $\mathcal{C}$ with coefficients in a field $k$ is a functor from $\mathcal{C}$, interpreted as a category, to the category of $k$-vector spaces. For the sake of generality, instead of the field $k$, we prefer in this article to work with any commutative ring $R$. Following the terminology of representation theory, we call a functor $\mathcal{C} \to R$-**Mod** an $R\mathcal{C}$-module. In this terminology, a persistence module is then a $k\mathcal{C}$-vector space. Note that the category of $R\mathbb{Z}^n$-modules is isomorphic to the category of $\mathbb{Z}^n$-graded modules over a polynomial ring in $n$ variables over $R$ (see [1, p. 78, Theorem 1]).

Persistence modules need not be finitely presented. For computational reasons, one has therefore introduced several notions of "tameness" for them. As a straightforward generalization of a positively determined module, we defined in [4, p. 22, Definition 4.1] an $R\mathcal{C}$-module $M$ to be $S$-*determined* if there

**CONTACT** Eero Hyry eero.hyry@tuni.fi Faculty of Information Technology and Communication Sciences, Tampere University, Kanslerinrinne 1 (Pinni B), Tampere, 33100, Finland.

exists a subset $S \subseteq \mathcal{C}$ such that $\mathrm{Supp}(M) \subseteq \uparrow S$, and for every $c \leq d$ in $\mathcal{C}$ the implication

$$S \cap \downarrow c = S \cap \downarrow d \;\Rightarrow\; M(c \leq d) \text{ is an isomorphism}$$

holds. For any $T \subseteq \mathcal{C}$, we use the usual notations

$$\uparrow T := \{c \in \mathcal{C} \mid t \leq c \text{ for some } t \in T\}$$

and

$$\downarrow T := \{c \in \mathcal{C} \mid c \leq t \text{ for some } t \in T\}$$

for the upset generated and the downset cogenerated by $T$, respectively.

As defined in [7, p. 24, Definition 4.1] an encoding of an $R\mathcal{C}$-module $M$ by a poset $\mathcal{D}$ is poset morphism $f\colon \mathcal{C} \to \mathcal{D}$ with an $R\mathcal{D}$-module $N$ such that the restriction $\mathrm{res}_f N$ is isomorphic to $M$. Suppose now that $S \subseteq \mathcal{C}$ is a finite set that is strongly bounded from above. The latter condition means that every finite subset $S \subseteq \mathcal{C}$ has a unique minimal upper bound in $\mathcal{C}$. We will consider the set $\tilde{S}$ of all minimal upper bounds of the subsets of $S$. We are going to define a functor $\alpha\colon \mathcal{C} \to \tilde{S}$ by mapping an element of $\mathcal{C}$ to the unique minimal upper bound of the elements of $S$ below it. In our main result, Theorem 3.4, we will then prove that $M$ is $S$-determined for some finite $S \subseteq \mathcal{C}$ if and only if $\alpha$ is an encoding of $M$.

As a consequence of Theorem 3.4 we can show in Theorem 5.4 that after adding infinitary points to $\mathbb{Z}^n$, finitely determined modules in fact become finitely presented. We follow here an idea due to Perling (see [8, p. 16]). We also show in Proposition 5.8 that our terminology is compatible with that of admissible posets used in [8].

## 2. Preliminaries

Throughout this article we use the terminology of category theory. We will always assume that $\mathcal{C}$ is a small category and $R$ a commutative ring. For any set $X$, we denote by $R[X]$ the free $R$-module generated by $X$. An $R\mathcal{C}$-module is a functor from $\mathcal{C}$ to the category of $R$-modules. A morphism between $R\mathcal{C}$-modules is a natural transformation. For more details on $R\mathcal{C}$-modules, we refer to [5] and [2].

Recall first that an $R\mathcal{C}$-module $M$ is called

- *finitely generated* if there exists an epimorphism

$$\bigoplus_{i \in I} R[\mathrm{Mor}_{\mathcal{C}}(c_i, -)] \to M,$$

where $I$ is a finite set, and $c_i \in \mathcal{C}$ for all $i \in I$;
- *finitely presented* if there exists an exact sequence

$$\bigoplus_{j \in J} R[\mathrm{Mor}_{\mathcal{C}}(d_j, -)] \to \bigoplus_{i \in I} R[\mathrm{Mor}_{\mathcal{C}}(c_i, -)] \to M \to 0,$$

where $I$ and $J$ are finite sets, and $c_i, d_j \in \mathcal{C}$ for all $i \in I$ and $j \in J$.

See, for example, [9].

Let $\varphi\colon \mathcal{S} \to \mathcal{C}$ be a functor between small categories. Recall that the *restriction* $\mathrm{res}_\varphi\colon R\mathcal{C}\text{-}\mathbf{Mod} \to R\mathcal{S}\text{-}\mathbf{Mod}$ is the functor defined by precomposition with $\varphi$, and the *induction* $\mathrm{ind}_\varphi\colon R\mathcal{S}\text{-}\mathbf{Mod} \to R\mathcal{C}\text{-}\mathbf{Mod}$ is its left Kan extension along $\varphi$. The induction is the left adjoint of the restriction. The counit of this adjunction gives us for every $R\mathcal{C}$-module $M$ the *canonical morphism*

$$\mu_M\colon\ \mathrm{ind}_\varphi \mathrm{res}_\varphi M \to M.$$

More explicitly, for any $R\mathcal{C}$-module $M$ and $R\mathcal{S}$-module $N$, we have the pointwise formulas

$$(\mathrm{res}_\varphi M)(s) = M(\varphi(s)) \quad \text{and} \quad (\mathrm{ind}_\varphi N)(c) = \operatorname*{colim}_{(t,u) \in (\varphi/c)} N(t)$$

for all $s \in \mathcal{S}$ and $c \in \mathcal{C}$. Here $(\varphi/c)$ denotes the slice category. Its objects are pairs $(s, u)$, where $s \in \mathcal{S}$ and $u\colon \varphi(s) \to c$ is a morphism in $\mathcal{C}$. For $(s, u), (t, v) \in \mathrm{Ob}(\varphi/c)$, a morphism $(s, u) \to (t, v)$ is a morphism

$f: s \to t$ in $S$ with $v\varphi(f) = u$. We will typically assume that $S$ is a full subcategory of $\mathcal{C}$ and that $\varphi$ is the inclusion functor. In this case, we use the notations $\mathrm{res}_S$ and $\mathrm{ind}_S$ instead of $\mathrm{res}_\varphi$ and $\mathrm{ind}_\varphi$. If $\mathcal{C}$ is also a poset, the latter formula yields

$$(\mathrm{ind}_S N)(c) = \operatorname*{colim}_{t\in S,\, t\leq c} N(t).$$

Let $\mathcal{C}$ be a small category and $S \subseteq \mathcal{C}$ a full subcategory. An $R\mathcal{C}$-module $M$ is said to be $S$-*generated* if the natural morphism

$$\rho_M\colon \bigoplus_{s\in S} M(s)[\mathrm{Mor}_{\mathcal{C}}(s,-)] \to M$$

is an epimorphism. Here

$$M(s)[\mathrm{Mor}_{\mathcal{C}}(s,-)] := M(s) \otimes_R R[\mathrm{Mor}_{\mathcal{C}}(s,-)],$$

where the tensor product is taken pointwise. Since the morphism $\rho_M$ factors through the canonical morphism $\mu_M$, we see that $M$ is $S$-generated if and only if $\mu_M$ is an epimorphism.

Following [3, p. 13, Proposition 2.14], we say that $M$ is $S$-*presented* if it is $S$-generated and the following condition holds: Given an exact sequence of $R\mathcal{C}$-modules

$$0 \to L \to N \to M \to 0,$$

where $N$ is $S$-generated, then $L$ is $S$-generated. It is shown in [3, p. 13, Proposition 2.14], that $M$ is $S$-presented if and only if $\mu_M$ is an isomorphism.

## 3. Modules over strongly bounded posets

In order to prove Theorem 3.4, we need to recall some order theory. In the following, $\mathcal{C}$ always denotes a poset.

**Notation 1.** Let $S \subseteq \mathcal{C}$ be a finite subset. We denote the set of minimal upper bounds of $S$ by $\mathrm{mub}(S)$. If $S$ is finite, we set

$$\hat{S} := \bigcup_{\emptyset\neq S'\subseteq S} \mathrm{mub}(S').$$

In other words, $\hat{S}$ is the set of minimal upper bounds of non-empty subsets of $S$.

We say that the poset $\mathcal{C}$ is *strongly bounded from above* if every finite $S \subseteq \mathcal{C}$ has a unique minimal upper bound in $\mathcal{C}$. If $\mathcal{C}$ is strongly bounded from above, then $\hat{S}$ is finite. The condition of $\mathcal{C}$ being strongly bounded from above is equivalent to $\mathcal{C}$ being a bounded join-semilattice. Also note that if $\mathcal{C}$ is strongly bounded from above, then $\mathcal{C}$ is weakly bounded from above and mub-complete, as defined in [4, p. 23, Definitions 4.5 and 4.6].

Let $\mathcal{C}$ be strongly bounded from above, and let $S \subseteq \mathcal{C}$ be a finite set. From now on, we consider $\mathrm{mub}(S)$ as an element of $\mathcal{C}$, and not as a (one element) set. In particular, every element of $\hat{S}$ is then of the form $\mathrm{mub}(S')$, where $S' \subseteq S$ is a non-empty subset. Viewing $\mathcal{C}$ as a join-semilattice, we have the join-operation

$$a \vee b := \mathrm{mub}(a, b) := \mathrm{mub}(\{a, b\}).$$

Extending this operation to finite sets, we get an operation that coincides with taking minimal upper bounds.

**Lemma 3.1.** *Let $\mathcal{C}$ be strongly bounded from above, and let $S \subseteq \mathcal{C}$ be a finite subset. Then $\hat{\hat{S}} = \hat{S}$.*

*Proof.* An element $s \in \hat{\hat{S}}$ may be written as

$$s = \mathrm{mub}(\mathrm{mub}(S_1), \dots, \mathrm{mub}(S_n)),$$

where $S_1, \dots, S_n$ are (finite) non-empty subsets of $S$. Since the join-operation is associative in join-semilattices, we see that

$$s = \bigvee_{i=1}^{n} (\bigvee S_i) = \bigvee (\bigcup_{i=1}^{n} S_i).$$

This implies that $s = \mathrm{mub}(S_1 \cup \dots \cup S_n)$, which belongs to $\hat{S}$ by definition. □

Assume that $\mathcal{C}$ is strongly bounded from above. Then $\mathcal{C}$ has a minimum element $\min(\mathcal{C}) = \mathrm{mub}(\emptyset)$. Let $S \subseteq \mathcal{C}$ be a finite subset. Denote

$$\tilde{S} := \hat{S} \cup \{\min(\mathcal{C})\}.$$

We define a poset morphism $\alpha_S \colon \mathcal{C} \to \tilde{S}$ by setting

$$\alpha_S(c) = \mathrm{mub}(S \cap \downarrow c)$$

for every $c \in \mathcal{C}$. In other words, $\alpha_S$ maps each $c \in \mathcal{C}$ to the minimal upper bound of the elements of $S$ below it. To show that $\alpha_S$ actually is a poset morphism, suppose that $c \le d$ in $\mathcal{C}$. Then $S \cap \downarrow c \subseteq S \cap \downarrow d$, which implies that $\alpha_S(c) \le \alpha_S(d)$.

**Proposition 3.2.** *Let $\mathcal{C}$ be strongly bounded from above, and let $S \subseteq \mathcal{C}$ be a finite subset. Then $\alpha_S = \alpha_{\hat{S}} = \alpha_{\tilde{S}}$.*

*Proof.* Using Lemma 3.1, we first note that $\tilde{\hat{S}} = \tilde{S}$ and $\tilde{\tilde{S}} = \tilde{S}$. Let $c \in \mathcal{C}$. We claim that

$$\mathrm{mub}(S \cap \downarrow c) = \mathrm{mub}(\hat{S} \cap \downarrow c) = \mathrm{mub}(\tilde{S} \cap \downarrow c).$$

The latter equation follows from the fact that for all subsets $T \subseteq \mathcal{C}$, we have $\mathrm{mub}(T) = \mathrm{mub}(T \cup \{\min(\mathcal{C})\})$. In particular, $\mathrm{mub}(T) = \min(\mathcal{C})$, if $T = \emptyset$.

For the first equation, since $S \subseteq \hat{S}$, we have $\mathrm{mub}(S \cap \downarrow c) \le \mathrm{mub}(\hat{S} \cap \downarrow c)$. On the other hand, $\hat{S} \cap \downarrow c$ is a subset of $\hat{S}$. Thus $\mathrm{mub}(\hat{S} \cap \downarrow c) \in \hat{\hat{S}} = \hat{S}$, where the equation follows from Lemma 3.1. By the definition of $\hat{S}$, we may now write

$$\mathrm{mub}(\hat{S} \cap \downarrow c) = \mathrm{mub}(s_1, \dots, s_n),$$

where $s_1, \dots, s_n \in S$. Furthermore, $\mathrm{mub}(\hat{S} \cap \downarrow c) \le c$, so we also have $s_1, \dots, s_n \le c$. This implies that

$$\mathrm{mub}(s_1, \dots, s_n) \le \mathrm{mub}(S \cap \downarrow c),$$

which completes the proof. □

Encouraged by Proposition 3.2, we will just write $\alpha$ instead of $\alpha_S$, if there is no risk of confusion. Before moving on to the main theorem of this section, we require one more lemma.

**Lemma 3.3.** *Let $\mathcal{C}$ be strongly bounded from above, and let $S \subseteq \mathcal{C}$ be a finite subset. Then $\hat{S} \cap \downarrow \alpha(c) = \hat{S} \cap \downarrow c$ for all $c \in \mathcal{C}$.*

*Proof.* Let $c \in \mathcal{C}$. We immediately see that $\hat{S} \cap \downarrow \alpha(c) \subseteq \hat{S} \cap \downarrow c$, because $\alpha(c) \le c$. Suppose that $d \in \hat{S} \cap \downarrow c$. We need to show that $d \le \alpha(c)$. This follows from Proposition 3.2, because now

$$\alpha(c) = \alpha_{\hat{S}}(c) = \mathrm{mub}(\hat{S} \cap \downarrow c).$$

□

Let $\mathcal{C}$ be strongly bounded from above, let $M$ be an $R\mathcal{C}$-module, and let $S \subseteq \mathcal{C}$ be a finite subset. The morphism $\alpha$ gives rise to a natural transformation

$$T_\alpha \colon \operatorname{res}_\alpha \operatorname{res}_{\tilde{S}} M \to M,$$

where for any $c \in \mathcal{C}$, $T_{\alpha,c}$ is the morphism

$$M(\alpha(c) \leq c)\colon (\operatorname{res}_\alpha \operatorname{res}_{\tilde{S}} M)(c) = M(\alpha(c)) \to M(c).$$

We are now able to prove our main result

**Theorem 3.4.** *Let $\mathcal{C}$ be strongly bounded from above, and let $M$ be an $R\mathcal{C}$-module. Given a finite subset $S \subseteq \mathcal{C}$, the following conditions are equivalent:*
*1) For all $c \leq d$ in $\mathcal{C}$,*

$$S \cap \downarrow c = S \cap \downarrow d \;\Rightarrow\; M(c \leq d) \text{ is an isomorphism;}$$

*2) $T_\alpha\colon \operatorname{res}_\alpha \operatorname{res}_{\tilde{S}} M \to M$ is an isomorphism;*
*3) $\alpha$ is an encoding of $M$.*
*If $\min(\mathcal{C}) \in S$, then condition 1) says that $M$ is $S$-determined.*

*Proof.* Suppose first that 1) holds. We can safely assume that $S$ includes the minimum element of $\mathcal{C}$, so that $\operatorname{Supp}(M) \subseteq \uparrow S = \mathcal{C}$. This will not affect the sets $\hat{S}$ or $\tilde{S}$, nor the functor $\alpha$. Therefore $M$ is $S$-determined. We have proved in [4, p. 25, Corollary 4.13] that an $S$-determined module is $\hat{\hat{S}}$-presented. Lemma 3.1 now tells us that $M$ is $\hat{S}$-presented. So $M \cong \operatorname{ind}_{\hat{S}} \operatorname{res}_{\hat{S}} M$. This implies that for $c \in \mathcal{C}$,

$$(\operatorname{res}_\alpha \operatorname{res}_{\tilde{S}} M)(c) = M(\alpha(c)) \cong \operatorname*{colim}_{d \leq \alpha(c),\, d \in \hat{S}} M(d).$$

Furthermore, by Lemma 3.3, we get

$$\operatorname*{colim}_{d \leq \alpha(c),\, d \in \hat{S}} M(d) = \operatorname*{colim}_{d \leq c,\, d \in \hat{S}} M(d) = (\operatorname{ind}_{\hat{S}} \operatorname{res}_{\hat{S}} M)(c) \cong M(c).$$

If 2) holds, we immediately see that the functor $\alpha$ with the $R\tilde{S}$-module $\operatorname{res}_{\tilde{S}} M$ is an encoding of $M$.

Finally, suppose that 3) is true. Assume that $c \leq d$ and $S \cap \downarrow c = S \cap \downarrow d$. We need to show that $M(c \leq d)$ is an isomorphism. Since $\alpha$ is an encoding of $M$, there exists an $R\tilde{S}$-module $N$ such that $\operatorname{res}_\alpha N \cong M$. Here $\operatorname{res}_\alpha N(c \leq d)$ is the morphism $N(\alpha(c) \leq \alpha(d))$. We note that

$$\alpha(c) = \operatorname{mub}(S \cap \downarrow c) = \operatorname{mub}(S \cap \downarrow d) = \alpha(d),$$

so the morphism $\operatorname{res}_\alpha N(c \leq d)$ is an isomorphism. Thus $M(c \leq d)$ is an isomorphism. Therefore 1) holds true. □

## 4. Adding infinitary points

One approach to understand $R\mathbb{Z}^n$-modules better is to expand the set $\mathbb{Z}^n$ to include points at infinity. This idea has been utilized by Perling in [8]. Set $\overline{\mathbb{Z}} := \mathbb{Z} \cup \{-\infty\}$. It is easy to see that $\overline{\mathbb{Z}}^n$ inherits a poset structure from $\mathbb{Z}^n$. Any $R\mathbb{Z}^n$-module $M$ can be naturally extended to an $R\overline{\mathbb{Z}}^n$-module $\overline{M}$ by setting

$$\overline{M}(c) = \lim_{d \geq c,\, d \in \mathbb{Z}^n} M(d)$$

for all $c \in \overline{\mathbb{Z}}^n$. More formally, this is the coinduction of $M$ with respect to the inclusion $\mathbb{Z}^n \to \overline{\mathbb{Z}}^n$. The functor $M \mapsto \overline{M}$ establishes an equivalence of categories between the category $R\mathbb{Z}^n$-**Mod** and its essential image in $R\overline{\mathbb{Z}}^n$-**Mod**.

Let $S \subseteq \overline{\mathbb{Z}}^n$ be a finite non-empty subset. We denote by $\operatorname{mlb}(S)$ the (unique) maximal lower bound of $S$. In this section, we will define a morphism $\beta$ "dual" to $\alpha$. The idea is to map an element to the maximal lower bound of the elements of $S$ above it. The morphism $\beta$ will play a crucial role in the proof of our main result, Theorem 5.4.

We restrict ourselves to *cartesian* subsets of $\overline{\mathbb{Z}}^n$, i.e. subsets of the form $S = S_1 \times \cdots \times S_n$, where $S_1, \ldots, S_n$ are subsets of $\overline{\mathbb{Z}}$. In this situation, we can calculate $\alpha$ and $\beta$ coordinatewise. We begin with the following observation.

**Proposition 4.1.** *Let $p_i \colon \overline{\mathbb{Z}}^n \to \overline{\mathbb{Z}}$ be the canonical projection for every $i \in \{1, \dots, n\}$, and let $S \subseteq \overline{\mathbb{Z}}^n$ be a finite non-empty subset. Then*
*1)* $\operatorname{mub}(S) = (\max(p_1(S)), \dots, \max(p_n(S)))$*;*
*2)* $\operatorname{mlb}(S) = (\min(p_1(S)), \dots, \min(p_n(S)))$*.*

*Proof.* Since both 1) and 2) are proved in the same way, we will only present the proof of 1) here. Let $i \in \{1, \dots, n\}$. The existence of $\max(p_i(S))$ follows from the fact that $p_i(S)$ is non-empty, linearly ordered and finite. Write

$$d = (d_1, \dots, d_n) := \operatorname{mub}(S).$$

We will show that $d_i = \max(p_i(S))$. First, since $d$ is an upper bound of $S$ and the canonical projection $p_i$ preserves order, we see that $d_i = p_i(d) \geq \max(p_i(S))$. Secondly, if $\max(p_i(S)) < d_i$, then

$$d' := (d_1, \dots, d_{i-1}, \max(p_i(S)), d_{i+1}, \dots, d_n)$$

is an upper bound of $S$ such that $d' < d$, contradicting the minimality of $d$. Thus $d_i = \max(p_i(S))$. □

Let $S := S_1 \times \cdots \times S_n \subseteq \overline{\mathbb{Z}}^n$ be a cartesian subset. We write

$$\overline{S} = \tilde{S}_1 \times \cdots \times \tilde{S}_n,$$

where $\widetilde{S_i} = S_i \cup \{-\infty\} \subseteq \overline{\mathbb{Z}}$ for all $i \in \{1, \dots, n\}$. Note that if $S$ is finite, then so is $\overline{S}$.

**Example 4.2.** Let $a \leq b$ in $\mathbb{Z}^n$. We write $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$. For the closed interval

$$[a, b] = \{c \in \mathbb{Z}^n \mid a \leq c \leq b\} = [a_1, b_1] \times \cdots \times [a_n, b_n],$$

we have

$$\begin{aligned}\overline{[a, b]} &= \widetilde{[a_1, b_1]} \times \cdots \times \widetilde{[a_n, b_n]} \\ &= \{(c_1, \dots, c_n) \mid a_i \leq c_i \leq b_i \text{ or } c_i = -\infty \ \ (i \in \{1, \dots, n\})\}.\end{aligned}$$

We now have

**Lemma 4.3.** *Let $S := S_1 \times \cdots \times S_n \subseteq \overline{\mathbb{Z}}^n$ be a finite cartesian subset, and let $T \subseteq S$ be a finite non-empty subset. Then*
*1)* $\operatorname{mub}(T) \in S$*;*
*2)* $\operatorname{mlb}(T) \in S$*;*
*3)* $\tilde{\overline{S}} = \overline{S}$*.*

*Proof.* To prove 1), let $p_i$ be the canonical projection $\overline{\mathbb{Z}}^n \to \overline{\mathbb{Z}}$ for all $i \in \{1, \dots, n\}$. From Proposition 4.1 1), we get that

$$\operatorname{mub}(T) = (\max(p_1(T)), \dots, \max(p_n(T))).$$

Thus $\operatorname{mub}(T) \in S$, because $p_i(T) \subseteq p_i(S) = S_i$ for all $i \in \{1, \dots, n\}$.

Next, the proof for 2) is done in the same way as 1), this time using Proposition 4.1 2).

Finally, for 3), we note that $\overline{S}$ is finite and cartesian, so 1) implies $\hat{\overline{S}} = \overline{S}$. Since $\overline{S}$ already contains the minimum element of $\overline{\mathbb{Z}}^n$, we get

$$\tilde{\overline{S}} = \hat{\overline{S}} \cup \{(-\infty, \dots, -\infty)\} = \overline{S} \cup \{(-\infty, \dots, -\infty)\} = \overline{S}.$$

□

Let $S := S_1 \times \cdots \times S_n \subseteq \overline{\mathbb{Z}}^n$ be a finite cartesian subset. Since $\tilde{\overline{S}} = \overline{S}$ by Lemma 4.3 3), we have a poset morphism $\alpha := \alpha_{\overline{S}} \colon \overline{\mathbb{Z}}^n \to \overline{S}$, where

$$\alpha(c) = \operatorname{mub}(\overline{S} \cap \downarrow c)$$

for all $c \in \overline{\mathbb{Z}}^n$. By Lemma 4.3 2), we now can define a "dual" poset morphism $\beta := \beta_S : \overline{S} \to S$ by setting

$$\beta(c) = \mathrm{mlb}(S \cap \uparrow c)$$

for all $c \in \overline{S}$. Here the set $S \cap \uparrow c$ is always non-empty, because $S$ is final in $\overline{S}$.

We can now give coordinatewise formulas for $\alpha$ and $\beta$.

**Proposition 4.4.** *We write $\alpha_i := \alpha_{\overline{S_i}}$ and $\beta_i := \beta_{S_i}$ for all $i \in \{1, \dots, n\}$. For $c := (c_1, \dots, c_n) \in \overline{\mathbb{Z}}^n$, we have*
*1) $\alpha(c) = (\alpha_1(c_1), \dots, \alpha_n(c_n))$;*
*2) if $c \in \overline{S}$, then $\beta(c) = (\beta_1(c_1), \dots, \beta_n(c_n))$.*

*Proof.* To prove 1), we will first show that

$$p_i(\overline{S} \cap \downarrow c) = \overline{S_i} \cap \downarrow c_i,$$

where $p_i : \overline{\mathbb{Z}}^n \to \overline{\mathbb{Z}}$ is the canonical projection for all $i \in \{1, \dots, n\}$. Since $p_i(\overline{S}) = \overline{S_i}$ and $p_i(\downarrow c) = \downarrow c_i$, we see that $p_i(\overline{S} \cap \downarrow c) \subseteq \overline{S_i} \cap \downarrow c_i$. For the other direction, suppose that $d \in \overline{S_i} \cap \downarrow c_i$. Then $d \leq c_i$, so we have an element

$$d' := (-\infty, \dots, -\infty, d, -\infty, \dots, -\infty) \in \overline{S} \cap \downarrow c$$

such that $p_i(d') = d$. Hence $p_i(\overline{S} \cap \downarrow c) = \overline{S_i} \cap \downarrow c_i$. Now, using this result and Proposition 4.1 1), we get

$$\begin{aligned} \alpha(c) &= \mathrm{mub}(\overline{S} \cap \downarrow c) \\ &= (\max(\overline{S_1} \cap \downarrow c_1), \dots, \max(\overline{S_n} \cap \downarrow c_n)) \\ &= (\alpha_1(c_1), \dots, \alpha_n(c_n)). \end{aligned}$$

For 2), the proof is similar. Let $c \in \overline{S}$. We will first show that

$$p_i(S \cap \uparrow c) = S_i \cap \uparrow c.$$

From $p_i(S) = S_i$ and $p_i(\uparrow c) = \uparrow c_i$, we see that $p_i(S \cap \uparrow c) \subseteq S_i \cap \uparrow c_i$. Next, suppose that $d \in S_i \cap \uparrow c_i$. Since $c \in \overline{S}$, there is an element $s := (s_1, \dots, s_n) \in S$ such that $s \geq c$. Because $d \geq c_i$ and $S$ is cartesian, we again have an element

$$d' := (s_1, \dots, s_{i-1}, d, s_{i+1}, \dots, s_n) \in S \cap \uparrow c$$

such that $p_i(d') = d$. Thus $p_i(S \cap \uparrow c) = S_i \cap \uparrow c$. To finish the proof, we use Proposition 4.1 2):

$$\begin{aligned} \beta(c) &= \mathrm{mlb}(S \cap \uparrow c) \\ &= (\min(S_1 \cap \uparrow c_1), \dots, \min(S_n \cap \uparrow c_n)) \\ &= (\beta_1(c_1), \dots, \beta_n(c_n)). \end{aligned}$$

□

We note that $\alpha$ and $\beta \circ \alpha$ are "continuous" in the following sense.

**Proposition 4.5.** *Let $c := (c_1, \dots, c_n) \in \overline{\mathbb{Z}}^n$.*
*1) If $N$ is an $R\overline{S}$-module, then*

$$\lim_{d \geq c,\, d \in \mathbb{Z}^n} N(\alpha(d)) \cong N(\alpha(c)).$$

*2) If $Q$ is an $RS$-module, then*

$$\lim_{d \geq c,\, d \in \mathbb{Z}^n} Q((\beta \circ \alpha)(d)) \cong Q((\beta \circ \alpha)(c)).$$

*Proof.* For 1), suppose that $N$ is an $R\overline{S}$-module. Let $c' := (c'_1, \dots, c'_n) \in \overline{\mathbb{Z}}^n$ as follows: For any $i \in \{1, \dots, n\}$, we set $a_i = \min(S_i \cap \mathbb{Z})$, if it exists, and

$$c'_i := \begin{cases} \max(c_i, 0), & \text{if } S_i \cap \mathbb{Z} = \emptyset; \\ \max(c_i, a_i - 1), & \text{otherwise.} \end{cases}$$

This guarantees that we always have $c \leq c'$ and $c' \in \mathbb{Z}^n$. With the notation from Proposition 4.4, we may write

$$\alpha(c) = (\alpha_1(c_1), \dots, \alpha_n(c_n)).$$

Let $i \in \{1, \dots, n\}$. If $S_i \cap \mathbb{Z} = \emptyset$, then $\alpha_i(c'_i) = -\infty = \alpha_i(c_i)$. Similarly, if $c'_i = a_i - 1$, then $\alpha_i(c'_i) = -\infty = \alpha_i(c_i)$. Thus $\alpha(c) = \alpha(c')$ in all cases. Since $\alpha$ is a poset morphism, we see that for all $d \in \mathbb{Z}^n$ such that $c \leq d \leq c'$,

$$\alpha(c) = \alpha(d) = \alpha(c'),$$

and therefore

$$N(\alpha(c)) = N(\alpha(d)) = N(\alpha(c')).$$

Furthermore, because the set $\{d \in \mathbb{Z}^n \mid c \leq d \leq c'\}$ is an initial subset of the set $\{d \in \mathbb{Z}^n \mid c \leq d\}$, we have

$$\lim_{d \geq c,\, d \in \mathbb{Z}^n} N(\alpha(d)) \cong \lim_{c \leq d \leq c',\, d \in \mathbb{Z}^n} N(\alpha(d)) \cong N(\alpha(c)).$$

Next, for 2), let $Q$ be an $RS$-module. Now $\operatorname{res}_\beta Q$ is an $R\overline{S}$-module, so by 1), we have

$$\lim_{d \geq c,\, d \in \mathbb{Z}^n} (\operatorname{res}_\beta Q)(\alpha(d)) \cong (\operatorname{res}_\beta Q)(\alpha(c)).$$

On the other hand, by definition, for all $e \in \overline{\mathbb{Z}}^n$,

$$(\operatorname{res}_\beta Q)(\alpha(e)) = Q(\beta(\alpha(e))) = Q((\beta \circ \alpha)(e)).$$

This means that we may write the above isomorphism as

$$\lim_{d \geq c,\, d \in \mathbb{Z}^n} Q((\beta \circ \alpha)(d)) \cong Q((\beta \circ \alpha)(c)).$$

□

**Corollary 4.6.** *Let $N$ be an $R\overline{\mathbb{Z}}^n$-module, and let $c \in \overline{\mathbb{Z}}^n$. Then*
*1)* $\lim_{d \geq c,\, d \in \mathbb{Z}^n} N(\alpha(d)) \cong N(\alpha(c))$*;*
*2)* $\lim_{d \geq c,\, d \in \mathbb{Z}^n} N((\beta \circ \alpha)(d)) \cong N((\beta \circ \alpha)(c))$*.*

*Proof.* For 1), we note that $\operatorname{res}_{\overline{S}} N$ is an $R\overline{S}$-module, where $(\operatorname{res}_{\overline{S}} N)(d) = N(d)$ for all $d \in \overline{S}$. We may then apply Proposition 4.5 1) to get the result. For 2), we use Proposition 4.5 2) on the $RS$-module $\operatorname{res}_S N$. □

## 5. Finitely determined modules

Let $M$ be an $R\mathcal{C}$-module. We say that $M$ is *pointwise finitely presented* if $M(c)$ is finitely presented for all $c \in \mathcal{C}$. Slightly generalizing the definition of Miller in [7, p. 25, Example 4.5], where $R = k$ is a field, we say that an $R\mathbb{Z}^n$-module $M$ is *finitely determined*, if $M$ is pointwise finitely presented, and for some $a \leq b$ in $\mathbb{Z}^n$, the convex projection $\pi : \mathbb{Z}^n \to [a, b]$ gives $M$ an encoding by the closed interval $[a, b] \subseteq \mathbb{Z}^n$. Here the convex projection $\pi$ takes every point in $\mathbb{Z}^n$ to its closest point in the interval $[a, b]$. If $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$, we have for any $c := (c_1, \dots, c_n) \in \mathbb{Z}^n$,

$$\pi(c) = (\pi_1(c_1), \dots, \pi_n(c_n)),$$

where

$$\pi_i(c_i) = \max(a_i, \min(c_i, b_i))$$

for all $i \in \{1, \dots, n\}$. Note that a pointwise finitely presented $R\mathbb{Z}^n$-module $M$ is finitely determined if and only if there exists a closed interval $[a, b] \subseteq \mathbb{Z}^n$ such that the morphisms $M(c \leq c + e_i)$ $(i = 1, \dots, n)$ are isomorphisms whenever $c_i$ lies outside $[a_i, b_i]$.

**Remark 1.** Let $M$ be an $R\mathbb{Z}^n$-module. Then $M$ is encoded by the closed interval $[a, b]$ with the convex projection $\pi : \mathbb{Z}^n \to [a, b]$ if and only if $M \cong \text{res}_\pi \, \text{res}_{[a,b]} \, M$. Indeed, if $M \cong \text{res}_\pi \, N$ for some $R[a, b]$-module $N$, then for all $c \in \mathbb{Z}^n$, we have

$$M(c) \cong (\text{res}_\pi \, N)(c) = N(\pi(c)) = N(\pi(\pi(c))) \cong M(\pi(c))$$

because for all $c \in \mathbb{Z}^n$, $\pi(\pi(c)) = \pi(c)$.

We would now like to investigate how the notion of finite determinacy relates to our notion of $S$-determinacy, when $S$ is finite and $M$ is pointwise finitely presented. While the requirement that $\text{Supp}(M) \subseteq \uparrow S$ does not necessarily hold for finitely determined modules, we do have the following:

**Proposition 5.1.** *Let $M$ be an $R\mathbb{Z}^n$-module, and $a, b \in \mathbb{Z}^n$ such that $a \leq b$. Set $u := (1, 1, \dots, 1) \in \mathbb{Z}^n$. If $M$ is $[a + u, b]$-determined, then $M$ has an encoding by the closed interval $[a, b]$ with the convex projection $\pi : \mathbb{Z}^n \to [a, b]$. The converse implication holds if* $\text{Supp}(M) \subseteq \uparrow a$.

*Proof.* For the first implication, suppose that $M$ is $[a + u, b]$-determined. We write $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$. Let $c := (c_1, \dots, c_n) \in \mathbb{Z}^n$. We note that if $c_i \leq a_i$ for some $i \in \{1, \dots, n\}$, then also $\pi_i(c_i) \leq a_i$, so that $c, \pi(c) \notin \text{Supp}(M)$. Otherwise $c > a$, in which case $\pi(c) \leq c$ and $[a+u, b] \cap \downarrow \pi(c) = [a + u, b] \cap \downarrow c$. Thus $M(\pi(c)) \to M(c)$ is an isomorphism by the definition of $[a + u, b]$-determined modules, and $M \cong \text{res}_\pi \, \text{res}_{[a,b]} \, M$.

To prove the converse, assume that $\text{Supp}(M) \subseteq \uparrow a$ and $M$ has an encoding by the closed interval $[a, b]$ with the encoding convex projection $\pi : \mathbb{Z}^n \to [a, b]$. Let $c := (c_1, \dots, c_n) \in \mathbb{Z}^n$. Suppose that $c_i < a_i$ for some $i \in \{1, \dots, n\}$. From the condition $\text{Supp}(M) \subseteq \uparrow a$, we see that $M(c) = 0$. Since $M$ is finitely determined, we also have $M(\pi(c)) = M(c) = 0$. Thus $M(c) = 0$ if $c_i \leq a_i$ for some $i \in \{1, \dots, n\}$. If this is not the case, we have $c \geq a + u$. Let $c \leq d$ in $\mathcal{C}$ such that $a + u \leq c \leq d$ and $[a + u, b] \cap \downarrow c = [a + u, b] \cap \downarrow d$. This implies that $\pi(c) = \pi(d)$, so $M(c \leq d)$ is an isomorphism. □

To proceed, we have to shift our focus to $R\overline{\mathbb{Z}}^n$-modules. Let $a \leq b$ in $\mathbb{Z}^n$. With the notation from Section 4, we will view the case $S = [a, b]$. In particular, we have $\alpha = \alpha_{\overline{[a,b]}}$ and $\beta = \beta_{[a,b]}$. Proposition 4.4 gives us formulas for $\alpha$ and $\beta$. If $c := (c_1, \dots, c_n) \in \overline{\mathbb{Z}}^n$ and $d := (d_1, \dots, d_n) \in \overline{[a, b]}$, then

$$\alpha(c) = (\alpha_1(c_1), \dots, \alpha_n(c_n)) \quad \text{and} \quad \beta(d) = (\beta_1(d_1), \dots, \beta_n(d_n)).$$

Here $\alpha_i := \alpha_{\overline{S_i}}$ and $\beta_i := \beta_{S_i}$ for all $i \in \{1, \dots, n\}$. Explicitly,

$$\alpha_i(c_i) = \begin{cases} -\infty, \text{ if } c_i < a_i; \\ c_i, \text{ if } a_i \leq c_i \leq b_i; \\ b_i, \text{ if } c_i > b_i \end{cases} \quad \text{and} \quad \beta_i(d_i) = \begin{cases} a_i, \text{ if } d_i = -\infty; \\ d_i, \text{ otherwise} \end{cases}$$

for every $i \in \{1, \dots, n\}$. The next proposition shows us that the composition $\beta \circ \alpha$ is an extension of the convex projection $\pi$ from $\mathbb{Z}^n$ to $\overline{\mathbb{Z}}^n$.

**Proposition 5.2.** *Let $\pi : \mathbb{Z}^n \to [a, b]$ be the convex projection. Then for any $c := (c_1, \dots, c_n) \in \mathbb{Z}^n$,*

$$\pi(c) = (\beta \circ \alpha)(c).$$

*Proof.* Suppose first that $n = 1$. Recall that $\pi(c) = \max(a, \min(c, b))$. Now there are three cases:

- If $c \in [a, b]$, then $(\beta \circ \alpha)(c) = \beta(c) = c = \pi(c)$;
- If $c < a$, then $(\beta \circ \alpha)(c) = \beta(-\infty) = a = \pi(c)$;
- If $c > b$, then $(\beta \circ \alpha)(c) = \beta(b) = b = \pi(c)$.

Suppose next that $n > 1$. Using Proposition 4.4, we may write

$$\alpha(c) = (\alpha_1(c_1), \dots, \alpha_n(c_n)) \quad \text{and} \quad \beta(d) = (\beta_1(d_1), \dots, \beta_n(d_n))$$

for all $d \in \overline{[a, b]}$. Similarly, recall that

$$\pi(c) = (\pi_1(c_1), \dots, \pi_n(c_n)).$$

It now follows from the case $n = 1$ that

$$\begin{aligned}(\beta \circ \alpha)(c) &= \beta(\alpha_1(c_1), \dots, \alpha_n(c_n)) \\ &= ((\beta_1 \circ \alpha_1)(c_1), \dots, (\beta_n \circ \alpha_n)(c_n)) \\ &= (\pi_1(c_1), \dots, \pi_n(c_n)) \\ &= \pi(c).\end{aligned}$$

□

**Remark 2.** In an effort to keep the notation simpler, we only defined $\beta$ for the elements in the image of $\alpha$. Of course, we could have defined $\beta$ in a fully dual fashion to $\alpha$, starting from posets that are strongly bounded from below, adding the point $\infty$ to $\mathbb{Z}$, and defining a set $\underline{S}$ dually to $\overline{S}$. This would have resulted in the situation where

$$(\beta|_{\overline{S}} \circ \alpha)(c) = (\alpha|_{\underline{S}} \circ \beta)(c) = \pi(c)$$

for all $c \in \mathbb{Z}^n$. In other words, the same result would have been achieved.

We saw in Remark 1 that if $M$ is encoded by a closed interval $[a, b]$ with the convex projection $\pi : \mathbb{Z}^n \to [a, b]$, we have $M(c) \cong M(\pi(c))$ for all $c \in \mathbb{Z}^n$. On the other hand, by Theorem 3.4, we have $\overline{M}(\alpha(c)) \cong \overline{M}(c)$ for all $c \in \overline{\mathbb{Z}}^n$ if $M$ is $S$-determined and $S \subseteq \overline{\mathbb{Z}}^n$ is finite. In preparation for the proof of Theorem 5.4, we will now show that a similar result applies to $\beta$ in both cases.

**Proposition 5.3.** *Set $u := (1, 1, \dots, 1) \in \mathbb{Z}^n$. Let $M$ be an $R\mathbb{Z}^n$-module, and let $c \in \overline{[a, b]}$.*

1) *If $M$ has an encoding by the closed interval $[a, b]$ with the convex projection $\pi : \mathbb{Z}^n \to [a, b]$, then $\overline{M}(c) \cong M(\beta(c))$.*
2) *If $\overline{M}$ is $\overline{[a + u, b]}$-determined, then $\overline{M}(c) \cong M(\beta(c))$.*

*Proof.* To show 1), suppose that $M$ has an encoding by the closed interval $[a, b]$ with the convex projection $\pi : \mathbb{Z}^n \to [a, b]$. Then, by the definition of $\overline{M}$,

$$\overline{M}(c) = \lim_{d \geq c,\ d \in \mathbb{Z}^n} M(d).$$

The encoding gives us $M(d) \cong M(\pi(d))$ for all $d \in \mathbb{Z}^n$. This implies that

$$\overline{M}(c) \cong \lim_{d \geq c,\ d \in \mathbb{Z}^n} M(\pi(d)).$$

We may now apply Corollary 4.6 to see that $\overline{M}(c) \cong M(\beta(\alpha(c)))$. Note that $c \in \overline{[a, b]}$ implies $\alpha(c) = c$. Thus $\overline{M}(c) \cong M(\beta(c))$.

Next, to prove 2), let $\overline{M}$ be $\overline{[a + u, b]}$-determined. Since $c \leq \beta(c)$, it is then enough to show that $\overline{[a + u, b]} \cap \downarrow c = \overline{[a + u, b]} \cap \downarrow \beta(c)$. We instantly have $\downarrow c \subseteq \downarrow \beta(c)$. For the other direction, let $d := (d_1, \dots, d_n) \in \overline{[a + u, b]} \cap \downarrow \beta(c)$. We want to show that $d \leq c$. Recall that we may write $\beta(c) =$

$(\beta_1(c_1), \dots, \beta_n(c_n))$, where

$$\beta_i(c_i) = \begin{cases} a_i, & \text{if } c_i = -\infty; \\ c_i, & \text{otherwise.} \end{cases}$$

for all $i \in \{1, \dots, n\}$. Suppose that $i \in \{1, \dots, n\}$. If $\beta_i(c_i) = c_i$, we have $d_i \leq \beta_i(c_i) = c_i$. Otherwise, if $\beta_i(c_i) = a_i$, we must have $d_i = c_i = -\infty$, because $d_i, c_i \in \overline{[a_i + 1, b_i]}$. We conclude that $d \leq c$. □

**Remark 3.** Let $M$ be a pointwise finitely presented $R\mathbb{Z}^n$-module and let $c \in \overline{\mathbb{Z}}^n$. If $M$ is finitely determined with the convex projection $\pi\colon \mathbb{Z}^n \to [a, b]$, then from the proof of Proposition 5.3, we have $\overline{M}(c) \cong M((\beta \circ \alpha)(c))$, so that $\overline{M}$ is pointwise finitely presented.

We are now ready to state

**Theorem 5.4.** *Let $M$ be a pointwise finitely presented $R\mathbb{Z}^n$-module. Then the following are equivalent:*
*1) $M$ is finitely determined;*
*2) $\overline{M}$ is $S$-determined for some finite $S \subseteq \overline{\mathbb{Z}}^n$;*
*3) $\overline{M}$ is finitely presented.*

*Proof.* We will first show the equivalence of 1) and 2). Note that for any finite subset $S \subseteq \overline{\mathbb{Z}}^n$, we can always find $a, b \in \mathbb{Z}^n$ such that $S \subseteq \overline{[a + u, b]}$. Consider the functor

$$\alpha' = \alpha_{\overline{[a+u,b]}}\colon \overline{\mathbb{Z}}^n \to \overline{[a + u, b]},$$

and denote its restriction to $\mathbb{Z}^n$ by $\overline{\alpha}$. By Theorem 3.4, $\overline{M}$ is $\overline{[a + u, b]}$-determined if and only if $\overline{M}$ is encoded by $\alpha'$. That is, $\overline{M} \cong \operatorname{res}_{\alpha'} N$ for some $R\overline{[a + u, b]}$-module $N$. By restricting to $\mathbb{Z}^n$, we see that

$$M \cong \operatorname{res}_{\mathbb{Z}^n} \operatorname{res}_{\alpha'} N = \operatorname{res}_{\overline{\alpha}} N,$$

so $\overline{\alpha}$ encodes $M$. Conversely, if $M$ is encoded by $\overline{\alpha}$, then $\overline{M}$ has an obvious encoding by $\alpha'$, because $\overline{\alpha}$ is a surjection on objects. Next, we note that the restriction of $\beta$ to $\overline{[a + u, b]}$,

$$\overline{\beta}\colon \overline{[a + u, b]} \to [a, b].$$

is an isomorphism of posets. Therefore $\overline{\beta} \circ \overline{\alpha}$ is an encoding of $M$ if and only if $\overline{\alpha}$ is an encoding of $M$. These conditions are equivalent to $M$ being finitely determined, because $\overline{\beta} \circ \overline{\alpha} = \pi$. Namely, for all $c \in \mathbb{Z}^n$, we have $(\overline{\beta} \circ \overline{\alpha})(c) = (\beta \circ \alpha')(c)$, where

$$\begin{aligned} (\beta \circ \alpha')(c)_i &= \begin{cases} \beta_i(-\infty), & \text{if } c_i = a_i, \\ \beta_i(\alpha_i(c_i)), & \text{else.} \end{cases} \\ &= \begin{cases} a_i, & \text{if } c_i = a_i, \\ \pi_i(c_i), & \text{else.} \end{cases} \\ &= \pi(c)_i \end{aligned}$$

for all $i \in \{1, \dots, n\}$.

Finally, we observe that the equivalence of 2) and 3) follows from the main result of our previous paper, [4, p. 25, Theorem 4.15]. For $\overline{\mathbb{Z}}^n$-modules, it states that being pointwise finitely presented and $S$-determined for some finite $S \subseteq \overline{\mathbb{Z}}^n$ is equivalent to being finitely presented. Also note Remark 3, which shows us that $\overline{M}$ is pointwise finitely presented. □

We are now able to give a "sharpened" version of Proposition 5.1.

**Corollary 5.5.** *If $M$ is an $R\mathbb{Z}^n$-module and $a, b \in \mathbb{Z}^n$ such that $a \leq b$, then the following are equivalent:*
*1) $M$ is encoded by the convex projection $\pi : \mathbb{Z}^n \to [a, b]$;*
*2) $\overline{M}$ is $\overline{[a+u, b]}$-determined, where $u := (1, \ldots, 1) \in \mathbb{Z}^n$.*

*Proof.* We showed in the proof of Theorem 5.4 that 2) implies 1). Conversely, suppose that 1) holds. Let $c \leq d$ in $\overline{\mathbb{Z}}^n$ such that $\overline{[a+u, b]} \cap \downarrow c = \overline{[a+u, b]} \cap \downarrow d$. Coordinatewise, for $i = \{1, \ldots, n\}$, this implies that either $c_i = d_i$, $b_i \leq c_i < d_i$ or $c_i < d_i \leq a_i$. In any case, $(\beta \circ \alpha)(c) = (\beta \circ \alpha)(d)$, so that

$$\overline{M}(c) \cong M((\beta \circ \alpha)(c)) = M((\beta \circ \alpha)(d)) \cong \overline{M}(d).$$

Thus $M(c \leq d)$ is an isomorphism, and $M$ is $\overline{[a+u, b]}$-determined. □

To demonstrate Theorem 5.4 and Corollary 5.5, it is convenient to take the point of view of topological data analysis, and consider the births and deaths of elements of a module. Given an $R\mathbb{Z}^n$ module $M$, one can track how an element $x \in \overline{M}(c)$, where $c \in \overline{\mathbb{Z}}^n$, evolves when mapped with the homomorphisms $M(c \leq c')$, $(c, c' \in \overline{\mathbb{Z}}^n)$. We say that the element $x$ is born at $c$ if it is not in the image of any morphism $M(c' \leq c)$, where $c' < c$. On the other hand, the element $x$ dies at $c''$ if $M(c \leq c'')(x) = 0$, but $M(c \leq c')(m) \neq 0$ for all $c \leq c' < c''$.

Consider now an $R\mathbb{Z}^2$-module $M$ that is finitely determined, and let $\pi : \mathbb{Z}^2 \to [a, b]$ be the accompanying convex projection. Note that no new elements are born or die in the leftmost edge or the bottom edge of the box $[a, b]$. This follows from the fact that every element on these two edges has already appeared infinite times before, and was born at some infinitary point. Let us write $a = (a_1, a_2)$. For example, if an element, say $x \in M((a_1, c))$, maps to zero on the leftmost edge of $[a, b]$, in $M((a_1, c+1))$, then $x \in \overline{M}((-\infty, c))$ will also map to zero in $\overline{M}((-\infty, c+1))$. Thus $x$ does not "die" at the point $(a_1, c+1)$, but rather at the infinitary point $(-\infty, c+1) \in \overline{[a+u, b]}$.

**Remark 4.** Let $M$ be an $R\mathbb{Z}^n$-module and $c \in \overline{\mathbb{Z}}^n$. Consider the natural homomorphism

$$\lambda_{\overline{M},c} \colon \operatorname*{colim}_{d \leq c, d \in \overline{[a+u,b]}} \overline{M}(d) \to \overline{M}(c).$$

Following [4, p. 15, Def. 3.6], we say that $c$ is a *birth* if $\lambda_{\overline{M},c}$ is a non-epimorphism, and a *death* if $\lambda_{\overline{M},c}$ is a non-monomorpism. Furthermore, suppose that $\overline{M}$ is $\overline{[a+u, b]}$-determined, and the births are "well-behaved" enough. That is, for any birth $c$, the module $\overline{M}(c)/\operatorname{Im}\lambda_{\overline{M},c}$ is projective. The latter of course holds if $R$ is a field. Then, as we discussed in [4, p. 21, Remark 3.27], births and deaths show the positions of the minimal generators and relations of $M$.

In the next example, we will demonstrate how, for a finitely determined module $M$, the extension $\overline{M}$ has births and deaths at infinitary points that guarantee the existence of a finite presentation of $\overline{M}$.

**Example 5.6.** Let $M$ be an $R\mathbb{Z}^2$-module that is defined on objects by

$$M(c) = \begin{cases} R, & \text{if } c \leq (0,0); \\ 0, & \text{otherwise,} \end{cases}$$

for all $c \in \mathbb{Z}^2$, and where a morphism $R \to R$ is always $\mathrm{id}_R$. Then $M$ is finitely determined with the convex projection $\pi : \mathbb{Z}^2 \to [(0,0), (1,1)]$. Now, by Remark 5.5, $\overline{M}$ is $\overline{[(1,1), (1,1)]}$-determined. Here $\overline{[(1,1), (1,1)]}$ is the set

$$\{(-\infty, -\infty), (1, -\infty), (-\infty, 1), (1,1)\}.$$

In particular, we have $\overline{M}((-\infty, -\infty)) = R$, and

$$\overline{M}((-\infty, 1)) = \overline{M}((1, -\infty)) = \overline{M}((1,1)) = 0.$$

Furthermore, by Theorem 5.4, $\overline{M}$ is now finitely presented. In more concrete terms, we have an exact sequence of $R\overline{\mathbb{Z}}^2$-modules

$$K \to N \to \overline{M} \to 0,$$

where

$$N = R[\mathrm{Mor}_{\overline{\mathbb{Z}}^2}((-\infty,-\infty),-)]$$

and

$$K = R[\mathrm{Mor}_{\overline{\mathbb{Z}}^2}((1,-\infty),-)] \oplus R[\mathrm{Mor}_{\overline{\mathbb{Z}}^2}((-\infty,1),-)].$$

Here $(-\infty,-\infty)$ is the only birth of $M$, while $(1,-\infty)$ and $(-\infty,1)$ are the deaths.

**Example 5.7.** If $k$ is a field, then it is well known that finitely generated $k\mathbb{Z}^n$-modules are finitely presented. This result, however, does not apply to $k\overline{\mathbb{Z}}^n$-modules. For a counterexample, consider a $k\mathbb{Z}^2$-module $M$, where

$$M((x,y)) = \begin{cases} k, & \text{if } x+y<0; \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $\overline{M}$ is finitely generated with its only birth in $(-\infty,-\infty)$. It is not finitely presented, since the deaths happen at points $(n,-n)$ for all $n \in \mathbb{Z}$.

Finally, we want to relate Theorem 5.4 to the work of Perling ([8]). Recall that a subset $L \subseteq \overline{\mathbb{Z}}^n$ is a join-sublattice if $\mathrm{mub}(S) \in L$ for every finite subset $S \subseteq L$. Note that this is equivalent to the condition that $L = \hat{L}$. Given a join-sublattice $L \subseteq \overline{\mathbb{Z}}^n$, following Perling in [8, pp. 16–19, chapter 3.1], we define the zip-functor

$$\mathrm{zip}_L \colon R\mathbb{Z}^n\text{-}\mathbf{Mod} \to RL\text{-}\mathbf{Mod}$$

and the unzip-functor

$$\mathrm{unzip}_L \colon RL\text{-}\mathbf{Mod} \to R\overline{\mathbb{Z}}^n\text{-}\mathbf{Mod}.$$

Contrary to Perling, we do not assume that $R$ is a field. The zip-functor maps an $R\mathbb{Z}^n$-module $M$ to the $RL$-module $\mathrm{res}_L \overline{M}$, whereas he unzip-functor maps an $RL$-module $N$ to an $R\overline{\mathbb{Z}}^n$-module $\mathrm{unzip}_L N$ defined by

$$(\mathrm{unzip}_L N)(c) = \begin{cases} N(\mathrm{mub}(L \cap \downarrow c)), & \text{if } L \cap \downarrow c \neq \emptyset; \\ 0, & \text{otherwise} \end{cases}$$

for all $c \in \overline{\mathbb{Z}}^n$. Note that $\mathrm{Supp}(\mathrm{unzip}_L N) \subseteq \uparrow L$.

**Remark 5.** It turns out that $\mathrm{unzip}_L$ is essentially the same thing as $\mathrm{res}_\alpha$, when $L$ is finite and $\alpha := \alpha_L$. There is the slight complication that $\mathrm{unzip}_L$ is defined for $RL$-modules, while $\mathrm{res}_\alpha$ is defined for $R\tilde{L}$-modules. We may, however, extend an $RL$-module $N$ to an $R\tilde{L}$-module $\tilde{N}$ by setting

$$\tilde{N}((-\infty,\dots,-\infty)) = 0,$$

if $(-\infty,\dots,-\infty) \notin L$, and $\tilde{N}(c) = N(c)$, otherwise. Having defined the module $\tilde{N}$ in this way, we see that $\mathrm{unzip}_L N \cong \mathrm{res}_\alpha \tilde{N}$.

Given an $R\mathbb{Z}^n$-module $M$, the join-sublattice $L$ is called *$M$-admissible* in [8, p. 18, Definition 3.4] if the condition $\overline{M} \cong \mathrm{unzip}_L \mathrm{zip}_L M$ is satisfied. This leads us to the following proposition.

**Proposition 5.8.** *Let $M$ be an $R\mathbb{Z}^n$-module, and $L$ a finite join-sublattice. Then $L$ is $M$-admissible if and only if $\overline{M}$ is $L$-determined.*

*Proof.* Let $c \in \overline{\mathbb{Z}}^n$. With the earlier notation, we see that

$$\mathrm{unzip}_L\, \mathrm{zip}_L\, M = \mathrm{unzip}_L\, \mathrm{res}_L\, \overline{M} \cong \widetilde{\mathrm{res}_\alpha\, \mathrm{res}_L\, \overline{M}},$$

where

$$(\widetilde{\mathrm{res}_\alpha\, \mathrm{res}_L\, \overline{M}})(c) = \begin{cases} (\mathrm{res}_\alpha\, \mathrm{res}_{\tilde{L}}\, \overline{M})(c), & \text{if } L \cap \downarrow c \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

Assume first that $\overline{M} \cong \mathrm{unzip}_L\, \mathrm{zip}_L\, M$. If $L \cap \downarrow c = \emptyset$, we have $\overline{M}(c) = 0$ by the definition of the functor $\mathrm{unzip}_L$. But in this case $\alpha(c) \leq c$, so that $L \cap \downarrow \alpha(c) = \emptyset$. Using the definition of $\mathrm{unzip}_L$ again, we get

$$(\mathrm{res}_\alpha\, \mathrm{res}_{\tilde{L}}\, \overline{M})(c) = \overline{M}(\alpha(c)) = 0.$$

On the other hand, if there is an element $d \in L \cap \downarrow c$, then, by the above formula, $\overline{M}(c) \cong (\mathrm{res}_\alpha\, \mathrm{res}_{\tilde{L}}\, \overline{M})(c)$. Thus,

$$\overline{M} \cong \mathrm{res}_\alpha\, \mathrm{res}_{\tilde{L}}\, \overline{M}$$

and $\mathrm{Supp}(\overline{M}) \subseteq \uparrow L$, so $\overline{M}$ is $L$-determined by Theorem 3.4.

Conversely, suppose that $\overline{M}$ is $L$-determined. By Theorem 3.4, we have $\overline{M} \cong \mathrm{res}_\alpha\, \mathrm{res}_{\tilde{L}}\, \overline{M}$ and $\mathrm{Supp}(\overline{M}) \subseteq \uparrow L$. The above formula shows us that

$$(\mathrm{unzip}_L\, \mathrm{zip}_L\, M)(c) = (\mathrm{res}_\alpha\, \mathrm{res}_{\tilde{L}}\, \overline{M})(c)$$

for all $c \in \uparrow L$. If $c \notin \uparrow L$, then $c \notin \mathrm{Supp}(\overline{M})$, which means that $\overline{M}(c) = 0$. In this case, we also have $(\mathrm{unzip}_L\, \mathrm{zip}_L\, M)(c) = 0$ by the definition of the functor $\mathrm{unzip}_L$. Thus we have an isomorphism

$$\overline{M} \cong \mathrm{unzip}_L\, \mathrm{zip}_L\, M.$$

□

## Disclosure statement

The authors report there are no competing interests to declare.

## References

[1] Carlsson, G., Zomorodian, A. (2007). The theory of multidimensional persistence. Proceedings of the Twenty-Third Annual Symposium on Computational Geometry, pp. 184–193.
[2] Dieck, T. T. (2006). *Transformation Groups and Representation Theory*, Vol. 766. Berlin: Springer.
[3] Djament, A. (2016). Des propriétés de finitude des foncteurs polynomiaux. *Fundam. Math.* 233:197–256.
[4] Hyry, E., Klemetti, M. (2022). Generalized persistence and graded structures. *Homol. Homotopy Appl.* 24(1):27–53.
[5] Lück, W. (2006). *Transformation Groups and Algebraic K-Theory*, Vol. 1408. Berlin: Springer.
[6] Miller, E. (2000). The Alexander duality functors and local duality with monomial support. *J. Algebra* 231(1): 180–234.
[7] Miller, E. (2020). Homological algebra of modules over posets. arXiv e-prints, arXiv-2008.
[8] Perling, M. (2013). Resolutions and cohomologies of toric sheaves: the affine case. *Int. J. Math.* 24(09):1350069.
[9] Popescu, N. (1973). *Abelian Categories with Applications to Rings and Modules*, Vol. 3. New York: Academic Press.