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Some Formulas Derived from Finite Integration

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Abstract

The finite sine series can sometimes be represented by a function of closed from. And it may conveniently work on the finite differece equation.

Some finite cosine series which can take functions of closed form, are formulated in this paper.

1. Mother function

Let take mother function $\boldsymbol{F}_{\alpha x}$:

$$\operatorname{coh}\left\{x\left(\alpha+\frac{m\pi}{n}i\right)\right\}=\boldsymbol{F}_{\alpha x},$$

where

$$m=1, 2, 3, \cdots, n$$

$$i=\text{imaginary unit},$$

then we have

$$\Delta^2\boldsymbol{F}_{\alpha\cdot x-1}=\cosh\left\{x\left(\alpha+\frac{m\pi}{n}i\right)\right\}\cdot\left\{2\cosh\left(\alpha+\frac{m\pi}{n}i\right)-2\right\},$$

in whic

$$\Delta^2F_{x-1}=F_{x+1}-2F_x+F_{x-1},$$

when $\tilde{\boldsymbol{F}}_{\alpha\cdot x}$ is expressed by

$$\tilde{\boldsymbol{F}}_{\alpha\cdot x}=\frac{\cosh\left\{x\left(\alpha+\frac{m\pi}{n}i\right)\right\}}{2\cosh\left(\alpha+\frac{m\pi}{n}i\right)-2},$$

which is written in

$$\Delta^2\tilde{\boldsymbol{F}}_{\alpha\cdot x-1}=\boldsymbol{F}_{\alpha\cdot x}.$$

So the finite integration of the above becomes

$$\mathfrak{S}\,\boldsymbol{F}_{\alpha\cdot x}\Delta x=\Delta\tilde{\boldsymbol{F}}_{\alpha\cdot x-1},$$

or

$$\mathfrak{S}\,\tilde{\boldsymbol{F}}_{\alpha\cdot x}\Delta x=\frac{\cosh\left\{x\left(\alpha+\frac{m\pi}{n}i\right)\right\}-\cosh\left\{(x-1)\left(\alpha+\frac{m\pi}{n}i\right)\right\}}{2\cosh\left(\alpha+\frac{m\pi}{n}i\right)-2}.$$

Considering that

$$\sum_{x=1}^{n-1} \boldsymbol{F}_{\alpha\cdot x} = \int_1^n \boldsymbol{F}_{\alpha\cdot x} \Delta x ,$$

we have at once

$$\sum_{x=1}^{n-1} \boldsymbol{F}_{\alpha\cdot x} = -\frac{1}{2}\left\{\cosh(n\alpha + m\pi i) + 1\right\} + \frac{\sinh(n\alpha + m\pi i)\sinh\left(\alpha + \frac{m\pi}{n} i\right)}{2\left\{\cosh\left(\alpha + \frac{m\pi}{n} i\right) - 1\right\}} .$$

The imaginary part of the above yields

$$\boldsymbol{I}\left\{\sum_{x=1}^{n-1} \boldsymbol{F}_{\alpha\cdot x}\right\} = -\frac{(-1)^m \sinh n\alpha \sin\frac{m\pi}{n}}{2\left(\cosh\alpha - \cos\frac{m\pi}{n}\right)} ,$$

from which we have

$$\sum_{x=1}^{n-1} \sinh x\alpha \cdot \sin\frac{m\pi}{n} x = -\frac{(-1)^m \sinh n\alpha \cdot \sin\frac{m\pi}{n}}{2\left(\cosh\alpha - \cos\frac{m\pi}{n}\right)} .$$

The inversion theorem leads to the following formula;

$$\sum_{m=1}^{n} \frac{(-1)^m \sin\frac{m\pi}{n} \cdot \sin\frac{m\pi}{n} x}{2\left(\cosh\alpha - \cos\frac{m\pi}{n}\right)} = -\frac{\sinh x\cdot\alpha}{\sinh n\alpha} . \tag{1}$$

In like manner, the real part can be written as follows;

$$\boldsymbol{R}_e\left\{\sum_{x=1}^{n-1} \boldsymbol{F}_{\alpha\cdot x}\right\} = -\frac{1}{2}\left\{(-1)^m \cosh n\alpha + 1\right\} + \frac{(-1)^m \sinh n\alpha \cdot \sinh\alpha}{2\left(\cosh\alpha - \cos\frac{m\pi}{n}\right)} ,$$

The inversion formula:

$$f(x) = \frac{2}{n} \sum_{m=0}^{n} \boldsymbol{R}_m[f(x)] \cos\frac{m\pi}{n} x ,$$

where

$$\boldsymbol{R}_0[f(x)] = \frac{1}{2}\left\{\boldsymbol{C}_0[f(x)] + \frac{1}{2} f(n) + \frac{1}{2} f(0)\right\}$$

$$\boldsymbol{R}_m[f(x)] = \boldsymbol{C}_m[f(x)] + \frac{1}{2}(-1)^m f(n) + \frac{1}{2} f(0)$$

$$\boldsymbol{R}_n[f(x)] = \frac{1}{2}\left\{\boldsymbol{C}_n[f(x)] + \frac{1}{2}(-1)^n f(n) + \frac{1}{2} f(0)\right\} ,$$

leads to the following results;

$$\frac{2}{n}\sum_{m=1}^{n-1}\frac{(-1)^m\cos\frac{m\pi}{n}x}{2\left(\cosh\alpha-\cos\frac{m\pi}{n}\right)}=\frac{\cosh x\alpha}{\sinh\alpha\cdot\sinh n\alpha}-\frac{1}{2n(\cosh\alpha-1)}-\frac{(-1)^{n-x}}{2n(\cosh\alpha+1)}, \tag{2}$$

which is differentiated with respect to α, to be as follows;

$$\frac{2}{n}\sum_{m=1}^{n-1}-\frac{(-1)^m\cos\frac{m\pi}{n}x\cdot\sinh\alpha}{2\left(\cosh\alpha-\cos\frac{m\pi}{n}\right)^2}=\frac{x\sinh x\alpha}{\sinh\alpha\cdot\sinh n\alpha}-\frac{n\cosh x\alpha\cdot\cosh n\alpha}{\sinh\alpha\cdot\sinh^2 n\alpha}-\frac{\cosh x\alpha\cdot\cosh\alpha}{\sinh^2\alpha\cdot\sinh n\alpha}-\frac{\sinh\alpha}{2n(\cosh\alpha-1)^2}-\frac{(-1)^{n-x}\sinh\alpha}{2n(\cosh\alpha+1)^2}. \tag{3}$$

2. Formulas of Polynomial Function

Letting α be zero in the Eqs. (2) and (3), we have

$$\frac{4}{n}\sum_{m=1}^{n-1}\frac{(-1)^m\cos\frac{m\pi}{n}x}{4\left(1-\cos\frac{m\pi}{n}\right)^2}=\frac{x^2}{12n}(2n^2+2-x^2)-\frac{1}{180n}(7n^2+17)\cdot(n^2-1)-\frac{1}{8n}\left\{1+(-1)^{n-x}\right\}, \tag{4}$$

and

$$\frac{2}{n}\sum_{m=1}^{n-1}\frac{(-1)^m\cos\frac{m\pi}{n}x}{2\left(1-\cos\frac{m\pi}{n}\right)}=\frac{1}{2n}\left\{x^2-\frac{n^2-1}{3}-\frac{1+(-1)^{n-x}}{2}\right\}. \tag{5}$$

Putting $n-x$ for x into the Eq. (5), we have

$$\frac{2}{n}\sum_{m=1}^{n-1}\frac{\cos\frac{m\pi}{n}(n-x)}{2\left(1-\cos\frac{m\pi}{n}\right)}=\frac{1}{2n}\left\{x^2-\frac{n^2-1}{3}-\frac{1+(-1)^{n-x}}{2}\right\}, \tag{6}$$

which becomes upon the action of Δ^2_{x-1}

$$-\frac{2}{n}\sum_{m=1}^{n-1}\cos\frac{m\pi}{n}(n-x)=\frac{1}{n}\left\{1+(-1)^{n-x}\right\}. \tag{7}$$

By the well known finite integration formulr, we obtain

$$\sum_{m=1}^{n-1}\cos\frac{m\pi}{n}(n-x)=\left.\frac{\sin\left\{\frac{\pi}{n}(n-x)\left(m-\frac{1}{2}\right)\right\}}{2\sin\frac{\pi}{2n}(n-x)}\right|_{1}^{n}=-\frac{1}{2}\left\{1+(-1)^{n-x}\right\},$$

which is same with the Eq. (7).

3. Advanced Formulas

Denoting that $D_m=2\left(1-\cos\frac{m\pi}{n}\right)$ and $\cosh\alpha=1+\xi+\eta i$, the equation (2) can be written as follows ;

$$\frac{2}{n}\sum_{m=1}^{n-1}\frac{(-1)^m(D_m+\xi+\eta i)}{(D_m+\xi)^2+\eta^2}\cos\frac{m\pi}{n}x$$
$$=\frac{\cosh x\alpha\cdot\sinh\bar{\alpha}\cdot\sinh n\bar{\alpha}}{\sinh\alpha\cdot\sinh\bar{\alpha}\cdot\sinh n\alpha\cdot\sinh n\bar{\alpha}}-\frac{1}{n(\xi^2+\eta^2)}-\frac{(-1)^{n-x}(\xi-\eta i+4)}{n\{(\xi+4)^2+\eta^2\}}.$$

The real part of the above, yields

$$\frac{2}{n}\sum_{m=1}^{n-1}\frac{(-1)^m(D_m+\xi)}{(D_m+\xi)^2+\eta^2}\cos\frac{m\pi}{n}x=\frac{1}{(\cosh 2\beta-\cos 2\gamma)(\cosh 2n\beta-\cos 2n\gamma)}$$
$$\times\left[\cosh\beta(n+x+1)\cos\gamma(n+1-x)-\cosh\beta(n-x-1)\cos\gamma(n-1+x)\right.$$
$$\left.+\cosh\beta(n+1-x)\cos\gamma(n+1+x)-\cosh\beta(n-1+x)\cdot\cos\gamma(n-1-x)\right]$$
$$-\frac{\xi}{n(\xi^2+\eta^2)}-\frac{(-1)^{n-x}(\xi+4)}{n\{(\xi+4)^2+\eta^2\}},\qquad (8)$$

in which

$$\alpha=\beta+\gamma i,$$

and the imaginary part,

$$\frac{2}{n}\sum_{m=1}^{n-1}\frac{(-1)^m\cos\frac{m\pi}{n}x}{(D_m+\xi)^2+\eta^2}=\frac{1}{\eta(\cosh 2\beta-\cos 2\gamma)\cdot(\cosh 2n\beta-\cos 2n\gamma)}$$
$$\times\left[\sinh\beta(n+1+x)\sin\gamma(n+1-x)-\sinh\beta(n-1-x)\sin\gamma(n-1+x)\right.$$
$$\left.+\sinh\beta(n+1-x)\sin\gamma(n+1+x)-\sinh\beta(n-1+x)\sin\gamma(n-1-x)\right]$$
$$+\frac{1}{n(\xi^2+\eta^2)}+\frac{(-1)^{n-x}}{n\{(\xi+4)^2+\eta^2\}}.\qquad (9)$$

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References

1) S. G. Nomachi: On Finite Fourier Sine Series with Respect to Finite Differences, the Memoirs of the Muroran Institute of Technology, Vol. 5, No. 1, 1965.
2) S. G. Nomachi: A Note on Finite Fourier Transforme Concerning Finite Integration, the Memori of the Murora Institute of Technology, Vol. 5, No. 2, 1966.
3) F. B. Hildebrand: Finite-Difference Equations and Simulations, Prentice-Hall, 1968.