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journal or publication title	Memoirs of the Muroran Institute of Technology. Science and engineering
volume	7
number	2
page range	523-536
year	1971-09-15
URL	http://hdl.handle.net/10258/3530

A Method of Solving Some Problems in Structural Mechanics by Means of Finite Integration Transforms

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Abstract

This paper presents inversion formulas coupled with a finite sine and cosine transforms which are defined by finite integration, together with the related formulas by which the finite defference equations can be solved in a similar way of the integral transform method.

As an illustrative example the elastic analysis of the Warren truss of n pannels is treated and the convergency of finite element method of triangular elements, which are applied to the plane stress problem and plate bending problem, is analytically confirmed by the aid of the presenting method.

1. Introduction

The definition of "Finite Integration", according to G. Boole[1], in the inverse operation of "Finite Difference". The finite integral of a function with a certain kernel defines a finite integral transform, similary the finite integration would yield a finite integration transform. It is a well-known fact that the integral transforms play an important part in the field of continuum mechanics, so the finite integration transforms are supposed to be a some tool for the stress analysis of the fram work structure, and grid work structure.

We can extend the method to the analytical evaluation of the finite difference system which is made from the continuum elastic body by the finite element method.

2. Inversion Formulas with Respect to Finite Sine and Cosine Series

(a) The Formulas for the Function of Integer x

$$\left.\begin{aligned} \boldsymbol{S}_i\left[f(x)\right] &= \sum_{x=1}^{n-1} f(x) \sin\frac{i\pi}{n}x \text{ [2),3)]}, \\ \boldsymbol{C}_i\left[f(x)\right] &= \sum_{x=1}^{n-1} f(x) \cos\frac{i\pi}{n}x . \end{aligned}\right\} \tag{2. 1}$$

We have the inversion formulas coupled with the above, as follows:

$$\left.\begin{aligned} f(x) &= \frac{2}{n}\sum_{i=1}^{n-1} \boldsymbol{S}_i\left[f(x)\right] \sin\frac{i\pi}{n}x , \\ f(x) &= \frac{2}{n}\sum_{i=1}^{n} \boldsymbol{R}_i\left[f(x)\right] \cos\frac{i\pi}{n}x , \end{aligned}\right\} \tag{2. 2}$$

where

$$\boldsymbol{R}_0\Big[f(x)\Big]=\frac{1}{2}\Big\{\boldsymbol{C}_0\Big[f(x)\Big]+\frac{1}{2}f(n)+\frac{1}{2}f(0)\Big\}.$$

$$\boldsymbol{R}_i\Big[f(x)\Big]=\boldsymbol{C}_i\Big[f(x)\Big]+(-1)^i\frac{1}{2}f(n)+\frac{1}{2}f(0),$$

$$\boldsymbol{R}_n\Big[f(x)\Big]=\frac{1}{2}\Big\{\boldsymbol{C}_n\Big[f(x)\Big]+(-1)^n\frac{1}{2}f(n)+\frac{1}{2}f(0)\Big\},$$

$$i=0,1\cdots,n;\quad x=0,1,\cdots,n.$$

(b) The Formulas for the Function of $x+\frac{1}{2}$

Let us introduce the symbolic notation as

$$\left.\begin{aligned}\bar{\boldsymbol{S}}_i\Big[f\Big(x+\frac{1}{2}\Big)\Big]&=\sum_{x=0}^{n-1}f\Big(x+\frac{1}{2}\Big)\sin\frac{i\pi}{n}\Big(x+\frac{1}{2}\Big),\\ \bar{\boldsymbol{C}}_i\Big[f\Big(x+\frac{1}{2}\Big)\Big]&=\sum_{x=0}^{n-1}f\Big(x+\frac{1}{2}\Big)\cos\frac{i\pi}{n}\Big(x+\frac{1}{2}\Big).\end{aligned}\right\}\tag{2. 3}$$

In a similar way, we have inversion formulas:

$$\left.\begin{aligned}f\Big(x+\frac{1}{2}\Big)&=\frac{2}{n}\sum_{i=1}^{n-1}\bar{\boldsymbol{S}}_i\Big[f\Big(x+\frac{1}{2}\Big)\Big]\sin\frac{i\pi}{n}\Big(x+\frac{1}{2}\Big)\\&\qquad+\frac{1}{n}(-1)^x\bar{\boldsymbol{S}}_n\Big[f\Big(x+\frac{1}{2}\Big)\Big],\\ f\Big(x+\frac{1}{2}\Big)&=\frac{2}{n}\sum_{i=2}^{n-1}\bar{\boldsymbol{C}}_i\Big[f\Big(x+\frac{1}{2}\Big)\Big]\cos\frac{i\pi}{n}\Big(x+\frac{1}{2}\Big)\\&\qquad+\frac{1}{n}\bar{\boldsymbol{C}}_0\Big[f\Big(x+\frac{1}{2}\Big)\Big],\\ &i=0,1,\cdots,n;\quad x=0,1,\cdots,n.\end{aligned}\right\}\tag{2. 4}$$

3. Related Formulas

For convenience sake, let us define the modified mean and the modified difference as follows

$$f(x+1)+f(x)=\nabla f(x),\qquad f(x+1)-f(x-1)=\varDelta f(x).$$

Applying the above formulas to the first, the second differences, the modified mean, and the modified difference, we find that for the sine transforms,

$$\boldsymbol{S}_i\Big[\varDelta^2 f(x-1)\Big]=-\sin\frac{i\pi}{n}\Big\{(-1)^i f(n)-f(0)\Big\}-D_i\boldsymbol{S}_i\Big[f(x)\Big],\tag{3. 1}$$

$$\boldsymbol{S}_i\Big[\varDelta f(x)\Big]=-2\sin\frac{i\pi}{n}\boldsymbol{R}_i\Big[f(x)\Big],\tag{3. 2}$$

$$\boldsymbol{S}_i\left[\Delta f\left(x-\frac{1}{2}\right)\right]=-2\sin\frac{i\pi}{n}\bar{\boldsymbol{C}}_i\left[f\left(x+\frac{1}{2}\right)\right], \tag{3. 3}$$

$$\boldsymbol{S}_i\left[\nabla f\left(x-\frac{1}{2}\right)\right]=2\cos\frac{i\pi}{2n}\bar{\boldsymbol{S}}_i\left[f\left(x+\frac{1}{2}\right)\right], \tag{3. 4}$$

$$\sum_{x=1}^{n-2}\Delta^2 f\left(x-\frac{1}{2}\right)\sin\frac{i\pi}{n}\left(x+\frac{1}{2}\right)=\sin\frac{i\pi}{2n}\left\{2f\left(\frac{1}{2}\right)-\Delta f\left(\frac{1}{2}\right)\right.$$
$$\left.-(-1)^i 2f\left(n-\frac{1}{2}\right)-\Delta f\left(n-\frac{3}{2}\right)\right\}-D_i\bar{\boldsymbol{S}}_i\left[f\left(x+\frac{1}{2}\right)\right], \tag{3. 5}$$

$$\sum_{x=1}^{n-2}\Delta f\left(x+\frac{1}{2}\right)\sin\frac{i\pi}{n}\left(x+\frac{1}{2}\right)=\sin\frac{i\pi}{2n}\left\{(-1)^i\Delta f\left(n-\frac{3}{2}\right)\right.$$
$$\left.-\Delta f\left(\frac{1}{2}\right)\right\}-2\sin\frac{i\pi}{n}\bar{\boldsymbol{C}}_i\left[f\left(x+\frac{1}{2}\right)\right], \tag{3. 6}$$

$$\bar{\boldsymbol{S}}_i\left[\Delta f(x)\right]=-2\sin\frac{i\pi}{2n}\boldsymbol{R}_i\left[f(x)\right], \tag{3. 7}$$

$$\bar{\boldsymbol{S}}_i\left[\nabla f(x)\right]=\sin\frac{i\pi}{2n}\left\{f(0)-(-1)^i f(n)\right\}+2\cos\frac{i\pi}{2n}\boldsymbol{S}_i\left[f(x)\right], \tag{3. 8}$$

and for the cosine transforms,

$$\boldsymbol{C}_i\left[\Delta^2 f(x-1)\right]=(-1)^i\Delta f(n-1)-\Delta f(0)-D_i\boldsymbol{R}_i\left[f(x)\right], \tag{3. 9}$$

$$\boldsymbol{C}_i\left[\Delta f(x)\right]=-(-1)^i\Delta f(n-1)-\Delta f(0)+\left(1+\cos\frac{i\pi}{n}\right)\left\{(-1)^i f(n)\right.$$
$$\left.-f(0)\right\}+2\sin\frac{i\pi}{n}\boldsymbol{S}_i\left[f(x)\right], \tag{3. 10}$$

$$\boldsymbol{C}_i\left[\Delta f\left(x-\frac{1}{2}\right)\right]=(-1)^i f\left(n-\frac{1}{2}\right)-f\left(\frac{1}{2}\right)+2\sin\frac{i\pi}{2n}\bar{\boldsymbol{S}}_i\left[f\left(x+\frac{1}{2}\right)\right], \tag{3. 11}$$

$$\boldsymbol{C}_i\left[\nabla f\left(x-\frac{1}{2}\right)\right]=-(-1)^i f\left(n-\frac{1}{2}\right)-f\left(\frac{1}{2}\right)+2\cos\frac{i\pi}{2n}\bar{\boldsymbol{C}}_i\left[f\left(x+\frac{1}{2}\right)\right], \tag{3. 12}$$

$$\sum_{x=1}^{n-2}\Delta^2 f\left(x-\frac{1}{2}\right)\cos\frac{i\pi}{n}\left(x+\frac{1}{2}\right)=-\cos\frac{i\pi}{2n}\left\{\Delta f\left(\frac{1}{2}\right)\right.$$
$$\left.-(-1)^i\Delta f\left(n-\frac{1}{2}\right)\right\}-D_i\bar{\boldsymbol{C}}_i\left[f\left(x+\frac{1}{2}\right)\right], \tag{3. 13}$$

$$\sum_{x=1}^{n-2} f\Delta\left(x+\frac{1}{2}\right)\cos\frac{i\pi}{n}\left(x+\frac{1}{2}\right)=-\cos\frac{i\pi}{2n}\left\{\nabla f\left(\frac{1}{2}\right)\right.$$
$$\left.-(-1)^i\nabla f\left(n-\frac{3}{2}\right)\right\}+2\sin\frac{i\pi}{n}\bar{\boldsymbol{S}}_i\left[f\left(x+\frac{1}{2}\right)\right], \tag{3. 14}$$

$$\bar{C}_i\left[\Delta f(x)\right]=-\cos\frac{i\pi}{2n}\left\{f(0)-(-1)^i f(n)\right\}+2\sin\frac{i\pi}{2n}\bar{S}_1\left[f(x)\right], \quad (3.15)$$

$$\bar{C}_i\left[\Delta f(x)\right]=2\cos\frac{i\pi}{2n}\boldsymbol{R}_i\left[f(x)\right], \quad (3.16)$$

where

$$D_i=2\left(1-\cos\frac{i\pi}{n}\right).$$

4. Analysis of the Warren Truss

As an example, let us consider the Warren truss as shown in Fig. 1, u and w denote the horizontal and vertical displacements at a nodal point respectively. A_l, A_u and A_d are the cross sectional areas of the lower chords, the upper

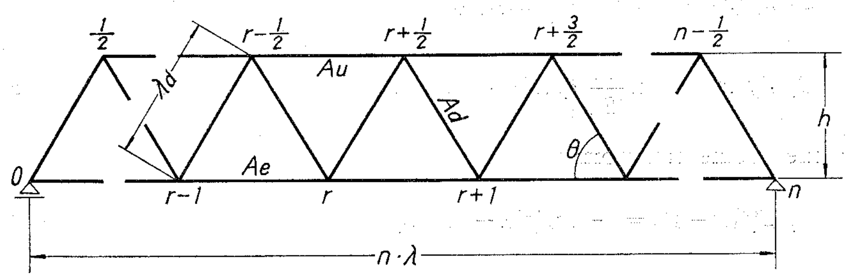

Fig. 1.

chords and the diagonals, and λ, λ_d represent the lengths of chord members and the diagonals.

Then the stresses of the members are related with the displacememts by the following equations

$$\left.\begin{aligned} S_{r\cdot r+1}&=S_{r+1\cdot r}=K_1(u_{r+1}-u_r),\\ S_{r+\frac{1}{2}\cdot r+\frac{3}{2}}&=S_{r+\frac{3}{2}\cdot r+\frac{1}{2}}=K_2(u_{r+\frac{3}{2}}-u_{r+\frac{1}{2}}),\\ S_{r\cdot r+\frac{1}{2}}&=S_{r+\frac{1}{2}\cdot r}=K_3\left\{(u_{r+\frac{1}{2}}-u_r)\alpha-(w_{r+\frac{1}{2}}-w_r)\beta\right\},\\ S_{r r-\frac{1}{2}}&=S_{r-\frac{1}{2}\cdot r}=K_2\left\{(u_r-u_{r-\frac{1}{2}})\alpha+(w_r-w_{r-\frac{1}{2}})\beta\right\}, \end{aligned}\right\} \quad (4.1)$$

where

$$K_1=\frac{EA_l}{\lambda},\quad K_2=\frac{EA_u}{\lambda},\quad K_3=\frac{EA_d}{\lambda},\quad \alpha=\cos\theta,\ \beta=\sin\theta.$$

And the equilibrium of forces at the nodes r and $r+\frac{1}{2}$ written in, for the horizontal components

$$S_{r\cdot r+1}-S_{r\cdot r-1}+(S_{r\cdot r+\frac{1}{2}}-S_{r\cdot r-\frac{1}{2}})\,\alpha+H_r$$
$$=K_1\Delta^2u_{r-1}+K_3\alpha^2(\nabla u_{r-}-2u_r)$$
$$-K_3\alpha\beta\Delta w_{r-\frac{1}{2}}+H_r=0\,, \qquad (4.2)$$

$$S_{r+\frac{1}{2}}-S_{r+\frac{1}{2}\cdot r+\frac{3}{2}}+(S_{r+\frac{1}{2}\cdot r+\frac{1}{2}}-S_{r\cdot r+\frac{1}{2}})\alpha+H_{r+\frac{1}{2}}$$
$$=K_2\Delta^2u_{r-\frac{1}{2}}+K_3\alpha^2(\nabla u_r-2u_{r+\frac{1}{2}})$$
$$+K_3\alpha\beta\Delta w_r+H_{r+\frac{1}{2}}=0 \qquad (4.3)$$

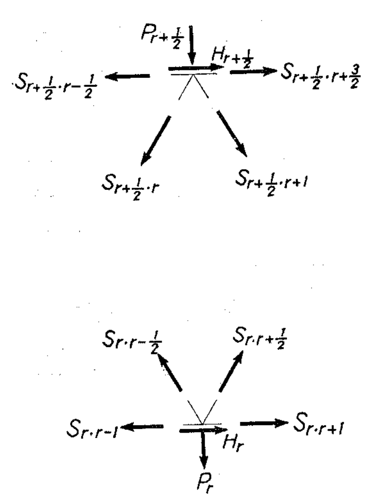

Fig. 2.

for the vertical components

$$(S_{r\cdot r+\frac{1}{2}}+S_{r\cdot r+\frac{1}{2}})\beta-P_r=K_3\alpha\beta\Delta u_{r-\frac{1}{2}}$$
$$-K_3\beta^2(\nabla w_{r-}-2w_r)-P_r=0 \qquad (4.4)$$

$$(S_{r+\frac{1}{2}\cdot r+1}+S_{r+\frac{1}{2}\cdot r})\beta+P_{r+\frac{1}{2}}=K_3\alpha\beta\Delta u_r$$
$$+K_3\beta^2(\nabla w_r-2w_{+r\frac{1}{2}})+P_{r+\frac{1}{2}}=0 \qquad (4.5)$$

in which H, P are the external forces acting at the nodal point.

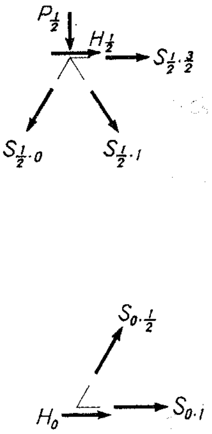

Fig. 3.

The equilibrium of forces at the nodal points 0, $\frac{1}{2}$ are

$$K_1\Delta u_0+K_3\alpha^2(u_{\frac{1}{2}}-u_0)-K_3\alpha\beta(w_{\frac{1}{2}}-w_0)+H_0=0, \qquad (4.6)$$
$$K_3\alpha\beta(u_{\frac{1}{2}}-u_0)-K_3\beta^2(w_{\frac{1}{2}}-w_0)+P_0=0\,, \qquad (4.7)$$
$$K_2\Delta u_{\frac{1}{2}}+K_3\alpha^2(\nabla u_0-2n_{\frac{1}{2}})+K_3\alpha\beta\Delta w_0+H_{\frac{1}{2}}=0\,, \qquad (4.8)$$
$$K_3\alpha\beta\Delta u_0+K_3\beta^2(\Delta w_0-2w_{\frac{1}{2}})+P_{\frac{1}{2}}=0\,, \qquad (4.9)$$

and the equations at the nodal points n, $n-\frac{1}{2}$ could easily be written in a similar way.

Applying the symbolic operators $\boldsymbol{S}_i$, $\boldsymbol{C}_i$, $\bar{\boldsymbol{S}}_i$, $\bar{\boldsymbol{C}}_i$ to the equations (4.2), (4.3), (4.4), (4.5) respectively and satisfying the boundary conditions that the truss is simply suppored, we have

$$\{K_1D_i+2K_3\alpha^2\}\,R_i[u_r]-2K_3\alpha^2\cos\frac{i\pi}{2n}\bar{C}_i[u_{r+\frac{1}{2}}]$$
$$+2K_3\alpha\beta\sin\frac{i\pi}{2n}\bar{\boldsymbol{S}}_i[w_{r+\frac{1}{2}}]=\boldsymbol{C}_i[H_r]\,, \qquad (4.10)$$

$$2K_3\beta^2\boldsymbol{S}_i[w_r]-2K_3\alpha\beta\sin\frac{i\pi}{2n}\bar{\boldsymbol{C}}_i[u_{r+\frac{1}{2}}]$$
$$-2K_3\beta^2\cos\frac{i\pi}{2n}\bar{\boldsymbol{S}}_i[w_{r+\frac{1}{2}}]=\boldsymbol{S}_i[P_r]\,, \qquad (4.11)$$

$$-2K_3\alpha^2\cos\frac{i\pi}{2n}\boldsymbol{R}_i[u_r]-2K_3\alpha\beta\sin\frac{i\pi}{2n}\boldsymbol{S}_i[w_r]$$
$$+\{K_2D_i+2K_3\alpha^2\}\bar{\boldsymbol{C}}_i[u_{r+\frac{1}{2}}]=\bar{\boldsymbol{C}}_i[H_{r+\frac{1}{2}}]\,, \qquad (4.12)$$

$$2K_3\alpha\beta\sin\frac{i\pi}{2n}\boldsymbol{R}_i[u_r]-2K_3\beta^2\cos\frac{i\pi}{2n}\boldsymbol{S}_i[w_r]$$
$$+2K_3\beta^2\bar{\boldsymbol{S}}_i[w_{r+\frac{1}{2}}]=\bar{\boldsymbol{S}}_i[P_{r+\frac{1}{2}}], \tag{4.13}$$

from which we find out the integration transforms of the displacements, and invert them into the actual displacements.

Taking the case when the concentrated load acts at the nodal point c, we obtain

$$\boldsymbol{C}_i[H_r]=\bar{\boldsymbol{C}}_i[H_{r+\frac{1}{2}}]=\bar{\boldsymbol{S}}_i[P_{r+\frac{1}{2}}]=0,$$
$$\boldsymbol{S}_i[P_r]=P\sin\frac{i\pi}{n}C,$$

from which the nodal displacements are written in the closed form, as follows

$$u_r=\frac{\alpha P}{K_1\cdot\beta}\cdot\begin{cases}\frac{c}{3n}\left\{(n^2-c^2)-3(n-r)^2\right\}, & r\geq c\\ \frac{n-c}{3n}\left\{3r^2-c(2n-c)\right\}, & r<c\end{cases} \tag{4.14}$$

$$u_{r+\frac{1}{2}}=\frac{\alpha P}{K_2B}\cdot\begin{cases}\frac{c}{3n}\left\{3(n-r)(n-r+1)-(n^2-c^2)\right\}, & r\geq c\\ \frac{n-c}{3n}\left\{c(2n-c)-3r(r+1)\right\} & r<c\end{cases} \tag{4.15}$$

$$w_r=\frac{2K_1-K_3\alpha^2}{K_1\cdot K_3\beta^2}P\cdot\begin{cases}\frac{c(n-r)}{n}\\ \frac{r(n-c)}{n}\end{cases}$$
$$+\frac{4\alpha^2(K_1+K_2)}{K_1\cdot K_3\cdot\beta^2}P\cdot\begin{cases}\frac{c(n-r)}{6n}\left\{r(2n-r)-c^2+1\right\}, & r\geq c\\ \frac{r(n-c)}{6n}\left\{c(2n-c)-r^2+1\right\}, & r<c\end{cases} \tag{4.16}$$

$$w_{r+\frac{1}{2}}=\frac{P}{K_3\cdot\beta^2}\cdot\begin{cases}\frac{c}{n}(2n-2r-1)\\ \frac{n-c}{n}(2r+1)\end{cases}$$
$$+\frac{\alpha^2(K_1+K_2)}{K_1\cdot K_2\beta^2}P\cdot\begin{cases}\frac{c}{3n}(2n-2r-1)\left\{r(2n-r-1)+(n-c^2)\right\}, & r\geq c\\ \frac{n-c}{3n}(2r+1)\left\{c(2n-c)-r(r+1)\right\}, & r<c\end{cases} \tag{4.17}$$

and the stresses of the members are written in

$$S_{r\cdot r+1}=\frac{\lambda P}{2h}\cdot\begin{cases}\dfrac{c}{n}(2n-2r-1), & r\geq c\\[2ex] \dfrac{n-c}{n}(2r+1), & r<c\end{cases}\tag{4.18}$$

$$S_{r+\frac{1}{2}\cdot r+\frac{1}{2}}=\frac{\lambda P}{h}\cdot\begin{cases}-\dfrac{c}{n}(n-r), & r\geq c\\[2ex] -\dfrac{r}{n}(n-c), & r<c\end{cases}\tag{4.19}$$

$$S_{r\cdot r+\frac{1}{2}}=\frac{P}{\beta}\cdot\begin{cases}\dfrac{c}{n}, & r\geq c\\[2ex] -\dfrac{n-c}{n}, & r<c\end{cases}\tag{4.20}$$

$$S_{r\cdot r-\frac{z}{1}}=\frac{P}{\beta}\cdot\begin{cases}-\dfrac{c}{n}, & r\geq c\\[2ex] \dfrac{n-c}{n}, & r<c.\end{cases}\tag{4.21}$$

5. Convergency of Finite Element Mthod by Means of Finite Integration Transforms

The most important question for the user of the finite element method, is whether the method yields sufficiently accurate results for his purpose. Numerous test calculations have been performed in order to compare results obtained by means of the finite element method with known analytical solution, but they have only partly answered the above question. There still remains the fundamental question whether the finest limit of the element can insure us the exact solution of the differential equation.

We will illustrate that the finite integration transforms can analitically examine the convergency of the finite element method.

(a) Plane Stress Problem

Let us take the triangular element as shown in Fig. 4, then the stiffness matrix $[K]$ is found as[4)]

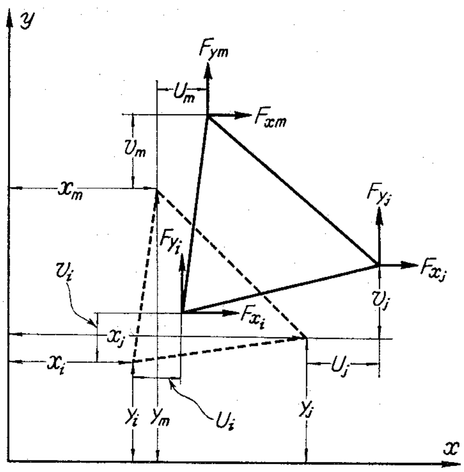

Fig. 4.

$$[K]=\begin{bmatrix} K_{ii} & K_{ij} & K_{im} \\ K_{ji} & K_{jj} & K_{jm} \\ K_{mi} & K_{mj} & K_{mm} \end{bmatrix}, \tag{5. 1}$$

where

$$[K_{rs}]=\frac{Et}{4\cdot A\cdot(1-\nu^2)}\begin{bmatrix} b_r b_s+\frac{1-\nu}{2}c_a c_s & \nu b_r c_s+\frac{1-\nu}{2}b_s c_r \\ \nu b_s c_r+\frac{1-\nu}{2}b_r c_s & c_r c_s+\frac{1-\nu}{2}b_r b_s \end{bmatrix},$$

$$b_i=y_j-y_m\,,\qquad c_i=x_m-x_j\,,$$

A, t; the area and the thickness of a triangular element respectively, from which we have for the case without body forces,

$$\{F\}^e=[K]\cdot\{\delta\}^e, \tag{5. 2}$$

where

$$\{F\}^e=\{F_{xi}\ F_{yi}\ F_{xj}\ F_{yi}\ F_{xm}\ F_{ym}\}^T,$$
$$\{\delta\}^e=\{u_i\ \ v_i\ \ u_j\ \ v_j\ \ u_m\ \ v_m\}^T.$$

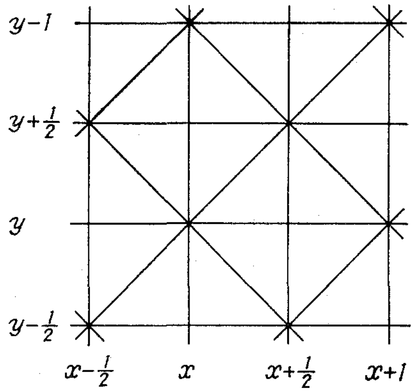

Fig. 5.

Assemble the triangular elements as shown in Fig. 5, then the equilibrium of forces in the x direction are expressed by, at the node (x, y)

$$\frac{4}{1-\nu^2}\left(u_{xy}-\frac{1}{2}\Delta_x u_{x-\frac{1}{2}\cdot y}\right)+\frac{2}{1+\nu}\left(u_{xy}-\frac{1}{2}\nabla_y u_{x\cdot y-\frac{1}{2}}\right)$$
$$-\frac{1}{2(1-\nu)}\Delta_x\Delta_y v_{x-\frac{1}{2}\cdot\cdot y-\frac{1}{2}}=0\,, \tag{5. 3}$$

at the node $\left(x+\frac{1}{2},\ y+\frac{1}{2}\right)$

$$\frac{4}{1-\nu^2}\left(u_{x+\frac{1}{2}\cdot y+\frac{1}{2}}-\frac{1}{2}\nabla_x u_{x\cdot y+\frac{1}{2}}\right)+\frac{2}{1+\nu}\left(u_{x+\frac{1}{2}\cdot y+\frac{1}{2}}\right.$$
$$\left.-\frac{1}{2}\nabla_y n_{x+\frac{1}{2}y}\right)-\frac{1}{2}\frac{1}{2(1-\nu)}\Delta_x\Delta_y v_{x\cdot y}=0\,, \tag{5. 4}$$

at the node $\left(x,\ y+\frac{1}{2}\right)$

$$\frac{4}{1-\nu^2}\left(u_{x\cdot y+\frac{1}{2}}-\frac{1}{2}\Delta_x n_{x-\frac{1}{2}y+\frac{1}{2}}\right)+\frac{2}{1+\nu}\left(u_{x\cdot y+\frac{1}{2}}-\frac{1}{2}\nabla_y u_{x\cdot y}\right)=0\,, \tag{5. 5}$$

at the node $\left(x+\frac{1}{2},\ y\right)$

$$\frac{4}{1-\nu^2}\left(n_{x+\frac{1}{2}\cdot y}-\frac{1}{2}\nabla_x u_{x\cdot y}\right)+\frac{2}{1+\nu}\left(u_{x+\frac{1}{2}\cdot y}-\frac{1}{2}\nabla_y u_{x+\frac{1}{2}\cdot y-\frac{1}{2}}\right)=0\,. \qquad (5.6)$$

Making the finite integration transforms from the equations (5.3)~(5.6), and combining them adequately, we have

$$\frac{1}{(1+\nu)(3-\nu)}\left\{4+\frac{D_i}{1-\nu}+\frac{1-\nu}{4}D_r\right\}\boldsymbol{R}_i\boldsymbol{S}_r[u_{x\cdot y}]$$
$$-\frac{4}{(1+\nu)(3-\nu)}\cos\frac{i\pi}{2n}\cos\frac{r\pi}{2m}\bar{\boldsymbol{C}}_i\bar{\boldsymbol{S}}_r[u_{x+\frac{1}{2}\cdot y+\frac{1}{2}}]$$
$$+\frac{1}{1-\nu}\sin\frac{i\pi}{2n}\sin\frac{r\pi}{2m}\bar{\boldsymbol{S}}_i\bar{\boldsymbol{C}}_r[u_{x+\frac{1}{2}\cdot y+\frac{1}{2}}]=0\,, \qquad (5.7)$$

$$\frac{1}{(1+\nu)(3-\nu)}\left\{4+\frac{D_i}{1-\nu}+\frac{1-\nu}{4}D_r\right\}\bar{\boldsymbol{C}}_i\bar{\boldsymbol{S}}_r[u_{x+\frac{1}{2}\cdot y+\frac{1}{2}}]$$
$$-\frac{4}{(1+\nu)(3-\nu)}\cos\frac{i\pi}{2n}\cos\frac{r\pi}{2m}\boldsymbol{R}_i\boldsymbol{S}_r[u_{x\cdot y}]$$
$$+\frac{1}{1-\nu}\sin\frac{i\pi}{2n}\sin\frac{r\pi}{2m}\boldsymbol{S}_i\boldsymbol{R}_r[u_{x\cdot y}]=0\,, \qquad (5.8)$$

in which boundary conditions are to be given as to eliminate the boundary values.

In a similar way, the equilibrium of forces in the y direction leads to the following results;

$$\frac{1}{(1+\nu)(3-\nu)}\left\{4+\frac{D_r}{1-\nu}+\frac{1-\nu}{4}D_i\right\}\bar{\boldsymbol{S}}_i\bar{\boldsymbol{C}}_r[u_{x+\frac{1}{2}\cdot y+\frac{1}{2}}]$$
$$-\frac{4}{(1+\nu)(3-\nu)}\cos\frac{i\pi}{2n}\cos\frac{r\pi}{2m}\boldsymbol{S}_i\boldsymbol{R}_r[u_{x\cdot y}]$$
$$+\frac{1}{1-\nu}\sin\frac{i\pi}{2n}\sin\frac{r\pi}{2m}\boldsymbol{R}_i\boldsymbol{S}_r[u_{x\cdot y}]=0\,, \qquad (5.9)$$

$$\frac{1}{(1+\nu)(3-\nu)}\left\{4+\frac{D_r}{1-\nu}+\frac{1-\nu}{4}D_i\right\}\boldsymbol{S}_i\boldsymbol{R}_r[v_{x\cdot y}]$$
$$-\frac{4}{(1+\nu)(3-\nu)}\cos\frac{i\pi}{2n}\cos\frac{r\pi}{2m}\bar{\boldsymbol{S}}_i\bar{\boldsymbol{C}}_r[v_{x+\frac{1}{2}\cdot y+\frac{1}{2}}]$$
$$+\frac{1}{1+\nu}\sin\frac{r\pi}{2n}\sin\frac{r\pi}{2m}\bar{\boldsymbol{C}}_i\bar{\boldsymbol{S}}_r[u_{x+\frac{1}{2}\cdot y+}]=0\,. \qquad (5.10)$$

Eliminating $\bar{\boldsymbol{C}}_i\bar{\boldsymbol{S}}_r[u_{x+\frac{1}{2}\cdot y+\frac{1}{2}}]$, $\bar{\boldsymbol{S}}_i\bar{\boldsymbol{C}}_r[v_{x+\frac{1}{2}\cdot y+\frac{1}{2}}]$ in the equations (5.7), (5.8) and (5.9), we have

$$\left\{4+\frac{D_i}{1-\nu}+\frac{1-\nu}{4}D_a\right\}\boldsymbol{R}_i\boldsymbol{S}_r[u_{x\cdot y}]-\frac{4\cos\frac{i\pi}{2n}\cos\frac{r\pi}{2m}}{\left\{4+\frac{D_i}{1-\nu}+\frac{1-\nu}{4}D_r\right\}}\cdot\left\{4\cos\frac{i\pi}{2n}\right.$$

$$\times \cos\frac{r\pi}{2m}\boldsymbol{R}_i\boldsymbol{S}_r[u_{x\cdot r}] - \frac{(1+\nu)(3-\nu)}{1-\nu}\sin\frac{i\pi}{2n}\sin\frac{r\pi}{2m}\boldsymbol{S}_i\boldsymbol{R}_r[u_{x\cdot y}]$$

$$+\frac{\dfrac{(1+\nu)(3-\nu)}{1-\nu}\sin\dfrac{i\pi}{2n}\sin\dfrac{r\pi}{2m}}{\left\{4+\dfrac{D_r}{1-\nu}+\dfrac{1-\nu}{4}D_i\right\}}\cdot\left\{4\cos\frac{i\pi}{2n}\cos\frac{r\pi}{2m}\boldsymbol{S}_i\boldsymbol{R}_r[v_{x\cdot y}]\right.$$

$$\left.-\frac{(1+\nu)(3-\nu)}{1-\nu}\sin\frac{i\pi}{2n}\sin\frac{r\pi}{2m}\boldsymbol{R}_i\boldsymbol{S}_r[u_{x\cdot y}]\right\}=0\,, \qquad (5.11)$$

which corresponds to the equilibrium of forces in the x direction.

Neglecting the higher order of D_i and D_r, we can write equation (5.11) in the following form;

$$\left\{\frac{D_i}{1-\nu^2}+\frac{D_r}{2(1+\nu)}\right\}\boldsymbol{R}_i\boldsymbol{S}_r[u_{x\cdot y}]+\frac{1}{2(1-\nu)}\sin\frac{i\pi}{n}\sin\frac{r\pi}{m}\boldsymbol{S}_i\boldsymbol{R}_r[v_{x\cdot y}]=0\,. \qquad (5.12)$$

Making the element be infinitly small, is equivalent to letting n and m be infinite, thus we have

$$D_i \doteqdot \left(\frac{i\pi}{n}\right)^2,\quad D_r \doteqdot \left(\frac{r\pi}{m}\right)^2,\quad \sin\frac{i\pi}{n}\doteqdot\left(\frac{i\pi}{n}\right),\quad \sin\frac{r\pi}{m}\doteqdot\left(\frac{r\pi}{m}\right).$$

In the other words, the more number of subdivision is used for the presented problem, the closer the finite integration transforms corresponding to it become the finite integral transforms. We therefore, write the equation (5.12) as follows

$$\frac{1}{(1-\nu^2)}\cdot\frac{\partial^2 u}{\partial x^2}+\frac{1}{2(1+\nu)}\cdot\frac{\partial^2 u}{\partial y^2}+\frac{1}{2(1-\nu)}\cdot\frac{\partial^2 v}{\partial x\partial y}=0\,, \qquad (5.13)$$

which is for the case of infinitesimal subdivision. Likewise, the equilibrium of forces in the y direction yields

$$\frac{1}{2(1-\nu)}\cdot\frac{\partial^2 u}{\partial x\partial y}+\frac{1}{2(1+\nu)}\cdot\frac{\partial^2 v}{\partial x^2}+\frac{1}{(1-\nu^2)}\cdot\frac{\partial^2 v}{\partial y^2}=0\,. \qquad (5.14)$$

A couple of equations (5.13) and (5.14) are well-known as the differential equations for the plane stress state. So it is concluded that this kind of subdivision method would lead us to the exact solution by letting the number of element be infinity.

(b) Bending Problem of Plate

A hybrid finite element method which was proposed by Kubo and Yoshida[5] will be taken into account. The notation $M=(M_x+M_y)/(1+\nu)$ leads to $M=-D(\partial^2 w/\partial x^2+\partial w^2/\partial y^2)$. M_i, M_j, M_k denote the values of M at the vertices on the triangular element i, j, k respectively, then the M-diagram is drawn as Fig. 6.

Assuming that the shearing forces along the sides of triangle are positive

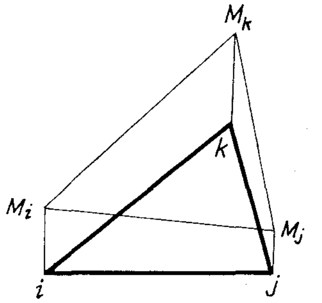

Fig. 6.

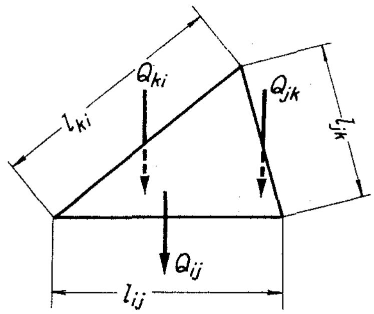

Fig. 7.

directing downward as shown in Fig. 7, we have

$$l_{ij}Q_{ij} = \frac{1}{4A_{ijk}}\left\{(l_{ij}^2+l_{jk}^2-l_{ki}^2)M_i+(l_{ij}^2+l_{kj}^2-l_{ij}^2)M_j-2l_{ij}^2M_k\right\},$$

in which A_{ijk} is the area of the triangle. These shearing forces may be replaced by the concentrated forces at the vertices as follows:

$$\overline{F}_i = \frac{1}{2}(l_{ij}Q_{ij}+l_{ik}Q_{ik}) = \frac{1}{8A_{ijk}}\left\{2l_{jk}^2M_i\right.$$
$$\left.+(l_{ij}^2-l_{jk}^2-l_{ki}^2)M_j+(l_{ki}^2-l_{ij}^2-l_{jk}^2)M_k\right\}.$$

The equilibrium of forces at the node i is expressed by

$$P_i = \sum \overline{F}_i \tag{5.15}$$

in the case of the distributed load, the equivalent vertex load should be introduced,

$$\overline{P}_i = \frac{A_{ijk}}{12}(2q_i+q_j+q_k)$$

where q_i, q_j, q_k denote the values of the distributed load at the vertices, and

$$P_i = \sum \overline{P}_i$$

which turns to

$$\sum \overline{P}_i = \sum \overline{F}_i \tag{5.16}$$

The similar discussion may be valied for the relation between the deflection and M,
thus

$$\overline{W}_i = \frac{A_{ijk}}{12D}(2M_i+M_j+M_k)$$

$$\overline{N}_i = \frac{1}{8A_{ijk}}\left\{2l_{jk}^2w_i+(l_{ij}^2-l_{jk}^2-l_{ki}^2)w_j+(l_{ki}^2-l_{ij}^2-l_{jk}^2)w_k\right\}$$

which is followed by

$$\sum \overline{W}_i = \sum \overline{N}_i \tag{5.17}$$

where w_i, w_j, w_k are the deflections at the vertices on the triangular element.

Setting the layout of the triangular element as shown in Fig. 8 we can write for the node (x, y) as

$$\sum \overline{F}_{x\cdot y} = \frac{1}{\sqrt{3}}(4M_{x\cdot y} - \Delta_x M_{x-1\cdot y} - \nabla_x \nabla_y M_{x-\frac{1}{2}\cdot y-\frac{1}{2}}),$$

$$\sum \overline{P}_{x\cdot y} = \frac{\sqrt{3}}{24} a^2 (8q_{x\cdot y} + \Delta_x^2 q_{x-1\cdot y} + \nabla_x \nabla_y q_{x-\frac{1}{2}\cdot y-\frac{1}{2}}),$$

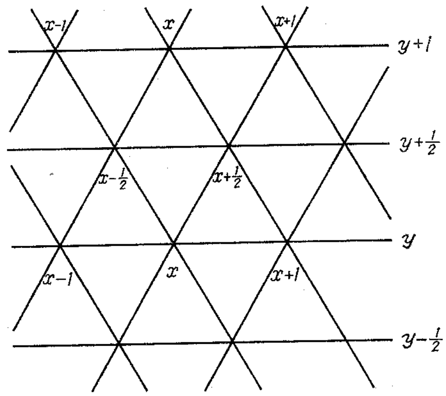

Fig. 8.

so the equation of the equilibrium of forces as follows

$$4M_{x\cdot y} - \Delta_x^2 M_{x-1\cdot y} - \nabla_x \nabla_y M_{x-\frac{1}{2}\cdot y-\frac{1}{2}} = \frac{a^2}{8}(8q_{x\cdot y} + \Delta_x^2 q_{x-1\cdot y} + \nabla_x \nabla_y q_{x-\frac{1}{2}\cdot y-\frac{1}{2}}), \tag{5.18}$$

from which the integration transform produces the following expression

$$(4-D_i)\boldsymbol{S}_i\boldsymbol{S}_r[M_{x\cdot y}] - 4\cos\frac{i\pi}{2n}\cos\frac{r\pi}{2m}\bar{\boldsymbol{S}}_i\bar{\boldsymbol{S}}_r[M_{x+\frac{1}{2}\cdot y+\frac{1}{2}}] = a^2\left(1-\frac{D_i}{8}\right)\boldsymbol{S}_i\boldsymbol{S}_r[q_{x\cdot y}] + \frac{a^2}{2}\cos\frac{i\pi}{2n}\cos\frac{r\pi}{2m}\bar{\boldsymbol{S}}_i\bar{\boldsymbol{S}}_r[q_{x+\frac{1}{2}\cdot y+\frac{1}{2}}]. \tag{5.19}$$

Likewise, at the node $\left(x+\frac{1}{2},\ y+\frac{1}{2}\right)$

$$(4+D_i)\bar{\boldsymbol{S}}_i\bar{\boldsymbol{S}}_r[M_{x+\frac{1}{2}\cdot y+\frac{1}{2}}] - 4\cos\frac{i\pi}{2n}\cos\frac{r\pi}{2m}\boldsymbol{S}_i\boldsymbol{S}_r[M_{x\cdot y}] = a^2\left(1-\frac{D_i}{8}\right)\bar{\boldsymbol{S}}_i\bar{\boldsymbol{S}}_r[q_{x+\frac{1}{2}\cdot y+\frac{1}{2}}] + \frac{a^2}{2}\cos\frac{i\pi}{2n}\cos\frac{r\pi}{2m}\boldsymbol{S}_i\boldsymbol{S}_r[q_{x\cdot y}]. \tag{5.20}$$

Equation (5.19) and (5.20) yield,

$$(12D_i + 4D_r + D_i^2 - D_i D_r)\boldsymbol{S}_i\boldsymbol{S}_r[M_{xy}] = a^2\left(6-\frac{1}{2}D_r-\frac{1}{8}D_i^2+\frac{1}{8}D_iD_r\right)\boldsymbol{S}_i\boldsymbol{S}_r[q_{x\cdot y}] + 6a^2\cos\frac{i\pi}{2n}\cos\frac{r\pi}{2m}\bar{\boldsymbol{S}}_i\bar{\boldsymbol{S}}_r[q_{x+\frac{1}{2}\cdot y+\frac{1}{2}}]. \tag{5.21}$$

Making subdivision infinitely small, and increasing their numbers to infinity, we can write

$$D_i \doteqdot \left(\frac{i\pi}{n}\right)^2, \quad D_r \doteqdot \left(\frac{r\pi}{m}\right)^2$$

$$\cos\frac{i\pi}{2n} \doteqdot 1-\frac{1}{8}\left(\frac{i\pi}{n}\right)^2, \quad \cos\frac{i\pi}{2m} \doteqdot 1-\frac{1}{8}\left(\frac{r\pi}{m}\right)^2.$$

We substitute the above into the equation (5.21), and neglect the higher order term, the equation (5.21) finally becomes

$$\left\{12\left(\frac{i\pi}{n}\right)^2+4\left(\frac{r\pi}{m}\right)^2\right\} \boldsymbol{S}_i\boldsymbol{S}_r[M_{x\cdot y}] = 12a^2\boldsymbol{S}_i\boldsymbol{S}_r[q_{x\cdot y}]. \tag{5.22}$$

Denoting l_x and l_y the lengthes of the plate in the x and y directions, we have

$$\sqrt{3}\,ma = l_y, \qquad na = l_x,$$

then equation (5.22) can be written as follows:

$$\left\{\left(\frac{i\pi}{l_x}\right)^2+\left(\frac{r\pi}{l_y}\right)^2\right\} \boldsymbol{S}_i\boldsymbol{S}_r[M_{xy}] = \boldsymbol{S}_i\boldsymbol{S}_r[q_{x\cdot y}], \tag{5.23}$$

which is equivalent to

$$\frac{\partial^2 M}{\partial x^2}+\frac{\partial^2 M}{\partial y^2} = -q \tag{5.24}$$

in a same way, we come to the expression

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2} = -\frac{M}{D} \tag{5.25}$$

Thus, the method considered is convergent to the differential equation of the bending of plate.

6. Conclusion

An analitical approach, the finite integration transforms, for finding the solution for regular structural lattices or the assemble of a regularly distributed finite element is presented.

The approach which the authors had partly presented can treat the problem of the regulaly distributed triangular net by the aid of a set of formulas regarding $\sin\frac{i\pi}{n}\left(x+\frac{1}{2}\right)$ and $\cos\frac{i\pi}{n}\left(x+\frac{1}{2}\right)$.

(Received May 20, 1971)

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