CORE
Metadata, citation and similar papers at core.ac.uk
Provided by Muroran-IT Academic Resource Archive

室蘭工業大学
学術資源アーカイブ
Muroran Institute of Technology Academic Resources Archive

Remarks to the paper "On Lie Derivatives in Areal Spaces"

著者	IGARASHI Takanori
journal or publication title	Memoirs of the Muroran Institute of Technology. Science and engineering
volume	6
number	3
page range	951-955
year	1969-07-15
URL	http://hdl.handle.net/10258/3474

Remarks to the paper "On Lie Derivatives in Areal Spaces"

Takanori Igarashi*)

Abstract

In the previous paper[3], the author extended the notion of Lie derivative to the areal space of general type, by the aid of theories[1),2)] mainly.

There, we treated p as m-ple element p^i_α in the fundamental function $F(x, p)$ of the areal space $A_n^{(m)}$..

However, the theory of the areal space was essentially started from the treatise p as m-dimensional "area"-element $p^{i[m]}$, so it is desirable that the theory of Lie derivative of $A_n^{(m)}$ is rewritten from this point of view. In this paper, we try to rewrite in the above-mentioned way.

More interesting results will be developed in the forth-coming paper.

1. By the reviewer[4], the summary of the previous paper is as follows:
The author considers an infinitesimal transformation of type

$$\bar{x}^i = x^i + \xi^i(x)\,dt \tag{1. 1}$$

which maps a point x of a surface $V_m : x^i = x^i(u^\alpha)$**) to a point $\bar{x}$ of a surface $\bar{V}_m : \bar{x}^i = \bar{x}^i(u^\alpha)$. Under this transformaion the m-ple element $p^i_\alpha \equiv \partial x^i/\partial u^\alpha$ is transformed to

$$\bar{p}^i_\alpha = p^i_\alpha + \xi^i_{,j} p^j_\alpha dt \quad \text{with} \quad \xi^i_{,j} \equiv \partial \xi^i/\partial x^j\,. \tag{1. 2}$$

When a geometric object $\Omega(x, p)$ is transformed to $\bar{\Omega}(\bar{x}, \bar{p})$ by (1. 1), the Lie derivative of Ω with respect to ξ is defined as

$$\pounds_\xi \Omega = \lim_{dt\to 0} \left\{\bar{\Omega}(\bar{x}, \bar{p}) - \Omega(\bar{x}, \bar{p})\right\}\Big/ dt\,. \tag{1. 3}$$

We call a transformation (1. 1) satisfying $\pounds_\xi F = 0$ an areal motion, because it does not change the area $S = \int\cdots\int F du^1 \cdots du^m$ of an m-dimensional surface in the space.

The main results of the paper are as follows.

(A). In order that the space admits an areal motion, it is necessary and sufficient that the Lie derivative of the metric m-tensor $g_{i[m],j[m]}$ vanishes.

(B). If the vector ξ^i in (1. 1) is transversal to p^i_α with respect to F, then the transformation (1. 1) is an areal motion.

---

*) 五十嵐敬典

**) Latin indices run over $1, 2, \cdots, n$; Greek indices over $1, 2, \cdots, m$ $(1 \leqq m < n)$ in section 1 and over 1, 2 in other sections.

(C). If the space $A_n^{(m)}$ admits an areal motion, then $\pounds_\xi C_{j,l}^{i\ \alpha}=0$.

If the space $A_n^{(m)}$ is of submetric class, the metric tensor g_{ij} can be introduced. Then we have:

(D). A motion is an areal motion.

(E). When the areal space $A_n^{(m)}$ admits a motion (1. 1), then $\pounds_\xi \tau^{*i}_{jk}=0$ and $\pounds_\xi C_{j,l}^{i\ \alpha}=0$.

2. In this section, we take in the areal space $A_n^{(2)}$ in place of $A_n^{(m)}$ for convenience. Under the tranformations (1.1) and (1.2), the bivector p^{ij} is transformed as

$$\bar{p}^{ij}=2\bar{p}^i_{[1}\bar{p}^j_{2]}=2(p^i_{[1}+\xi^i_{,|k|}p^k_{1}dt)(p^j_{2]}+\xi^j_{,|l|}p^l_{2]}dt)$$
$$=p^{ij}+2\xi^{[i}_{,h}p^{|h|j]}dt+\xi^i_{,k}\xi^j_{,l}p^{kl}dt^2,$$

therefore, the variations of x^i and p^{ij} are represented as follows;

$$\delta x^i\equiv\bar{x}^i-x^i=\xi^i dt \tag{2. 1}$$

$$\delta p^{ij}\equiv\bar{p}^{ij}-p^{ij}=2\xi^{[i}_{,h}p^{|h|j]}dt+\xi^i_{,k}\xi^j_{,l}p^{kl}dt^2. \tag{2. 2}$$

If a contravariant vector $X(x^h, p^{ij})$ is transformed to $\bar{X}(\bar{x}^h, \bar{p}^{ij})$ by (1.1), then

$$\overset{v}{d}X^i=X^i_{,h}\delta x^h+X^i_{;kl}\delta p^{kl}$$
$$=X^i_{,h}\xi^h dt+X^i_{;kl}\{2\xi^{[k}_{,h}p^{|h|l]}dt+\xi^k_{,r}\xi^l_{,s}p^{rs}dt^2\}. \tag{2. 3}$$

Now, in the other hand, we interpret that (1.1) is an infinitesimal coordinate transformation, then

$$\frac{\partial\bar{x}^i}{\partial x^j}=\delta^i_j+\xi^i_{,j}dt,\qquad \frac{\partial x^i}{\partial\bar{x}^j}=\delta^i_j-\xi^i_{,j}dt,\ \cdots \tag{2. 4}$$

neglecting higher order terms with respect to dt.

If the contravariant vector X^i is transformed by the coordinate transformation (1. 1), then

$$\bar{X}^i=\frac{\partial\bar{x}^i}{\partial x^j}X^j=(\delta^i_j+\xi^i_{,j}dt)X^j=X^i+\xi^i_{,j}X^j dt, \tag{2. 5}$$
$$\overset{m}{d}X^i\equiv\bar{X}^i-X^i=\xi^i_{,j}X^j dt.$$

Substituting (2. 3) and (2. 5) into the definition (1. 3), we have

$$\pounds_\xi X^i=X^i_{,h}\xi^h+2X^i_{;kl}\xi^{[k}_{,h}p^{|h|l]}-\xi^s_{,h}X^h. \tag{2. 6}$$

This is the new definition of Lie derivative of contravariant vector X^i with respect to ξ. For a covariant vector $Y(x^h, p^{ij})$ and for a tensor $T(x^h, p^{ij})$ of 1–1 type, we have analogously that

$$\pounds_\xi Y_i=Y_{i,h}\xi^h+2y_{i;kl}\xi^{[k}_{,h}p^{|h|l]}+\xi^h_{,i}Y_h, \tag{2. 7}$$

$$\pounds_\xi T^i_j=T^i_{j,h}\xi^h+2T^i_{j;kl}\xi^{[k}_{,h}p^{|h|l]}-\xi^i_{,h}T^h_j+\xi^h_{,j}T^i_h. \tag{2. 8}$$

These expressions (2. 6), (2. 7) and (2. 8) are new definition of Lie derivative in $A_n^{(2)}$ with use of p^{ij}.

3. In the previous paper, the author defined the Lie derivative of the contravariant vector $X(x^h, p^{ij})$ in the form;

$$\mathfrak{L}_\xi X^i = X^i_{,h}\xi^h + X^{i\alpha}_{;l}\xi^l_{,h}p^h_\alpha - \xi^i_{,h}X^h . \tag{3. 1}$$

The term $X^{i\alpha}_{;l}(\xi^l_{,h}p^h_\alpha$ is rewritten as follows;

$$\begin{aligned} X^{i\alpha}_{;l}\xi^l_{,h}p^h_\alpha &= X^i_{;jk}p^{jk\alpha}{}_{;l}\xi^l_{,h}p^h_\alpha \\ &= X^i_{;jk}(\delta^j_l\delta^\alpha_1 p^k_2 + p^j_1\delta^k_l\delta^\alpha_2 - \delta^j_l\delta^\alpha_2 p^k_1 - p^j_2\delta^k_2\delta^\alpha_1)\,\xi^l_{,h}p^h_\alpha \\ &= X^i_{;jk}\left\{\xi^j_{,h}(p^h_1p^k_2 - p^h_2p^k_1) - \xi^k_{,h}(p^h_1p^j_2 - p^h_2p^j_1)\right\} = 2X^i_{;jk}\xi^{[j}_{,h}p^{|h|k]} . \end{aligned}$$

Hence, we can conclude as follows:

Lemma 1. The new definition (2. 6) of Lie derivative is coinside with the definition (3. 1) of the previous paper.

Moreover, we have the following:

Lemma 2. The Lie derivative defined by (2. 6) and (2. 7) satisfies the Leibnitz' rule, that is,

$$\mathfrak{L}_\xi(X^i Y_i) = (\mathfrak{L}_\xi X^i)\, Y_i + X^i(\mathfrak{L}_\xi Y_i) . \tag{3. 2}$$

Proof) On account of (2. 8), we can see

$$\begin{aligned} \mathfrak{L}_\xi(X^i Y_i) &= (X^i Y_i)_{,h} + 2(X^i Y_i)_{;kl}\xi^{[k}_{,h}p^{|h|l]} - \xi^i_{,h}(X^h Y_i) + \xi^h_{,i}(X^i Y_h) \\ &= (X^i_{,h}\xi^h + 2X^i_{;kl}\xi^{[k}_{,h}p^{|h|l]} - \xi^i_{,h}X^h)\, Y_i \\ &\quad + X^i(Y_{i;k}\xi^k + 2Y_{i;kl}\xi^{[k}_{,h}p^{|h|l]} + \xi^h_{,i}Y_h) . \end{aligned}$$

On the end of this section, we consider two infinitesimal transformation;

$$\bar{x}^i = x^i + \xi^i(x)\,dt , \qquad \bar{x}^i = x^i + \eta^i(x)\,dt .$$

If we operate $\mathfrak{L}_\xi$ and $\mathfrak{L}_\eta$ successively, then, after somewhat complicated calculations,

$$\begin{aligned} \mathfrak{L}_\eta\mathfrak{L}_\xi X^i - \mathfrak{L}_\xi\mathfrak{L}_\eta X^i &= X^i_{,h}(\xi^h_{,j}\cdot\eta^j - \xi\eta^h_{,j}) \\ &\quad + 2X^i_{;rs}(\xi^{[r}_{,h,j}\eta^{|j|} + \xi^{[r}_{,j}\cdot\eta^{|j|}_{,h} - \xi^{|j|}_{,h}\eta^{[r}_{,j} - \xi^{|j|}\eta^{[r}_{,h,j})\,p^{|h|s]} \\ &\quad - (\xi^i_{,h,j}\eta^j + \xi^i_{,j}\eta^j_{,h} - \xi^j_{,h}\eta^i_{,j} - \xi^j\eta^i_{,h,j})\,X^h , \end{aligned}$$

hence,

$$\mathfrak{L}_\eta\mathfrak{L}_\xi X^i - \mathfrak{L}_\eta\mathfrak{L}_\xi X^i = \mathfrak{L}_\zeta X^i , \tag{3. 3}$$

$$\mathfrak{L}_\zeta X^i = X^i_{,h}\zeta^h + 2X^i_{;rs}\zeta^{[r}_{,h}p^{|h|s]} - \zeta^i_{,h}X^h , \tag{3. 4}$$

where we put

$$\zeta^i \equiv \xi^i_{,j}\eta^j - \xi^j\eta^i_{,j} = \mathfrak{L}_\eta\xi^i = -\mathfrak{L}_\xi\eta^i$$

These facts tell us the following:

Theorem 1. If (1. 1) belongs to a transformation group, that is, ξ^i, η^i, ζ^i, $\cdots$ are elements of an r-parameter group of transformation, then $\mathfrak{L}$'s in (2. 6) are r infinitesimal operators of an r-parameter group of transformations and (3. 3) with (3. 4) holds good.

4. A. Kawaguchi and Y. Katsurada[1] defined a line-metric connection in the areal space $A_n^{(2)}$ in the form;

$$DX^i = X^i_{/h}dx^h + X^i_{/\!/kl}\omega^{kl}, \tag{4.1}$$

where

$$X^i_{/h} \equiv X^i_{,h} - X^i_{;kl}B^{kl}_h + \Gamma^{*i}_{jh}X^j, \tag{4.2}$$

$$X^i_{/\!/h} \equiv FX^i_{,kl} + FC^i_{j,kl}X^j, \tag{4.3}$$

and

$$B^{kl}_h \equiv 4p^{[k}_{[1}B^{l]}_{2]h}, \qquad B^l_{ah} \equiv \Gamma^{*l}_{jk}p^j_a.$$

Sice the transformation vector ξ^i depends only on position x, so

$$\xi^i_{/h} = \xi^i_{,h} - \Gamma^{*i}_{jh}\xi^j. \tag{4.4}$$

From (4.2) and (4.4), the expression of defintion (2.6) is rewritten such that

$$\begin{aligned}\pounds_\xi X^i = {} & (X^i_{/h} + X^i_{;kl}B^{kl}_h - \Gamma^{*i}_{jh}X^j)\,\xi^h \\ & + 2X^i_{;kl}(\xi^{[k}_{/h} - \Gamma^{*|k}_{jh}\xi^{|j|})\,p^{|h|l]} - (\xi^i_{/h} - \Gamma^{*i}_{jh}\xi^j)\,X^h,\end{aligned}$$

and by means of

$$\begin{aligned}B^{kl}_h &= 4p^{[k}_{[1}B^{l]}_{2]h} = p^k_1B^l_{2h} - p^l_1B^h_{2h} + p^l_2B^k_{1h} - p^k_2B^l_{1h} \\ &= 2\Gamma^{*[k}_{jh}p^{|h|l]},\end{aligned} \tag{4.5}$$

we obtain finally

$$\pounds_\xi X^i = X^i_{/h}\xi^h + 2X^i_{;kl}\xi^{[k}_{/h}p^{|h|l]} - \xi^i_{/h}X^h. \tag{4.6}$$

Now, we apply the Lie derivative (4.5) to the fundamental function

$$\pounds_\xi F = F_{/h} - F_{;rs}B^{rs}_h. \tag{4.7}$$

If we recall the relation $F_{;rs} = 2F^{[1}_{;[r}p^{2]}_{s]}$ and (4.5), we can easily see that

$$F_{;rs}B^{rs}_h = F^\alpha_{;l}B^l_{\alpha h}.$$

Accordingly, we can say that the expression of the Lie derivative (4.7) of the fundamental functions is coinside with that of the previous paper.

Between the metric bitensor $g_{ij,kl}$ and F, there is a relation

$$g_{ij,kl}p^{ij}p^{kl} = 4F^2.$$

Differentiating both sides of this relation by p^{ij}, we have

$$g_{ij,kl;rs}p^{ij}p^{kl} + g_{ij,kl}\delta^{ij}_{rs}p^{kl} + g_{ij,kl}p^{ij}\delta^{kl}_{rs} = 8FF_{;rs}$$

and making use of $g_{ij,kl;rs}p^{kl}=0$, $g_{ij,kl}p^{kl}=2G_{ij}$ and $g_{ij,kl}p^{ij}=g_{kl,ij}p^{ij}=2G_{kl}$, and putting $F_{/h}=0$, finally we have

$$\pounds_\xi F = \frac{1}{2F}\,G_{rs}\xi^{[r}_{/h}p^{|h|s]}. \tag{4.8}$$

In view of (4. 8), we can conclude the following:

Theorem 2. If the transversal bivector G_{ij} vanishes, then the space $A_n^{(2)}$ admits an areal motion.

Acknowledgement: The present author wishes to express his best thanks to Prof. A. Kawaguchi for his cordial encouragement.

Mathematical Seminar of Muroran Institute of Technology, Muroran, Japan.

(Received Apr. 30, 1969)

References

1) A. Kawaguchi and Y. Katsurada: On areal space IV, Connection parameters in an areal space of general type. Tensor Vol. 1 (1951), 137–156.
2) K. Yano: The theory of Lie derivatives and its applications, North-Holland Pub., Amsterdam. (1957).
3) T. Igarashi: On Lie derivatives in areal spaces, Tensor Vol. 18 (1967), 205–211.
4) Mathematical Review, 36 (1968), 3302.