室蘭工業大学
学術資源アーカイブ
Muroran Institute of Technology Academic Resources Archive

A Note on Finite Fourier Transforms Concerning Finite Integration

著者	NOMACHI Sumio
journal or publication title	Memoirs of the Muroran Institute of Technology
volume	5
number	2
page range	711-718
year	1966-08-25
URL	http://hdl.handle.net/10258/3271

A Note on Finite Fourier Transforms Concerning Finite Integration

Sumio G. Nomachi*

Abstract

Since the differential equation has a basic similarity with the finite difference equation, it is expectable that the principle of an analitical method which is good for the one should analogically be valid for the other. On this standpoint the discussion is focused to the finite sine and cosine transforms which play an important part in the boundary problems of the differential equations, in this paper. The inversioe theorem of finite cosine series is discussed first, and the method of finite sine and cosine transforms for solving simultaneous finite difference equations is presented.

1. The inversion Theorem of Finite Cosine Series Concerning Finite Integration

Let x denote a variable defined by integer from zero to n, then any function $f(x)$ apparently makes sense only when x takes the prescribing integer. If $f(x)$ never takes infinite or multi value, $n+1$ functions different from each other, which are continuously single valued from zero to n:

$$F_r(x) \qquad (r=0, 1, \cdots, n)$$

can be combined as follows:

$$\sum_{r=0}^{n} A_r F_r(x) = f(x), \tag{1}$$

so long as

$$(F_r(x)) \fallingdotseq 0,$$

where $r, x=0, 1, 2, \cdots, n$.

If $F(x)$ is confined to be cosine functions, the expression (1) is written in

$$A_0 + \sum_{r=1}^{r=n} A_r \cos\frac{r\pi x}{n} = f(x). \tag{2}$$

Finite integration of x from 1 to $n-1$, transforms (2) into

$$\mathop{S}_1^n A_0 \Delta x + \mathop{S}_1^n \sum_{r=1}^{n} A_r \cos\frac{r\pi x}{n} \Delta x = \mathop{S}_1^n f(x)\cdot\Delta x,$$

which yields

$$A_0(n-1) - \frac{1}{2}\sum_{r=1}^{n} A_r\left\{(-1)^r + 1\right\} = \mathop{S}_1^n f(x)\cdot\Delta x$$

and accordingly

---

* 能 町 純 雄

$$A_0 n-\frac{1}{2}\{f(n)+f(0)\}=\sum_{x=1}^{n-1}f(x)\,. \tag{3}$$

Multiplication of (2) and $\cos m\pi x/n$, through finite integration by x from 1 to $n-1$, yields

$$-\frac{1}{2}A_0\{(-1)^m+1\}+A_m\frac{n}{2}-\sum_{r=1}^{n}\frac{A_r\{(-1)^{r+m}+1\}}{2}=\sum_{x=1}^{n-1}f(x)\cos\frac{m\pi x}{n}\,,$$

where $m\neq 0,\ n$, because

$$\begin{aligned}&\mathop{S}_{1}^{n}\cos\frac{r\pi x}{n}\cos\frac{m\pi x}{n}\Delta x\\&=\frac{1}{4}\left[\frac{\sin\pi\dfrac{(r-m)}{n}\left(x-\dfrac{1}{2}\right)}{\sin\dfrac{\pi}{2n}(r-m)}+\frac{\sin\dfrac{\pi(r+m)}{n}\left(x-\dfrac{1}{2}\right)}{\sin\dfrac{\pi(r+m)}{2n}}\right]_1^n\\&=\frac{1}{4}\left[\frac{\sin\pi(r-m)}{\sin\dfrac{\pi}{2n}(r-m)}\cos\frac{\pi(r-m)}{2n}-2(-1)^{m+r}-2\right]\end{aligned} \tag{4}$$

which is followed by

$$\mathop{S}_{1}^{n}\cos\frac{r\pi x}{n}\cos\frac{m\pi x}{n}\Delta x=\begin{cases}\dfrac{1}{2}\left(n-(-1)^{m+r}-1\right) & (m=r)\\ -\dfrac{1}{2}\{(-1)^{m+r}+1\} & (m\neq r)\,.\end{cases}$$

So that

$$A_m\frac{n}{2}-\frac{1}{2}\{f(n)(-1)^m+f(0)\}=\sum_{x=1}^{n-1}f(x)\cos\frac{m\pi x}{n}\,. \tag{5}$$

When $m=n$, the same procedure as the above produces a different result, because

$$\lim_{r\to n}\frac{\sin\pi(r+n)}{\sin\dfrac{\pi}{2n}(r+n)}=\lim_{\varepsilon\to 0}\frac{\sin\pi(2n-\varepsilon)}{\sin\dfrac{\pi}{2n}(2n-\varepsilon)}=-2n$$

from which (4) yields

$$\mathop{S}_{1}^{n}\cos\frac{r\pi x}{n}\cos\pi x\cdot\Delta x=\begin{cases}n-\dfrac{1}{2}\{(-1)^{n+r}+1\} & (n=r)\\ -\dfrac{1}{2}\{(-1)^{n+r}+1\} & (n\neq r)\,.\end{cases}$$

Consequently

$$A_n n-\frac{1}{2}\{f(n)(-1)^n-f(0)\}=\sum_{x=1}^{n-1}f(x)\cos\pi x\,. \tag{6}$$

These results are summed up to be the inversion formula as

$$\left.\begin{aligned} & A_0+\sum_{m=1}^{n} A_m \cos\frac{m\pi x}{n}=f(x)\,, \\ & \text{where} \\ & A_0=\frac{1}{n}\left[\sum_{x=1}^{n-1} f(x)+\frac{1}{2}\{f(n)+f(0)\}\right], \\ & A_m=\frac{2}{n}\left[\sum_{x=1}^{n-1} f(x)\cos\frac{m\pi x}{n}+\frac{1}{2}\{f(n)(-1)^m+f(0)\}\right], \\ & A_n=\frac{1}{n}\left[\sum_{x=1}^{n-1} f(x)\cos\pi x+\frac{1}{2}\{f(n)(-1)^n+f(0)\}\right], \end{aligned}\right\} \qquad (7)$$

while the inversion formula regarding the finite sine series takes a simpler form than of the finite cosine series: that is

$$\left.\begin{aligned} & \sum_{m=1}^{n-1} A_m \sin\frac{m\pi x}{n}=f(x)^{1)} \\ & A_m=\frac{2}{n}\sum_{x=1}^{n-1} f(x)\sin\frac{m\pi x}{n} \end{aligned}\right\} \qquad (8)$$

which can not be valid for $x=0$, and $x=n$, when $f(n)\neq 0$ and $f(0)\neq 0$.

2. Finite Fourier Transforms of Particular Differences

Since the 2nd and 4th difference frequently appear in engineering problems, and they are sometimes accompanied by the difference

$$f(x+1)-f(x-1)=\Delta f(x+1)+\Delta f(x)\,,^*$$

the finite sine and cosine transforms corresponding to the above mentioned differences will be introduced.

$$\mathop{S}_{1}^{n} \Delta^r f_{(x-r+1)} \sin\frac{m\pi x}{n}\Delta x=\mathop{S}_{0}^{n} \Delta^r f_{(x-r+1)} \sin\frac{m\pi x}{n}\Delta x$$

$$\begin{aligned} \mathop{S}_{0}^{n} \Delta^r f_{(x-r+1)} \sin\frac{m\pi}{n}x\cdot\Delta x &= \Delta^{r-1} f_{(x-r+1)} \sin\frac{m\pi}{n}x\Big]_0^n \\ &\quad -2\sin\frac{m\pi}{2n}\Delta^{r-2} f_{(x-r+2)}\cos\frac{m\pi}{n}\left(x+\frac{1}{2}\right)\Big]_0^n \\ &\quad -4\left(\sin\frac{m\pi}{2n}\right)^2 \mathop{S}_{0}^{n} \Delta^{r-2} f_{(x-r+3)} \sin\frac{m\pi}{n}(x+1)\Delta x \end{aligned}$$

which, with the consideration of

$$\begin{aligned} \mathop{S}_{0}^{n} \Delta^{r-2} f_{(x-r+3)} \sin\frac{m\pi}{n}(x+1)\cdot\Delta x &= \mathop{S}_{1}^{n+1} \Delta^{r-2} f_{(x-r+3)} \sin\frac{m\pi x}{n}\Delta x \\ &= \mathop{S}_{0}^{n} \Delta^{r-2} f_{(x-r+2)} \sin\frac{m\pi x}{n}\Delta x\,, \end{aligned}$$

* $\Delta f(x)=f(x+1)-f(x)$

yields

$$\left.\begin{aligned}
&\mathop{\text{\large S}}_{0}^{n} \Delta^r f_{(x-r+1)} \sin\frac{m\pi}{n}x\cdot\Delta x = \Delta^{r-1} f_{(x-r+1)} \sin\frac{m\pi}{n}x\Big]_0^n \\
&\quad -2\sin\frac{m\pi}{2n}\Delta^{r-2} f_{(x-r+2)}\cos\frac{m\pi}{n}\left(x+\frac{1}{2}\right)\Big]_0^n \\
&\quad -4\sin^2\frac{m\pi}{2n}\Delta^{r-3} f_{(x-r+3)}\sin\frac{m\pi}{n}x\Big]_0^n \\
&\quad +8\sin^3\frac{m\pi}{2n}\Delta^{r-4} f_{(x-r+4)}\cos\frac{m\pi}{n}\left(x+\frac{1}{2}\right)\Big]_0^n \\
&\quad +16\sin^4\frac{m\pi}{2n}\mathop{\text{\large S}}_{1}^{n}\Delta^{r-4} f_{(x-r+4)}\sin\frac{m\pi}{n}(x+1)\cdot\Delta x\,.
\end{aligned}\right\} \tag{9}$$

From the above it follows that

$$\begin{aligned}
\sum_{x=1}^{n-1}\Delta^2 f_{(x-1)}\sin\frac{m\pi}{n}x &= -\sin\frac{m\pi}{n}\left\{(-1)^m f(n)-f(0)\right\} \\
&\quad -2\left(1-\cos\frac{m\pi}{n}\right)\sum_{1}^{n-1} f(x)\sin\frac{m\pi}{n}x\,,
\end{aligned} \tag{10}$$

$$\begin{aligned}
\sum_{x=1}^{n-1}\Delta^4 f_{(x-2)}\sin\frac{m\pi}{n}x &= -\sin\frac{m\pi}{n}\left\{(-1)^m\Delta^2 f(n-1)-\Delta^2 f(0)\right\} \\
&\quad +4\sin^2\frac{m\pi}{2n}\sin\frac{m\pi}{n}\left\{(-1)^m f(n)-f(0)\right\} \\
&\quad +16\sin^4\frac{m\pi}{2n}\sum_{1}^{n-1} f(x)\sin\frac{m\pi}{n}x\,.
\end{aligned} \tag{11}$$

which are the relations between the finite sine transforms of the 2nd and 4th difference of a function.

The finite cosine tranforms of the 2nd difference of the function is defined as

$$\begin{aligned}
\mathop{\text{\large S}}_{1}^{n} \Delta^2 f(x-1)\cos\frac{m\pi}{n}x\cdot\Delta x &= \Delta f(n-1)(-1)^m - \Delta f(0)\cos\frac{m\pi}{n} \\
&\quad +2f(n)\sin\frac{m\pi}{2n}\cdot\sin\frac{m\pi}{n}\left(n+\frac{1}{2}\right) \\
&\quad -2f(1)\sin\frac{m\pi}{2n}\sin\frac{m\pi}{n}\left(1+\frac{1}{2}\right)-4\sin^2\frac{m\pi}{2n} \\
&\quad \times\mathop{\text{\large S}}_{1}^{n} f(x+1)\cos\frac{m\pi}{n}(x+1)\Delta x\,,
\end{aligned} \tag{12}$$

the left side of which is found by finitely integrating by part, and the series expression becomes

$$\begin{aligned}
\sum_{x=1}^{n-1}\Delta^2 f(x-1)\cos\frac{m\pi}{n}x &= \Delta f(n-1)(-1)^m - \Delta f(0) - \left\{f(n)(-1)^m + f(0)\right\} \\
&\quad \times\left\{1-\cos\frac{m\pi}{n}\right\} - 4\sin^2\frac{m\pi}{2n}\sum_{x=1}^{n-1} f(x)\cos\frac{m\pi}{n}x\,.
\end{aligned} \tag{13}$$

The substitution of $\Delta^2 f(x-1)$ for $f(x)$ in the above yields

$$\sum_{x=1}^{n-1} \Delta^4 f(x-2)\cos\frac{m\pi}{n}x$$

$$= \Delta^3 f(n-2)(-1)^m - \Delta^3 f(0) - \left\{\Delta^2 f(n-1)(-1)^m + \Delta^2 f(0)\right\}\left\{1-\cos\frac{m\pi}{n}\right\}$$

$$-4\sin^2\frac{m\pi}{2n}\left\{\Delta f(n-1)(-1)^m - \Delta f(0)\right\} + 4\sin^2\frac{m\pi}{2n}\left(1-\cos\frac{m\pi}{n}\right)$$

$$\times\left\{f(n)(-1)^m + f(0)\right\} \quad +16\sin^4\frac{m\pi}{2n}\sum_{x=1}^{n-1} f(x)\cos\frac{m\pi}{n}x\,. \tag{14}$$

The same finite integration of $f(x+1)-f(x-1)$ are related to the finite sine or cosine transforms of $f(x)$ as follows,

$$\mathop{S}_{1}^{n}\left(\Delta f(x)+\Delta f(x-1)\right)\cdot\sin\frac{m\pi x}{n}\Delta x$$

$$= \left(f(x)+f(x-1)\right)\sin\frac{m\pi}{n}x\Big]_1^n - 2\sin\frac{m\pi}{2n}$$

$$\times \mathop{S}_{1}^{n}\left(f(x+1)+f(x)\right)\cos\frac{m\pi}{n}\left(x+\frac{1}{2}\right)\Delta x\,, \tag{15}$$

which can evidently be tranformed into the series expression :

$$\sum_{1}^{n-1}\left(\Delta f(x)+\Delta f(x-1)\right)\sin\frac{m\pi x}{n}$$

$$= -\left(f(1)+f(0)\right)\sin\frac{m\pi}{n} - \sum_{x=1}^{n-1}\left(f(x+1)\sin\frac{m\pi}{n}(x+1) - f(x)\sin\frac{m\pi}{n}x\right)$$

$$+\sum_{x=1}^{n-1}\left(f(x+1)\sin\frac{m\pi}{n}x - f(x)\sin\frac{m\pi}{n}(x+1)\right).$$

The above equation reduces to the formula :

$$\sum_{x=1}^{n-1}\left(f(x+1)-f(x-1)\right)\sin\frac{m\pi}{n}x$$

$$= -\sin\frac{m\pi}{n}\left\{f(n)(-1)^m + f(0)\right\} - 2\sin\frac{m\pi}{n}\sum_{x=1}^{n-1} f(x)\cos\frac{m\pi}{n}x\,. \tag{16}$$

Replacing $\sin m\pi x/n$ by $\cos m\pi x/n$ in (15) leads to

$$\mathop{S}_{1}^{n}\left(\Delta f(x)+\Delta f(x-1)\right)\cos\frac{m\pi}{n}x\Delta x$$

$$= \left\{f(x)+f(x-1)\right\}\cos\frac{m\pi}{n}x\Big]_1^n + 2\sin\frac{m\pi}{2n}$$

$$\times \mathop{S}_{1}^{n}\left(f(x+1)+f(x)\right)\sin\frac{m\pi}{n}\left(x+\frac{1}{2}\right)\Delta x$$

which can be written in

$$\sum_{x=1}^{n-1}\left\{\Delta f(x)+\Delta f(x+1)\right\}\cos\frac{m\pi}{n}x$$
$$=\left\{f(n)+f(n-1)\right\}(-1)^m-\left\{f(1)+f(0)\right\}\cos\frac{m\pi}{n}$$
$$-\sum_{x=1}^{n-1}f(x+1)\cos\frac{m\pi}{n}(x+1)+\sum_{x=1}^{n-1}f(x)\cos\frac{m\pi}{n}x$$
$$+\sum_{x=1}^{n-1}f(x+1)\cos\frac{m\pi}{n}x-\sum_{x=1}^{n-1}f(x)\cos\frac{m\pi}{n}(x+1)\,.$$

It follows from the above that

$$\sum_{x=1}^{n-1}\left\{f(x+1)+f(x-1)\right\}\cos\frac{m\pi}{n}x$$
$$=-\left\{\Delta f(n-1)(-1)^m+\Delta f(0)\right\}+\left(1+\cos\frac{m\pi}{n}\right)\left\{f(n)(-1)^m-f(0)\right\}$$
$$+2\sin\frac{m\pi}{n}\sum_{x=1}^{n-1}f(x)\sin\frac{m\pi}{n}x \tag{17}$$

which, in particular case of $m=0$ and $m=n$, yields

$$\sum_{x=1}^{n-1}\left\{f(x+1)-f(x-1)\right\}=f(n)+f(n-1)-f(1)-f(0)$$

and

$$\sum_{x=1}^{n-1}\left\{f(x+1)-f(x-1)\right\}=-\Delta f(n-1)(-1)^n+\Delta f(0)$$

respectively.

3. Finite Transformation Method for Solving a Simultaneous Finite Difference Equations

Let us introduce here a couple of finite difference equations

$$\Delta^2\phi(x-1)+a\phi(x)-b\left\{\psi(x+1)-\psi(x-1)\right\}=p_x\,, \tag{18}$$

$$\Delta^2\psi(x-1)+c\psi(x)+e\left\{\phi(x+1)-\phi(x-1)\right\}=0\,, \tag{19}$$

of which boundary conditions satisfy

$$\phi(n)=\phi(0)=0\,,$$
$$\Delta\psi(n-1)-\frac{c}{2}\psi(n)-e\Delta\phi(n-1)=0\,, \tag{20}$$
$$-\Delta\psi(0)-\frac{c}{2}\psi(0)-e\Delta\phi(0)=0\,.$$

Supposing $\psi(x)$ denote the slope of joint x and $\phi(x)$, its deflection, the expressions (18), (19), and (20) can easily be translated into the equilibrium equations appropriate to the bending problem of the continuous beam elastically fixed at each joint.

The finite sine transform of (18) and the finite cosine transform of (19) by means of the finite integration by x from 1 to $n-1$, are written in

$$\Phi(m)(a-D^2)-2b\sin\frac{m\pi}{n}\cdot\Psi(m)-\sin\frac{m\pi}{n}\left\{(-1)^m\phi(n)-\phi(0)\right\}=\bar{P}(m), \tag{21}$$

$$\Psi(m)(c-D^2)-2e\sin\frac{m\pi}{n}\cdot\Phi(m)+\left\{\Delta\phi(n-1)(-1)^m-\Delta\phi(0)\right\}$$
$$-\frac{c}{2}\left\{\phi(n)(-1)^m+\phi(0)\right\}-e\left\{\Delta\phi(n-1)(-1)^m+\Delta\phi(0)\right\}=0, \tag{22}$$

which and the conditions (20) yield the results

$$\Phi(m)=\frac{\bar{P}(m)(c-D^2)}{(a-D^2)(c-D^2)-4be\sin^2\frac{m\pi}{n}} \tag{23}$$

$$\Psi(m)=\frac{2b\bar{P}(m)\sin\frac{m\pi}{n}}{(a-D^2)(c-D^2)-4be\sin^2\frac{m\pi}{n}} \tag{24}$$

where

$$\Phi(m)=\frac{2}{n}\sum_{x=1}^{n-1}\phi(x)\sin\frac{m\pi x}{n}, \qquad \bar{P}(m)=\frac{2}{n}\sum_{x=1}^{n-1}P_x\sin\frac{m\pi x}{n},$$

$$\Psi(m)=\frac{2}{n}\left\{\frac{1}{2}\phi(n)(-1)^m+\frac{1}{2}\phi(0)+\sum_{x=1}^{n-1}\phi(x)\cos\frac{m\pi x}{n}\right\},$$

$$D^2=2\left(1-\cos\frac{m\pi}{n}\right), \qquad 1\leqq x:\ \text{integer}\leqq n-1\,.$$

The substitution of m for 0 and n into (22) leads to

$$c\Psi(0)+\left(\Delta\phi(n-1)-\Delta\phi(0)\right)-\frac{c}{2}\left\{\phi(n)+\phi(0)\right\}-e\left\{\Delta\phi(n-1)+\Delta\phi(0)\right\}=0,$$

$$c\Psi(n)+\left(\Delta\phi(n-1)(-1)^n-\Delta\phi(0)\right)-\frac{c}{2}\left\{\phi(n)(-1)^n+\phi(0)\right\}$$
$$-e\left\{\Delta\phi(n-1)(-1)^n+\Delta\phi(0)\right\}=0,$$

to which application of (20) yields

$$\Psi(0)=\Psi(n)=0\,.$$

The inversion formulas, therefore, give the solution as follows:

$$\phi(x)=\frac{2}{n}\sum_{m=1}^{n-1}\frac{\bar{P}(m)(c-D^2)}{(a-D^2)(c-D^2)-4be\sin^2\frac{m\pi}{n}}\cdot\sin\frac{m\pi x}{n},$$

$$\phi(x)=\frac{2}{n}\sum_{m=1}^{n-1}\frac{2b\sin\frac{m\pi}{n}\bar{P}(m)}{(a-D^2)(c-D^2)-4be\sin^2\frac{m\pi}{n}}\cdot\cos\frac{m\pi x}{n},$$

which, by virtue of the relation between the inversion function and its finite transformation with respect to the finite integration[2], may be expressed by the closed form.

(Received Apr. 30, 1966)

References

1) W. Klemp: Ein neues Verfahren für Tragerrostberechnung, Beton und Stahlbetonbau, 21. Jahrg. Heft 1, (1956), S. 15-19.

2) S. G. Nomachi: On Finite Sine Series with Respect to Finite Differences, the Memoirs of Muroran Inst. of Tech., **5**, 1 (1965), p. 187-202.